Lagrange multiplier functionals and their applications in symplectic geometry and string topology

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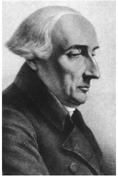
1. Lagrange multiplier functionals

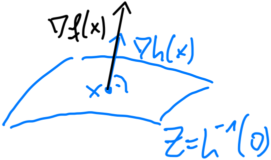
Lagrange multipliers



Joseph-Louis Lagrange 1804:
critical points of f subject to constraint $h = 0$
$\leftrightarrow$ crit. pts. of the **Lagrange multiplier functional**

$$F(x, \lambda) = f(x) - \langle \lambda, h(x) \rangle.$$

Proof: $(x, \lambda) \in \mathrm{Crit}(F)$
$\iff df(x) = \langle \lambda, dh(x) \rangle$ and $h(x) = 0$
$\iff x$ critical point of f on $Z = h^{-1}(0)$
if 0 is a regular value of h. $\square$



- $f : X \to \mathbb{R}$ with $X = \mathbb{R}^n$, manifold, Banach manifold, ...
- $h : X \to V$ with $V = \mathbb{R}, \mathbb{R}^k$, Banach space, ...
- Lagrange multiplier $\lambda \in V^*$ topological dual space

Example

Example (Eigenvalues)

X complex Hilbert space,
$A : X \to X$ self-adjoint bounded linear operator,

$$f, h : X \to \mathbb{R}, \qquad f(x) = \langle x, Ax\rangle, \qquad h(x) = \|x\|^2 - 1.$$

Critical points of the restriction of f to the unit sphere $S = h^{-1}(0)$
$\leftrightarrow (x, \lambda) \in X \times \mathbb{R}$ satisfying $\|x\| = 1$ and $Ax = \lambda x$
$\leftrightarrow$ Lagrange multiplier λ is an eigenvalue of A with eigenvector x.

If A is compact (e.g. if X is finite dimensional), then f attains its maximum and minimum on S and it follows that $\|A\|$ or $-\|A\|$ is an eigenvalue.

Hessian

Exercise

If X and V are finite dimensional, then the Hessian of F at a critical point (x, λ) is given by

$$\mathrm{Hess}\, F(x, \lambda) = \begin{pmatrix} \mathrm{Hess}\, f(x) & dh(x)^* \\ dh(x) & 0 \end{pmatrix},$$

and $\mathrm{Hess}(f|_Z)(x)$ have the same nullity and signature (number of positive minus number of negative eigenvalues).

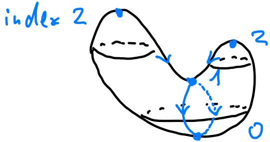
In particular, the Hessian of F is never positive or negative definite.
$\rightsquigarrow$ Its critical points cannot be detected by direct maximization or minimization methods.
$\rightsquigarrow$ Indirect variational methods.

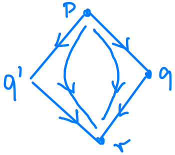
Morse homology

M closed finite dimensional manifold,

$\varphi : M \to \mathbb{R}$ Morse function,

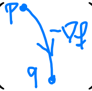
$\mathrm{Crit}_k(\varphi) := \{\text{critical points of index } k\}$.





$$MC_k(\varphi) := \langle \mathrm{Crit}_k(\varphi) \rangle_{\mathbb{Z}} \xrightarrow{\partial} MC_{k-1}(\varphi)$$

$$\partial p := \sum_q \# \left\{ \quad \right\} \cdot q$$



$$\partial \circ \partial = 0 \rightsquigarrow MH_k(\varphi) := \frac{\ker(\partial : MC_k \to MC_{k-1})}{\mathrm{im}(\partial : MC_{k+1} \to MC_k)} \cong H_k(M)$$

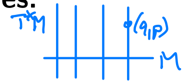
Expectation: $MH_*(F) \cong MH_*(f|_Z)$ if both are graded by the signature rather than the Morse index.

Problem: Possible escape of gradient trajectories to infinity.

2. Rabinowitz Floer homology

Hamiltonian systems

Liouville manifold (W, λ): manifold W with 1-form μ such that $\omega = d\mu$ is symplectic and convex at infinity. **Examples:**



- $\mathbb{C}^n, \mu = \frac{1}{2}\sum_j (x_j dy_j - y_j dx_j)$,
- cotangent bundles T^*M of closed manifolds, $\mu = \sum_j p_j dq_j$.

Hamiltonian function $H : W \to \mathbb{R}$
$\rightsquigarrow$ Hamiltonian vector field X_H defined by $dH = \omega(\cdot, X_H)$

1-periodic solutions $x : S^1 = \mathbb{R}/\mathbb{Z} \to W$ of the Hamiltonian system $\dot{x} = X_H(x)$ are the critical points of the Hamiltonian action

$$\mathcal{A}_H : C^\infty(S^1, W) \to \mathbb{R}, \qquad \mathcal{A}_H(x) = \int_x \mu - \int_0^1 H(x)dt.$$

Rabinowitz action functional

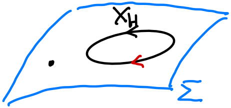
Suppose that 0 is a regular value of H. The Rabinowitz action functional is the Lagrange multiplier functional

$$\mathcal{A}^H : C^\infty(S^1, W) \times \mathbb{R} \to \mathbb{R}, \qquad \mathcal{A}^H(x,\lambda) = \int_x \mu - \lambda \int_0^1 H(x)dt.$$

$$(x,\lambda) \in \mathrm{Crit}(\mathcal{A}^H) \iff \dot{x} = \lambda X_H(x) \text{ and } \int_0^1 H(x)dt = 0$$
$$\iff \dot{x} = \lambda X_H(x) \text{ and } H(x(t)) \equiv 0$$

since $H(x(t)) \equiv \mathrm{const}$ by the first equation (energy conservation), hence $H(x(t)) \equiv 0$ by the second equation.

So there are 3 types of critical points of $\mathcal{A}^H$:



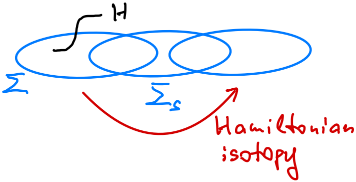
1. $\lambda > 0$: orbits $t \mapsto x(t/\lambda)$ of X_H of period λ,
2. $\lambda < 0$: such orbits run backwards,
3. $\lambda = 0$: constant loops on $\Sigma = H^{-1}(0)$.

Rabinowitz Floer homology

Rabinowitz Floer homology $RFH_*(\Sigma)$:= Morse homology of $\mathcal{A}^H$ for a defining Hamiltonian H with $\Sigma = H^{-1}(0)$.

Wishful Thinking

(a) $RFH_*(\Sigma)$ is well-defined and independent of H;
(b) $RFH_*(\Sigma_0) \cong RFH_*(\Sigma_1)$ for a smooth family of hypersurfaces Σ_s, $s \in [0,1]$;
(c) $RFH_*(\Sigma) = 0$ if Σ is displaceable from itself by a Hamiltonian isotopy;
(d) $RFH_*(\Sigma) \cong H^{n-*}(\Sigma)$ if Σ carries no periodic orbits of X_H.



This cannot be true because there exist compact hypersurfaces in $\mathbb{C}^n$ without periodic orbits (V. Ginzburg 1995)!

Theorem 1 (C, Frauenfelder 2009)

(a) $RFH_*(\Sigma)$ *is well-defined and independent of* H;
(b) $RFH_*(\Sigma_0) \cong RFH_*(\Sigma_1)$ *for a smooth family of hypersurfaces* Σ_s, $s \in [0,1]$;
(c) $RFH_*(\Sigma) = 0$ *if* Σ *is displaceable from itself by a Hamiltonian isotopy;*
(d) $RFH_*(\Sigma) = H^{n-*}(\Sigma)$ *if* Σ *carries no periodic orbits of* X_H

provided that Σ, Σ_s *are of exact contact type (*Σ *carries a contact form* α *such that* $\alpha - \mu|_\Sigma$ *is exact), or more generally stable and tame.*

3. Poincaré duality for loop spaces

String topology

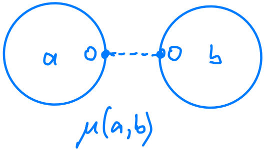
M closed connected (oriented) manifold of dimension n,
$\Lambda = C^\infty(S^1, M)$ free loop space $\supset \Lambda_0$ constant loops.

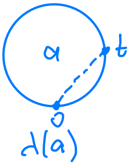
M. Chas, D. Sullivan 1999: string topology operations, e.g.

- loop product $\mu = \bullet$ on $H_*\Lambda$ of degree $-n$,
- loop coproduct λ on $H_*(\Lambda, \Lambda_0)$ of degree $1-n$,

dual

- cohomology product $\circledast = \lambda^\vee$ on $H^*(\Lambda, \Lambda_0)$ of degree $n-1$ (also called Goresky–Hingston product) .

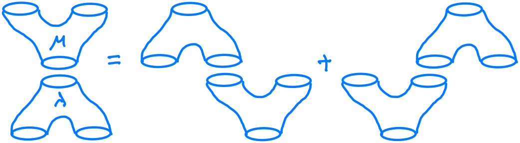




Puzzles in string topology

Puzzle 1: Can μ and λ be defined on the same space? If yes, what algebraic structure do they define? For example, does the following relation conjectured by D. Sullivan (Sullivan's relation) hold:

$$\lambda\mu = (1 \otimes \mu)(\lambda \otimes 1) + (\mu \otimes 1)(1 \otimes \lambda)\,?$$



Puzzle 2 (N. Hingston): Many results concerning $\bullet$ and $\circledast$ arise in dual pairs. For example, the **critical levels**

$$\mathrm{Cr}(X) = \inf\{a \in \mathbb{R} \mid X \in \mathrm{im}(H_*\Lambda^{<a} \to H_*\Lambda)\},$$
$$\mathrm{cr}(x) = \sup\{a \in \mathbb{R} \mid x \in \mathrm{im}(H^*(\Lambda, \Lambda^{<a}) \to H^*(\Lambda, \Lambda_0))\}$$

for $X \in H_*\Lambda$ and $x \in H^*(\Lambda, \Lambda_0)$ satisfy the dual inequalities

$$\mathrm{Cr}(X \bullet Y) \leq \mathrm{Cr}(X) + \mathrm{Cr}(Y), \qquad \mathrm{cr}(x \circledast y) \geq \mathrm{cr}(x) + \mathrm{cr}(y).$$

Can this be explained by some kind of “Poincaré duality”?

Rabinowitz loop homology

It turns out that all the puzzles get naturally resolved in terms of the Rabinowitz Floer homology of the unit sphere cotangent bundle $S^*M = \{(q, p) \in T^*M \mid |p| = 1\}$, the Rabinowitz loop homology

$$\widehat{H}_*\Lambda := RFH_*(S^*M).$$

Relation to loop homology and cohomology

Theorem 2 (C, Frauenfelder, Oancea 2010)

Rabinowitz loop homology is related to ordinary loop (co)homology by a commuting diagram with exact rows

$$
\begin{array}{ccccccccc}
 & & & & & & H^{1-*}(\Lambda, \Lambda_0) & & \\
 & & & & & \overset{i}{\swarrow} & \downarrow{\scriptstyle j} & & \\
\cdots H^*\Lambda & \xrightarrow{\varepsilon} & H_*\Lambda & \xrightarrow{\iota} & \widehat{H}_*\Lambda & \xrightarrow{\pi} & H^{1-*}\Lambda & \xrightarrow{\varepsilon} & \cdots \\
 & & \downarrow{\scriptstyle q} & \overset{p}{\swarrow} & & & & & \\
 & & H_*(\Lambda, \Lambda_0) & & & & & &
\end{array}
$$

Product and coproduct on Rabinowitz loop homology

Theorem 3 (C, Hingston, Oancea 2020)

$\widehat{H}_*\Lambda$ carries a natural degree $-n$ product $\boldsymbol{\mu}$ and a natural degree $1-n$ coproduct $\boldsymbol{\lambda}$ such that in the commuting diagram

$$
\begin{array}{ccccccc}
 & & & & (H^{1-*}(\Lambda,\Lambda_0), \lambda^\vee = \circledast) & & \\
 & & & \swarrow^{i} & \downarrow^{j} & & \\
\xrightarrow{\varepsilon} (H_*\Lambda, \mu = \bullet) & \xrightarrow{\iota} & (\widehat{H}_*\Lambda, \boldsymbol{\mu}, \boldsymbol{\lambda}) & \xrightarrow{\pi} & (H^{1-*}\Lambda, \mu^\vee) & \xrightarrow{\varepsilon} & \\
\downarrow^{q} & \swarrow^{p} & & & & & \\
(H_*(\Lambda,\Lambda_0), \lambda) & & & & & &
\end{array}
$$

the maps ι, i intertwine the products $\mu = \bullet$, $\boldsymbol{\mu}$, $\lambda^\vee = \circledast$, and the maps p, π intertwine the coproducts λ, $\boldsymbol{\lambda}$, $\mu^\vee$.

Thus $\boldsymbol{\mu}$ extends both products $\bullet$ and $\circledast$,
and $\boldsymbol{\lambda}$ extends λ and $\mu^\vee$!

Algebraic structure on Rabinowitz loop homology

Define the degree shifted (co)homology groups

$$\widehat{\mathbb{H}}_*\Lambda := \widehat{H}_{*+n}\Lambda, \qquad \widehat{\mathbb{H}}^*\Lambda := \widehat{H}^{*+n}\Lambda.$$

Theorem 4 (C, Hingston Oancea 2022)

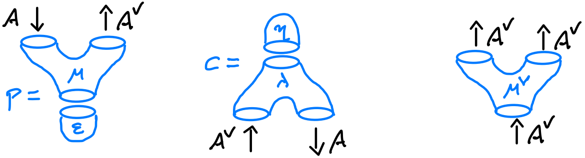
The product $\boldsymbol{\mu}$ (of shifted degree 0) and coproduct $\boldsymbol{\lambda}$ (of shifted degree $1-2n$) make $\widehat{\mathbb{H}}_*\Lambda$ a commutative and cocommutative graded Frobenius algebra.

Algebraic structure on Rabinowitz loop homology

Graded Frobenius algebra: graded module A endowed with

- an associative degree zero product $\boldsymbol{\mu}$ with unit $\boldsymbol{\eta}$,
- a coassociative coproduct $\boldsymbol{\lambda}$ with counit ε

such that the pairing $\boldsymbol{p} = (-1)^{|\boldsymbol{\lambda}|}\varepsilon\boldsymbol{\mu}$ is symmetric and induces an isomorphism $\vec{\boldsymbol{p}} : A \xrightarrow{\cong} A^\vee$. Equivalently, the copairing $\boldsymbol{c} = \boldsymbol{\lambda}\eta$ is symmetric and induces an isomorphism $\vec{\boldsymbol{c}} : A^\vee \xrightarrow{\cong} A$.



Poincaré duality for Rabinowitz loop homology

Theorem 5 (C, Hingston Oancea 2022)

The dual operations $\boldsymbol{\lambda}^\vee, \boldsymbol{\mu}^\vee$ make $\widehat{\mathbb{H}}^*\Lambda$ also a graded Frobenius algebra, and there exists a Poincaré duality isomorphism

$$PD : (\widehat{\mathbb{H}}_*\Lambda, \boldsymbol{\mu}, \boldsymbol{\lambda}) \xrightarrow{\cong} (\widehat{\mathbb{H}}^{1-2n-*}\Lambda, \boldsymbol{\lambda}^\vee, \boldsymbol{\mu}^\vee).$$

At zero energy this specializes to ordinary Poincaré duality on S^*M (not M!)

Three proofs of Poincaré duality.

1. In the original definition of Rabinowitz Floer homology via the involution $(x, \lambda) \mapsto (\bar{x}, -\lambda)$, $\bar{x}(t) = x(-t)$ (without products)
2. Via a long exact sequence relating $\widehat{H}_*\Lambda$ and $\widehat{H}^*\Lambda$.
3. In terms of the graded Frobenius algebra structure: $PD = \vec{\boldsymbol{p}} = \vec{\boldsymbol{c}}^{-1}$

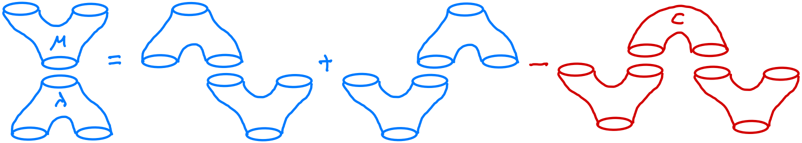
□

Puzzles resolved

Puzzle 1: The loop product μ and coproduct λ naturally extend to operations $\boldsymbol{\mu}, \boldsymbol{\lambda}$ on the common space $\widehat{\mathbb{H}}_*\Lambda$, where they define the structure of a graded Frobenius algebra.

Sullivan's relation appears as part of this structure with an extra term involving the copairing $\boldsymbol{c} = \boldsymbol{\lambda\eta}$ (which is responsible for Poincaré duality!):

$$\boldsymbol{\lambda\mu} = (1 \otimes \boldsymbol{\mu})(\boldsymbol{\lambda} \otimes 1) + (\boldsymbol{\mu} \otimes 1)(1 \otimes \boldsymbol{\lambda}) - (\boldsymbol{\mu} \otimes \boldsymbol{\mu})(1 \otimes \boldsymbol{c} \otimes 1).$$



Puzzles resolved

Puzzle 2: Each pair of N. Hingston's results about $\bullet$ and $\circledast$ extends to a pair of results about $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$ which are related by Poincaré duality $(\widehat{\mathbb{H}}_*\Lambda, \boldsymbol{\mu}) \cong (\widehat{\mathbb{H}}^{1-2n-*}\Lambda, \boldsymbol{\lambda}^\vee)$. While the original results had topological proofs, the extended results have symplectic proofs.

Outlook: other Lagrange multiplier functionals

Constrained Lagrangian systems ($\lambda \sim$ constraint force)

Eulers equations of hydrodynamics ($\lambda =$ pressure)

Gauge theories ($\lambda =$ gauge fixing boson)

Symplectic vortex equations ($\lambda =$ connection 1-form)

Symplectic field theory ($\lambda \sim \mathbb{R}$-component in symplectization)

Thank you!

Sketch of proof of Theorem 1.

Define the L^2-metric on $C^\infty(S^1, W) \times \mathbb{R}$

$$m_{(x,\lambda)}\Big((\hat{x}_1, \hat{\lambda}_1), (\hat{x}_2, \hat{\lambda}_2)\Big) = \int_0^1 \omega(\hat{x}_1, J\hat{x}_2)dt + \hat{\lambda}_1\hat{\lambda}_2$$

for an ω-compatible almost complex structure J on W. Then gradient flow lines of $\mathcal{A}^H$ are maps $(u, \lambda) : \mathbb{R} \to C^\infty(S^1, W) \times \mathbb{R}$ satisfying

$$\partial_s u + J(u)\big(\partial_t u - \lambda X_H(u)\big) = 0, \qquad \partial_s \lambda + \int_0^1 H(u)dt = 0$$

(coupled system of an elliptic PDE and a non-local ODE)

Three potential sources of noncompactness for its solutions:

1. explosion of the gradient of $u \rightsquigarrow$ excluded by exactness of ω
2. escape of u to infinity $\rightsquigarrow$ prevented by convexity at infinity
3. escape of the Lagrange multiplier λ to $\pm\infty \rightsquigarrow$ prevented by assumptions on $\Sigma = H^{-1}(0)$

□

Applications of RFH in symplectic geometry

- Existence of periodic orbits on stable hypersurfaces in $\mathbb{C}^n$ (Weinstein conjecture)
- Non-existence of exact contact embeddings of unit cotangent bundles into $\mathbb{C}^n$ (generalizing Gromov's theorem that there are no exact Lagrangian embeddings into $\mathbb{C}^n$)
- Non-displaceability results, e.g. in cotangent bundles
- Non-stability of hypersurfaces at Mañé's critical values (with G. Paternain)
- Existence of leafwise intersections (P. Albers, U. Frauenfelder)
- Periodicity of symplectic homology of Brieskorn manifolds (P. Uebele)

...

Applications of Poincaré duality for loop spaces

- Behaviour of critical levels with respect to products;
- graded open-closed TQFT structure on homology of free and based loop space;
- BV operator and string point invertibility of constant rank one symmetric spaces;
- duality between fastest growth of the index and slowest growth of index+nullity for iterations of a closed geodesic;

  . . .