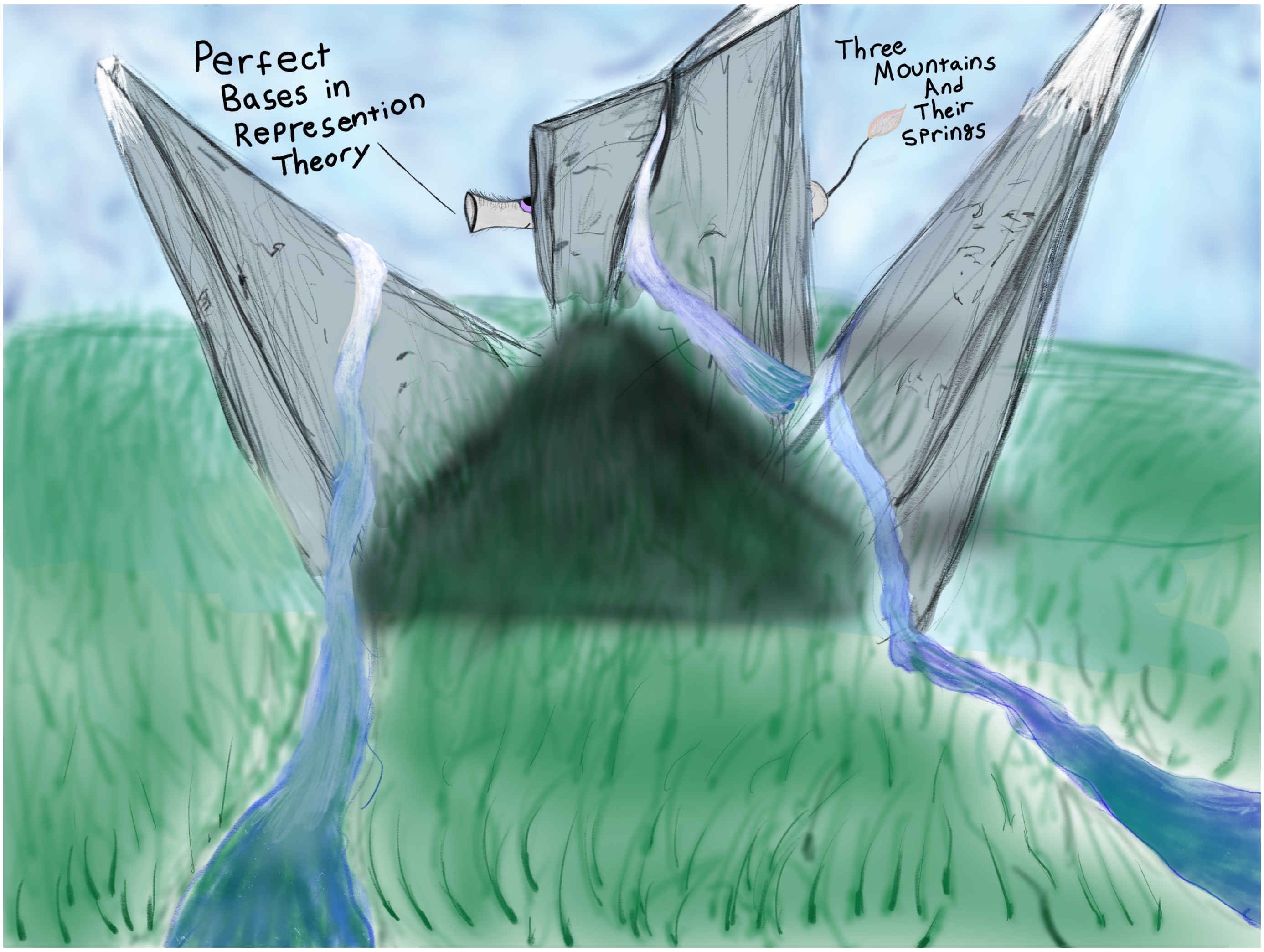

Perfect
Bases in
Represention
Theory
Three
Mountains
And
Their
Springs

G semisimple group $/ \mathbb{C}$ eg $G = SL_n \mathbb{C}$

Irreducible representations $V(\lambda)$, $\lambda \in P_+$

P_+ = dominant weights $\subset P$ = wt lattice

eg $\{\lambda_1 \geq \cdots \geq \lambda_n\} \subset \mathbb{Z}^n / \mathbb{Z}$

$$V(\lambda) = \bigoplus_{\mu \in P} V(\lambda)_\mu \qquad \text{weight space decomposition}$$

$$V(\lambda) \otimes V(\mu) = \bigoplus_{\nu \in P_+} V(\nu)^{\oplus c^{\nu}_{\lambda \mu}}$$

Problem

- Determine combinatorial formulae for weight and tensor product multiplicities.

First solved by Littelmann, Berenstein-Zelevinsky

- Find bases for $V(\lambda)$ compatible with weight and tensor product multiplicities.

Lemma

$$\mathrm{Hom}_{g}(V(\nu), V(\lambda) \otimes V(\mu)) \longrightarrow V(\lambda)_{\nu - \mu}$$
$$\phi \longmapsto (1 \otimes v_\mu^*)(\phi(v_\nu))$$

is injective with image $\{v : e_i^{\langle \rho, \mu \rangle + 1}(v) = 0 \quad \forall i \in I\}$

A basis for V is called <u>good</u> if it is a weight basis and is compatible with kernels of powers of e_i, for all $i \in I$.

A basis B for V is called <u>perfect</u> if it is good and for each $i \in I$, $b \in B$ either $e_i b = 0$ or $\exists\ \tilde{e}_i(b) \in B$ s.t. $e_i b = \varepsilon_i(b)\, \tilde{e}_i(b) + \ldots$ smaller nilpotency degree wrt e_i

A perfect basis B carries a combinatorial structure called a <u>crystal</u>:

$\mathrm{wt} : B \to P \qquad \tilde{e}_i : B \dashrightarrow B \qquad \varepsilon_i : B \to \mathbb{N}$

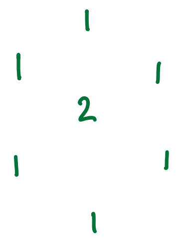
<u>Theorem</u> [Berenstein – Kazhdan]
If B, B' are two perfect bases for the same representation, then $B \cong B'$ as crystals.

<u>Question</u>

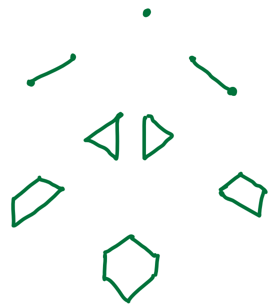
① How can we construct perfect bases?
② What are the combinatorics of perfect bases?

There are many combinatorial models, but today we will focus on Mirkovic-Vilonen polytopes [Anderson, K] closely related to Lusztig, Berenstein-Zelevinsky.

Eg adjoint rep SL_3



wt diagram



MV polytopes

There are three known constructions of perfect bases

① Lusztig's canonical basis - quantum groups, categorification, ...

② Lusztig's semicanonical basis - geometry of quiver varieties

dual

③ Mirkovic-Vilonen basis - geometry of affine Grassmannians

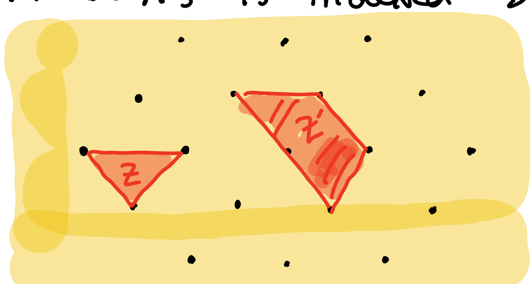
All these bases flow down from high (geometric) mountains.

The MV basis

$G^\vee$ Langlands dual group $\quad Gr = G^\vee(\mathbb{C}((t)))/G^\vee(\mathbb{C}[[t]])$

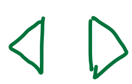
$P = \mathrm{Hom}(\mathbb{C}^\times, T^\vee)$, so $\mu \in P$ gives $t^\mu \in Gr$

MV basis is indexed by certain $Z \subset Gr$



$Pol(Z) = \mathrm{Conv}(\mu : t^\mu \in Z)$

the moment polytope of Z



correspond to $\mathbb{P}^2$'s

__Theorem__ [K]

The map $Z \mapsto \mathrm{Pol}(Z)$ gives a bijection between the MV cycles and an explicit set of polytopes called MV polytopes.

Semicanonical basis

Assume now G is simply-laced.

(I, Q) choice of orientation of Dynkin diagram

$\bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet$ $\qquad \overline{Q}$ doubled quiver

$$\Lambda = \mathbb{C}\overline{Q} / \sum a^* a - a a^*$$

A Λ-module is $M = \bigoplus_{i \in I} M_i$ with $M_i \rightleftarrows M_j$

Given a Λ-module M, $\underline{\dim}\, M = \sum_i (\dim M_i)\, \alpha_i$

$\nu = \sum_{i \in I} v_i \alpha_i \qquad \Lambda(\nu) =$ variety of Λ-module structures on $\oplus \mathbb{C}^{v_i}$ $\subset \bigoplus_{i \to j \in \overline{Q}} \mathrm{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{v_j})$

a reducible affine variety

__Theorem__ [Baumann-K-Tingley]

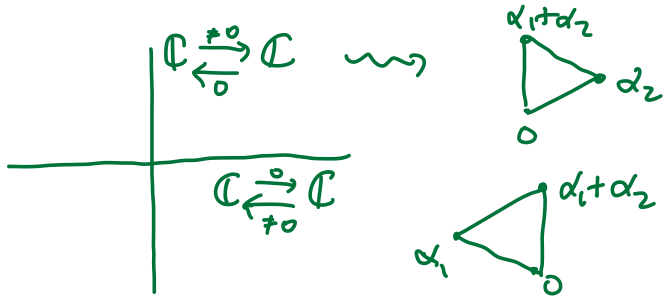
The map $M \longmapsto \mathrm{Pol}(M)$ gives a bijection

$$\bigcup_{\nu} \mathrm{Irr}\, \Lambda(\nu) \xrightarrow{\sim} \text{MV polytopes}$$

__Eg__

$\bullet \underset{B}{\overset{A}{\rightleftarrows}} \bullet$

$AB = BA = 0$



Finer invariants

We have seen

$$\begin{pmatrix}\text{complicated}\\ \text{geometry}\end{pmatrix} \rightrightarrows \begin{pmatrix}\text{basis}\\ \text{vectors}\end{pmatrix} \longrightarrow \begin{pmatrix}\text{MV polytopes}\\ \text{crystal}\end{pmatrix}$$

For $G = SL_2, SL_3, SL_4$ (maybe SL_5), there is a unique perfect basis, but not for other groups.

To tell apart different basis elements we need a more refined invariant that the MV polytope.

Measure supported on the MV polytope $\overset{FT}{\rightsquigarrow}$ rational function

Z MV cycle $\quad i_{t^0}(Z) \in H^\bullet_{T^\vee}(pt)_{loc} \cong \mathbb{C}(\mathfrak{t})$

$T^\vee \circlearrowright Z$ $\quad$ equivariant multiplicity (Brion)

measures the equivariant topology of Z near t^0

For M a Λ-module, we consider composition series

$0 \subset M' \subset \cdots \subset M^m = M$ s.t. $M^k/M^{k-1} \cong S_{i_k}$ is simple

Given $\underline{i} = (i_1, \ldots, i_m)$ let $F_{\underline{i}}(M) =$ all comp. series of type $\underline{i}$

We will study $\chi(F_{\underline{i}}(M))$ top. Euler char.

$$\bar{D}_{\underline{i}} = \frac{1}{\alpha_{i_1}(\alpha_{i_1}+\alpha_{i_2})\cdots(\alpha_{i_1}+\cdots\alpha_{i_m})} \in \mathbb{C}(\mathfrak{t})$$

$$\bar{D}(M) = \sum_{\underline{i}} \chi(F_{\underline{i}}(M))\, D_{\underline{i}} \in \mathbb{C}(\mathfrak{t})$$

Theorem [Baumann-K-Knutson]
Let Z be an MV cycle, M a generic Λ-module.
If $b_Z = b_M$ then $i_p(Z) = \overline{D}(M)$

MV basis vector — semicanon. basis vector

Corollary [BKK + Dranowski - Morton-Ferguson]
For SL_6, there is an example of an MV cycle Z and a generic Λ-module M s.t. $b_Z \neq b_M$.