Counting lattice points in moduli spaces of quadratic differentials (after a joint work with V. Delecroix, É. Goujard and A. Zorich)

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Moduli space of quadratic differentials

A holomorphic (resp. meromorphic) quadratic differential q on a smooth complex curve C of genus g is a holomorphic (resp. meromorphic) section of $T^*C^{\otimes 2}$. The *moduli space* of quadratic differentials with $n \geq 0$ simple poles is

$$\mathcal{Q}_{g,n} = \{(C,q)|C \in \mathcal{M}_{g,n}, q \in H^0(C, T^*C^{\otimes 2} \otimes \mathcal{O}(x_1 + \ldots + x_n))\}$$

i.e. $\mathcal{Q}_{g,n} \cong T^*\mathcal{M}_{g,n}$.

The space $\mathcal{Q}_{g,n}$ is naturally stratified according to the sets of multiplicities of zeros of q.

The *principal stratum* $\mathcal{Q}(1^{4g-4+n}, -1^n)$ (here we assume that both zeros and poles of q are labeled) is a (ramified) cover of its image in $\mathcal{Q}_{g,n}$ of degree $(4g-4+n)!$ (the image is open and dense in $\mathcal{Q}_{g,n}$).

Period (or *homological*) coordinates on $\mathcal{Q}(1^{4g-4+n}, -1^n)$ are introduced via the canonical 2-fold cover

$$\widehat{C} = \{(x, \omega(x)) \mid x \in C,\ \omega(x) \in T_x^* C,\ \omega(x)^2 = q(x)\}$$

ramified precisely over zeros and poles of q. The curve $\widehat{C}$ is smooth of genus $4g - 3 + n$, and ω is a holomorphic 1-form on $\widehat{C}$. Decompose

$$H_1(\widehat{C}, \mathbb{C}) = H_1^+(\widehat{C}, \mathbb{C}) \oplus H_1^-(\widehat{C}, \mathbb{C})$$

into the sum of even and odd subspaces with respect to the action of the covering involution of $\widehat{C}$, and put

$$H_1^-(\widehat{C}, \mathbb{Z}) = H_1(\widehat{C}, \mathbb{Z}) \cap H_1^-(\widehat{C}, \mathbb{C}).$$

The *period map* $\mathcal{Q}(1^{4g-4+n}, -1^n) \to H^1_-(\widehat{C}, \mathbb{C})$ is defined by

$$(C, q) \mapsto \int_\alpha \omega, \quad \alpha \in H_1^-(\widehat{C}, \mathbb{Z}),$$

and provides coordinates on $\mathcal{Q}(1^{4g-4+n}, -1^n)$.

The *Masur-Veech volume form* dV is the linear volume form on $H^1_-(\widehat{C}, \mathbb{C})$ normalized by the condition $\mathrm{Vol}(H^1_-(\widehat{C}, \mathbb{C})/L) = 1$, where $L = \mathrm{Hom}(H_1^-(\widehat{C}, \mathbb{Z}), \mathbb{Z} \oplus \sqrt{-1}\mathbb{Z})$.

The volume form dV induces volume forms on the level sets

$$\mathcal{Q}_{g,n}^{A=a} = \{(C, q) \in \mathcal{Q}_{g,n} \mid A(C, q) = a\}$$

of the area function $A(C, q) = \int_C |q|$. By definition,

$$\begin{aligned}\mathrm{Vol}\, \mathcal{Q}(1^{4g-4+n}, -1^n) &= \mathrm{Vol}\, \mathcal{Q}^{A=1/2}(1^{4g-4+n}, -1^n) \\ &= 2(6g - 6 + 2n)\, \mathrm{Vol}\, \mathcal{Q}^{A\le 1/2}(1^{4g-4+n}, -1^n).\end{aligned}$$

Square-tiled surfaces, multicurves, and stable graphs

A *square-tiled surface* C is a connected oriented compact surface without boundary built of squares of size $\frac{1}{2} \times \frac{1}{2}$. Each square has a pair of opposite sides called *horizontal* and another pair of sides called *vertical*, so that horizontal sides are glued to horizontal ones, and vertical sides to vertical ones.

Quadratic differential dz^2 on each square is compatible with gluing and endows C with a complex structure and a meromorphic quadratic differential q with at most simple poles.

Fact: There is a bijection between the set of square-tiled surfaces in the principal stratum $\mathcal{Q}(1^{4g-4+n}, -1^n)$ and the lattice $L = \mathrm{Hom}(H_1^-(\widehat{C}, \mathbb{Z}), \mathbb{Z} \oplus \sqrt{-1}\mathbb{Z})$ in $H^1_-(\widehat{C}, \mathbb{C})$. In particular,

$$\begin{aligned}&\mathrm{Vol}\,\mathcal{Q}(1^{4g-4+n}, -1^n)\\ &= 2(6g-6+2n) \lim_{N\to\infty} \frac{|ST(\mathcal{Q}(1^{4g-4+n}, -1^n), 2N)|}{N^{6g-6+2n}}.\end{aligned}$$

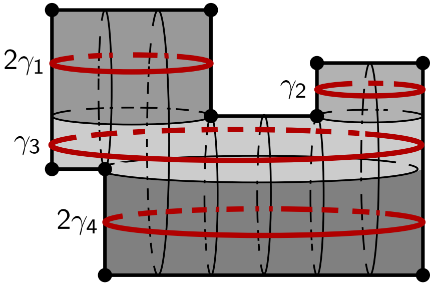

Figure 1: A square-tiled surface in $\mathcal{Q}(1^3, -1^7)$ made up of 54 squares with 3 conical singularities of angle 3π (corresponding to simple zeros of q) and 7 conical points of angle π (corresponding to simple poles of q).

A square-tiled surface admits a decomposition into maximal horizontal cylinders encoded by $\gamma = \sum_{i=1}^{k} h_i \gamma_i$, where γ_i are the waist curves and h_i are the cylinder heights (in units of $1/2$).

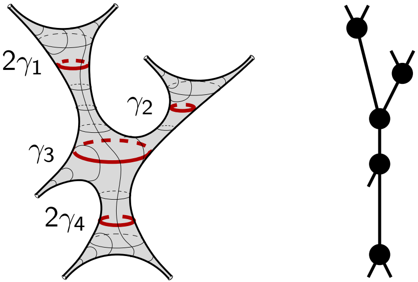

Figure 2: The multicurve and stable graph associated with the square-tiled surface on Fig. 1.

Passing from the flat metric $|q|$ on C to the compatible hyperbolic metric with cusps at the poles of q, we get a geodesic *multicurve* $\gamma = \sum_{i=1}^{k} h_i\gamma_i$, where γ_i are simple closed geodesics and h_i are the corresponding multiplicities.

To each square-tiled surface or, equivalently, each multicurve $\gamma = \sum_{i=1}^{k} h_i \gamma_i$ we associate its *stable graph* $\Gamma(\gamma)$:

- vertices of $\Gamma(\gamma)$ represent the components of $C \setminus \{\gamma_1 \cup \ldots \cup \gamma_k\}$ labeled with the genus of the corresponding component;
- edges of $\Gamma(\gamma)$ correspond to the curves γ_i and connect the vertices representing the components of $C \setminus \{\gamma_1 \cup \ldots \cup \gamma_k\}$ adjacent to γ_i;
- $\Gamma(\gamma)$ has n "legs" (or half-edges) labeled from 1 to n, where the ith leg is incident to the vertex that represents the component that contains the i-th pole of q;
- at each vertex v the stability condition $2g(v) - 2 + n(v) > 0$ is satisfied, where $g(v)$ is the genus assigned to v and $n(v)$ is the degree (or valency) of v.

The genus of a stable graph Γ is defined as $g = \sum_{v \in V(\Gamma)} g(v) + b_1(\Gamma)$, where $V(\Gamma)$ is the set of vertices and $b_1(\Gamma)$ is the first Betty number of the graph Γ).

For a pair of non-negative integers g and n with $2g-2+n>0$, denote by $\mathcal{G}_{g,n}$ the set of isomorphism classes of stable graphs of genus g with n legs.

For a stable graph Γ in $\mathcal{G}_{g,n}$, consider the subset $ST_{\Gamma,h}(\mathcal{Q}(1^{4g-4+n},-1^n),2N)$ of square-tiled surfaces with at most $2N$ squares, having Γ as the associated stable graph and $h=(h_1,\dots,h_k)$ as the set of heights of the cylinders. Its contribution to $\operatorname{Vol}\mathcal{Q}(1^{4g-4+n},-1^n)$ is

$$\operatorname{Vol}(\Gamma,h)=2d\cdot\lim_{N\to\infty}\frac{|ST_{\Gamma,h}(\mathcal{Q}(1^{4g-4+n},-1^n),2N)|}{N^d},$$

where $d=6g-6+2n$. The limit exists, is positive, and

$$\operatorname{Vol}\mathcal{Q}(1^{4g-4+n},-1^n)=\sum_{\Gamma\in\mathcal{G}_{g,n}}\sum_{h\in\mathbb{N}^{|E(\Gamma)|}}\operatorname{Vol}(\Gamma,h),$$

where $E(\Gamma)$ is the number of edges of Γ.

Formula for Masur–Veech volumes

Put

$$N_{g,n}(b_1,\ldots,b_n) = \frac{1}{2^{5g-6+2n}} \sum_{d_1,\ldots,d_n} \frac{b_1^{2d_1}\ldots b_n^{2d_n}}{d_1!\ldots d_n!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1}\ldots\psi_n^{d_n},$$

where $\psi_1, \ldots, \psi_n$ are the tautological classes on the Deligne–Mumford moduli space $\overline{\mathcal{M}}_{g,n}$ and $d_1 + \ldots + d_n = 3g - 3 + n$.

Consider the linear operator Y_h defined on monomials by

$$Y_h : \prod_{i=1}^{k} b_i^{m_i} \longmapsto \prod_{i=1}^{k} \frac{m_i!}{h_i^{m_i+1}}$$

and extended to arbitrary polynomials in $b_1, \ldots, b_k$ by linearity.

For a stable graph Γ with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$ we associate a homogeneous polynomial P_Γ of degree $6g-6+2n$ by

$$P_\Gamma(b_1,\dots,b_k) = c(\Gamma)\cdot \prod_{e\in E(\Gamma)} b_e \cdot \prod_{v\in V(\Gamma)} N_{g(v),n(v)}(\boldsymbol{b}_v)\,,$$

where $k=|E(\Gamma)|$ and

$$c(\Gamma) = \frac{2^{6g-5+2n}\cdot(4g-4+n)!}{(6g-7+2n)!}\cdot\frac{1}{2^{|V(\Gamma)|-1}}\cdot\frac{1}{|\operatorname{Aut}\Gamma|}\,.$$

Teorem 1. *The Masur–Veech volume of the principal stratum is*

$$\operatorname{Vol}\mathcal{Q}(1^{4g-4+n},-1^n) = \sum_{\Gamma\in\mathcal{G}_{g,n}}\ \sum_{h\in\mathbb{N}^{|E(\Gamma)|}} Y_h(P_\Gamma)\,.$$

Alternative approach: Chen–Möller–Sauvaget

Statistical geometry of random multicurves

Let C be a hyperbolic surface of genus g with n cusps. Let $\gamma = \sum_{i=1}^{k} h_i \gamma_i$ be a multicurve on C consisting of pairwise disjoint primitive simple closed geodesics γ_i. Denote by ℓ the hyperbolic length function, and put $L = \sum_{i=1}^{k} h_i \, \ell(\gamma_i)$ to be the total length of γ.

Denote by $\mathcal{ML}_{g,n}(\mathbb{Z})$ the set of integer points in the space of measured laminations on C. Two multicurves have the same topological type if they belong to the same orbit of the mapping class group $\mathrm{Mod}_{g,n}$ in $\mathcal{ML}_{g,n}(\mathbb{Z})$. By definition, the asymptotic probability that a random multicurve belongs to the orbit $\mathrm{Mod}_{g,n} \cdot \gamma$ is

$$P_{g,n}(\gamma) = \lim_{L \to \infty} \frac{|\{\gamma' \in \mathrm{Mod}_{g,n} \cdot \gamma \mid \ell(\gamma') \leq L\}|}{|\{\gamma' \in \mathcal{ML}_{g,n}(\mathbb{Z}) \mid \ell(\gamma') \leq L\}|}.$$

The following is a refinement of a result of Mirzakhani:

Theorem 2. *The asymptotic probability* $P_{g,n}(\gamma)$ *is given by*

$$P_{g,n}(\gamma) = \frac{\mathrm{Vol}(\Gamma, h)}{\mathrm{Vol}\,\mathcal{Q}(1^{4g-4+n}, -1^n)},$$

where Γ *is the stable graph corresponding to the multicurve* γ.

For $n = 0$, $g \geq 2$ there is a single topological type of non-separating simple closed geodesics γ_0 as in Fig. 3 and $[g/2]$ topological types of separating closed geodesics $\gamma_1, \dots, \gamma_{[g/2]}$ as in Fig. 4, where γ_i cuts the complex curve C into two parts of genera i and $g - i$ respectively.

Corollary.

$$\frac{\sum_{i=1}^{[g/2]} P_{g,0}(\gamma_i)}{P_{g,0}(\gamma_0)} \approx \sqrt{\frac{2}{3\pi g}} \cdot \frac{1}{2^{2g}} \quad \text{as } g \to \infty.$$

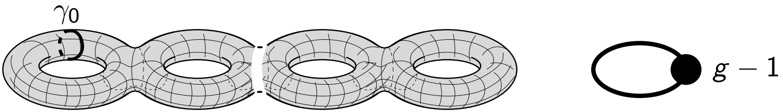

Figure 3: Non-separating curve γ_0 and the associated stable graph Γ_0.

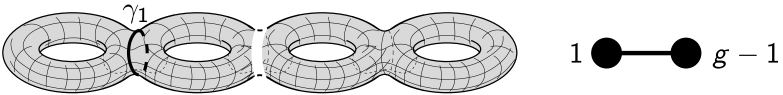

Figure 4: Separating curve γ_1 and the associated stable graph Γ_1.

It means that on a compact hyperbolic surface of large genus non-separating simple closed curves are exponentially more frequent than separating ones.

Statistical geometry of multicurves on surfaces of large genus is discussed in detail in [DGZZ, Invent. Math. (2022)].

Square-tiled surfaces and enumeration of meanders

A *meander* is a configuration in the plane that consists of a straight line and a simple closed curve transversely intersecting it, considered up to isotopy. Enumeration of meanders is a long-standing difficult combinatorial problem.

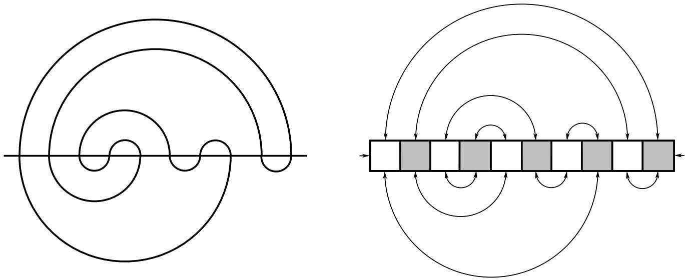

Figure 5: A meander with 10 crossings and 6 minimal arcs (left), and the corresponding square-tiled surface in $\mathcal{Q}(1^2, 0, -1^6)$ (right),where pairs of sides connected with arrowed arcs are identified.

Let $\mathcal{M}(N)$ be the number of meanders with $2N$ crossings. Conjecturally,

$$\mathcal{M}(N) \approx \text{const} \cdot R^N N^\alpha \quad \text{as } N \to \infty.$$

An arc is called *minimal* if it connects two adjacent intersections (the maximal arc connecting the first and the last intersections, if present, is treated as a minimal arc as well). Denote by $\mathcal{M}_n(N)$ the number of meanders with $2N$ crossings and n minimal arcs.

To each meander with $2N$ crossings and n minimal arcs we associate a genus 0 square-tiled surface made up of $2N$ squares with one horizontal and one vertical cylinders of maximal circumference, n simple poles and a marked point (this correspondence is generically two-to-one for large N).

We have

$$|ST(\mathcal{Q}(1^{n-4},0,-1^n),2N)| = c(n)\,\frac{N^d}{2d} + \underset{N\to\infty}{o}(N^d),$$

$$|ST_1(\mathcal{Q}(1^{n-4},0,-1^n),2N)| = c_1(n)\,\frac{N^d}{2d} + \underset{N\to\infty}{o}(N^d),$$

$$|ST_{1,1}(\mathcal{Q}(1^{n-4},0,-1^n),2N)| = c_{1,1}(n)\,\frac{N^d}{2d} + \underset{N\to\infty}{o}(N^d),$$

where $d = 2n-5$. In particular,

$$\mathcal{M}_n(N) = \frac{2c_{1,1}}{n!(n-4)!}\frac{N^d}{2d} + \underset{N\to\infty}{o}(N^d)\,.$$

Here

$$c(n) = 8\left(\frac{\pi^2}{2}\right)^{n-3}$$

is the Masur–Veech volume of $\mathcal{Q}(1^{n-4},0,-1^n)$,

$$c_1(n) = 4\binom{2n-4}{n-2},$$

and

$$\frac{c_{1,1}(n)}{c_1(n)} = \frac{c_1(n)}{c(n)}.$$

As a conclusion, we get

Theorem 3.

$$\mathcal{M}_n(N) = \frac{4}{n!(n-4)!}\left(\frac{2}{\pi^2}\right)^{n-3}\binom{2n-4}{n-2}^2 \frac{N^{2n-5}}{4n-10} + \underset{N\to\infty}{o}(N^d).$$

These techniques are also applicable to asymptotic enumeration of pairs of transversal multicurves on surfaces of arbitrary genus satisfying certain topological restrictions [DGZZ, to appear].