Stable homotopy groups of spheres and motivic homotopy theory

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July 13, 2022

- Dedicated to Mark Mahowald

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- Outline:
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- This talk is partially supported by the NSF

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M: closed manifold, dim $= 3$, $\pi_0 M = \pi_1 M = 0$.
Is M homeomorphic to S^3?

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- $n \geqslant 5$, Smale (smooth), Newman, Connell. 1960's.

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1. For which n, is there a unique smooth structure on S^n?
2. How many smooth structures are there on S^n?

Kervaire–Milnor $n \geqslant 5$

- Θ_n = smooth structures on S^n
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Theorem (Kervaire–Milnor)

For $n \geqslant 5$, the subgroup Θ_n^{bp} is cyclic,

$$|\Theta_n^{bp}| = \begin{cases} 1, & \text{if } n \text{ is even,} \\ 1 \text{ or } 2, & \text{if } n = 4k+1, \\ b_k, & \text{if } n = 4k-1. \end{cases}$$

$b_k = 2^{2k-2}(2^{2k-1}-1)\cdot$ *the numerator of* $\frac{4B_{2k}}{k}$,
B_{2k}: *Bernoulli number.*

Theorem (Kervaire–Milnor)

(continued) Suppose $n \geqslant 5$.

1. *For* $n \not\equiv 2 \pmod 4$, *there is an exact sequence*

$$0 \to \Theta_n^{bp} \to \Theta_n \to \pi_n/J \to 0.$$

π_n: *n-th stable homotopy groups of spheres,*
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Theorem (Browder, Barratt–Jones–Mahowald–Tangora, Hill–Hopkins–Ravenel)

$\Phi_n \neq 0$ *if and only if* $n = 2, 6, 14, 30, 62$ *and possibly 126.*

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- S^{125}: not unique, Hurewicz image of *tmf* (the spectrum of topological modular forms).

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- S^6, S^{12}: Kervaire–Milnor
- S^{56}: Isaksen
- no more: confirmed by Behrens, Hill, Hopkins, Mahowald, Quigley for more than half of the even dimensions.

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- (Toda) *EHP*-(spectral) sequence: up to 19-stem (unstable).

Stable range computations

- (Adams) Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{A_*}^{s,t}(\mathbb{F}_p, \mathbb{F}_p) \Longrightarrow \pi_{t-s}(S^0)_p^\wedge$$

$A_* = \mathrm{H}\mathbb{F}_{p*}\mathrm{H}\mathbb{F}_p$: dual Steenrod algebra

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- (Novikov) Adams–Novikov spectral sequence

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MU: complex cobordism spectrum

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- **The Second Mahowald Uncertainty Principle**:
  Any method that computes nontrivial differentials in such a spectral sequence will leave infinitely many differentials undecided.

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& \pi_*(S^0)^\wedge_p & \\
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- Jump of filtrations!

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Theorem (Miller)

Adams d_2 differentials $\longleftrightarrow$ algebraic Novikov d_2 differentials

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- About dimension $p^3(2p-2)$

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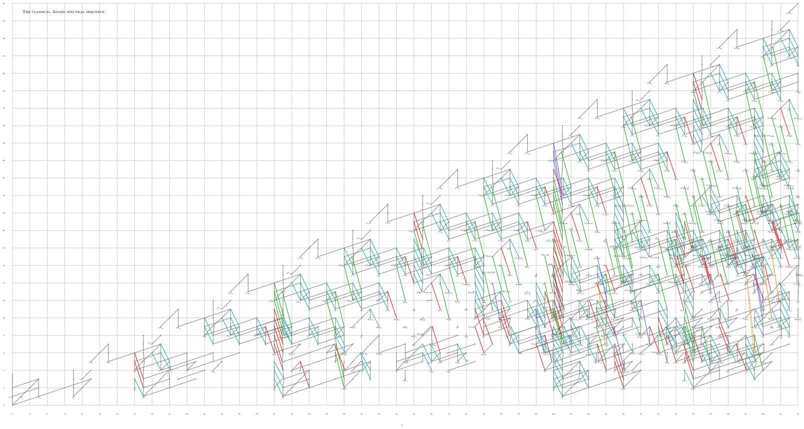
  - (Burklund–Isaksen–Xu)
    Pstrągowski's synthetic homotopy theory

Classical Adams Spectral Sequence up to 90-stem



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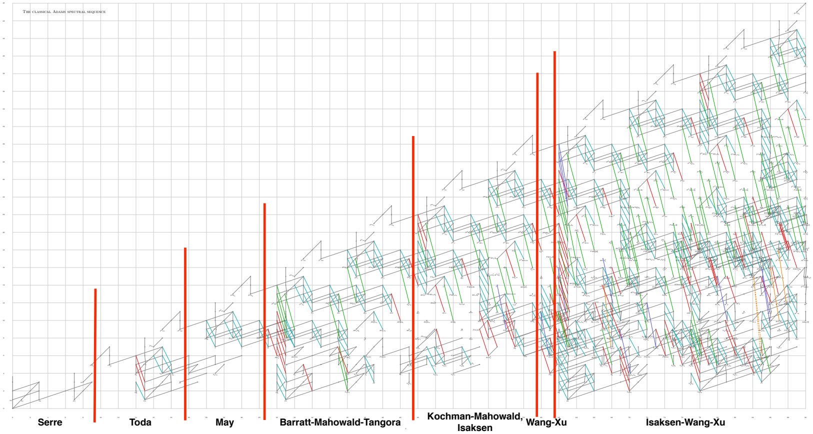
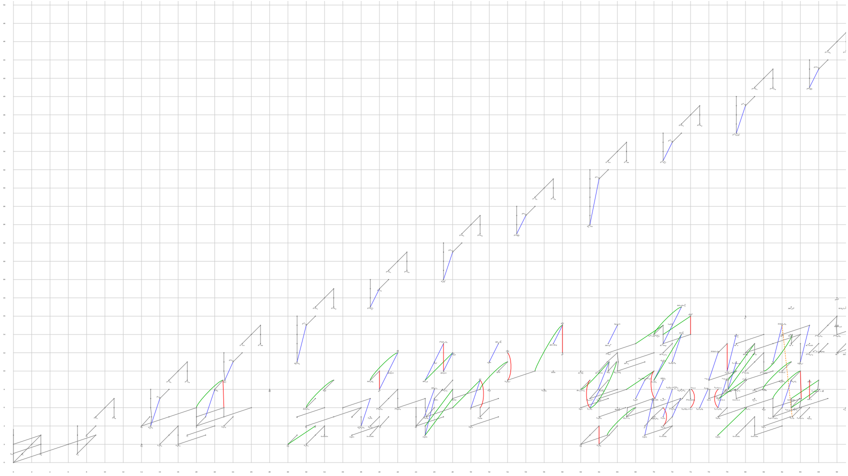


Classical Adams E_∞-page up to 90-stem



Motivic homotopy theory

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	SH	SH(k)
building blocks	simplices	simplices, smooth varieties
model category	presheaves over Δ	simplicial presheaves over $Sm(k)$
topology	trivial	Nisnevich/étale
homotopy	$[0,1]$	$\mathbb{A}^1$
basic spheres	S^1	$S^{1,0}$,$S^{1,1} = \mathbb{G}_m$
stabilization	invert S^1	invert both $S^{1,0}$ and $S^{1,1}$
sphere spectrum	S^0	$S^{0,0}$
homotopy groups	π_*	$\pi_{*,*}$

Motivic Stable Homotopy Groups of Spheres

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Motivic analogue of classical computational tools exist:

- motivic dual Steenrod algebra $A_{*,*}^{mot}$
- motivic Adams spectral sequence
- motivic Adams–Novikov spectral sequence

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motivic $\widehat{S^{0,0}}/\tau$-method

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Theorem (Gheorghe–Wang–Xu)

The above two spectral sequences are isomorphic.

τ as a deformation parameter

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There is an equivalence of stable ∞-categories:

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Theorem (Miller)

Adams d_2 differentials $\longleftrightarrow$ algebraic Novikov d_2 differentials

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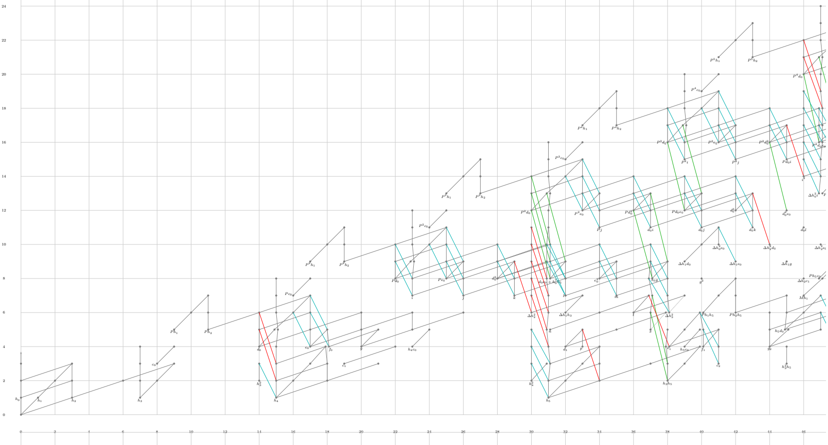
Re-compute early range very effectively

Classical Adams spectral sequence



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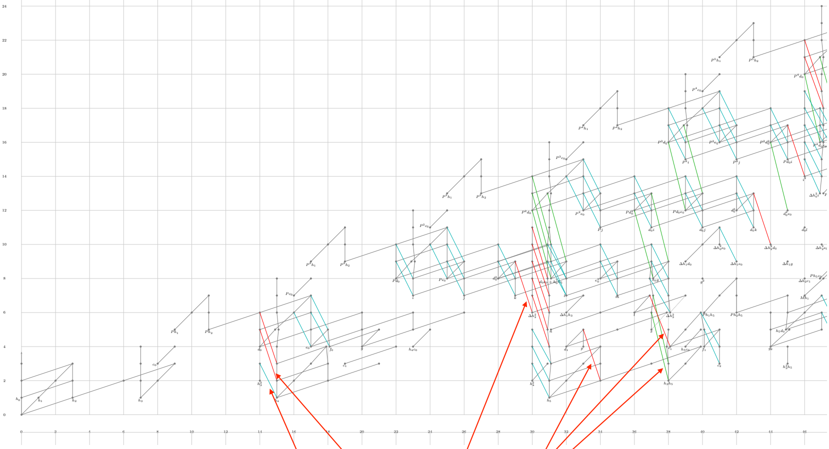
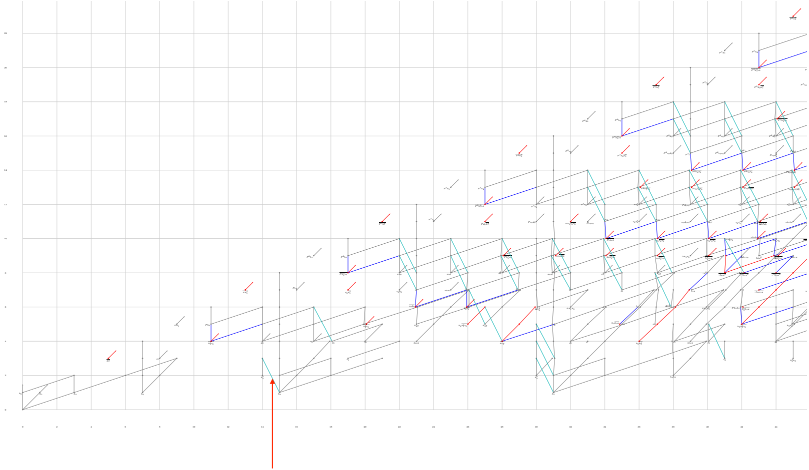
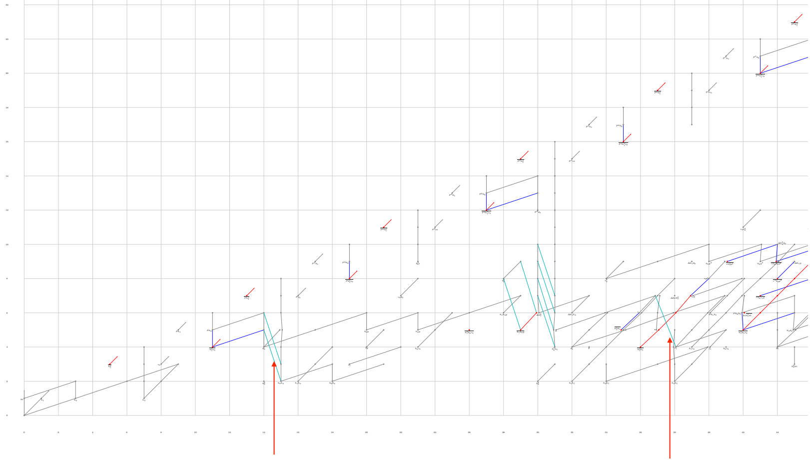


Harder Differentials

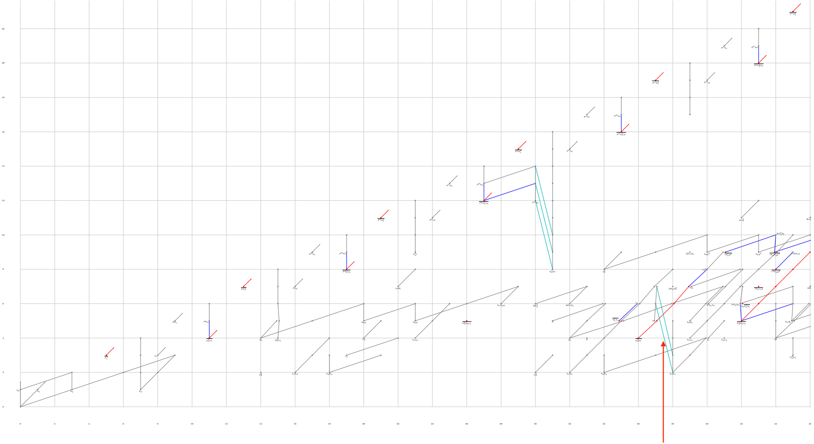
Motivic E_2-page of $\widehat{S^{0,0}}/\tau$



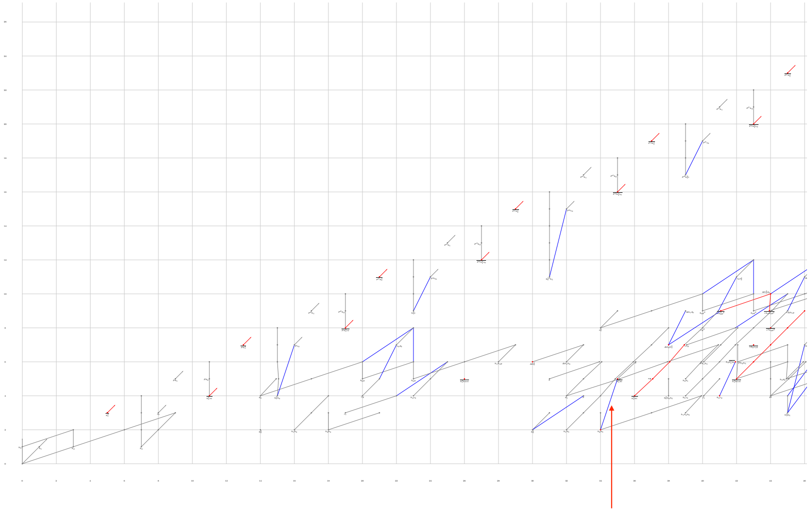
Motivic E_3-page of $\widehat{S^{0,0}}/\tau$



Motivic E_4-page of $\widehat{S^{0,0}}/\tau$



Motivic E_∞-page of $\widehat{S^{0,0}}/\tau$



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So it does not violate the Second Mahowald Uncertainty Principle!

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These equivalences are independent of the base field k!

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Postnikov–Whitehead tower for $S^{0,0}$ w.r.t. the Chow t-structure:

$$\begin{tikzcd}
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Computing $\pi_{*,*}\widehat{S^{0,0}}$ over k

Apply the motivic Adams spectral sequences:

$$\begin{tikzcd}
 & {} \arrow[d] & & \\
 & \mathbf{motASS}(S^{0,0}_{c\geqslant 2}) \arrow[r] \arrow[d] & \mathbf{motASS}(S^{0,0}_{c=2}) \arrow[r, equal] & \mathbf{algNSS}((\mathrm{MGL}_{*,*})_{c=2}) \\
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Question (Automated Differential Computation Question)

Are there effective algorithms for Adams d_3 or even d_4-differentials?

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In Adams filtrations 1 and 2, the New Doomsday Conjecture is essentially equivalent to the Hopf invariant one problem and the Kervaire invariant one problem respectively.

Thank you!