Homological algebraic geometry

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Homological algebraic geometry

Algebraic geometry is a fusion of geometry and algebra:

- geometry gives intuition and a way of thinking,
- algebra provides machinery for proving theorems.

Likewise, homological algebraic geometry is a fusion of

- algebraic geometry with
- homological algebra.

The main object of HAG is the derived category of coherent sheaves.

Other nicknames : categorical, derived, noncommutative, ...

Why do we study HAG?

- New interesting "noncommutative varieties" to be studied.
- Many classical results become more straightforward and better behaved when considered from the viewpoint of HAG.

§1. Derived category

Brief history

A very incomplete timeline:

- A. Grothendieck devised derived categories for proving duality;
- J.-L. Verdier (student of Grothendieck) developed the notion in the 60's;
- A. Beilinson in the late 70's observed that the derived category of a projective space has a particularly nice structure;
- M. Kapranov in the late 80's found that the derived categories of quadrics and Grassmannians have a similar structure;
- A. Bondal and D. Orlov in the 90's initiated a fullscale study of derived categories and proved many important results, presented in their ICM talk in 2002.

Definition

The derived category of an algebraic variety X is defined as the Verdier quotient

$$\mathbf{D}(X) := \mathrm{Com}(X)/\,\mathrm{Acycl}(X).$$

- $\mathrm{Com}(X)$ is the category of complexes of vector bundles over X

$$\dots \to E^{i-1} \xrightarrow{d^{i-1}} E^{i} \xrightarrow{d^{i}} E^{i+1} \to \dots,$$

  where d^i are fiberwise linear morphisms such that $d^i \circ d^{i-1} = 0$ for all i,
- $\mathrm{Acycl}(X) \subset \mathrm{Com}(X)$ is the subcategory of acyclic complexes, i.e.,

$$\mathrm{Acycl}(X) := \{(E^\bullet, d^\bullet) \mid \mathcal{H}^i(E^\bullet) := \mathrm{Ker}(d^i)/\,\mathrm{Im}(d^{i-1}) = 0 \text{ for all } i\}.$$

The bounded derived category:

$$\mathbf{D}^{\mathrm{b}}(X) := \{E \in \mathbf{D}(X) \mid \mathcal{H}^i(E) = 0 \text{ for } |i| \gg 0\}.$$

Geometric objects in the derived category

Basic geometric objects are represented in the derived category:

- A vector bundle E corresponds to the complex

$$\dots \to 0 \to E \to 0 \to \dots$$

- Morphisms of varieties $f\colon X \to Y$ give derived pullback and pushforward functors

$$f^*\colon \mathbf{D}(Y) \to \mathbf{D}(X), \qquad f_*\colon \mathbf{D}(X) \to \mathbf{D}(Y).$$

  Under appropriate finiteness assumptions they preserve boundedness; this holds, e.g., when both X and Y are smooth and proper.

Triangulated structure

The derived category is triangulated, i.e.,

- Morphisms $\mathrm{Hom}(F_1, F_2)$ in $\mathbf{D}(X)$ are vector spaces.
- The composition $\mathrm{Hom}(F_2, F_3) \otimes \mathrm{Hom}(F_1, F_2) \to \mathrm{Hom}(F_1, F_3)$ is bilinear.
- $\mathbf{D}(X)$ is endowed with an action of $\mathbb{Z}$ by shift autoequivalences:

$$[k] \colon \mathbf{D}(X) \to \mathbf{D}(X), \qquad k \in \mathbb{Z},$$

  such that $[k_1] \circ [k_2] = [k_1 + k_2]$.
- Any morphism $\phi \colon F_1 \to F_2$ extends to a distinguished triangle

$$F_1 \xrightarrow{\ \phi\ } F_2 \longrightarrow \mathrm{Cone}(\phi) \longrightarrow F_1[1].$$

- If $F_1 \to F_2 \to F_3 \to F_1[1]$ is distinguished, one gets long exact sequences

$$\dots \to \mathrm{Hom}(F, F_1) \to \mathrm{Hom}(F, F_2) \to \mathrm{Hom}(F, F_3) \to \mathrm{Hom}(F, F_1[1]) \to \dots,$$
$$\dots \to \mathrm{Hom}(F_1[1], F) \to \mathrm{Hom}(F_3, F) \to \mathrm{Hom}(F_2, F) \to \mathrm{Hom}(F_1, F) \to \dots.$$

- Other axioms ...

§2. Noncommutative varieties

Semiorthogonal decompositions

A semiorthogonal decomposition is a splitting of $\mathbf{D}^{\mathrm{b}}(X)$ into simpler pieces.

Definition

A semiorthogonal decomposition

$$\mathbf{D}^{\mathrm{b}}(X) = \langle \mathcal{A}, \mathcal{B} \rangle$$

is a pair of triangulated subcategories $\mathcal{A}, \mathcal{B} \subset \mathbf{D}^{\mathrm{b}}(X)$ such that

- $\mathrm{Hom}(B, A) = 0$ for all $A \in \mathcal{A}$, $B \in \mathcal{B}$;
- for any $F \in \mathbf{D}^{\mathrm{b}}(X)$ there is a distinguished triangle

$$B \to F \to A \to B[1], \qquad A \in \mathcal{A},\ B \in \mathcal{B}.$$

Similarly, one defines semiorthogonal decompositions with many components.

Semiorthogonal components of $\mathbf{D}^{\mathrm{b}}(X)$ are noncommutative varieties.

Exceptional objects

Definition

$E \in \mathbf{D}^{\mathrm{b}}(X)$ is an exceptional object if $\dim \mathrm{Hom}(E, E[k]) = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$

Example

$H^{>0}(X, \mathcal{O}_X) = 0 \iff$ any line bundle is exceptional.

With an exceptional object $E \in \mathbf{D}^{\mathrm{b}}(X)$ one associates subcategories

- triangulated envelope $\langle E \rangle := \{\oplus(E[i]^{\oplus n_i})\} \simeq \mathbf{D}^{\mathrm{b}}(\mathrm{point})$,
- right orthogonal $E^{\perp} := \{F \mid \mathrm{Hom}(E[k], F) = 0\ \forall k\}$,
- left orthogonal ${}^{\perp}E := \{F \mid \mathrm{Hom}(F, E[k]) = 0\ \forall k\}$,

that combine into semiorthogonal decompositions

$$\mathbf{D}^{\mathrm{b}}(X) = \langle E, {}^{\perp}E \rangle \qquad \text{and} \qquad \mathbf{D}^{\mathrm{b}}(X) = \langle E^{\perp}, E \rangle.$$

Categorically, an exceptional object is a "noncommutative embedding of a point".

Exceptional collections

Iterating this construction, we obtain the notion of an exceptional collection.

Definition

A collection $E_1, \dots, E_n \in \mathbf{D}^{\mathrm{b}}(X)$ is an exceptional collection if

- each object E_i is exceptional and
- $\mathrm{Hom}(E_i, E_j[k]) = 0$ for $i > j$ and all k.

An exceptional collection induces a semiorthogonal decomposition

$$\mathbf{D}^{\mathrm{b}}(X) = \langle \mathcal{A}, E_1, \dots, E_n \rangle, \qquad \text{where } \mathcal{A} := \langle E_1, \dots, E_n \rangle^{\perp}.$$

An exceptional collection is full if $\mathcal{A} = 0$, i.e., $\mathbf{D}^{\mathrm{b}}(X) = \langle E_1, \dots, E_n \rangle$.

Example (Beilinson, '78)

$$\mathbf{D}^{\mathrm{b}}(\mathbb{P}^n) = \langle \mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n) \rangle.$$

Residual categories

Typically, we have an exceptional collection and its complement. Let

- $X \subset \mathbb{P}^n$, a hypersurface of degree d, or
- $X \subset \mathbb{P}^n$, a complete intersection of type $(d_1, \dots, d_k)$, $d := \sum d_i$,

and $d \le n$ (Fano condition). Then

$$\mathbf{D}^{\mathrm{b}}(X) = \langle \mathcal{R}_X, \mathcal{O}_X, \mathcal{O}_X(1), \dots, \mathcal{O}_X(n-d) \rangle,$$

where $\mathcal{R}_X := \langle \mathcal{O}_X, \mathcal{O}_X(1), \dots, \mathcal{O}_X(n-d) \rangle^\perp$ is the residual category.

$\mathcal{R}_X$ is an interesting example of a noncommutative variety.

Example (Bondal–Orlov, '95)

If X is a smooth complete intersection of type $(2, 2)$ in $\mathbb{P}^{2g+1}$ then

$$\mathcal{R}_X \simeq \mathbf{D}^{\mathrm{b}}(C_g),$$

where C_g is a hyperelliptic curve of genus g.

Serre functor

Serre duality: if X is smooth projective and $\omega_X := \det(\Omega^1_X)$, then

$$\mathrm{Hom}(F, G)^\vee \cong \mathrm{Hom}(G, F \otimes \omega_X[\dim X]) \qquad \text{for } F, G \in \mathbf{D}^{\mathrm{b}}(X).$$

Definition ([Bondal–Kapranov, '89])

A Serre functor of a triangulated category $\mathcal{T}$ is an autoequivalence $\mathbf{S}_{\mathcal{T}}$ such that

$$\mathrm{Hom}(F, G)^\vee \cong \mathrm{Hom}(G, \mathbf{S}_{\mathcal{T}}(F)).$$

- If a Serre functor exists, it is unique.
- The Serre functor exists for all smooth and proper noncommutative varieties.
- The Serre functor of $\mathbf{D}^{\mathrm{b}}(X)$ has the form

$$\mathbf{S}_{\mathbf{D}^{\mathrm{b}}(X)}(F) = F \otimes \omega_X[\dim X].$$

Thus, the Serre functor of $\mathbf{D}^{\mathrm{b}}(X)$ encodes the canonical class and dimension of X.

Examples of residual categories

Let $X \subset \mathbb{P}^n$ be a hypersurface of degree $d \le n$. Then

- if $d = 1$ then $\mathcal{R}_X = 0$;
- if $d = 2$ then $\mathcal{R}_X \simeq \mathbf{D}^{\mathrm{b}}(\mathcal{C}\ell_0)$, where $\mathcal{C}\ell_0$ is the even part of the (noncommutative!) Clifford algebra of the quadratic form of X; in particular $\mathbf{S}_{\mathcal{R}_X} \cong \mathrm{id}$;
- if $d = 3$ then $\mathbf{S}_{\mathcal{R}_X} \cong \left[\frac{\dim(X)+2}{3}\right]$ or $\mathbf{S}^3_{\mathcal{R}_X} \cong [\dim(X) + 2]$;
- if $d \ge 4$ then $\mathbf{S}^d_{\mathcal{R}_X} \cong [(\dim(X) + 2)(d - 2)]$.

Most noncommutative varieties among $\mathcal{R}_X$ have fractional dimension .

Remark ([-, Perry, '21])

The residual category of a complete intersection has more complicated structure: it is "stratified" with strata of different fractional dimensions.

Residual category of a cubic fourfold

Let $X \subset \mathbb{P}^5$ be a hypersurface of degree 3. Then

$$\mathbf{D}^{\mathrm{b}}(X) = \langle \mathcal{R}_X, \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle.$$

The noncommutative variety $\mathcal{R}_X$ is particularly interesting.

- $\mathcal{R}_X$ is a noncommutative K3 surface ;
  - $\mathbf{S}_{\mathcal{R}_X} \cong [2]$, similarly to the case of K3 surfaces;
  - for special X one may have $\mathcal{R}_X \simeq \mathbf{D}^{\mathrm{b}}(S)$, where S is a K3 surface;
  - for a very general X the category $\mathcal{R}_X$ is not commutative.
- $\mathcal{R}_X$ gives rise to various hyper-Kähler varieties.
- $\mathcal{R}_X$ (conjecturally) encodes birational properties of X.

Conjecture

A smooth cubic hypersurface $X \subset \mathbb{P}^5$ is rational if and only if $\mathcal{R}_X \simeq \mathbf{D}^{\mathrm{b}}(S)$.

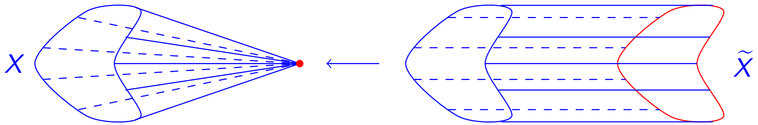
§3. Classical versus homological

Resolutions of singularities

If X is singular, a resolution is a proper morphism $\pi\colon \widetilde{X} \to X$ such that

- $\widetilde{X}$ is smooth;
- there is an open subset $U \subset X$ such that $\pi^{-1}(U) \cong U$.

Example



On the level of categories we have adjoint functors

$$\mathbf{D}^{\mathrm{perf}}(X) \xrightarrow{\pi^*} \mathbf{D}^{\mathrm{perf}}(\widetilde{X}) = \mathbf{D}^{\mathrm{b}}(\widetilde{X}) \xrightarrow{\pi_*} \mathbf{D}^{\mathrm{b}}(X),$$

and if the singularites of X are rational, we have $\pi_* \circ \pi^* \cong \mathrm{id}$.

Categorical resolutions

Definition ([–, '08])

A categorical resolution of singularities of X is a triple $(\mathcal{D}, \pi^*, \pi_*)$, where

- $\mathcal{D}$ is a smooth and proper noncommutative variety;
- $\pi^* \colon \mathbf{D}^{\mathrm{perf}}(X) \to \mathcal{D}$ and $\pi_* \colon \mathcal{D} \to \mathbf{D}^{\mathrm{b}}(X)$ are adjoint functors;
- $\pi_* \circ \pi^* \cong \mathsf{id}$.

Theorem ([–, Lunts, '15])

Any separable scheme of finite type over a field of characteristic zero has a categorical resolution of singularities.

- Categorical resolutions exist in higher generality.
- All singularities are "derived rational".

Simultaneous resolutions of singularities

Let $f\colon X \to B$ be a flat proper morphism, smooth over $B \setminus \{o\}$ for $o \in B$.
A simultaneous resolution of singularities is a resolution $\pi\colon \widetilde{X} \to X$
such that $\widetilde{X}$ is smooth over B, i.e.,

- $\widetilde{X}$ is smooth, and
- $\widetilde{X}_o := (f \circ \pi)^{-1}(o)$ is smooth.

Usually one also assumes that π is an isomorphism over $B \setminus \{o\}$.

Theorem ([Brieskorn, '70], [Tjurina, '70])

If $f\colon X \to B$ is a deformation of a surface X_o with rational double points, there is a finite covering $B' \to B$ such that $X' := X \times_B B' \to B'$ has a simultaneous resolution of singularites.

This result does not extend to higher dimensions.

Simultaneous categorical resolutions

Let $f\colon X \to B$ be as before.

Definition ([–, '22])

A simultaneous categorical resolution of singularities of $f\colon X \to B$ is a triple $(\mathcal{D}, \pi^*, \pi_*)$, where

- $\mathcal{D}$ is a smooth and proper over B noncommutative variety;
- $\pi^*\colon \mathbf{D}^{\mathrm{perf}}(X) \to \mathcal{D}$, $\pi_*\colon \mathcal{D} \to \mathbf{D}^{\mathrm{b}}(X)$ are B-linear adjoint functors;
- $\pi_* \circ \pi^* \cong \mathrm{id}$.

Theorem ([–, '22])

If B is a smooth curve, X and X_o have an ordinary double point at $x \in X_o$, and $\dim(X_o)$ is even then f has a simultaneous categorical resolution.

Thus, $\mathcal{D}$ is a smooth and proper over B noncommutative variety;
it provides a smooth extension of $X \setminus X_o \to B \setminus \{o\}$ across the point o.

Categorical absorption of singularities

Definition ([–, Shinder, '22])

A semiorthogonal component $\mathcal{P} \subset \mathbf{D}^{\mathrm{b}}(X_o)$ absorbs singularites of X_o if both orthogonals $\mathcal{P}^{\perp} \simeq {}^{\perp}\mathcal{P} \subset \mathbf{D}^{\mathrm{b}}(X_o)$ are smooth and proper.

Example

Assume Y is smooth and proper and $Z \subset Y$ is a singular local complete intersection of codimension 2. Then $X_o := \mathrm{Bl}_Z(Y)$ is singular and

$$\mathbf{D}^{\mathrm{b}}(X_o) = \langle \mathbf{D}^{\mathrm{b}}(Y), \mathbf{D}^{\mathrm{b}}(Z) \rangle.$$

Thus $\mathcal{P} := \mathbf{D}^{\mathrm{b}}(Z) \subset \mathbf{D}^{\mathrm{b}}(X_o)$ absorbs singularities of X_o with

$$\mathcal{P}^{\perp} \simeq {}^{\perp}\mathcal{P} \simeq \mathbf{D}^{\mathrm{b}}(Y).$$

This example shows the idea: absorption is a "categorical contraction" of $\mathbf{D}^{\mathrm{b}}(X_o)$ to a smooth and proper noncommutative variety.

Absorption for nodal varieties

Theorem ([–, Shinder, '22])

Let X_o be a variety with a single ordinary double point $x \in X_o$.
Let $\pi \colon \widetilde{X}_o \to X_o$ be the blowup of x with the exceptional divisor $E \subset \widetilde{X}_o$.
Assume there is an exceptional object $\mathcal{E}$ on $\widetilde{X}_o$ such that $\mathcal{E}|_E$ is a spinor bundle.
Then the subcategory $\mathcal{P} := \langle \pi_* \mathcal{E} \rangle$ absorbs singularities of X_o.

Assume, moreover, $f \colon X \to B$ is a smoothing *of X_o, i.e.,*

- $X_o \cong f^{-1}(o) \xrightarrow{\iota} X$;
- *X is smooth and f is smooth over $B \setminus o$;*

If $\dim(X_o)$ is odd then $\iota_* \pi_* \mathcal{E} \in \mathbf{D}^{\mathrm{b}}(X)$ is exceptional,

$$\mathbf{D}^{\mathrm{b}}(X) = \langle \iota_* \pi_* \mathcal{E}, \mathcal{D} \rangle,$$

and $\mathcal{D}$ is a smooth and proper over B noncommutative variety.

Again, $\mathcal{D}$ provides a smooth extension of $X \setminus X_o \to B \setminus \{o\}$ across the point o.

Thanks for your attention!