Scaling limits and universality of Ising and dimer models

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Based on joint works with

V. Mastropietro, F. Toninelli, B. Renzi

Outline

1. Introduction and overview
2. Weakly non-planar dimer models
3. Proof ideas

Statistical mechanics and universality

Universality in equilibrium statistical mechanics: **robustness** of macroscopic behavior of interacting many-body w.r.t. microscopic details of the model.

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Wilsonian RG provides right framework for studying scaling limit at T_c and prove universality.

Idea: integrate out the small-scale d.o.f., rescale, define flow of effective Hamiltonian. Show that there exists a choice of T_c at which flow converges to non-trivial (conformal inv.) FP.

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Strongest results concern Ising and dimer models: in both cases, standard models solvable in terms of Pfaffians or **determinants** (exact solution $\Leftrightarrow$ **free Fermi** gas – provides bulk scaling limit of some correlations 'easily').

Known results on critical 2D Ising and dimers

Universality of scaling limit proven w.r.t

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   (Kenyon, Smirnov, Sheffield, Okounkov, Chelkak, Hongler, Izyurov, Dubedat, Duminil-Copin, Aggarwal, ...)
   - scaling limit of spin/energy correl. and of interfaces (Ising)
   - GFF scaling limit of height correlations (dimers)
   - conformal covariance w.r.t. Riemann mapping of domain
   - universality w.r.t. lattice

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2. 'Interacting' or weakly non-planar case
   (Spencer, Mastropietro, Benfatto, Falco, Giuliani, Greenblatt, Toninelli, Aizenman–Duminil-Copin–Tassion–Warzel, ...)
   - scaling limit of energy correl. in plane, torus, cylinder (Ising)
   - universal sub-leading corrections to critical free energy (Ising)
   - GFF scaling limit of height correlations (dimers)
   - universal scaling relations (dimers)

Irrelevant and marginal, standard and weak universality

Ising and dimer cases deeply different: Wilsonian RG predicts that weak short range perturbations are irrelevant for **Ising** and marginal for **dimers**, as well as for coupled pairs of Ising layers (**AT, 8V, 6V**, ...)

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Universality in naive sense fails, right notion is weak universality (Kadanoff): model characterized by **universal scaling relations**, all critical exponents can be deduced by just one of them.

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- We define height difference via paths avoiding to pass under non-planar edges.
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- Resulting picture compatible with bosonization.

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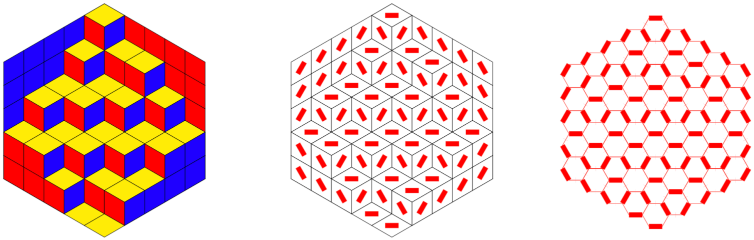

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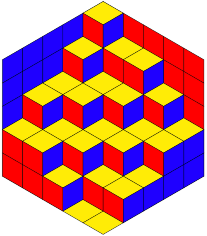

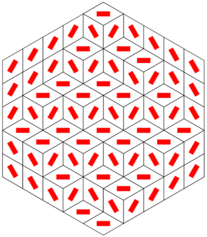

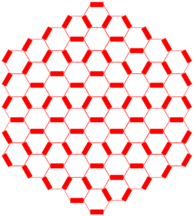

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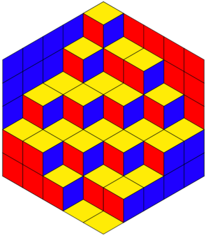

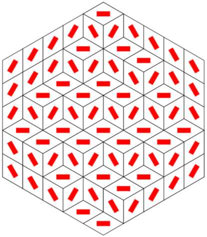

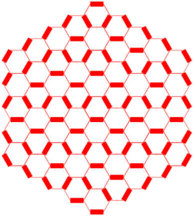

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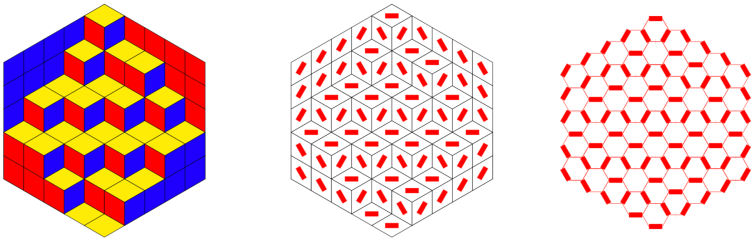

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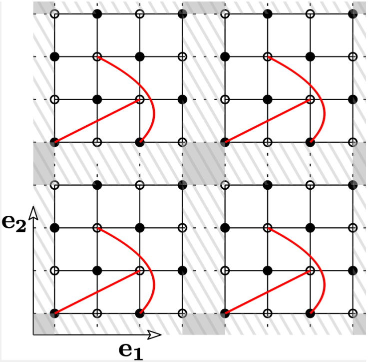
Universality of liquid/rough phase?

Weakly non-planar dimers



We consider an $Lm \times Lm$ portion of $\mathbb{Z}^2$ w. periodic b.c., called $G_L = (V_L, E_L^0)$, consisting of L^2 cells of size m^2, $m \geq 4$ even

In each cell we arbitrarily:

1. add non-planar bonds connecting b to w,
2. assign positive weights $\tilde{t}_e$ to all edges,
3. rescale by λ weights of long edges.

Then repeat periodically over cells.

Probability measure of weakly non-planar dimers

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Probability weight of $D \in \Omega_L$:

$$\mathbb{P}_{L,\lambda}(D) = Z_{L,\lambda}^{-1} \prod_{e \in D} t_e$$

with $Z_{L,\lambda} = \sum_{D \in \Omega_L} \prod_{e \in D} t_e$ the partition function.

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Goal: discuss large scale properties of

$$\mathbb{P}_\lambda = \lim_{L \to \infty} \mathbb{P}_{L,\lambda}$$

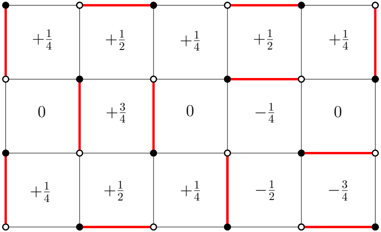
for $\{t_e\}_{e \in E_L^0}$ chosen so that $\mathbb{P}_0$ is in the liquid phase.

Height function

At $\lambda = 0$ exhaustive classification of phases in terms of fluctuation properties of the height:

$$h(\eta) - h(\xi) = \sum_{e \in C_{\xi \to \eta}} \sigma_e(\mathbb{1}_e - 1/4)$$

where $\sigma_e = \pm 1$ if e crossed with white on the right/left.



Liquid, gaseous and frozen phases

Look at fluctuations of $h(\eta) - h(\xi)$ w.r.t. $\mathbb{P}_0$:

- frozen phase: $h(\eta) - h(\xi)$ deterministic
- gaseous phase: bounded $\mathrm{Var}(h(\eta) - h(\xi)) \neq 0$
- liquid phase: unbounded $\mathrm{Var}(h(\eta) - h(\xi))$

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Phases characterized in terms of $\kappa(p) = \det \hat{K}(p)$, $\hat{K}(p)$ Fourier symbol of Kasteleyn matrix $K(b, w)$.

Liquid phase $\Leftrightarrow$ $\kappa(p)$ has two simple zeros.

Corresponding set of $\{t_e\}_{e \in E^0}$ open and non-trivial.

Liquid phase at $\lambda = 0$

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Dimer correlations expressed in terms of the inverse Kasteleyn's matrix: if $e = (b, w)$ and $e' = (b', w')$:

$$\mathbb{E}_0(\mathbb{1}_e \mathbb{1}_{e'}) = K(b, w) K(b', w') \det \begin{pmatrix} K^{-1}(w, b) & K^{-1}(w, b') \\ K^{-1}(w', b) & K^{-1}(w', b') \end{pmatrix}$$

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If $w = (x, \ell)$ and $b = (y, \ell')$:

$$K^{-1}(w,b) = \int_{[-\pi,\pi]^2} \frac{dp}{(2\pi)^2} [\hat{K}(p)^{-1}]_{\ell,\ell'} e^{-ip\cdot(x-y)}$$

Liquid phase at $\lambda = 0$

In particular, if $\kappa(p)$ has two simple zeros p_0^ω, $K^{-1} \propto (\text{dist})^{-1}$ at large distances, from which:

$$\mathbb{E}_0(\mathbb{1}_e; \mathbb{1}_{e'}) = \sum_{\omega=\pm} \frac{K^0_{\omega,j,\ell} K^0_{\omega,j',\ell'}}{(\phi^0_\omega(x-x'))^2} + \sum_{\omega=\pm} \frac{H^0_{-\omega,j,\ell} H^0_{\omega,j',\ell'}}{|\phi^0_\omega(x-x')|^2} e^{2ip_0^\omega \cdot (x-x')} + O(|x|^{-3}),$$

where $\boxed{\phi^0_\omega(x) = \beta^0_\omega x_1 - \alpha^0_\omega x_2}$ with

$$\alpha^0_\omega = \partial_{k_1} \kappa(p_0^\omega), \qquad \beta^0_\omega = \partial_{k_2} \kappa(p_0^\omega).$$

Liquid phase at $\lambda = 0$

One can also compute height fluctuations:

$$\mathbb{E}_0\big(h(\eta_x) - h(\eta_y); h(\eta_w) - h(\eta_z)\big) = \sum_{\substack{e \in C_{\eta_x \to \eta_y} \\ e' \in C_{\eta_w \to \eta_z}}} \sigma_e \sigma_{e'} \mathbb{E}_0(\mathbb{1}_e; \mathbb{1}_{e'})$$

where η_x is a representative face of the cell B_x.

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Deforming paths and using asympt. of dimer corr. one finds (Kenyon, Kenyon-Okounkov-Sheffield)

$$\mathbb{E}_0\big(h(\eta_x) - h(\eta_y); h(\eta_w) - h(\eta_z)\big) = \\ = \frac{1}{2\pi^2}\mathrm{Re}\log\frac{\phi^0_+(z-x)\phi^0_+(w-y)}{\phi^0_+(z-y)\phi^0_+(w-x)}.$$

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Building upon this: GFF scaling limit of the height.

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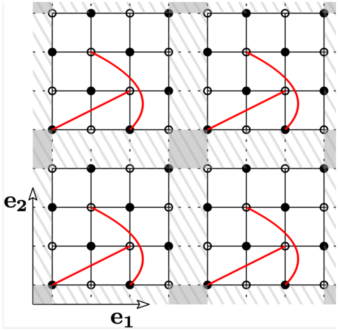

Does variance of height diff. diverge logarithm.? Is scaling limit still GFF?

Main result, I: interacting dimer-dimer correlation

Theorem 1 [G.-Mastropietro-Toninelli (2015, 2017, 2020)], [G.-Renzi-Toninelli (2022)]

Let $\{t_e\}_{e\in E_L^0}$ be s.t. $\kappa(p)$ has two simple zeros $p_0^{\pm}$. Then, for λ small enough, if e, e' are in cells B_x, $B_{x'}$, of type (j,ℓ), (j',ℓ'):

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where: $\phi_\omega(x)=\beta_\omega x_1-\alpha_\omega x_2$ and $K_{\omega,j,\ell}$, $H_{\omega,j,\ell}$, α_ω, β_ω, p^ω, ν are all analytic in λ. Moreover, $\nu = 1 + c_1\lambda + \cdots$ and, generically, $c_1 \neq 0$.

Height fluctuations

From $\mathbb{E}_\lambda(\mathbb{1}_e; \mathbb{1}_{e'})$ can compute height covariance:

$$\mathbb{E}_\lambda\big(h(\eta_x)-h(\eta_y); h(\eta_w)-h(\eta_z)\big) = \sum_{\substack{e\in C_{\eta_x\to\eta_y}\\ e'\in C_{\eta_w\to\eta_z}}} \sigma_e\sigma_{e'}\mathbb{E}_\lambda(\mathbb{1}_e; \mathbb{1}_{e'})$$

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Deforming the paths and using Thm. 1 gives

$$\mathbb{E}_\lambda\big(h(\eta_x) - h(\eta_y); h(\eta_w) - h(\eta_z)\big) = \\ = \frac{A(\lambda)}{2\pi^2} \operatorname{Re} \log \frac{\phi_+(z-x)\phi_+(w-y)}{\phi_+(z-y)\phi_+(w-x)}.$$

thanks to special relation between $K_{\omega,j,\ell}$ and $\alpha_\omega, \beta_\omega$

(recall: $A(0) \equiv 1$, indep. of $\{t_e\}$)

Main result, II: GFF scaling limit

Theorem 2 [G.-Mastropietro-Toninelli (2015, 2017, 2020)], [G.-Renzi-Toninelli (2022)]

Same hypotheses as previous theorem. Then, for any $\psi \in C_0^\infty(\mathbb{R}^2; \mathbb{R})$ of zero average and $\alpha \in \mathbb{R}$,

$$\lim_{\epsilon \to 0} \mathbb{E}_\lambda\left(e^{i\alpha h_\epsilon(\psi)}\right) = e^{\frac{\alpha^2}{4\pi^2} A(\lambda) \int_{\mathbb{R}^2} dx \int_{\mathbb{R}^2} dy\, \psi(x)\psi(y) \log|\phi_+(x-y)|}$$

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In general, $A(\lambda)$ depends on $\{t_e\}_{e\in E_L}$. Moreover,

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3. Thm.1 & 2 can be read by saying that
   - $h(\eta(x)) \leftrightarrow$ GFF $\phi(x)$ with stiffness A
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4. Expectation of $e^{i\alpha h_\epsilon(\psi)}$ is technically similar to spin-spin correl. in weakly non-planar Ising models. Could strategy of Thm.2 be used to prove universality of 1/8 critical exponents in Ising models with finite range interactions?

Outline

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5. Effective theory is UV regularized version of Luttinger model; it can be studied by multiscale analysis via comparison with IR reference model

Proof's strategy, II

6. Theory of reference model developed by Benfatto-Mastropietro. Key point: vanishing of Beta function $\Rightarrow$ strength of quartic interaction tends to $\lambda_{-\infty} \neq 0$ at large distances.

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   It implies asymptotic vanishing of Beta function for dimer model, too $\Rightarrow$ anomalous exponent ν
8. To prove $A = \nu$ compare asymptotic WIs of ref. model with exact lattice WIs of dimer model following from $\sum_{e \to x} \mathbb{1}_e = 1$. From this we find that A/ν is protected by symmetry, no dressing.

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- Related results, via similar methods, for: non-planar Ising, Ashkin-Teller, $8V$, $6V$, XXZ

Open problems and perspectives

- Compute $\mathbb{E}_0\left(e^{i\alpha(h(\eta_x)-h(\eta_y))}\right)$ w.o. coarse graining (and possibly monomer-monomer correl.)
- Understand KPZ-type fluctuations at the boundary between liquid and frozen region
- Understand effect of boundaries, compute boundary critical exp.
- Compute scaling limit in domains of arbitrary shape, prove conformal covariance
- Rough phase of 3D Ising w. Dobrushin b.c.
- In Ising, scaling limit of spin-spin correlations

Thank you!