Weingarten calculus and its applications

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1. The Weingarten formula: a formula to compute integrals of polynomial functions w.r.t. the Haar measure on compact groups.
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The Haar measure on compact groups

- A compact group G admits a unique left and right invariant probability measure μ_G, the *Haar measure*:
  $\mu_G(G) = 1$, and for any Borel subset A of G and $g \in G$,

  $$\mu_G(Ag) = \mu_G(gA) = \mu_G(A)$$

  $(Ag = \{hg, h \in A\}; gA = \{gh, h \in A\})$.
- left & right invariance + uniqueness of $\mu_G \implies \mu_G(A^{-1}) = \mu_G(A)$.
- How to integrate functions w.r.t μ_G?

Polynomial functions on a matrix group

- We focus on matrix unitary compact groups

$$G \subset \mathcal{U}_n \subset \mathcal{M}_n(\mathbb{C}) = \mathbb{R}^{2n^2}.$$

  $U = (u_{ij})_{i,\in\{1,\dots n\}} \in G$; we view $u_{ij} : G \to \mathbb{C}$ and their conjugates as variables of a polynomial function.

- Moment formulation:

$$\mathbb{E}(X_1^{k_1} \dots X_n^{k_n}), k_1, \dots, k_n \geq 0$$

  for a random variable $X = (X_1, \dots, X_n) \in \mathbb{R}^n$

- w.r.t μ_G, u_{ij} is a complex random variable. We are interested in computing the moments of

$$(Re(u_{ij}), Im(u_{ij}))_{i,j\in\{1,n\}}$$

The Haar measure

- Reformulation with conjugates: compute

$$\int_{U\in G} u_{i_1 j_1}\dots u_{i_k j_k}\overline{u_{i'_1 j'_1}\dots u_{i'_{k'} j'_{k'}}}d\mu_G(U).$$

- Remark: the moments determine the measure (Riesz' theorem + Stone-Weierstrass).
- Matrix notation: compute

$$Z_G^{k,k'} = \int_{U\in G} U^{\otimes k}\otimes \overline{U}^{\otimes k'} d\mu_G(U) \in \mathcal{M}_n(\mathbb{C})^{\otimes k+k'}.$$

The Haar measure

- $V = \mathbb{C}^n$ the fundamental representation of G,
  $\overline{V}$ the contragredient representation.
- For any representation (ρ, W) of G,
  $Fix_\rho(G, W) = \{x \in W, \forall U \in G, \rho(U)x = x\}$.
- **Proposition:** The matrix $Z_G^{k,k'}$ is the orthogonal projection onto $Fix(G, V^{\otimes k} \otimes \overline{V}^{\otimes k'})$.

Fundamental integration formula

- $(E_1, \ldots, E_n)$: canonical o.n.b of $V = \mathbb{C}^n$.
  For $I = (i_1, \ldots, i_k) \in \{1, \ldots n\}^k$, let $E_I = e_{i_1} \otimes \ldots \otimes e_{i_k}$ be the canonical o.n.b. of $V^{\otimes k}$.
- Repeat for $V^{\otimes k} \otimes \overline{V}^{\otimes k'}$: Let $I = (i_1, \ldots, i_k, i'_1, \ldots, i'_{k'})$, $J = (j_1, \ldots, j_k, j'_1, \ldots, j'_{k'})$ be $k + k'$-indices, i.e. elements of $\{1, \ldots, n\}^{k+k'}$.
- Let $y_1, \ldots, y_l$ generate $Fix(G, V^{\otimes k} \otimes \overline{V}^{\otimes k'})$,
  $Gr = (g_{ij})_{i,j \in \{1,\ldots,l\}} = (< y_i, y_j >)_{ij}$ be its Gram matrix;
  $W = (w_{ij})_{ij}$ pseudo-inverse of Gr.
- **Theorem** (Integration formula)

$$\int_{U \in G} u_{i_1 j_1} \ldots u_{i_k j_k} \overline{u_{i'_1 j'_1} \ldots u_{i'_{k'} j'_{k'}}} d\mu_G = \sum_{i,j \in \{1,\ldots,l\}} \langle E_I, y_i \rangle \langle y_j, E_J \rangle w_{ij}$$

Historical remarks and comments

- w_{ij} is called the Weingarten function. It was first described by 't Hooft and Weingarten in the 70's for large unitary and orthogonal groups. Rediscovered/reused sporadically for 20 years.
- For interesting applications to be derived, the following conditions must be met:
  1. $y_1, \ldots, y_l$ must be easy to describe (Schur-Weyl, Tannaka-Krein duality).
  2. Gr should be easy to compute – and if possible, meaningful (geometric interpretation, loops, etc)
  3. $\langle E_I, y_i \rangle$ should be easy to compute (generalized Kronecked delta)
  4. Computing/understanding w_{ij} is the hard part (...how to compute the inverse of a matrix...?)

Specific choice of groups: $\mathcal{O}_n$ and $\mathcal{U}_n$

- Let $P_2(k)$ be the collection of *pair partitions* on $\{1, \ldots, k\}$ (empty if k is odd, and has $1 \cdot 3 \cdot \ldots \cdot (k-1) = k!!$ elements if k is even). A typical $\pi \in P_2(k)$: $\pi = \{V_1, \ldots, V_{k/2}\}$.
- Let $\delta_{\pi,I}$ be the multi-index Kronecker function whose value is 1 if, for any block $V = \{k < k'\}$ of π, $i_k = i_{k'}$, and zero in all other cases. For example:

$$\delta_{\cap\cap,I} = \delta_{i_1,i_2}\delta_{i_3,i_4}$$

- Likewise, we call $E_\pi = \sum_I E_I \delta_{\pi,I} \in V^{\otimes k}$. For example:

$$E_{\cap\cap} = \sum_{i,j} E_i \otimes E_i \otimes E_j \otimes E_j \in V^{\otimes 4}$$

Specific choice of groups: $\mathcal{O}_n$ and $\mathcal{U}_n$

The entries of Gr have a simple geometric interpretation:

$$\langle E_\pi, E_{\pi'} \rangle = n^{\mathrm{loops}(\pi,\pi')},$$

and we have

$$\langle E_I, E_\pi \rangle = \delta_{\pi,I}.$$

- *The orthogonal case:* For $\mathcal{O}_n$, $E_\pi, \pi \in P_2(k)$ is a generating family of the image of $Z^{k,0}_{\mathcal{O}_n}$
- *The unitary case:* Let $2k' = k$. The subset of $P_2(k)$ of pair partitions such that each block pairs one of the first k' elements with one of the last k' elements (in bijection with $S_{k'}$) is the generating family of the image of $Z^{k',k'}_{\mathcal{U}_n}$.

Representation theoretic formulas (unitary case)

- For $\sigma \in S_k$, let $\#\sigma$ the number of cycles in its cycle product decomposition. Define the function

$$G = \sum_{\sigma \in S_k} n^{\#\sigma} \lambda_\sigma \in \mathbb{C}[S_k],$$

and its pseudo-inverse $W = G^{-1} = \sum_{\sigma \in S_k} w(\sigma)\lambda_\sigma$.

- **Theorem:** (C, Śniady) G is positive in $\mathbb{C}[S_k]$, and

$$\mathrm{Wg}_U(n, \tau\sigma^{-1}) := w(\sigma, \tau) = w(\tau\sigma^{-1})$$

has the following character expansion:

$$\mathrm{Wg}_U(n, \sigma) = \frac{1}{k!^2} \sum_{\lambda \vdash k} \frac{\chi_\lambda(e)^2 \chi_\lambda(\sigma)}{dim V_\lambda}.$$

Combinatorial formulations

- Let $P(\sigma, l)$ to be the set of solutions to the equation $\sigma = (i_1 j_1) \dots (i_l j_l)$ with $i_p < j_p$, $j_p \leq j_{p+1}$.
- The number of solutions to this problem is related to *Hurwitz numbers*.
- **Theorem:** (C, Matsumoto; Novak) We have the expansion

$$\mathrm{Wg}_U(n, \sigma) = n^{-k} \sum_{l \geq 0} \# P(\sigma, l)(-n^{-1})^l. \tag{1}$$

- The orthogonal and symplectic cases were done by (C, Matsumoto) [representation theoretic expansions, and combinatorial expansions]

Digression: the quantum group case

- Woronowicz introduced compact matrix quantum groups in the 80's. He proved the existence and uniqueness of a Haar measure (non-constructive, non-explicit except for characters).
- The Weingarten formula extends mutatis mutandis to Woronwicz's quantum groups (Banica, C)
- In the case of O_n^+, U_n^+, we obtained free Borel type theorems, i.e. asymptotic freeness results for entries of free quantum orthogonal groups (Banica, C)
- The character expansion does not extend to quantum compact groups. However, the combinatorial expansion does in some cases – e.g. O_n^+ (Brannan, C). Pair partitions are replaced by *non-crossing pair partitions*.
- We obtain explicit formulas to solve a problem of Jones of positivity of coefficients of the dual of the Temperley Lieb basis (Brannan, C).

Leading order Asymptotics of Wg ($\mathcal{U}_n$ case)

- The full cycle explicit formula (C 2003):

$$Wg(n, (1 \cdots k)) = \frac{(-1)^{k+1} c_k}{(n-k+1)\dots(n+k-1)},$$

where $c_k = \frac{1}{k+1}\binom{2k}{k}$ is the Catalan number.

- In addition, Wg is almost multiplicative in the following sense: if σ is a disjoint product of two permutations $\sigma = \sigma_1 \sqcup \sigma_2$ then

$$Wg(n, \sigma) = Wg(n, \sigma_1) Wg(n, \sigma_2)(1 + O(n^{-2}))$$

- [Biane 1998] The leading order is given by Biane-Speicher's Moebius function $\mathrm{Moeb} : \sqcup_{k \geq 1} S_k \mapsto \mathbb{Z} - \{0\}$ satisfying

$$Wg(n, \sigma) = n^{-k-|\sigma|} \mathrm{Moeb}(\sigma)(1 + O(n^{-2})).$$

Applications of the asymptotics (a subjective selection)

1. *Pointwise, leading order*: asymptotic freeness, Quantum Information Theory.
2. *Arbitrary order*: matrix integrals + tensors
3. *Uniform*: QIT
4. *Centered*: operator norm convergence + Random RT.

Asymptotic freeness (pointwise, leading order)

- For matrix models $A_i^{(n)} \in \mathcal{M}_n(\mathbb{C})$, $i \in I$ assume the existence of $\lim_n trA_{i_1}^{(n)} \dots A_{i_l}^{(n)}$ for any sequence $i_1, i_2, \dots \in I$
  This is called an *asymptotic distribution* in Voiculescu's sense of non-commutative distributions.
- Make the same assumption for an additional family $B_i^{(n)}$.
- **Theorem:** (Voiculescu 82, 98)
  Almost surely, the same holds true for the extended family $(A_i^{(n)})_{i\in I} \sqcup (UB_j^{(n)}U^*)_{j\in J}$ after a global rotation of B by a Haar unitary matrix:
  it admits an asymptotic distribution governed by *asymptotic freeness*.
  The most general versions of this result can be proved with Weingarten calculus.

Asymptotic freeness: quantum (pointwise, leading order)

- In $\mathcal{M}_n(\mathbb{C}) \otimes \mathfrak{U}(GL_n(\mathbb{C}))^{\otimes 2}$, consider

$$A_n^{(1)} = \sum_{ij} E_{ij} \otimes e_{ij} \otimes 1 \ , \ A_n^{(2)} = \sum_{ij} E_{ij} \otimes 1 \otimes e_{ij}.$$

- **Theorem** (C, Novak, Śniady):
Consider two sequences of characters $\chi_{1,n}, \chi_{2,n}$ associated to irreducible f.d. representations of $GL_n(\mathbb{C})$ and assume that $A_n^{(1)}, A_n^{(2)}$ have a joint limiting distribution w.r.t to $tr \otimes \chi_{1,n} \otimes \chi_{2,n}$. Then, $A_m^{(1)}, A_n^{(2)}$ are asymptotically free.
- Weingarten calculus for this purpose generalizes character techniques developed by Biane in the 90's and gives new information about the statistical properties of factors in the decomposition of two irreducible representation of $GL_n(\mathbb{C})$.

Quantum Information (pointwise, leading order)

- Matrix model:
  $k \in \mathbb{N}$, $t \in (0,1)$ fixed. For each n, $U_n : \mathbb{C}^{p_n} \to \mathbb{C}^k \otimes \mathbb{C}^n$ a random Haar isometry, with $p_n/(nk) \sim t$. Let $\Phi : M_{p_n}(\mathbb{C}) \to M_n(\mathbb{C})$ given by $\Phi(X) = Tr_k(U_n^* X U_n)$.
  Let *Bell* be the Bell state on $M_{p_n}(\mathbb{C})^{\otimes 2}$ – the o.n. projection onto $\sum E_i \otimes E_i$
- **Theorem** (C, Nechita)
  Almost surely, as $n \to \infty$, the random matrix $\Phi \otimes \overline{\Phi}(\mathrm{Bell}) \in M_{n^2}(\mathbb{C})$ has non-zero eigenvalues converging towards

$$\gamma^{(t)} = \left(t + \frac{1-t}{k^2}, \underbrace{\frac{1-t}{k^2}, \ldots, \frac{1-t}{k^2}}_{k^2-1 \text{ times}} \right).$$

  This result plays an important role in the understanding of the sub-additivity of the minimum output entropy.

Higher order asymptotic freeness (higher order)

- Let C_k be the *classical cumulants in non-normalized traces* of random matrices.
  Assume the existence of
  $\lim_n n^{2k-2} C_k(A^{(n)}_{i_{11}} \dots A^{(n)}_{i_{l_1 1}}, \dots, A^{(n)}_{i_{1k}} \dots A^{(n)}_{i_{l_k k}})$ for any sequence of indices. This set of limits is called the *higher order limit*.
  Make the same assumption for an additional family B.
- **Theorem** (C, Mingo, Speicher, Śniady)
  The extended family $(A^{(n)}_i)_{i \in I} \sqcup (UB^{(n)}_j U^*)_{j \in J}$ admits a higher order limit. In addition, there exists a combinatorial rule to construct the joint asymptotic correlations from the asymptotic correlations of each family.
  This rule extends freeness, and is called *higher order freeness*.

Matrix integrals and random tensors (higher order)

- **Theorem** (C, 2003) This implies the formal convergence of arbitrary matrix integrals in non-commuting polynomials.
- Remark: this extends convergence results for HCIZ integrals.
- **Theorem** (C, Gurau, Lionni, 2020 + WIP) This extends to random tensors $U = U_1 \otimes \ldots \otimes U_D$ where $U_i \in M_n(\mathbb{C})$ are iid.
- This is linked to a new notion of generalized Hurwitz numbers.

Uniform estimates

- **Theorem** (C, Matsumoto, 2018) For any $\sigma \in S_k$ and $n > \sqrt{6}k^{7/4}$,

$$\frac{1}{1 - \frac{k-1}{n^2}} \leq \frac{n^{k+|\sigma|} Wg(n, \sigma)}{\mathrm{Moeb}(\sigma)} \leq \frac{1}{1 - \frac{6k^{7/2}}{n^2}}.$$

- Uniform estimates has applications in Quantum information (Area law, C, Perez-Garcia, Gonzalez-Guillen – see previous works with Nechita, Zyczkowski in this domain too)

Centered version

- *Centering* of a random variable X: $[X] = X - E(X)$.
  We can write a Weingarten formula:

$$\mathbb{E}\prod_{t=1}^{T}[\prod_{l=1}^{k_t} U_{x_{tl}y_{tl}}^{\varepsilon_{tl}}] = \sum_{\sigma,\tau\in P_2(k_1+\ldots+k_T)} \delta_{\sigma,x}\delta_{\tau,y} Wg(\sigma,\tau;k_1,\ldots,k_T),$$

  where the function Wg depends on the pairings and the partition.
- A block of the partition is *lonesome* with respect to the pairing (σ,τ) iff the group generated by σ,τ stabilizes it.
  Theorem: (C, Bordenave)
  Wg decays as n^{-k} where
  $k = (k_1 + \ldots + k_T)/2 + d(\sigma,\tau) + 2\#\text{lonesome blocks}$, and this estimate is uniform on $k \sim Poly(n)$.

Strong Asymptotic freeness (Centering)

- **Definition:** (Voiculescu) Let $X_1^{(n)}, \ldots, X_d^{(n)}$ be elements of a tracial NCPS $(A^{(n)}, \tau^{(n)})$ (Non-Commutative Probability Space).
  Let $X_1, \ldots, X_d$ be elements of a tracial NCPS (A, τ).
  Convergence in *NC distribution* holds iff for any NC polynomial P in d variables and its adjoint,

$$\tau^{(n)} P(X_i^{(n)}) \to \tau P(X_i)$$

- **Definition:** (C, Male) If, in addition,

$$||P(X_i^{(n)})|| \to ||P(X_i)||,$$

  then one speaks of *strong convergence* (or convergence in operator norm).

Strong Asymptotic freeness (Centering)

- Fix integers q_+, q_-
  Theorem: (C, Bordenave) $(\overline{U}_i^{\otimes q_-} \otimes U_i^{\otimes q_+})_{i=1,\dots,d}$ are strongly asymptotically free as $n \to \infty$ on the orthogonal of fixed point spaces.
- The same holds true for random orthogonal matrices.
- Remark: this has multiple consequences in representation theory. The main obstruction to strong convergence (in operator norm) seem to be the presence of a trivial representation in a sequence of representations.

Outline of the proof

- *The moment method*:
  For a positive matrix,
  $$n^{-1} Tr(A^l) \leq ||A||^l \leq Tr(A^l).$$
  Taking the l-th root,
  $$n^{-1/l}(Tr(A^l))^{1/l} \leq ||A|| \leq (Tr(A^l))^{1/l}.$$
  If $n^{-1/l} \sim 1$, i.e. $l >> \log n$, then the trace estimate is robust.
- The moment method with NC polynomials is too hard.
- Doing it with matrix valued linear polynomials in the variables
  $$A = \sum a_i \otimes U_i^{(n)}$$
  (*the linearization trick*) is *equivalent* easier, but still too hard.

Non-Backtracking theory

- Let $(b_1, \ldots, b_l) \in \mathcal{B}(\mathcal{H})$. The index set is endowed with an involution $i \mapsto i^*$ (and $i^{**} = i$ for all i). The *non-backtracking operator* associated to the ℓ-tuple of matrices $(b_1, \ldots, b_l)$ is

$$B = \sum_{j \neq i^*} b_j \otimes E_{ij} \in \mathcal{B}(\mathcal{H} \otimes \mathbb{C}^l) \tag{2}$$

- The matrix B works very well with moment techniques thanks to its non-backtracking structure.
  This strategy is viable is we manage to relate the spectrum of B to that of $A = \sum a_i \otimes U_i^n$

Non-Backtracking theory

- The relation is as follows
  Theorem: (Bordenave, C)
  Let $\lambda \in \mathbb{C}$ satisfy $\lambda^2 \notin \cup_{i\in\{1,\ldots,l\}} \mathrm{spec}(b_i b_{i^*})$. Define the operator A_λ on $\mathcal{H}$ through

  $$A_\lambda = b_0(\lambda) + \sum_{i=1}^{\ell} b_i(\lambda)\,, \qquad b_i(\lambda) = \lambda b_i(\lambda^2 - b_{i^*} b_i)^{-1}$$

  and

  $$b_0(\lambda) = -1 - \sum_{i=1}^{\ell} b_i(\lambda^2 - b_{i^*} b_i)^{-1} b_{i^*}.$$

  Then $\lambda \in \sigma(B)$ if and only if $0 \in \sigma(A_\lambda)$.
- This relation can be inverted and allows to do the calculations with the help of Weingarten estimates.

Concluding remarks

This was just a subjective overview of recent applications and developments of Weingarten calculus.
Other applications:

- Finance (Saad,...)
- Algebra: e.g. Alon Tarski conjecture (Landsberg Kumar).
- Discrete group theory (Magee Puder)
- Random geometry (Magee,...)
- AI (Hayase,...)

Thank you!