Hydrodynamic stability at high Reynolds number and Transition threshold problem

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Reynolds experiment in 1883

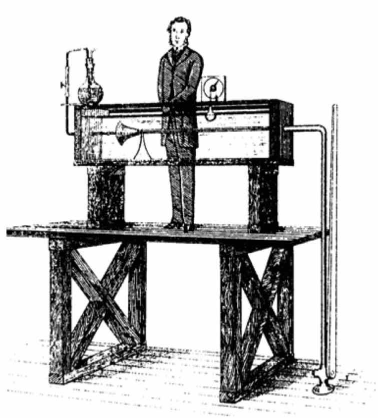
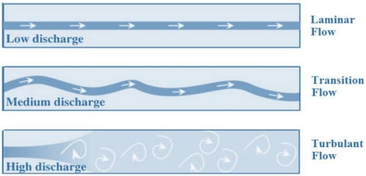


Figure: Reynolds experiment

The state of the flow is determined by a dimensionless constant called **Reynolds number**:

$$Re = \rho LU/\mu$$

where ρ density, L diameter, U velocity, μ viscosity.



Mathematical model

Navier-Stokes equations:

$$(NS) \quad \begin{cases} \partial_t v - \nu \Delta v + v \cdot \nabla v + \nabla P = 0 \quad \text{in} \quad \mathbb{R}_+ \times \Omega, \\ \nabla \cdot v = 0 \quad \text{in} \quad \mathbb{R}_+ \times \Omega, \\ v = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

where v is the velocity, P is the pressure, and $\nu = Re^{-1}$ is the viscosity coefficient.

Fluid domain Ω:

- Channel domain:

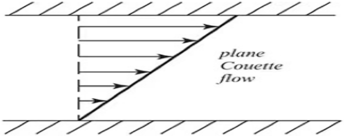
$$\Omega = \Big\{(x, y, z) : x \in \mathbb{T} \text{ or } \mathbb{R},\ z \in \mathbb{T},\ y \in (-1, 1)\Big\}.$$

- Pipe domain:

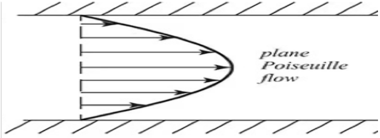
$$\Omega = \Big\{x = (x, y, z) : r = \sqrt{x^2 + y^2} < 1,\ z \in \mathbb{T} \text{ or } z \in \mathbb{R}\Big\}.$$

Examples of laminar flow

- Plane Couette flow: $(y, 0, 0)$



- Plane Poiseuille flow:$(1 - y^2, 0, 0)$



- Pipe Poiseuille flow:$(0, 0, 1 - r^2)$

Our task is to study the stability and instability of laminar flows at high Reynolds number.

Eigenvalue analysis

Consider the linearized NS system around the laminar flow:

$$\partial_t u - \mathcal{L}_\nu u = 0.$$

Seek the solution $e^{\lambda t} U$, where U solves the eigenvalue problem:

$$\mathcal{L}_\nu U = \lambda U.$$

The system is stable if Re$\lambda < 0$ and unstable if Re$\lambda > 0$.

- Plane Couette flow: stable for any Reynolds number(*Romanov, Funk. Anal. 1973*).
- Plane Poiseuille flow: stable for Reynolds number less than 5772(*Orszag, JFM 1971*).
- Pipe Poiseuille flow: stable at high Reynolds number(*Chen-Wei-Zhang, CPAM online*).

  Conjecture: *Pipe Poiseuille flow is stable for any Reynolds number*.

Subcritical transition

Experiment and numerical simulation indicated that these flows could transit to turbulence when Reynolds number exceeds a certain critical number. For example,

- Plane Couette flow: transition at $Re = 350$.
- Plane Poiseuille flow: transition at $Re = 1000$.

This transition number is much smaller than the critical number predicted by eigenvalue analysis. This kind of transition is called **subcritical transition**(Sommerfeld paradox). Understanding this transition mechanism is of great interest in fluid mechanics.

Transition threshold problem

To shed some light on the transition mechanism, Trefethen et al(*Science 1993*) proposed **Transition threshold problem**:

How much disturbance will lead to the instability of the flow and the dependence of disturbance on Reynolds number?

Mathematical formulation(*Bedrossian-Germain-Masmoudi, BAMS 2019*):

Given a norm $\|\cdot\|_X$, *find a* $\beta = \beta(X)$ *such that*

$$\|u_0\|_X \ll Re^{-\beta} \Longrightarrow \textit{Stability},$$
$$\|u_0\|_X \gg Re^{-\beta} \Longrightarrow \textit{Instability}.$$

The exponent β is referred to as the **transition threshold.**

Numerics and asymptotic analysis results

The following table shows numerical result (*Lundbladh et al, Transition, Turbulence and Combustion 1994*) and asymptotic analysis result (*Chapman, JFM 2002*) for Couette flow and Poiseuille flow:

Laminar flow	Perturbation	Numerical analysis	Asymptotic analysis
Couette flow	streamwise perturbation	$\beta = 1$	$\beta = 1$
	oblique perturbation	$\beta = \frac{5}{4}$	$\beta = 1$
Poiseuille flow	streamwise perturbation	$\beta = \frac{7}{4}$	$\beta = \frac{3}{2}$
	oblique perturbation	$\beta = \frac{7}{4}$	$\beta = \frac{5}{4}$

3-D Couette flow in $\Omega = \mathbb{T} \times \mathbb{R} \times \mathbb{T}$:

- If X is Gevrey class, then $\beta \le 1$(*Bedrossian-Germain-Masmoudi, Mem AMS 2021*).
- If $X = H^N$, then $\beta \le \frac{3}{2}$(*Bedrossian-Germain-Masmoudi, Ann Math 2017*).
- If $X = H^2$, then $\beta \le 1$(*Wei-Zhang, CPAM 2021*).

3-D Couette flow in $\Omega = \mathbb{T} \times [-1, 1] \times \mathbb{T}$:

- If $X = H^2$, then $\beta \le 1$(*Chen-Wei-Zhang, Mem AMS in press*).

3-D Kolmogorov flow in $\Omega = \mathbb{T}_{2\pi\delta} \times \mathbb{T} \times \mathbb{T}, \delta < 1$:

- If $X = H^2$, then $\beta \le \frac{7}{4}$(*Li-Wei-Zhang, CPAM 2020*).

Key factors influencing the threshold

The following factors play a crucial role in determining the transition threshold:

- **3-D lift-up**: instability mechanism.
- **Boundary layer**: wall modes and also central modes for the Poiseuille flow.
- **Inviscid damping**: vorticity mixing
- **Enhanced dissipation**: vorticity mixing
- **Null structure of nonlinear terms**

The main difficulty of this problem is to reveal how these bad effects and good effects influence nonlinear stability via complex nonlinear interactions.

Linear inviscid damping

Consider the linearized 2-D Euler equation around shear flow $(U(y), 0)$ in a finite channel:

$$\partial_t \omega + \mathcal{L}\omega = 0,$$

where $\omega = \partial_x v^2 - \partial_y v^1$ is the vorticity and

$$\mathcal{L} = U(y)\partial_x + U''(y)\partial_x(-\Delta)^{-1}.$$

In particular, for the Couette flow, we have

$$\partial_t \omega + y\partial_x \omega = 0.$$

In 1907, Orr found that the velocity could decay to zero as $t \to \infty$. This phenomenon is so called **inviscid damping**, which is similar to **Landau damping** in plasma physics.

Linear inviscid damping: monotone flow

Theorem. (*Wei-Zhang-Zhao, CPAM 2018*)

Let $U(y) \in C^4([0,1])$ be a monotone function. Assume that the linearized operator $\mathcal{L}$ has no embedding eigenvalues. If $\int_{\mathbb{T}} \omega_0(x,y)dx = 0$ and $P_d\omega_0 = 0$, where P_d is the spectral projection to $\sigma_d(\mathcal{L})$, then it holds that

1. if $\omega_0(x,y) \in H_x^{-1}H_y^1$, then

$$\|v(t)\|_{L^2} \le C\langle t\rangle^{-1}\|\omega_0\|_{H_x^{-1}H_y^1};$$

2. if $\omega_0(x,y) \in H_x^{-1}H_y^2$, then

$$\|v^2(t)\|_{L^2} \le C\langle t\rangle^{-2}\|\omega_0\|_{H_x^{-1}H_y^2}.$$

Remark. *The spectral assumption holds automatically when the flow has no inflection points.*

Linear inviscid damping: symmetric flow

Consider a class of symmetric flow(Poiseuille flow y^2):

$U(y) = U(-y)$, $U'(y) > 0$ for $y > 0$, $U'(0) = 0$ and $U''(0) > 0$.

Theorem.(*Wei-Zhang-Zhao, Ann PDE 2019*)

Assume that $\mathcal{L}$ has no embedding eigenvalues.
If $\int_{\mathbb{T}} \omega_0(x, y)dx = 0$ and $P_d\omega_0 = 0$, then it holds that

$$\|v(t)\|_{L^2} \leq C\langle t\rangle^{-1}\|\omega_0\|_{H_x^{-1/2}H_y^1},$$

$$\|v^2(t)\|_{L^2} \leq C\langle t\rangle^{-2}\|\omega_0\|_{H_x^{1/2}H_y^2}.$$

Remark. *For general non-monotone shear flows, we can prove linear inviscid damping in the sense of* $\|\widehat{v}(\cdot, \alpha, \cdot)\|_{L^2_{t,y}} \leq C$ *for any* $\alpha \neq 0$.

Linear inviscid damping: Kolmogorov flow

Consider the Kolmogorov flow $U(y) = \sin y$ or $\cos y$ in $\Omega = \left\{(x,y) : x \in \mathbb{T}_{2\pi\delta}, y \in \mathbb{T}\right\}$ with $\delta < 1$.

Theorem.(*Wei-Zhang-Zhao, Adv Math 2020*)

If $\int_{\mathbb{T}} \omega_0(x,y)dx = 0$, then it holds

$$\|v(t)\|_{L^2} \le C\langle t\rangle^{-1}\|\omega_0\|_{H_x^{-1/2}H_y^1},$$

$$\|v^2(t)\|_{L^2} \le C\langle t\rangle^{-2}\|\omega_0\|_{H_x^{1/2}H_y^2}.$$

Remark. *For non-monotone shear flows, except the vorticity mixing, a new dynamical phenomenon called* **vorticity depletion**(*Bouchet-Morita, Physics D 2010*) *has a crucial effect:*

$$\lim_{t\to\infty} \omega(t,x,y_c) = 0 \quad \textit{when } U'(y_c) = 0.$$

See [Wei-Zhang-Zhao, Ann PDE 2019] for rigorous proof.

Linear inviscid damping: methods of the proof

The key ingredient of the proof is to solve the inhomogeneous Rayleigh equation: for $c = c_r \pm i\epsilon, c_r \in \mathrm{Ran}U, \epsilon > 0$

$$(U - c)(\Phi'' - \alpha^2\Phi) - U''\Phi = F, \quad \Phi(-1) = \Phi(1) = 0.$$

- **Direct method**(*Wei-Zhang-Zhao, CPAM 2018*): construct linearly independent solutions for homogeneous Rayleigh equation.
- **Compactness method**(*Wei-Zhang-Zhao, Ann PDE 2019*):

$$\|\Phi\|_{H^1(-1,1)} \leq C\|F\|_{H^1(-1,1)}.$$

  Contradiction-compactness argument and blow-up analysis near critical points of $U(y)$.
- **Vector field method**(*Wei-Zhang-Zhu, CMP 2020*):

$$[\partial_t + U\partial_x, X] = 0, \quad X = \frac{1}{U'}\partial_y + t\partial_x.$$

Nonlinear inviscid damping

Nonlinear inviscid damping in Gevrey class 2:

- Couette flow(*Bedrossian-Masmoudi, Publ Math IHES 2015*)
- Stable monotone shear flow(*Ionescu-Jia, arXiv 2020 Acta Math*)
- Stable monotone shear flow(*Masmoudi-Zhao, arXiv 2020*)

Negative results:

- Existence of steady non-shear solution near Couette flow in Sobolev space(*Lin-Zeng, ARMA 2011*).
- Nonlinear instability in Gevrey class 2+(*Deng-Masmoudi, arXiv 2018 CPAM*).

Linear enhanced dissipation

The linearized 2-D NS equation around Couette flow:

$$\partial_t \omega - \nu \Delta \omega + y \partial_x \omega = 0, \quad \omega(0) = \omega_0.$$

If $x \in \mathbb{T}$ and $y \in \mathbb{R}$, then we have

$$\widehat{\omega}(t, k, \eta) = e^{-\nu \int_0^t (k^2 + (\eta - k\tau)^2) d\tau} \widehat{\omega}_0(k, \eta).$$

Due to $\int_0^t (k^2 + (\eta - k\tau)^2) d\tau \geq c k^2 t^3$, if $\int_{\mathbb{T}} \omega_0(x, y) dx = 0$, then

$$\|\omega(t)\|_{L^2} \leq C e^{-c\nu^{\frac{1}{3}} t} \|\omega_0\|_{L^2}.$$

This decay rate $\nu^{\frac{1}{3}}$ is much bigger than the diffusion rate ν. This phenomenon is so called **enhanced dissipation**.

LInear enhanced dissipation

The linearized 2-D NS equation around Kolmogorov flow:

$$\partial_t \omega + \mathcal{L}_\nu(t)\omega = 0, \quad \omega(0) = \omega_0,$$

where

$$\mathcal{L}_\nu(t) = -\nu\Delta + \cos y \partial_x (1 + \Delta^{-1}).$$

Beck and Wayne's conjecture(*Proc Roy Soc Edinburgh Sect A 2013*):

If $\int_{\mathbb{T}_{2\pi\delta}} \omega_0(x, y) dx = 0$ and $\delta < 1$, then it holds that

$$\|\omega(t)\|_{L^2} \leq Ce^{-c\nu^{\frac{1}{2}} t} \|\omega_0\|_{L^2}.$$

Linear enhanced dissipation

Three approaches solving Beck and Wayne's conjecture:

1. **Wave operator method**(*Wei-Zhao-Zhang, Adv Math 2020*): construct a wave operator D so that

$$D\cos y\Big(1+(\partial_y^2-\alpha^2)^{-1}\Big)\omega=\cos yD\omega.$$

2. **Resolvent estimate method** (*Li-Wei-Zhang, CPAM 2020*):

$$\|(L_\nu-i\lambda)w\|_{L^2}\geq C\nu^{\frac{1}{2}}|\beta|^{\frac{1}{2}}(1-\beta^{-2})\|w\|_{L^2},$$

where $|\beta|>1$ and

$$L_\nu w=-\nu\partial_y^2 w+i\beta\cos y\Big(w+\varphi\Big),\quad (\partial_y^2-\beta^2)\varphi=w.$$

Enhanced dissipation estimate by Gearhart-Prüss type lemma for m-accretive operator(*Wei, SCM 2020*).

3. **Hypocoercivity method**(*Wei-Zhang, SCM 2019*):

$$\partial_t\omega + \nu(-\partial_y^2 + \beta^2)\omega - i\beta B\omega = 0, \quad B = \cos y\Big(1 + (\partial_y^2 - \beta^2)^{-1}\Big).$$

The key idea is to introduce the energy functional

$$\Phi(t) = E_0(t) + \alpha_0\nu t E_1(t) + \beta_0\nu t^2\mathcal{E}_1(t) + \gamma_0\nu t^3\mathcal{E}_2(t),$$

where $\alpha_0, \beta_0, \gamma_0$ are suitable positive constants and

$$E_0(t) = \|\omega(t)\|_*^2, \quad E_1(t) = \|\partial_y\omega(t)\|_*^2, \quad \mathcal{E}_2(t) = \|\omega(t)\|_*^2 - \|B\omega(t)\|_*^2,$$
$$\mathcal{E}_1(t) = \mathrm{Re}\langle\partial_y\omega(t), iC\omega(t)\rangle_*, \quad C = -[\partial_y, B],$$

with new inner product $\langle u, w\rangle_* = \langle u, w - (\beta^2 - \partial_y^2)^{-1}w\rangle$.

Remark. *An independent proof (Ibrahim-Maekawa-Masmoudi, Ann PDE 2019); Compactness and resolvent method for general monotone flows(Chen-Wei-Zhang, preprint).*

Chapman toy model

Consider a toy model introduced by Chapman(*JFM 2002*):

$$\begin{cases} \dfrac{d\psi_1}{dt} + \epsilon\psi_1 = \phi_2^2, \\ \dfrac{d\phi_1}{dt} + \epsilon\phi_1 - \psi_1 = 0, \\ \dfrac{d\psi_2}{dt} + \delta\psi_2 = \phi_1\phi_2, \\ \dfrac{d\phi_2}{dt} + \delta\phi_2 - \psi_2 = 0. \end{cases}$$

- In physics, ψ_1: streamwise vorticity, ϕ_1: streamwise streak, (ψ_2, ϕ_2): oblique modes.
- Enhanced dissipation: $0 < \epsilon \ll \delta \ll 1$(Couette flow $\delta = \epsilon^{\frac{1}{3}}$).
- ϕ_1 undergoes a transient growth due to lift-up term ψ_1:

$$|\phi_1(t)| \lesssim te^{-\epsilon t} \lesssim \epsilon^{-1} \quad \text{for} \quad t \leq \epsilon^{-1}.$$

Chapman toy model: scaling analysis

We introduce the following scaling:

$$t = \epsilon^{-1}\hat{t}, \quad \phi_1 = \delta^2\hat{\phi}_1, \quad \psi_1 = \epsilon\delta^2\hat{\psi}_1, \quad \phi_2 = \epsilon\delta\hat{\phi}_2, \quad \psi_2 = \epsilon\delta^2\hat{\psi}_2.$$

The rescaled toy system takes as follows

$$\begin{cases} \dfrac{d\hat{\psi}_1}{d\hat{t}} + \hat{\psi}_1 = \hat{\phi}_2^2, \\ \dfrac{d\hat{\phi}_1}{d\hat{t}} + \hat{\phi}_1 - \hat{\psi}_1 = 0, \\ \dfrac{\epsilon}{\delta}\dfrac{d\hat{\psi}_2}{d\hat{t}} + \hat{\psi}_2 = \hat{\phi}_1\hat{\phi}_2, \\ \dfrac{\epsilon}{\delta}\dfrac{d\hat{\phi}_2}{d\hat{t}} + \hat{\phi}_2 - \hat{\psi}_2 = 0. \end{cases}$$

We rewrite the system of oblique modes as follows

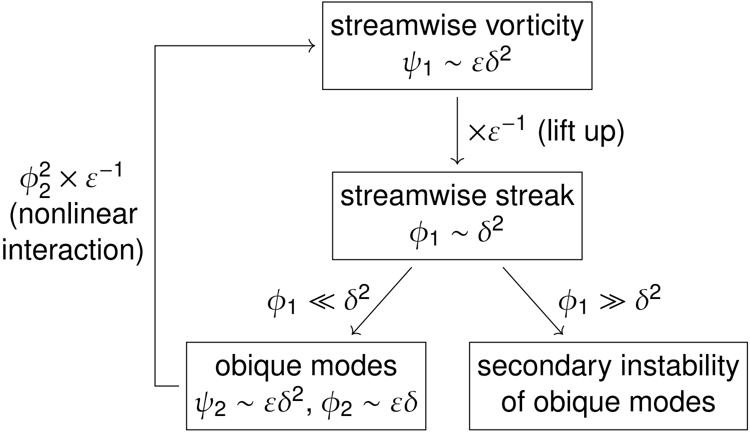
$$\partial_t \begin{pmatrix} \hat{\psi}_2 \\ \hat{\phi}_2 \end{pmatrix} = \frac{\delta}{\epsilon} \begin{pmatrix} -1 & \hat{\phi}_1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \hat{\psi}_2 \\ \hat{\phi}_2 \end{pmatrix}.$$

Secondary instability of oblique modes:

- if $\hat{\phi}_1 > 1$, the system has an unstable eigenvalue.
- if $\hat{\phi}_1 < 1$, the system is stable and the oblique modes will rapidly decay to zero due to $\frac{\delta}{\epsilon} \gg 1$.

This scaling analysis indicates that the transition amplitude of this toy model is $\epsilon\delta^2$.

Transition route: *Streamwise vorticity*→ *Streamwise streak*→ *Secondary instability of oblique modes*.



Perturbation NS system

We introduce the perturbation around Couette flow $(y, 0, 0)$:

$$v = (y, 0, 0) + u, \quad \nabla \times v = (0, 0, -1) + \omega.$$

Consider the coupled system of $(\Delta u^2, \omega^2)$:

$$\begin{cases} \partial_t(\Delta u^2) - \nu\Delta^2 u^2 + y\partial_x \Delta u^2 = F_1, \\ \partial_t \omega^2 - \nu\Delta\omega^2 + y\partial_x\omega^2 + \partial_z u^2 = F_2, \\ u^2(t, x, \pm 1, z) = \partial_y u^2(t, x, \pm 1, z) = \omega^2(x, \pm 1, z) = 0, \end{cases}$$

where

$$F_1 = -(\partial_x^2 + \partial_z^2)(u \cdot \nabla u^2) + \partial_y\Big[\partial_x(u \cdot \nabla u^1) + \partial_z(u \cdot \nabla u^3)\Big],$$
$$F_2 = -\partial_z(u \cdot \nabla u^1) + \partial_x(u \cdot \nabla u^3).$$

Perturbation NS system

Let $k^2=\alpha^2+\beta^2$. Then $(u^2_{\alpha\beta},\omega^2_{\alpha\beta})$ satisfies

$$\begin{cases}\Big(\partial_t+i\alpha y+\nu(-\partial_y^2+k^2)\Big)(-\partial_y^2+k^2)u^2_{\alpha\beta}=F_1^{\alpha,\beta},\\ \Big(\partial_t+i\alpha y+\nu(-\partial_y^2+k^2)\Big)\omega^2_{\alpha\beta}+i\beta u^2_{\alpha\beta}=F_2^{\alpha,\beta}.\end{cases}$$

When $k\neq 0$, nonlinear terms $F_1^{\alpha\beta}$ and $F_2^{\alpha\beta}$ behave as

$$\begin{aligned}F_1^{\alpha\beta}\sim&\Big\{(u^2u^2_{yyy}+u^2_yu^2_{yy})+(\omega^2u^2_{yy}+u^2\omega^2_{yy}+u^2_y\omega^2_y)\\ &+(u^2u^2_y+\omega^2\omega^2_y+u^2\omega^2)\Big\}_{\alpha\beta},\\ F_2^{\alpha\beta}\sim&\Big(\omega^2\omega^2+u^2u^2_{yy}+u^2\omega^2_y+u^2_y\omega^2\Big)_{\alpha\beta}.\end{aligned}$$

For $F_1^{\alpha\beta}$, the middle part is the worst nonlinear interaction.

Secondary instability of wall mode

Consider the linearized NS system

$$\begin{cases}\Big(\partial_t + i\alpha y + \nu(-\partial_y^2 + k^2)\Big)(-\partial_y^2 + k^2)u_{\alpha\beta}^2 = 0,\\ \Big(\partial_t + i\alpha y + \nu(-\partial_y^2 + k^2)\Big)\omega_{\alpha\beta}^2 + i\beta u_{\alpha\beta}^2 = 0.\end{cases}$$

Seek the solution $u_{\alpha\beta}^2 = e^{-i\alpha\lambda t}v, \omega_{\alpha\beta}^2 = e^{-i\alpha\lambda t}\eta$. Then (v, η) solves the following eigenvalue system($R = \nu^{-1}$):

$$\begin{cases}i(\lambda - y)(\partial_y^2 - k^2)v + \dfrac{1}{R\alpha}(\partial_y^2 - k^2)^2 v = 0,\\ -i(\lambda - y)\eta - \dfrac{1}{R\alpha}(\partial_y^2 - k^2)\eta + \dfrac{i\beta}{\alpha}v = 0.\end{cases}$$

Question: *how large the perturbation induced by nonlinear interactions could excite unstable eigenvalues?*

Secondary instability of wall mode

Following Chapman's asymptotic analysis, we consider a class of eigenvalues with imaginary part of order $(\alpha R)^{-\frac{1}{3}}$. This class is called **wall modes**, whose eigenfunctions are localized in a region of order $(\alpha R)^{-\frac{1}{3}}$ near the boundary.

To excite unstable eigenvalues, we introduce a perturbation F into the eigenvalue equation of v:

$$i(\lambda - y)(\partial_y^2 - k^2)v + \frac{1}{R\alpha}(\partial_y^2 - k^2)^2 v = F.$$

For wall modes, in the inner layer, we have

$$\begin{aligned} &i(\lambda - y)(\partial_y^2 - k^2)v + \frac{1}{R\alpha}(\partial_y^2 - k^2)^2 v \\ &\quad \sim (\alpha R)^{-\frac{1}{3}}(\alpha R)^{\frac{2}{3}} v + (R\alpha)^{-1}(R\alpha)^{\frac{4}{3}} v \sim (\alpha R)^{\frac{1}{3}} v. \end{aligned}$$

The worst nonlinear interaction:

$$F \sim \bar{\eta} v_{yy} \sim \bar{\eta} (\alpha R)^{\frac{2}{3}} v, \quad \bar{\eta} \sim \omega_{0\beta}^2.$$

To excite unstable eigenvalues, F should have the same scale as the left hand side. So, $\bar{\eta}$ should be of order $(\alpha R)^{-\frac{1}{3}}$ in the inner layer, which requires $\bar{\eta} \sim O(1)$ in the outer layer due to $\partial_y \bar{\eta} \sim \bar{\eta}$ and $\bar{\eta}(-1) = 0$.
On the other hand, due to the lift-up effect, we have

$$R\bar{v} \sim \bar{\eta} = O(1) \Longrightarrow \bar{v} \sim R^{-1}.$$

This scaling analysis suggests that the transition threshold $\beta = 1$ for the Couette flow.

Theorem.(*Chen-Wei-Zhang, Mem AMS in press*)

There exist constants $c, C > 0$ independent of ν so that if $\|u_0\|_{H^2} \le c_0\nu$, then it holds

1. Uniform bounds of streamwise modes:

$$\|\bar{u}^1(t)\|_{H^2} + \|\bar{u}^1(t)\|_{L^\infty} \le C\nu^{-1}\min(\nu t + \nu^{2/3}, e^{-\nu t})\|u_0\|_{H^2},$$
$$\|\bar{u}^2(t)\|_{H^2} + \|\bar{u}^3(t)\|_{H^1} + \|(\bar{u}^2, \bar{u}^3)(t)\|_{L^\infty} \le Ce^{-\nu t}\|u_0\|_{H^2}.$$

2. Uniform bounds of oblique modes:

$$\|(\partial_x, \partial_z)\partial_x u_{\neq}(t)\|_{L^2} + \nu^{1/6}\|(u^1_{\neq}, u^3_{\neq})(t)\|_{L^\infty} \le Ce^{-c\nu^{1/3}t}\|u_0\|_{H^2},$$
$$\|(\partial_x, \partial_z)\nabla_{x,y,z}u^2_{\neq}(t)\|_{L^2} + \|u^2_{\neq}(t)\|_{L^\infty} \le Ce^{-c\nu^{1/3}t}\|u_0\|_{H^2}.$$

Here $\bar{u} = \int_{\mathbb{T}} u dx$ and $u_{\neq} = u - \bar{u}$.

To control Δu^2 and ω^2, we need to consider the linearized NS system with Navier-slip boundary condition:

$$\begin{cases} \partial_t \omega - \nu(\partial_y^2 - k^2)\omega + i\alpha y\omega = i\alpha f_1 + \partial_y f_2 + i\beta f_3, \\ \omega|_{y=\pm 1} = 0, \quad \omega|_{t=0} = \omega_{in}, \end{cases} \tag{1}$$

and with non-slip boundary condition:

$$\begin{cases} \partial_t \omega - \nu(\partial_y^2 - k^2)\omega + i\alpha y\omega = i\alpha f_1 + \partial_y f_2 + i\beta f_3, \\ (\partial_y^2 - k^2)\varphi = \omega, \quad \partial_y \varphi|_{y=\pm 1} = \varphi|_{y=\pm 1} = 0, \\ \omega|_{t=0} = \omega_{in}. \end{cases} \tag{2}$$

Key ingredients(I): space-time estimates

For the linearized system (2), we prove that

$$
\begin{aligned}
&|\eta|\|e^{a\nu^{\frac{1}{3}}t}(\partial_y,k)\varphi\|_{L^\infty L^2}+\nu^{\frac{1}{4}}\|e^{a\nu^{\frac{1}{3}}t}\omega\|_{L^\infty L^2}+|\alpha k|^{\frac{1}{2}}\|e^{a\nu^{\frac{1}{3}}t}(\partial_y,k)\varphi\|_{L^2L^2}\\
&\quad+\nu^{\frac{3}{4}}\|e^{a\nu^{\frac{1}{3}}t}\partial_y\omega\|_{L^2L^2}+\nu^{\frac{1}{2}}|k|\|e^{a\nu^{\frac{1}{3}}t}\omega\|_{L^2L^2}\\
&\leq C\Big(|k|^{-1}\|\partial_y\omega_{in}\|_{L^2}+\|\omega_{in}\|_{L^2}\Big)+C\nu^{-\frac{1}{2}}\|e^{a\nu^{\frac{1}{3}}t}(f_1,f_2,f_3)\|_{L^2L^2}.
\end{aligned}
$$

Remarks.

- The rapid decay $e^{a\nu^{\frac{1}{3}}t}$ is due to enhanced dissipation.
- $|\alpha k|^{\frac{1}{2}}\|e^{a\nu^{\frac{1}{3}}t}(\partial_y,k)\varphi\|_{L^2L^2}$ is due to inviscid damping.
- The loss $\nu^{\frac{1}{4}}$ in front of $\|e^{a\nu^{\frac{1}{3}}t}\omega\|_{L^\infty L^2}$ is due to the boundary layer effect.
- The proof is based on resolvent estimate method developed in [*Chen-Li-Wei-Zhang, ARAM 2020*].

Key ingredients(II): exclude secondary instability

Consider the linearized NS system around the flow $\big(V(y,z),0,0\big)$, which is a small perturbation of Couette flow, i.e.,

$$\|V-y\|_{H^4} \le \varepsilon_0, \quad V(y,z)-y|_{y=\pm 1}=0,$$

with ε_0 small enough but independent of ν. We denote

$$\mathbb{A}_{\nu,V}u=\mathbb{P}\Big(\nu\Delta u - V\partial_x u - \big(\partial_y V(u^2+\kappa u^3),0,0\big)\Big),$$

here $\mathbb{P}$ is the Leray projection and $\kappa=\partial_z V/\partial_y V$. The linearized NS system takes

$$\partial_t u_{\neq} - \mathbb{A}_{\nu,V}u_{\neq}=F.$$

The key point is to exclude the existence of unstable eigenvalues of $\mathbb{A}_{\nu,V}$.

Key ingredients(II): exclude secondary instability

Motived by [*Wei-Zhang, CPAM 2021*], **the key idea** is to introduce $W = u^2 + \kappa u^3$ and $U = u^3$. The problem is reduced to solving the following OS system:

$$\begin{cases} -\nu\Delta W + i\alpha(V(y,z)-\lambda)W + (\partial_y + \kappa\partial_z)p^{L1} \\ \qquad\qquad = G_1 - \nu(\Delta\kappa)U - 2\nu\nabla\kappa\cdot\nabla U, \\ -\nu\Delta U + i\alpha(V(y,z)-\lambda)U + \partial_z p^{L1} = G_2, \\ \Delta p^{L1} = -2i\alpha\partial_y VW, \quad W = \partial_y W = U = 0 \quad \text{on } y = \pm 1, \end{cases}$$

where $\lambda \in \mathbb{R}$. It holds that

$$\begin{aligned} &\nu^{\frac{1}{3}}\Big(\|\partial_x^2 U\|_{L^2}^2 + \|\partial_x(\partial_z - \kappa\partial_y)U\|_{L^2}^2\Big) + \nu\Big(\|\nabla\partial_x^2 U\|_{L^2}^2 + \|\nabla\partial_x(\partial_z - \kappa\partial_y)U\|_{L^2}^2\Big) \\ &\quad + \nu^{\frac{1}{3}}\|\partial_x\nabla W\|_{L^2}^2 + \nu\|\partial_x\Delta W\|_{L^2}^2 + \nu^{\frac{5}{3}}\|\partial_x\Delta U\|_{L^2}^2 \\ &\le C\nu^{-1}\Big(\|\nabla G_1\|_{L^2}^2 + \|\partial_x G_2\|_{L^2}^2\Big). \end{aligned}$$

(1) Energy functional of streamwise modes.

A key decomposition: $\overline{u}^1 = \overline{u}^{1,0} + \overline{u}^{1,\neq}$ with

$$\begin{cases}(\partial_t - \nu\Delta)\bar{u}^{1,0} + \bar{u}^2 + \bar{u}^2\partial_y\bar{u}^{1,0} + \bar{u}^3\partial_z\bar{u}^{1,0} = 0,\\ (\partial_t - \nu\Delta)\bar{u}^{1,\neq} + \bar{u}^2\partial_y\bar{u}^{1,\neq} + \bar{u}^3\partial_z\bar{u}^{1,\neq} + \overline{u_{\neq}\cdot\nabla u^1_{\neq}} = 0,\\ \bar{u}^{1,0}|_{y=\pm1} = \bar{u}^{1,\neq}|_{y=\pm1} = 0,\\ \bar{u}^{1,0}|_{t=0} = 0, \quad \bar{u}^{1,\neq}|_{t=0} = \bar{u}^1(0).\end{cases}$$

The main reason making this decomposition is to avoid estimating high order derivatives of oblique modes.

Key ingredients(III): energy functional

The energy E_1 of $\bar{u}^1$ is defined by

$$E_1 = \|\bar{u}^{1,0}\|_{L^\infty H^4} + \nu^{-1}\|\partial_t \bar{u}^{1,0}\|_{L^\infty H^2} + \nu^{-\frac{1}{2}}\|\partial_t \bar{u}^{1,0}\|_{L^2 H^3}$$
$$+ \nu^{-2/3}\Big(\|\bar{u}^{1,\neq}\|_{L^\infty H^2} + \nu^{\frac{1}{2}}\|\nabla \bar{u}^{1,\neq}\|_{L^2 H^2}\Big).$$

The energy E_2 of $(\bar{u}^2, \bar{u}^3)$ is defined by

$$E_2 = \|\Delta \bar{u}^2\|_{L^\infty L^2} + \nu^{\frac{1}{2}}\|\nabla\Delta \bar{u}^2\|_{L^2 L^2} + \nu^{\frac{1}{2}}\|\Delta \bar{u}^2\|_{L^2 L^2} + \nu^{-\frac{1}{2}}\|\partial_t \nabla \bar{u}^2\|_{L^2 L^2}$$
$$+ \|\nabla \bar{u}^3\|_{L^\infty L^2} + \nu^{\frac{1}{2}}\|\Delta \bar{u}^3\|_{L^2 L^2} + \nu^{\frac{1}{2}}\|\nabla \bar{u}^3\|_{L^2 L^2} + \nu^{-\frac{1}{2}}\|\partial_t \bar{u}^3\|_{L^2 L^2}$$
$$+ \|\min((\nu^{\frac{2}{3}} + \nu t)^{\frac{1}{2}}, 1 - y^2)\Delta \bar{u}^3\|_{L^\infty L^2}$$
$$+ \nu^{-\frac{1}{2}}\|\min((\nu^{\frac{2}{3}} + \nu t)^{\frac{1}{2}}, 1 - y^2)\nabla\partial_t \bar{u}^3\|_{L^\infty L^2}$$
$$+ \nu^{\frac{1}{2}}\|\min((\nu^{\frac{2}{3}} + \nu t)^{\frac{1}{2}}, 1 - y^2)\nabla\Delta \bar{u}^3\|_{L^2 L^2}.$$

The estimates of E_1, E_2 are based on direct energy estimate for the system of streamwise modes.

(2) Energy functional of oblique modes(semilinear part):

$$E_3 = E_{3,0} + E_{3,1},$$

where

$$\begin{aligned} E_{3,0} =& \nu^{\frac{1}{2}} \| e^{2\epsilon\nu^{\frac{1}{3}}t} (\partial_x, \partial_z) \Delta u^2_{\neq} \|_{L^2L^2} + \nu^{\frac{3}{4}} \| e^{2\epsilon\nu^{\frac{1}{3}}t} \nabla \Delta u^2_{\neq} \|_{L^2L^2} \\ &+ \| e^{2\epsilon\nu^{\frac{1}{3}}t} (\partial_x, \partial_z) \nabla u^2_{\neq} \|_{L^\infty L^2} + \| e^{2\epsilon\nu^{\frac{1}{3}}t} \partial_x \nabla u^2_{\neq} \|_{L^2L^2} \\ &+ \| e^{2\epsilon\nu^{\frac{1}{3}}t} (\partial_x^2 + \partial_z^2) u^3_{\neq} \|_{L^\infty L^2} + \nu^{\frac{1}{2}} \| e^{2\epsilon\nu^{\frac{1}{3}}t} (\partial_x^2 + \partial_z^2) \nabla u^3_{\neq} \|_{L^2L^2}, \\ E_{3,1} =& \nu^{\frac{1}{3}} \Big(\| e^{2\epsilon\nu^{\frac{1}{3}}t} \nabla \omega^2_{\neq} \|_{L^\infty L^2} + \nu^{\frac{1}{2}} \| e^{2\epsilon\nu^{\frac{1}{3}}t} \Delta \omega^2_{\neq} \|_{L^2L^2} \Big). \end{aligned}$$

The estimate of E_3 is based on space-time estimates for the linearized NS system (1) and (2).

(3) Energy functional of oblique modes(quasilinear part):

$$E_4 = \nu^{1/6}\|e^{3\epsilon\nu^{1/3}t}\partial_x^2 u^2_{\neq}\|_{L^2L^2} + \nu^{1/6}\|e^{3\epsilon\nu^{1/3}t}\partial_x^2 u^3_{\neq}\|_{L^2L^2}.$$

This part is crucial to control nonlinear interactions with lift-up effect such as $\overline{u}^1\partial_x u^j_{\neq}$ and $u^j_{\neq}\partial_j\overline{u}^1 (j = 2, 3)$:

$$e^{2\epsilon\nu^{1/3}t}\left|\overline{u}^1\partial_x u^j_{\neq}\right| \lesssim (\nu t)e^{2\epsilon\nu^{1/3}t}\left|\partial_x u^j_{\neq}\right| \lesssim e^{3\epsilon\nu^{1/3}t}\left|\partial_x u^j_{\neq}\right|.$$

The estimate of E_4 is based on space-time estimates for the coupled system of (U, W).

In conclusion:

$$E_1 \sim o(1), \quad E_2 \sim o(\nu), \quad E_3 \sim o(\nu), \quad E_4 \sim o(\nu).$$

Open problems

- Plane Couette flow in $\mathbb{R} \times [-1, 1] \times \mathbb{T}$: **conjecture:** $\beta < 1$.
- Plane Poiseuille flow in $\mathbb{T} \times [-1, 1] \times \mathbb{T}$: **conjecture:** $\beta = \frac{3}{2}$.
- Kolmogorov flow in $\mathbb{T}_{2\pi\delta} \times \mathbb{T} \times \mathbb{T}, \delta < 1$: **conjecture:** $\beta = \frac{3}{2}$. Known result $\beta \leq \frac{7}{4}$(*Li-Wei-Zhang, CPAM 2020*).
- Pipe Poiseuille flow:
  - Experiment result(*Hof-Juel-Mullin, PRL 2004*): $\beta = 1$;
  - Numerical result(*Mellibovskya-Meseguer, Phys Fluids 2007*): $\beta = 1$;
  - Asymptotic analysis result: $\beta = 1$.

  Conjecture: $\beta = 1$.
- Other physical system such as Boussinesq system, MHD, Compressible NS etc.

Thanks a lot for your attention!