Research Report

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This report describes the progress achieved during my research visit to the University of Salerno, which was supported by the Abel Visiting Scholarship Program. The visit started on November 15 and ended on December 15. During that time, I worked with my host, Professor Costantino Delizia, and one of his postdoctoral researchers, Carmine Monetta.

Our project involved the study of finite groups through particular subsets. Specifically, in group theory there is an approach to the investigation of global properties of groups based on the consideration of their 2-generated subgroups. This point of view is especially relevant with respect to nilpotence and solubility of groups. One interesting subject is the characterization of finite groups in terms of their 2-generated soluble subgroups. To approach this goal, for an element $x \in G$, we define

$$\mathrm{Sol}_G(x) := \{g \in G \mid \langle x, g \rangle \text{ is soluble}\},$$

which in general is just a subset and not a subgroup of G. In [1, 2], a new outlook was established for characterizing a group through these subsets. For instance, in [1], my coauthors and I were curious about how restrictions on the structure of the sets $\mathrm{Sol}_G(x)$ for $x \in G$, influence the structure of G. Delizia, Monetta and I further pursued this approach to gain some results about the structure of finite groups. In the following, I make a list of some results we obtained during this visit:

(1) Let G be a finite group and let $x \in G$ be such that $\mathrm{Sol}_G(x)$ is a subgroup. Then either G is soluble or $|\mathrm{Sol}_G(x)|$ is not a power of an odd prime.

(2) For a finite group G, let p be a prime and let x be an element of G of order p. Assume that $P = \langle x \rangle$ and $|\mathrm{Sol}_G(x)| \leq p^2$. Then $\mathrm{Sol}_G(x)$ is the normalizer $N_G(P)$ of P.

(3) Given an insoluble group G, let x be an element of G. Then $|\mathrm{Sol}_G(x)|$ is not the square of a prime. This generalizes a result in [1].

(4) For an insoluble group G and an element $x \in G$, we have $|\mathrm{Sol}_G(x)| \neq 8$.

(5) Let G be a finite group and let M be a nilpotent subgroup of G of odd order. If M is not properly contained in any soluble subgroup of G, then $G = M$.

(6) Let G be a finite group and let $x \in G$. If $[a, b, c] = 1$ for every $a, b, c \in \mathrm{Sol}_G(x)$, then G is nilpotent of class at most 2.

In light of items (1) and (4) above, it is natural to make the following conjecture.

Conjecture. *Given an insoluble group G and an element $x \in G$, $|\mathrm{Sol}_G(x)|$ cannot be a power of a prime.*

My colleagues Delizia, Monetta and I are continuing our collaboration in order to make progress on this conjecture and to more generally further this line of research.

References

[1] B. Akbari, M. L. Lewis, J. Mirzajani and A. R. Moghaddamfar, *The solubility graph associated with a finite group*, International Journal of Algebra and Computation, 30(8)(2020), 1555–1564.

[2] D. Hai-Reuven, *Nonsolvable graph of a finite group and solvabilizers*, preprint (2013), arXiv:1307.2924v1 [math. GR].