ICM 2022

Introduction to Decoupling in Fourier Analysis

What is Fourier analysis?

In Fourier analysis, we write a function f as

$$f(x) = \sum_n \hat{f}(n) e^{2\pi i n x}.$$

Why? The building blocks behave nicely with respect to

- Differentiation: $\frac{d}{dx} e^{2\pi i n x} = 2\pi i n e^{2\pi i n x}$.
- Translation: $e^{2\pi i n (x + x_0)} = e^{2\pi i n x_0} e^{2\pi i n x}$.

Many problems that involve derivatives or translation-structure of the real line connect naturally with Fourier analysis.

A problem with Fourier analysis

In Fourier analysis, we write a function f as

$$f(x) = \sum_n \hat{f}(n) e^{2\pi i n x}.$$

This representation can be hard to work with.

To find $f(2)$ we have to add up many terms.
They have positive and negative parts.
It's hard to tell if $f(2)$ is positive or negative.
It's hard to tell if $f(2)$ is big or small.

We will see a deep open problem in a minute.

Decoupling is a recently developed set of tools that helps transfer information about $\hat{f}$ into information about f.

Decoupling has led to the solution of several longstanding problems in harmonic analysis, PDE, and analytic number theory.

Introduced by Wolff (2000).
Breakthrough by Bourgain-Demeter (2014).

Plan for the day

Introduce one old problem which has been solved using decoupling.

Talk through some of the ideas of the proof.
(Draw lots of pictures.)

Fourier analysis and diophantine equations

In the circle method, the number of solutions of a diophantine equation can be encoded using Fourier analysis.

Sample problem (raised by Hardy-Littlewood).

Let $HL_{s,k}(N)$ be the number of integer solutions of

$$n_1^k + \ldots + n_s^k = n_{s+1}^k + \ldots + n_{2s}^k, \text{ with } 1 \leq n_i \leq N \qquad (HL).$$

For fixed s, k, what are the asymptotics of $HL_{s,k}(N)$ as $N \to \infty$?

Fourier analysis and diophantine equations

In the circle method, the number of solutions of a diophantine equation can be encoded using Fourier analysis.

Notation: $e(x) = e^{2\pi i x}$.

Let $h(x) = h_{k,N}(x) := \sum_{n=1}^{N} e(n^k x)$.

Proposition $\int_0^1 |h_{k,N}(x)|^{2s} dx = HL_{s,k}(N)$, the number of integer solutions of

$$n_1^k + ... + n_s^k = n_{s+1}^k + ... + n_{2s}^k, \text{ with } 1 \leq n_i \leq N \qquad (HL).$$

On the next slide, we sketch the proof of the Proposition. It is a good example of how Fourier analysis interacts nicely with the addition structure of the real line.

Proof sketch of Proposition

If $m \in \mathbb{Z}$, then

$$\int_0^1 e(mx)dx = \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{if } m \neq 0 \end{cases}$$

Let $h(x) := \sum_{n=1}^{N} e(n^k x)$.

$|h|^{2s} = h^s \overline{h}^s = \sum_{n_1,\dots,n_{2s}=1}^{N} e((n_1^k + \dots + n_s^k - n_{s+1}^k - \dots - n_{2s}^k)x)$.

$\int_0^1 |h|^{2s} = \sum_{n_1,\dots,n_{2s}=1}^{N} \int_0^1 e((n_1^k + \dots + n_s^k - n_{s+1}^k - \dots - n_{2s}^k)x)dx$,

the number of integer solutions of

$$n_1^k + \dots + n_s^k = n_{s+1}^k + \dots + n_{2s}^k, \text{ with } 1 \leq n_i \leq N$$

Fourier analysis and diophantine equations

$HL_{s,k}(N)$ is the number of integer solutions of

$$n_1^k + ... + n_s^k = n_{s+1}^k + ... + n_{2s}^k, \text{ with } 1 \leq n_i \leq N \qquad (HL).$$

We understand the asymptotics of $HL_{s,k}(N)$ as $N \to \infty$ in the following cases:

- If $k = 2$, classical.
- If s is much bigger than k. (Hardy-Littlewood, Vinogradov, ...)

For many other values of s, k, the asymptotics are poorly understood.

Example: $k = 3$, $s = 3$.

Fourier analysis and diophantine equations

$$h_{k,N}(x) := \sum_{n=1}^{N} e^{2\pi i n^k x}. \qquad (*)$$

Proposition $HL_{s,k}(N) = \int_0^1 |h_{k,N}(x)|^{2s} dx$

$(*)$ is the Fourier series of $h_{k,N}$. But it is difficult to convert this explicit Fourier series into accurate information about $h_{k,N}(x)$.

Deep open problems:

- Estimate the order of magnitude of $|h_{3,N}(\sqrt{2})|$.
- Estimate the order of magnitude of $\int_0^1 |h_{3,N}(x)|^6 dx$.

Vinogradov system

In the 1930s, Vinogradov studied the number of solutions of the following system of equations:

$$n_1^j + ... + n_s^j = n_{s+1}^j + ... + n_{2s}^j \text{ for all } 1 \le j \le k. \qquad (V)$$

(Note: $2s$ variables and k equations.)

$J_{s,k}(N) =$ the number of integer solutions of (V) with $1 \le n_i \le N$.

Vinogradov proved good estimates for $J_{s,k}(N)$ for some k and s. He applied these estimates to the Hardy-Littlewood problem above and other problems in number theory.

On the next slide, I'll show one of the results he got in this way.

Asymptotics for Hardy-Littlewood problem

$HL_{s,k}(N)$ is the number of integer solutions of

$$n_1^k + ... + n_s^k = n_{s+1}^k + ... + n_{2s}^k, \text{ with } 1 \leq n_i \leq N \qquad (HL).$$

We understand the asymptotics of $HL_{s,k}(N)$ in the following cases.

- $k = 2$. Classical.
- $s > 2^k$, Hardy-Littlewood-Hua.
- $s > Ck^2 \log k$. Vinogradov.
- $s > k^2/2 + \text{ lower order terms}$. Current record.

Vinogradov system

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(Note: $2s$ variables and k equations.)

$J_{s,k}(N) =$ the number of integer solutions of (V) with $1 \leq n_i \leq N$.

Vinogradov proved good estimates for $J_{s,k}(N)$ for some k and s.

In the last decade, mathematicians have proven good estimates for all k and s.

Sharp estimates for Vinogradov system

$$n_1^j + \ldots + n_s^j = n_{s+1}^j + \ldots + n_{2s}^j \text{ for all } 1 \leq j \leq k. \qquad (V)$$

$J_{s,k}(N) =$ the number of integer solutions of (V) with $1 \leq n_i \leq N$.

Theorem (Sharp estimate for Vinogradov system)

For every $\epsilon > 0$, there is a constant $C(\epsilon, k)$ so that

$$J_{s,k}(N) \leq C(\epsilon, k) N^{\epsilon} \left(N^s + N^{2s - \frac{k(k+1)}{2}} \right).$$

This upper bound is sharp up to the factor $C(\epsilon, k) N^{\epsilon}$.

Sharp estimates for Vinogradov system

$$n_1^j + ... + n_s^j = n_{s+1}^j + ... + n_{2s}^j \text{ for all } 1 \leq j \leq k. \qquad (V)$$

$J_{s,k}(N) =$ the number of integer solutions of (V) with $1 \leq n_i \leq N$.

Theorem (Sharp estimate for Vinogradov system)

$$J_{s,k}(N) \leq C(\epsilon, k) N^\epsilon \left(N^s + N^{2s - \frac{k(k+1)}{2}} \right).$$

There are several proofs of this theorem.

- Wooley, k=3, efficient congruencing.
- Bourgain-Demeter-G, all k, decoupling.
- Wooley, all k, efficient congruencing.
- Guo-Li-Yang-Zorin-Kranich, all k, 10 pages, combined ideas.

Goals of the talk

Main goal: Describe some main ideas of the proof(s).

- All proofs involve complex formulas and computations. But we will try to explain the ingredients of the computations without writing long formulas.
- I will focus on the decoupling proof, but I will try to make some comments that apply to all the proofs.

Hardy-Littlewood vs. Vinogradov

Hardy-Littlewood: $n_1^k + ... + n_s^k = n_{s+1}^k + ... + n_{2s}^k$.

$2s$ variables. One equation.
Open problem for many values of k, s.

Vinogradov: $n_1^j + ... + n_s^j = n_{s+1}^j + ... + n_{2s}^j$ for all $1 \leq j \leq k$.

$2s$ variables. k equations.
Understood for all k, s up to factor $C_\epsilon N^\epsilon$.

Why is the Vinogradov system easier to understand?

Roughly: in the Vinogradov system, it is possible to combine information from many different scales (of N).
Vinogradov used this idea in his work in the 1930s.
Recent work in the area carries the multiscale idea even further.

Fourier analysis and the Vinogradov system

$$n_1^j + ... + n_s^j = n_{s+1}^j + ... + n_{2s}^j \text{ for all } 1 \leq j \leq k. \qquad (V)$$

$J_{s,k}(N) =$ the number of integer solutions of (V) with $1 \leq n_i \leq N$.

Can write $J_{s,k}(N)$ as an integral

$J_{s,k}(N) = \int_{[0,1]^k} |f(x)|^{2s} dx$

where $f(x)$ has a nice Fourier series.

CAN'T estimate $|f(x)|$ pointwise. That would be at least as hard as full understanding of Hardy-Littlewood problem.

CAN estimate $\int_{[0,1]^k} |f(x)|^p dx$ for any p.

Comments on the proof

In the dcoupling proof, we estimate $\int_{[0,1]^k} |f(x)|^p dx$ using purely analytic methods.
The ingredients are things like

- Orthogonality
- Holder's inequality
- Elementary geometry
- Induction on scales. (Or combining information from many scales.)

It is perhaps surprising that these ingredients are enough to prove sharp estimates for the Vinogradov system.

Comments on the proof 2

The decoupling proof is purely analytic. The ingredients are things like

- Orthogonality
- Holder's inequality
- Elementary geometry
- Induction on scales. (Or combining information from many scales.)

The induction on scales is crucial. It plays a crucial role in Vinogradov's work and in all the proofs of the sharp bounds for Vinogradov system.

Goal: Explain what we mean by induction on scales and discuss why it is helpful.

Comments on the proof 3

The decoupling proof is also quite visual (or geometric).

We will draw pictures.

At the beginning we choose a coordinate system that makes the pictures nice.

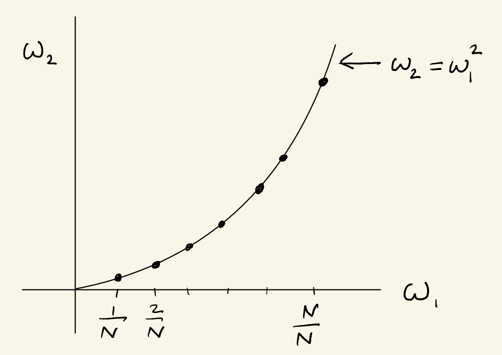
To keep the discussion simple, we will focus on dimension $k = 2$ and we will prove a weaker estimate. The discussion will illustrate some of the main tools in the proof.

Our exponential sum

$x = (x_1, x_2) \in \mathbb{R}^2$.

$f(x) = \sum_{n=1}^{N} e\left(\frac{n}{N} x_1 + \frac{n^2}{N^2} x_2\right)$.

The frequencies $\left(\frac{n}{N}, \frac{n^2}{N^2}\right)$ lie on a parabola:



Our exponential sum

$f(x) = \sum_{n=1}^{N} e\left(\frac{n}{N}x_1 + \frac{n^2}{N^2}x_2\right).$

(Here $x \in \mathbb{R}^2$.)

We write $Q_S(x)$ for a square of side length S centered at x.

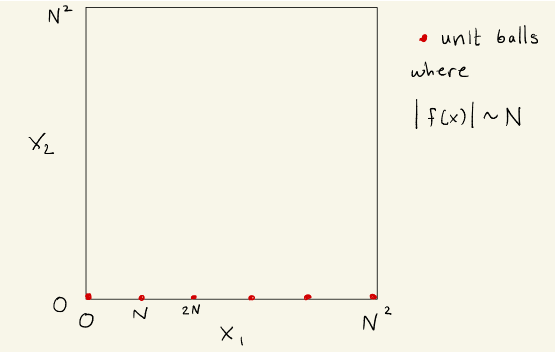
To get sharp estimates for Vinogradov system of degree 2, need to bound

$\int_{Q_{N^2}(0)} |f(x)|^6 dx.$

Our exponential sum

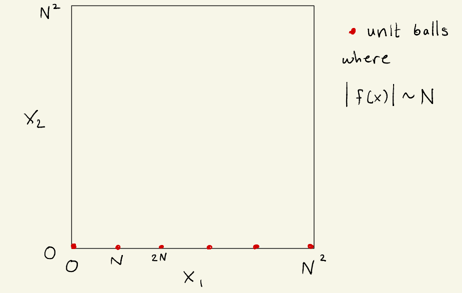
$$f(x) = \sum_{n=1}^{N} e\left(\frac{n}{N}x_1 + \frac{n^2}{N^2}x_2\right).$$

- $f(0) = N$.
- Because of orthogonality, $|f(x)| \leq 10\sqrt{N}$ for most points x.
- $f(x)$ is N-periodic in x_1 variable.
- $|f(x)|$ is roughly constant on each unit square.



Our exponential sum

$$f(x) = \sum_{n=1}^{N} e\left(\frac{n}{N}x_1 + \frac{n^2}{N^2}x_2\right).$$



$U_\lambda(f) := \{x : |f(x)| > \lambda\}$. $|U|$ = measure of U.

Theorem (Baby version of main theorem)

$$|U_{N/10}(f) \cap Q_{N^2}| \leq C_\epsilon N^{1+\epsilon}.$$

Notation to remember:

- $e(x) = e^{2\pi ix}$.
- $Q_S(x)$ is a cube of side S centered at x.
- $U_\lambda(f) := \{x : |f(x)| > \lambda\}$.
- $|U| =$ measure of U.

$f(x) = \sum_{n=1}^{N} e\left(\frac{n}{N}x_1 + \frac{n^2}{N^2}x_2\right)$.

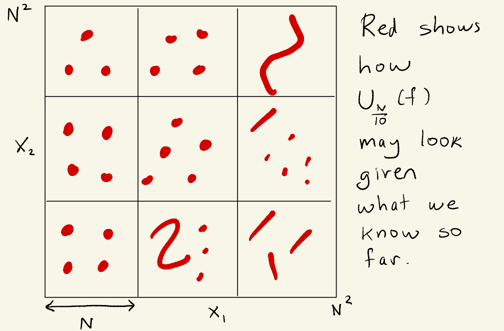
Goal: $|U_{N/10}(f) \cap Q_{N^2}| \leq C_\epsilon N^{1+\epsilon}$.

We will now introduce one tool at a time and see how close we get to our goal.

$f(x) = \sum_{n=1}^{N} e\left(\frac{n}{N}x_1 + \frac{n^2}{N^2}x_2\right)$.
Goal: $|U_{N/10}(f) \cap Q_{N^2}| \leq C_\epsilon N^{1+\epsilon}$.

Tool 1: Orthogonality.
The functions $\{e\left(\frac{n}{N}x_1 + \frac{n^2}{N^2}x_2\right)\}_{n=1}^{N}$ are orthogonal on each Q_N.

So $|U_{N/10}(f) \cap Q_N(x)| \leq CN$ for any x.
So $|U_{N/10}(f) \cap Q_{N^2}(0)| \leq CN^3$.



$f(x) = \sum_{n=1}^{N} e\left(\frac{n}{N} x_1 + \frac{n^2}{N^2} x_2\right).$

Tool 2: Pieces of the sum. If $I \subset \{1, ..., N\}$,

$f_I(x) = \sum_{n \in I} e\left(\frac{n}{N} x_1 + \frac{n^2}{N^2} x_2\right).$

We can partition $\{1, ..., N\}$ into intervals I of length L, and then

$f(x) = \sum_{I \text{ length } L} f_I(x).$

All theorems about Vinogradov system involve estimates for f_I for many different I with many different length scales L.

This idea goes back to Vinogradov.

$f(x) = \sum_{n=1}^{N} e\left(\frac{n}{N}x_1 + \frac{n^2}{N^2}x_2\right).$

Tool 2: Pieces of the sum. If $I \subset \{1, ..., N\}$,

$f_I(x) = \sum_{n \in I} e\left(\frac{n}{N}x_1 + \frac{n^2}{N^2}x_2\right)$. $f(x) = \sum_{I \text{ length } L} f_I(x)$.

Lemma If $x \in U_{N/10}(f)$, then $x \in U_{L/20}(f_I)$ for most I.

Proof idea.

- $|f(x)| \leq N$.
- $|f_I(x)| \leq L$.
- The number of I is N/L.

So if $|f(x)| = N$, then $|f_I(x)| = L$ for every I.
If $|f(x)|$ is close to N, then $|f_I(x)|$ is close to L for most I. □

$$f(x) = \sum_{n=1}^{N} e\left(\frac{n}{N}x_1 + \frac{n^2}{N^2}x_2\right).$$

Tool 2: Pieces of the sum. If $I \subset \{1, ..., N\}$,

$$f_I(x) = \sum_{n \in I} e\left(\frac{n}{N}x_1 + \frac{n^2}{N^2}x_2\right). \quad f(x) = \sum_{I \text{ length } L} f_I(x).$$

Lemma If $x \in U_{N/10}(f)$, then $x \in U_{L/20}(f_I)$ for most I.

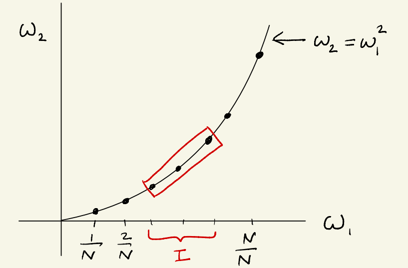
Questions:

- What can we say about shape of each set $U_{L/20}(f_I)$?
- What can we say about how these sets overlap?

$f(x) = \sum_{n=1}^{N} e\left(\frac{n}{N}x_1 + \frac{n^2}{N^2}x_2\right)$.
Goal: $|U_{N/10}(f) \cap Q_{N^2}| \leq C_\epsilon N^{1+\epsilon}$.

Tool 3: The shape of f_I.

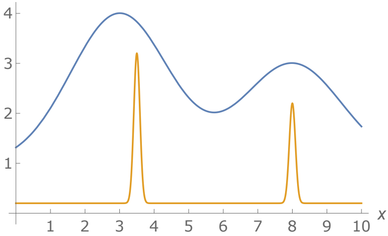
The set of frequencies $\{(\frac{n}{N}, \frac{n^2}{N^2})\}_{n\in I}$ lie in a small box.



In other words, $\hat{f}_I$ is supported in the red box. What does this tell us about f_I?

Warmup problem

Suppose that g is a function on $\mathbb{R}$ and $\hat{g}$ is supported in $[-1/2, 1/2]$. What could g look like?



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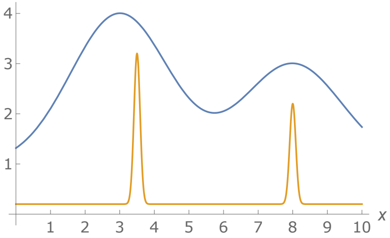
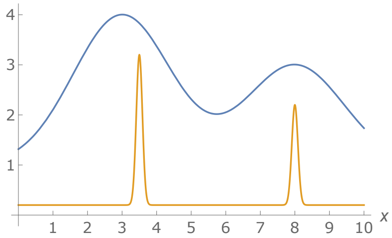


Answer: The blue function has $\hat{g}$ supported in $[-1/2, 1/2]$.
The orange function does not.
The sharp peaks in the orange function require a large support in Fourier space.

Warmup problem

Suppose that $g : \mathbb{R} \to \mathbb{R}$ and $\hat{g}$ is supported in $[-\frac{1}{2}, \frac{1}{2}]$.
What could g look like?



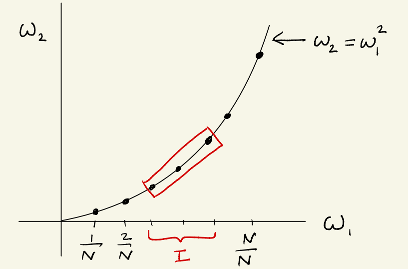
Theorem. (Shannon-Nyquist) If $\hat{g}$ supported in $[-\frac{1}{2}, \frac{1}{2}]$, then g can be recovered from $g(n)$ for $n \in \mathbb{Z}$.

Heuristic: If $\hat{g}$ is supported in $[-1/2, 1/2]$, then g is roughly constant on each interval of length 1.

$f(x) = \sum_{n=1}^{N} e\left(\frac{n}{N}x_1 + \frac{n^2}{N^2}x_2\right)$.
Goal: $|U_{N/10}(f) \cap Q_{N^2}| \leq C_\epsilon N^{1+\epsilon}$.

Tool 3: The shape of f_I.

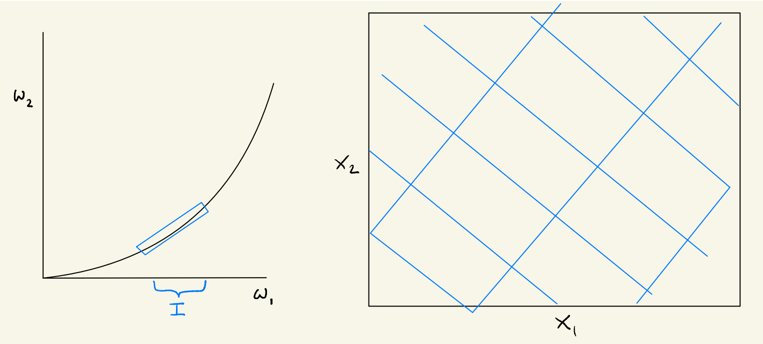
The set of frequencies $\{(\frac{n}{N}, \frac{n^2}{N^2})\}_{n \in I}$ lie in a small box.



In other words, $\hat{f}_I$ is supported in the red box. What does this tell us about f_I?

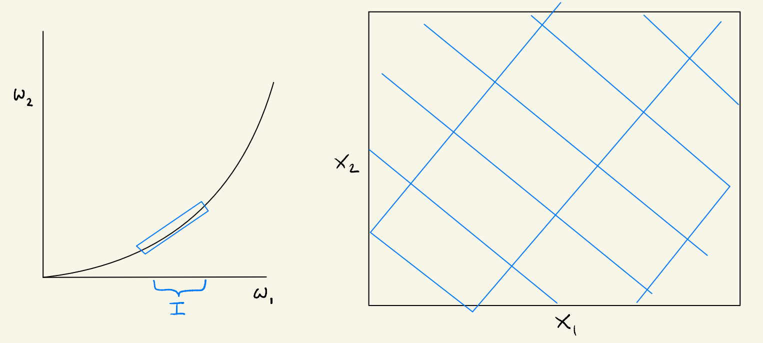
Tool 3: The shape of f_I.

The set of frequencies $\{(\frac{n}{N}, \frac{n^2}{N^2})\}_{n \in I}$ lie in a small box.
T_I is a tiling of $\mathbb{R}^2$ by rectangles that are dual to the small box.



Then $|f_I(x)|$ is roughly constant on each rectangle in the tiling.

Tool 3: The shape of f_I.
The set of frequencies $\{(\frac{n}{N}, \frac{n^2}{N^2})\}_{n \in I}$ lie in a small box.
T_I is a tiling of $\mathbb{R}^2$ by rectangles that are dual to the small box.



The box on the left has dimensions $\frac{L}{N} \times \frac{L^2}{N^2}$.
Each tile on the right has dimensions $\frac{N}{L} \times \frac{N^2}{L^2}$.
The long axis of the tile corresponds to the short axis of the box on the left.

Recap and combine Tools 1-3

$f(x) = \sum_{n=1}^{N} e\left(\frac{n}{N}x_1 + \frac{n^2}{N^2}x_2\right)$.
Goal: $|U_{N/10}(f) \cap Q_{N^2}| \leq C_\epsilon N^{1+\epsilon}$.

Tool 1: The exponentials in the sum are orthogonal on each Q_N.
Can compute $\int_{Q_N} |f_I|^2 dx$.

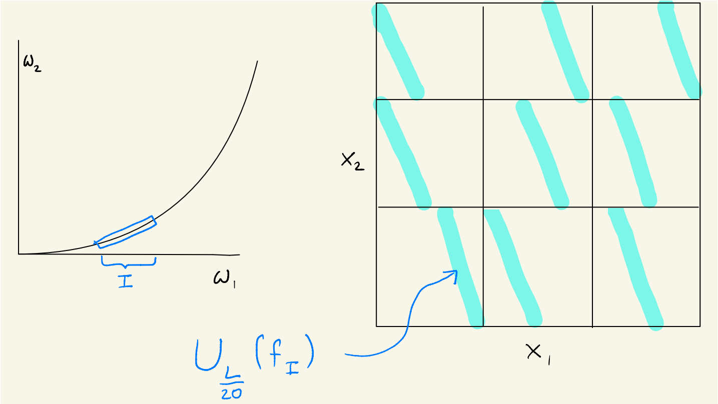
Tool 2: Look at f_I for many different intervals. $I \subset \{1, ..., N\}$ interval of length L.
If $x \in U_{N/10}(f)$, then $x \in U_{L/20}(f_I)$ for most I.

Tool 3: The shape of f_I. $|f_I|$ roughly constant on each rectangular tile of a tiling that is "dual to I".
So $U_{L/20}(f_I)$ is a union of these dual rectangles.

Let us combine all these tools and see what we can figure out about f_I when $L = N^{1/2}$. We would like to understand $U_{L/20}(f_I)$.

Recap and combine Tools 1-3

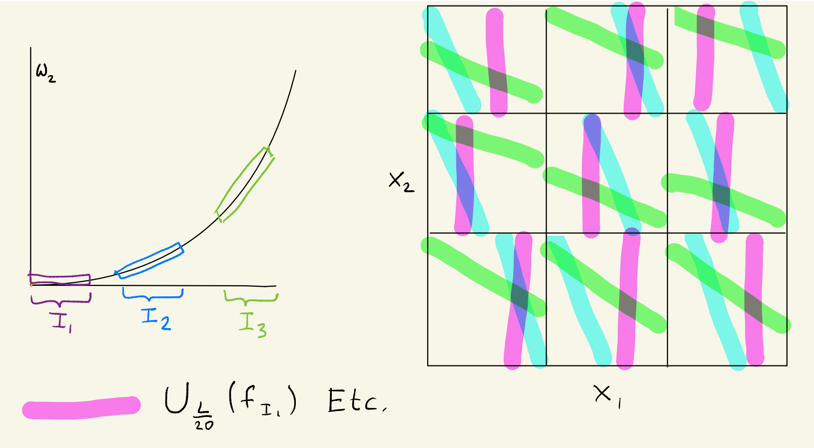
We study f_I where I has length $L = N^{1/2}$.
Tool 1 (orthogonality) tells us that $|U_{L/20}(f_I) \cap Q_N| \leq CN^{3/2}$.
Tool 3 tells us that $U_{L/20}(f_I)$ is organized into $N^{1/2} \times N$ rectangles.
So there are $O(1)$ rectangles in each Q_N.



$f(x) = \sum_{n=1}^{N} e\left(\frac{n}{N}x_1 + \frac{n^2}{N^2}x_2\right)$.

Goal: $|U_{N/10}(f) \cap Q_{N^2}| \leq C_\epsilon N^{1+\epsilon}$.

Tool 4: Transversality. For different I, the rectangles are oriented in different directions.



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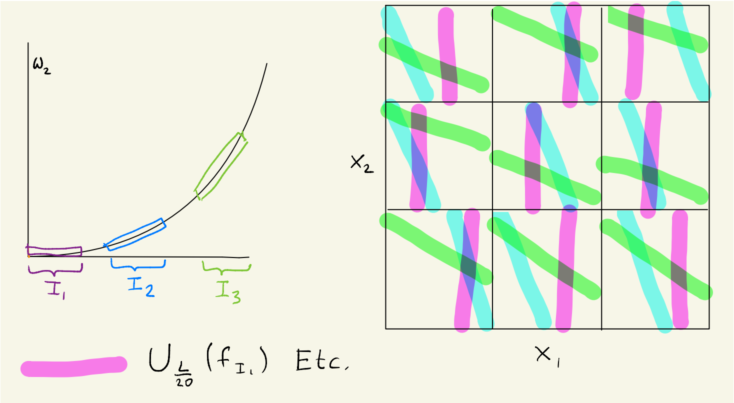
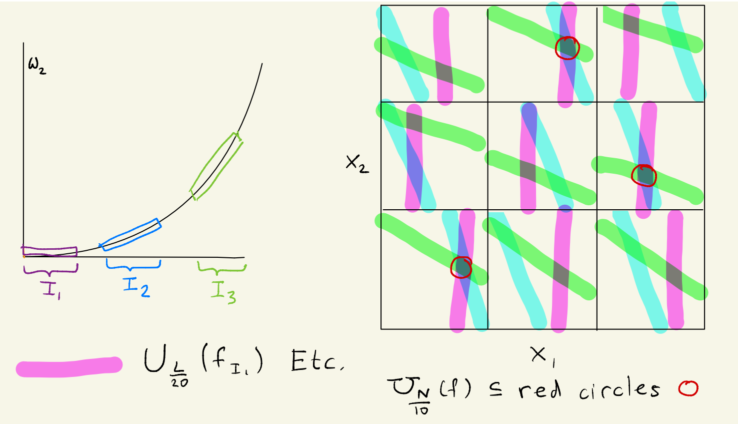
Tool 4: Transversality.



Recall: If $x \in U_{N/10}(f)$, then $x \in U_{L/20}(f_I)$ for most I.

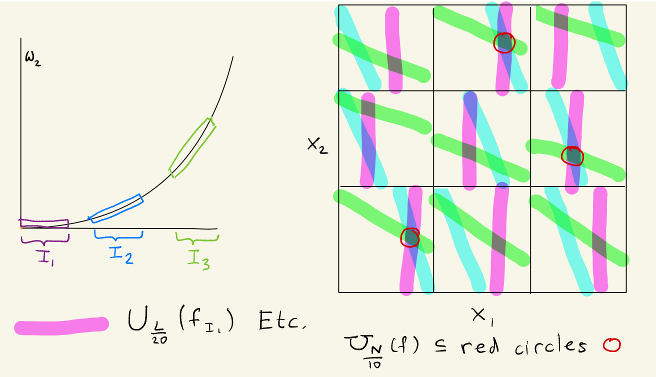
Tool 4: Transversality. For different I, the rectangles are oriented in different directions.



Recall: If $x \in U_{N/10}(f)$, then $x \in U_{L/20}(f_I)$ for most I.

So $U_{N/10}(f) \cap Q_N$ is contained in $O(1)$ smaller squares $Q_{N^{1/2}}$.

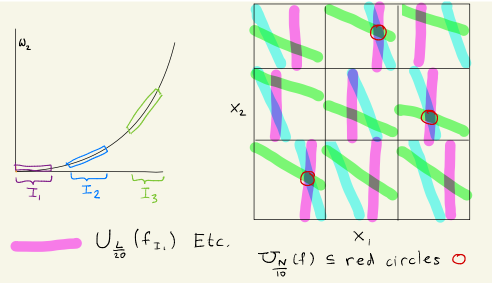
Tool 4: Transversality.



$U_{N/10}(f) \cap Q_N$ is contained in $O(1)$ smaller squares $Q_{N^{1/2}}$.
Can use the same method to study f inside each of these smaller squares.

$|U_{N/10}(f) \cap Q_N| \leq N^{\epsilon}$. So $|U_{N/10}(f) \cap Q_{N^2}| \leq N^{2+\epsilon}$.

Historical remarks



Transversality depends on the curvature of the parabola.
Stein began a program to investigate this connection between curvature and Fourier analysis in the 1960s.
It was developed by many people.

The argument we just sketched is due to Wolff and Bennett-Carbery-Tao.
We will call it the orthogonality/transversality argument.

Taking stock

$f(x) = \sum_{n=1}^{N} e\left(\frac{n}{N}x_1 + \frac{n^2}{N^2}x_2\right)$.
Goal: $|U_{N/10}(f) \cap Q_{N^2}| \leq C_\epsilon N^{1+\epsilon}$.

Just orthogonality: $|U_{N/10}(f) \cap Q_{N^2}| \leq CN^3$.
Orthogonality/transversality: $|U_{N/10}(f) \cap Q_{N^2}| \leq CN^{2+\epsilon}$.

This was the best estimate available in harmonic analysis before Bourgain-Demeter.

Decoupling applies these tools at many different scales.

$f_N(x) = \sum_{n=1}^{N} e\left(\frac{n}{N}x_1 + \frac{n^2}{N^2}x_2\right)$.
Goal: $|U_{N/10}(f) \cap Q_{N^2}| \leq C_\epsilon N^{1+\epsilon}$.

Tool 5: Induction on scales.

We discussed estimates for f_I and for $U_{L/20}(f_I)$.
This is actually similar to our original problem.

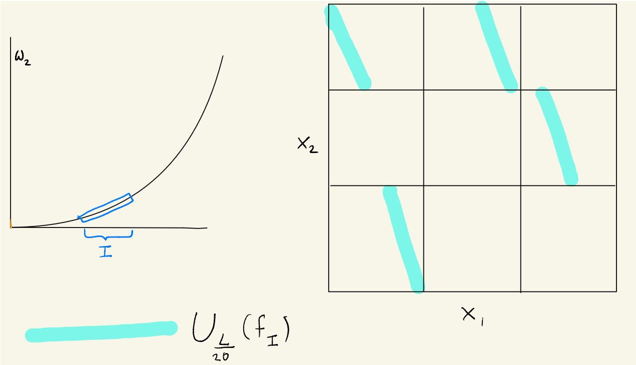
For each I of length L, there is a linear change of variables that converts f_I to f_L, our original function but with L in place of N. So our previous tools give estimates about $U_{L/20}(f_I)$.

$f_N(x) = \sum_{n=1}^{N} e\left(\frac{n}{N}x_1 + \frac{n^2}{N^2}x_2\right).$

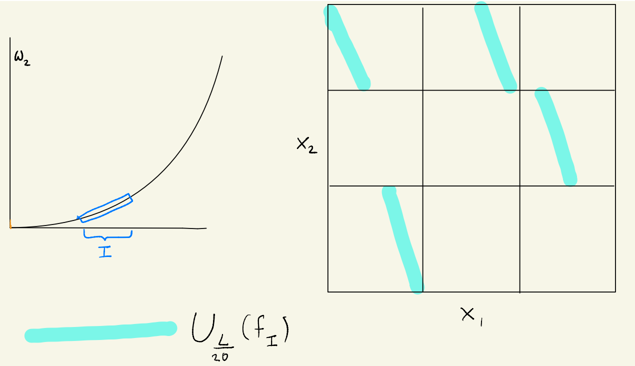
Goal: $|U_{N/10}(f) \cap Q_{N^2}| \leq C_\epsilon N^{1+\epsilon}$.

Look again at f_I with length $L = N^{1/2}$.

- Local orthogonality bounds $|U_{L/20}(f_I) \cap Q_N|$ for each Q_N.
- Induction on scales bounds $|U_{L/20}(f_I) \cap Q_{N^2}|$.



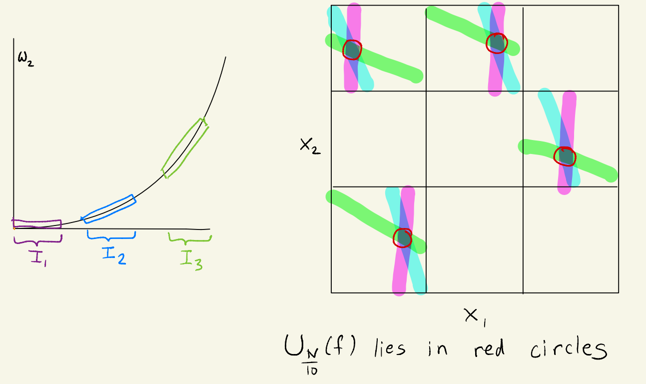
- Local orthogonality bounds $|U_{L/20}(f_I) \cap Q_N|$ for each Q_N.
- Induction on scales bounds $|U_{L/20}(f_I) \cap Q_{N^2}|$.



The two bounds are complementary.
Induction gives a better bound on the number of tiles in $U_{L/20}(f_I)$.
Local orthogonality controls how the tiles pack.

We add this new information into the transversality method.

Old: Local orthogonality bounds $|U_{L/20}(f_I) \cap Q_N|$ for each Q_N.
New: Induction on scales bounds $|U_{L/20}(f_I) \cap Q_{N^2}|$.



This improves bound for $|U_{N/10}(f) \cap Q_{N^2}|$ all the way to the goal.

Recap of our approaches

1. Just orthogonality.

$$|U_{N/10}(f) \cap Q_{N^2}| \leq CN^3.$$

2. Orthogonality and transversality.
(Harmonic analysis of the 70s, 80s, 90s.)

$$|U_{N/10}(f) \cap Q_{N^2}| \leq C_\epsilon N^{2+\epsilon}.$$

3. Orthogonality and transversality and induction on scales.
(Decoupling theory of Wolff and Bourgain-Demeter.)

$$|U_{N/10}(f) \cap Q_{N^2}| \leq C_\epsilon N^{1+\epsilon}.$$

Sharp up to factor of N^ϵ.

Reflecting on the induction step

Why does the induction step help so much?

In the orthogonality/transversality argument, we considered f_I at scale $L = N^{1/2}$.
We used transversality between the rectangles at that scale.

When we use induction and unwind the induction, the argument involves scales L^{α} for a dense set of $\alpha \in [0, 1]$.

This is a common feature of all proofs of sharp bounds for the Vinogradov system. Each proof

- uses f_I at a dense set of scales.
- takes advantage of some type of transversality at all those scales.

The current landscape of the field

Decoupling has changed the landscape of the field.

Combining information from many scales is more powerful than anyone had realized.

People have been thinking a lot about:

- What other problems can we solve by combining information from many scales?
- How can we combine information from many scales in a systematic way?
- What problems are out of reach of this multiscale method?