The History and Historiography of the Discovery of Calculus in India

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Outline of the talk & Acknowledgements

- Outline
  1. Introduction
  2. Discovery of Calculus (where and when?) in India
  3. History (context + content + trajectory taken)
  4. Historiography (study of the recorded history)
  5. Conclusion
- Acknowledgments
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Introduction

Introduction – General (mis)conception about ancient India

Some describe the land to be filled with various superstitions; Nurturing performance of a variety of sacrifices for propitiating nature and reaping benefits from that.

For others India is a land of spiritual seekers, practising penance; Believing in other world; immersed in philosophical discussion, etc.

- People also know India for fine arts, crafts, architecture, etc.
- How many of us know about great mathematicians of India?

David Mumford highlighting outstanding Indian contributions

The Fields medalist and past IMU president (1995–98) David Bryant Mumford, commences his review of Kim Plofker's book Mathematics in India as follows:

> *Did you know that Vedic priests were using the so called Pythagorean theorem to construct their fire altars in 800 BCE?; that the differential equation for the sine function, in finite difference form, was described by Indian mathematician-astronomers in the fifth century CE?; and that "Gregory's" series* $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \ldots$ *was proven using the power series for arctangent and, with ingenious summation methods, used to accurately compute* π *in southwest India in the fourteenth century? If any of this surprises you, ...*

Quote from David Mumford's Review (pub. in Notices of the AMS, 2010)

Only a fraction of this *has become generally known to mathematicians in the West. Too many people still think that mathematics was born in Greece and more or less slumbered until the Renaissance.*

It is right time that the full story of Indian mathematics from Vedic times through 1600 became generally known. I am not minimizing the genius of the Greeks and their wonderful invention of pure mathematics, but other people have been doing math in different ways and they have often attained the same goals independently. Rigorous mathematics in the Greek style should not be seen as the only way to gain mathematical knowledge .... The muse of mathematics can be wooed in many different ways and her secrets teased out of her.

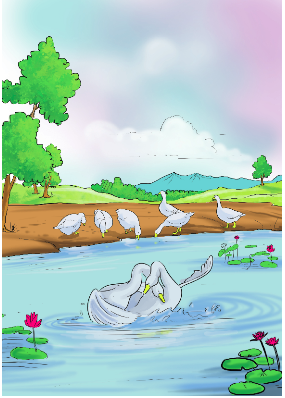
this –> the **major** contributions of Indians to Mathematics!

this –> generally presented in the form of metrical composition!

Quadratic equation in a beautiful metrical form

Bhāskara (12th century astronomer) using the beautiful *vasantatilakā* metre provides an enticing description of the flora and fauna:

bāle marālakulamūladalāni sapta
tīre vilāsabharamantharagāṇyapaśyam ।
kurvacca kelikalahaṃ kalahaṃsayugmaṃ
śeṣaṃ jale vada marālakulapramāṇam ॥



If x^2 denotes the total number of geese, then the given problem when expressed in modern notation translates to:

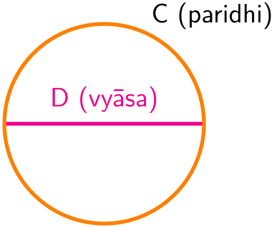
$$x^2 - 7\,\frac{x}{2} = 2.$$

Solving this equation, we have $x^2 = 16$.

Mādhava Infinite series for π (found in *Yuktidīpikā* of Śaṅkaravāriyar)

व्यासे वारिधिनिहते रूपहृते व्याससागराभिहते ।
त्रिशरादिविषमसङ्ख्याभक्तम् ऋणं स्वं पृथक् क्रमात् कुर्यात् ॥

vyāse vāridhinihate rūpahṛte vyāsasāgarābhihate ।
triśarādiviṣamasaṅkhyābhaktam ṛṇaṃ svaṃ pṛthak kramāt kuryāt ॥



(रूप = 1; हृत = divided; ऋणं = minus स्वं = plus)
(व्यास = diameter; वारिधि = 4; निहत = multiplied)

- *viṣamasaṅkhyābhaktam* → Divided by odd numbers
- *triśarādi* → 3, 5, etc. [successively]

$$Paridhi(C) = 4 \times Vy\bar{a}sa \times \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots\right)$$

This is really a beautiful equation, with all the odd numbers appearing as unit fractions in the RHS with alternating sign!

Steady increase in mathematical sophistication!

- From Vedic times, the development of mathematics in India has witnessed a steady progress till the beginning of 19th century.
- Acknowledging this, Mumford observes:

  > *Just as in Indian mathematics, there is a steady increase in sophistication over the centuries, culminating in dramatic advances in Kerala, most strikingly, Nīlakaṇṭha in the fifteenth century … a "virtually" heliocentric model remarkably better than Ptolemy's.*

- What are they?

  1. Finding the infinite series for $\frac{\pi}{4}$
  2. Obtaining fast convergent approximations of it.
  3. Discovering the series for sine and cosine functions.
  4. Obtaining the derivative of sine-inverse function, and so on.

- Aren't they **remarkable** discoveries during the 14–15th century?

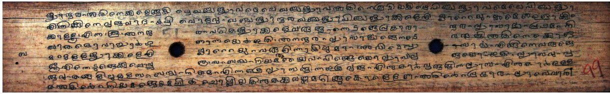
Calculus in Kerala

Where in India did this dramatic advance happen?

Where in India? – On the banks of the river Bharatapuzha!

- View of the river Bharatapuzha, also called Nila – the banks of which are filled with coconut trees (Kerala).
- The **birth** of infinitesimal Calculus took place here — during the 14th century, almost three centuries before its inception in the West!.

- Their findings got recorded in palm leaf manuscripts having 7–8 lines per folio whose dimensions are roughly 1.5×12 inches as depicted.



Two major figures – associated with the advent of calculus

- The major figures generally associated with in the advent of calculus:
  - Isaac Newton (1642–1727)
  - Gottfried Wilhelm Leibniz (1646–1716)[1]
- John von Neumann (1903–1957), the renowned Hungarian mathematician, and a polymath wrote:

  > The calculus was the first achievement of modern mathematics, and it is difficult to overestimate its importance.

- Two other figures worth mentioning (from the ancient and modern periods) are:
  - Archimedes of Syracuse (287-212 BCE), and
  - Pierre de Fermat (1601-1665), who is known for determining slopes of tangents and areas under curves.

[1] One of the papers published by Leibniz is 1684 contains the word 'calculi' (a system of calculation), and this has got stuck. to refer to this discipline today.

Gaṇitayuktibhāṣā — 'The first textbook on Calculus'

- Calculus "essentially" refers to the 'language' that got developed primarily to study the rates of change of quantities — though the notion has evolved over time.
- This tool has enabled us to describe the dynamics of the physical world effectively.
- Besides physicists, this language has also been well-explored by engineers, economists, biologists, statisticians, and so on.
- Coming to speak of the **origins** of this language:
  - We find reputed scientists and historians of science describing the 16th century text ***Gaṇitayuktibhāṣā*** as 'The first textbook on Calculus'. (Divakaran)
  - We also find articles expressing reservations regarding this attribution.
- My aim during the talk, is to take the audience on an excursion into the past, to show how the subject evolved over centuries in India.

History

Discovery of the series for $\frac{\pi}{4}$

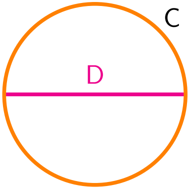
Finding the sum of a geometric series

Brief history of π (till Āryabhaṭa)

What is the relation between the **circumference** (C) and the **diameter** (D) ?

Why this notation π? (the word peripherion in Greek: $\pi\epsilon\rho\iota\phi\epsilon\rho\epsilon\iota\alpha$).

Who obtained	When	Value	Accuracy
Babylonian	2000 BCE(?)	$3 + \frac{1}{8}$	1
Śulbasūtrakāras	$<$ 800 BCE(?)	3.08	1
Archimedes	250 BCE	$3\frac{10}{71} < \pi < 3\frac{1}{7}$	3
Ptolemy	150 CE	3.14166	3
Liu Hui	263 CE	$3.1416 = \frac{3927}{1250}$	4
Zu Chongzhi	480 CE	$\frac{355}{113}$	6
Āryabhaṭa	499 CE	$3.1416 = \frac{62832}{20000}$	4



- Interestingly, after Āryabhaṭa $\rightarrow$ c. 1600. Mādhava completely missing!
- Could the omission be due to **ignorance**? **Possibly**, but NOT **always**!

Approximation for π by Āryabhaṭa

- Āryabhaṭa (c. 499) gives an approximate value for π in the following verse:

  चतुरधिकं शतं अष्टगुणं द्वाषष्टिस्तथा सहस्राणाम्।
  अयुतद्वयविष्कम्भस्य आसन्नो वृत्तपरिणाहः॥

  caturadhikaṃ śatam aṣṭaguṇaṃ dvāṣaṣṭistathā sahasrāṇām।
  ayutadvayaviṣkambhasya āsanno vṛttapariṇāhaḥ॥

  One hundred plus four multiplied by eight and added to sixty-two thousand: This is the approximate measure of the circumference of a circle whose diameter is twenty-thousand.

- Thus as per the above verse $\pi = \frac{C}{D} \approx \frac{62832}{20000} = 3.1416.$
- Note the use of the word āsanna –> close by/approximate.
- How could have Āryabhaṭa obtained this value? (polygon doubling)

Earlier rational approximations to irrational quantities

- The following *sūtra* from *Bodhāyanaśulbasūtra* gives an approximation to $\sqrt{2}$:
  प्रमाणं तृतीयेन वर्धयेत्, तच्च चतुर्थेन, आत्मचतुस्त्रिंशोनोनेन, सविशेषः ।
  pramāṇaṃ tṛtīyena vardhayettacca caturthena ātmacatustriṃśonena। *saviśeṣaḥ*।

  The measure [of the side] is to be increased by its third and this [third] again by its own fourth less the thirty-fourth part [of the fourth]. That is the approximate diagonal (saviśeṣa).

$$\sqrt{2} \approx 1+\frac{1}{3}+\frac{1}{3\times 4}\left(1-\frac{1}{34}\right) \quad (1)$$
$$= \frac{577}{408} = 1.414215686 \qquad \text{(correct to 5 decimals)}$$

- What is noteworthy here is the use of the word सविशेषः in the *sūtra*, which literally means 'that which has some speciality' (speciality $\equiv$ being approximate)

Precedents to the work on infinitesimals by the Kerala School

- Mumford in his review remarks that the "discovery of the finite difference equation for sine led Indian mathematicians eventually to the full theory of calculus for polynomials and for sine, cosine, arcsine, and arctangent functions".
- In Indian astronomy, the motion of a planet is computed by applying two corrections:
  - the *manda-saṃskāra* (corresponds to the equation of centre) and
  - the *śīghra-saṃskāra* conversion of the "heliocentric" $\rightarrow$ "geocentric")
- While discussing the computation of planetary longitudes, Bhāskara (c. 1150), clearly introduces the notion of instantaneous velocity of a planet (*tātkālika-gati*).
- He also gives precise mathematical procedure to compute it – which obviously involves finding the derivative of sine functions.
- Much **before** Bhāskara, Āryabhaṭa II (c. 950) in his *Mahā-siddhānta*, presents similar mathematical relations in connection with computing the daily motion of planets.

Do the popular books present the right history?

A few years back Alex Bellos, a British journalist, having interviewed me, handed a over a book authored in 2010 (just before departing).

For two thousand years the only way to pinpoint pi was by using polygons. But, in the seventeenth century, Gottfried Leibniz and John Gregory ushered in a new age of pi appreciation with the formula:

$$\frac{\text{pi}}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \ldots$$

// Historically incorrect

In other words, a quarter of pi is equal to one minus a third plus a fifth minus a seventh plus a ninth and so on, alternating the addition and subtraction of unit fractions of the odd numbers as they head to infinity. Before this point scientists were aware only of the scattergun randomness of pi's decimal expansion. Yet here was one of the most elegant and

According to Bellos, the new age of appreciation is only after Leibniz;
Was he **ignorant** of the Kerala mathematician's contributions? NOT sure!

Derivation of the infinite series for π

The derivation is simple — based on the identification of similar triangles.

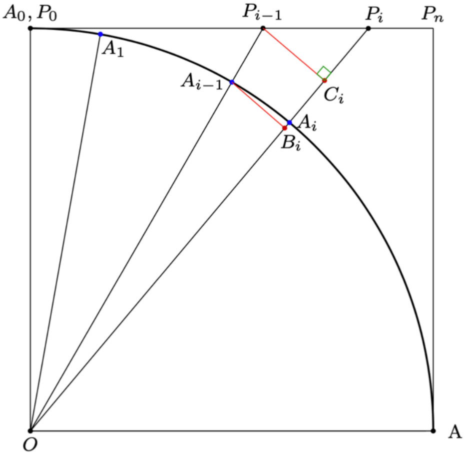
The triangles $OP_{i-1}C$ and $OA_{i-1}B$ are similar. Hence,

$$\frac{A_{i-1}B_i}{OA_{i-1}} = \frac{P_{i-1}C_i}{OP_{i-1}} \qquad (2)$$

Similarly triangles $P_{i-1}C_iP_i$ and P_0OP_i are similar. Hence,

$$\frac{P_{i-1}C_i}{P_{i-1}P_i} = \frac{OP_0}{OP_i} \qquad (3)$$

Here $OP_i = k_i$ are called the *karṇas*.



Derivation of the infinite series for π

From these two relations we have,

$$A_{i-1}B_i = P_{i-1}P_i \times \frac{OA_{i-1}}{OP_{i-1}} \times \frac{OP_0}{OP_i} = \left(\frac{r}{n}\right) \times \frac{r}{k_{i-1}} \times \frac{r}{k_i} = \left(\frac{r}{n}\right)\left(\frac{r^2}{k_{i-1}k_i}\right). \quad (4)$$

It is $\left(\frac{r}{n}\right)$ that is refered to as *khaṇḍa* in the text. The text also notes that, when the *khaṇḍas* **become small** (or equivalently n becomes large), the Rsines can be taken as the arc-bits itself.

परिधिखण्डस्यार्धज्या $\to$ परिध्यंश एव
i.e., $A_{i-1}B_i \to A_{i-1}A_i$

(local approximation by linear functions i.e., tangents/differentiation)

Derivation of the infinite series for π (contd.)

Though the value of $\frac{1}{8}$th of the circumference has been obtained as

$$\frac{C}{8} = \left(\frac{r}{n}\right)\left[\left(\frac{r^2}{k_0 k_1}\right) + \left(\frac{r^2}{k_1 k_2}\right) + \left(\frac{r^2}{k_2 k_3}\right) + \cdots + \left(\frac{r^2}{k_{n-1} k_n}\right)\right], \tag{5}$$

there may not be much difference in approximating it by the following expression:

$$\frac{C}{8} = \left(\frac{r}{n}\right)\left[\left(\frac{r^2}{k_0^2}\right) + \left(\frac{r^2}{k_1^2}\right) + \left(\frac{r^2}{k_2^2}\right) + \cdots + \left(\frac{r^2}{k_{n-1}^2}\right)\right] \tag{6}$$

or

$$\frac{C}{8} = \left(\frac{r}{n}\right)\left[\left(\frac{r^2}{k_1^2}\right) + \left(\frac{r^2}{k_2^2}\right) + \left(\frac{r^2}{k_3^2}\right) + \cdots + \left(\frac{r^2}{k_n^2}\right)\right] \tag{7}$$

The difference between the two would be $\left(\frac{r}{n}\right)\left(\frac{1}{2}\right)$ since $\left(k_0^2, k_n^2 = r^2, 2r^2\right)$.

Now the text notes –

खण्डस्य अल्पत्ववशात् तदन्तरं शून्यप्रायमेव । *(since $\frac{r}{n}$ is infinitesimally small)*

Derivation of the infinite series for π (contd.)

- Therefore, the expression that is to be evaluated now becomes,

$$\frac{C}{8} = \left(\frac{r}{n}\right)\left[\left(\frac{r^2}{k_1^2}\right) + \left(\frac{r^2}{k_2^2}\right) + \left(\frac{r^2}{k_3^2}\right) + \cdots + \left(\frac{r^2}{k_n^2}\right)\right] \tag{8}$$

- Alternatively,

$$\frac{C}{8} = \sum_{i=1}^{n} \frac{r}{n}\left(\frac{r^2}{k_i^2}\right) \qquad \textit{summming up/integration} \tag{9}$$

- The Kerala mathematicians used an ancient Indian (ingenious) identity.
- This has to do with turning an algebraic expression $\rightarrow$ infinite series.
- In modern parlance this would be called the **binomial expansion**.

Trick: Turn a finite number into an infinite series

- Consider the product $a\left(\frac{c}{b}\right)$ $(a, b, c: +\text{ve integers})$; Two distinct possibilities:
- **Case i:** $(c > b)$. In this case we rewrite the product in the following form

$$a\left(\frac{c}{b}\right) = a + a\,\frac{(c-b)}{b}. \tag{10}$$

- **Case ii:** $(c < b)$. In this case we rewrite the product as

$$a\left(\frac{c}{b}\right) = a - a\,\frac{(b-c)}{b}. \tag{11}$$

- In the expression $a\frac{(b-c)}{b}$, if we want to replace the division by b by division by c, then we have to make a subtractive correction which amounts to the following equation.

$$a\frac{(b-c)}{b} = a\frac{(b-c)}{c} - a\frac{(b-c)}{c} \times \frac{(b-c)}{b}. \tag{12}$$

Binomial series expansion

If we again replace the division by the divisor b by the multiplier c,

$$
\begin{aligned}
a\,\frac{c}{b} &= a - \left[a\frac{(b-c)}{c} - a\frac{(b-c)}{c} \times \frac{(b-c)}{c} \times \frac{c}{b}\right] \\
&= a - \left[a\frac{(b-c)}{c} - \left[a\frac{(b-c)^2}{c^2} - \left(a\frac{(b-c)^2}{c^2} \times \frac{(b-c)}{b}\right)\right]\right] \qquad (13)
\end{aligned}
$$

Thus, after taking m *śodhya-phala*-s we get

$$
\begin{aligned}
a\,\frac{c}{b} &= a - a\frac{(b-c)}{c} + a\left[\frac{(b-c)}{c}\right]^2 - \ldots + (-1)^{m-1} a \left[\frac{(b-c)}{c}\right]^{m-1} \\
&\quad + (-1)^m a \left[\frac{(b-c)}{c}\right]^{m-1} \frac{(b-c)}{b}. \qquad (14)
\end{aligned}
$$

The text notes that there is logically no end to the process; But, we can terminate the process when the terms become small enough. Since $b - c < c$, successive correction terms keep $\downarrow$.

Derivation of the infinite series for π

$$
\begin{aligned}
\frac{C}{8} &= \sum_{i=1}^{n}\left[\frac{r}{n}-\frac{r}{n}\left(\frac{k_i^2-r^2}{r^2}\right)+\frac{r}{n}\left(\frac{k_i^2-r^2}{r^2}\right)^2-\frac{r}{n}\left(\frac{k_i^2-r^2}{r^2}\right)^3+\ldots\right]\\
&= \left(\frac{r}{n}\right)[1+1+\ldots+1]-\left(\frac{r}{n}\right)\left(\frac{1}{r^2}\right)\left[\left(\frac{r}{n}\right)^2+\left(\frac{2r}{n}\right)^2+\ldots+\left(\frac{nr}{n}\right)^2\right]\\
&\quad+\left(\frac{r}{n}\right)\left(\frac{1}{r^4}\right)\left[\left(\frac{r}{n}\right)^4+\left(\frac{2r}{n}\right)^4+\ldots+\left(\frac{nr}{n}\right)^4\right]\\
&\quad-\left(\frac{r}{n}\right)\left(\frac{1}{r^6}\right)\left[\left(\frac{r}{n}\right)^6+\left(\frac{2r}{n}\right)^6+\ldots+\left(\frac{nr}{n}\right)^6\right] + \ldots . \qquad (15)
\end{aligned}
$$

If we take out the powers of *bhujā-khaṇḍa* $(\frac{r}{n})$, the summations involved are that of even powers of the natural numbers, namely *edādyekottara-varga-saṅkalita*, $1^2+2^2+\ldots+n^2$, *edādyekottara-varga-varga-saṅkalita*, $1^4+2^4+\ldots+n^4$, and so on.

Derivation of the infinite series for π

Kerala astronomers used mathematical induction to show that, for large n

$$\sum_{i=1}^{n} i^k \approx \frac{n^{k+1}}{k+1}. \qquad \text{a crucial result !} \tag{16}$$

Thus, we arrive at the result

$$\frac{C}{8} = r\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right), \tag{17}$$

which is given in the form

$$Paridhi = 4 \times Vy\bar{a}sa \times \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots\right)$$

The excitement that Leibniz had in discovering this series!

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots \tag{18}$$

- The beauty of the above series cannot be overemphasised. It arises from the sheer simplicity and absolutely trivial pattern. The terms are the reciprocals of the odd integers with alternating signs.
- This apparently **innocuous** expression sums to, of all things, $\frac{\pi}{4}$. Being excited by the discovery, trying to communicate the significance of this discovery — perhaps in an attractive way — Leibniz notes:

  it was now proved for the first time that the area of a circle was exactly equal to a series of rational quantities.

- One may quibble with his use of "exactly," used here.
- "But it is hard to argue with his enthusiasm" — says William Dunham.

Rave reviews received by Leibniz (which Mādhava too deserves!)

- What lies at the heart of the derivation of this series for $\frac{\pi}{4}$ by Leibniz is the so called Transmutation theorem.
- It has to do with finding areas beneath curves — which was a hot topic in the middle of the seventeenth century — normally denoted by $\int_a^b y\,dx$.
- The name 'transmutation' indicates that the original integral $\int y\,dx$ has been transformed into a sum of the new integral $\frac{1}{2}\int z\,dx$ and the constant.
- Leibniz communicated the result to the stalwart Huygens,[2] about which the latter notes:

  a discovery always to be remembered among mathematicians

- What would have been Huygens's surprise if he were to **know** ...

[2]Christiaan Huygens (1629–1695) was a Dutch mathematician, physicist, astronomer and inventor, who is regarded as one of the great scientists of all time.

Though beautiful, series is of no value! (as a numerical appoximator of π)

- The series for $\frac{\pi}{4}$ is an excrutiatingly slowly converging series. To obtain value of π correct to 4-5 decimal places we need to consider almost a million terms.
- To circumvent this problem, Mādhava seems to have found an ingenious way — a technique called "*antyasaṃskāra*."
- It essentially consists of –
  - Terminating the series at any particular term — once you get boredom (जामितया).
  - Make an estimate of the remaining terms in the series.
  - Apply it (*+vely/-vely*) to the value obtained by summation.
- This technique to estimate the remaining terms has proved to be quite effective!
- Even if one considers a few terms (say 20), we are able to get π values accurate to 8-9 decimal places.

Expression for the "remainder" terms obtained by *antyasaṃskāra*

यत्सङ्ख्यात्र हरणे कृते निवृत्ता हृतिस्तु जामितया ।
तस्या ऊर्ध्वगता या समसङ्ख्या तद्दलं गुणोऽन्ते स्यात् ॥ (तद्दलं = half of that)
तद्वर्गो रूपयुतो हारो व्यासाब्धिघाततः प्राग्वत् ।
ताभ्यामाप्तं स्वमृणे कृते घने क्षेप एव करणीयः ॥
लब्धः परिधिः सूक्ष्मः बहुकृत्वो हरणतोऽतिसूक्ष्मः स्यात् ॥

- *yatsaṅkhyayātra haraṇe* → After dividing by a certain number of terms (p)
- *nivṛttā hṛtistu* → if the division process is terminated (p is our choice); Why?
- *jāmitayā* → being bored (due to slow-convergence)

$$\text{Remainder term} = \frac{\left(\frac{p+1}{2}\right)}{(p+1)^2+1}$$

- *labdhaḥ paridhiḥ sūkṣmaḥ* → the circumference obtained is accurate

When does the end-correction give exact result ?

The discussion by Śaṅkara Vāriyar is almost in the form of a engaging dialogue between the teacher and the taught and commences with a pertinent question:.

कथं पुनरत्र मुहुर्विषमसङ्ख्याहरणेन लभ्यस्य परिधेः आसन्नत्वम् अन्त्यसंस्कारेण आपाद्यते? उच्यते — तत्र तावदुक्तरूपस्संस्कारः सूक्ष्मो न वेति प्रथमं निरूपणीयम्। तदर्थं ययाकयाचिद् विषमसंख्यया हरणे कृते पृथक् संस्कारं कुर्यात्। अथ तदुत्तरविषमसंख्याहरणानन्तरं च पृथक् संस्कारं कुर्यात्। एवं कृते लब्धौ परिधी यदि तुल्यौ भवतः तर्हि संस्कारः सूक्ष्म इति निर्णीयताम्। कथम्?

How is it that you get the value close to the circumference by using *antyasaṃskāra*, instead of repeatedly dividing by odd numbers? This is being explained — ... Having done this, if the circumferences obtained are equal, then the accuracy of the correction can be ascertained [by yourself]. How?

When does the end-correction give exact result ?

Argument: If the correction term $\frac{1}{a_{p-2}}$ is applied after odd denominator $p-2$ (with $\frac{p-1}{2}$ is odd), then

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots - \frac{1}{p-2} + \frac{1}{a_{p-2}}. \tag{19}$$

On the other hand, if the correction term $\frac{l}{a_p}$, is applied after the odd denominator p, then

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots - \frac{1}{p-2} + \frac{1}{p} - \frac{1}{a_p}. \tag{20}$$

If the correction terms are exact, then both should yield the same result. That is,

$$\frac{1}{a_{p-2}} = \frac{1}{p} - \frac{1}{a_p} \quad \text{or} \quad \frac{1}{a_{p-2}} + \frac{1}{a_p} = \frac{1}{p}, \tag{21}$$

is the condition for the end-correction to lead to the exact result.

End-correction in the infinite series for π

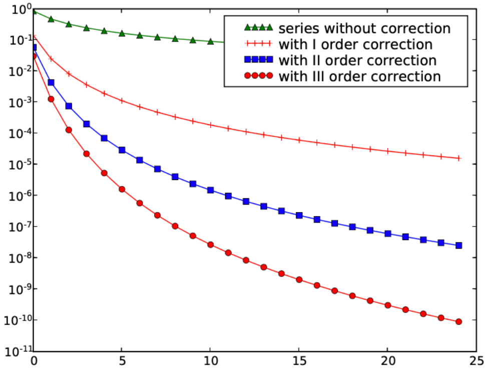
Optimal choice for error-minimizaion ?

- It is first observed that we cannot satisfy this condition trivially by taking $a_{p-2} = a_p = 2p$. For, the correction has to follow a uniform rule of application and thus, if $a_{p-2} = 2p$, then $a_p = 2(p+2)$;
- We can, however, have both a_{p-2} and a_p close to $2p$ as possible. Hence, as first (order) estimate one tries with, "double the even number above the last odd-number divisor p", $a_p = 2(p+1)$.
- But, it can be seen right away that, the condition for accuracy is **not exactly satisfied**. The measure of inaccuracy (*sthaulya*) $E(p)$ is introduced, and is estimated

$$E(p) = \left(\frac{1}{a_{p-2}} + \frac{1}{a_p}\right) - \frac{1}{p}\ .$$

- The objective is to find the correction denominators a_p such that the inaccuracy $E(p)$ is minimised.

Error-minimization in the evaluation of π



- The graph depicts how swiftly the accuracy improves with the application of correction terms.
- Three correction terms:

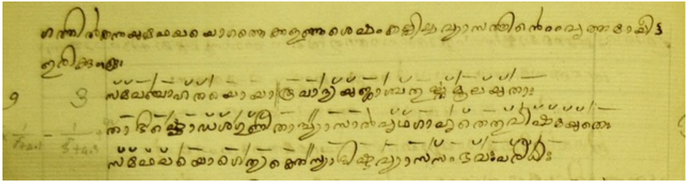
$$a_p(1) = 2(p+2)$$

$$a_p(2) = (2p+2) + \frac{4}{(2p+2)}$$

$$a_p(3) = (2p+2) + \cfrac{4}{2p+2+\cfrac{16}{2p+2}}$$

Elegant verses codifying the expression for series

- Here is a clip from the manuscript of Charles Whish (commentary *Kriyākalāpa*)



- Represented in Devanāgarī script and modern notation:

  समपञ्चाहतयो या रूपाद्ययुजाः चतुर्घ्नमूलयुताः ।
  ताभिश्षोडशगुणिताद्व्यासाद् पृथगाहते तु विषमयुतेः । समफलयोगे त्यक्ते ...

$$C = 16 \times D\left(\frac{1}{1^5 + 4.1} - \frac{1}{3^5 + 4.3} + \frac{1}{5^5 + 4.5} - \frac{1}{7^5 + 4.7} + \cdots\right) \qquad (22)$$

What is the nature of this mysterious number π?

Nīlakaṇṭha's discussion of irrationality of π in his commentary to *Āryabhaṭīya*

- While discussing the value of π Nīlakaṇṭha observes:

  परिधिव्यासयोः सङ्ख्या-सम्बन्धः प्रदर्शितः ।...
  आसन्नः, आसन्नतयैव अयुतद्वयसङ्ख्याविष्कम्भस्य इयं परिधिसङ्ख्या उक्ता । कुतः पुनः वास्तवीं सङ्ख्याम् उत्सृज्य आसन्नैव इहोक्ता ? उच्यते । तस्या वक्तुमशक्यत्वात् । कुतः ?

  The relation between the circumference and the diameter was expressed. ...

  Approximate: This value [62,832] was stated to be nearly the circumference of a circle having a diameter of 20,000. "Why then has an approximate value been mentioned here leaving behind the actual value?" It is explained [as follows]. Because it (the exact value) cannot be stated. Why?

What if I reduce the unit of measurement?

येन मानेन मीयमानो व्यासः निरवयवः स्यात्, तेनैव मीयमानः परिधिः पुनः सावयव एव स्यात् । येन च मीयमानः परिधिः निरवयवः तेनैव मीयमानो व्यासोऽपि सावयव एव; इति एकेनैव मीयमानयोः उभयोः क्वापि न निरवयवत्वं स्यात् । महान्तम् अध्वानं गत्वापि अल्पावयवत्वम् एव लभ्यम् । निरवयवत्वं तु क्वापि न लभ्यम् इति भावः ॥

...Thus when both [the diameter and circumference] are measured by the same unit, they cannot both be specified [as integers] without [fractional] parts. Even if you go a long way (i.e., keep reducing the unit employed), the fractional part [in specifying one of them] will only become very small. A situation in which there will be no [fractional] part (i.e., both the diameter and circumference can be specified in terms of integers) is impossible, and this is what is the import [of the expression *āsanna*].

However small the unit be, the two quantities will never become commensurate – is indeed a very important statement.

Sum of an infinite geometric series

- Nīlakaṇṭha in his commentary to *Āryabhaṭīya* presents an interesting approximation for the arc of a circle in terms of the sine and the versine.
- In this connection, he also presents a detailed demonstration of how to sum an infinite geometric series.
- The specific geometric series that arises in the above context is:

$$\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \ldots + \left(\frac{1}{4}\right)^n + \ldots = \frac{1}{3}. \qquad (\text{valid in } \lim_{n\to\infty})$$

- Here, we shall present an outline of Nīlakaṇṭha's argument.
- This gives a cue to understand as to how the notion of limit was present.
- Approach of Nīlakaṇṭha very different from that of Archimedes.

How to recognise that the difference tends to zero?

This is demonstrated as follows. Consider the following sequence of expressions.

$$\begin{aligned} \frac{1}{3} &= \frac{1}{4} + \frac{1}{(4.3)}, \\ \frac{1}{(4.3)} &= \frac{1}{(4.4)} + \frac{1}{(4.4.3)}, \\ \frac{1}{(4.4.3)} &= \frac{1}{(4.4.4)} + \frac{1}{(4.4.4.3)}, \ldots \end{aligned} \tag{23}$$

Finding the difference (Δ) between LHS and RHS, Nīlakaṇṭha notes:

$$\Delta = \frac{1}{3} - \left[\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \ldots + \left(\frac{1}{4}\right)^n\right] = \left(\frac{1}{4}\right)^n \left(\frac{1}{3}\right). \tag{24}$$

- As $n \uparrow$ the difference $\Delta \downarrow$;
- Becomes extremely small, but never zero!.

Intriguing question that Nīlakaṇṭha poses & the Notion of limit

The question that Nīlakaṇṭha poses as he commences his detailed discussion on the sum of geometric series is very important and pertinent.

> कथं पुनः तावदेव वर्धते तावद्वर्धते च ?
>
> *kathaṃ punaḥ tāvadeva vardhate tāvadvardhate ca ?*
>
> How does one know that [the sum of the series] increases only upto that [limiting value] and that it certainly increases up to that [limiting value]?

Cauchy's (1821) definition of limit:

> *If the successive values attributed to the same variable approach indefinitely a fixed value, such that finally they differ from it by as little as one wishes, this latter is called the limit of all the others.*[3]

[3] Cauchy, *Cours d'Analyse*, cited by Victor J. Katz, *A History of Mathematics*, Addison Wesley Longman, New York 1998, p. 708.

The lineage of Kerala astronomers and mathematicians

- **Mādhava** (c. 1340–1420) — pioneer of the Kerala School of Mathematics.
- **Parameśvara** (c. 1380–1460) — disciple of Mādhava, great observer & prolific writer.
- **Nīlakaṇṭha Somayājī** (c. 1444–1550) — monumental contributions *Tantrasaṅgraha* and *Āryabhaṭīyabhāṣya*.
- **Jyeṣṭhadeva** (c. 1530) — author of the celebrated *Yuktibhāṣa*.
- **Śaṅkara Vāriyar** (c. 1500–1560) — well known for his commentaries.
- **Putumana Somayājī** (c. 1550) — author of several works, but well known for his *Karaṇapaddhati*.
- **Acyuta Piṣāraṭi** (c. 1550–1621) — a disciple of Jyeṣṭhadeva and a polymath – teacher of Melpattur Nārāyaṇa.
- **Sankara Varman** (1774–1839) — author of *Sadratnamālā*

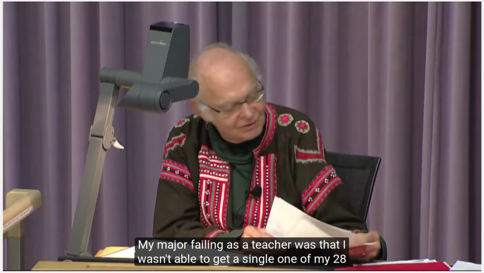
Folio of a long palm leaf manuscript — roughly 1/2 displayed!

Doing serious history of science is not an easy job. It will be

- quite fascinating, but really time consuming!
- could be extremely challenging at times (deciphering text/ content), &
- of course, highly rewarding experience — once we reach the end of the tunnel!

The thrill in investigating historical material

- Donald Knuth, renowned computer scientist, is the author of *The Art of Computing*.
- In one of his talks, he makes this interesting remark, highlighting the fascination in one has in working with History of Science!



(My major failing as a teacher was that I wasn't able to get a single one of my 28 PhD students to realise what a thrill it is to work on source material!)

Historiography

Recount – Few signal achievements of Kerala mathematicians

- The "Newton" series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots, \tag{25}$$

- The "Gregory-Leibniz" series [4]

$$Paridhi = 4 \times Vy\bar{a}sa \times \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right) \tag{26}$$

- The derivative of sine inverse function

$$\frac{d}{dt}\left[\sin^{-1}\left(\frac{r}{R}\sin M\right)\right] = \frac{\frac{r}{R}\cos M\,\frac{dM}{dt}}{\sqrt{1-\left(\frac{r}{R}\sin M\right)^2}} \tag{27}$$

- These and other remarkable results were discovered during 14th–15th cent.

[4] The quotes indicate compromise between current use and historical accuracy.

About Whish's paper: George Hyne's letter to John Warren

I have great pleasure in communicating the Series, to which I alluded ...

$$C = 4D\left(1 - \frac{1}{3} + \frac{1}{5} - \ldots\right) \quad , \tag{28}$$

$$C = \sqrt{12D^2} - \frac{\sqrt{12D^2}}{3.3} + \frac{\sqrt{12D^2}}{3^2.5} - \frac{\sqrt{12D^2}}{3^3.7} + \cdots , \tag{29}$$

$$C = 2D + \frac{4D}{(2^2-1)} - \frac{4D}{(4^2-1)} + \frac{4D}{(6^2-1)} - \cdots \tag{30}$$

$$C = 8D\left[\frac{1}{(2^2-1)} + \frac{1}{(6^2-1)} + \frac{1}{(10^2-1)} + \cdots\right]. \tag{31}$$

$$C = 8D\left[\frac{1}{2} - \frac{1}{(4^2-1)} - \frac{1}{(8^2-1)} - \frac{1}{(12^2-1)} - \cdots\right]. \tag{32}$$

$$C = 3D + \frac{4D}{(3^3-3)} - \frac{4D}{(5^3-5)} + \frac{4D}{(7^3-7)} - \cdots \tag{33}$$

$$C = 16D\left(\frac{1}{1^5+4.1} - \frac{1}{3^5+4.3} + \frac{1}{5^5+4.5} - \cdots\right) \tag{34}$$

I am, my dear Sir, most sincerely, your's,

MADRAS, 17th August 1825. G. HYNE.

George Hyne's note to John Warren

I owe the following Note to Mr. Hyne's favour.

> *The Hindus never invented the series; it was communicated with many others, by Europeans, to some learned Natives in modern times. Mr. Whish sent a list of the various methods of demonstrating the ratio of the diameter and circumference of a Circle employed by the Hindus to the literary society, being impressed with the notion that they were the inventors.*
> *I requested him to make further inquiries, and his reply was, that he had reasons to believe them entirely modern and derived from Europeans, observing that not one of those used the Rules could demonstrate them. Indeed the pretensions of the Hindus to such a knowledge of geometry, is too ridiculous to deserve refutation.*

A summary of the exchanges in 1820s

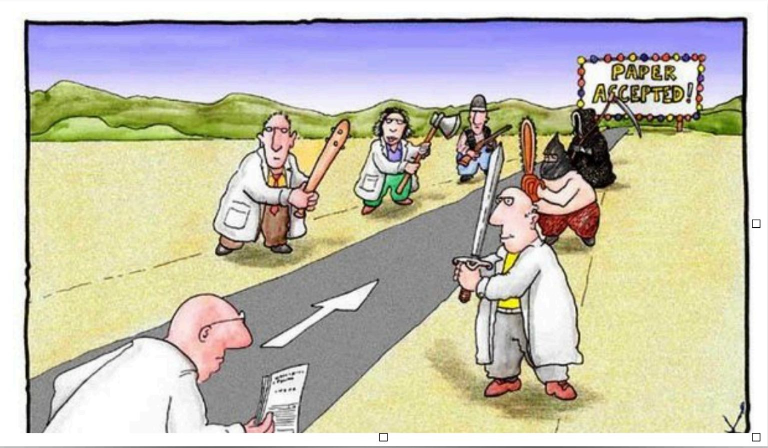
Based on the nature of exchanges recorded by Warren in 1825 in his *Kālasaṅkalita*, it is quite clear that:

1. Whish was convinced that the infinite series were discovered by the “Natives”.
2. Hyne was convinced that the infinite series were NOT discovered by the “Natives” but was only borrowed, and that the Hindus were merely pretending as originators of the series.
3. Warren decides to go with the opinion of Hyne, though initially he felt that the latter’s argument is not “conclusive”.

Under such circumstances, one could only imagine how challenging it would have been for Whish (who was hardly 30 years old), to swim against the current, and place on record his own understanding regarding the Indian work on the infinite series.

Remarkable perseverance, assiduity and tenacity of Whish!

- Certainly not an easy task to wear down his far more senior opponents.
- Hats off to Whish for the outstanding courage displayed by him!



More striking and intriguing development (supress/dimiss)

- There seems to have been a kind of consensus among the European indologists and historians to undermine and suppress it for almost a hundred years.
- Either the work itself was not referenced in their writings, or the work gets dismissed.
- For instance, Augustus De Morgan and scholar administrators such as Charles P Brown criticise in far more strong terms than Hyne — by castigating the Kerala works on infinite series as "hoax" and "forgery".

- Not providing reference is certainly not out of ignorance.
- It seems to be clearly a volitional act!

No reference, though germane to the discussion

- Geroge Thibaut[5] in his scholarly monograph (in German) on Indian Astronomy, Astrology and Mathematics[6] makes note of 1827 article of Whish, on the Greek origin of the Hindu Zodiac.
- He mysteriously fails to mention this 1834 paper of Whish.

Werk J. Warrens — Kālasaṃkalita betitelt — welches eine Fülle von Belehrung über kalendarische und chronologische, und überhaupt astronomische, Berechnungen enthält, besonders nach den südindischen Methoden. Eine 1827 in Madras veröffentlichte Abhandlung von C. M. Whish ist die erste Arbeit, die sich ausführlicher auf den vermutlichen Einfluss der griechischen Astronomie und Astrologie auf Indien einlässt.

Figure: A clip of the relevant section from Thibaut's volume

[5] Born in Germany in 1848, studied in England, and was posted in India in 1875.

[6] George Thibaut, *Astronomie Astrologie und Mathematik*, Strassbourg 1899, p.2.

Conclusion

Concluding remarks

- The study of the ancient scientific texts is essentially trying to understand the evolution of ideas. Cannot be considered obsolete!.
- The work *Gaṇita-yukti-bhāṣā* of Jyeṣṭhadeva is indeed a landmark in the history of mathematics. It has been described as the first textbook on calculus!
- This helps us in understanding how the major discoveries in the foundations of calculus, mathematical analysis, etc., did take place in Kerala (14-16 century).
- Besides arriving at the infinite series, the Kerala astronomers also manipulated with them to obtain several rapidly convergent versions!
- The simple and intuitive ways in which some of the results have been arrived at by the ancients are truly astonishing!
- We demonstrated it for π series. There are many other things!

Concluding Remarks

- Speaking of the ingenuity of Mādhava and his contributions, Mumford notes:

  ... Mādhava of Saṅgamagrāma, who lived from approximately 1350 to 1425. It seems fair to me to compare him with Newton and Leibniz. The high points of their mathematical work were the discoveries of the power series expansions of arctangent, sine, and cosine. By a marvellous and unique happenstance, there survives an informal exposition of these results with full derivations, written in Malayalam, the vernacular of Kerala, by Jyeṣṭhedeva perhaps about 1540. This book, the Gaṇita-Yukti-Bhāṣā, ... gives a unique insight into Indian methods. Simply put, these are recursion, induction, and careful passage to the limit.

- CM Whish in 1820s brings this to the attention of the West.

Concluding Remarks

- It was pointed out that Whish's historic paper, though authored in 1820s got published in JRAS posthumously, after a struggle of almost a decade.
- Notwithstanding this publication, some western scholars brazenly make dismissals.

  Hindus may have inherited some of the bare facts of Greek science, but not the Greek critical acumen. ***Fools*** *rush in where angels fear to tread.* (T. Dantzig, 1930)

- Making unverifiable claims or exaggerated claims is clearly unscientific. But what is scientific about suppressing or making uninformed, blind rejection?
- Whish in his paper mentions: "A further account of the Yucti-Bhāshā, ...with the series for the sines, cosines, and their demonstrations, will be given in a separate paper". Unfortunately, he did not survive to publish this.
- This could have silenced all those who doubted.
- Good news: Things seem to be changing for the better!

Concluding Remarks

- For most of us trained completely in the modern scheme of education and ways of thinking, it may be hard to imagine doing mathematics without the 'luxury' of expressing things 'neatly' in symbolic forms.
- It is equally hard to think of expressing power series for trigonometric functions, derivatives of functions, and the like, **purely in metrical** forms.
- But that is how knowledge seems to have been preserved and handed down from generation to generation in India for millennia. It only proves the point:
  - equations may be handy but not essential;
  - notations may be useful, but not indispensable.
- It may be good to recall Niels Bohr's[7] Principle of **Complementarity**;
- After all, mathematics is mathematics;
  Irrespective of how, where and why it is practiced!

[7] As the lecture is being delivered from the University of Copenhagen.

Thanks!

Thank You !