Entropy in Mean Curvature Flow

Lu Wang

Yale University

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Outline

Background for mean curvature flow and entropy

Sharp lower bounds on entropy

Topological stability for entropy

A family of hypersurfaces in Euclidean space evolves under mean curvature flow if the velocity of every point on the evolving hypersurface is given by the mean curvature.

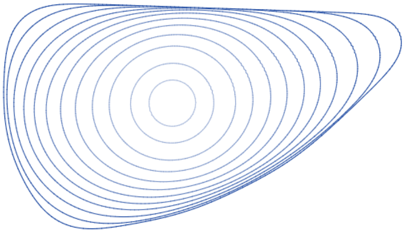

Figure: Convergence of a convex curve to a round circle under curve shortening flow, by courtesy of David Eppstein, via Wikimedia Commons

All blowups of a mean curvature flow at a given singularity are modeled by self-similarly shrinking solutions to the flow

$$M_t = \sqrt{-t}\, M_{-1} \quad \text{for } t < 0;$$

in that case M_{-1} is called a self-shrinker. (Brakke, Huisken, Ilmanen, and White)

The simplest examples:

static planes, shrinking spheres and cylinders

are conjectured to be the only ones arising in the flow starting from a generic closed surface. (Angenent–Chopp–Ilmanen, Huisken)

Given $x_0 \in \mathbb{R}^{n+1}$ and $t_0 > 0$, define the functional $F_{x_0,t_0}(\Sigma)$ of a hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ by

$$F_{x_0,t_0}(\Sigma) = (4\pi t_0)^{-\frac{n}{2}} \int_\Sigma e^{-\frac{|x-x_0|^2}{4t_0}}\, dV$$

A critical point of F_{x_0,t_0} is exactly the time $t = -t_0$ slice of a self-similarly shrinking solution to the mean curvature flow

$$M_t = \sqrt{-t}\,(M_{-1} - x_0) + x_0 \quad \text{for } t < 0.$$

Following Colding–Minicozzi, define the entropy $\lambda(\Sigma)$ of Σ by

$$\lambda(\Sigma) = \sup_{x_0, t_0} F_{x_0, t_0}(\Sigma).$$

The entropy has the properties:

- It is invariant under rigid motions and dilations.
- It is nonincreasing under mean curvature flow.
- The critical points of entropy are self-shrinkers.

The only entropy stable self-shrinkers with polynomial volume growth are:

hyperplanes, the round sphere, and generalized cylinders.

(Colding–Minicozzi)

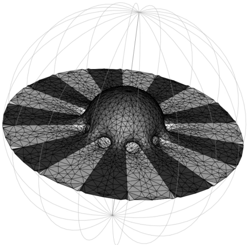

Figure: A truncated self-shrinker conjectured by Ilmanen and shown to exist by Kapouleas–Kleene–Møller and X.H. Nguyen

All smooth embedded self-shrinking curves in $\mathbb{R}^2$ are straight lines and the round circle. (Abresch–Langer)

Any simple, closed, smooth curve γ in $\mathbb{R}^2$ flows smoothly, eventually becomes convex and disappears in a round point. (Grayson, Gage–Hamilton)

Thus $\lambda(\gamma) \geq \lambda(\mathbb{S}^1)$ with equality if and only if γ is a round circle.

Two conjectures of Colding–Ilmanen–Minicozzi–White:

Conjecture A. For $n \leq 6$, there exists $\epsilon = \epsilon(n) > 0$ so that if $\Sigma \subset \mathbb{R}^{n+1}$ is a nonflat self-shrinker not equal to the round sphere, then $\lambda(\Sigma) \geq \lambda(\mathbb{S}^n) + \epsilon$.

Conjecture B. For $n \leq 6$, if M is a closed hypersurface in $\mathbb{R}^{n+1}$, then $\lambda(M) \geq \lambda(\mathbb{S}^n)$.

Given n, there exists $\epsilon = \epsilon(n) > 0$ so that if $\Sigma \subset \mathbb{R}^{n+1}$ is a *closed* self-shrinker not equal to the round sphere, then

$$\lambda(\Sigma) \geq \lambda(\mathbb{S}^n) + \epsilon.$$

(Colding–Ilmanen–Minicozzi–White)

There exists $\delta > 0$ so that if $\Sigma \subset \mathbb{R}^3$ is a self-shrinker not equal to a plane, round sphere or cylinder, then

$$\lambda(\Sigma) \geq \lambda(\mathbb{S}^1) + \delta.$$

(Bernstein–W.)

Ideas of proof: argue by contradiction

Perturb Σ to one side to produce a rescaled mean convex hypersurface $\tilde{\Sigma}$.

When Σ is closed, the flow starting from $\tilde{\Sigma}$ may develop spherical or cylindrical singularities.

When Σ is a noncompact surface, the flow starting from $\tilde{\Sigma}$ may either develop cylindrical singularities, or become star-shaped and so, by Brendle, Σ is a plane.

Given n, if M is a closed hypersurface in $\mathbb{R}^{n+1}$, then

$$\lambda(M) \geq \lambda(\mathbb{S}^n)$$

with equality if and only if M is a round sphere.
(Bernstein–W. for $n \leq 6$, J. Zhu for general n)

Key observation:

Terminal singularities of low entropy are "collapsing".

If M is a closed hypersurface in $\mathbb{R}^4$ with

$$\lambda(M) \leq \lambda(\mathbb{S}^2),$$

then M is smoothly isotopic to the standard 3-sphere.
(Bernstein–W., Chodosh–Choi–Mantoulidis–Schulze)

4**D Smooth Schoenflies Conjecture** in low-entropy setting

Question A. How does mean curvature flow resolve a conical singularity?

Question B. How to tackle the nonuniqueness of the mean curvature flow starting from a cone?

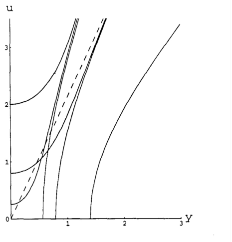

Figure: Cross-sections of self-similarly expanding evolutions by Angenent–Chopp–Ilmanen

Given a cone $C \subset \mathbb{R}^{n+1}$ with smooth embedded link, consider a mean curvature flow $\{M_t\}_{t\in(0,T)}$ so that $M_t \to C$ as $t \to 0$.

If M_t is trapped between two self-similarly expanding solutions to the flow

$$M_t^{\pm} = \sqrt{t}\, M^{\pm}, \quad M_t^{\pm} \to C \quad \text{as } t \to 0,$$

then all blowups at the vertex of the cone are modeled by self-similarly expanding solutions to the flow. (Bernstein–W.)

Idea: a notion of relative entropy motivated by self-expanders

For $2 \leq n \leq 6$, if $C \subset \mathbb{R}^{n+1}$ is a cone with smooth embedded link and if

$$\lambda(C) < \lambda(\mathbb{S}^{n-1}),$$

then all self-expanders asymptotic to C are in the same smooth isotopy class. (Bernstein–W.)

Idea: construct Morse flow lines for the expander functional

Summary:

- On the one hand, entropy is a useful quantity in the study of singularities for mean curvature flow;
- On the other, mean curvature flow is a tool to study entropy as a natural measure of geometric complexity.

Future directions:

- Classify self-shrinkers of low entropy in dimension ≥ 3, or more generally ancient solutions to the mean curvature flow of low entropy. (e.g., Choi–Haslhofer–Hershkovits)
- Sharp lower bounds on entropy for submanifolds of higher codimensions (e.g., Colding–Minicozzi)