SYMBOLIZING MATHEMATICAL REALITY INTO BEING —

or

How Mathematical Discourse and Mathematical Objects Create Each Other

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The Question of Meaning in Virtual Reality Discourse

According to Bertrand Russell (1904), "Mathematics may be defined as a subject in which we never know what we are talking about, nor whether what we are saying is true" (p. 84). Thus, for Russell, "not knowing what we are talking about" is the unique characteristic of mathematics, something that sets mathematical discourse apart from other discursive formations. To have a better grasp of what Russell might have had in mind, let us compare the following two utterances:

(a) The expressions "The founder of psychoanalysis" and "Sigmund Freud" mean the same because they refer to the same person.

(b) The symbols "$^{2}/_{3}$" and "$^{12}/_{18}$" mean the same because they refer to the same number.

The two sentences are similar in many ways. Both speak about something being signified by something else. Both lead the attention of the reader from one kind of thing to another: from objects that refer to something (the name "Sigmund Freud," the symbol "$^{2}/_{3}$") to those that are referred to (the person, the number). Thus, on the face of it, the way to understand a sentence and to decide about its truth value should be basically the same in the two cases. Summoning his or her knowledge of the history of modern psychological thought or of mathematics, one has to verify that in each of the sentences the two referring objects (e.g., "Sigmund Freud" and "The founder of psychoanalysis") have the same object as their referent.

Those, however, who take the verifying business seriously will soon discover a subtle difference between the two instances. In *(a)*, all one has to do to convince an interlocutor of the equivalence of the two expressions is to point out the person who is supposed to be their common referent. This straightforward method would not work for *(b)*. Unlike the Austrian psychiatrist, rational numbers are not palpable objects that can be seen, heard and touched. Thus, the only way to prove the equivalence of the symbols "$^{2}/_{3}$" and "$^{12}/_{18}$" is to use an indirect procedure, such as the one that shows how the fraction $^{2}/_{3}$ may be obtained from $^{12}/_{18}$ by dividing the numerator and denominator by the same number. This method is neither immediate nor does it display the object whose existence is presupposed by the sentence. Throughout the whole procedure, the implied entity remains behind the scene and its existence can only be inferred from the processes performed on its "representations."

To put it more succinctly, whereas the referent of "Sigmund Freud" and of "The founder of psychoanalysis" may be identified "ostensively"[1] (e.g., by showing Freud's portrait), the object allegedly referred to by the symbols "$^{2}/_{3}$" and "$^{12}/_{18}$" can at best be imagined. Unlike the former, the latter cannot be communicated to others just by pointing to something. In this case, a straightforward appeal to one's senses would be useless[2]. The common referent of the symbols "$^{2}/_{3}$" and "$^{12}/_{18}$" is an elusive entity, the ontological status of which has been puzzling philosophers for ages.

One may highlight the difference between *(a)* and *(b)* metaphorically, by saying that the two sentences belong to fundamentally different types of discourse: *(a)* comes from Actual Reality (AR) discourse while *(b)* belongs to Virtual Reality (VR) discourse. This figurative description underlines the basic disparity in the ways the meaning of the two types of sentences is constructed and communicated. To put it in a somewhat simplistic way, Actual Reality communication may be perceptually mediated by the objects which are being discussed, while in the Virtual Reality discourse perceptual mediation is scarce and is only possible with the help of what is understood as symbolic substitutes of objects under consideration. This description should not be read as a statement on an ontological status of the "realms" underlying the two types of discourse. In introducing the metaphor of the two realities, I was psychologically, rather than philosophically, minded. That is, the distinction was drawn with an eye to the differing actions and experiences of the participants of the discourses rather than to ontological questions.

While seen in this light, the Virtual Reality metaphor not only gives an idea about the rather unique mode of communication characteristic to mathematics, but also conveys a message as to the particular rights and obligations the mathematical discourse confers upon the participants. The demands seem more obvious than the rights. Those who really wish to communicate, not being able to rely on their senses, have to use all their mental faculties in an attempt to reconstruct for themselves the realm within which the moves of their interlocutors make sense. On the other hand, VR discourse frees its participants from responsibilities typical of the AR discourse. The latter kind of discourse mediates concrete human actions and, as such, may have a tangible influence on people's physical condition and on their environment. Think, for example, about possible effects of a medical prescription on a patient or about the influence of an architectural plan of a house on the lives of its inhabitants. The VR discourse, in contrast, being much less likely to have an immediate impact on the actual reality, creates an atmosphere of freedom which for some students is a source of enjoyment while for some others seems rather disturbing. For learners of the latter type, lack of real-life responsibility and constraints may mean lack of importance, and thus lack of interest.

To sum up, the mathematical discourse was presented here as a VR discourse which can be defined in opposition to the AR discourse. The distinction, drawn along the experiential axis, proved quite fundamental. In light of this, the question may rightly be asked why the AR discourse was juxtaposed to mathematical discourse in the first place. Indeed, why AR discourse and not, say, scientific or literary discourse?

AR discourse may not be the only one against which the particularities of mathematical discourse would stand in full relief, but for my present purpose it is certainly the most appropriate. First and foremost, in the fervor of pinpointing differences, we should not overlook striking similarities. One example substantiating this claim has already been given in the form of the sentences *(a)* and *(b)* above. Clearly, the two are almost identical structurally. They both speak about two objects being, in fact, the same entity. This surface similarity is so great that we had to make a certain effort to notice and understand the way in which they differ. An additional glimpse at mathematical textbooks or classroom exchanges should be enough to convince ourselves that the similarity is quite general and that mathematical discourse, almost in all its manifestations, is structurally very close to AR discourse. Indeed, in descriptions of mathematical processes and objects, the same linguistic forms are used as in AR narrative, even though the AR narrative is concerned with physical procedures performed on concrete material objects.

Another way to characterize the similarity between AR and VR discourses is to say that, in both, *the use of symbols is often mediated by objects*. Within our present discussion, in which a psychological perspective has been adopted, object-mediation means a certain distinctive way of manipulating symbols, of solving problems and of communicating — a mode which reminds us of what can be observed when people talk about physical objects, whether actually present or only recalled. The claim that a discourse is object-mediated signifies that in their attempt to communicate, interlocutors can help themselves with a reference to entities conceived as external to, and independent of, the discourse itself (this is what people do, for example, when discussing the appropriateness of a given kind of fabric for a certain type of garment; or when they judge the applicability of a given function as a model for a certain biological phenomenon). My use of the phrase 'conceived as external' rather than 'which are external' with regard to the mediating entities (objects) comes to emphasize that in the present discussion, the question of the 'real' ontological status of these entities is of no relevance, if not entirely out of place. It is very important to understand that in our context, the word "objects" does not stand alone and does not signal an existence of special entities which regulate the discourse. In phrases such as “the discourse is object-mediated”, “the student has constructed a mathematical object” or “a new object emerged (in discourse)” its use is essentially metaphorical. The term “object-mediation” refers, metaphorically, to a certain set of discursive competencies and underlying psychological processes, and should by no means be read as a statement on the nature or the extra-discursive existence (or non-existence) of the mediating entities.

A detailed description of this mode of symbol use and discursive competencies will be given in a due time. Here, I will restrict myself to illustrating the idea with the example presented in Figure 1. A twelve-year-old boy named George, who has learned about linear functions but has never before tackled a linear equation, is given a problem that he is only able to solve thanks to his ability to "see" more than algebraic expressions themselves. He speaks and acts as if the symbols were mere representations of some invisible objects. He clearly refers to a structure which cannot be seen on the paper. What is remarkable about George's behavior is that he is able to manipulate this structure "in his head" exactly as he does in the case of concrete objects. His actions may be compared, for example, to those of a person who speaks about moving furniture in a room and tries to figure out the final outcome. As can be learned from mathematicians' testimonies (Sfard 1994a), what George is able to attain by "drawing in his head" is often achieved by expert practitioners by imagining objects that cannot be actually drawn on paper or even on a computer screen.

As can be seen from this example and from those that follow, object-mediated use of mathematical symbols is the use that gives the VR discourse the leading characteristics of AR discourse: object-mediated mathematical discourse is conducted as if "with an eye to" a certain entity that keeps this discourse in focus and integrated. The expression "in focus" refers to a discourse in which the participant has a clear sense of knowing what his or her utterances are all about. A discourse is "integrated" if there is a general agreement between interlocutors about the focus of the conversation. In AR discourse, thanks to objects-mediation, that is, to the perceptual accessibility of concrete objects to which participants refer, both of these features are attained without particular difficulty. Indeed, when the discourse is about “a table,” there is little danger that the interlocutors would have doubts about the focus of the conversation (and this is true even if the table in question cannot be seen at the moment). Focus and integration are much more difficult to attain in mathematical discourse, in which object-names and symbols bring almost no perceptual hints as to the relevant aspects of the situation — those aspects which should be attended to in the conversation. Thus, for example, young students asked to formulate a question for which the answer could be "$^2/_9$" would often reply with something like the following: "The cake was divided into nine equal parts; children ate seven of them. How many parts were left?"[3] This is a classical example of an out-of-focus response, which shows that "two ninths" does not yet refer to a well-defined mathematical object for the students; rather, it signifies a certain type of situation.

Let me add that the decision to contrast mathematics with AR discourse has also to do with a possible developmental connection between the two. Because of the above-mentioned similarities in structure and in use on the one hand, and in light of the primary, basic character of AR discourse on the other hand, one has good reason to hypothesize that AR discourse may be the earliest link in the long developmental chain from which mathematical discourse eventually emerges. The hypothesis of a developmental connection between AR and VR discourses will be examined in the reminder of this paper with particular attention to mathematical symbols and to the role they play in the construction of mathematical meaning.

This example comes from a study recently completed by Carolyn Kieran and me in Montreal. In our experiment, twelve-year-olds took their first steps in algebra. Our approach was functional, and the learning was massively supported with computer graphics. In the final interview, a boy named George was asked to solve the equation

$$7x+4 = 5x+8.$$

Here is our exchange with George:

G: Well, you could see, it would be like,... Start at 4 and 8, this one would go up 7, hold on, 8 and 7, hold on... no, 4 and 7; 4 and 7 is 11.... they will be equal at 2 or 3 or something like that.
I: How are you getting that 2 or 3?
G: I am just graphing in my head.

FIG. 1. Object-mediated mathematical discourse — a sample.

What is in the symbol and how does it get there?

The Problem

When stating that in mathematics we do not know what we are talking about, Russell certainly did not intend to say that mathematical symbols have no meaning or that mathematicians talk about nothing. Indeed, it seems that his only claim was that, unlike in AR discourse, the participants of the mathematical discourse have no direct access to this special "something" which is supposed to be signified by mathematical expressions. Using the traditional language of theories of meaning, one may say that the question Russell was posing was that of the "referent": mathematical symbols refer to something — but to what?

Let me put the problem in clearer terms. Traditionally, symbol has been understood to be an entity that points to another entity (the referent). Encouraged by Eco (1976), one of the central figures in today's semiotics, we may say that symbol "is a lie" because it stands for something else. This classical dichotomy imposes a series of questions about the nature of the entities to which mathematical symbols are supposed to refer. What is the ontological status of these entities? Where do they come from? How can one get hold of them (or construct them)? These are the most prominent of the issues to be dealt with.

In the former section I was arguing that the words ‘mathematical object’ are only meaningful in certain phrases and have, at best, some metaphorical meaning. As a consequence, I objected to ontological questions such as those asked above. This objection, far from being immediately obvious, is nevertheless an inevitable conclusion from centuries long debates on the nature of the relation between symbol and its referent. Before we continue our deliberations on the object-mediation in mathematical discourse, it would be useful to have a quick look at the way this debate evolved.

Parting from the Referent: A Brief History

What follows is a story of the gradual shift from the objectivist symbol/referent dichotomy to the relativist theories of signification based on penetrating insights into socio-cultural mechanisms of meaning production. Three basic positions on the nature of meaning have been proposed over the centuries. First, there is the classical realist/objectivist viewpoint according to which the referents of mathematical symbols have a real existence of a kind, whereas seeking the objective truth about these abstract entities is the mission of the mathematician. For the followers of Plato and Descartes it was, therefore, only natural to view meaning as independent of, and primary to, symbols which, consequently, were regarded as mere vehicles for forwarding an external semantic "load" from one person to another (cf. Reddy, 1979). Taking the realist position as a point of departure, it was only natural to ask what should come first in the process of learning: the knowledge of mathematical objects or the use of the symbols representing these objects. Should the student ‘have the idea’ of, say, negative numbers, before she can actually talk about them or symbolize them? The problem is a theoretical one, but with a potential for crucially important pedagogical implications. Many would say that, even if not absolutely necessary, this previous knowledge of the object denoted by the new symbol would be as helpful to learners of mathematics as having an experience of a physical object is useful to a child who learns the word signifying this object. The belief that, indeed, the knowledge of the world is helpful in the acquisition of language has been aptly expressed in Bruner's Alerting Hypothesis: "it is immeasurably valuable, in learning a code, to know already what the code stands for" (Bruner, 1983, p. 29). It is only natural that this maxim would be extended by realistically-minded educators to the learning of mathematics.

The realist/objectivist position has been questioned first by the founders of the constructivist movement, who denied the possibility of a God's eye view and transmitted the power of creation from God and nature to the individual human mind; and then by the interactionists, who claimed that "the meaning... is derived from, or arises out of, the social interaction that one has with one's fellows" (Blumer, 1969, p. 2). Both these changes of view were concurrent with a shift of general attention from ontological questions to the mechanisms underlying human experience. The old conceptions of symbolic content as an externally-determined "cargo" carried by the signifier gave way to the vision of meaning as lying "in the eyes of the beholder" or in an inter-personal sphere. From now on, one of the focal issues will be the question to what degree the experience (the knowledge) of mathematical object depends on symbolic representation of this object.

Be the constructivist and the interactionist paradigms as far removed from the objectivist framework as they might, they still have proven to be compatible, at least initially, with Platonism and with theories of meaning that have sustained a clear-cut distinction between symbols and meaning[4]. These theories have tended to admit that meaning, whether received or constructed, is quite independent of symbols and may, in principle, develop prior to the introduction of any kind of notation. This applies, of course, to mathematical objects — this principal manifestation of mathematical meaning. There is therefore no reason why the maxim of 'objects before symbols' should not find its advocates also among constructivists and interactionists. Indeed, many would agree with Thompson's advice that, as teachers, we should better "focus on having students use signs and symbols only when they (students) have something to say through them (symbols)" (Thompson & Sfard, 1994, p. 8).

Further revision of the objectivist doctrine and the advent of semiotics put a question mark upon the clear-cut distinction between the symbol and its referent and eventually led to a re-conceptualization of the issue of the construction of meaning in general, and of mathematical objects in particular. The conception of a sign and its meaning as independent entities was replaced with the claim of an indissoluble unity of the two. This was the message brought almost simultaneously, although quite independently and in different ways, by the founders of semiotics, Saussure (1986) and Peirce (1931-1935)[5]. It was also promoted by the Russian school of thought that originated in the work of Vygotsky (1962, 1987).

The rejection of the classical dualist view expressed itself, above all, in the Saussurian change of terminology. Defining the notion of sign as denoting an indissoluble union of signified and signifier, Saussure exorcised the idea of symbol as an empty container and of an independent referent which fills it with meaning. The emphasis here is at the untenability of the traditional dichotomy of sign and meaning. A similar idea is conveyed by Vygotsky through the metaphor of decomposing water into its components: just like there is no possibility to learn about properties of water by simply studying properties of oxygen and hydrogen, there is no way to understand human conceptual thinking by severing signifieds from signifiers and by their separate investigations (Vygotsky, 1987, p. 45).

This re-conceptualization of the relation between semiotic form and content made it necessary to revise common conceptions about mechanisms underlying production of meaning. The claim that one cannot consider signifieds independently of signifiers implies untenability of the belief in 'meaning before symbols'. It is only natural that Vygotsky (1962) thought of language, and symbols at large, as having constitutive, rather than only representational, role: "Thought is not merely translated in words; it comes into existence through them." (p. 125)[6], he claimed. Approximately at the same time Peirce began to pursue the idea of meaning as originating in the intricate interplay of signifiers. The American semiotician viewed the signification and construction of meaning as an ongoing process in which an <u>interpretant</u> of one sign becomes a <u>representamen</u> of another (the interpretant and representamen are Peirce's terms for two out of his three components of sign; the last element in the triple is object; see Peirce, 1955). Thus, for example, in response to the symbol (representamen) x^2 , one may draw a parabola. The parabola would be an interpretant of the sign. While seeing the graph, another person may say, "This function has no negative values." This utterance is an interpretant of the parabola which, in the context of the new sign (the utterance), functions as a representamen[7] . The motif of such "cyclic," hierarchic signification recurs in works of the French semiotician and psychoanalyst Lacan (1977) who may be viewed as both a successor and a reformer of Saussure. In Lacan's writings, one finds the idea of a sign turning into a signified of another sign. Thus, for instance, in the above example, the symbol x^2, in itself a signifier (of the basic quadratic function), may be viewed as a signified of parabola. In turn, the parabola may become a signified of the expression "the basic quadratic curve." Excellent illustrations of such “chains of signification” were given by Walkerdine (1988) and by Cobb, Gravemeijer, Yackel, McClain and Whitenack (1997).

The notion of meaning as emerging from an interplay of signs was a significant step toward freeing the theory of meaning from the need to consider externally-given, pre-existing referents. The anti-referent position has been explicitly declared in this way or another by almost all contemporary semioticians (cf. Gottdiener, 1995). Thus, for example, French thinker Michel Foucault (1972, 1973) objects to "treating discourses as groups of signs (signifying elements referring to contents or representations)" and prefers to treat them "as practices that systematically form the objects of which they speak" (p. 49). Referring specifically to the theme of madness, he explains that its "object" (mental illness), like all the other objects around which discourses are built,

> was constituted by all that was said in all the statements that named it, divided it up, described it, explained it, traced its developments, indicated its various correlations, judged it, and possibly gave it speech by articulating , in its name, discourses that were to be taken as its own.

For Foucault the conclusion is obvious: "there can be no question of interpreting discourse with a view to writing a history of the referent" (Foucault, 1972, p. 47.).

Dreyfus and Rabinow (1982) translate this position into "practical" advice by stating that "The tendency to think of language in terms both of referents, and of words pointing to objects, must be resisted" (p. 62). If not referents, however, then what? How can one investigate the issue of meaning when, on the one hand, the idea of a referent as a meaning-rendering device has been given up and, on the other hand, the conception of meaning as beginning and ending in the free play of signifiers (as Derrida (1976) and other deconstructivists would have it) does not seem to tell all of the story? One viable alternative to both of these positions may be found in Wittgenstein's famous statement, "the meaning of a word is its *use* in the language " (Wittgenstein, 1953, Remark 43, p. 20). The use, in turn, is not unrestricted and arbitrary. It has a well-defined, although infinitely complex, set of rules which, by determining discursive forms, also determines their meaning.

The assumption that meaning has much to do with linguistic use is central to this paper. It is also important to stress, however, that the position adopted along these pages is much less formal and less restrictive than implied by the equation of "meaning = use in language." While there is full agreement as to the utmost importance of linguistic use, there is also a conviction that from psychological point of view, the issue of meaning cannot be captured through linguistic analysis alone. In such an analysis, the gestalt effect, which seems to be the gist of the *experience of meaningfulness* (and of "having a mathematical object"), is inevitably lost. The experiential facet of meaning includes imagery and emotions, among others. As Lotman observed while referring to the exclusive use of linguistic analysis, "if we put together lots of veal cutlets, we do not obtain a calf. But if we cut up a calf, we obtain lots of veal cutlets" (quoted in Eco, 1990). To say it differently (and in a less "bloody" way!), understanding the parts of a whole, which is the kind of understanding we gain while analyzing use, does not translate automatically into an understanding of the whole. The collection of pieces is not enough to reconstruct the “living creature”, this unique experience of "seeing" a meaning of a symbol.

In the beginning was the word: The role of signifier

In the rest of this paper, I will be grappling with the question what it means to participate in a discourse on mathematical objects and how this participation is made possible by an appropriate use of mathematical symbols. The central thesis which will be expounded and examined along these pages is that mathematical discourse and its objects are mutually constitutive: It is the discursive activity, including its continuous production of symbols, that creates the need for mathematical objects; and these are mathematical objects (or rather the object-mediated use of symbols) that, in turn, influence the discourse and push it into new directions.

Before I start explaining this thesis and examining its implications, I must make some lexical preparations. The vocabulary with the help of which we all use to speak on issues of meaning still brings to mind the old symbol/referent dichotomy and is therefore somehow incompatible with the general spirit of the present dispute. The objectivist approach, which rendered the referent a separate existence and primacy over the symbol, is immortalized in the notions that are still in use. Whether we talk about symbols, signs or representations, we always imply the existence of another entity for which the entity at hand (a mark on paper, a word) is intended to be a discursive replacement. Indeed, each one of these terms invites the follow-up phrase "of something." For instance, a symbol is always a symbol *of something*, as are sign and representation. Thus, if there is room for the word *representation* in the present framework, it is under the condition that we interpret it in a novel way, tacitly agreeing that representations are not necessarily born as such, and more often than not earn their "representational" status only much later, if at all[8].

To be able to tackle the process through which the objects "represented" by the symbols come *retroactively* into being I need a vocabulary which can be trusted to minimize the impact of the objectivist metaphors for meaning. Not having much choice, I decided to use the words sign, signifier and signified in the sense similar to those of Saussure and Lacan. The word 'sign' should be understood as anything experienced as meaningful, whether it is a spoken word, a written symbol, or any other artifact used for communication. The words 'signifier' and 'signified' refer to two inseparable aspects of human relation to signs: the former implies that a sign must have a perceptually accessible form and the latter makes it clear that from the user's point of view there is more to the sign than what meets the eye (or ear). In this paper I will be referring mainly to three types of mathematical signifiers: names (words), algebraic symbols, and graphs.

Having made these lexical clarifications we are now in position to delve into a more detailed analysis of the role of signifiers in the development of discourses on numbers, on functions and on sets. Within our present framework, naming and symbolizing can no longer be considered "baptizing an object" (Wittgenstein, 1953, p. 38) — as merely introducing linguistic pointers to already existing entities. Rather, they must be viewed as crucial components of processes of creation. While generally true, nowhere is this claim more prominent than in VR discourse. Unlike in AR discourse, in which the way toward a new concept (object) may sometimes begin not with the help of symbolic tools but with a visually accessible physical object, in VR discourse name and symbol are the only public means for focusing attention — indeed, for creating the discourse which, in its turn, creates the object that we speak about. Thus, we can metaphorically say that in the process of learning mathematics, the role of discourse in creating the virtual reality of mathematics is no less significant than the role of this reality in creating the discourse.

As may be learned from the history of mathematics, the introduction of new mathematical signifier is rarely preceded by formal definitions. Moreover, when a new symbol is introduced, the mathematicians may be unable to give any description at all of the "object represented by" this symbol. Contrary to what seems to be a quite common conviction, mathematical names and symbols may become a regular part of mathematical discourse at the stage mathematicians have only a very faint notion of their signifieds.

In this paper, I am trying to promote the idea that the introduction of a new structural signifier, and thus the creation of a new discursive focus, can be seen as an act of conception of a new mathematical object. The words "structural signifier" refer to those mathematical symbols that appear in propositions dealing with objects — as opposed to propositions that focus on, say, operations. Thus, for example, while +, -, *, √ and $^{d}/_{dx}$ play the role of operational signifiers in most of their appearances (such as in "In order to find the area of a rectangle, you have to perform the multiplication: *width * length*"), the symbols -3 and *f(x)* would more often than not be used as structural (such as in "-3 < 0" or in "Multiply *f(x)* by 7"). The distinction, therefore, is made with regard to the linguistic uses of the symbols. This means that the suggested categorization of symbols is not absolute: the distinction between structural and operational signifiers is relative to the discursive context in which they are employed and the same symbol may sometimes be used as operational, and sometimes as structural. As was argued elsewhere (e.g., Sfard, 1991; Sfard & Linchevski, 1994), most of the known mathematical symbols display such structural/operational dualism (this dualism has been epitomized in the notion of "procept" introduced by Gray and Tall (1994)). It should be understood, therefore, that while using the adjectives 'structural' and 'operational' with reference to a symbols, I am hinting at the way it is used in the discourse under consideration.

It can be shown that many important historical developments in mathematics occurred due to ontological turns that followed an introduction of structural symbols into a discourse which until now focused on operations. This is certainly what happened when such symbols as -3 and *i* were proposed. But the lives of negative and complex numbers began even earlier than that. These two types of numbers emerged from symbolic operations such as those which can be presented by the expressions 5 - 8 or √-1. These operations led to talks about objects which did not seem to exist; they enabled discursive activities where the interlocutors seemed to be manipulating these non-existent entities. Surprisingly, the discursive manipulations over the "empty symbols" occasionally produced perfectly legitimate objects. (This is what happens when, for example, Cardan's algorithms are applied to such an equation as $x^3 = 15x+4$, producing √-121 as an intermediate result and $x = 4$ as a final solution.) Sometimes, the structural upheaval in discourse would be an almost inevitable result of the inner dynamics of the existing symbolic system. Talk about negative numbers could begin by admitting into mathematics non-standard combinations of existing symbols (e.g., 5-8), which owed their existence to the fact that the rules of operating over the known symbols could be usefully extended. As is customary in mathematics, the expression "5-8" itself could be used both operationally, as denoting an operation, and structurally, as signifying an object (the result of an operation). The fact, however, that the same signifier had to be employed in two seemingly incompatible roles, operational and structural, certainly aggravated the difficulty of *reification* (transition to the structural mode). Indeed, process/object duality defies our perceptually-conceived intuitions. In the perceptual reality, processes and objects are two different things. The questions "But what is [the result of] √-2?" or "How much is √-2?," which we can often hear from students, show that the learners of mathematics may experience exactly this kind of difficulty. No wonder, then, that special symbols were evidently necessary to play the role of objects, as opposed to symbols that signified the operations. The structural turn in discourse was completed and the new types of numbers were generally accepted only after the numerals preceded by minus were introduced to denote "the result" of some subtractions and the letter *i* was proposed by Euler to stand for "the result" of square root extraction from -1.

The history of negative and complex numbers shows how a symbolically-induced ontological change in mathematical discourse may bring about the emergence of new mathematical objects. As another example, let me take the concept of function. The biography of this notion has been analyzed many times in many different ways. Here, it will be dealt with from semiotical point of view[9]. Looking at the earliest events in the history of function through the lens of the theories of meaning summarized above, one immediately notices that this central mathematical concept, like many others, began its life as a symbol, and as quite an empty symbol at that. ("Empty" does not imply "meaningless"; it only says that those who agreed to use the symbol and knew how to do so were, nevertheless, puzzled about the nature of its signified; the signified was supposed to be an object, but they could not figure out the nature of this object.) Let us examine, for example, the first two definitions of function that are known in the history of mathematics presented in Figure 2.

Jean Bernoulli, 1718:
One calls here Function of a variablea quantity composed in any manner whatever of this variable and of constants.
Leonard Euler, 1748:
A function of a variable quantity is an analytical expression composed in any manner from that variable quantity and numbers or constant quantities.
(after Kleiner, 1989)

FIG. 2. Early definitions of function.

In order to understand the point I wish to make here, one has to look *at* the language used in the two definitions rather than *through* this language. This change from looking-through to looking-at may be metaphorically compared to the shift of focus that occurs when one switches from looking through a window to looking at the window (cf. Lave & Wenger, 1991 on the use of tools). The surface features of the phrases used by both Bernoulli and Euler are what indicate the ontological status of function. Both of these phrases contain the same message: they point to certain *symbolic expressions* as *genus proximum* of function. To put it in a simpler language, function is identified with a certain type of algebraic symbol, and not with anything that might be represented by this symbol. This is obviously true for Euler's definition, which explicitly equates function with an *analytical expression*. Some of the readers may remain unconvinced about the accuracy of this statement with regard to Bernoulli's definition because of the use it makes of the term “quantity.” (Ontologically speaking, quantity is not a symbol.) These readers are advised to consider the fact that this quantity is said to be composed of variables and constants. Since the distinction between "variable" and "constant" can only have sense with respect to algebraic symbols, one has good reasons to conclude that, indeed, Bernoulli’s definition is similar in its implicated ontology to that of Euler, even though its message is conveyed in a less direct way.

The emergence of the notion of function has, therefore, much to do with the advent of algebraic symbolism. By the end of the sixteenth century, and due mainly to the work of Vieta and Descartes, "analytical expressions" involving letters, numerals, and operators began deluging mathematical texts in spite of wide-spread doubts about their exact meaning and about the validity of their use (see e.g., Kline, 1980). The doubts, it seems, stemmed precisely from the fact that the introduction of the symbolism preceded a good idea of what the symbols stand for, and about the exact nature of the rules of their use. As a result, the newborn algebra was treated with caution. Newton, who used it quite extensively, described it as the “analysis of bunglers.”

Let me dwell a little longer on the nature of these historical doubts. On the face of it, there is no reason why the symbols themselves could not serve as objects called "functions." This is, evidently, what Euler believed possible when he conferred the title of function on the multifarious algebraic expressions that, in the late sixteenth century, grew out of the mathematical discourse as if by themselves. If his definition were accepted, the transformation of the discourse from purely operational to structural would be accomplished in no time. After all, anything one wants to know about mathematical objects — their properties and the ways in which they can be manipulated — could be easily derived from the former "language games." Indeed, the rules of use imposed themselves upon the users. Why is it, therefore, that mathematicians could not, in fact, relax and acknowledge these new symbols as being the thing itself, not standing for anything else? Why couldn't they do what Hilbert and other formalists tried to do over two hundred years later? What were they looking for when complaining that they were compelled to deal with "empty" symbols?

It is interesting to note that it was Euler himself who questioned his definition of 1748. Seven years after he equated function with "analytic expression," he announced that one is dealing with function whenever "some quantities depend on others in such a way that if the latter are changed that former undergo changes themselves." He went on to formulate a new definition: "If... x denotes a variable quantity then all the quantities which depend on x in any manner whatever, or are determined by it, are called its functions" (Ruthing, 1984, pp. 72-73). The ontology of this new description is dramatically different from that of the old one. This time, there is no mention of symbols. Rather than being a mark on paper, function presents itself now as a disembodied abstract entity, existing independently of its perceptually accessible "avatars". It is enlightening to look at the events which brought about this conceptual turn. They reveal the reason why mathematicians cannot accept the idea that symbols, as such, are the objects they are talking about.

It was his famous dispute with d'Alembert on the possible solutions of the problem of vibrating string (see e.g., Kleiner, 1989) that eventually forced Euler to reconsider his idea of function. While analyzing the possible solutions to the problem, Euler faced a dilemma. There were two basic assumptions that appeared to contradict each other. On the one hand, each state of the vibrating string, thus every continuous line in a plane, was supposed to correspond to a function; on the other hand, there were shapes which could not be described by any (single) "analytical expression." There were only two ways out of this quandary: either some states of the string were not functional, or the definition of function as an analytic expression had to be changed. Euler chose the latter solution and decided that, in order to broaden his conception of function, he must make it independent of any kind of symbolism. (As an aside, let me note that d'Alembert went the other way, that is, he decided to adhere to the idea of function as an expression.) It was, therefore, the attempt to bridge the two different kinds of symbolism — algebraic and graphical — that brought about a conclusion that one needs a symbol-independent entity if one wants to account for the complex relationships between the symbols themselves.

Once again, we are witnessing here the effects of metaphorical projection from AR discourse to VR discourse. In order to account for the isomorphism between symbolic systems, one brings into play a mathematical object which unifies the respective symbols the way that Sigmund Freud, the person, unifies the pictures presented in Figure 3b. It is for a similar reason that teachers and textbook authors feel compelled to talk to children about function when they try to account for symbolic equivalencies, such as $2+x+3x = 2+4x$ or $3(x+2) = 3x+6$. They need these intangible objects in order to answer the question "What is it that remains the same when the symbols themselves change?" The question, in fact, could be reformulated: "What is it that makes two symbolic expressions such as $3(x+2)$ and $3x+6$ fully interchangeable in mathematical discourse?" It is this discursive interchangeability, therefore, that makes the introduction of mathematical objects indispensable[10].

1.

x	*y*
-1	1
0	0
1	1
2	4
3	9
4	16

2. $y = x^2$

3.

```
read(x);
x:=x*x;
write('f(x)=', x);
```

4.

a. VR discourse: All the above symbols are representations of the same function: the basic quadratic function.

b. AR discourse: All the above pictures are representations of the same person, Sigmund Freud.

FIG. 3. Symbols as representations in VR discourse and in AR discourse.

Let me finish with a general remark on possible reasons for the introduction of structural symbolism. Whether we are talking about numbers or functions, these symbols played a crucial role in turning the discourse onto itself and making its former practices an object of reflection. In all the cases discussed above, the new signifiers did fulfill the mission for which they were meant: they helped to thematize the former discursive processes and turn them into the focus the mathematical conversation. In this way, they catalyzed a new discursive formation and raised the discourse to a meta-level where mathematical processes previously merely executed became a reified object of study. At this higher level, the process could be reflected upon and combined into ever more complex units. Because the discourse itself was new, however, the signifieds of the new signifiers were yet to be built. The process of construction would often lead into directions unforeseen by those who introduced the signifier in the first place. It is in this sense, therefore, that the phrase "signifier before signified" is being used in this paper. Although meaning construction is an ongoing process with no clear beginning and no end, there are, nevertheless, certain events which influence the course of further development in a crucial way and may, therefore, count as beginnings of a "new chapter." These are events which re-direct the discourse and bring a new discursive focus. Introduction of a new signifier is one of these events. In the long run, it is bound to bring about the development of a new signified. The exact course of this development, especially as it takes place in the process of learning mathematics, is the topic of the next section.

Pump mechanism: How signified is built out of signifier

Notwithstanding the generally-accepted view of learning as construction, there is a substantial difference between historical and individual symbolizing. While mathematicians were the first inventors of mathematical symbols and the pioneer builders of the structural discourse, today's student is usually thrown straight into a pre-determined mathematical conversation, governed by a set of ready-made rules. In this discourse, many symbols are readily available and their use is structural from the moment of their introduction. In other words, the structural interpretation of symbols is imposed by the way in which the symbols are used — the ontological assumptions are implicit in the expert-participants' language. In most cases, the act of introduction is performed by a teacher. No special effort has to be made by the latter in order to evoke the metaphor of, say, -3 as an object. This metaphor is inherent in the language the teacher uses. The way in which -3 is incorporated into mathematical discourse puts it naturally into one category with the numbers the student already has used before. The realization of this fact may well be the beginning of the dialectic process of object construction which is going to be carried out from now on. This process will be fueled and shaped by social interactions all along the way. Aside, perhaps, from the extremely rare cases of exceptional creativity (and the radical interactionists are likely to argue against the very possibility of this exception), eventual transition to object-mediated use of symbols is only made possible by students' ongoing exchange with others. The "others" may be teachers, peers, parents or the authors of the texts the student is reading.

Introduction of negative numbers, as it is usually done in schools, is a good illustration. When the negative numbers are first mentioned in the process of teaching, they are usually defined by a reference to some symbols. Figure 4 presents a sample of an introduction to signed numbers taken from a Hebrew textbook. If the look-at-the-language approach is assumed, one immediately realizes that negative numbers are presented here as new symbols. Indeed, they are introduced as names for points on an extended number line (the points themselves are not the numbers!). In view of the previous claims on the use of symbolism and object-mediation, this approach, even if seemingly undesirable, could not, in fact, be easily modified.

Let's choose a point on a straight line and name it "zero." Let's choose a segment and call it "the unit of length." Let's place the unit head-to-tail repeatedly on the line to the right of the point "zero." The points made this way will be denoted by 1, 2, 3 and so on (or +1, +2, +3, ...). To the left of the point "zero," we put the unit segment head-to-tail again and denote the points obtained in this way with *numbers* -1, -2, -3,... The set of *numbers* created in this way is called the set of negative numbers.

From Mashler, M. (1976). Algebra for seventh grade. Hakibutz Hameuhad. (Translation from Hebrew and italics — A.S.)

FIG. 4. Introduction of negative numbers in a Hebrew textbook.

Indeed, as strange as it may seem, this state of affairs is not but one possibility out of many. In view of the basic assumptions about meaning adopted in this study, at certain levels of VR discourse this order of things — signifiers before signified — is the only possibility. If we agree with Wittgenstein and Foucault that discourse creates its own objects and that discursive relations and uses of symbols are perhaps most crucial components of what we previously called the signified, then there are no mathematical objects without designated signifiers; or, to put it differently, before a symbol enters the language and becomes a full-fledged element of the discourse, there is simply no object to talk about.

All of this means that any discourse, and mathematical discourse in particular, suffers from an inherent circularity. Signifieds can only be built through discursive use of the signifiers, but at the same time, the existence of these signifieds is a prerequisite for the meaningful use of the signifiers.This circularity, seemingly a serious trap for the prospective participants of the discourse, is in fact the driving force behind its incessant growth. This is what fuels the process of co-emergence in which the new discursive practices and the new signified spur each other's development. In this process, the discursive forms and the meaning, as practiced and experienced by interlocutors, are like two legs which make moving forward possible due to the fact that they are never in exactly the same place, and at any given time one of them is ahead of the other.

In this section I will take a closer look at this intricate, somehow circular, process of object construction which follows introduction of a new structural signifier. This process consists, basically, of two stages. These stages differ from each other by the type of use that is made of the symbol under consideration. The first hint of the distinction I have in mind may be found in the study carried out early in this century by two psychologists, Woodrow and Lowell (1916), and repeated later with similar results by many others (Brown & Berko, 1960; Ervin, 1961; Palermo, 1963; Palermo & Jenkins, 1963). The researchers tested the free associations arising in children and adults in reaction to simple everyday words. They found an interesting age-related difference: the children tended to make associations with words which would usually follow the stimulus word in a sentence; by contrast, the majority of adults thought of words with a related or opposite meaning. Thus, for example, the young subjects would follow "table" with "eat," and "deep" with "hole," while adults associated "table" with "chair," and "deep" with "shallow." One may say that the words presented to the children activated in them ready-made linguistic templates in which the stimulus word and the response word represented different parts of speech and together formed a meaningful phrase. Adults, on the other hand, associated words belonging to the same semantic categories, words that fit the same "slots" within propositional structures. To put it differently, the children's associations brought syntactic complements of the stimulus words while grown-ups' associations led to syntactic equivalents. Using Sausserian language, one may say that the children associated *syntagmatically*, whereas the adults associated *paradigmatically*[11] .

This interesting result indicates a substantial difference between the way children and adults control the use of words in language. Children's use of words may be termed *templates-driven*, whereas that of the adults is mediated by their former knowledge and by their ability to classify words into semantic categories. This difference is the basis for my characterization of the two stages of meaning production in general, and of the development of object-mediated use in particular[12].

In this paper, the focus is on structural signs. According to what has been said before, an act of introducing a new structural name or a new symbol cannot be viewed as a "finishing touch" in the creation of a new mathematical object. On the contrary, this operation is often a beginning rather than an end of the process. As mentioned before, when the symbol of fraction, say $^7/_9$, is presented to children for the first time, one can hardly expect the young students to have the underlying abstract object, that is to view the symbol as but an 'avatar' of another, intangible, entity. In spite of this, the students do not remain idle until the the new mathematical object is fully formed. They do use the new signifier even though they may still only be able to see the symbol and not what should be seen *through* the symbol. However, the way they use the symbol at this early stage is quite different from the way they will do so later, when the symbol develops in their minds as a fully-fledged, signifier-signified couple. It will be the central thesis of this paper that *when structural symbols (symbols that refer to objects) are introduced for the first time, their use is mainly templates-driven,* *and it is only some time later that it becomes object-mediated.* Moreover, I will claim that this order of things, far from being undesirable, is inherent in the logic of the process of symbolization. If this sounds somehow too strong, let me add that in certain cases the period of templates-driven use may be so brief that it goes virtually unnoticed[13]. In some other instances object-mediation would be very slow to develop, and in most extreme cases it would remain out of reach forever.

Of particular interest to us in the present discussion is the phase in which discoursive practices outgrow the meaning, which takes place during the period immediately following the introduction of a new signifier. This act of introduction creates a “semantic space” yet to be filled with meaning. The signifier enters the language game before the rules of this game have been sufficiently specified and before the signifier have acquired the power of evoking a familiar experiential resonance. “Filling in” the semantic space occurs in a gradual manner. At the first stage, the symbol starts establishing its place within the discourse in the only way possible — by following in the footsteps of some other, already well-established, element of this discourse. If sneaking into old discursive "slots" is an opening move, slipping out of these slots, due to some basic misfit, marks the beginning of a new stage — a stage at which new discursive forms are generated. These new forms are what eventually defines the symbol's unique use and, thus, its particular meaning. The meaning of mathematical symbols may come from within the mathematical discourse itself — from gradual realization of its relations with other mathematical signs. Such an intra-discursive creation of meaning is known only too well to today's mathematicians. However, the meaning of a symbol may often originate in the inter-discursive space, in the process of carrying a symbol over from one discourse to another. In the latter case, the links with the AR discourse are the most significant. Endogenous mathematical signification — creation of new signifieds that occur exclusively within the closed system of mathematical discourse — may be a common practice since the advent of the formalist movement, but for many centuries the links between the AR and VR discourses were a central source of mathematical meaning. They may still be indispensable in today's mathematics classroom. As will be shown in the next sections, the mechanisms of reification (Sfard, 1991, 1992) and of metaphorical projection (Johnson, 1987; Lakoff, 1987; Sfard, 1997) underlie the creation of mathematical chains of signification, both those internal to mathematical discourse and those that link the latter to AR discourse.

This whole process, drawn here in very rough strokes, brings to mind the mechanism of a pump. Introduction of a symbol is like lifting the piston in that it creates a new semantic space. The gradual emergence of object-mediation is analogous to the procedure of filling in the space thus created. Based on this metaphor, I will talk about the process of symbolization as governed by the pump mechanism. It is through an intermittent creation of a space "hungry" for new objects and through its subsequent replenishment with new discursive forms and relations that the participants of mathematical discourse steadily expand its limits. In the next section I will focus on the phase of "lifting the piston" -- the phase of creating semantic space by the introduction of new structural signifiers. This is the stage in which the use of a new symbol is governed by old rules. The subsequent stage in which the new sign gets life of its own and becomes an integrated signifier-signified unit will be presented in the section that follows. Theoretical reflections on both of these phases will be illustrated, among others, with vignettes taken from my recent studies.

Meaning From Symbols, Act One: Creating Object Space

Templates-Driven Use

Summing up what has been said up to this point, the VR discourse of mathematics may be viewed as an autopoietic system, which is continually self-producing[14]. The discursive practices and mathematical objects have been presented as being mutually constitutive and as being in a constant dialectic process of co-emergence. As is always the case with autopoietic systems, by adopting this model we doom ourselves to the dilemma: How does the ongoing process of co-emergence begin? According to what has been said, structural symbols cannot become fully meaningful before they are used; however, how can one use a symbol before it becomes meaningful?[15] On the face of it, reducing meaning to use, as Wittgenstein suggested, may help us out of the entanglement. In the reductionist case, in which there is no longer any use-meaning dichotomy, the question of "What comes first?" seems to lose its grounds. However, the problem still exists except that this time it is translated into the question of "How does old use produce new use?"

That new uses indeed originate in old uses there can be no question. To illustrate this claim, let us help ourselves with thought experiment.

Thought experiment. You have never heard the word *krasnal* before, but you have just read the sentence:

A krasnal woke up and got up from his bed.

Does the sentence tell you what the krasnal means? Certainly not. However, the fact that you do not know the meaning of the word does not leave you unable to make certain uses of this word. Indeed, after hearing only one accidental sentence about krasnals, you are certainly able to take some steps toward incorporating it into a discourse. Even though no one told you what krasnals are, you may be expected to be able to distinguish between meaningful and absurd sentences about krasnals, and even say a few things about krasnals which you believe must be true. Did you shrug at my use of "what" rather than "who" in this last sentence? Well, this is the first evidence of your having recourse to certain well-established propositional templates. You expect "krasnal" to point to a person rather than to a thing, thus, for you the use of the word "krasnal" along with the word "what" is incorrect. Let us test it further.

Which of the following propositions seem to you to be correct and meaningful sentences about krasnals, and which of them do not?

Yesterday, a krasnal went to a supermarket.

A krasnal was divided by three and then squared.

Some of the krasnals were cheerful, some of them sang.

This krasnal is raised by public subscription.

A krasnal begins at 5:30 pm.

This krasnal is younger than this one.

All these expressions are syntactically-correct propositions, but only some of them seem meaningful (possibly true). Now, can you complete the following sentence in a meaningful way?

Krasnal A is wise, whereas krasnal B is.....

Finally, try to construct a possibly meaningful sentence about krasnals yourself. Build one you believe cannot be meaningful.

The example we have just seen shows that the way out of the autopoietic circle leads through old habits — through language games which are already well known and deeply rooted. Indeed, it is only natural that in order to circumvent the dilemma of having to use new words before we are aware of their unique uses, we resort to uses with which we are already familiar. We do it by putting the new words and symbols into slots of well-known, remembered-by-heart, propositional templates (e.g., "___ is younger than ____",

"____ is cheerful") into which they seem to fit. (It is interesting to explore the sources of our intimations about this linguistic fit. This will be done in a moment.) This is why I called this early kind of use "templates-driven"[16] .

Let me immediately follow this decision about terminology with the following disclaimer: "templates-driven" does not mean meaningless. Rather, we are talking here about a certain special, even if not most effective, type of meaning. (Indeed, since, for pragmatic purposes, meaning can be equated to use and since we are talking here about a kind of reasonable use, it clearly follows that we are also talking about a kind of meaning.) As was shown in the thought experiment above, this is the kind of meaning which we may acquire through an exposure to just one sentence containing the new name. It expresses itself in our ability to make some limited linguistic use of the new signifier. This use is characterized by rigidity. The new signifier does not have existence of its own and it can only appear within certain ready-made phrases. Often, we use and complete such phrases mechanically just as we complete a familiar melodic piece after hearing its first notes. There are no general rules for the use, only a number of concrete cases. Thus, at this point the signifier certainly does not have a well-developed signified. I am tempted to paraphrase Derrida's famous maxim and say that at the stage of templates-driven use, the signifier is the thing itself[17].

The ability to make use of symbols that were never seen before was observed many times by my colleagues and me in studies with children who were not yet acquainted with algebra but, nevertheless, displayed a certain intuitive understanding of algebraic symbolism (see e.g., Sfard & Linchevski, 1994). At a further stage, when faced for the first time with an expression $2x+3x=$, many students would spontaneously complete it to the proposition $2x+3x=5x$. As reported by Demby (1994), students usually explain their decision by saying, "Two apples and three apples make five apples." Clearly, the new symbol, x, was substituted here instead of "apples" and thus x took the role of a label rather than of a number. This tendency to view algebraic variables as fitting label slots in non-mathematical discourse may account for a frequently observed error of completing such expressions as $2x+3$ to $2x+3=5x$. This error was observed many times in the study Carolyn Kieran and I carried out in Montreal. We conjectured that the mistaken completion may have been due to the underlying reliance on the template of "Two ____ and three more make five ____ " (e.g., Two apples and three more make five apples). The plausibility of this explanation was then reinforced by the fact that when the multiplication sign was explicitly written in the expression ($2\cdot x+3$), the error usually disappeared. Indeed, the multiplication sign re-directs the student to a different discourse and associates the expression with a cluster of mathematical, rather than everyday, templates. In the arithmetical templates, a slot on either side of an operator can only be filled with a number. Thus, once the multiplication sign appeared, it became clear that the *x* is a replacement for a number and that 2*x*+3 should be used according to rules differing from those that had been initially assumed.

Interestingly, at a later stage in the study, our twelve-year-old Montreal students who were just introduced to algebra provided an explicit confirmation of this interpretation in a classroom discussion that developed around the expression “15000-300*w*.” This expression was constructed by the children and was supposed to present one person's dwindling savings as a function of time measured in weeks (thus the letter *w*). A difference of opinions developed when one of the students requested inclusion of 300*w* in brackets, hinting that otherwise, in the absence of the multiplication sign, the *w* might be interpreted as "weeks" rather than as "number of weeks." The excerpt in Figure 5 speaks for itself and requires no comment. I added italics in order to stress those students' utterances which were most revealing.

[95] Teacher: Would anyone do anything differently? Martha?
[96] Martha: I'd do 15 thousand minus brackets 300 and number of weeks.
......
[100] Teacher: ... All right. Do we need brackets around this?
[101] A student: No.
[102] Teacher: Why not?
......
[104] Simon: Yes, you do, *because you have to know that there's an operation. A person, now, he'll probably think 300 weeks, not 300 times weeks.*
[105] Teacher: OK, anyone who now knows algebra will know there is an operation.
....
[110] Simon: Well, how do you know...
[111] Martha (?): So, do you need the brackets or not?
[112] Simon: Maybe he is trying to say 300 weeks.
[113] Stephanie: Yes...

FIG. 5. A classroom discussion on the need of brackets in the expression “15000-300*w*.”

Later in our course, we saw how the letter-as-a-label use hindered the understanding of the rule 2*x*+3*x*=5x as a special case of distributive law application. Obviously, one does not "take a common divisor out of brackets" when adding apples! Even the teacher seemed startled when we first mentioned the connection between the formula and the rule.

The fact that the introduction of a multiplication sign brought about such a radical change in the further use of the formula brings into full relief another important feature of the templates-driven use. *Templates come in clusters*. As we have seen in the thought experiment, certain uses of words go along with some other uses, yet clearly disagree with still others. This phenomenon underlies our ability to judge meaningfulness of the propositions about krasnals. We could see how the very first use of the word directed us to a particular kind of discourse — to a cluster of templates which seemed to come together with the one underlying the original proposition. To put it more simply, the very first use already *framed* the discourse and greatly delineated the set of possible linguistic applications.

Co-occurrence of templates is a matter of former experience and habit. The bond between the co-occuring templates is not necessarily that of logical dependence. The templates in a given cluster have been brought together in former uses, where their slots were filled with well-established signs. (It seems that certain prototypical uses are particularly forceful in this respect. In the case of discourse about numbers, all types of meaningful propositions about, say, a prototypical number 5 will tend to cluster together also when the signifier “5” is replaced with a new symbol.) Templates cumulated around each sign, and as time went by the clusters thus formed acquired a cohesion which still keeps them together when a new signifier is substituted for the old. This helps to account for the fact that mathematicians knew how to manipulate such symbols as –3 or √-1 even before the negative and complex numbers entered their discourse as fully-fledged mathematical objects.

To give yet another example, let me recount a very restricted informal experiment that I recently carried out among my colleagues. I approached them with the following sentence:

$$a \text{ and } b \text{ are pexons and } a+b=c.$$

"Pexon" is a word I invented myself so, naturally, none of my respondents could know any more about it than was available through this one sentence. Nevertheless, when asked to decide among other propositions containing the word “pexon,” which could, under certain circumstances, be true or which could only be meaningless, they had their answers and none of them claimed that the task was impossible. First and foremost, nobody seemed to doubt the fact that pexons belong to a mathematical discourse and that they signify an object (rather than, say, an operation or a relation). Moreover, it was evident that most respondents associated it with algebraic or maybe even with arithmetic templates. Pexons were expected to appear within propositions that contain operations of all kinds, even operations that were not explicitly mentioned in the original sentence ("John often makes mistakes when multiplying pexons"). Similarly, it did not seem impossible that pexons might be combined with the inequality symbol ("$a > b$"). On the other hand, and in contrast to krasnals, pexons appeared unlikely to go along with adjectives such as "wise" or "cheerful" or with verbs such as "sing" or "type." Notably, all the respondents univocally claimed that the proposition, "If *a* and *b* are pexons then $a+b$ is a pexon," must be true. A similar conviction may well be the source of the idea of negative number. If the first two slots in a template of the form "___ - ____ = ____" are filled with numbers, then the sign in the last slot must also be a number. When negative numbers are introduced as mere symbols (historically, as signifiers that replace expressions such as 5-8; in today's classroom as "names of points" on the number line), it is their placement within arithmetic templates that frames them as numbers (conceived as abstract objects).

As can be seen from the examples already given, the old templates for new uses may come from many different kinds of discourse. Thus, for example, the word "function" appeared initially in the context of computational processes on the one hand (as we have seen, Bernoulli and Euler associated it with variable quantities), and in relation to geometric shapes on the other hand (Leibnitz applied it in the context of curves in the plane; compare Kleiner, 1989; Sfard, 1991). This multiplicity of discursive sources will play a key role at the later stage. As a result of a crossbreeding between discourses, a new, unique kind of discourse will eventually emerge. It will include language games which only the new signifiers can play. Sometimes, the discursive crossbreeding will be a relatively straightforward cumulative process, with an immediate acceptance of the new transplant and no adverse side effects. Other times, when there is a clash between the uses within different discourses (as happened many times in the history of function, before the geometric and the algebraic discourses agreed to combine), it will require a total restructuring of all the discourses involved.

Another important aspect to stress is that the new uses may be endogenous — coming from purely mathematical templates (e.g., the uses of a variable originating in arithmetic discourse); and they may be exogenous — formed by habits acquired outside mathematical discourse (e.g., templates within which a letter is a label). Clearly, wherever new uses have their roots, the mechanism at work is that of a metaphorical projection (Johnson, 1987; Lakoff & Johnson, 1980; Sfard, 1994a, 1997). The family resemblance between AR and VR discourses is the recurring theme in this paper. The virtual reality of mathematics that emerges from VR discourse is made in the image of actual reality. In light of the above discussion, the reason for this seems to be clear. AR discourse, due to the fact that it certainly was here before all other kinds of discourse, seems to be the primary source of all the templates we use and of all our linguistic habits and, therefore, may be regarded as a great grandmother of all the other discourses.

Let me now make a summarizing comment. Throughout this section, I have been describing and analyzing processes that take place following the introduction of a new mathematical signifier. I tried to make my point with a fairly artificial example — that of the word krasnal, the meaning of which has been significantly framed by a single sentence. In this final remark, I wish to stress the fact that the artificiality of the example only fortifies its message. Unlike in a real learning situation, the sentence about krasnal came as if out of nowhere, completely detached from the ongoing discourse, and thus almost entirely context-free. Even so, it managed to evoke responses which show that already in this very limited and uninformative appearance it did not remain devoid of any meaning. Let us now consider the fact that real learning is never a matter of one isolated sentence, and whatever new symbols we encounter, they usually appear within the context of some ongoing discourse. In this situation, the mechanism that proved quite effective in the least favorable circumstances may be expected to become truly powerful. After all, this is the mechanism which underlies children's learning of their mother tongue. Let me also stress that when I am talking about context, I mean more than purely linguistic elements of the discourse. The general circumstances in which the new signifier appears may play a crucial role in the process of meaning construction. Thus, for example, when the term color is encountered within the context of a physics lesson, it is understood differently than when it is introduced in, say, an art lesson. Except for the dramatic difference in the immediate treatment of the concept, these two situations provide non-comparable sets of expectations as to the possible uses of the notion. Moreover, the word "color" itself, when introduced as a scientific concept (in the sense of Vygotsky), is already laden with meaning constructed through its use in everyday discourse. When it comes to construction of a new meaning within new discursive context, this former meaning is both a prop and a hurdle. The same mechanism, through which contexts shape new meaning, helps overcome polysemy, which is so ubiquitous in mathematics[18]. To sum up, if a single sentence proves so powerful as a "meaning activator," the effectiveness of a rich, thoughtfully-engineered discursive context can hardly be overestimated.

Before closing this section, let me raise again the issue of the social nature of the processes that have been presented. The mechanism of templates-driven meaning construction is only effective because it is accompanied by an ongoing discursive interaction. It is through this interaction that public signifiers are turned into private signs. Conversational feedback plays a central role in preparing discursive and experiential background for the introduction of the sign. It is essential in the act of bringing the signifier to the learner, and it remains crucially important during the subsequent meaning-building. In particular, the templates-driven use becomes increasingly successful due to its being regulated by discursive negotiations.

Expectations and Verifications

One consequence of the things said in the last section is that the very first use of a new signifier frames the discourse, that is, has the power of directing us toward certain uses rather than toward other uses. Together with old templates come old uses, old meanings. The new and the old are now linked together for better and for worse. The effects of the permanent link between the old and the new will be scrutinized in this section.

We may say that we are dealing here with the issue of *expectations* and *verifications*. With the first appearance of a new signifier, certain metaphors come into play and some expectations as to the nature of its signified are conceived. From then on, we will be testing the expectations, sometimes finding that they were justified and other times proving that they were untenable. As will be shown in the present section, the expectations may come from the very first use of the new signifier, or they may arise from associations evoked by the signifier itself. In any case, the mechanism of metaphorical projection from familiar to unknown is at work.

In her recent book devoted to framing in discourse, Tannen (1993) emphasizes the centrality of expectations in shaping both our perception and our abstract thinking. As she explains, talk about expectations deals, in fact, with the well-known truth that old knowledge shapes new knowledge.

> I have been struck lately by the recurrence of a single theme in a wide variety of contexts: the power of expectation.... The emphasis on expectation seems to corroborate a nearly self-evident truth: in order to function in the world, people cannot treat each new person, object, or event as unique and separate. The only way we can make sense of the world is to see the connections between things, and between present things and

> things we have experienced before or heard about. These vital connections are learned as we grow up and live in a given culture. As soon as we measure a new perception against what we know of the world from prior experience, we are dealing with expectations. (pp. 14-15)

Expectations are known also as *prejudgements*, *prejudices*, *intimations* or *intuitions*. The motif of meaning constructed through the to-and-fro movement between what we expect and what we find goes back to Heidegger and Gadamer, on the one hand, and to Bartlett, Piaget and Vygotsky on the other hand. In the immediate context of mathematics, it became known as an issue of conjecturing vs. proving/refuting (Lakatos, 1976; Lampert, 1990). In the present paper the focus is on linguistically-induced expectations[19] .

Ontological framing and reification. If I do not wish to end up writing a book, I should refrain from discussing of the possible types of expectations and framing. Indeed, if framing (or expecting) is understood to be directing toward particular discursive formations and hinting at the statements that may be made about the signifier, then the issue is much too rich and complex to be treated in passing. Different discourses overlap and interact with each other in such intricate ways that any attempt to isolate and categorize them is doomed to failure. (Compare Wittgenstein's reflections on language games.) Moreover, any sign belongs simultaneously to many different discourses, and these discourses, in turn, may be classified and stratified according to such diverse criteria as theme, level (e.g., one may distinguish between object-level and meta-level), ontological status of the focal notions, etc. Let me only tackle this last type of framing, since it is of particular importance for the present discussion. In the following paragraph, I will elaborate on the theme which has earlier been dealt with only in passing — that of the "ontological upheaval" caused by an appearance of a new structural signifier.

It has already been said that, when it comes to mathematical objects, introduction of a new name should be viewed as an act of conception rather than of baptism. Indeed, naming plays a most crucial role in the process of reification. Introduction of nouns into those places in which, until now, people had only been talking about processes (e.g., counting or subtracting 8 from 5) re-focuses the discourse. If there was nothing but a certain process up to this point, now the attention may split between the process and its hitherto nonexistent product (a natural number, a negative number). This ontological shift from an operational to a structural focus is well felt, for example, in the transition from the expression "These things cost five dollars" to the expression "The cost of these things is five dollars." It is echoed in the more abstract example — the transition from "The class numbers twenty students" to "The number of students in the class is twenty." Since nouns fit those template slots that are reserved for objects (as opposed to verbs or adjectives that refer speakers and listeners to different ontological categories), it is expected that the new signified will have the properties of an object: existence, permanence, manipulability. The anticipation of these features is reinforced repeatedly by the way in which the new name is used within language. The templates activated by nouns are direct descendants of templates coming from AR discourse, in which, more often than not, nouns are names of perceptually-accessible material objects. As has been mentioned before, the effect of transferring templates from discourse to discourse is known as metaphor. We may say, therefore, that what we call "mathematical objects" are metaphors resulting from certain linguistic transplants. As may have been already learned from our discussion about function, introduction of a new mathematical symbol is often enough to spur an ontological shift in the discourse — and to bring about reification. In the case of function, focusing the discourse on "analytical expression" was a decisive step in reification of computational processes which had been in use for many centuries. Introduction of the name "function" gave a further push in the direction of the structural approach.

To summarize, templates we use bring with them ontological messages. If we write a new name or a new symbol in the slot reserved for objects — the new signifier will eventually spur an emergence of a new mathematical object. The history of negative and complex numbers, of functions and of groups, may certainly be read this way.

<u>Expectations that work — building new uses for a new signifier.</u> Let me now turn to an example which will show the dialectic process of expecting and verifying in an action. The episode is taken from my recent study in which a mathematically-precocious 14-year-old student — let us call him Dan — learned a number of new mathematical notions.

The aim of the experiment was to try to understand more clearly the discursive construction of mathematical objects and, more specifically, to expose the linguistic elements of this mechanism. I wanted to see the "wheels of the symbolism" turning on their own. To put it differently, I strove to see how much could be attained by a formal introduction of symbols, unsupported by a meaningful context. This is why I created a learning situation, which by constructivist standards (in fact, any standards), must appear extremely "unfriendly." I created teaching material in which new symbols were introduced within a sterile context of formal manipulations. Examples of operations that could be performed on these symbols were the only available source of their meaning. In this experiment, therefore, the new concept was presented to the student as purely artificial in the Vygotskian sense, that is as one that grows neither from a network of already-constructed concepts nor from its earlier "spontaneous" version[20]. Thus, while solving the problems that were presented to him after a brief exposure to the examples, Dan could only rely on deductive reasoning and on his linguistic associations. In this clinically-sterile setting, I had hoped to be able to find out how much could be achieved through "symbol games" alone, unsupported by links to previous knowledge or by relation to student's needs. It was Dan's job to build these links for himself, while it was my job as a researcher to watch the processes through which these links were constructed — if they were constructed at all.[21]

The first of a series of new mathematical notions to which Dan was exposed in the course of the experiment is presented in Figure 6. Since the intention of the study was to observe an intra-mathematical production of meaning, the whole process began with a formal introduction of a new signifier. In the experiment, I acted both as an instructor and a researcher. During ten one-hour-long meetings, I observed Dan closely while he worked his way toward meaning.

During the present meeting we will define addition and multiplication between *pairs of whole numbers*. Here are a few examples:

a. (2,3)·(5,4) = (10, 12) b. (11,2)·(5,6) = (55,12)

c. (3,4)+(2,5) = (23, 20) d. (5,2)+(1,7) = (37, 14)

1. Complete:

(a,b)·(c,d) =

(a,b)+(c,d) =

2. Compute:

a. (1,3)·(2,5) b. (5,1)+(2,3) c. (3,5)+(7,5)

d. (2,15)·(10,3) e. (8,3)+(0,5) f. (7,8)+(3,12)

g. (5,4) : (1,2)   h. (8,15) : (2,3)   i. (11,9)-(5,3)

3. Complete:

(a,b):(c,d) =

(a,b)-(c,d) =

FIG. 6. Introducing calculus of whole-number pairs.

The new signifiers introduced during the first meeting were pairs of whole numbers which could be multiplied and added in certain well-defined ways[22]. A careful reader will immediately recognize the pairs as another "representation" of rational numbers. For Dan, however, the isomorphism with rational numbers remained unnoticed until the third meeting. Therefore, for more than two hours he acted as a *tabula rasa* as far as the new signifiers were concerned. It is important to stress that during the first two meetings I refrained from referring to the pairs as "numbers," lest this particular name frame the discourse in ways that might distort the processes I wished to observe. I wanted Dan to arrive at certain conclusions totally on his own and not just because some particular kinds of behavior could be expected from objects called numbers. The dialogue between Dan and me was in Hebrew.

After Dan discovered the general formulas for addition and multiplication (problem 1 in Figure 6) and applied them to a number of concrete cases (items *a* -- *f* in problem 2), he unexpectedly encountered the operation of division (*g*) which had not been introduced to him, thus far. The conversation that followed is presented in Figure 7.

[1] D: ... I have a problem here.
[2] A: ?
[3] D: Am I supposed to try now?
[4] A: Do what you can. Write this down [points to item 2*g*], read it.
[5] D: OK, five coma two divided by... I think I will do the same [as in the case of multiplication], only I will change to division instead of multiplication, 'cause the symbol of operation here...
[6] A: What do you mean? What are you going to do?
[7] D: Five divided by one, coma, four divided by two, equals five coma two [writes: (5,4):(1,2) = ($^5/_1$, $^4/_2$) = (5, 2)].
[8] A: OK, now I would like you to explain why you did what you did.
[9] D: My reason... In principle, I wouldn't... It is only because you gave me these operations... In fact, I wanted to remark already in the beginning that you shouldn't have used the symbols of addition and

multiplication 'cause it is confusing.. . 'cause here [points to the inside of brackets] we use multiplication as one operation, and here in the equality [points to the multiplication sign appearing outside the brackets, between two pairs] we used it for a different operation. But since you gave me this in that form, I tried to solve it according to what I know.. that division is an inverse of multiplication. So I just did reverse operations...

FIG. 7. Dan defines division between whole-number pairs.

As can be seen, Dan did not have much difficulty deciding how division should be performed. Moreover, he was also very eloquent about the reasons for his decisions. In [9], after criticizing the teacher (me) for using the familiar multiplication sign to denote a non-standard kind of operation (between pairs), he stated that this was that very sign which had made him act the way he did. ("Since you gave me this in that form [with multiplication sign], I tried to solve it according to what I know.. that division is an inverse of multiplication.") In an exchange that took place a few minutes later, after Dan had successfully verified his result by multiplying (1,2) and (5, 2), he explicitly confirmed the role of signifier-induced expectations (Figure 8, utterances [25] and [27] - [29]).

[24] A: Listen, I defined the addition and the multiplication as I wanted. I had my reasons to do it the way I did, but I will keep them to myself for now. The question is... could you do the same when defining the division? Were you free to define it as you wanted?

[25] D: No, I was restricted by my associations.

[26] A: What do you mean? What kind of associations?

[27] D: That this sign is a multiplication...

[28] A: And this one is division? ...

[29] D: Yes, and for all I know, they are related.

FIG. 8. Dan explains why he defined division the way he did.

It is noteworthy that Dan's decision was grounded exclusively in linguistically-evoked expectations. The appearance of signs with some previous meaning was the only reason for the way in which he chose to broaden the use of the new signifiers. Rather than stick to deduction (which, in this case, would leave him empty-handed because of the insufficiency of the information at hand), he decided to rely on intuition and analogy.

What Dan created constituted a consistent whole. It was now up to me as a teacher to confirm his interpretation or try to change it. The role of the social aspect in the process of sign-building stands in full relief again. My instructional interaction with Dan was an interplay between Dan's individual constructions and my regulatory interventions.

Expectations that do not work. After demonstrating the strength of linguistically-driven expectations, let me turn to the obvious pitfalls of projecting the old to the new. First, by activating old uses, the new signifiers may lead to beliefs that obstruct creation of new meaning, and create interdiscursive contradictions. Second, the expectations may be superficial and fuzzy, so that their implications are difficult to implement or test.

The overprojection of old uses results in the phenomena known as misconceptions. This may be best illustrated by the example of the notion of infinity. One may envision the following scenario. A person first becomes familiar with utterances of the form, "Function f grows infinitely." This may well be the phrase through which "infinity" makes its first appearance. At this point the new signifier has no existence of its own. The basic meaningful units are the expressions "function f" and "grows infinitely." And then, borrowing the template "Function f grows/tends to _____" from the discourse on functions and numbers ("Function f grows/tends to a number y_0 (when x tends to x_0)), the learner would say, "Function f grows/tends to infinity" or even in the symbolic form, "Function f grows/tends to $\propto$." Once inserted into a slot originally meant for numbers, the word "infinity" and the symbol "$\propto$" have a tendency to sneak into any place destined for numbers. Thus, since the phrase "Function f tends to a number y_0" may be translated into "The limit of function f equals y_0," it seems only natural to say, "The limit of function f equals $\propto$." Here, because of a clear ontological shift (from operational "grows infinitely" to the structural "equals (is) infinity"), the name "infinity" and the symbol "$\propto$" acquire a life of their own and start acting in language as signifiers of an independently existing object. This is a perfect example of hypostasis — bringing a new mathematical object into existence just by a change in the rules of the language game. To this point, everything seems fine. However, if not restricted, the expectation that $\propto$ should fit any slot meant for numbers would soon produce statements creating intra-discursive anomalies and contradictions. The common error $^{\propto}/_{\propto} = 1$ is a good example.

[1] A: Good... now, would you, please, do [problem 2] i [(11,9)-(5,3)=]?

[2] D: OK, This is already more complicated. Would you mind if I made side notes?

[3] A: On the contrary, suit yourself.

[4] D: (a,d) minus (c,d) equals... This is a problem... a problem... there are... there are many more possibilities for the inverse operation and I have to check them.

[5] A: Fine. Let's see. Continue.

[6] D: I will try to do [it]. Suppose, *a* equals... equals *a* divided by *d* minus *b* divided by *c*, coma, *b* divided by *d* [writes ($^a/_d$ -$^b/_c$, $^b/_d$)], and I reverse all the operations that are here.

FIG. 9. Dan looks for a definition of subtraction — first trial.

The fact that using an old template is a "package deal" finds its other expression in the common expectation that whatever appears within expressions with arithmetic operators must also be applicable in utterances about quantities and magnitudes. Thus, the fact that complex numbers cannot be ordered appears counterintuitive.

The other weakness of expectations — the one resulting from their blurred inexact nature — may be illustrated with yet another episode taken from the study with Dan. I have just shown how the appearance of known signs (operators '·' and ':') enabled Dan to act in a meaningful way in an unknown situation (division of pairs of whole numbers). Dan's expectation that division should be "an inverse of multiplication" proved sufficient as a basis for constructing a working definition of this new operation. As may be seen from the excerpt in Figure 9, this was not the case with the operation of subtraction. The mention of subtraction (the appearance of the sign "-") invoked the phrase, "Subtraction is the inverse of addition" but did not give precise directives about the way the term "inverse" should be applied. Thus, Dan's first impulse was to reverse anything that could be reversed — all the operations on numbers that appeared in the formula for addition of the pairs. After all, reversing the component operations did work in the case of division. The only difficulty in the present case was that there seemed to be more possibilities for combining different reversals.

By testing the suggested formula in a concrete case, Dan soon realized that reversing of all the operations did not work — it did not result in a pair of whole numbers which, when added to the subtrahend would produce the minuend. Thus, he ventured a new conjecture, as presented in Figure 10.

More often than not, the certain difficulty stemming from an inexact nature of expectations is not an insurmountable obstacle. Substantial progress may be made either in a gradual way, by a succession of trials and errors, or in one big step — by translating the anticipation into an algorithm for finding a working definition. The latter is how Dan eventually overcame the present difficulty. He translated the claim about the relation between addition and subtraction into a symbolic statement:

$$(a,b) - (c,d) = (x,y) \quad \textit{iff} \quad (x,y) + (c,d) = (a,b)$$

and then, after applying the formula for addition, solved the resulting equations for x and y.

[56] A: Yes, good. It seems that the conjecture did not prove itself, did it? What next? Where were we?
[57] D: I have to derive the operation of subtraction.
......
[66] D: Now, I have an idea.
[67] A: A brilliant idea, of course...
[68] D: No, I am not sure. Now, when I have seen ... When I tried to perform this operation here, I reversed all the signs and the right-hand side answer was correct, the answer that did not require reversal of the operation of addition. So perhaps I only have to reverse the multiplication sign in order to get...
[69] A: Namely?...
[70] D: Namely, that I will do (a,b) minus (c,d) equals a divided by d plus b divided by c coma b divided by d. [writes ($^a/_d$ +$^b/_c$, $^b/_d$)]
[71] A: So the difference between this and what we had before is that we now have plus instead of minus?
[72] D: Yes.

FIG. 10. Dan looks for a definition of subtraction — second trial

Can Definitions Help?

The limitations of reasoning based on analogy are obvious. It is an ambition of mathematicians to ground the mathematical discourse in more reliable principles. Indeed, mathematical discourse may be the only one that aspires to derive decisions about co-occurrence of different templates (and utterances) from purely deductive considerations. Here, an introduction of a new signifier is usually accompanied (or at least, according to the rules of the mathematical discourse, should be accompanied) by a *definition*. Like any other utterance, a definition is a statement that has the power of activating clusters of other utterances. The way it operates, however, is intended to be quite special. Definition is an utterance that formally-minded mathematicians would like to regard as the sole generator of the entire discourse involving the signifier which it defines. Mathematicians recognize as legitimate only those utterances that may be presented as a result of formal deductive derivation from definitions, or at least may be shown to be consistent with these definitions[23] .

Thus, those who reduce meaning to linguistic use would claim that definitions fully determine the meaning of signifiers. As teachers and researchers, we know only too well that, in practice, this is hardly the case. Definitions certainly help in establishing meaning of signifiers, but they do not tell the whole story. It is a well-known fact that people often hold to certain beliefs about scientific and mathematical signs in spite of the dictum of definitions. On the face of it, it may seem truly surprising that students may have robust preconceptions about a signifier they have never seen before. The following example shows, however, that even in the case of mathematical inventions, the mechanism of expectations is not only possible, but may even play a very central role. In fact, the power of framing and analogy may be greater than that of deductive reasoning.

In a new mathematics text for schools that I was recently asked to review, I found a curious redundancy. Aiming at the concept of derivative, the authors first introduced the concept of the slope of a graph at a point (presented as the slope of the tangent to the graph at that point). Then, in the very next sentence, they defined a derivative at a point as the slope of the graph of the function at this point. At the meeting with the project team, I pointed out the double naming (slope, derivative) of exactly the same thing and asked the authors for the reason underlying their decision to pass through slope on their way toward derivative.

This question incited a fierce discussion. Quite clearly, all the participants felt intuitively that, in spite of the mathematical equivalence of "slope" and "derivative," these two do not bring the same message. One of the speakers, G., put it this way:

> No, they are not the same... 'cause slope is the basic notion and derivative exists only as a function [rather than in its local sense of "derivative in a point"]. *Slope tells them* [the students] *something*... They know this is a number from the beginning. *You tell them "slope" and they see something — an angle*. "Derivative" does not tell them anything. It's a function dependent on [another] function. Yeah, everything must start with the slope. [With slope] the kids understand something. *They have something in their head*. [Translation from Hebrew and italics — A.S.]

What G. was clearly trying to say is that the different names have a different framing power. They evoke different expectations and connect the new notion to different discursive formations and different template clusters. With the term "slope," one is directed toward the geometric context within which the new notion comes complete with connections to other related notions and with well-developed mental images. In the case of derivative, all these helpful links and pictures are initially absent. The difference may be as great as expressing itself in the possibility that within one frame the notion will be semantically mediated while in the other its use will continue to be template-driven.

This example brings quite a number of noteworthy messages. First, it shows in a persuasive way that definitions are not the only source of signifieds. Definitions as such introduce new signs in a skeletonized form, with almost no links to previous discourses. Indeed, cutting such links seems to be the very essence of formalization and abstraction. Using expression Bruner (1987) has applied in a different context we may say that formal mathematical definitions are "all reference and no sense"[24] (p. 119). These definitions, therefore, may be very well-suited as tools for testing hypotheses and expectations, but they are not equally effective as generators of the expectations themselves. Thus, contrary to what some people may believe, definitions cannot be regarded as always effective short cuts to meaning[25].

More often than not, the signifiers themselves become more or less powerful meaning-activators, their signifying strength depending on their history. Thus, the name "slope" obviously brings a much richer semantic heritage than the word "derivative"[26] . The same distinctions can often be made when it comes to mathematical symbols. For example, the symbol $^a/_b$ for a rational number is certainly a more powerful-meaning generator than the symbol (a,b). The former brings an immediate association with division, thus connecting rational numbers to the computational processes which constitute their operational source.

The final implication regards the nature of the meaning. An important moral of the story of slope and derivative is that there is much more to meaning than logically-regulated use. The example casts doubt on the reductionist position which tries to do without reference to anything but forms of use. Except for saying that different names activate different discourses, G. was stressing that they evoke different experiential resonance. According to G., the word "slope" brings about a spectrum of mental phenomena, including spontaneous associations and "seeing something" (a mental image of an angle), whereas derivative has almost no such effect. Thus, in spite of the fact that "slope" and "derivative" are mathematically indistinguishable and have exactly the same definition, the former is believed to be readily accessible to the student as a signifier-signified pair, whereas the latter is initially no more than an empty signifier. The emergence of these and other aspects of meaning will be dealt with in the next section.

Objects from Symbols, Act Two: Filling in the Semantic Space

Object Mediation

In the templates-driven phase, the use of a new symbol may be quite effective and a person may have little difficulty communicating with others, but this ability is not accompanied by an awareness of the reasons why things work. Using Vygotsky's language (1987), one may say that in this phase, names and symbols are but "signs-for-others"[27]. It is important to stress again that "templates-driven" does not mean meaningless, and that the period of templates-driven use may be extremely brief. When it comes to learning, it seems likely that the length of this period may be regulated by instructional intervention.

When employed in the templates-driven way, the signifier is not yet conceived by its user as standing for something else. In fact, the user may not even perceive the sign as a self-sustained entity. The basic meaningful units are all those more-or-less constant phrases within which the sign makes its appearances. It is usually only much later that the name or symbol gains independence. The liberation of the signifier from the confinement of constant phrases is usually accompanied by a reference to entities different from the signifier itself. Thus, for example, a student presented with an equation, $3(x+1)-2 = 3x+5$, may transform it into $3x+1=3x+5$ and then say that "The equation has no solution because the two sides represent linear functions the graphs of which do not cross each other." Here, the mediating entities are functions, which, according to the way the word "function" is used in the sentence, are independent objects that should not be identified with the signifiers ($3x+1$, $3x+5$, graph) themselves.

Let us take a closer look at the main characteristics of object mediation.

Flexibility and generality of use. As was just mentioned, the salient attribute of this new stage in the life of a structural sign is that its use becomes flexible. The sign gets life of its own and the user is now able to incorporate the signifier into utterances which he or she has never heard before. This change may be compared to the transition from a mechanical use of a musical instrument, say piano, to the use based on a knowledge of music. A person who has no musical education but who has learned to play a few melodies by memorizing the sequence of necessary key-strokes may be able to repeat these melodies at any time, and may even enjoy this greatly, but she will probably not be able to play a new melody, even if she can sing it. For such a person, finding the necessary key-strokes would be an uphill struggle. Others, who can decompose melodies into basic elements, understand the principles of harmony and know how the elementary sounds may be produced by appropriate actions, are able to compose, or at least reconstruct by themselves, melodies they have never played before. This ability to create novel uses is often regarded as an ultimate criterion of meaningfulness and understanding. As Rotman (1994) put it while summarizing Wittgenstein's (1978) observations on meaning, "It is the move to new cases, and the subsequent use of the label [signifier] that constitutes its meaning" (p. 24).

The symbol starts functioning as a representation. Another salient phenomenon indicative of the phase of object-mediation is a change in the role of a symbol: rather than being the object of discourse (as, for example, analytic expression in Euler's first definition of function), the symbol now becomes a representation of another entity. This new identity is conferred on the signifier by the way it is used. The "analytic expression," for instance, turns into a means of talking about a certain abstract object ("a set of ordered pairs," for example) which may also be represented with other symbols (e.g., a curve in the plane). From now on, if we wish to be precise, we should no longer say, "This is a quadratic function," while pointing to the symbol x^2 or to a parabola; rather, now we should say that "This is an expression of (or a graph of) a quadratic function," or "This expression (graph) represents a quadratic function."

The distinction between "the object itself" and its representation is more difficult in the case of numbers, but it is still possible. Nowehere is the sense of the split made more evident than in the following statement by the French mathematician Alain Connes who complains about people's frequent inability to see the 'obvious' difference between a number and a numeral:

> …it would be wrong to attribute significance to the numerals that appear in the number. (Consider that very soon we will be celebrating the year 2000. The importance of this number is a purely cultural phenomenon: in mathematics, the number 2000 is utterly devoid of interest!). (Changeux & Connes, 1995, p. 13) [28]

As aptly expressed by Cassirer (1957), the developmental importance of the mental split between the signifier and the object signified can hardly be overestimated. The transition from signifier-as-an-object-in-itself to signifier-as-a-representation-of-another-object is a quantum leap in a subject's consciousness:

> The ontogenic development shows that wherever the function of representation stands out *as such*, where, instead of giving himself wholly to the actuality, [to] the simple presence of sensuous content, man succeeds in taking it as a representation of another, he has achieved entirely new level of consciousness...
> When the representative function of names has thus dawned on a child, his whole inner attitude toward reality has changed — a fundamentally new relation between subject and object has come into being. (pp. 112, 113)

In this context, it is important to stress two things. First, the change in the role of the symbol — from the-thing-in-itself to a representation — would usually not happen unless there are other symbols that can be regarded as signifying the same entity. The features that make us consider two symbols as referring to the same third object will be discussed below. Second, once the symbol takes the role of representation, the whole discourse undergoes a modification. The old ways of expressing mathematical truths give way to new formulations. One glimpse into any old mathematical text would be enough to realize how far-reaching this change may be. Thus, the ongoing mathematical conversation, perhaps more than any other type of discourse, is similar to a living organism that incessantly grows and mutates without losing its identity.

The transition from signifier-as-an-object-in-itself to signifier-as-a-representation can be spurred by the teacher who insists on using language that underlines the separate existence of signifier and signified. Thus, rather than say, "Function x^2," the teacher would say, "Function *represented by* x^2." However, the awkwardness of such language, whose effectiveness is highly doubtful, may be too high a price to pay. From my experience, the mental split between signifier and signified can hardly occur without a certain experience of the signified (see the discussion of the experience of the signified below), and this experience is not a matter of language alone.

<u>The symbol is often inadvertently exchanged with other signifiers</u>. One of the most telling indications of semantic mediation is a phenomenon of a change in wording which may often be witnessed when people try to re-capitulate a former statement or when they are asked to use a particular word or symbol

in a sentence. For example, Dan (see the previous section), when requested to compose a sentence using the words "one and a half," replied:

Mother gave Yossi a shekel and a half[29] .

In spite of the explicit request, the words "one and a half" did not appear in the utterance. Similarly, in another study, an eleven-year-old boy who was asked to construct a sentence with the word "zero" said:

I gave all my apples to friends, so now I have nothing.

The mechanism of semantic mediation is obvious here. In both cases, the students did not look at the words but rather through them, to the entities which these words seem to represent (the quantities one-and-a-half and zero, respectively). Without noticing, they then made a transition to different words which, for them, represented the same entities but for some reasons were more easily applicable within sentences. Incidentally, it is probably not by chance that in both above cases the utterances constructed by the children came from AR discourse. The physical reality is the greatest provider of object mediation. It abounds in perceptually-accessible objects that are ideal to think with and to use as signifieds.

Economy of expression. As a result of object mediation, there is a substantial gain in the efficiency of discourse: much more can be said now with one signifier. While assuming the role of a mere "representation," the given symbol becomes, in a sense, equivalent to other signifiers (think, for example, about algebraic expressions and graphs), and thus, what until now had to be said twice — for each of the different signifiers separately — from now on may be said just once. Due to the mediation of an abstract object supposedly referred to by both of these symbols (function, in the case of the expression and graph), what is said with one signifier usually has meaning for the other.

Experience of the signified (abstract object). One can hardly argue with the claim that there is more to "grasping a meaning" than can be found through purely structural linguistic analysis. We have seen this in the discussion of “slope” and “derivative” reported above. In fact, convincing evidence for this can be found everywhere: from well-controlled clinical studies to field observations and to mathematicians' testimonies. Repeatedly, people report having vivid images of abstract objects, thinking without words and having recourse to metaphorically-transformed perceptual experiences (Hadamard, 1949; Johnson, 1987; Lakoff, 1987; Sfard, 1994a). As in a virtual reality game, the presence of mathematical objects may seem to a person to be real even if it is difficult for him or her to communicate it to others. So it seems to be in Connes’ case, judging from his statement on numbers and numerals above. Indeed, the way he speaks shows that for him there is no much difference between numbers and physical objects when it comes to questions of existence: both types of entities exist independently of human mind. To the psychologist, the experiential aspect of meaning seems too central to be neglected only because it does not yield easily to scientific investigation.

To summarize, the transition from template-driven to object-mediated use of a symbol is a rich and multi-facet event. The remainder of this section is devoted to observations on how it happens. I have already mentioned two possible sources of object mediation. One of them is located within the closed system of mathematical discourse, and the other crosses its boundaries and reaches as far as AR discourse. We now take a closer look at each.

Intra-Discursive Creation of Mathematical Objects

In the present section I will focus on the ways in which endogenous mathematical meaning is constructed. I will show how the semantic space created by the introduction of a new signifier may be replenished with content coming from within the mathematical discourse itself.

As has been mentioned above, one typical indication of object-mediation is the use of a number of signifiers as if they were but different representations of the same entity. All this begins with the awareness of some sort of "kinship" between signs. Such awareness may come about spontaneously or may be evoked by others (the teacher, for example). It is important to stress that, more often than not, it is prior to any mention of an object ("referent"), and, therefore, may be viewed as a direct reason for its emergence. The already analysed development of the notion of function is a good example.

There is also another type of kinship between symbols, one that expresses itself in an isomorphism of discursive formations built around these symbols. By "isomorphism of discourses" I mean a relations-preserving correspondence between the two discourses. This kind of relationship exists, for example, between a discourse that evolves around algebraic expressions with one variable and one that concerns two-dimensional curves. Indeed, any utterance about expressions (e.g., "$2x+1=26-3x$ when $x=5$") may be mapped onto an utterance about curves (e.g., "The straight lines with the slopes 2 and -3 and y-intercepts 1 and 26, respectively, cross each other at the point (5,11)") in such a way that the logical relationships within each one of the two discursive formations are preserved. Once again, creation of an intermediate entity (function) which may be regarded as a common referent of the two symbols — the algebraic formula and the curve — is a feasible way to account for the isomorphism.

To summarize, endogenous semantic mediation in general, and mathematical objects in particular, arise between symbols as a result of an attempt to account for discursive equivalencies and isomorphisms. (The adjective "discursive" has been added to stress that these equivalencies and isomorphisms are phenomena that express themselves in the discursive use of signifiers). It is noteworthy that the way we deal with linguistic equivalencies and isomorphisms in mathematical discourse is parallel to the manner in which these phenomena are accounted for in AR discourse. It is also worth mentioning, however, that the order of things seems to be reversed. Whereas in AR discourse perception of an object can sometimes be primary to any linguistic treatment, the abstract objects of mathematics seem to be secondary to the awareness of discursive equivalence of a number of signifiers. This may be viewed as additional evidence of the perceptual sources of all human thinking and the metaphorical, embodied nature of imagination (Johnson, 1987; Lakoff, 1987; Sfard, 1994a, 1997).

As an illustration of the intra-discursive creation of object-mediation, I will refer to another example from my study with Dan (see earlier discussion in this chapter). As the reader may recall, I presented Dan with new mathematical symbols — pairs of whole numbers on which some operations, called addition and multiplication, had been defined. Dan successfully constructed subtraction and division as reversals of these two operations. The resulting structure was that of rational numbers, but Dan had not been told that at any stage. Moreover, in spite of his intensive work with the pairs, he remained unaware of the isomorphism for the duration of the first meeting. By the end of the meeting, a problem appeared: some of the divisions and subtractions yielded pairs of fractional, rather than whole, numbers (e.g., $(5,3):(1,4)=(5,{}^{3}/_{4})$). According to the definition, these were not legitimate members of the new structure. After a brief discussion in which possible solutions were considered, the instructor suggested:

> How about the following idea: We will not distinguish anymore between all the possible pairs. We shall see some of them as equal. Some of them will be considered equivalent to each other. There will be pairs that, from our point of view, will be the same, even if they are written with different numbers. For example, (3,2) and (6,4) for us will be the same [writes: (3,2)=(6,4)][30].

From this, Dan concluded that two pairs should be considered as equal if the relation between them could be presented in the following way:

> [1] the numbers [in the original pair] are multiplied… the two numbers are multiplied by the same number [and this is how the second pair is obtained].

This is a purely operational description: Dan identified the transformation that has to be performed on a pair in order to turn it into an equivalent pair. In the attempt to trace the subsequent transition to a structural definition, it is most enlightening to follow gradual changes in Dan's use of language. Even before the end of the first meeting Dan declared:

> [2] We can decide that pairs are equal according to the ratio... to the ratio of the two numbers.

The change with respect to utterance [1] above is quite significant — and most telling. The transformation of a pair into an equivalent pair has been reified into a mathematical object called 'ratio'. Rather than consider the two pairs alone (as in [1]), Dan helps himself with this third entity. This new object will be the touchstone of the equality between pairs. Dan's use of the words "according to" shows, however, that at this early stage, although it is obvious that the ratio is strongly related to the pairs, these pairs are not yet regarded as representations of the ratio.

During the second meeting, I could witness a gradual metamorphosis of the whole-number-pairs from objects-in-themselves into representations of some other objects. First, when recalling the decision made during the previous meeting, Dan told me:

> [3] We said that some pairs of numbers will be equal because the ratio between the first and the second number is equal...

This is but a repetition of [2]. However, a few moments later, the formulation changed. After I suggested that we may call the pairs "numbers" and asked Dan whether this proposal seemed to him acceptable, Dan replied:

> [4] Yes, I think it is possible [to call the pairs "numbers"] because what really counts is the ratio between the two numbers in the pair, and I think that in fact we could write this [writes $^8/_5$] instead of this [writes (8,5)]. We could write it as eight divided by five?

Here, for the first time, Dan identified the pairs with the ratio. This, however, was put in an interesting and telling way. Rather than say that a pair (8,5) *is* the ratio $^8/_5$, he talked about replacement of the pair by the ratio. Thus, Dan pointed to a very close relationship between the pair and the ratio — so close that one may be replaced by the other. However, this still was not necessarily a relation of signifier-signified.

The transition to this relationship happened some time later, when on a certain occasion Dan unexpectedly declared:

> [5] You just took numbers that I know and you represented them in a different way.

This time, Dan said explicitly that the pairs are representations of some third entity. This other entity is a number already known to Dan from his past learning. This, incidentally, is how the chain of signification builds: the familiar numbers (of the form $^a/_b$), which once began their life as mere signifiers, are now fully-fledged signs which, in their turn, become signifieds of the new signifiers (the pairs (*a*,*b*)).

Several aspects of this story merit special attention. For Dan, the symbols $^2/_5$, $^{11}/_8$, etc., are full of meaning. They bring with them a rich network of both intra- and inter-discursive relationships and a long history of abstract objects that evolved through the process of reification from the processes of division. (The symbol "/" within the rational number sign commemorates this history.) By assuming the role of "a representation" of rational numbers, the pairs (*a*,*b*) inherit all of this meaning. Above all, they become reconnected to the operational origins of the notion. Dan's later utterances make it quite clear that in his eyes the two symbols, $^a/_b$ and (*a*,*b*), differed in their roles. Indeed, there could be little doubt that for him, the pairs

(a,b) acted as signifiers while the ratios $^{a}/_{b}$ functioned as signifieds. When requested to perform any operation or investigation on the whole-number-pairs, Dan invariantly switched to ratios. He did it even in a recall task, where he was asked to look at a series of pairs and recapitulate them later.

As may be seen from episode [6], which took place by the end of the second meeting, the bond between the signifier (the pair) and the signified (the rational number) soon became so strong that Dan would not distinguish between them anymore. The pair (a,b) and the rational number $^{a}/_{b}$ were now referred to as the same (this is true at least in the case of the pairs of the form (o, b) and the number 0).

[6] D: The moment I write zero coma something, (0,b) for example, and I change it to $^{0}/_{b}$ which, according to what we said, is equal [to the pair (0,b)] then this is automatically equal to 0.

A: What does it tell you about the pairs?

D: That all the pairs with left component equal to 0 are equal.

A: Equal to each other?

D: Equal to 0.

To summarize, the above example instantiates the process of intra-discursive construction of mathematical object. Once certain signifiers were declared equivalent (equal), the need was created for an object that would give meaning to this equivalence. The object was eventually found[31] and from then on the equivalent signifiers played the role of representations.

Inter-Discursive Creation of Mathematical Objects: Connecting VR Discourse to AR Discourse

In the last section we have seen how mathematical discourse produces its own meanings. One can hardly overestimate the importance of another process leading to the emergence of object mediation: metaphorical projection from one discourse to another. As was noted before, AR discourse occupies a special position. I have already discussed some distinct structural similarities between AR and VR discourses. The focus in this section will be on another kind of link between the two: the link created by names and symbols which are shared by both discourses. Such symbols are well known to all of us and can be found in abundance in everyday language. The names and symbols of positive integers and rational numbers, which are almost as common in everyday talk as they are in a classroom-mathematical discourse, are the simplest examples[32].

Naturally, mathematical signifiers are not always a shared property of AR and VR discourses. The existence of some of them, at least for some people, may be fully confined to mathematical discourse. This is certainly the case for such notions as matrix or complex number, provided the user knows nothing about their scientific applications. Some other mathematical signs are not mathematical at all for a person who is only able to use them within AR discourse. Much has been said lately about different brands of AR discursive formations which evolve around signifers shared with mathematical discourse. There is talk about "everyday mathematics," "street mathematics," "market-place mathematics" and "ethno-mathematics" (Lave, 1988; Nunes, Schliemann, & Carraher, 1993; Saxe, 1988; Walkerdine, 1988). While marveling at Brazilian street vendors' ability to make appropriate calculations when selling their goods, and at similar skills displayed by uneducated women buying provisions for their families, the writers emphasize that the same people are usually stymied when faced with a requirement to make paper-and-pencil numerical manipulations. Clearly, for those who only know how to use numbers in everyday discourse, the number is not the same object as it is for the learner of mathematics in school. Although the same names appear in both discourses, these names lead to entirely different connotations and, thus, constitute different signs. While a street seller thinks of numbers in terms (and images) of coins, school children identify numbers with written number-symbols. To put it differently, if an abstract mathematical object which functions as a signified of, say, the term “five” is a totality of symbols, linguistic uses, and experiences related to this signifier, then the Brazilian vendor and the European pupil refer to completely different objects using the same number-name.

For many people, a signifier — either mathematical or otherwise — must find its place in AR discourse in order to be meaningful. More than that, some researchers would claim that AR discourse is the only possible source of meaning for mathematics. This claim is usually supported with argument about the situatedness of learning (Lave, 1988; Walkerdine, 1988; Brown et. al., 1989; Lave & Wenger, 1991[33]). In this paper I have been trying to show that this position may be too extreme. Arguments have been brought forward to show that the intra-discursive production of mathematical meaning is not just possible; sometimes, it is the essence of mathematical creation. Nevertheless, it is now time to stress the importance of meaning which comes from outside mathematical discourse.

If a sign exists and functions in both AR and VR discourses, it is likely to be meaningful in the latter mainly due to the perceptual-world connotations it brings with it from the former. Thus, the symbols $^2/_3$ and 17 carry with them the metaphor of physical quantities (cf. Sfard, 1994b, 1997), and this metaphor is what mediates and regulates the uses of the signifiers both in AR and in VR discourses. If we take the ability to use a mathematical sign in AR discourse as a touchstone of understanding, then a small test I carried out recently among my university students has shown that not many of them could boast of having an object-mediated understanding of the concept of negative and irrational numbers. First, eighteen students were asked to construct sentences with the numbers 1.5, $^2/_3$, and -3. Later, they were requested to compose questions, the answers to which could be $^7/_9$, -2, and $\sqrt{2}$. In both cases, they were encouraged to look for utterances with "everyday content." While both tasks — construction of everyday sentences and questions — proved easy in the case of the positive rational numbers 1.5, $^2/_3$, $^7/_9$ (see Figure 11), for the rest of the numbers not many everyday utterances were provided. Students' responses were taken mostly from mathematical discourse itself. The few "everyday uses" of negative numbers were made solely in the context of temperature, latitude and bank overdraft. In the first two examples, the number was applied as a label rather than as a measure of quantity. Most of the "everyday" questions to which the answer was supposed to be -2 suffered from out-of-focus syndrome; that is, although the negative quantity was somehow involved in the situation presented in the question, the actual answer to the question should be 2 rather than -2 (see the example in Figure 11).

In my experiment with Dan, the request to compose sentences and questions with numbers of different kinds recurred several times. The student repeatedly stressed his difficulty with negative, irrational and complex numbers (the latter were introduced to him beginning with the fourth meeting as pairs of real numbers and under the name "novel numbers"). He explicitly complained about his inability to compose "everyday" sentences and questions for these numbers. Here are representative samples of his utterances:

Remark: the numbers in brackets show the percentage of students who constructed this type of example. In the second task, the students were encouraged to compose more than one question for each item, thus the percentages do not add to 100.

	AR discourse	VR discourse
SENTENCES		

1.5:	I drunk a glass and a half of water (67%).	If one multiplies 1.5 by $^2/_3$, one gets 1. (332%)
$^2/_3$:	Dan was hungry and he ate $^2/_3$ of a pie. (75%)	$^2/_3$ is a rational number. (25%)
-3:	The temperature went down to -3. (42%)	-2 is greater than -3. (58%)
QUESTIONS		
$^7/_9$:	The pie was divided into 9 parts. I ate two of them. How much of the pie was left? (75%)	What is the result of $^3/_9$ + $^4/_9$? (92%)
-2:	label: temperature went down 12 degrees from 10 degrees. What is the temperature now? (42%) out of focus: How much money do you owe [*sic*!] to John? (25%)	What is the solution of the equation $3x+9=3$? (100%)
√2:	What is the length of the hypotenuse in the right-angle triangle with sides equal 1? (25%)	What is the solution of the equation $x^2=2$? (75%) (16% -- no response)

FIG. 11. Examples of sentences and questions composed by the students.

- on "novel" (complex) number (2,3), meeting 7: I still don't know how to write (2,3) in a sentence. The two [real numbers] I can … I can say this as a mathematical sentence.
- on -2, meeting 9: A question: "What is 1 minus 3?" [pause] … All I can think about is in a mathematical context.
- on √2, meeting 9: A question: "Give an example of an irrational number"… [pause] I don't really have any question to ask.

[1]	A: Why do you smile?
[2]	D: Because I don't.... I have no sense of this number, I don't...

[3] A: Try something...

[4] D: "Dan ate √2 candies from the bag...

[5] A: Great.

[6] D: Not really.

[7] A: Oh?

[8] D: It's difficult to explain...

......

[35] A: You have seen that there is a segment the length of which is √2. Doesn't it give you some sense of √2?

[36] D: It helps a bit, but I still can't... I can't imagine a segment the length of which is √2.

[37] A: What do you need to be able to imagine this?

[38] D: I have to know what is √ 2. I have to present it as a rational number.

[38] A: Why? What is the advantage of rationals over irrationals?

[39] D: That you can grasp them.... That I can imagine without much difficulty what this number is... An example from real life... I have to see it as a number. π is easier for me because I can see it as 3.14, etc.

[40] A: Does 3.14 give you anything which you don't have in the case of √2?

......

[50] D: An exact measure of the number. 3.14... I can now take a ruler, measure 3.14 and draw this interval... But even if I take the most precise ruler in the world, I can't draw a segment the length of which is √2.

FIG. 12. Numbers must have a precise magnitude.

Some other utterances have shown that the only link to AR discourse could come from the relation of order. Dan said this quite explicitly during the ninth meeting, while explaining the difficulty he experienced when trying to compose a sentence with √2 (Figure 12).

Evidently, the symbol √ was for Dan still operational rather than structural and, in order to account for its structural use, Dan needed object-mediation which could only come from AR discourse. His concept of number was inextricably tied to AR notions of quantity. This is why the numbers that could not be easily located between rational numbers (e.g., irrational, complex) did not count for Dan as "true" numbers. When I asked, during the ninth meeting, what the word "number" meant to him, the exchange presented in Figure 13 took place.

[1] D: OK. Number is something we use for counting... We use them to assess quantity.

[2] A: The rational number can measure quantity?

[3] D: Yes, you can say…

[4] A: And the novel [complex] numbers can?

[5] D: The "left-hand numbers" [this is what we called the complex numbers of the form (a,0), which Dan has identified as "another representation" of "ordinary numbers"] can, but the others can't.

[6] A: So, are these numbers or not?

[7] D: [hesitates]... No...

FIG. 13. Numbers measure quantity.

The inability to incorporate negative, irrational and complex numbers into AR discourse was for Dan an evidence that "these are not true numbers." Several times he made a distinction between the numbers that "exist in the world" and those which are but man-made. During the eighth meeting, he tried to explain his inability to incorporate negative numbers into AR discourse, as presented in Figure 14.

The claim about "unnaturalness" of irrational, negative and complex numbers may be seen as an indication of Dan's uncertainty about the legitimacy and meaningfulness of these numbers. All of this means that Dan's sense of understanding numbers, like that of seventeenth and eighteenth-century mathematicians, depended very strongly on his ability to use them in AR discourse. This example illustrates the point I am trying to make in this section. It shows that AR discourse is a very important, and sometimes irreplaceable, source of mathematical meaning. To put it differently, in certain circumstances transition to object mediation becomes possible only due to the "import" of signifieds from AR discourse.

[1] D: Minus is something that people invented. I mean... we don't have anything in the environment to show it. I can't think about anything like that.

[2] A: Is everything that regards numbers invented by people?

[3] D: No, not everything…

[4] A: For instance?

[5] D: For example, the basic operation of addition, one plus one [is two] and according to the logic of the world this cannot be otherwise.

[6] A: And half plus one-third equals five sixths. Does it depend on us, humans or...

[7] D: Not on us. You can show it in the world.

[8] A: I see... and 5 minus 8 equals -3. It's us or not us?

[9] D: It's us.

[10] A: Why?

[11] D: Because in our world there is no example for such a thing.

FIG. 14. Negative numbers are human invention.

Making Word Flesh -- Some Didactic Considerations

Summary

In this paper, mathematical discourse has been presented as an autopoietic system in which an intricate interplay of signifiers is a principal source of meaning. Although the discourse on numbers, functions, sets, and the like bears striking structural resemblance to the discourse on physical object, there is no point in talks about objects of mathematics as entities existing independently of the discourse itself. In fact, the very idea of mathematical object is only metaphorical, and it refers to a particular use of signifiers. To be a fully-fledged participant of mathematical discourse means, among others, to be able to use mathematical names and symbols with proficiency and with experiences similar to those which are characteristic of the use of structural signifiers in communication on material objects. However, because of a relative scarcity of perceptual support, the abilty to make such an object-mediated use of mathematical names and symbols is not easy to attain.

In spite of the difficulty, new structural signifiers are being introduced to mathematical discourse all the time. The eventual gains of such additions are significant enough to recompense the effort necessary to develop object-mediated use of the new symbols. By changing the ontological status of focal mathematical notions from operational to structural (or from referring to processes to referring to objects), new signifiers enable turning the discourse onto itself, that is they make it possible to initiate a higher-level mathematical discourse – a discourse on the discursive procedures developed and practiced so far.

There is an inherent circularity in the process of construction of such a higher-level mathematical discourse. On the one hand, introduction and use of the structural signifier have a constitutive role in establishing the objects of this discourse; on the other hand, participants' sense of existence of these objects is a condition for the effective use of the signifiers. This circularity is a source of additional difficulty, but is also the driving force behind the discursive growth: The lack of equilibrium between discursive practices and the participants' sense of the objects of the discourse is what impels this growth.

The way out of the circularity leads through the stage of templates-driven use of new signifiers, that is use which is guided by old discursive habits and forms. The employment of new signifiers within previously established discursive templates is regulated through expectations and verifications. A former experience with signifiers deemed to be somehow analogous to the new ones engenders expectations about possible discursive uses of these new names or symbols. These expectations are then tested and, if confirmed, the given templates are added to the repertoire of the new signifier's possible uses; if refuted, the templates are deleted from the list. The confirmations and refutations are a product of either communicative interactions or deductive considerations. The object-mediated use will eventually develop endogenously, as a result of attempts to account for intra-discursive equivalencies and isomorphisms between symbolic systems; and exogenously, with the support of meanings developed by applying the signifier in other discourses.

Conclusions

After the detailed discussion of the mechanisms which underlie the growth of mathematical discourse, it is time now to give some thought to the practical implication. Those who renounce the traditional vision of symbols as coming to capture ready-made meaning and to represent mind-independent intangible objects are likely to be critical also about traditional instructional practices.

As was mentioned before, it is widely believed these days that mathematical symbols should not be introduced before the concepts represented by these symbols are already understood, at least partially. However, if one accepts the thesis that there can be no talk about mathematical objects without mathematical symbols, then the request to prepare an easy, meaningful landing for symbolic newcomers seems inherently difficult — if not outright impossible. Thus, instead of asking, "What comes first," we would rather give a thought to the question of how to orchestrate and facilitate the back and forth movement between symbols and meanings. This seems to be the only way to cope with the problem of circularity that is inherent in discourse creating its own objects.

The roots of all the language games people play seem to go back to AR discourse. However, sequences of template transplants and chains of signification forged on the way from AR to VR discourse of mathematics may be so long that their AR sources are no longer visible from the distant VR end. They may be much too extensive to be fully re-constructed during ten or twelve years of schooling. Indeed, as long as mathematical contents taught in today's high schools remain basically as they are, it seems hardly possible that the starting point of every instructional sequence is placed within AR discourse, or that the bridges between AR and VR discourses are constantly kept in sight when mathematical symbols are being manipulated. As I already mentioned, this is something that many mathematical educators find increasingly difficult to accept[34]. There can be little doubt that they have good reasons for their protests against mathematics for its own sake. As has been shown in the last section, what seems natural and fully understandable for a mathematician brought up in a strict formalist tradition, may be as inconceivable for the student as it was for the Medieval and Renaissance mathematicians who rejected the notions of negative and imaginary numbers because of the impossibility of incorporating them into AR discourse. Today's mathematicians may pay lip service to the requirement of AR usefulness by saying that AR applications "will be found one day." The necessity to postpone the re-linking with AR discourse (possibly *ad infinitum*) does not affect their work and does not disturb their peace of mind. For the student, such a delay may be highly consequential.

The problem seems quite complex indeed: only too often, linking VR and AR discourses is either too difficult to be feasible, or is simply impossible. However, two didactic avenues offer a certain promise. First, the bridging may sometimes be done in reverse order — something that may also save time and effort. Indeed, if we begin with purely mathematical discourse and then return to AR discourse already endowed with new mathematical signifiers, these new signifiers would often serve as looking glasses through which the tangible world would readily seem more orderly than before. As Miller (1991) put it, "...we do not take signifiers as that which describes reality. We take signifiers as what enters the real to structure it" (p. 32). This, in fact, is to say that starting with mathematical signifiers may sometimes be the right thing to do. Not only abstract mathematical objects, but also real-world structures, do not come into existence by themselves, but rather have to be "symbolized into being." The claim that the structure is symbol-independent, objective, and just waiting to be captured in symbols sounds equally unconvincing as Michelangelo Buonarroti's modest assertion that his only role as a sculptor was to expose the forms which were "already there" in the stone.

For the sake of skeptics who doubt the possibility of meaningful activity with links to AR discourse temporarily suspended, let me remark that mathematical ideas which are firmly situated in mathematical context and are well-connected to previously-constructed concepts may appear as meaningful and relevant as those which grow from real-life situations. Here, the emphasis is, of course, on the issue of understanding and accepting the rules of purely mathematical discourse — something which is quite difficult to attain and is, therefore, rarely found in schools. Another point to stress is the issue of context. Whether "real-life" or purely mathematical, it is the context that makes budding ideas meaningful and helps establish object mediation. If we recall how much had been achieved by Dan in the clinically-sterile situation in which the links to his former knowledge were deliberately withheld, we have every reason to assume that in a richer context his progress could be even more impressive.

Yet another argument in support of the possibility of learning mathematics intra-discursively has to do with the ways of incorporating new signifiers into mathematical discourse. Although in no situation should we expect the new mathematical objects to emerge without symbolical support, the symbols themselves do not have to be arbitrarily introduced. In the optimal case, within an appropriate context, they can be expected to grow as if "of themselves," just as in the late-sixteenth century algebraic expressions emerged almost against mathematicians' better judgment out of the simple idea (which, nevertheless, required an exceptional ingenuity to be invented) of denoting unknown and given quantities with letters. The symbolic turnaround was triggered by the need to make the growingly complex discourse on numerical computations more manageable, integrated and focused. Today's student may be similarly motivated while engaging in the activity of symbolizing.

The other promising direction is one that has opened up with the advent of computerized virtual reality. This special technological advance has brought an artificial extension to tangible reality. By rendering figments of human imagination perceptually real, it has increased dramatically the assortment of objects particularly well-equipped to play the role of mediators within mathematical discourse. It seems quite plausible that great parts of mathematical reality, which till now could only be imagined, will soon materialize on the computer screen. If such a turn indeed takes place, its impact on students' ability to talk mathematics may be immense. Because of perceptual mediation, the learner will now be able to engage in VR discourse the way he or she conducts AR discourse. Giving mathematical objects “flesh” may be expected to substantially increase students' understanding of mathematics. This is what one can learn, for example, from the story told by Davis and Hersh (1981) on the impact of computer-generated images of "three-dimensional slices" of a four-dimensional cube. One of the authors reports that there was a quantum increase in his understanding when, after the images had been manipulated and explored for a while, "[t]he hypercube leaped into palpable reality" (p.404).

Whatever path we chose to usher young people into the world of mathematics, we may be better off if we think of this world as symbolized into being rather than merely represented with symbols. If we accept this, then one of the immediate conclusions will be almost identical with what may be learned from Shaw's “Pygmalion”: If we wish our students to feel at ease in the virtual world of mathematics, we have to follow in Professor Higgins' footsteps and see to it that young people learn to speak mathematical language fluently and with a proper, “object-mediated” accent.

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[1] As argued by Wittgenstein (1953), the apparent straightforwardness of ostensive definitions is misleading and, in fact, it is impossible to determine the meaning of words by simply pointing to their referents. Please note, however, the difference between the present use of the term “ostensive” and the use made by Wittgenstein in “Philosophical Investigations”. While Wittgenstein is concerned with the logical soundness of the idea of “ostensive definition” (which he rightly views as problematic), I am dealing with an entirely

different and more practical issue of the means which keeps a discourse in focus and which makes the different utterances meaningful to the discourse participants.

[2] At this point, one may object by saying that the equivalence of the symbols "$^{2}/_{3}$" and "$^{12}/_{18}$" can be demonstrated by pointing to two pieces of a whole (of a pie, say), one of them being composed of two-thirds of the whole and the other of twelve eighteenth parts, and by showing that these two pieces are equal. However, none of the particular pieces of a particular whole may be regarded as a proper referent of the symbols, since in this case, the referent is conceived of as a universal entity.

[3] The example is taken from my experiments with students of different ages. The experiment has shown that this kind of utterance is very frequent, and not only among young children.

[4] Radical constructivists, who take a clear anti-objectivist postion on epistemological issues, are likely to say that ontological questions are of no relevance to their project. Most of them would simply refuse to make any ontological commitments (cf. Von Galsersfeld, 1991, 1995; see also Cobb, 1994, 1995; Cobb, Jaworski, & Presmeg, 1995.), which means that they would not describe themselves as either realists or non-realists. It also implies, however, that if a constructivist wishes to take an ontological position, realism is a viable option.

[5] French linguist Saussure lived from 1857 to 1913, and the American philosopher Peirce was born in 1839 and died in 1914.

[6] In Vygotsky, 1987, this sentence appears in a different translation: **"**Thought is not expressed but completed in the word." (p. 250). The message about the priority of word with respect to meaning becomes slightly weakened. Without consulting the Russian original I would opt for the earlier translation since it seems to me more in tune with the rest of Vygotsky's writings.

[7]Please note that in semiotics every linguistic expression, as well as every action, thought or feeling, counts as a sign. Thus, an utterance "This function has no negative values" can be treated as a sign, or as representamen.

[8] While I was embarking on the present project, these limitations of the existing terminology gave me quite a headache. I put much effort into an attempt to find a word which would denote a written or spoken sign without the connotation of something standing for something else. But in vain. None of the candidates — be it representation, symbol, sign, ideogram, or mark — proved free of the unwanted sematic load. One word

that looked more promising than the others, from this point of view, was the term "inscription." I rejected this possibility because of the restriction to the written signs that it clearly imposes. It may well be that my problem was aggravated by the limitation of my English. I invite the readers to think about better alternatives. In the meantime, I have to settle on the generally accepted terminology, hoping that I will manage to convey some innovative messages in spite of the unwanted connotations it brings. Incidentally, this little story may serve as a case study for those interested in the dynamics of discourse evolution. It brings into full relief the phenomenon of language constraints influencing conceptual change.

[9] The analysis which will be presented here should not be regarded as an alternative to these earlier analyses, but rather as a complement to them.

[10] Another possible way to define the equivalence is to specify the operations which transform formulae into equivalent ones. If this approach is to work, the rules of transformations must be introduced axiomatically. Thus, rather than trying to explain the roots of the observed linguistic equivalence, the axiomatic treatment carries the message of the arbitrariness of the rules of algebra.

[11] Syntagmatic relationship is the relationship that occurs between words which can be combined together to create a sentence. Thus, the words *she*, *is*, and *hungry* are syntagmatically related. The relationship is paradigmatic if, from a grammatical point of view, the words may replace each other in a sentence. Thus, *he* and *she*, or *hungry* and *heavy* are paradigmatically related.

[12] All this brings to mind Vygotsky's (1978) claim that, "*For the young child, to think means to recall; but for the adolescent, to recall means to think*" (p. 51; italics in the original). What Vygotsky seems to be saying is that children's thinking is grounded in collections of concrete instances, whereas adolescents and adults have recourse to abstraction which effectively mediates the use (and recall) of concrete instances. Thus, thinking-by-recall means, among others, solving problems with the help of concrete recipes fitting these particular problems. Recalling-by-thinking refers to summoning general algorithms and heuristics to tackle the concrete cases at hand. In the present paper, this difference is not considered to be solely age-related. On the contrary, it is expected to be found repeatedly in the processes of symbolization, whether the person who symbolizes is a child or an adult.

[13]. Moreover, one cannot exclude the possibility of some *experience of mathematical objects* prior to the introduction of any symbolism. According to mathematicians' testimonies (Hadamard, 1949; Sfard, 1994a), a

sensation of mathematical objects would sometimes appear without any symbolical support, simply as a result of a long period of particularly intensive thinking. It is evident, however, that even among mathematicians this is a rather rare event and, in any case, one which, in the absence of symbolism, cannot be adequately communicated and studied.

[14] For discussion of autopoietic systems see Maturana & Varela, 1987.

[15] We seem doomed to entanglement in vicious circles. The present quandary immediately brings to mind the other circles known from literature: the hermeneutic circle (see e.g., Bauman, 1978) and the learning paradox (see e.g., Bereiter, 1985). The present dilemma is, in fact, equivalent to the circle of reification, as presented in Sfard, 1991.

[16] As was remarked by a reader of the early draft of this paper, the templates-driven use may be mediated by imagery — by sometimes quite vivid mental images evoked by the first utterance. As reported by this reader, the sentence, "A krasnal woke up and got up from his bed," brought to mind a cartoon-like vision of a little creature getting up from a bed.

[17] Derrida (1976) declared that in spite of our need of an "absolute object" which would safely put an end to the free play of signifiers, we have no choice but to admit that there is "nothing but the text" and that "the thing itself is the sign" (pp. 48-49).

[17] The word 'polysemy' refers to the phenomenon of the same signifier having a number of different meaning (signifieds). Mathematics, which has been described by Poincare (1929) as "the science that calls different things the same names," seems to be afflicted by the phenomenon of polysemy in a particularly acute way.

[19] Those who identify meaning with linguistic use would say that there can be almost no other source for our expectations. I use this opportunity again to warn against such reductionism.

[20] Although I prefer, as probably most of the readers do, to see this "clinical" situation as purely theoretical and far removed from the reality of today's classrooms, the sad truth is that many would recognize it as only too familiar. Even if it becomes more and more rare in schools, it is still quite frequent in colleges and at universities. I am also sure that the conversation between Dan and me, which resulted from my experimental script, is likely to be considered by some people as a classical mathematical discourse — the kind of

discourse that is generated by those who transmit mathematics to others by lecturing or through professional mathematical texts.)

[21] It is important to remark that Dan made truly impressive progress in this extremely "unfriendly" situation. Not every student could be expected to reach the point he did. It seems that Dan was exceptionally motivated. His outstanding ability to cope with mathematical problems was evidently a very important element of his self-image, and this made him a willing participant in the kind of discourse we led. Dan seemed to recognize the situation as normal, and he complied with the rules of the symbolic game, never questioning them or wondering about them.

[22] In full accordance with the theory of chains of signification, the new signifier made a reference to previously-established signs.

[23] To put it in a slightly weaker form, a definition of a signifier S regulates the entire discourse about S by being a touchstone against which the admissibility of another statement about S is tested.

[24] The terms *sense* and *reference* are to be understood, after Frege, as referring to “relation to other things and other concepts” and to “extension of the concept to the world of instances”, respectively (Frege’s original German terms, often used by those who quote him, are *Bedeutung* for reference and *Sinn* for sense).

[25] Bakhtin spoke to this effect when he contrasted “contextual meaning” with definitions which only “contain potential meaning” (Bakhtin, 1986, p. 145)

[26] In our work as researchers, the awareness of a potential impact of names and symbols makes us very careful in choosing terminology for new theoretical frameworks. It has already been observed that a researcher is an analogical reasoner who, in creating new theories, avails herself of metaphors (Knorr, 1980). The decisions about names and symbols may have far-reaching consequences. A felicitous choice, through a chain of associations with previous meanings, will bring important new insights; an unlucky choice will result in a cluster of counterproductive connotations.

[27] As opposed to "signs-in-themselves" and later "signs-for-oneself." Vygotsky made this distinction when analyzing development of concepts.

[28] Our languge does not always help in separating the number and the numeral the way Connes does. We say "78 is a two-digit number" whether we mean the signifier (which is sometimes called “numeral”) or the the number itself – the abstract object supposedly represented by this signifier (a counterpart of this expression in

AR discourse would sound rather funny; to say "78 is a two-digit number" is like saying that "Shakespeare is an eleven-letter person."). This difficulty to separate the signifier from the signified in speech and in writing is a permanent reminder that "referents" have no existence without symbols.

[29] *Shekel* is an Israeli monetary unit.

[30] As can be seen here, the idea of equivalence of symbols was suggested and then defined by the instructor rather than being intuitively raised by the learner himself. However, in spite of the lack of spontaneity in the opening move, I believe that the processes I was able to observe later were not much different from those that can be seen in more "natural" circumstances.

[31] In this case, it was not built totally on its own because some previously-constructed objects could be used as signifiers. While talking about "constructing objects," I intend to say that the signifiers "referring" to these objects have already entered the phase of semantically mediated use, that is the phase characterized by all the phenomena presented in a previous section (including the experiential aspects).

[32] Of course, the uses of these signifiers may be quite different in the different discourses. Here is, for example, a sentence I read in the British Guardian (24.4.95), describing the celebrations of Shakespeare's birthday: "Six or 700 celebrants walked through the shrine in Henley Street." Here, 700 is used not as an indivisible number symbol, but as a shorthand for the words "seven hundred," so that each one of the two components has its own independent role within the sentence.

[33] In fact, the sweeping call to teach mathematics through solving real-life problems may be a product of a misinterpretation of the doctrine of situatedness. The whole point about this doctrine is that situation in which problems are being solved frames these problems in ways irreproducible in other situations. Thus, all those characteristics of a problem which are so helpful to its solving in a supermarket or at home are inevitably lost when the problem is transferred to school. Moreover, nothing can be done about it since the loss is due to the fact that simply cannot be changed: that the school is not a supermarket.

[34] The "extremists" would be most happy if the intra-mathematical creation of meaning was banned from schools altogether. However, rejecting endogenous mathematical meaning would be tantamount to the rejection of major portions of mathematics itself. Today's mathematicians might say that it would be banning all mathematics.