On the Complexity of Quantum Many Body Systems

Umesh V. Vazirani
U.C. Berkeley

Quantum states are exponentially complex

n particles

$$State = \Psi = \sum_x \alpha_x |x\rangle = \begin{pmatrix} \alpha_{0000} \\ \alpha_{0001} \\ . \\ . \\ \alpha_{1111} \end{pmatrix}$$

all n-bit strings

Quantum states are exponentially complex

n particles

C^{2^n}

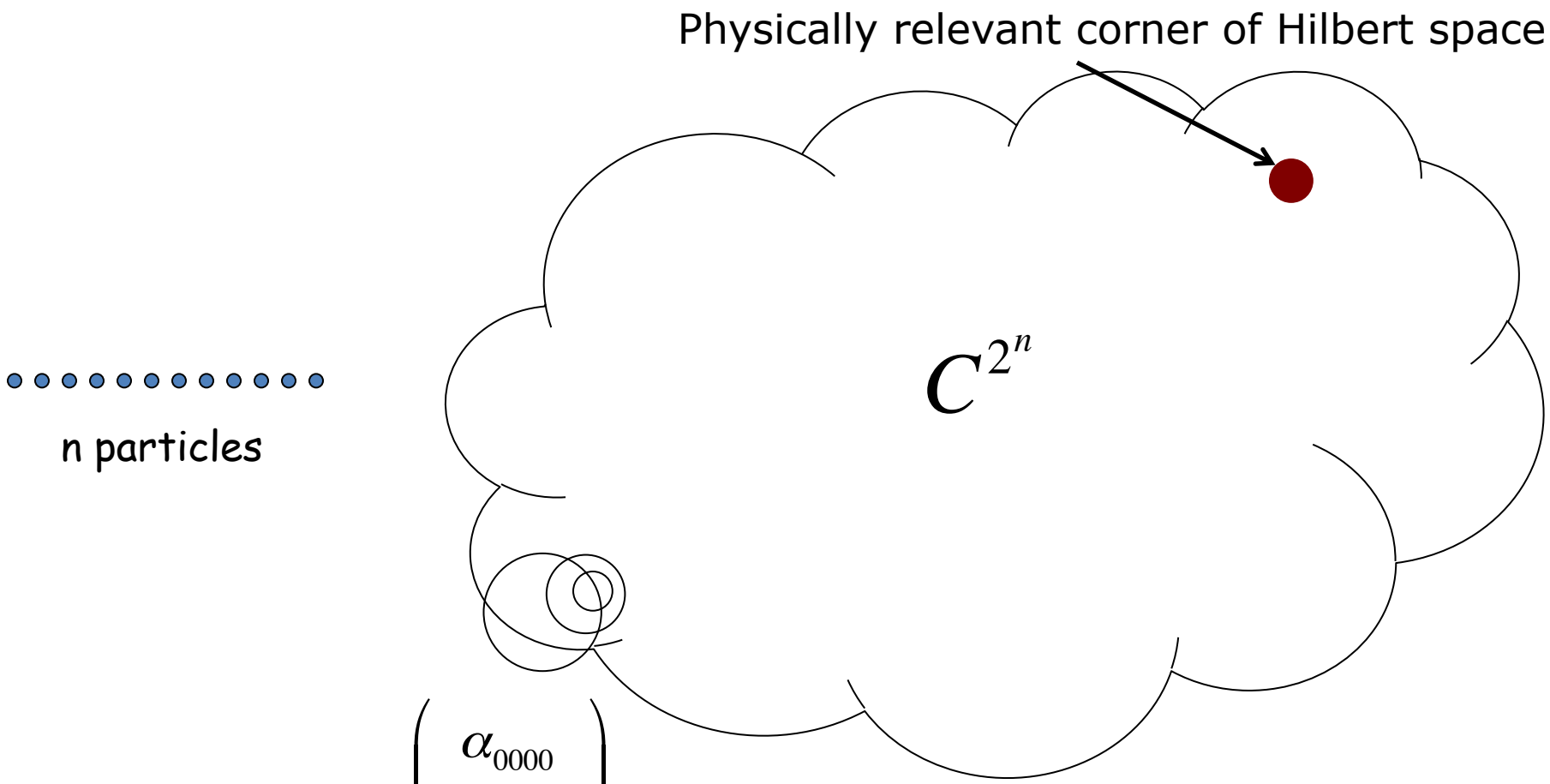
$$State = \Psi = \sum_x \alpha_x |x\rangle = \begin{pmatrix} \alpha_{0000} \\ \alpha_{0001} \\ . \\ . \\ \alpha_{1111} \end{pmatrix}$$

all n-bit strings

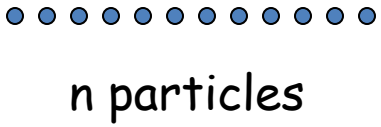
Quantum Entanglement:
Doesn't suffice to describe each individual qubit

Given this exponential complexity of quantum systems, how is it possible to do quantum many-body physics?

Condensed Matter Physics



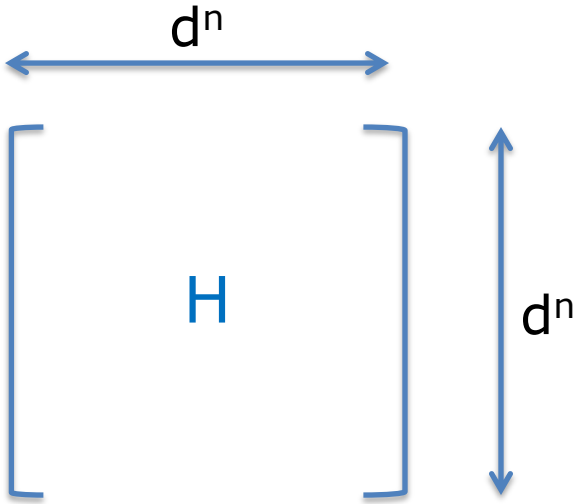
$$State = \Psi = \sum_x \alpha_x |x\rangle = \begin{pmatrix} \alpha_{0000} \\ \alpha_{0001} \\ . \\ . \\ \alpha_{1111} \end{pmatrix}$$



n particles

Each particle is a d-level system

Hilbert space associated with n-particle system is C^{d^n}

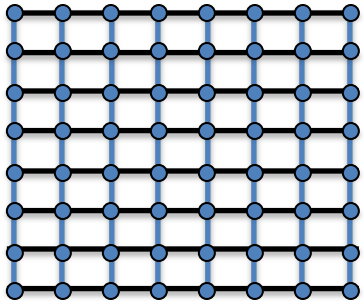


Hamiltonian H, $d^n x d^n$ Hermitian matrix

Schrodinger Eqn: $\frac{d\psi}{dx} = H\psi$

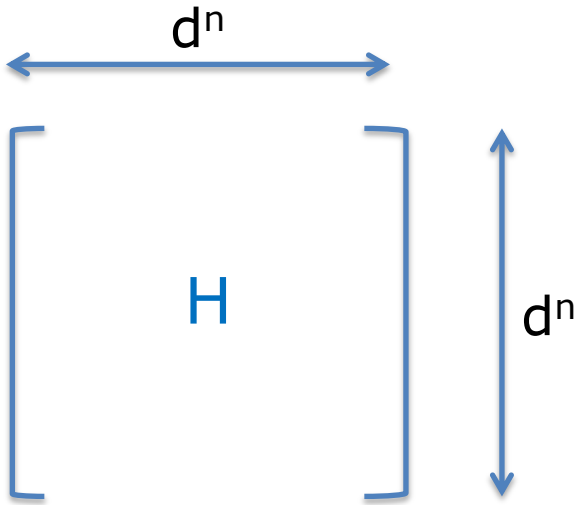
Condensed matter physics: physics determined by ground and low energy states of H

E_0 = minimum eigenvalue
Low energy state = eigenvector with small eigenvalue.



$$H = H_1 + \cdots + H_m$$

Each H_i is $d^2 x d^2$, positive, norm ≤ 1



H_1 acts on C^{d^n} as $H_1 \otimes I$

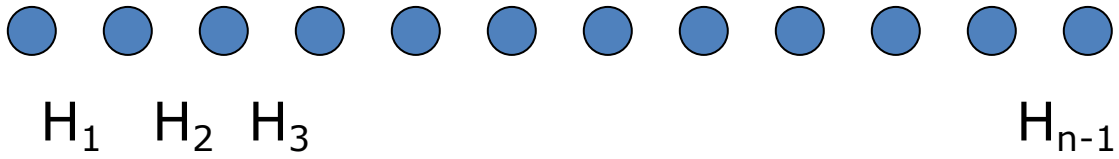
$H = H_1 + H_2 + \ldots H_m$

Energy of $|\psi\rangle = \sum_i \langle\psi|H_i|\psi\rangle$

Given compact representation of $d^n x d^n$ matrix H (specified by terms H_i), output compact descriptions of eigenvectors with small eigenvalues.

Why should these eigenvectors have compact descriptions?

Computational Condensed Matter Physics



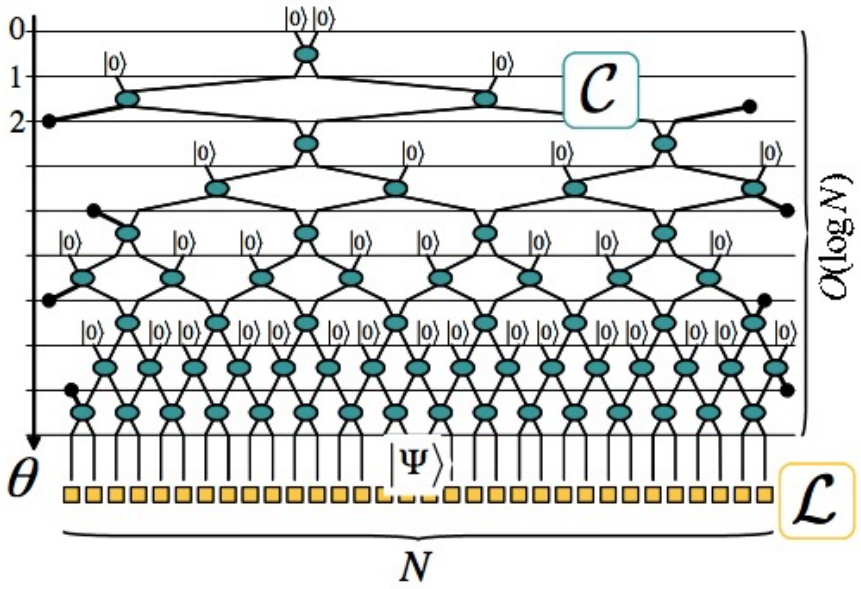
- DMRG (Density Matrix Renormalization Group) [White ’92] has been remarkably successful in practice for 1D quantum systems. Quickly outputs compact representation of ground/low energy state.

- Doesn’t always work. Provably hard examples known [Eisert ’06]

2D Quantum Systems

- [Vestraete, Cirac ’04] Projected entangled pair states (PEPS)

- [Vidal ’06] Multi-scale Entanglement Renormalization Ansatz (MERA).



Quantum Hamiltonian Complexity

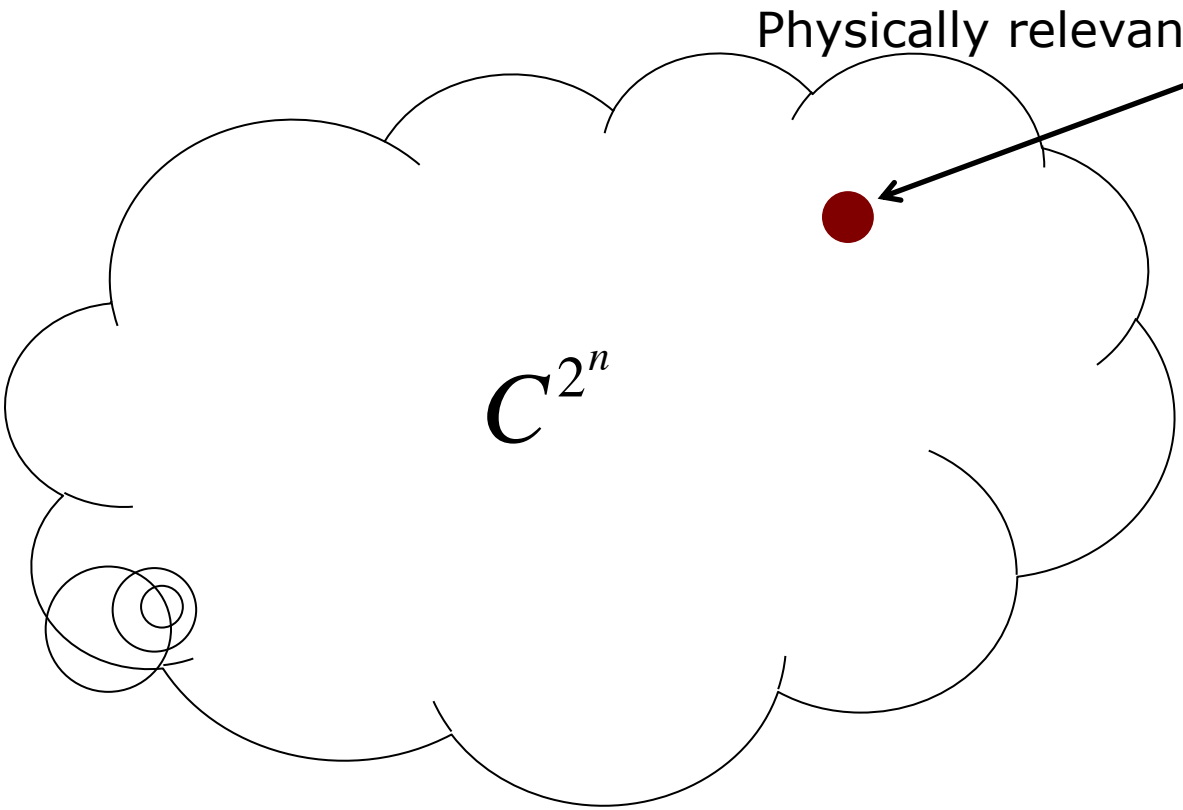
- Formalization of these questions using concepts from quantum computation/quantum information theory.
- Studies the computational complexity of quantum states, and algorithms for computing them
- Lies at intersection of quantum computation and condensed matter physics

Two sides of Quantum Hamiltonian Complexity

1. Intractability
   [Kitaev ’99] QMA-hard to find ground states or compute ground energy
   Refined to 2-local Hamiltonians, 2D or even 1D lattices
   QMA = quantum analog of NP.
   Conjecture: QMA-hard implies no compact classical description

2. Compact description of ground/low energy states for large class of Hamiltonians (1D)
   Efficient algorithms for computing these states

Exponential dimension ➔ basis independent treatment won't work



Tensor product structure to underlying Hilbert space that the Hamiltonian respects. This structure provides foothold that makes these questions tractable, starting with a measure of complexity of a state …

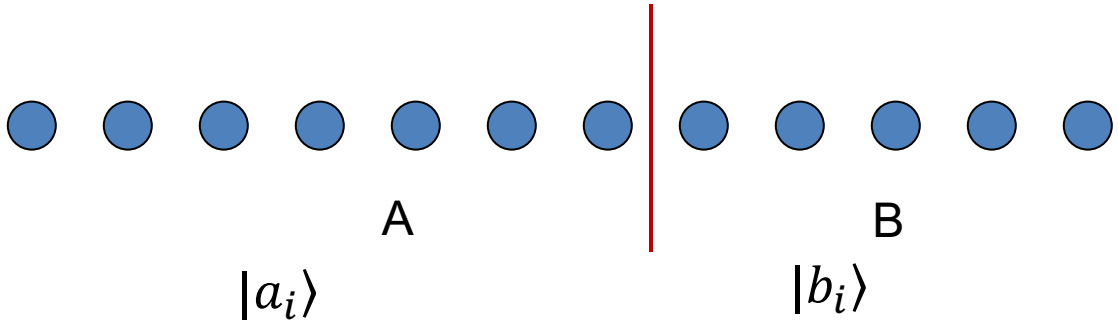
Entanglement

$$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

- Cannot describe state of multi-particle system by describing state of each particle

- The state of the composite system is totally determined, while each subsystem is completely random

Entanglement Rank (Entropy)



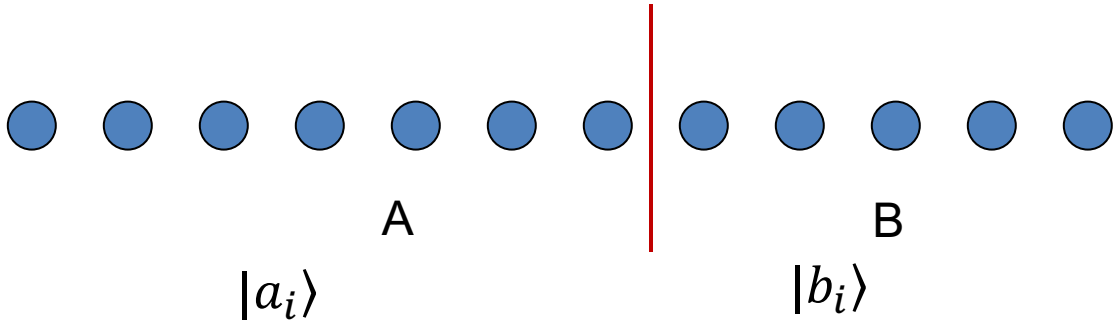
Product state: $|a_i\rangle \otimes |b_i\rangle$

Rank k state: $|\psi\rangle = \sum_{i=0}^{k-1} ci|a_i\rangle \otimes |b_i\rangle$ $\quad ci \geq 0, \ \sum[c_i]^2 = 1$

More generally, given $|\phi\rangle$ interested in min rank $|\psi\rangle$:

$|\psi\rangle \sim |\phi\rangle$ in trace norm.

Entanglement Rank (Entropy)



Product state: $|a_i\rangle \otimes |b_i\rangle$

Rank k state: $|\psi\rangle = \sum_{i=0}^{k-1} ci|a_i\rangle \otimes |b_i\rangle$ $\quad ci \geq 0, \ \sum[c_i]^2 = 1$

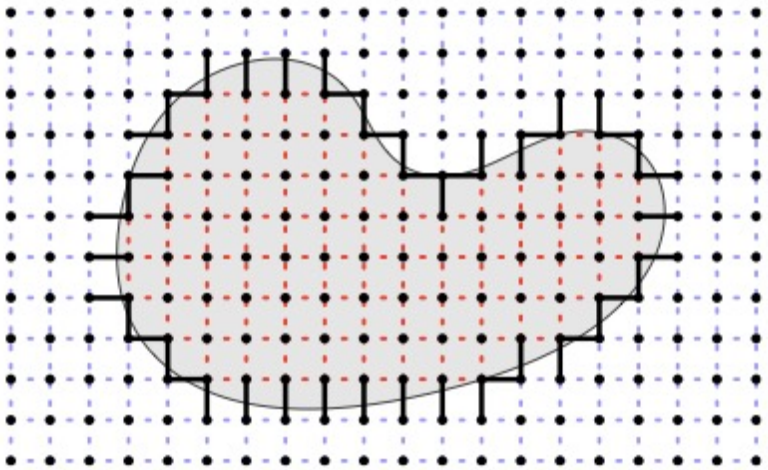
More generally, given $|\phi\rangle$ interested in min rank $|\psi\rangle$:
$|\psi\rangle \sim |\phi\rangle$ in trace norm.

Schmidt decomposition: $\{|a_i>\}$, $\{|b_i>\}$ orthonormal sets

Entanglement entropy = $H(\{c_i^2\}) = \sum c_i^2 \log 1/c_i^2$

Area Law

For gapped local Hamiltonians $H = H_1 + \ldots + H_m$, entanglement entropy of the ground state scales like surface area, rather than volume.



Spectral gap = $E_1 - E_0 = \epsilon$

Inspired by Holographic Principle: Black hole entropy scales like surface area.

[Vidal, Latorre, Rico, Kitaev '02]

Matrix Product States

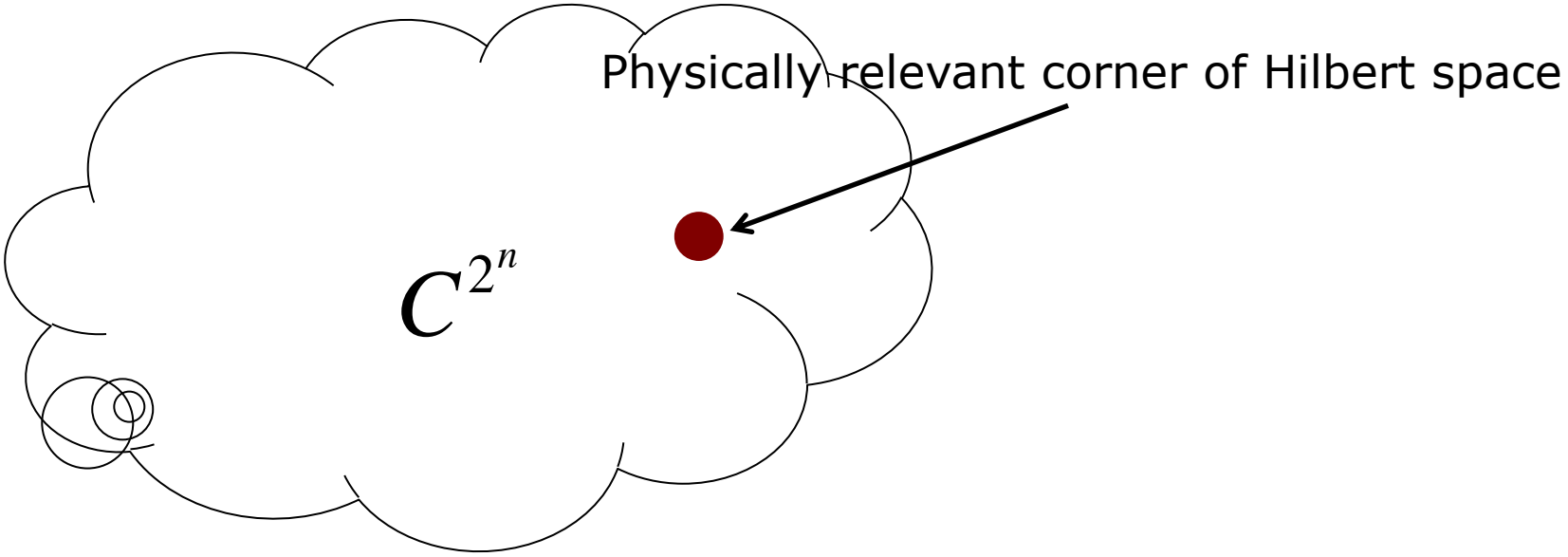
Maximally entangled pair of b-level particles: $\sum_{i=0}^{b-1} \frac{1}{\sqrt{b}} |i\rangle |i\rangle$

Map from $C^b x C^b$ to C^d

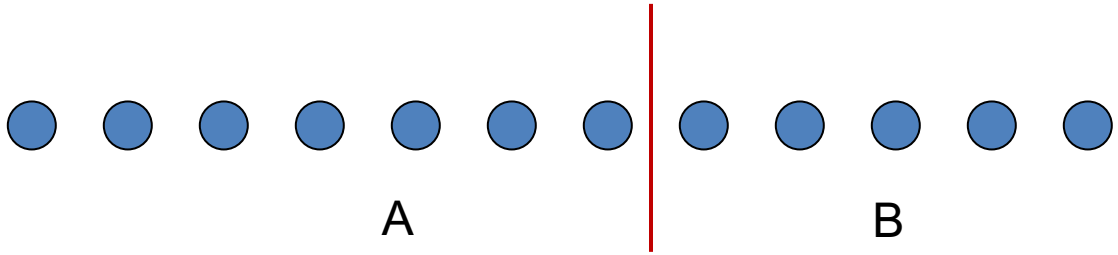
An MPS (Matrix Product State) is a Tensor Network

- Any state with entanglement rank b can be written as a bond b MPS
- Can do standard linear algebra efficiently: addition, inner products, applying linear operators in $O(nb^2)$ time.

Outline of talk



- Ground state of gapped 1D Hamiltonian satisfies an area law, and has a succinct MPS representation.
- Rigorous Renormalization Group: an efficient algorithm for computing ground and low energy states. As a corollary show that degenerate ground spaces and low energy states satisfy an area law, and have succinct MPS.
- Discuss 2D and other open questions in QHC.



1D area law equivalent to saying that there is a product state $|a_i\rangle \otimes |b_i\rangle$ with constant overlap with the ground state:

$$|\Omega\rangle \approx \sum_{i=0}^{k-1} ci|a_i\rangle \otimes |b_i\rangle$$

Random product state has exponentially small overlap with ground state.

Area Law for 1D Gapped Hamiltonians

[Hastings 2007] $S_{1D} = O(\exp(1/\epsilon \log d)$

Used quantum mutual information and
Lieb-Robinson bound on entanglement spread

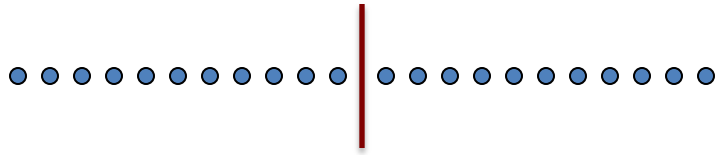
Combinatorial Techniques:

[Aharonov, Arad, Landau, V 2009]

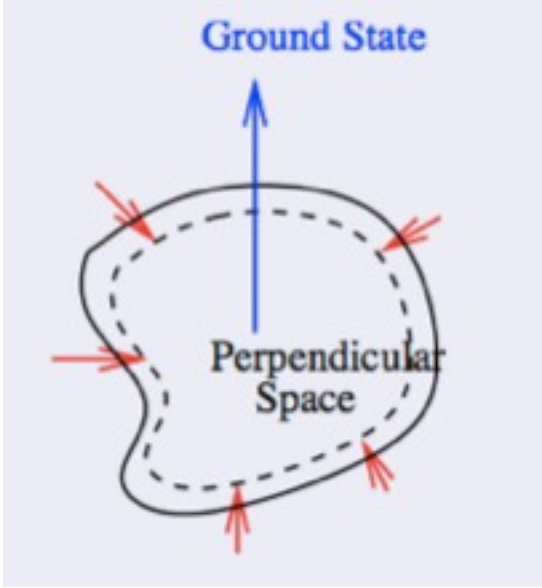
…

[Arad, Landau, V 2012]

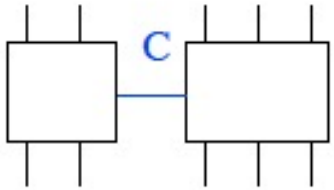
AGSP: Approximate Ground State Projector



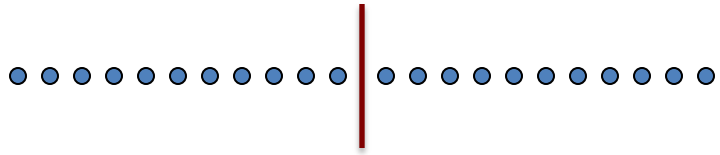
An AGSP is a "low complexity" operator K that approximately projects onto the ground state.

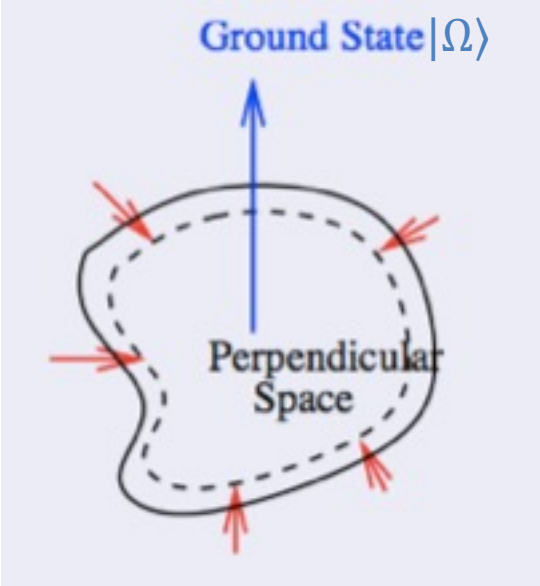


An operator on $\mathcal{H}_1 \otimes \mathcal{H}_2$ of the form $\sum_1^C A_i \otimes B_i$ will be said to have entanglement rank C.



AGSP: Approximate Ground State Projector

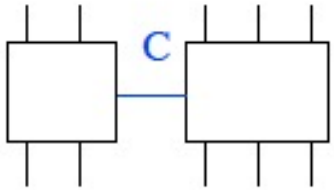




An AGSP is a ”low complexity” operator K that approximately projects onto the ground state:

- $K|\Omega\rangle = |\Omega\rangle$
- Shrinks orthogonal space by $\Delta < 1$
- Has low entanglement rank D: $D\Delta << 1$

An operator on $\mathcal{H}_1 \otimes \mathcal{H}_2$ of the form $\sum_1^C A_i \otimes B_i$ will be said to have entanglement rank C.



Improvement Lemma

K = (D, Δ)-AGSP with DΔ < ½

Claim: ∃ product state $|\psi\rangle = |L\rangle \otimes |R\rangle$: $\langle\psi\,|\Omega\rangle \geq \frac{1}{\sqrt{2D}}$

Proof:

$|\Omega\rangle = \sum a_i\,|L_i\rangle \otimes |R_i\rangle$

a = largest Schmidt coefficient/vector $|L\rangle \otimes |R\rangle$

K($|L\rangle \otimes |R\rangle$) has Schmidt rank D, and one of its Schmidt vectors must have the desired overlap:

K($|L\rangle \otimes |R\rangle$) = a $|\Omega\rangle$ + s$|\Omega^{\perp}\rangle$

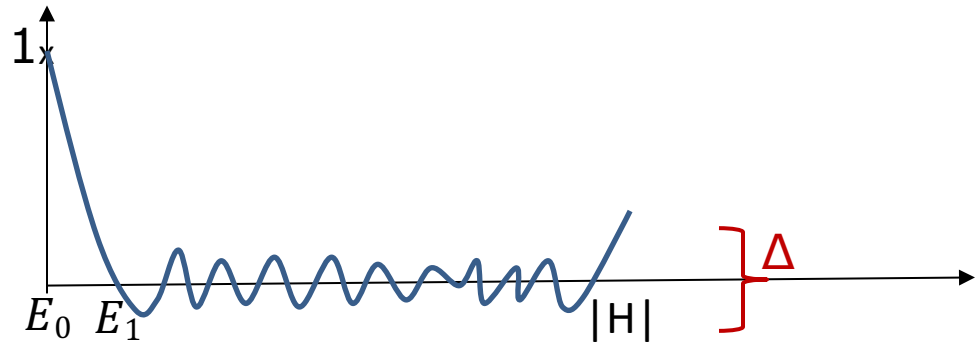
Squared inner product with $|\Omega\rangle$ at most Da^2

$\frac{a^2}{a^2+s^2} \leq Da^2$ i.e. $D(a^2+s^2) \geq 1$. But $D\,s^2 \leq$ ½

So $D\,a^2 \geq$ ½. i.e. $a \geq \frac{1}{\sqrt{2D}}$

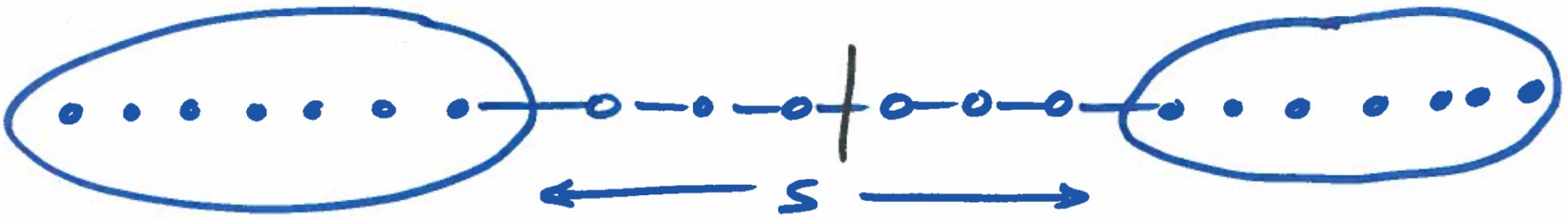
AGSP construction – quick sketch

K = polynomial in H



- Use Chebyshev polynomial to minimize degree
- Degree a function of |H|. So truncate H to reduce norm.
- Can bound entanglement rank as a function of degree

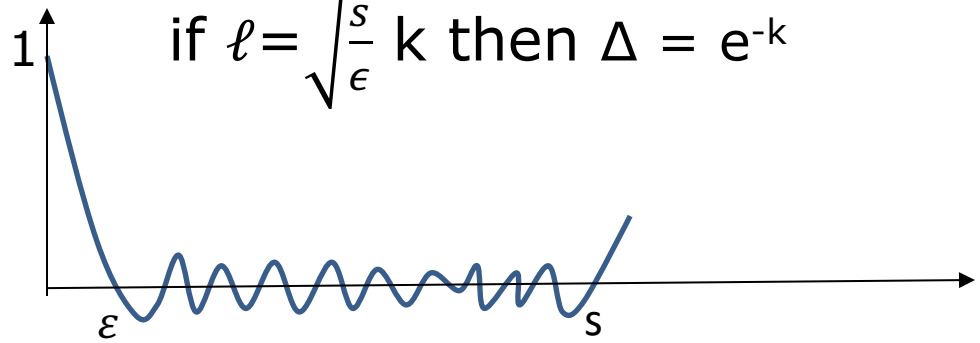
AGSP Construction



$H' = H_L + H_1 + \ldots + H_s + H_R$

H' has spectral gap ϵ

$K = C_\ell(H')$, where C_ℓ is a scaled Chebyshev polynomial



Theorem: Choose $s = O\left(\frac{log^2 d}{\epsilon}\right)$ then K = (D, Δ)-AGSP,

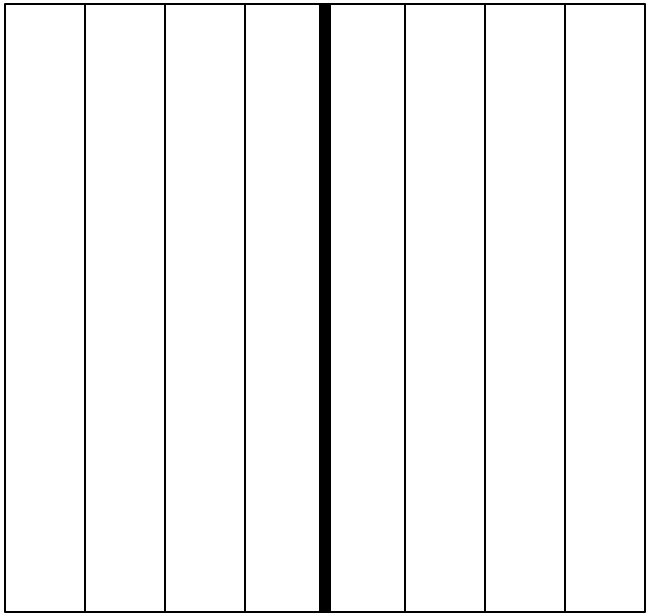
with log $D = O\left(\frac{log^3 d}{\epsilon}\right)$. Area law with $S_{1D} = O\left(\frac{log^3 d}{\epsilon}\right)$

Area laws for higher dimensional lattices

- Any improvement of exponent in 1D bound, $S_{1D} = O(\log^3 d/\varepsilon)$ would yield non-trivial result for 2D.

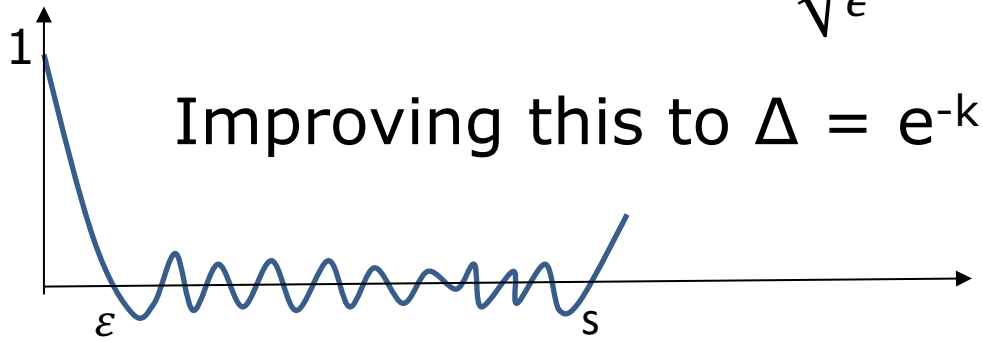
- $S_{1D} = O(\log d/\varepsilon)$ implies area law for every dimension

$$O(\log D) = O(\log d^B) = O(B \log d)$$



- In Chebyshev construction,

$\Delta = e^{-k}$ for degree $\ell = \sqrt{\frac{s}{\epsilon}}\, k$



Improving this to $\Delta = e^{-k}$ for degree $\ell = \sqrt{\frac{sk}{\epsilon}}$ would give area law for every dim

Efficient Algorithms

Polynomial time algorithm for computing ground state of gapped 1D Hamiltonian

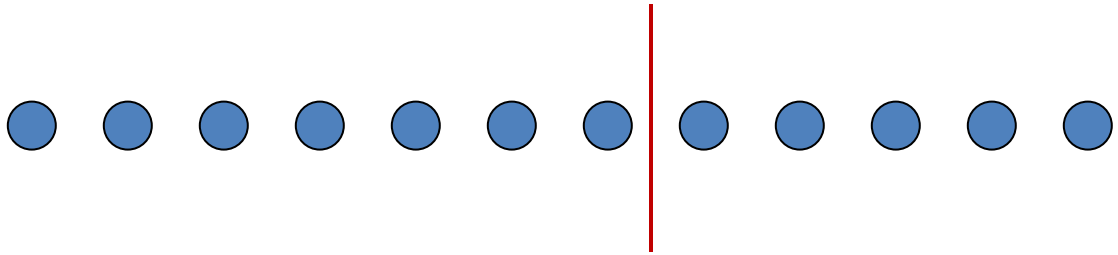
- Running time high degree polynomial
- Unpractical

[Landau, V, Vidick *Nature Physics 2015]*

Rigorous RG Algorithm for computing low energy states [Arad, Landau, V, Vidick *Comm in Math Physics 2017]*

- [Roberts, Vidick, Motrunich *Phys Rev B 2017*]
  [Bloch et al 2020]
  Implementions that suggest RRG can sometimes give more reliable results than DMRG near first order phase transitions and in topological phases, since DMRG's local update procedure can fail to adequately explore near degenerate manifolds.

- Proves area laws and succinct MPS representation for low energy and degenerate ground states



Suppose Hamiltonian H has a degenerate ground space

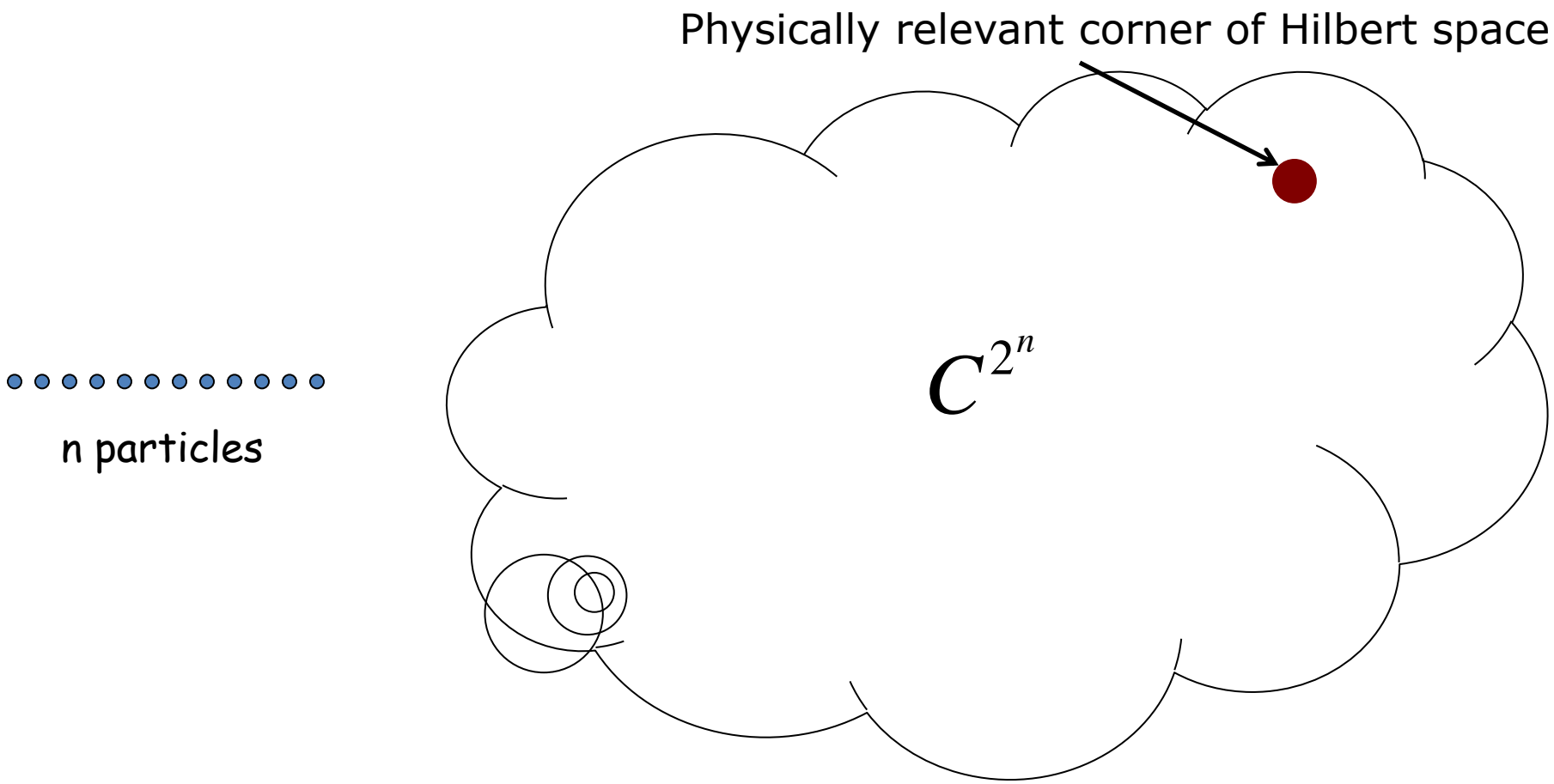
Existing area law proof shows that for a given cut there exists some vector in ground space with small entanglement rank (entropy) across this cut

Hard to extend argument to all vectors in ground space

Doesn’t show that there is a particular state with small entanglement rank (entropy) across all cuts

Key to overcoming this obstacle is to approach the whole ground state at the same time

Efficient Algorithms

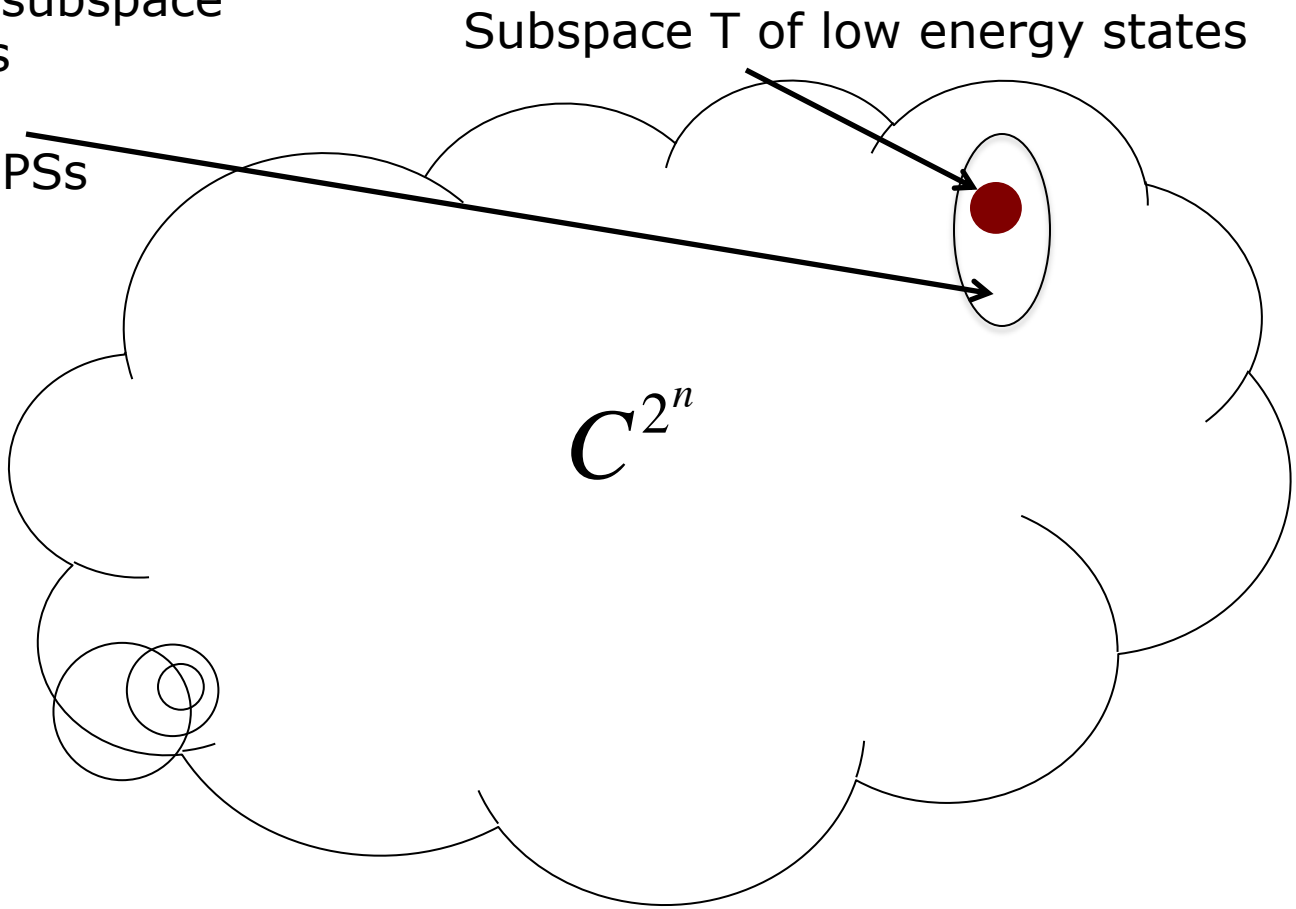


Cannot use usual algorithms for computing eigenvectors, since most of the vectors in the space do not have compact description

What we would like!

Quickly identify a small subspace in which the solution is guaranteed to lie

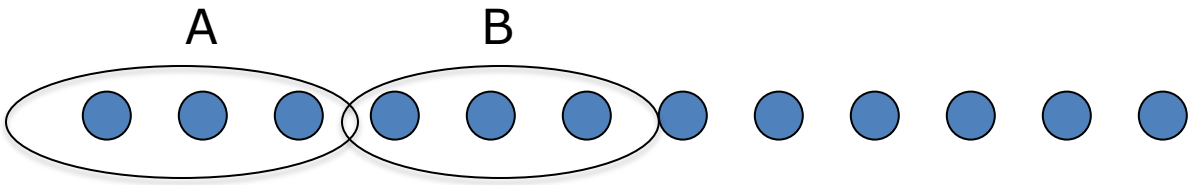
Described by succinct MPSs



$H = H_1 + H_2 + \ldots H_{n-1}$

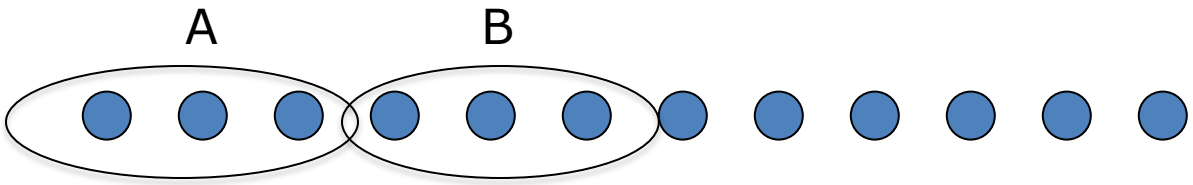
Once this is done, can compute eigenvectors quickly!

Algorithmic Framework



- Locally process a few particles – what should a partial solution look like? Viable set
- How to combine partial solutions for A and B?
- Algorithmic design primitives for viable sets.
- Resulting algorithm has a hierarchical structure.

Algorithmic Framework



- Locally process a few particles – what should a partial solution look like? Viable set

Answer: it should be a subspace!

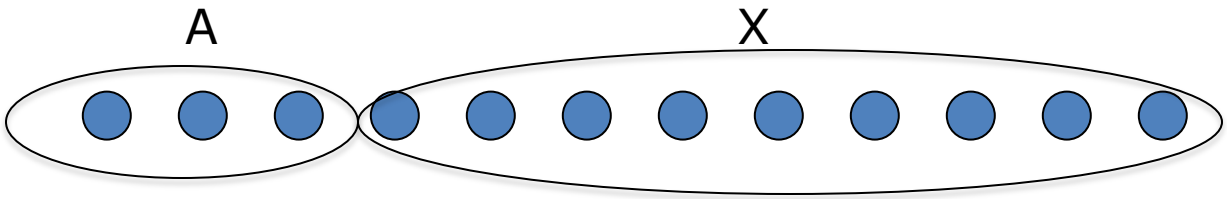
Suppose we are trying to find unique ground state $|\psi>$

What does a partial solution on the first few particles look like?

If $|\psi\rangle = \sum c_i |a_i\rangle \otimes |b_i\rangle, \quad c_i \geq 0$

then partial solution looks like S = span{$|a_i>$}

Local Approach



T = Target subspace of low energy states

Ideally: Identify a subspace $S \subseteq A$ such that $T \subseteq S \otimes X$

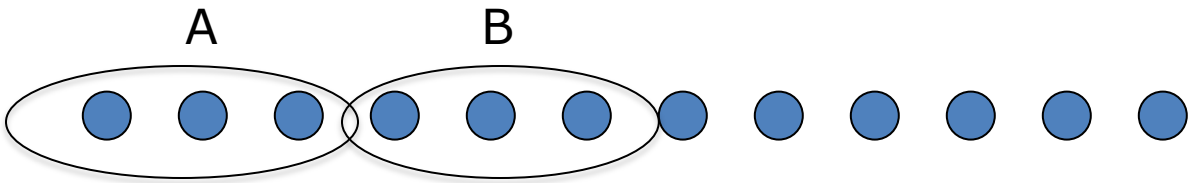
Definition: We will say that S is a δ −viable set if

$$P_T P_{S \otimes X} P_T \geq (1 - \delta) P_T$$

δ = error. $(1 - \delta)$ = overlap

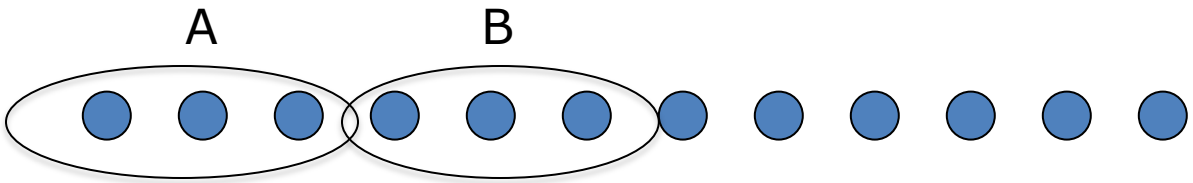
Want: dim(S) small, δ small.

RRG Step



1. Tensoring
   If $S_1 \subseteq A$ and $S_2 \subseteq B$ are δ-viable with $\mathrm{Dim}(S_i)=s$
   then $S_1 \otimes S_2$ is 2δ-viable with $\mathrm{Dim}(S_1 \otimes S_2) = s^2$

RRG Step

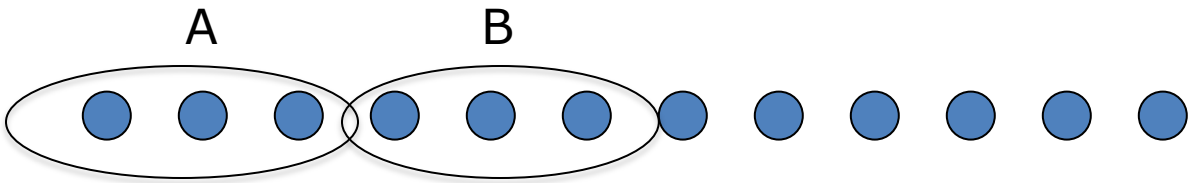


1. Tensoring
   If $S_1 \subseteq A$ and $S_2 \subseteq B$ are δ-viable with $\mathrm{Dim}(S_i)=s$
   then $S_1 \otimes S_2$ is 2δ-viable with $\mathrm{Dim}(S_1 \otimes S_2) = s^2$

2. Random projection: Can tradeoff dimension for overlap

- S is δ-viable i.e. overlap = 1- δ, and Dim(S) = s,
- Let R be a random subspace of S of dimension r.
  Then R has overlap ~ $(1-\delta)\, r/s$

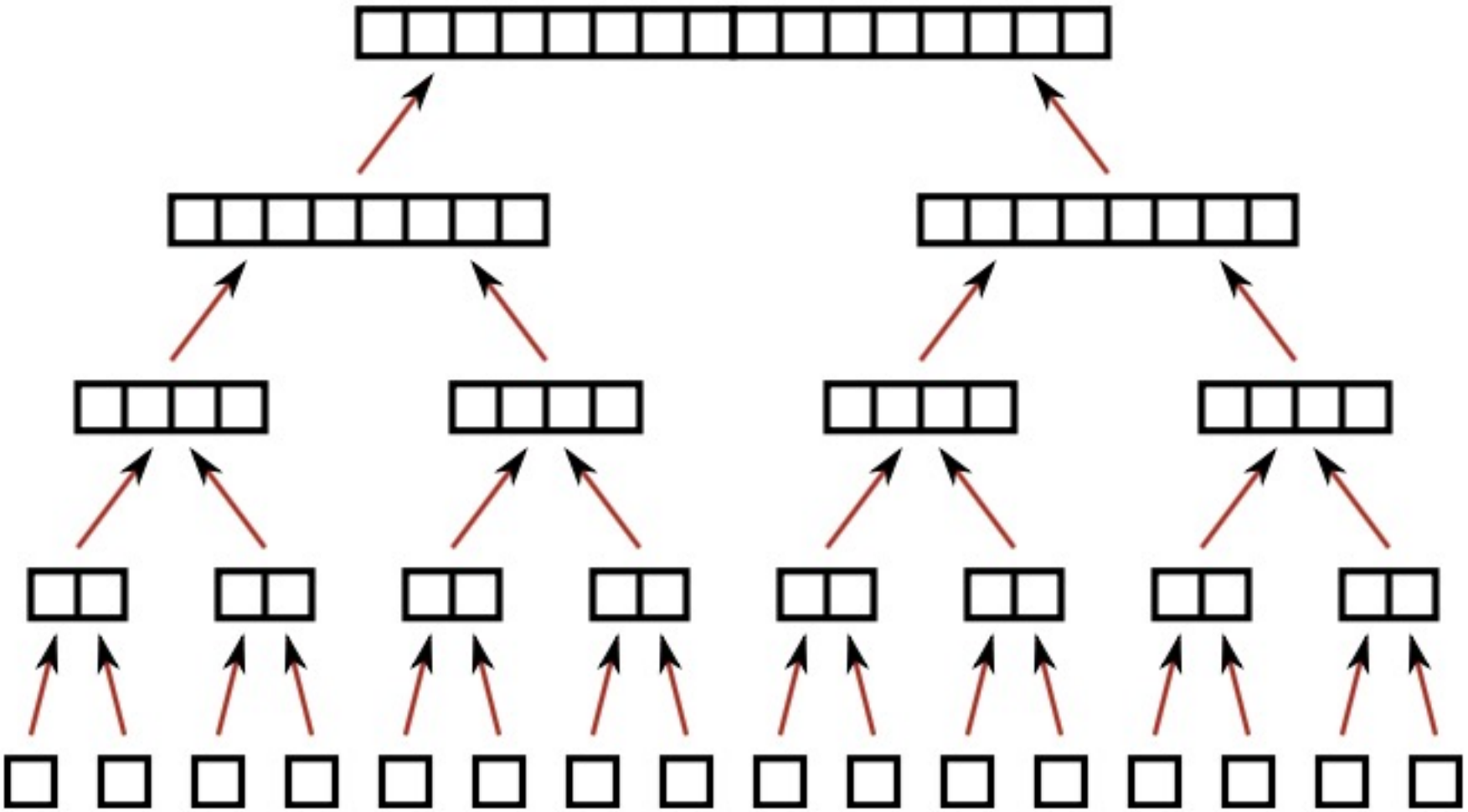
RRG Step



1. Tensoring
   If $S_1 \subseteq A$ and $S_2 \subseteq B$ are δ-viable with $\mathrm{Dim}(S_i)=s$ then $S_1 \otimes S_2$ is 2δ-viable with $\mathrm{Dim}(S_1 \otimes S_2) = s^2$

2. Random projection: Can tradeoff dimension for overlap
   - S is δ-viable i.e. overlap = 1- δ, and Dim(S) = s,
   - Let R be a random subspace of S of dimension r. Then R has overlap ~ $(1-\delta)\, r/s$

3. Error Reduction: large reduction in error at expense of small small blowup in dimension!
   - If S is δ-viable and Dim(S) = s. Then error reduction yields S′ which is $\Delta/(1-\delta)^2$-viable with $\mathrm{Dim}(S') = D^2 s$.
   - Error reduction carried out by applying D-Δ AGSP, which satisfies $D^{16}\Delta << 1$



Comments on RRG Algorithm

- Must modify AGSP construction to ensure:
  1. Bond dimension away from special cuts is O(poly(n))
  2. Computationally efficient

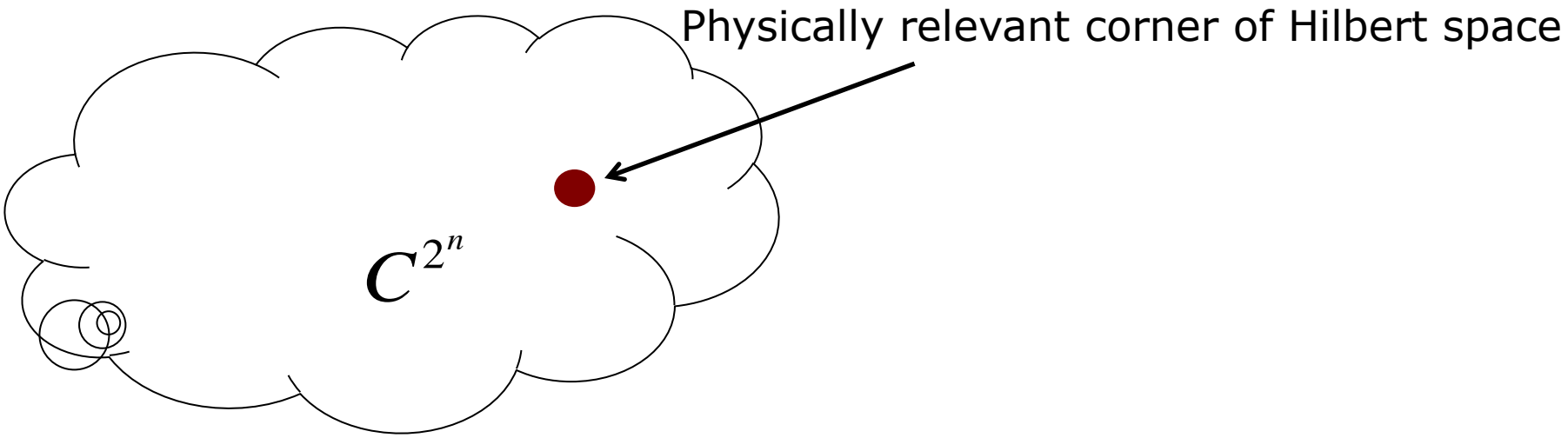
- Must pay attention to complexity of describing each element of viable set as MPS – bond trimming

- For frustration-free Hamiltonians with unique ground state, the algorithm runs in near linear time, under a certain conjecture about bond trimming.

- Since algorithm is efficient, it proves that degenerate ground and low energy state satisfy an area law and have succinct MPS descriptions

- Heuristic implementations [Roberts, Vidick, Motrunich '17], [Block et al '20] suggest RRG does better than DMRG near first order phase transitions and topological phases

Discussion and Open Questions



- Area laws and succinct MPS for 1D systems

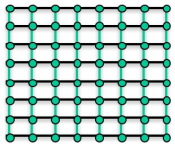
Open questions:

- Area law for 2D and higher dimensional systems
- Succinct tensor network descriptions
- Efficient algorithms for finding succinct descriptions

[Anshu, Arad, Gosset '20] Recent progress on area law for frustration-free 2D Hamiltonians with local gap.

- Based on robust polynomials, RRG type arguments
- $\ell = \sqrt{\frac{s}{\epsilon}}\,\mathrm{k}$ vs $\ell = \sqrt{\frac{sk}{\epsilon}}$ bound for 1D

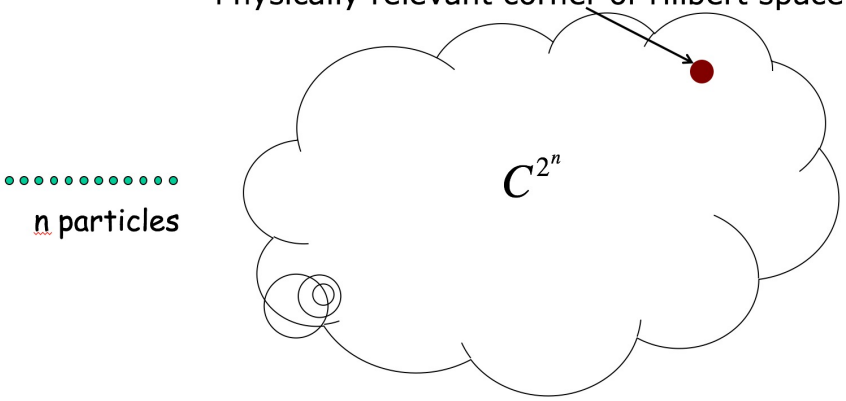
Quantum Inspired Classical Algorithms



$$H = H_1 + \cdots + H_m$$

Ground and low-energy states of local Hamiltonian

Physically relevant corner of Hilbert space

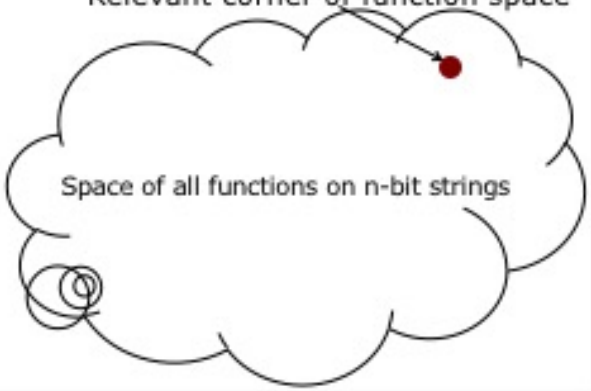


Deep Neural Nets

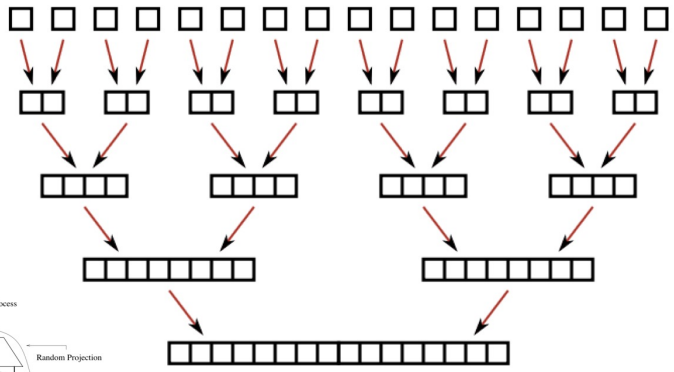
Missing a theoretical understanding of why they work

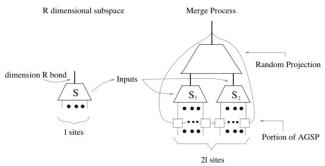
Learning a function f:{0,1}^{n} → {0,1}^{n}

Relevant corner of function space



Rigorous RG: Provably efficient algorithm for computing low energy states of 1D systems
[Arad, Landau, V, Vazirani 2016]





Tensor Networks, RG and Deep Nets

Relationship between convolutional deep nets and variant of tree tensor networks
[Levine, Sharir, Cohen, Shashua 2018]