Face numbers: the upper bound side of the story

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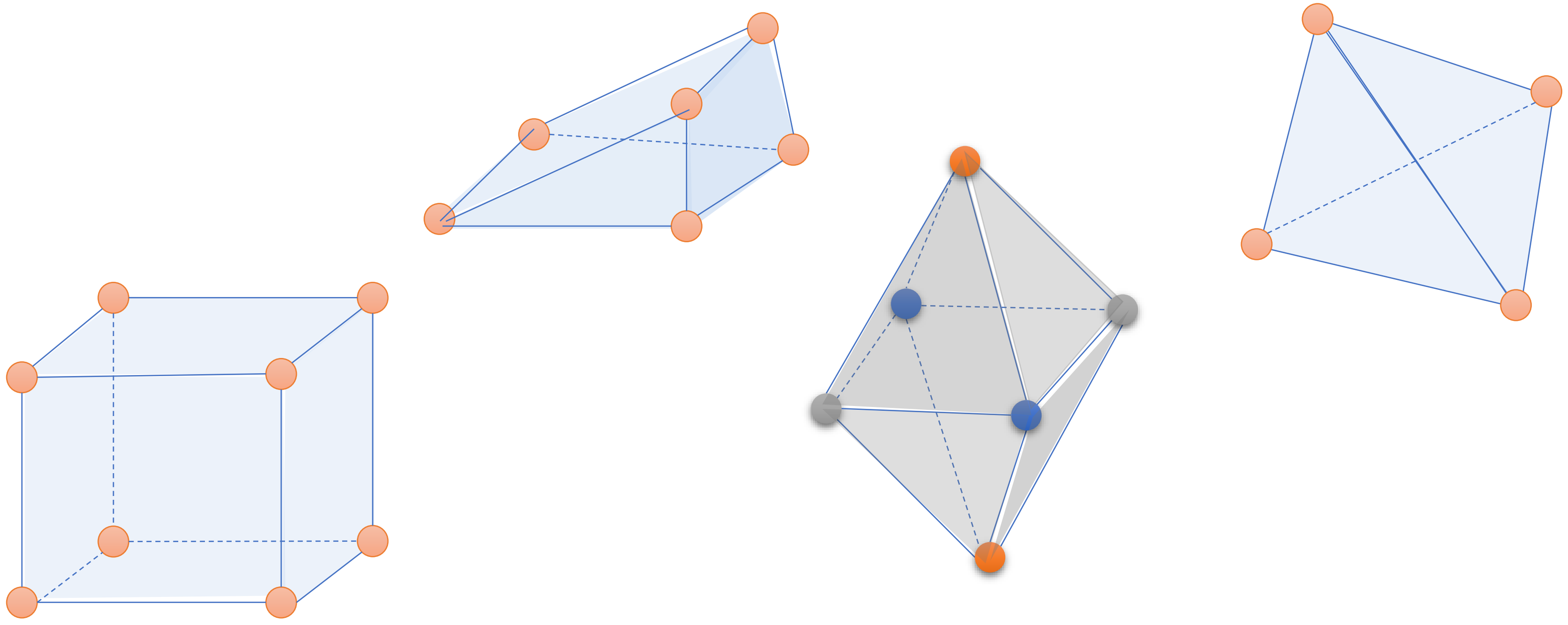
I. Basics on Polytopes

A polytope is the convex hull of *finitely many* points in $\mathbb{R}^d$

Equivalently, it is a bounded intersection of *finitely many* closed half-spaces in $\mathbb{R}^d$

Example: a simplex is the convex hull of affinely independent points

Polytopes are studied and have applications in combinatorics, discrete geometry, optimization, analysis, statistics, …



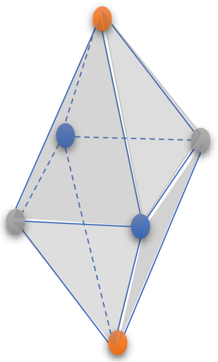
Faces and face numbers

- The dimension of a polytope P is the dimension of its affine hull
- A face of P is the intersection of P with a supporting hyperplane
- A face F of P is itself a polytope; it is an i-face if $\dim F = i$

Given a *d*-polytope we can count how many vertices, edges, 2-faces,..., $(d-1)$-faces (also known as facets) it has:

$f_i(P) :=$ # of i–faces of P $\quad f(P) = (f_0, f_1, \dots, f_{d-1})$

Example:

$f($  $) = (6, 12, 8)$

Motivation --- the Upper Bound Problem

- What is the largest number of i-faces that a d-polytope with n vertices can have?
- Connection to optimization: the dual form of this question is "What is the largest number of vertices that a d-polytope defined by n linear constraints can have?"

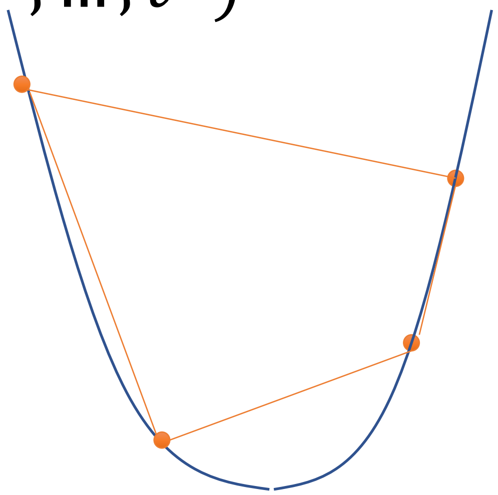
To start, we need to look for polytopes with many faces

This leads us to the cyclic polytopes (discovered and rediscovered by Carathéodory, Gale, Motzkin,…)

Cyclic polytopes

Moment curve: $M = M_d : \mathbb{R} \to \mathbb{R}^d$

$$t \mapsto (\mathrm{t}, \mathrm{t}^2, \mathrm{t}^3, \ldots, t^d)$$



Let $t_1 < t_2 < \cdots < t_n \in \mathbb{R}$

The cyclic polytope, $C(d, n)$, is defined as

$$\mathrm{conv}(M(t_1), M(t_2), \ldots, M(t_n))$$

Properties of cyclic polytopes

- $C(d,n)$ is a d-dimensional simplicial polytope on n vertices (i.e., all facets are simplices)
- The combinatorial type of $C(d,n)$ is independent of the choice of $t_1, t_2, \ldots, t_n$: there is a complete characterization of the vertex sets of facets due to Gale (Gale's evenness condition)
- $C(d,n)$ is $\lfloor \frac{d}{2} \rfloor$**-neighborly**: every set of $\leq \lfloor \frac{d}{2} \rfloor$ vertices forms the vertex set of a face. Thus

$$f_{i-1}(C(d,n)) = \binom{n}{i} \quad \forall i \leq \lfloor \frac{d}{2} \rfloor$$

A digression: simplicial complexes and simplicial spheres

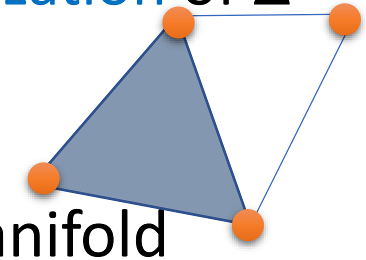
Def $\Delta \subseteq 2^V$ is a **simplicial complex** on a finite vertex set V if

- $\{v\} \in \Delta \quad \forall v \in V$
- $F \in \Delta,\ G \subset F \Rightarrow G \in \Delta$

Elements of V are vertices, elements of Δ are faces

A face F is an i-face if $|F| = i + 1$; the number of i-faces is $f_i(\Delta)$

Simplicial complex Δ $\rightarrow$ Topological space $||\Delta||$ = the geometric realization of Δ



Δ is a simplicial $(d-1)$-sphere if $||\Delta||$ is homeomorphic to S^{d-1}

Δ is a simplicial manifold if $||\Delta||$ is homeomorphic to a (closed) manifold

The Upper Bound Theorems

The Upper Bound Conjecture

Motzkin, 1957: Among all d-polytopes with n vertices, the cyclic polytope simultaneously maximizes all the face numbers $f_i(P) \leq f_i(C(d,n))\ \forall i$

V. Klee, 1964: Among all Eulerian complexes of dimension $d-1$ with n vertices, the boundary complex of the cyclic polytope simultaneously maximizes all the face numbers

Eulerian complexes include all simplicial spheres, all odd-dim manifolds, all even-dim manifolds with Euler char 2

The Upper Bound Theorems:

- P. McMullen, 1970: The UBC holds for all polytopes
- R. Stanley, 1975: The UBC holds for all simplicial spheres
- N, 1998: The UBC holds for all Eulerian manifolds

Comments

McMullen's proof uses *shellability* of polytopes

Stanley's proof relies on the theory of Stanley-Reisner rings and specifically on the properties of Cohen-Macaulay rings

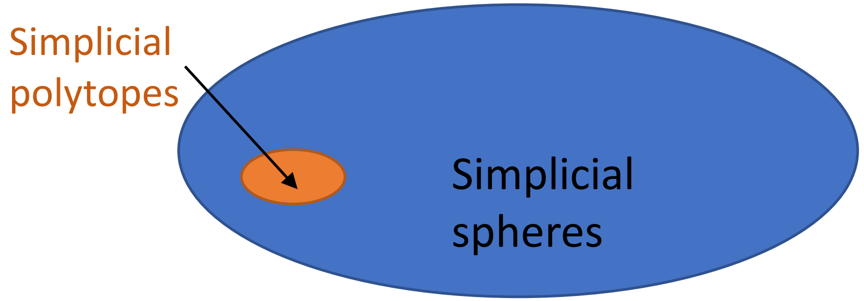
The proof for manifolds relies on the properties of Buchsbaum rings

$C(d,n)$ in the statement of the UBT can be replaced with any $\lfloor\frac{d}{2}\rfloor$-neighborly d-polytope or $(d-1)$-sphere with n vertices. This leads us to the question of how many $\lfloor\frac{d}{2}\rfloor$-neighborly $(d-1)$-spheres with n vertices are there?

II. There are many more spheres than polytopes

- Let $c(d,n)$ = # of simplicial d-polytopes with n (labeled) vertices
- Let $s(d,n)$ = # of simplicial $(d-1)$-spheres with n (labeled) vertices

Steinitz's theorem implies that $c(3,n) = s(3,n)$



Theorem (Goodman-Pollack; Alon 1986)

For $d \geq 4$, $c(d,n) = 2^{\Theta(n \log n)}$, i.e.,

$$2^{a_d n \log n} \leq c(d,n) \leq 2^{A_d n \log n} \text{ for some constants } a_d, A_d > 0.$$

Theorem (Kalai, 1988; Pfeifle-Ziegler, 2004; Nevo-Santos-Wilson, 2016)

For $d \geq 4$, $2^{\Omega\left(n^{\lfloor \frac{d}{2} \rfloor}\right)} \leq s(d,n) \leq 2^{O\left(n^{\lfloor \frac{d}{2} \rfloor} \log n\right)}$ (e.g., $2^{\Omega(n^2)} \leq s(4,n) \leq 2^{O(n^2 \log n)}$)

What proportion of $\boldsymbol{d}$-polytopes are $\lfloor\frac{\boldsymbol{d}}{\boldsymbol{2}}\rfloor$-neighborly ?

Results of Shemer and Padrol indicate that "most of d-polytopes are $\lfloor\frac{d}{2}\rfloor$-neighborly as $n \to \infty$":

Theorem (Shemer, 1982; Padrol, 2013)

- There are $2^{\Theta(n \log n)}$ $\lfloor\frac{d}{2}\rfloor$-neighborly simplicial d-polytopes with n vertices.

  Padrol's lower bound on the number of $\lfloor\frac{d}{2}\rfloor$-neighborly d-polytopes with n vertices is currently the best known lower bound on $c(d,n)$

- There are at least $2^{\Omega(n \log n)}$ $\lfloor\frac{d}{2}\rfloor$-neighborly $(d-1)$-spheres with n vertices arising from non-realizable oriented matroids

What proportion of $(\boldsymbol{d-1})$-spheres are $\lfloor\frac{\boldsymbol{d}}{\boldsymbol{2}}\rfloor$-neighborly ?

Let $\mathrm{sn}(d,n)$ = # of $\lfloor\frac{d}{2}\rfloor$-neighborly simplicial $(d-1)$-spheres with n vertices

Conjecture (Kalai, 1988) For all $d \geq 4$, $\lim_{n\to\infty} \frac{\log sn(d,n)}{\log s(d,n)} = 1$

Theorem (N-Zheng, 2021+) There are many neighborly spheres: for all $d \geq 5$,

$$\mathrm{sn}(d,n) \geq 2^{\Omega\left(n^{\left\lfloor\frac{d-1}{2}\right\rfloor}\right)}$$

The proof is by construction based on Kalai's squeezed balls --- certain subcomplexes of the boundary complex of $C(d,n)$

[For comparison, recall that $2^{\Omega\left(n^{\left\lfloor\frac{d}{2}\right\rfloor}\right)} \leq \mathrm{s}(d,n) \leq 2^{\Omega\left(n^{\left\lfloor\frac{d}{2}\right\rfloor}\log n\right)}$]

Summary so far

- For $d \geq 4$, there are many more simplicial $(d-1)$-spheres than d-polytopes
- Nonetheless, d-polytopes with n vertices and $(d-1)$-spheres with n vertices (and even Eulerian $(d-1)$-manifolds with n vertices) satisfy the **same** Upper Bound Theorem

In fact, recent very exciting news (due to Adiprasito, and Papadakis and Petrotou) is that the set of f-vectors of simplicial $(d-1)$-spheres coincides with the set of f-vectors of simplicial d-polytopes

- The maximizers are given by $\lfloor \frac{d}{2} \rfloor$-neighborly d-polytopes and $\lfloor \frac{d}{2} \rfloor$-neighborly $(d-1)$-spheres
- There are many $\lfloor \frac{d}{2} \rfloor$-neighborly d-polytopes; there are also many $\lfloor \frac{d}{2} \rfloor$-neighborly $(d-1)$-spheres

III. Cs polytopes and cs spheres

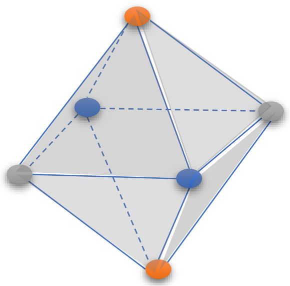
- A polytope $P \subset \mathbb{R}^{\mathrm{d}}$ is centrally symmetric if $x \in P \Leftrightarrow -x \in P$
- A simplicial sphere Δ is centrally symmetric if $\exists \varphi: V \to V$ such that

$$\varphi(F) \in \Delta, \qquad \varphi(\varphi(F)) = F, \text{but } \varphi(F) \neq F \quad \forall \emptyset \neq F \in \Delta$$

(The vertices v and $\varphi(v)$ are called antipodal)

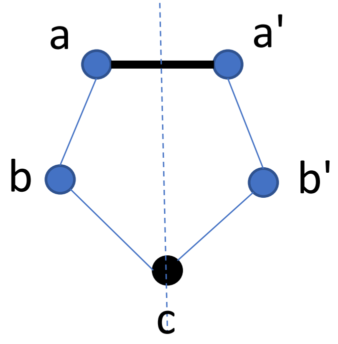
Note: if Δ is cs, then v and $\varphi(v)$ are **not** connected by an edge!

Example



$\varphi(v) = -v$

Non-example



The Upper Bound Problem for cs polytopes and spheres

Problems:

- What restrictions does being cs impose on the f-vectors?
- More specifically, what is the largest number of i-faces that a cs d-polytope with n vertices can have? What is the largest number of i-faces that a cs $(d-1)$-sphere with n vertices can have?

Motivation:

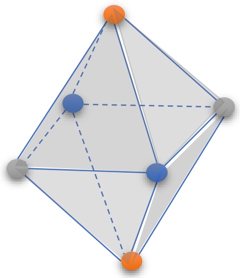
Donoho, and Rudelson and Vershynin observed that cs polytopes with **many** faces have applications in sparse signal reconstruction and coding theory

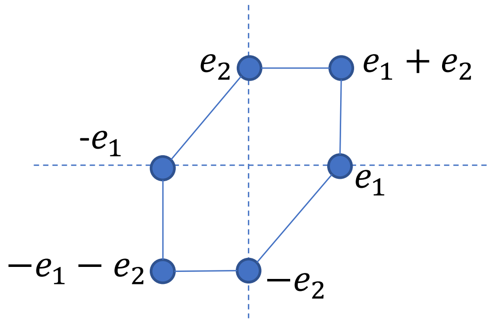
Cs neighborliness

A cs simplicial sphere Δ is cs-k-neighborly if every set of $\leq k$ vertices of Δ *no two of which are antipodal* is the vertex set of a face of Δ

Examples:

1) the d-dimensional cross-polytope $C_d^* :=\ \mathrm{conv}(\pm \mathrm{e}_1, \pm \mathrm{e}_2, \ldots, \pm \mathrm{e}_\mathrm{d})$ is cs-d-neighborly





2) McMullen-Shephard, 1968: $\mathrm{conv}(\pm \mathrm{e}_1, \pm \mathrm{e}_2, \ldots, \pm \mathrm{e}_\mathrm{d}, \pm(e_1 + \cdots + e_d))$ is cs-$\lfloor \frac{d}{2} \rfloor$-neighborly

How neighborly can a cs polytope be?

Keeping the cyclic polytope in mind, we might expect the answer to be $\left\lfloor \frac{d}{2} \right\rfloor$.

However:

Theorem (McMullen-Shephard, 1968; $d = 4$ case is due to Grünbaum, 1967)

A cs d-polytope with $\geq 2(d+2)$ vertices cannot be cs-$\left(\left\lfloor \frac{d+1}{3} \right\rfloor + 1\right)$-neighborly (e.g., a cs 4-polytope with 12 vertices cannot be cs-2-neighborly)

Theorem (Burton, 1991) A cs d-polytope with a sufficiently large number (about d^d) of vertices cannot be even cs-2-neighborly

Cs-2-neighborliness of cs polytopes

Theorem Let $d \geq 3$ be any integer.

1) [Linial-N, 2006] A cs d-polytope with 2^d or more vertices cannot be even cs-2-neighborly

The proof is based on the volume argument going back to Danzer-Grünbaum

2) [N, 2018] There exists a cs d-polytope with $2^{d-1}+2$ vertices that is cs-2-neighborly.

Embed the $(d-1)$-cube C_{d-1} in $\mathbb{R}^d$ as $[-1,1]^{d-1} \times \{0\}$ and perturb its vertices using the d-th dimension

Thus, the maximum number of vertices that a cs-2-neighborly d-polytope can have lies in $[2^{d-1}+2, 2^d-2]$.

Open: What is this number? [For $d = 3, 4$, it is $2^{d-1}+2$.]

From non-neighborliness to f-numbers

What is the value of

$\mathrm{fmax}(d, n; 1) := \max\{f_1(P) : P \text{ is a cs polytope}, \dim \mathrm{P} = d, f_0(P) = n)$?

By non cs-2-neighborliness, if $n \geq 2^d$, then $\mathrm{fmax}(d, n; 1) < \binom{n}{2} - \frac{n}{2}$

Theorem [Barvinok-N, 2008; Barvinok-Lee-N, 2013]

For an even $d \geq 4$,

$$\left(1 - 3 \cdot \left(\sqrt{3}\right)^{-d}\right)\binom{n}{2} \leq \mathrm{fmax}(d, n; 1) \leq \left(1 - 2^{-d}\right)\frac{n^2}{2}$$

For $d = 4$, $\frac{3}{4} \cdot \frac{n^2}{2} - O(n) \leq \mathrm{fmax}(4, n; 1) \leq \frac{15}{16} \cdot \frac{n^2}{2}$

Wide open: what is the value of $\mathrm{fmax}(4, n; 1)$?

IV. cs neighborliness of cs spheres

We saw that cs-neighborliness of cs polytopes is quite restricted

What about *cs simplicial spheres*?

Theorem (Adin, 1991; Stanley) Among all cs simplicial $(d-1)$-spheres on n vertices, a cs-$\lfloor \frac{d}{2} \rfloor$-neighborly simplicial sphere simultaneously maximizes all the face numbers, *assuming such a sphere exists*

Does it exist?

Grünbaum, late 60s: there is a cs simplicial 3-sphere with 12 vertices that is cs-2-neighborly.

Jockusch, 1995: For every $m \geq 4$, there exists a cs simplicial 3-sphere with $2m$ vertices that is cs-2-neighborly.

Lutz, 1999: there is a cs simplicial 5-sphere with 16 vertices that is cs-3-neighborly; there is also a cs simplicial 7-sphere with 18 vertices that is cs-4-neighborly

(In contrast, by [McMullen-Shephard, 1968] there are no such cs polytopes)

It does exist!

Theorem (N-Zheng, 2020)

For every $d \geq 4$ and $m \geq d$, **there is** a cs $(d-1)$-sphere with $2m$ vertices, Δ_m^{d-1}, that is cs-$\lfloor \frac{d}{2} \rfloor$-neighborly

(In fact, for $m \gg d$, there are at least two non-isomorphic constructions)

This together with Adin's and Stanley's work leads to

The Upper Bound Theorem for cs spheres Among all cs simplicial $(d-1)$-spheres on $2m$ vertices, Δ_m^{d-1} simultaneously maximizes all the face numbers

Summary and Open problems

The f-vectors of simplicial spheres/polytopes without symmetry satisfy the same UBT.

The situation for cs spheres and cs polytopes is drastically different: the Upper Bound Problem for cs simplicial spheres is now completely resolved, but for cs *polytopes*, there is not even a plausible Upper Bound Conjecture

In fact, we don't even know

- What is the maximum possible number of edges that a cs 4-polytope with $2m$ vertices can have?
- What is the maximum possible number of vertices that a cs-2-neighborly d-polytope can have?

There are many more remaining mysteries, but let me stop here

THANK YOU!