Introduction to Gauge Theories

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This report is divided into three parts. In the first part, we describe some nonlinear equations of classical physics. Certain of these equations, for example the Yang–Mills equations, have a geometrical significance. Their solutions are harmonic connection forms on a principal bundle; the structure group of the bundle is called the “gauge group” in physics.

The purely mathematical questions concerning the solutions to these equations have an interest of their own. In the second part of this report, however, we shall sketch the relation between these solutions and physics. Here it becomes necessary to introduce the notion of quantization. The classical equations we consider do not have a direct interpretation in physical terms; rather they yield insight into quantum field theory. In order to explain this connection, we shall describe the constructive field program. Many steps in this program have been established over the past ten years; now the construction of quantized gauge theories (i.e. quantization of geometry) poses interesting, new problems.

Finally, we shall return in part three to the interpretation of the solutions described in part one. We sketch how these solutions may provide an explanation of quark confinement, i.e. the lack of observation of the quark particles, suggested by Gell–Mann and others to be the basic building blocks of nuclear particles.

1. Classical nonlinear equations from physics.

I. 1 Some examples. Let us begin with typical equations from physics. We let $M$ denote a subset of $\mathbb{R}^4$ on which the equations are defined, and denote $x \in M$....

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by space-time coordinates \( x = (t, \vec{x}) = (t, x_1, x_2, x_3) \). If we wish to ensure that solutions have limits at infinity, we replace \( \mathbb{R}^4 \) by the four sphere \( S^4 \). We occasionally denote coordinates \( x_\mu, \mu = 0, 1, 2, 3 \), where \( x_0 = t \).

(i) **Nonlinear wave equations.** Let \( P(s) \) denote a real polynomial of \( s \in \mathbb{R}^1 \). The equation

\[
\varphi_{tt} - \sum_{j=1}^{3} \varphi_{x_j x_j} + P'(\varphi) = 0
\]

for the function \( \varphi = \varphi(x) \) is a nonlinear wave equation with potential energy density \( P(\varphi(x)) \). Equations of the type (1) play a role in the description of meson interactions, as well as in the classical nonlinear oscillation of a membrane. Such equations have been studied extensively.

(ii) **Maxwell’s equations.** The equations of classical electrodynamics, i.e. Maxwell’s equations, describe a two-form \( F \), the electromagnetic field,

\[
F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu.
\]

(Here and in the following we use a summation convention.) The Maxwell equations are

\[
dF = 0, \quad d*F = *J,
\]

where \( * \) denotes Hodge duality, and where \( J \) is the “current” one form. Given \( J \), these equations are linear. In case that the form \( F \) is exact, i.e. there is a one form \( A \) such that

\[
F = dA,
\]

then the equation \( dF = 0 \) is automatically satisfied. In physics, \( A \) is called the “potential.” Thus the solution of Maxwell’s equations is reduced to the solution of the equation

\[
d* dA = *J.
\]

Any exact differential may be added to \( A \) yielding the same \( F \), so the solution to (5) is not unique,

\[
A' = A + dA, \quad F' = F.
\]

In physics, the arbitrariness (5') in \( A \) is called a “gauge transformation”, and Maxwell’s electromagnetic field is the simplest example of a gauge theory.

To concretely identify these equations, write Maxwell’s equations in standard form, where the electric field \( \vec{E} \) and magnetic field \( \vec{B} \) have components defined as various components \( F_{\mu\nu} \) of \( F \). Writing these components as an antisymmetric matrix,

\[
F_{\mu\nu} = \begin{pmatrix}
0 & E_1 & E_2 & E_3 \\
-E_1 & 0 & B_3 & -B_2 \\
-E_2 & -B_3 & 0 & B_1 \\
-E_3 & B_2 & -B_1 & 0
\end{pmatrix}
\]
Then the equation \( dF = 0 \) can be written

\[
\text{div} \, \mathcal{B} = 0, \quad \frac{\partial \mathcal{B}}{\partial t} + \text{curl} \, \mathcal{E} = 0,
\]

while \( d\ast F = \ast J \) is the pair of equations

\[
\text{div} \, \mathcal{E} = \rho, \quad \text{curl} \, \mathcal{B} - \frac{\partial \mathcal{E}}{\partial t} = \mathcal{J},
\]

where \( J = \rho \, dt + \sum_{i=1}^{3} J_i \, dx^i \).

(iii) **Pure Yang–Mills equations.** These equations are a nonlinear generalization of the Maxwell equations. The generalization is most easily understood in a geometrical context. We can regard the potential \( A \) of Maxwell’s equations as a connection form on a circle bundle over \( M \). Locally the bundle is \( M \times U(1) \), and the connection form \( A = A_{\mu}(x) \, dx^\mu \) has sections with values in the Lie algebra \( \mathfrak{u}(1) \) of the circle group \( U(1) \). The electromagnetic field \( F = dA \) is the curvature defined by \( A \). To obtain a Yang–Mills theory, we replace the circle bundle by a principal bundle \( P(M, G) \). Locally \( P \) is the product \( M \times G \), and the case \( G = U(1) \) reduces to Maxwell’s equations above. In general, we take \( G \) to be a compact Lie group.

The potential \( A \) is now defined to be a connection on \( P \), i.e. to be a Lie algebra valued one form, \( A = A_{\mu}(x) \, dx^\mu \), where the sections \( A_{\mu}(x) \) belong to the Lie algebra \( \mathfrak{g} \) of \( G \). A connection defines parallel transport of the fibre over a point \( x \in M \), and a covariant exterior derivative \( D_A \) on Lie algebra valued forms, cf. [8], [25].

Such a differential form \( \theta \) is tensorial if it transforms under the adjoint action of \( G \), namely \( \theta \mapsto \theta^g = g^{-1} \theta g \), where \( g : M \to G \). Then \( D_A \) is covariant in the sense that for tensorial \( \theta \),

\[
(D_A \theta)^g = D_{A^g} \theta^g,
\]

where

\[
A^g = g^{-1} A g + g^{-1} dg.
\]

The transformation (9) is the generalization of the gauge transformation (5'); here the exact differential \( dA \) is generalized to the cocycle \( g^{-1} dg \). In case \( G = U(1) \), the formulas reduce to \( D = d, \ A^g = A' \), and \( \theta^g = \theta \). Groups \( G \neq U(1) \) enter physics as “gauge groups” of particles. An extremely important example is the group of internal “color” symmetry, \( G = SU(3) \).

The generalization of Maxwell’s equation (3) is obtained by replacing \( d \) by the covariant derivative \( D_A \), namely the requirement that

\[
D_A F = 0, \quad D_A \ast F = \ast J.
\]

Often these equations are considered in the “free space” case, meaning \( J \equiv 0 \). We restrict our attention to this problem. Thus we study \( DF = 0, \ D \ast F = 0 \), which can be regarded as the requirement that \( F \) be harmonic, \( \Delta_A F = 0 \), where \( \Delta_A \) is the covariant operator \( \Delta_A = D_A D_A^\ast + D_A^\ast D_A \).
The assumption that $F$ is the curvature form defined by the connection $A$ on $P$, automatically assures that $D_A F = 0$. In fact, this requirement means that

$$(11) \quad F = D_A A = dA + 1/2[A, A].$$

Then, although $D_A^2 F \neq 0$, in general, it is an identity that $D_A^2 A = D_A F = 0$. In fact this identity is the Jacobi identity of differential geometry, the generalization of $d^2 = 0$ to covariant differentiation.

With the assumption (11), the Yang–Mills equations are reduced to finding a curvature $F$ satisfying

$$(12) \quad D_A \ast F = 0.$$ 

This equation, because of (11), can be regarded as a nonlinear system for $A$, namely

$$D_A \ast D_A A = 0.$$ 

In terms of components,

$$(13) \quad \partial_\mu F_{\mu \nu} + [A_\mu, F_{\mu \nu}] = 0,$$

with $F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. Here $\partial_\mu$ denotes $\partial/\partial x^\mu$.

(iv) Particle physics. The actual interaction of particles is described in physics by a Lagrangian combining the nonlinear wave equation, the Yang–Mills theory and a current $J$ arising from a Dirac field. The mathematical abstraction of isolating parts of the complete system above is made to simplify the analysis. Ultimately the complete coupled system is of interest to physics. We now restrict attention to pure Yang–Mills theories with $J = 0$.

1.2. Action functionals. Equations as (1) or (12) above can be derived from a variational principle ("action principle" in physics). The standard action functional $\mathcal{A}$ is defined as the integral of the Lagrangian density

$$(14) \quad \mathcal{A} = \int_M \mathcal{L} d\vec{x} \, dt.$$ 

In the case of equation (1) above,

$$(15) \quad \mathcal{L} = -1/2 \partial_\mu \phi \partial^\mu \phi - P(\phi).$$

In the case of the Yang–Mills equations (15), we let $A$ be a connection and define

$$(16) \quad \mathcal{L}(A) = \text{Tr} F_{\mu \nu} F^{\mu \nu},$$

$$\mathcal{A} = -\|F\|^2 = \text{Tr} \int_M F \wedge \ast F.$$

In each case, the metric to raise or lower indices has the Minkowski signature $(-++++)$. The flat metric $ds^2 = -dt^2 + (d\vec{x})^2$ (special relativity) has been studied extensively; a general pseudo-Riemannian metric is the case of general relativity. We restrict attention here to the flat case. The requirement that $\mathcal{A}$ has vanishing first variation, is equivalent to the equations of motion (1) or (15). In classical physics the action plays no more fundamental role; however in section II we discover that the action has another, central significance in quantization.
Associated with each action functional above is a second functional, the *Euclidean action*
\[
\mathcal{A}_E = \int_M \mathcal{L}_E \, d\tilde{x} \, dt.
\]
Here $\mathcal{L}_E$ differs from $\mathcal{L}$ only in that it is calculated with the Euclidean (flat) metric with signature $(++++)$. Here again, the requirement that the first variation of $\mathcal{A}_E$ vanishes is equivalent to variational equations. Unlike the case above, these equations are elliptic (rather than hyperbolic) on account of the metric. For the two cases above, they are
\[
(1') \quad \Delta \varphi + P'(\varphi) = 0,
\]
\[
(12') \quad D^* F = 0,
\]
where $^*$ depends upon the metric, and $\Delta$ is the Laplace operator.

In physics we are used to the role of classical hyperbolic equations of propagation. For the remainder of this talk, we consider the elliptic Yang–Mills case (12'). One might ask, "Why?" Aside from the mathematical appeal of the problem, there is an important role for these equations in physics: The elliptic problem provides a basis for understanding the quantization of the hyperbolic, classical equation. Here we study the mathematics of the elliptic, classical problem. In §§ II, III we discuss the application.

### I.3. Classical elliptic problems

We return to the Yang–Mills equations (12') with action $\mathcal{A}_E$ of the form (16). Two cases:

**Case A.** $F \in L_2$, $^* F = \pm F$. This is the case of instantons (antiinstantons). The first explicit example of such a connection form was discovered in 1975 by Polyakov and coworkers [4]. Recently, Atiyah–Hitchin and independently Manin–Drinfeld gave explicit formulas for all solutions [2], [10], see also [6]. The condition $F = \pm ^* F$ means that any curvature (which must satisfy $DF=0$) automatically satisfies $D^* F = 0$. The condition $^* F = \pm F$ becomes the integrability condition for a complex structure, and the classifications of these structures is dealt with by methods of algebraic geometry. See Atiyah in these PROCEEDINGS.

It has been conjectured that *all* solutions with $F \in L_2$ must satisfy $^* F = \pm F$. In that case, all square-integrable solutions would already be known. A partial proof of this conjecture (i.e. that local maxima of $\mathcal{A}_E$ satisfy $^* F = \pm F$) has been announced recently [5].

**Case B.** $F \notin L_2$. In case $F \notin L_2$, then $^* F \neq \pm F$, in general. This problem is no longer purely algebraic, but involves hard analysis. These solutions appear to have interesting interpretations and applications. We shall concentrate here entirely on $F \notin L_2$. Only partial results are known for $F \notin L_2$, and many general questions remain open.

The non-$L_2$ character of $F$ arises from a singular set $S$ for $F$. We can regard (12') as equation on $M = \mathbb{R}^4 \setminus S$, with specified growth near $S$. Alternatively we
can consider (12') as an equation for generalized functions defined on \( \mathbb{R}^4 \) with singular support \( S \). The simplest cases are:

(i) \( S = \) finite set of points (dim \( S \)=0). In this case we interpret the solutions as having point charges at \( S \). In known examples these charges are called "merons" and \( F \) has poles at \( S \).

(ii) \( S = \) curve (dim \( S \)=1). This case of "line charges" can also be interpreted as dipole charges. This case has only been studied in detail for \( G=U(1) \) and on a lattice [15].

(iii) \( S = \) 2-surface. In this case the topology of \( \mathbb{R}^4 \setminus S \) may be nontrivial, for example the fundamental group of \( \mathbb{R}^4 \setminus S \) may not be the identity. In this case the Yang–Mills connections are said to have "vortices" on \( S \). For lack of space, however, we shall not discuss them here. See [22], [11], [18].

1.4. Parallel transport. An important notion in the physical interpretation of gauge theories is the holonomy, defined by the connection \( A \) on the bundle \( P \). Let \( C \) be a closed curve in \( M = \mathbb{R}^4 \setminus S \), starting and ending at \( x \). A point \( g \) in the fibre over \( x \) is moved by parallel transport along \( C \) into a point \( g' \). Thus by varying \( g \), parallel transport along \( C \) yields a mapping of the fibre over \( x \) into itself. This mapping \( U(C) \) is by definition an element of the holonomy group of \( A \) with respect to \( P \), and with base point \( x \). (If \( C_1, C_2 \) are two such curves, and \( C = C_1 \circ C_2 \) is their composition, then \( U(C) = U(C_1) U(C_2) \).) The character, \( \text{Tr} U(C) \), is in fact independent of the base point.

As parametric representation for \( C \), take \( s \in [0,1] \), \( s \mapsto x(s) \in C \). The usual expression for \( U(C) \) is the solutions to the differential equation

\[
\frac{dV(s)}{ds} = A(X(s))V(s), \quad V(0) = I,
\]

where \( X(s) \) is the tangent vector to \( C \) at \( x(s) \). There is a special formula in the physics literature for \( U(C) \), namely

\[
U(C) = P \exp \left( \int_C A \right).
\]

Here \( P \) denotes a "path ordered" or "multiplicative" integral. This notation is motivated by the fact that for smooth \( C \),

\[
U(C) = \lim_{n \to \infty} \exp (n^{-1}A_n) \exp (n^{-1}A_{n-1}) \ldots \exp (n^{-1}A_1),
\]

where \( A_k = A(X(k/n)) \).

1.5. \( S = \) points in a plane. We consider the special case where the singular set \( S \) consists of points lying in a place. Meron connections are characterized vanishing Chern class on \( M \),

\[
\text{Tr} \, F \wedge F = 0.
\]
REMARK. All known examples have the property that for a curve \( C \) in the place of \( S \), the holonomy is

\[ U(C) = (-1)^n, \]

where \( n \) is the number of points enclosed by \( C \).

We first consider the subcase that all the points of \( S \) lie on a line \( l \). The construction of the connection \( A \) then simplifies by the use of cylindrical symmetry to eliminate redundant coordinates. In the case the group \( G=SU(2) \), the elliptic system \( D*F=0 \) of 12 equations simplifies to a single equation. We describe this case in a series of steps.

**Step 1. Reduction** [16]. Choose the \( t \) coordinate axis along \( l \). Map the upper half plane \( z=t+i|\tilde{x}| \) onto the unit disc \( D \) in the standard conformal way and consider the equation

\[ \Delta \Psi = \Psi^3 - \Psi \]

in \( D \) with \( \Psi = \pm 1 \) on \( \partial D \). The discontinuities of \( \Psi \) occur exactly on the image on \( \partial D \) of the singular set \( S \). We now denote this image by \( S \). Here \( \Delta \) is the Laplace–Beltrami operator on \( D \) with constant negative curvature,

\[ \Delta = \frac{1}{4}(1-|\tilde{x}|^2)\partial \tilde{\partial}. \]

**Theorem 1.** Every solution to \((21)\) yields an \( SU(2) \) Yang–Mills connection on \( M \), given by

\[ A = \frac{1}{2}(\Psi+1)|\tilde{x}|^{-2}(\tilde{x} \times d\tilde{x}) \cdot i\tilde{\sigma}. \]

Here \( i\tilde{\sigma} \) are Lie algebra generators for \( SU(2) \).

**Step 2. Existence;** Jonsson, Zirilli, McBryan, Hubbard [24]. Let \( C \) denote the \( C^\infty \) functions on the interior of \( D \) which are continuous on \( D \setminus S \) and which satisfy the boundary condition for \( \Psi \) as described above.

**Theorem 2.** There exists a solution \( \Psi \in C \) to \((21)\).

The proof of this theorem involves the adaptation of variational methods to singular problems. The naive action functional is not defined on the solution \( \Psi \) to \((21)\). Let \( S_\varepsilon \) denote an \( \varepsilon \)-neighborhood of \( S \) in \( D \), and define

\[ \mathcal{A}_\varepsilon(\Psi) = \int_{D \setminus S_\varepsilon} \left[ |\partial \Psi|^2 + 2(1-|\Psi|^2)^{-2} (\Psi^3-1)^2 \right] d\tilde{x} d\tilde{z}. \]

Although \( \mathcal{A}_\varepsilon(\Psi) \) has no limit as \( \varepsilon \to 0 \), there exists a constant \( a_\varepsilon \) independent of \( \Psi \) such that

\[ \mathcal{A}_{\text{Ren}}(\Psi) = \lim_{\varepsilon \to 0} (\mathcal{A}_\varepsilon(\Psi) - na_\varepsilon), \quad n = \text{cardinality } S, \]

exists for \( \Psi \in C \). The resulting “renormalized” functional \( \mathcal{A}_{\text{Ren}} \) is bounded from below and can be minimized to obtain a solution to \((21)\). Uniqueness is still an interesting open question.
I mention two steps to generalize these results: In the case $G=\text{SU}(3)$, there are solutions which are not $\text{SU}(2)$ imbeddings. These solutions satisfy two coupled equations, as has been shown by Imbrie [23]. For general points $S$ in a plane, Taubes [30] has reduced the 12-equation $\text{SU}(2)$ system to a 4-equation system in a space of constant negative curvature. While only partial existence results have been established in this case, presumably existence can be proved. The boundary conditions will automatically yield the holonomy (20).

There are many open questions of interest both for geometry and for physics.

II. The constructive field program. Main goals of the constructive field program are to quantize classical equations, to prove existence of solutions to these equations (quantum fields) and to establish properties of these solutions. Ultimately, of course, one hopes to shed new light on physics, by dealing with issues such as those discussed in § III.

Quantum fields are not completely understood either as laws of nature or from a mathematical point of view. For this reason, mathematicians sometimes fail to appreciate the fundamental role that quantum fields play in physics. Physicists, however, are convinced that quantum fields provide an accurate picture of nature. The basic reasons are persuasive and close to the heart of physics. First, the rules for calculation given in field theory texts explain a variety of phenomena, from elementary particles, to atoms to macroscopic matter. Also, these rules yield numbers far more precise than any other physical theory—numbers which can be compared with observations made in the most accurate experiments on nature. In the case of the magnetic moment of the electron, for example, we have eleven decimal place agreement between calculational rules and observations. Both experiment and calculation in this case have been developed over a thirty year period.

This very success, however, places a great constraint on possible mathematical formulations of quantum field theory. In particular, when specialized to the case of these perturbation calculations, the mathematical theory must predict and reproduce the existing rules. This is the case with constructive field models, in that existing models have asymptotic expansions which agree with the rules (perturbation theory) of the textbooks. Presently, we have examples in two and three space-time dimensions, and a framework for a theory in four dimensions. Much of this has been joint work with J. Glimm and by our collaborators.

With this background, let us begin our discussion of quantization. The problem is to construct a one parameter unitary group $U(t)=\exp(-itH)$ on a Hilbert space $\mathcal{H}$, such that the fields are linear operators on $\mathcal{H}$. Also

\begin{equation}
\varphi(t+t_0) = U(t)^{-1}\varphi(t_0)U(t)
\end{equation}

is the solution to the quantized field problem with initial data $\varphi(t_0)$.

A number of methods have been developed in the last ten years to solve such problems. I shall describe only one quantization procedure, starting from the Euclidean action functional $\mathcal{A}_g$. Basically it generalizes the Feynman–Kac repre-
sentation for the kernel of the heat operator \( \exp(tA-tV) \), written as a Wiener integral. Ultimately it yields a formula for \( \exp(-tH) \).

For quantum fields one must replace Wiener integration by a measure on some space of generalized functions \( \mathcal{F} \). We can take these functions as sections of a bundle associated with \( P \), and take \( M=R^4 \). In the case of the nonlinear wave equations, for example, we take \( \mathcal{F}=\mathcal{P}'(M) \), the space of tempered distributions.

Without giving details, one defines a translation invariant probability measure \( d\mu \) on \( \mathcal{F} \), which formally has density \( \exp(\mathcal{A}_R) \). Generally, the measure has the form

\[ d\mu = \lim_{\varepsilon \to 0} Z(\varepsilon)^{-1} \exp(\mathcal{A}_R) \prod_{x \in M(\varepsilon)} dg(x). \]

Here \( Z(\varepsilon) \) is a normalizing factor, and \( M(\varepsilon) \) denotes a discretized \( M \) (e.g. a lattice approximation with lattice spacing \( \varepsilon \)). Also \( dg(x) \) denotes Haar measure on the fibre over \( x \). The hard work of constructive field theory goes into the estimates proving existence and establishing properties of such measures.

The construction of Osterwalder and Schrader [28], [27] gives simple sufficient conditions for the measure \( d\mu \) to yield a quantum field theory. Under these conditions there exists a Hilbert space \( \mathcal{H} \), a canonical projection \( \pi \) from \( L_2(d\mu) \) to \( \mathcal{H} \), and a commutative diagram which defines the Hamiltonian \( \mathcal{H} \), cf. also [19].

\[ \begin{array}{ccc}
L_2(d\mu) & \xrightarrow{t\text{-translation}} & L_2(d\mu) \\
\downarrow \pi & & \downarrow \pi \\
\mathcal{H} & \xrightarrow{e^{-tH}} & \mathcal{H}
\end{array} \]

This is the quantization of \( \mathcal{A} \), since in the perturbation limit it agrees with the standard physics textbook quantization of \( \mathcal{A} \).

The most powerful method to analyze \( d\mu \) is based on expansions. In particular, these expansions have the character

\[ \text{(28) Quantum Theory=Classical + Fluctuation,} \]

\[ d\mu = d\mu_{\text{classical}} \times d\mu_{\text{fluctuation}}. \]

Here \( d\mu_{\text{classical}} \) is the Gaussian measure about a stationary point of \( \mathcal{A}_R \). Thus the classical Euclidean solutions of § I provide the zero order term in the expansion, and they yield approximate quantum fields.

Using these expansions, going by the generic title of cluster expansions, existence theorems and many properties of quantum field examples have been established. I mention in particular two results established since 1974, namely existence of dimension \( d=3 \) quantum fields [11], [26] and nonuniqueness (existence of phase transitions) for \( d=2 \) quantum fields [21], cf. [9]. Both these subjects have developed into whole areas, cf. [19], [20], [13], [12] for references.
Osterwalder and Seiler have established the existence of lattice gauge models [29]. However, little has been proved in the continuum limit. The analysis of classical Euclidean solution is the first of many steps in this constructive program.

**III. Quark confinement.** If quarks are the basic building blocks of protons, neutrons, mesons, etc., why are they not observed in the products of high energy particle collisions? This is one major unanswered question-of-principle in particle theory. Physicists propose that for nonabelian groups $G$, a gauge theory can provide the answer to this puzzle. The method is to show that pairs (and certain triples) of quarks are attracted to each other by a potential $V(L)$ which increases with distance $L$ between the quarks. Thus if one attempts to separate the quarks, the attractive force between them increases, keeping them in a bound state (quark confinement). One would then only observe these bound states (protons, neutrons, mesons, etc.).

We now outline an argument that

\[ V(L) \sim \alpha L, \quad L \to \infty, \quad \alpha > 0. \]

An asymptotically linear potential (29) would ensure confinement. We reduce our discussion of (29) to three basic hypotheses. Establishing these three properties of a quantized gauge theory would yield a mathematical proof of (29). While these particular hypotheses are tentative and may not be correct, it is likely that a variation of the theme is correct. We present this argument because it illustrates a potentially important physical application of classical solutions to the Euclidean Yang–Mills problem, such as those described in § I. The proposal given here appears in [18], [17]. Other proposals can be found in [7], [22], [18].

The first hypothesis is that $V(L)$ can be computed from the expectation of the holonomy group. Let $C$ be an $L \times T$ rectangle and define

\[ V(L) = -\lim_{T \to \infty} \frac{1}{T} \ln \int \text{Tr} \ U(C) \, d\mu. \]

K. Wilson [31] suggested that $V(L)$, defined by (30) and computed in a pure gauge theory, is the physically correct potential between heavy quarks in a gauge theory including both quarks and gauge particles (gluons). One can plausibly justify this assumption, so we accept (30) as our potential.

Our second hypothesis is that the solutions with holonomy (20), i.e. $U(C)=(-I)^n$, dominate the contribution to the integral in (30). In this case,

\[ \int U(C) \, d\mu \sim \text{Pr}(C)_+ - \text{Pr}(C)_-, \quad T \to \infty, \]

where $\text{Pr}_\pm$ are the probabilities in measure $d\mu$ that $C$ encloses an even or an odd number of points.

The third hypothesis is that the distribution of the number of points inside $C$ obeys a Bernoulli-type distribution law with a given density. (Such a statistical
property is commonly true in statistical physics, e.g. in Ising-type models.) By this assumption

\[(32) \quad \Pr(C) + \Pr(C) = \exp[-O(\text{Number of Points in } C)] \]

\[= \exp[-O(\text{Area } C)] \]

\[\sim \exp(-\alpha TL), \]

which ensures (29).

We thus confront the most challenging open mathematical problem: To understand qualitatively and quantitatively quantum fields in dimension four.

References


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