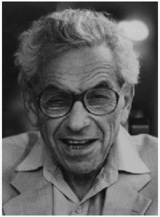
Restricted extremal problems in hypergraphs



Mathias Schacht

Universität Hamburg

When Erdős forgot to ask the general question

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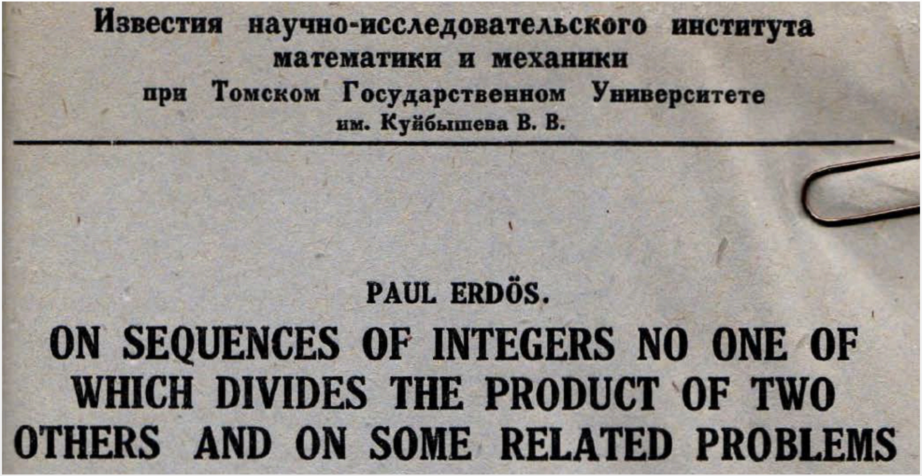

Известия научно-исследовательского института математики и механики при Томском Государственном Университете им. Куйбышева В. В.

PAUL ERDÖS.

ON SEQUENCES OF INTEGERS NO ONE OF WHICH DIVIDES THE PRODUCT OF TWO OTHERS AND ON SOME RELATED PROBLEMS

When Erdős forgot to ask the general question

The argument was really based upon the following theorem for graphs. Let $2k$ points be given. We split them into two classes each containing k of them. The points of the two classes are connected by segments such that the segments form no closed quadrilateral. Then the number of segments is less than $3k^{3/2}$. Putting $k = n^{1/3}$ we obtain our result.

Turán's problem

For a graph F and $n \in \mathbb{N}$ set

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- set of possible Turán-densities $\Pi = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{t-2}{t-1}, \dots\right\}$

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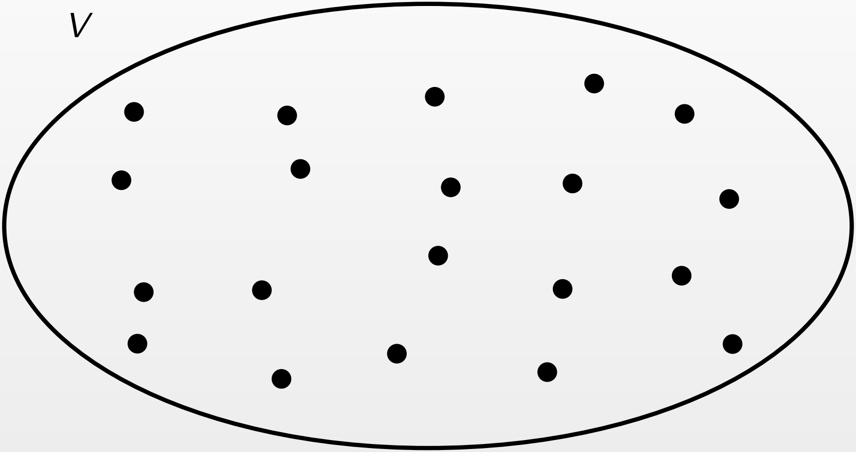

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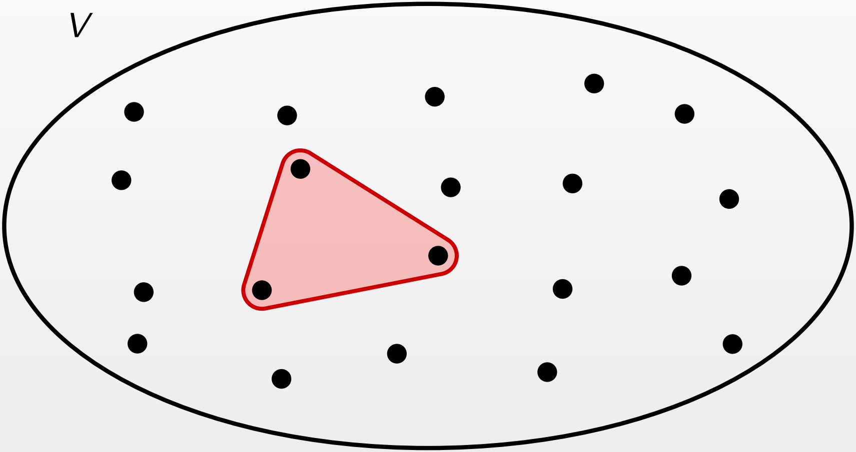

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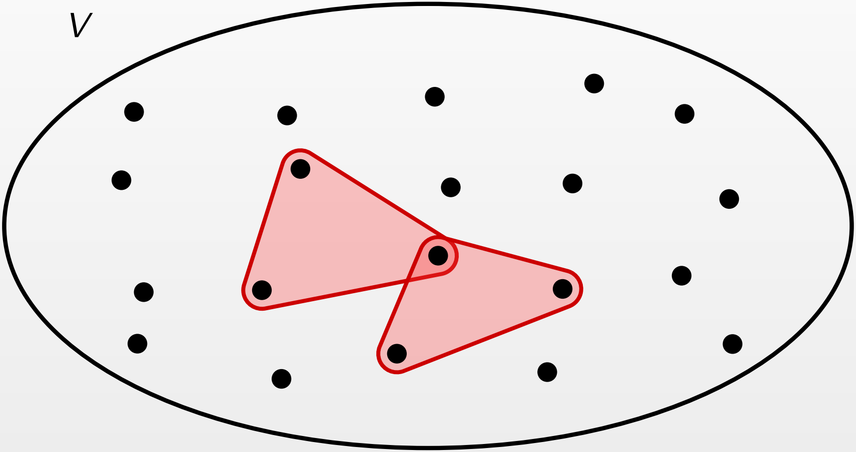

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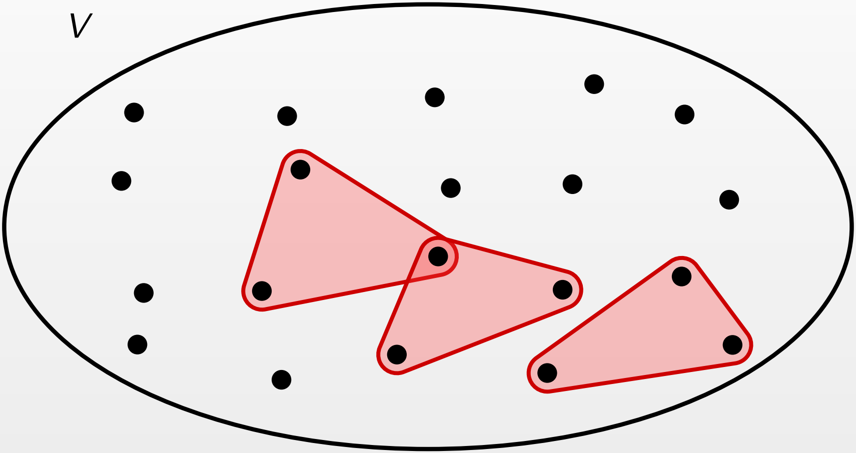

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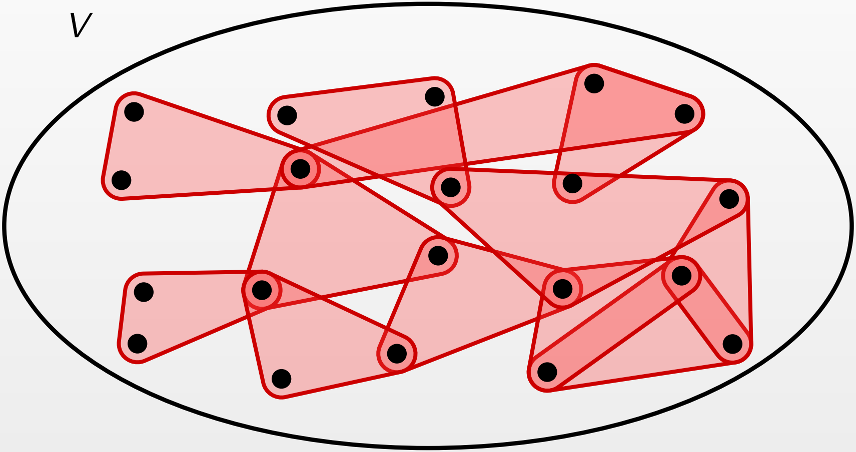

Turán-type problems for hypergraphs

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Example:

$$\pi(\mathrm{Fano}) = \tfrac{3}{4}$$

de Caen & Füredi 2000
Füredi & Simonovits 2005
Keevash & Sudakov 2005
Reiher & Bellmann 2019

Prominent open problems

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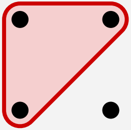

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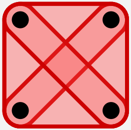

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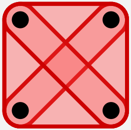

$$5/9 = 0.\overline{5} \leqslant \pi(K_4^{(3)}) \leqslant 0.5615$$

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- we shall consider different notions of uniformly dense hypergraphs

Uniformly dense graphs

Definition

A graph $G = (V, E)$ is *(ε, p)-bidense*, if

$$e_G(U, W) \geqslant p\,|U|\,|W| - \varepsilon\,|V|^2$$

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However:

$$\pi_{\mathrm{u.d.}}(F) = 0 \text{ for every graph } F$$

Weakly dense hypergraphs

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A 3-uniform hypergraph $H = (V, E)$ is *weakly* (ε, p)*-dense*, if for all sets X, Y, $Z \subseteq V$ we have

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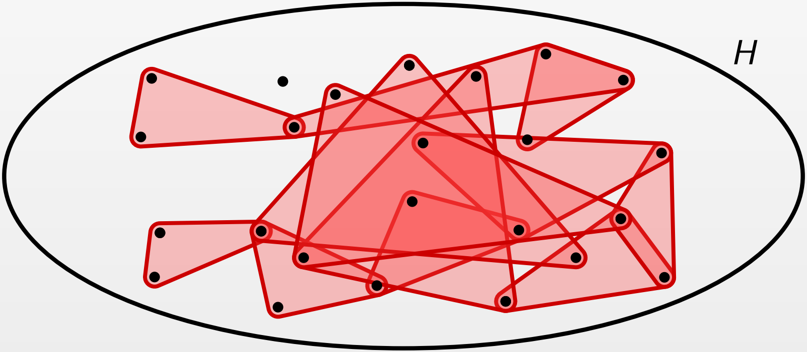

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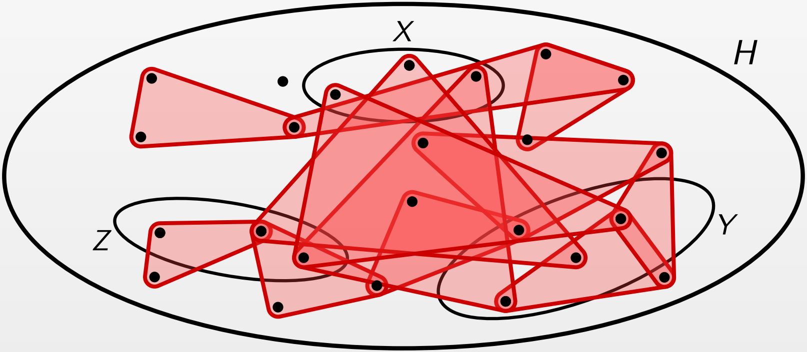

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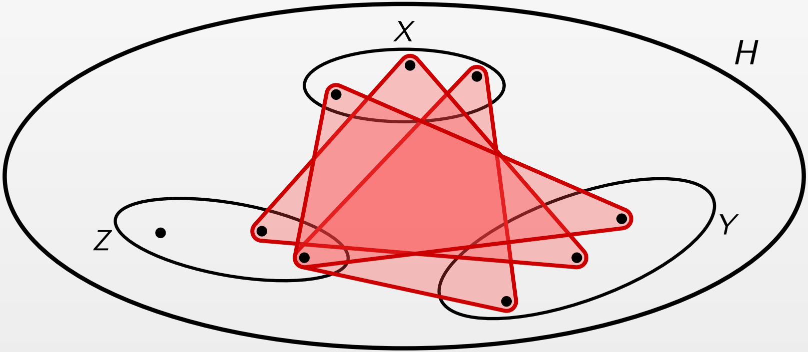

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- The three dots $\therefore$ in the subscript of $\pi_{\therefore}$ stand symbolically for the possible witness sets X, Y, and Z.

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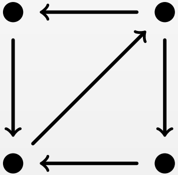

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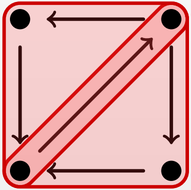

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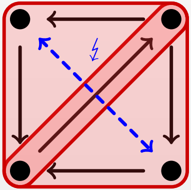

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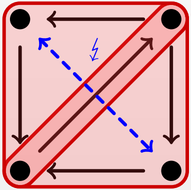

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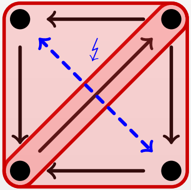

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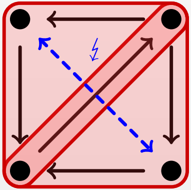

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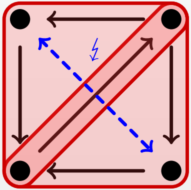

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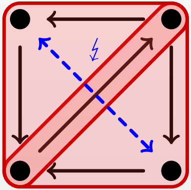

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- **Open problem (Erdős):** Density $> 1/4$ guarantees some non-bipartite link?

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  $\Rightarrow$ the right link of every vertex is bipartite

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- let $H_\varphi = (V, E)$ with $\{i < j < k\} \in E \iff \varphi(ij) \neq \varphi(ik)$

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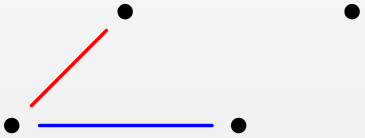

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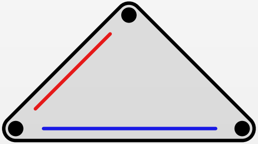

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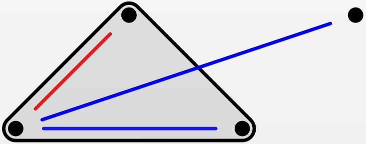

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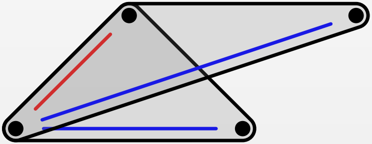

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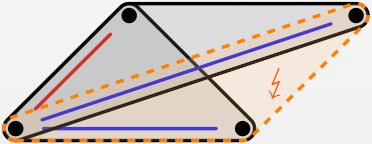

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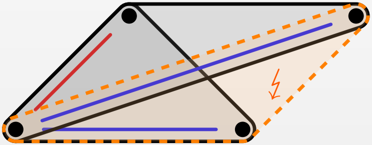

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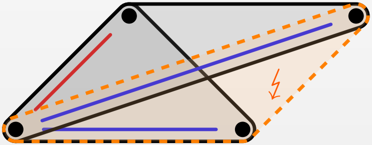

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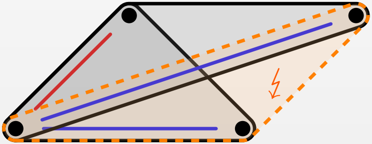

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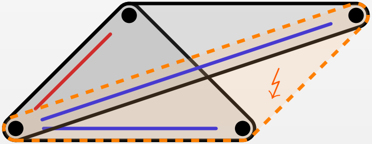

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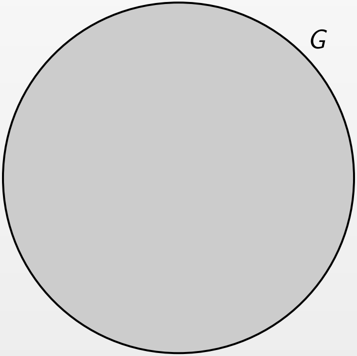
- These examples led to the more technical notions in the hypergraph regularity projects of Gowers and Rödl et al.
- Hypergraph regularity method turned out to be a useful tool for addressing extremal problems for uniformly dense hypergraphs

Szemerédi's regularity lemma

Regularity Lemma (informal version)

Every large graph G can be decomposed into "relatively few," mostly random-like (uniform edge distribution) bipartite subgraphs.



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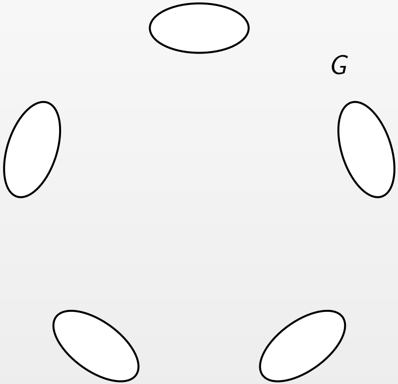

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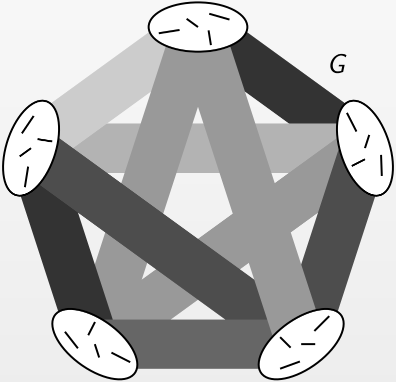

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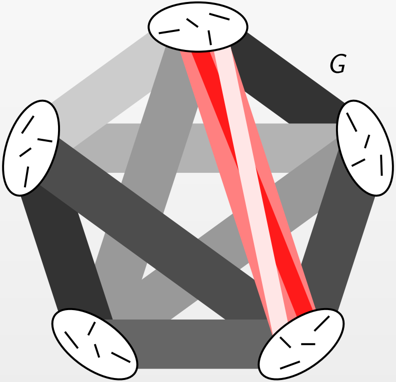
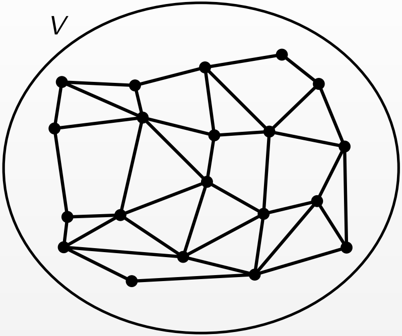


Regularity for 3-uniform hypergraphs



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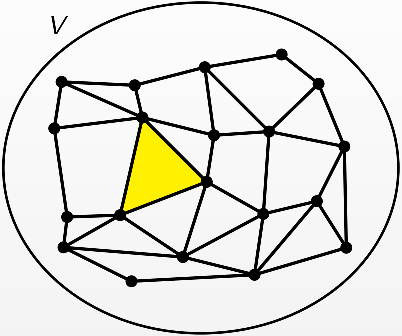

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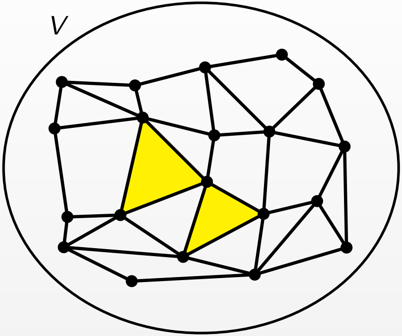

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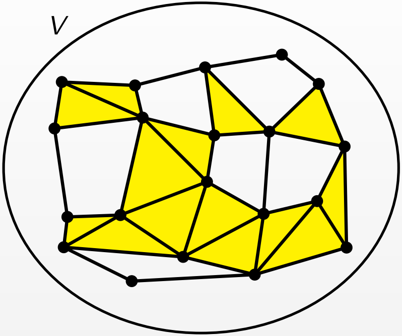

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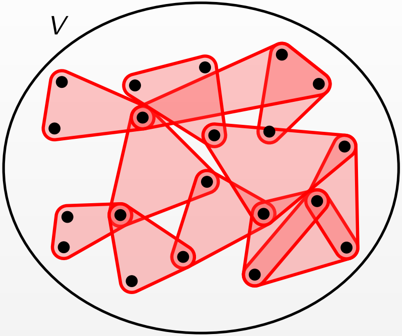

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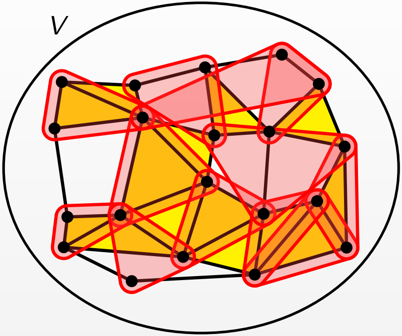

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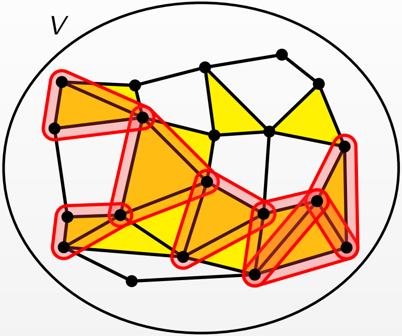

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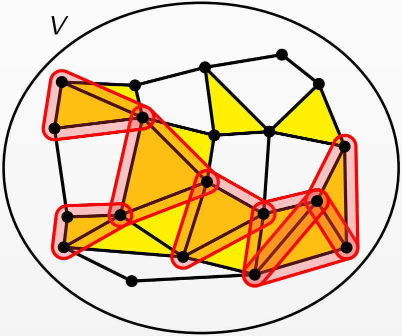

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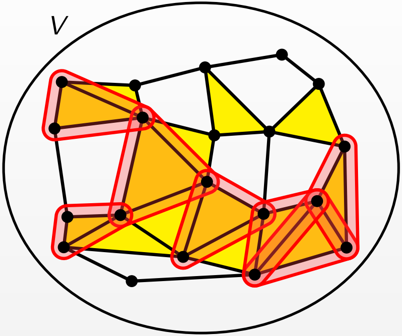

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- Hypergraph regularity lemma provides partition $G_1 \,\dot\cup\, \dots \,\dot\cup\, G_\ell$ of $V^{(2)}$ so that H is ε-regular w.r.t. most triads $G_i \,\dot\cup\, G_j \,\dot\cup\, G_k$

Results and problems for weakly dense hypergraphs

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Strengthening the denseness assumption

- weakly dense $\Longleftrightarrow$ density induced on large vertex sets X, Y, Z

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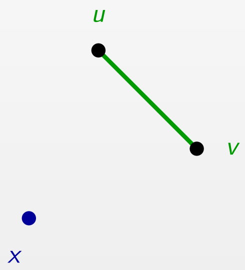
Given a 3-uniform hypergraph $H = (V, E)$ and $X \subseteq V$ and $P \subseteq V \times V$ set



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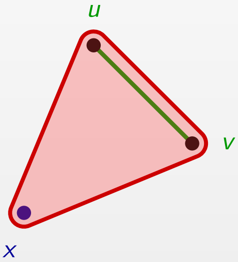

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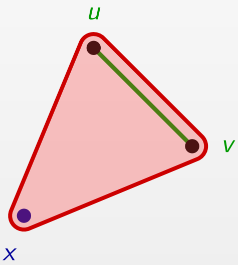
i.e., the number (counted with multiplicity) of hyperedges $e \in E$ “supported by” vertices from X and pairs from P.



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A 3-uniform hypergraph $H = (V, E)$ is $(\varepsilon, p, \therefore)$-*dense*, if for every $X \subseteq V$ and every $P \subseteq V \times V$ we have

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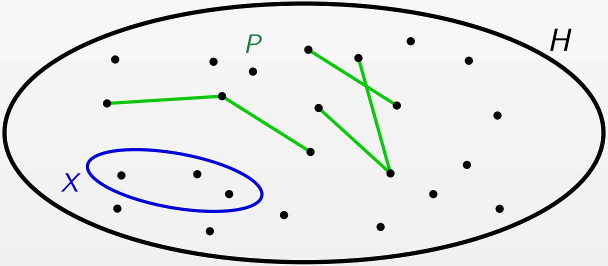

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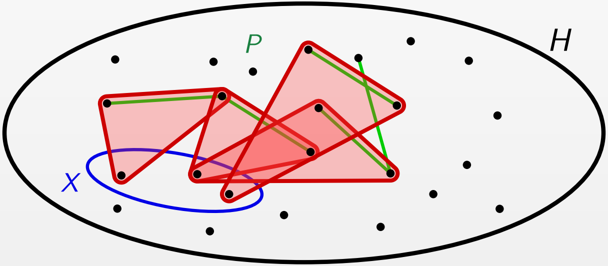

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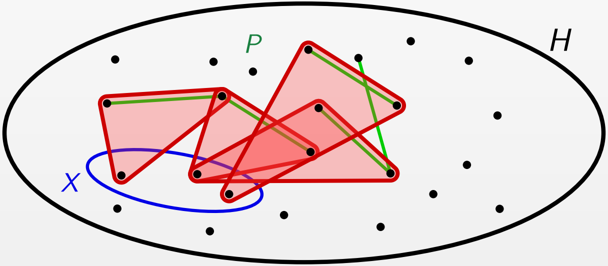

- $\therefore$-dense $\Longrightarrow$ weakly dense (by setting $P = Y \times Z$)

Strengthening the denseness assumption

- weakly dense $\iff$ density induced on large vertex sets X, Y, Z

Definition ($\therefore$-dense)

A 3-uniform hypergraph $H = (V, E)$ is *$(\varepsilon, p, \therefore)$-dense*, if for every $X \subseteq V$ and every $P \subseteq V \times V$ we have

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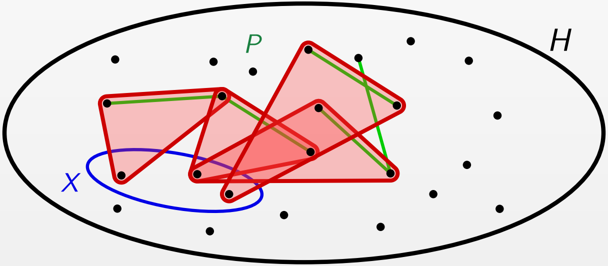

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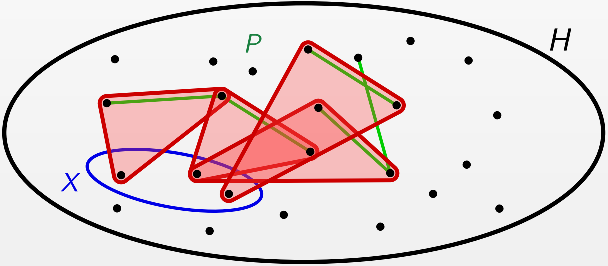

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- $\therefore$-dense $\iff$ "localised pair-degree" $\iff$ "min-degree in reduced hypergraph"
- for this notion with Reiher and Rödl '16 we could show $\pi_{\therefore}(K_4^{(3)}) = 1/2$

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Theorem (Reiher, Rödl & Sch. '18)

1. *For every $t \geqslant 2$ we have*

$$\pi_{\wedge}(K^{(3)}_{2^t}) \leqslant \frac{t-2}{t-1},$$

which is tight for $t = 2, 3, 4$.

2. $0 = \pi_{\wedge}(K^{(3)}_4) < \frac{1}{3} \leqslant \pi_{\wedge}(K^{(3)}_5) \leqslant \pi_{\wedge}(K^{(3)}_6) = \pi_{\wedge}(K^{(3)}_7) = \pi_{\wedge}(K^{(3)}_8) = \frac{1}{2}$

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New result

$$\frac{1}{3} \leqslant \pi_{\wedge}(K_5^{(3)}) \leqslant \frac{1}{2}$$

New result

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Lower bound for $K_5^{(3)}$:

- random $\varphi\colon V^{(2)} \longrightarrow \mathbb{Z}/3\mathbb{Z}$
- $xyz \in E \iff \varphi(xy) + \varphi(xz) + \varphi(yz) \equiv 1$

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Remarks:

- proof is based on hypergraph regularity method

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Remarks:

- proof is based on hypergraph regularity method
- analysis of the structure of "holes" in an alleged counterexample
- **Open problems:** determine $\pi_{\therefore}(K_9^{(3)})$ and $\pi_{\therefore}(K_{10}^{(3)})$

Thank you very much for your attention