Hodge theory, between algebraicity and transcendence

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ICM 2022

Hodge theory in a nutshell: a linearization principle

- Algebraic variety = space of solutions of a system of algebraic equations, e.g.

$$X/k = \{\underline{z} = [z_0, \dots, z_n] \in \mathbb{P}^n_k \mid f_1(\underline{z}) = \cdots = f_r(\underline{z}) = 0\},$$

where the f_i's are homogeneous polynomials with coefficients in a field k.

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- Hodge theory is the art of "linearizing" such algebraic varieties when $k = \mathbb{C}$.

$$\underbrace{X/\mathbb{C}}_{\text{smooth projective variety}} \quad \rightsquigarrow \quad \underbrace{(H^\bullet_B(X^{\mathrm{an}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}, F^\bullet)}_{\text{filtered complex vector space}}$$

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$$\underbrace{f : X \to S}_{\text{smooth projective}} \quad \rightsquigarrow \quad \underbrace{\Phi : S^{\mathrm{an}} \to \Gamma \backslash D}_{\mathbb{C}-\text{analytic period map}}$$

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$$\begin{array}{ccc} H^\bullet(X^{\mathsf{an}}, \Omega^\bullet_{X^{\mathsf{an}}}) & \xrightarrow{\cong} & H^\bullet_{dR}(X^{\mathsf{an}}, \mathbb{C}) \\ \cong \uparrow & & \downarrow \cong \\ H^\bullet(X/\mathbb{C}, \Omega^\bullet_{X/\mathbb{C}}) & \xrightarrow{\cong} & H^\bullet_B(X^{\mathsf{an}}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \end{array}$$

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- $Z^p(X)_{\mathbb{Q}} \overset{\text{cycle map}}{\to} F^p \cap H^{2p}_B(X^{\mathrm{an}}, \mathbb{Q})$

Hodge structure and Mumford-Tate group

Theorem (Hodge, Frölicher, Deligne)

$V = (V_{\mathbb{Z}} := H_B^i(X^{\mathrm{an}}, \mathbb{Z}), F^\bullet)$ *is a polarizable* $\mathbb{Z}$*-Hodge structure of weight* i*:*

(a) $V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} = F^p \oplus \overline{F^{i+1-p}} (\iff V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{p+q=i} (F^p \cap \overline{F^q}))$.

(b) $Q : V_{\mathbb{Z}} \otimes_{\mathbb{Z}} V_{\mathbb{Z}} \to \mathbb{Z}$, $(-1)^i$*-symmetric,* $Q_{\mathbb{C}}(F^p, F^{i+1-p}) = 0$ *and* $Q_{\mathbb{C}}(Cv, \overline{v}) > 0$.

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  $\mathbf{G}_V$ is the Tannaka group of $\langle V_{\mathbb{Q}}^{\otimes} \rangle \subset \mathbb{Q}$HS; equivalently, the fixator in $\mathbf{GL}(V_{\mathbb{Q}})$ of all Hodge tensors in $V^{\otimes}$.

Hodge theory in families

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with $\nabla F^\bullet \subset F^{\bullet-1} \otimes \Omega^1_S$ (Griffiths transversality).

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- Period map:

  $$\begin{array}{ccccc} \widetilde{S^{\mathrm{an}}} & \xrightarrow{\tilde{\Phi}} & D = \mathbf{G}(\mathbb{R})/M & \overset{\text{open}}{\hookrightarrow} & D^{\vee} = \mathbf{G}(\mathbb{C})/P \text{ flag variety} \\ \downarrow{\scriptstyle \pi} & & \downarrow & & \\ S^{\mathrm{an}} & \xrightarrow[\Phi]{} & \Gamma\backslash D & & \end{array}$$

  where $\mathbf{G}$ is the generic Mumford-Tate group, and $\Gamma = \mathbf{G}(\mathbb{Z})$.

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where $\mathbf{G}$ is the generic Mumford-Tate group, and $\Gamma = \mathbf{G}(\mathbb{Z})$.

- The period map Φ is $\mathbb{C}$-analytic, and severely constrained:

$$d\Phi(TS_s^{\text{an}}) \subset \mathfrak{g}^{-1,1}_{\Phi(s)} \subset T_{\Phi(s)}(\Gamma \backslash D) = \mathfrak{g}^{-1,1}_{\Phi(s)} \oplus \cdots \oplus \mathfrak{g}^{-l,l}_{\Phi(s)}.$$

$$l = \text{level}(\mathbb{V}).$$

Hodge loci

- $$\begin{aligned}\mathrm{HL}(S,\mathbb{V}^{\otimes}) &= \{s \in S^{\mathrm{an}} \,|\, \mathbb{V}_s \text{ admits "exceptional" Hodge tensors}\} \\ &= \{s \in S^{\mathrm{an}} \,|\, \mathbf{G}_s \subsetneq \mathbf{G} \text{ generic Mumford-Tate group}\}\end{aligned}$$

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- Cartesian diagram:

$$\begin{array}{ccc} S^{\mathsf{an}} & \xrightarrow{\quad\Phi\quad} & \Gamma\backslash D \\ \uparrow & & \uparrow \\ \mathsf{HL}(S, \mathbb{V}^{\otimes}) & \longrightarrow & \bigcup_{(\mathbf{G}', D') \subsetneq (\mathbf{G}, D)} \Gamma'\backslash D' \end{array}$$

- $f : \mathcal{E} \to S = \{\text{elliptic curve}\}$ universal elliptic curve

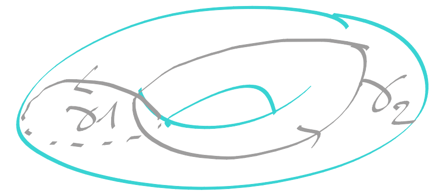
- $\left\{ \begin{array}{l} \text{Elliptic curve } E \\ H_1(E^{an}, \mathbb{Z}) = \mathbb{Z}\gamma_1 \oplus \mathbb{Z}\gamma_2 \end{array} \right\} \xrightarrow{\tilde{\Phi}} \mathbb{H}^{\pm} = \mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R}) \subset \mathbb{P}^1(\mathbb{C})$

$$E \longmapsto \frac{\int_{\gamma_1} \omega_E}{\int_{\gamma_2} \omega_E} \qquad = [\mathbb{C}\omega_E \subset \mathbb{C}\gamma_1^* \oplus \mathbb{C}\gamma_2^*]$$

$$S^{an} = \{\text{Elliptic curve}_E\}^{an} \xrightarrow{\Phi} GL(2,\mathbb{Z}) \backslash \mathbb{H}^{\pm} = \mathbb{H}^+ / SL(2,\mathbb{Z})$$

Riemann

j



- $\mathrm{End}_{\mathbb{Z}HS}\, H^1(E_\tau, \mathbb{Z}) = \begin{cases} \text{order in } \mathbb{Q}(\tau) & \text{if } \tau \text{ imaginary quadratic} \\ \mathbb{Z} & \text{otherwise} \end{cases}$

$\rightsquigarrow G_{E_\tau} = \begin{cases} \mathrm{Res}_{\mathbb{Q}(\tau)/\mathbb{Q}} \mathbb{G}_m & \text{if } \tau \text{ imaginary quadratic} \\ GL_{2,\mathbb{Q}} & \text{otherwise} \end{cases}$ generic Mumford group   level $(V) = 1$

- $HL(S, f) = \{ j(\tau), \tau \text{ imaginary quadratic} \} \subset \overline{\mathbb{Q}} \subset \mathbb{C} = S^{an}$

Hodge theory is at heart transcendental...

- The proof that $V = (V_{\mathbb{Z}} := H^i_B(X^{\mathrm{an}}, \mathbb{Z}), F^\bullet)$ is a polarizable $\mathbb{Z}$HS of weight i is transcendental.

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- If X/K, $K \subset \mathbb{C}$ number field,

$$H^\bullet_{dR}(X/K) \otimes_K \mathbb{C} \simeq H^\bullet_B(X^{\mathrm{an}}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}.$$

$\rightsquigarrow k_X := \langle \text{periods of } X/K \rangle \subset \mathbb{C}$.

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- $\mathbb{V}$ $\mathbb{Z}$VHS $\rightsquigarrow \Phi : S^{\mathrm{an}} \to \Gamma\backslash D$.

  If level$(\mathbb{V}) = 1$ then $\Gamma\backslash D$ is a Shimura variety and Φ is algebraic; but as soon as level$(\mathbb{V}) > 1$ then $\Gamma\backslash D$ has no algebraic structure and Φ is a mere complex analytic map.

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(Weil 1979): If $f : X \to S$, the Hodge conjecture implies that $\mathsf{HL}(S, \mathbb{V}^{\otimes})$ is a countable union of algebraic subvarieties of S.

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Theorem (Cattani-Deligne-Kaplan '95)

Let $\mathbb{V}$ be a polarizable $\mathbb{Z}$VHS on a smooth quasi-projective variety S. Then $\mathrm{HL}(S, \mathbb{V}^{\otimes})$ is a countable union of irreducible algebraic subvarieties of S: the special subvarieties of S for $\mathbb{V}$.

Tame geometry

- "If $\text{level}(\mathbb{V}) > 1$ then $\Gamma \backslash D$ has no algebraic structure and $\Phi : S^{\text{an}} \to \Gamma \backslash D$ is a mere complex analytic map."

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- To be discarded: $\Gamma =$ graph of $(x \mapsto \sin \frac{1}{x}), x > 0$.

$$\overline{\Gamma} = \mathsf{I} \cup \Gamma$$

  Γ is not tame for at least three reasons:
  (a) $\overline{\Gamma}$ is connected but not arc-connected;
  (b) $\dim \partial\Gamma = \dim \Gamma$;
  (c) $\Gamma \cap \mathbb{R}$ is "not of finite type".

Tame geometry

- A structure is a collection $\mathcal{S} = (S_n)_{n\in\mathbb{N}}$, where S_n is a set of subsets of $\mathbb{R}^n$ such that:
  (1) algebraic subsets of $\mathbb{R}^n$ belong to S_n.
  (2) S_n is stable under intersection, finite union and complement.
  (3) $A \in S_p, B \in S_q \Rightarrow A \times B \in S_{p+q}$.
  (4) If $p : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is a linear projection and $A \in S_{n+1}$ then $p(A) \in S_n$.

  The elements of S_n are called the $\mathcal{S}$-definable subsets of $\mathbb{R}^n$.
  $f : A \to B$ is $\mathcal{S}$-definable if A, B and $\Gamma(f)$ are $\mathcal{S}$-definable.

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- Examples:
  - $\mathbb{R}_{\text{alg}}$
  - $\mathbb{R}_{\mathcal{F}}$ for $\mathcal{F}$ a collection of functions $f : \mathbb{R}^n \to \mathbb{R}$ and of subsets of $\mathbb{R}^n$ (e.g. $\mathbb{R}_{\exp}$, $\mathbb{R}_{\text{an}}$, $\mathbb{R}_{\sin}$).

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  - $\mathbb{R}_{\mathsf{alg}}$
  - $\mathbb{R}_{\mathsf{an}}$ (Łosajiewicz, Gabrielov)
  - $\mathbb{R}_{\exp}$ (Khovanskii, Wilkie),
  - $\mathbb{R}_{\mathsf{an},\exp}$ (Miller-Van den Dries)

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- Globalization: $\mathcal{S}$-definable topological spaces

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Theorem (Pila-Wilkie '06)

Let $Z \subset \mathbb{R}^n$ be definable in some o-minimal structure.
Let $Z^{\mathsf{alg}} \subset Z$ be the union of all positive-dimensional connected semi-algebraic subsets of Z. Then:

$$\forall \varepsilon > 0,\ \exists C_\varepsilon > 0 \ / \ \left|\{x \in (Z - Z^{\mathsf{alg}}) \cap \mathbb{Q}^n,\ H(x) \leq T\}\right| < C_\varepsilon T^\varepsilon \ .$$

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Theorem (Peterzil-Starchenko '09, o-minimal Chow)

Let S be a quasi-projective variety over $\mathbb{C}$, e.g. $S = \mathbb{C}^n$. Let $Z \subset S^{\mathrm{an}}$ be a closed analytic subset. If Z is definable in some o-minimal structure extending $\mathbb{R}_{\mathrm{an}}$ then $Z \subset S$ is algebraic.

Tame geometry of period maps

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Theorem (Bakker-K.-Tsimerman '20)

$\Gamma\backslash D$ has a canonical structure of $\mathbb{R}_{\mathrm{alg}}$-definable manifold.
Each $\Gamma'\backslash D' \subset \Gamma\backslash D$ coming from $(\mathbf{G}', D') \subset (\mathbf{G}, D)$ is a definable subspace.

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Let $\mathbb{V}$ be a polarizable $\mathbb{Z}$VHS on a smooth quasi-projective variety S. Then $\mathrm{HL}(S, \mathbb{V}^{\otimes})$ is a countable union of irreducible algebraic subvarieties of S: the special subvarieties of S for $\mathbb{V}$.

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Theorem (Brunebarbe-Bakker-Tsimerman)

Images of period maps have a natural algebraic structure.

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- Bi-algebraic format: the diagram

$$\begin{array}{ccccc} \widetilde{S^{\mathsf{an}}} & \xrightarrow{\tilde{\Phi}} & D & \overset{\mathsf{open}}{\hookrightarrow} & D^\vee \text{ flag variety} \\ {\scriptstyle \pi}\downarrow & & & & \\ S^{\mathsf{an}} & & & & \end{array}$$

emulates an algebraic structure on $\widetilde{S^{\mathsf{an}}}$:

Hodge theory as bi-algebraic geometry

- Bi-algebraic format: the diagram

$$\begin{array}{ccccc} \widetilde{S^{\text{an}}} & \xrightarrow{\tilde{\Phi}} & D & \xhookrightarrow{\text{open}} & D^{\vee} \text{ flag variety} \\ \downarrow{\scriptstyle \pi} & & & & \\ S^{\text{an}} & & & & \end{array}$$

emulates an algebraic structure on $\widetilde{S^{\text{an}}}$:

- $Y \subset \widetilde{S^{\text{an}}}$ is algebraic for Φ if $Y = \tilde{\Phi}^{-1}(\text{algebraic in } D^{\vee})^0$.
  $W \subset S$ is bi-algebraic for Φ if W is algebraic and $W = \pi(Y)$, with $Y \subset \widetilde{S^{\text{an}}}$ algebraic.

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- We want to study the transcendence of π with respect to the algebraic structure on S and the emulated one on $\widetilde{S^{\mathrm{an}}}$.
- Generalizes the case of tori, abelian varieties, Shimura varieties.

Hodge theory and functional transcendence

Proposition (K.-Otwinowska '21)

Let $\Phi : S^{\mathsf{an}} \to \Gamma\backslash D$ be a period map. The bi-algebraic subvarieties of S for Φ are the weakly special ones.

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Theorem (Ax-Schanuel conjecture for $\mathbb{Z}$VHS, conjectured by K. '17; Bakker-Tsimerman '19)

Let $Z \subset S \times D^\vee$ be a closed algebraic subvariety.

(a) *If the intersection of Z^{an} with $\Delta := S^{\mathrm{an}} \times_{\Gamma\backslash D} D$ is atypical, i.e.*

$$\mathrm{codim}_{S^{\mathrm{an}} \times D} Z^{\mathrm{an}} \cap \Delta < \mathrm{codim}_{S^{\mathrm{an}} \times D} Z^{\mathrm{an}} + \mathrm{codim}_{S^{\mathrm{an}} \times D} \Delta \ ,$$

then $p(Z^{\mathrm{an}} \cap \Delta)$ is contained in a strictly weakly special subvariety of S.

(b) *In particular: if $Z \subset \widetilde{S^{\mathrm{an}}}$ is algebraic then $\overline{p(Z)}^{\mathrm{Zar}}$ is weakly special (Ax-Lindemann conjecture for $\mathbb{Z}$VHS).*

Distribution of the Hodge loci

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Theorem (K-Otwinowska '21)

Assume for simplicity that $\mathbf{G}^{\mathsf{ad}}$ *is simple.*
Either $\mathsf{HL}(S, \mathbb{V}^{\otimes})_{\mathsf{pos}}$ *is Zariski-dense in* S*, or it is algebraic.*

Hodge loci as (a)typical intersections: conjectures

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- A special subvariety $Z = \Phi^{-1}(\Gamma_Z \backslash D_Z)^0 \subset S$ is said atypical if $\operatorname{codim}_{\Gamma\backslash D} \Phi(Z^{\mathsf{an}}) < \operatorname{codim}_{\Gamma\backslash D} \Phi(S^{\mathsf{an}}) + \operatorname{codim}_{\Gamma\backslash D} \Gamma_Z \backslash D_Z$.

  Otherwise it is typical.

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Conjecture (Zilber-Pink for $\mathbb{Z}$VHS; K'17; Baldi-K-Ullmo)

(1) $\mathsf{HL}(S, \mathbb{V}^{\otimes})_{\mathsf{atyp}}$ *is algebraic.*

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This implies:

Conjecture (André-Oort for $\mathbb{Z}$VHS; K'17)

If S contains a Zariski-dense set of CM-points for $\mathbb{V}$, then

(a) $\mathrm{level}(\mathbb{V}) = 1$, *i.e.* $\Gamma\backslash D$ *is a Shimura variety;*

(b) $\Phi : S^{\mathrm{an}} \to \Gamma\backslash D$ *is a dominant algebraic map.*

Hodge loci as (a)typical intersections: results

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Theorem (Baldi-K-Ullmo)

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(2) *Suppose in addition that* $\mathbf{G}^{\mathsf{ad}}$ *is simple. Then* $\mathsf{HL}(S, \mathbb{V}^{\otimes})_{\mathsf{pos}}$ *is algebraic.*

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Corollary (Baldi-K-Ullmo)

Let $f : H_{n,d} \to U_{n,d} \subset \mathbb{P}H^0(\mathbb{P}^{n+1}_{\mathbb{C}}, \mathcal{O}(d))$ *be the family of smooth hypersurfaces of degree* d *in* $\mathbb{P}^{n+1}_{\mathbb{C}}$*.*
If $n \geq 3$ *and* $d > 5$ *then* $\mathsf{HL}(U_{n,d}, f)_{\mathsf{pos}} \subset U_{n,d}$ *is algebraic.*

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Theorem (Baldi-K-Ullmo)

If $\mathsf{HL}(S, \mathbb{V}^{\otimes})_{\mathsf{typ}} \neq \emptyset$ *(hence* $\mathsf{level}(\mathbb{V}) = 1$ *or* 2*) then* $\mathsf{HL}(S, \mathbb{V}^{\otimes})$ *is analytically dense in* S^{an}*.*

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Let $\mathbb{V} \to S$ *be a* $\mathbb{Z}$*VHS defined over a number field* $K \subset \mathbb{C}$*. Then*

(1) *any special subvariety of* S *for* $\mathbb{V}$ *is defined over* $\overline{\mathbb{Q}}$*;*

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Theorem (K-Otwinowska-Urbanik '20)

(a) *Suppose that* $\mathbf{G}^{\mathrm{ad}}$ *is simple. Then the conjecture above holds true for the maximal special subarieties of positive period dimension. In particular if* $\mathrm{level}(\mathbb{V}) \geq 3$ *then* $\mathrm{HL}(S, \mathbb{V}^{\otimes})_{\mathrm{pos}}$ *is algebraic, defined over* $\overline{\mathbb{Q}}$*.*

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Theorem (Kreutz)

Let $(\mathbb{V}^{\sigma})_{\sigma}$ be a (de Rham) motivic variation of Hodge structure on S. Suppose that $\mathbf{G}^{\mathrm{ad}}$ is simple. Then any maximal special subvariety $Y \subset S$ of positive period dimension for $\mathbb{V}$ is absolutely special.