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Numerical Stability of Algorithms at Extreme Scale and Low Precisions

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Slides available at https://bit.ly/icm-22

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The Limits of What We Can Compute

$n \times n$ matrix prob.: rounding error bound $f(n)u$.

- ★ **Problem dimension** n
- ★ **Unit roundoff** u

both getting **larger**.

Increasingly **mixed precision world**:
$u < u_1 < u_2 \cdots$.

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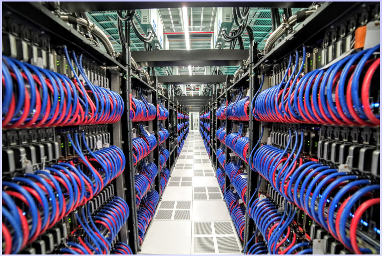
- **What can we guarantee about the computed solution?**

TOP500: June 2022

- **Frontier** at Oak Ridge.
- AMD EPYC 64C 2GHz,
  AMD Radeon Instinct GPU.
  8,730,112 cores
- Peak $>$ 1.5 exaflops.
- IEEE **double** $u \approx 10^{-16}$,
  half $u \approx 10^{-3}$ or $u \approx 10^{-4}$.



	Rate	n
HPL	1.10 exaflops	2.4×10^7
HPL-AI	6.86 "**exaflops**"	2.7×10^7

Petaflops $= 10^{15}$ *flops*, *Exaflops* $= 10^{18}$ *flops*

Growth of Problem Size in TOP500

Dimension of matrix for #1 machine.

Machine	Date	n
Fugaku	2020	2.0×10^7
Jaguar	2010	6.3×10^6
ASCI RED	2000	3.6×10^5
CM-5/1024	1993	5.2×10^4

- Growing by roughly a **factor 10 every decade**.

Today's Floating-Point Arithmetics

		Bits			
Type	Name	Signif. (t)	Exp.	Range	$u = 2^{-t}$
Quarter	fp8-e5m2	3	5	$10^{\pm 5}$	1.2×10^{-1}
Quarter	fp8-e4m3	4	4	$10^{\pm 2}$	6.2×10^{-2}
Half	bfloat16	8	8	$10^{\pm 38}$	3.9×10^{-3}
Half	fp16	11	5	$10^{\pm 5}$	4.9×10^{-4}
Single	fp32	24	8	$10^{\pm 38}$	6.0×10^{-8}
Double	fp64	53	11	$10^{\pm 308}$	1.1×10^{-16}

- Last three types are IEEE standard.
- fp8 types introduced on NVIDIA H100 (2022).

Backward Error Analysis for LU Factorization

Let $\gamma_n = \dfrac{nu}{1-nu} = nu + O(u^2)$.

Theorem

Computed solution $\widehat{x}$ to $Ax = b$ where $A \in \mathbb{R}^{n\times n}$ satisfies

$$(A + \Delta A)\widehat{x} = b, \quad |\Delta A| \le \gamma_{3n}|\widehat{L}||\widehat{U}|.$$

Then for $n \approx 10^7$:

- in IEEE double precision, $nu \approx 2.3 \times 10^{-9}$.
- in IEEE single precision, $nu \approx 1.25$.

Sharper Bound

Proof uses $A + \Delta A_1 = \widehat{L}\widehat{U}$, where (recall $\gamma_n \approx nu$),

$$|\Delta A_1| \le \begin{bmatrix} \gamma_1 & \gamma_1 & \cdots & \cdots & \gamma_1 \\ \gamma_1 & \gamma_2 & \cdots & \cdots & \gamma_2 \\ \vdots & \vdots & \ddots & \dots & \vdots \\ \vdots & \vdots & \ddots & \gamma_{n-1} & \gamma_{n-1} \\ \gamma_1 & \gamma_2 & \cdots & \gamma_{n-1} & \gamma_n \end{bmatrix} \circ |\widehat{L}||\widehat{U}|. \qquad (*)$$

Not fruitful to try to use $(*)$.

Low Precision in Deep Learning

- "We find that very low precision is sufficient not just for running trained networks but also for training them." —**Courbariaux, Benji & David** (2015)

- "Deep learning models ... are very tolerant of reduced-precision computations."—**Dean (2019)**.

$$|\,\mathrm{fl}(x^T y) - x^T y| \le nu|x|^T|y|.$$

fp16:	$nu = 1$ for $n = 2048$
bfloat16:	$nu = 1$ for $n = 256$

- Yet deep learning successfully uses half precision.

The (Partial) Explanation

- **Inner products** not computed in the obvious way but are **blocked** $\Rightarrow$ much smaller error bounds possible.
- We use **blocked algorithms**.
- **Hardware features** automatically boost accuracy.
- The rounding error bounds are **worst-case** and **very pessimistic**. **Probabilistic error bounds** are more insightful.

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Blocking is done for speed but also improves accuracy.

Blocked Inner Products: 2 Pieces

Original

$$s = \sum_{i=1}^{n} x_i y_i \;\Rightarrow\; |s - \widehat{s}| \le nu|x|^T|y|.$$

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Blocked, 2 pieces

Let $n = 2b$.

$$\begin{aligned} s_1 &= x(1\colon b)^T y(1\colon b) \\ s_2 &= x(b+1\colon n)^T y(b+1\colon n) \\ s &= s_1 + s_2 \end{aligned}$$

$$|s - \widehat{s}| \le \left(\frac{n}{2} + 1\right) u|x|^T|y|.$$

Blocked Inner Products; k Pieces

Original

$$s = \sum_{i=1}^{n} x_i y_i \;\Rightarrow\; |s - \widehat{s}| \le nu|x|^T|y|.$$

Blocked, k pieces

Let $n = kb$.

$$s_i = x((i-1)b + 1\colon ib)^T y((i-1)b + 1\colon ib),\; i = 1\colon k$$
$$s = s_1 + s_2 + \cdots + s_k$$
$$|s - \widehat{s}| \le \left(\frac{n}{k} + k - 1\right) u|x|^T|y|.$$

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$$s = s_1 + s_2 + \cdots + s_k$$
$$|s - \widehat{s}| \le \left(\frac{n}{k} + k - 1\right) u|x|^T|y|.$$

Optimal $k = \sqrt{n}$:

$$|s - \widehat{s}| \le 2\sqrt{n}\,u|x|^T|y|.$$

Block Summation

Recursive summation of $x_1, \dots, x_n$:

$s = 0$
for $i = 1\colon n$, $s = s + x_i$, end

Standard block summation:

1. sum blocks of size b by recursive summation: $(b-1)n/b = n - n/b$ additions
2. sum n/b partial sums by recursive summation: $n/b - 1$ additions

Idea: use a **more accurate method** in step 2.
E.g., recursive summation at *higher precision*, *compensated summation*.

FABsum

Blanchard, H & Mary (2020).

Input: n-vector x, block size b,
algs **FastSum**, **AccurateSum**.

1: **for** $i = 1: n/b$ **do**
2: Compute $s_i = \sum_{j=(i-1)b+1}^{ib} x_j$ with **FastSum**.
3: **end for**
4: Compute $s = \sum_{i=1}^{n/b} s_i$ with **AccurateSum**.

- **FastSum** is doing $n - n/b$ additions.
- **AccurateSum** is doing $n/b - 1$ additions.

FABsum Error Bound

$$\textbf{FastSum}: \quad \widehat{s} = \sum_{i=1}^{n} x_i(1+\mu_i^f), \quad |\mu_i^f| \le \epsilon_f(n),$$

$$\textbf{AccurateSum}: \quad \widehat{s} = \sum_{i=1}^{n} x_i(1+\mu_i^a), \quad |\mu_i^a| \le \epsilon_a(n).$$

Theorem

The computed $\widehat{s}$ from ***FABsum*** *satisfies*

$$\widehat{s} = \sum_{i=1}^{n} x_i(1+\mu_i),$$

$$|\mu_i| \le \epsilon(n,b) = \epsilon_f(b) + \epsilon_a(n/b) + \epsilon_f(b)\epsilon_a(n/b).$$

Error Bound for Recursive/Compensated

Take **FastSum** = recursive summation,
AccurateSum = compensated summation. Then

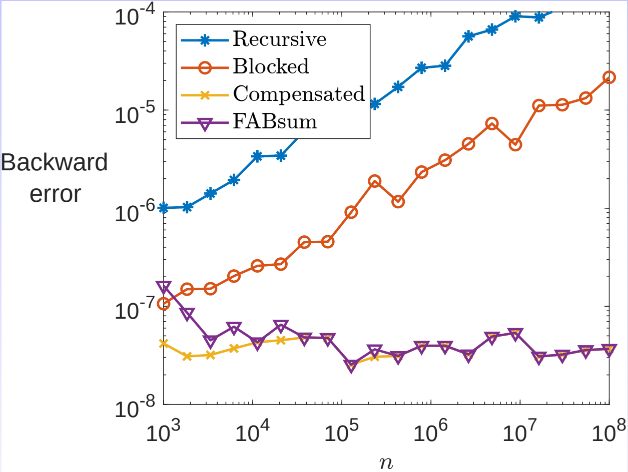
$$\epsilon(n,b) = (b+1)u + \left[4n/b + 2 + (b-1)^2 + 2(b-1)\right] u^2 + O(u^3).$$

Recall error bound is

- $nu + O(u^2)$ for recursive summation,
- $(n/b)u + O(u^2)$ for blocked summation.

FABsum error bound **independent of** n to first order.

Random Uniform [0, 1], $b = 128$, fp32



Blocked Matrix Multiplication

Let $A, B \in \mathbb{R}^{n\times n}$ be partitioned into $b \times b$ blocks A_{ij} and B_{ij}, where $p = n/b$ is assumed to be an integer. This algorithm computes $C = AB$.

for $i = 1\colon p$
    for $j = 1\colon p$
        $C_{ij} = 0$
        for $k = 1\colon p$
            $X = A_{ik}B_{kj}$
            $C_{ij} = C_{ij} + X$
        end
    end
end

- Compare $c_{rs} \leftarrow c_{rs} + a_{rk}b_{ks}$.

Blocked Algorithms

LAPACK philosophy: blocked matrix factorizations with a block size $b = 128$ or $b = 256$.
$\Rightarrow$ **Reduction in error bounds by factor b.**
At block level, apply block inner products giving further reduction!

- LAPACK manual states error bounds $p(n)u$, where "*$p(n)$ is a modestly growing function of n*".
- Standard NLA refs don't mention b in error bounds.
  - Optimizing constants not the point (Wilkinson).
  - Constants depend on the block alg.
  - Analysis is more complicated.

Extended Precision Registers

- **Intel x86-64** processors include 80-bit floating point registers with 64-bit significand (but not used by SSE2).
- Registers have $u = 2^{-64}$ rather than $u = 2^{-53}$ for double precision. Error bounds smaller by a factor up to $2^{11} = 2048$.
- **Caveat**: extra precision registers can lead to strange rounding effects, including double rounding!

Fused Multiply-Add (FMA)

Computes $x + yz$ at same speed as "$+$" or "$*$" with just one rounding error.

Without an FMA,

$$\mathrm{fl}(x + yz) = \big(x + yz(1+\delta_1)\big)(1+\delta_2), \quad |\delta_1|, |\delta_2| \le u,$$

but **with an FMA**

$$\mathrm{fl}(x + yz) = (x + yz)(1+\delta), \quad |\delta| \le u.$$

Error bounds for inner product-based computations **reduced by a factor 2.**

Mixed Precision Block FMA

Precisions u_{low} (fp8, bfloat16, fp16), u_{high} (fp16, fp32).

Dimensions:

$$\underbrace{D}_{b_1 \times b_2} = \underbrace{C}_{b_1 \times b_2} + \underbrace{A}_{b_1 \times b}\,\underbrace{B}_{b \times b_2}.$$

Precisions:

$$\underbrace{D}_{u_{\mathrm{low}} \text{ or } u_{\mathrm{high}}} = \underbrace{C}_{u_{\mathrm{low}} \text{ or } u_{\mathrm{high}}} + \underbrace{A}_{u_{\mathrm{low}}}\;\underbrace{B}_{u_{\mathrm{low}}}.$$

Computation:

$$\mathrm{fl}_{\mathrm{high}}\Big(C + \mathrm{fl}_{\mathrm{high}}(AB)\Big).$$

Can chain: $C \leftarrow C + AB$.

Block FMA Hardware

Year	Device	Dimensions	u_{low}	u_{high}
2020	Google TPU v4i	$128 \times 128 \times 128$	bfloat16	fp32
2017	NVIDIA V100	$4 \times 4 \times 4$	fp16	fp32
2019	ARMv8.6-A	$2 \times 4 \times 2$	bfloat16	fp32
2020	NVIDIA A100	$8 \times 8 \times 4$	bfloat16	fp32
		$8 \times 8 \times 4$	fp16	fp32
		$8 \times 4 \times 4$	TFloat-32	fp32
		$2 \times 4 \times 2$	fp64	fp64

Note

- Not necessarily IEEE compliant.
- Very fast throughput (*"one result per cycle"*) compared with none block-FMA arithmetic.

Error Analysis of Block FMAs

Blanchard, H , Lopez, Mary, & Pranesh (2020).

Analysis of algs for **matrix mult** $C = AB$ based on block FMA. *Inherently multiprecision.*

For $A, B \in \mathbb{R}^{n\times n}$ using chained block $b \times b$ FMAs,

$$|C - \widehat{C}| \le f(n, b, u_{\mathrm{low}}, u_{\mathrm{high}})|A||B|,$$

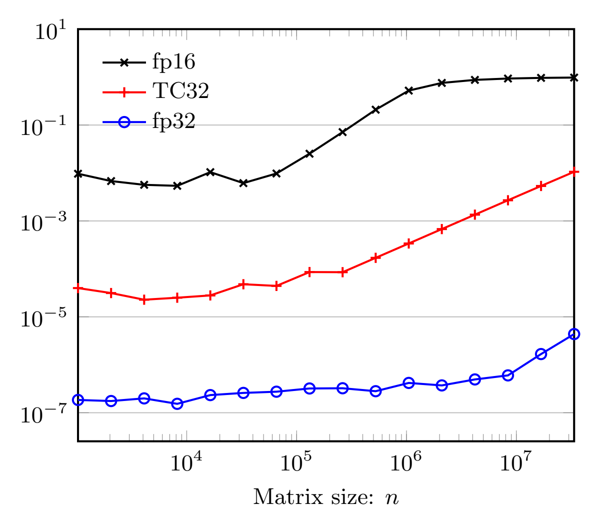
where with A, B given in u_{high}, $f(\cdot)$ is

Standard in precision u_{low}	$(n+2)u_{\mathrm{low}}$
Block FMA, u_{high} internally	$2u_{\mathrm{low}} + nu_{\mathrm{high}}$
Standard in precision u_{high}	nu_{high}

- Similar results for **LU factorization** and $Ax = b$.

NVIDIA V100

- Matrix entries are rand unif $[0, 10^{-3}]$.
- In fp32, cmp'wise error $\max_{i,j} |\widehat{C} - C|_{ij}/(|A||B|)_{ij}$:



Probabilistic Error Analysis

Rounding error bounds above are **worst-case**.

> ***"To be realistic, we must prune away the unlikely. What is left is necessarily a probabilistic statement."***
>
> ***— Stewart, 1990***

Statistical Effects

"In general, the statistical distribution of the rounding errors will reduce considerably the function of n occurring in the relative errors. We might expect in each case that this function should be replaced by something which is no bigger than its square root."

— Wilkinson, 1961

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>
> ***— Wilkinson, 1961***

Limitations of central limit theorem argument

- Rounding errors **independent** random variables of **mean zero**.
- Applies only to **first-order** part of error.
- n is **sufficiently large**.

Standard Tool for Rounding Error Analysis

Theorem

If $|\delta_i| \le u$ for $i = 1 : n$ and $nu < 1$ then

$$\prod_{i=1}^{n}(1 + \delta_i) = 1 + \theta_n,$$

where

$$|\theta_n| \le \gamma_n := \frac{nu}{1 - nu} = nu + O(u^2).$$

- The basis of most rounding error analyses.
- We seek an analogous result with a smaller, but **probabilistic**, bound on θ_n.

Assumptions for Probabilistic Analysis

Model M

- **Rounding error bound**:

  $$fl(x \,\mathrm{op}\, y) = (x \,\mathrm{op}\, y)(1+\delta), \quad |\delta| \le u, \quad \mathrm{op} \in \{+,-,*,/\}.$$

- **Mean independence**:
  The computation generates $\delta_1, \delta_2, \dots$ that are random variables of mean zero such that

  $$\mathbb{E}(\delta_{k+1} \mid \delta_1, \dots, \delta_k) = \mathbb{E}(\delta_{k+1}) = 0.$$

- Weaker than assuming the δ_i are independent.
- The δ_i need not be from same distribution.

Probabilistic Analysis

Theorem (Connolly, H & Mary, 2021)

Let $\delta_1, \ldots, \delta_n$ satisfy Model M. For any constant $\lambda > 0$,

$$\prod_{i=1}^{n}(1+\delta_i) = 1 + \theta_n, \quad |\theta_n| \le \widetilde{\gamma}_n(\lambda) \approx \lambda\sqrt{n}u,$$

holds with probability at least $1 - 2\exp(-\lambda^2/2)$.

- Proof uses martingales.
- Valid for all n.
- Valid to all orders.
- Explicit probability $P(\lambda)$ (pessimistic).
- Earlier result by **H & Mary (2020)** assumes indep.

Inner Products

Theorem

Let $s = x^T y$, where $x, y \in \mathbb{R}^n$. Under Model M, the computed $\widehat{s}$ satisfies

$$\widehat{s} = (x + \Delta x)^T y,$$
$$|\Delta x| \le \widetilde{\gamma}_n(\lambda)|x| \approx \lambda\sqrt{n}u|x|,$$

with probability at least $1 - 2n\exp(-\lambda^2/2)$.

Similar results by **H & Mary (2020)**, **Ipsen & Zhou (2020)**.

Linear Systems

Theorem

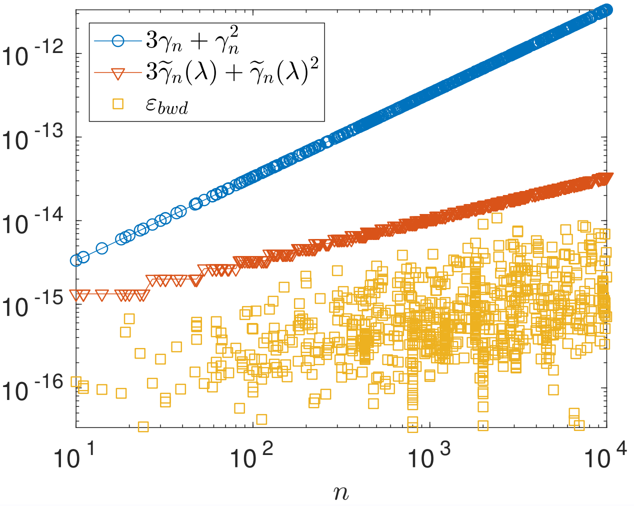
Under Model M, the computed solution $\widehat{x}$ to $Ax = b$ from LU factorization satisfies

$$(A + \Delta A)\widehat{x} = b, \quad |\Delta A| \le \big(3\widetilde{\gamma}_n(\lambda) + \widetilde{\gamma}_n(\lambda)^2\big)|\widehat{L}||\widehat{U}|,$$

with probability at least $1 - 2n^3/3 \exp(-\lambda^2/2)$.

Real-Life Matrices

Solution of $Ax = b$ (fp64), b from Uniform $[0, 1]$,
for 943 matrices from **SuiteSparse** collection ($\lambda = 1$).



Probabilistic QR Error Bound

Theorem (Connolly & H, 2022)

Under Model M and a technical assumption, for the computed $\widehat{R} \in \mathbb{R}^{m\times n}$ *from Householder QR on* $A \in \mathbb{R}^{m\times n}$ $(m \ge n)$, $\exists$ *orthogonal* $Q \in \mathbb{R}^{m\times m}$ *such that*

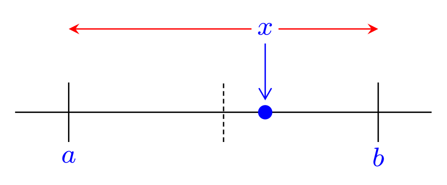
$$A + \Delta A = Q\widehat{R},$$
$$\|\Delta a_j\|_2 \le c\lambda\sqrt{mn}\,u\|a_j\|_2 + O(u^2), \quad j = 1\colon n,$$

holds with probability at least $1 - 2mn\exp(-\lambda^2)$.

- Uses a matrix concentration inequality of **Tropp (2012)**.
- Worst-case bound has *mnu*.
- Square rooting of constant applies to Givens QR, too.

Stochastic Rounding

Forsythe (1950), ..., **Croci et al. (2022)**.



Theorem (Connolly, H & Mary, 2021)

The rounding errors $\delta_1, \delta_2, \dots$ *from stochastic rounding are rand. vars of mean* 0 *s.t.* $\mathbb{E}\big(\delta_k \mid \delta_1, \dots, \delta_{k-1}\big) = \mathbb{E}(\delta_k) = 0$.

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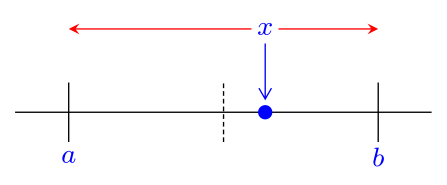

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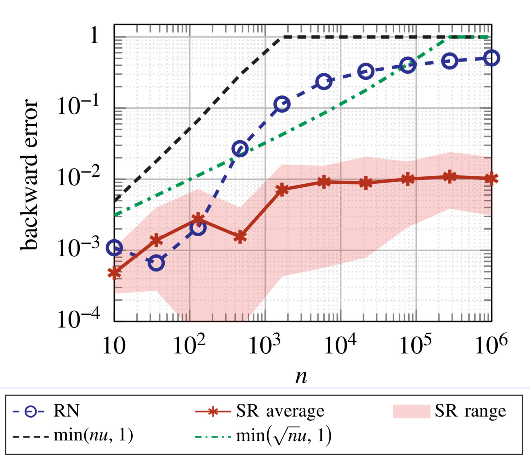
Stochastic rounding always satisfies the assumptions!

For SR, we can *always* replace nu by $\sqrt{nu}$ in a worst-case rounding error bound to obtain a probabilistic error bound.

Stagnation

Harmonic sum $\sum_{k=1}^{n} 1/k$ in fp16.



Stochastic rounding avoids *stagnation*!

Random Data

Model M'

- d_j, $j = 1\colon n$, are independent random variables from a distribution of mean μ_x s.t. $|d_j| \le \xi_d$, $j = 1\colon n$.
- $\mathbb{E}(\delta_k \mid \delta_1, \dots, \delta_{k-1}, d_1, \dots, d_n) = \mathbb{E}(\delta_k) = 0$.

Theorem (H & Mary, 2020)

If $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$ Model M', with means μ_A, μ_B and bounds ξ_A, ξ_B, and let $C = AB$. Under Model M',

$$\max_{i,j} |(C - \widehat{C})_{ij}| \le \big(\lambda |\mu_A \mu_B| n^{3/2} + (\lambda^2 + 1)\xi_A \xi_B n\big) u + O(u^2)$$

with probability at least $P(\lambda) = 1 - 2mnp\exp(-\lambda^2/2)$.

Putting It All Together

- Block algs reduce error bound by factor b.
- For blocking at multiple levels, the reduction factors can accumulate.
- Extended precision registers and (block) FMAs give automatic accuracy boost.
- Block size $b = 256$ and 80-bit registers reduces error bound by factor $256 \times 2048 = 5.2 \times 10^5$.
- Prob error anal. says "$f(n)u \to \sqrt{f(n)}u$".
- Prob. error anal. applies to blocked algs. Error constant $(b + n/b)u$ for a blocked inner product translates to $(\sqrt{b} + \sqrt{n/b})u$ in a prob. bound.

Conclusions

- Classical analyses **no longer guarantee the numerical stability** of classical algorithms for all n and u of interest.
- **Block algs** (designed for speed) & **hardware features** give significant accuracy boosts.
- New **probabilistic bounds** show "$f(n)u \to \sqrt{f(n)}u$". Even these bounds often very pessimistic.
- We often do better than we can currently explain.

Slides at **`https://bit.ly/icm-22`**

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