1. Introductory remarks. The past 25 years have witnessed a remarkable change in the field of algebraic geometry, a change due to the impact of the ideas and methods of modern algebra. What has happened is that this old and venerable sector of pure geometry underwent (and is still undergoing) a process of arithmetization. This new trend has caused consternation in some quarters. It was criticized either as a desertion of geometry or as a subordination of discovery to rigor. I submit that this criticism is unjustified and arises from some misunderstanding of the object of modern algebraic geometry. This object is not to banish geometry or geometric intuition, but to equip the geometer with the sharpest possible tools and effective controls. It is true that the lack of rigor in algebraic geometry has created a state of affairs that could not be tolerated indefinitely. Effective controls over the free flight of geometric imagination were badly needed, and a complete overhauling and arithmetization of the foundations of algebraic geometry was the only possible solution. This preliminary foundational task of modern algebraic geometry can now be regarded as accomplished in all its essentials.

But there was, and still is, something else and more important to be accomplished. It is a fact that the synthetic geometric methods of classical algebraic geometry, operating from a narrow and meager algebraic basis and faced by the extreme complexity of the problems of the theory of higher varieties, were gradually losing their power and in the end became victims to the law of diminishing returns, as witnessed by the relative standstill to which algebraic geometry came in the beginning of this century. I am speaking now not of the foundations but of the superstructure which rests on these foundations. It is here that there was a distinct need of sharper and more powerful tools. Modern algebra, with its precise formalism and abstract concepts, provided these tools.

An arithmetic approach to the geometric theories which we were fortunate to inherit from the Italian school could not be undertaken without a simultaneous process of generalization; for an arithmetic theory of algebraic varieties cannot but be a theory over arbitrary ground fields, and not merely over the field of complex numbers. For this reason, the modern developments in algebraic geometry are characterized by great generality. They mark the transition from classical algebraic geometry, rooted in the complex domain, to what we may now properly designate as abstract algebraic geometry, where the emphasis is on abstract ground fields.

2. Revision of the concept of a variety. My object is to present some of the fundamental ideas of abstract algebraic geometry. I must begin with the very
concept of a variety, since the arithmetic point of view led to a subtle revision of this concept and revealed some of its aspects that were not visible in the classical case. What I want to discuss in this connection concerns the following two topics: (1) the set-theoretic modifications in our conception of a variety as a set of points, modifications which were made methodologically necessary by the introduction of the well-known notion of a general point of an irreducible variety, due to Emmy Noether and van der Waerden; (2) the distinction between absolute and relative properties of a variety, a distinction which was made possible only by the admission of arbitrary ground fields.

If we wish to arrange matters so that the general point of an irreducible variety be an actual point of the variety, we must allow point coordinates which are elements of some transcendental extension of the ground field. Furthermore, in the theory of algebraic correspondences it is essential to operate simultaneously with any finite number of independent general points of one and the same variety. It follows that we must have a reservoir of infinitely many transcendental for the point coordinates in our geometry. For these reasons, it was found convenient, following Busemann and André Weil, to fix once and for always a universal coordinate domain; this is to be an algebraically closed field having infinite transcendence degree over the particular ground field \( k \) in which we happen to be interested. Once this universal domain has been fixed, only such ground fields will be allowable which are subfields of the universal domain and over which the universal domain has infinite transcendence degree. We deal then with projective spaces over the universal domain, and all our varieties will be immersed in these spaces.

This being so, if \( k \) is any allowable ground field and if a variety \( V \) admits a system of defining equations with coefficients in \( k \), then \( k \) is said to be a field of definition of \( V \). Naturally, any variety \( V \) has infinitely many fields of definition. A property of \( V \) is relative or absolute according as it does or does not depend on the choice of the field of definition of \( V \). For example, irreducibility of a variety is a relative property. But we also have the so-called absolutely irreducible varieties which are irreducible over each one of their fields of definition. The concept of the general point \((x)\) of an irreducible variety \( V/k \) is a relative concept. On the other hand, the dimension of that irreducible variety \( V/k \), i.e., the transcendence degree of the function field \( k(x) \) of \( V/k \), is an absolute concept.

A necessary and sufficient condition that a variety \( V \) be absolutely irreducible is that it be irreducible over some algebraically closed field of definition. An equivalent condition is the following: an irreducible variety \( V/k \) is absolutely irreducible if the ground field \( k \) is quasi-maximally algebraic in the function field \( k(x) \) of \( V/k \), i.e., if every element of \( k(x) \) which is separably algebraic over \( k \) belongs to \( k \). In theory, it would be sufficient to restrict the study of varieties to absolutely irreducible varieties, since any variety has a unique representation as a sum of absolutely irreducible varieties. However, in practice, and especially in the foundations, such a restriction introduces unnecessary complications.
It may be advisable to give a special name to those varieties which admit every (allowable) ground field as field of definition. Obviously, these are the varieties which are defined over the prime field of the given characteristic \( p \). I propose to call them \textit{universal varieties}. The projective space and the Grassmannian varieties are examples of universal varieties. Another important class of universal varieties is obtained by considering the set of all algebraic varieties, of a given order and dimension, in the \( n \)-dimensional universal projective space and introducing in that set an algebro-geometric structure based on the Chow coordinates of a cycle. The study of these varieties (of which the Grassmannian varieties are special cases) is closely connected with the outstanding problem of developing a theory of algebraic equivalence of cycles on a given variety, and will no doubt be a fundamental object of future research.

The definition of a variety as a set of points having coordinates in the universal domain has some startling, and perhaps unpleasant, set-theoretic implications. We have populated our varieties with points having coordinates which are transcendental over \( k \). Thus, if \( x \) and \( y \) are independent variables the pair \((x, y)\) is a legitimate point of the plane; and—what is worse—if \( x' \) and \( y' \) are other independent variables, then \((x', y')\) is another point of the plane, quite distinct from the point \((x, y)\). This is shocking, especially if we recall that our universal domain has infinite transcendence degree and that consequently we have created infinitely many replica of that ghostlike point \((x, y)\). However, we are dealing here with a methodological fiction which is extremely useful in proving very real theorems. For instance, the entire theory of specializations is based on this set-theoretic conception of a variety, and the entire elementary theory of algebraic correspondences can be developed on that basis in the most effortless and simple fashion. Furthermore, most results concerning irreducible subvarieties of a given variety can be best expressed and derived as results concerning the general points of these subvarieties.

Nevertheless fiction remains fiction even if it is useful, and I feel that perhaps our varieties have altogether too many points to be good geometric objects. As the theory progresses beyond its foundational stage, some cuts and reductions may become necessary. Thus, one may begin first of all by eliminating isomorphic replica of points, by identifying points which are isomorphic over the given ground field \( k \). Or one may restrict the coordinate domain to the algebraic closure of \( k \). Or one may do both of these things at the same time. I have no strong convictions on these issues, and I am quite content in leaving their settlement to the future development of the theory of algebraic varieties. But to round up this discussion, let me indicate briefly some topological aspects of these issues.

Given a variety \( V \) and given any field \( k \) (not necessarily a field of definition of \( V \)), there is a natural topology on \( V \), \textit{relative to} \( k \): it is the topology in which the closed sets are intersections of \( V \) with varieties which are defined over \( k \). In particular, if \( V \) itself is defined over \( k \), then the closed sets on \( V \) are the subvarieties of \( V \) which are also defined over \( k \), and in terms of this topology
general points and specialization of points are easily defined. Thus, a general point of an irreducible variety $V/k$ is a point whose closure is the entire variety $V$; a point $Q$ is the specialization of a point $P$, over $k$, if $Q$ belongs to the closure of $P$. It is clear that in this topology even the weakest separation axioms are not valid. The only points of $V$ which constitute closed sets are the points having coördinates which are pure inseparable over $k$. Hence $V$ is not a $T_1$-space. It is not even a $T_0$-space, for if $P$ and $Q$ are $k$-isomorphic points, then each belongs to the closure of the other. However, if we identify $k$-isomorphic points of $V$, we restore the separation axiom $T_0$. If, moreover, we restrict the coördinate domain to the algebraic closure of the ground field $k$, then $V$ becomes a $T_1$-space.

An even more radical revision of the concept of a variety has been offered by André Weil. His so-called abstract varieties are not defined as subsets of the projective space, but are built out of pieces of ordinary varieties, pieces that must hang together in some well-defined fashion. It is still an open question whether the varieties of Weil can be embedded in the projective space.

In all that precedes I have used deliberately the term "general point" rather than that of "generic point". When the Italian geometers speak of a property enjoyed by the generic point of an irreducible variety, they mean a property that is enjoyed by all points of $V$, except perhaps those which belong to some proper subvariety of $V$. It is clear that this is not equivalent to saying that the general point of $V$ has that given property. There is equivalence if and only if we are dealing with a property of points that can be expressed by equations and inequalities (with coefficients in $k$) connecting the point coördinates. But not every algebro-geometric property is of this category. For instance, it is possible to define algebraically the notion of analytical irreducibility of a variety $V$ at a point. Now if $W$ is, say, an algebraic curve on $V$, $V$ may be analytically reducible at the generic algebraic point of the curve and analytically irreducible at all the general points of the curve. From our point of view, according to which $W$ consists of both algebraic and transcendental points, either one of the following statements is false: (1) $V$ is analytically irreducible at the generic point of $W$; (2) $V$ is analytically reducible at the generic point of $W$. In the complex domain this corresponds to the following state of affairs: at the generic (complex) point of the curve $W$ the variety $V$ decomposes into several analytical branches, but these branches are permuted transitively along closed paths traced on the Riemann surface of the curve $W$. This is a good example of the difference between the meanings of general and generic.

I shall pass now, without delay, to more concrete topics dealing with the major developments in abstract algebraic geometry. Roughly speaking, these major developments come under the following headings: (1) theory of specializations; (2) normal varieties; (3) analytical methods; (4) theory of valuations; (5) Abelian varieties. Time will not allow me to discuss the very general and elegant theory of Abelian varieties which we now possess and for which André
Weil is entirely responsible. Let me, then, first make a few remarks about the theory of specializations.

3. The theory of specializations. Specialization arguments in abstract algebraic geometry are the arithmetic substitute, or analogue, of continuity arguments of classical algebraic geometry, and have been largely developed by van der Waerden. The theory of specializations centers around one basic fact, concerning extensions of specializations: if an algebraic function \( f \) is defined at the general point of an irreducible variety \( V/k \), then it is possible to extend the domain of the function \( f \) to the entire variety \( V \), including therefore also those points of \( V \) at which the explicit expression of \( f \) is indeterminate. In this statement, the function \( f \) need not be a numerical function; the values of \( f \) may be points of another variety \( V' \), and when that is so, we are dealing with an algebraic correspondence between \( V \) and \( V' \). It is well-known that the theorem on extensions of specializations is equivalent to the existence of resultant systems in elimination theory. Without advocating the elimination of elimination theory, it may be pointed out that also the Hilbert Nullstellensatz, \textit{in its homogeneous form}, can be used as a foundation for the theory of specializations (and hence also of algebraic correspondences). In fact, the Nullstellensatz provides a key to the whole of the elementary theory of algebraic varieties, including such topics as the dimension theory, the principal ideal theorem (in its geometric formulation), the decomposition of a variety under ground field extensions, etc.

In my forthcoming Colloquium book, this part of the theory of varieties is built entirely around the Hilbert Nullstellensatz.

One of the most important applications of the theory of specializations was the development of the general intersection theory. This application is due to van der Waerden and André Weil, with the work of Severi serving as a geometric background for the general plan of this undertaking. At present, then, we have a complete intersection theory which is valid for any nonsingular variety over an arbitrary algebraically closed ground field. A parallel development is the intersection theory for algebraic manifolds due to Chevalley. Chevalley’s theory is an outstanding example of the arithmetization of some of the concepts and methods of the theory of analytic functions which are used in algebraic geometry. As far as the local analytical treatment is concerned, modern algebra has provided us with the necessary tools. I refer to the theory of local rings and their completions, due to Krull and Chevalley and further enriched by important contributions by I. S. Cohen and P. Samuel.

The present intersection theories all have an absolute character, since they refer to an algebraically closed ground field. It is still an open question whether there exists a consistent \textit{relative} intersection theory, i.e., an intersection theory relative to a given ground field. The fact that there exists such a thing as the \textit{relative order} of a variety seems to indicate the possible existence of a relative intersection theory. Another unsolved question is whether there exists a rea-
sonable intersection theory on varieties which have singularities, for instance—and above all—on normal varieties. The example of algebraic cones shows that in the case of singular varieties one may have to use fractional intersection numbers and—more generally—fractional cycles.

4. Normal varieties. I shall now discuss briefly the concept of a normal variety, especially from the standpoint of the theory of specializations. This concept, which is purely arithmetic in character, turned out to be a useful contribution even to classical algebraic geometry.

An irreducible variety $V/k$ is said to be normal at a point $Q$ if the local ring of $V$ at that point is integrally closed. The variety $V$ is normal if it is locally normal at every point. Normality is a relative property. The normalization of a variety $V$ consists in passing from $V$ to a birationally equivalent variety $\overline{V}$ such that: (1) $\overline{V}$ is normal; (2) the birational transformation between $V$ and $\overline{V}$ has no fundamental points on $V$, i.e., to every point of $\overline{V}$ there corresponds on $\overline{V}$ at most a finite number of points. By these two conditions the normal variety $\overline{V}$ is uniquely determined by $V$, to within a regular birational transformation (a birational transformation is regular if it is $(1,1)$ without exceptions and if corresponding points have the same local ring). Normal varieties were originally introduced in connection with the problem of the resolution of singularities, for a normal variety of dimension $r$ has the property that its singular locus is of dimension at most $r - 2$. This property of normal varieties is connected with the well-known fact that as far as the minimal prime ideals are concerned, the ideal theory of integrally closed Noetherian domains does not differ essentially from the classical ideal theory of Dedekind domains. At any rate, the process of normalization does have the effect of resolving all the singular loci of $V$, of dimension $r - 1$.

But there are other properties of normal varieties which are of particular interest for the theory of specializations, and hence also for the general theory of algebraic correspondences. Suppose that $V$ is a normal variety and that $T$ is a birational transformation of $V$ into some other variety $V'$. Then the following theorem holds: if to a given point $Q$ of $V$ there corresponds on $V'$ more than one point, then (1) the point $Q$ is fundamental for the birational transformation $T$, i.e., to $Q$ there corresponds on $V'$ an infinite set of points, and (2) the set of points of $V'$ which correspond to $Q$ is a variety, all irreducible components of which are of positive dimension. The really significant and nontrivial part of this theorem is the second part. It is this that I have given in a Transactions paper as the “main theorem” on birational transformations and for which I gave a short and simple proof, based on valuation theory and the theory of local rings, in a recent note in Proc. Nat. Acad. Sci. U. S. A. In terms of specializations, the above “main theorem” signifies, roughly speaking, that, in the case of normal varieties, the presence of an isolated specialization implies the uniqueness of that specialization. As was pointed out by André Weil, this property of normal varieties leads at once to the proof of the uniqueness of the intersection mul-
tiplicity of two varieties at a common isolated intersection. The uniqueness of intersection multiplicity is, on the other hand, the crucial point of the whole of intersection theory, and the fact that this point can be disposed of in such a casual manner by the use of a general theorem on normal varieties illustrates the usefulness of the concept of normality. The treatment of the intersection theory would be further simplified, in fact the whole theory would become almost trivial, if one could prove the normality of any complete algebraic system of cycles in the projective space, i.e., the normality of the Chow-van der Waerden representative variety of such a system. It would even be sufficient to prove that this variety is analytically irreducible everywhere, for in that case the normalization process would lead to another representative variety, whose points are still in (1,1) correspondence with the cycles of the system (without exceptions) but which is normal.

Another important aspect of normal varieties has to do with the theory of complete linear systems. Any normal variety \( V \) has the following characteristic property: the hypersurfaces of a sufficiently high order \( n \) cut out on \( V \) a complete linear system \( | nC | \), where \( C \) is any hyperplane section of \( V \). This result, in conjunction with Hilbert's postulation formula and the existence of derived normal models, leads at once to an expression of the dimension of the complete system \( | nC | (n \text{ large}) \), whether \( V \) is normal or not. In a joint paper of Muhly and myself, now in course of publication in Trans. Amer. Math. Soc., we define the virtual arithmetic genus \( p(V) \) as the constant term in Hilbert's postulation formula of \( V \). In the case of algebraic surfaces, we prove that, in any birational class \( \{ V \} \) of normal surfaces \( V \), the numerical character \( p(V) \) is a monotone nonincreasing function of \( V \), with respect to the following partial ordering of the class: \( V < V' \) if the birational transformation from \( V \) to \( V' \) is single-valued without exceptions. An essential ingredient of the proof is the remark that, since a normal surface \( V \) has only a finite number of singular points, the generic hyperplane section of \( V \) is normal (since it is a curve free from singularities). Now the normality of a generic hyperplane section of any normal variety has recently been established by Seidenberg. In virtue of this interesting result of Seidenberg, the monotone character of the virtual arithmetic genus \( p(V) \) can now be regarded as established for normal varieties of any dimension. In particular, \( p(V) \) is invariant under regular birational transformations.

Added in proof: In view of the technical difficulties of the proof of Seidenberg's theorem, we point out that the results of our joint paper with Muhly do not actually require Seidenberg's theorem. All that is needed is the following statement: the general hyperplane section of a normal variety \( V/k \) \((k \text{ algebraically closed})\) is absolutely normal. This statement is an immediate consequence of the theorem of Bertini, of the classical Jacobian criterion for simple points, and of Weil's characterization of absolutely normal varieties.

In the joint paper of Muhly and myself it is proved that (a) the function \( p(V) \), defined in a given birational class of varieties of dimension \( \leq 3 \), has a
minimum; (b) this minimum is reached for the nonsingular varieties of the class; (c) this minimum is equal to the effective arithmetic genus of the field of algebraic functions determined by the given birational class. Each of these statements represents an unsolved problem for varieties of dimension greater than three.

5. Holomorphic functions and the principle of degeneration. The "main theorem" on birational transformations is a special case of a much more general "connectedness theorem" on algebraic correspondences, a theorem which in its turn contains as a special case a principle of degeneration for varieties over arbitrary ground fields. The proof of this theorem is based on a theory of abstract holomorphic functions which I have developed in a paper now in course of publication in the Memoirs of the American Mathematical Society, and which represents an extension of the analytical methods of abstract algebraic geometry from a local theory to a theory in the large. The use of normal varieties is essential in this theory. I shall now briefly outline the geometric background and the underlying ideas of this work.

In the classical case, the principle of degeneration (first formulated by Enriques) asserts that if an irreducible variety \( V \) varies continuously and degenerates in the limit into a reducible variety \( V_0 \), then this limit variety is connected. This principle is almost self-evident, since \( V_0 \) is a continuous image of the irreducible—and therefore connected—variety \( V \). I say "almost evident", because in order to assert that \( V_0 \) is a continuous map of \( V \) it would be necessary to show that the continuous variation of \( V \) can be accompanied by a continuous deformation of \( V \) into \( V_0 \). The existence of such a deformation has always been taken more or less for granted.

Now, the principle of degeneration can easily be transformed into an equivalent statement in which no reference is made to continuity or limits and which therefore makes sense also in the abstract case. First of all, if \( V \) is a variety and \( k \) is any ground field (not necessarily a field of definition of \( V \)), then the expression "\( V \) is connected over \( k \)" has a meaning, since \( V \) has a natural topology over \( k \). In particular, if \( k \) is an algebraically closed field of definition of \( V \) and if \( V \) is connected over \( k \), then it is easy to see that \( V \) is connected over every one of its fields of definition. We say then that \( V \) is absolutely connected. This definition can then be extended in an obvious fashion to effective algebraic cycles, i.e., to formal linear combinations, with positive integral coefficients, of absolutely irreducible varieties of the same dimension \( r \). Now let \( M \) be an irreducible algebraic system of \( r \)-dimensional cycles, and suppose that \( M \) is defined and irreducible over a given ground field \( k \). Then I prove the principle of degeneration under the following form:

\[ \text{If the general cycle of } M/k \text{ is absolutely irreducible, then every cycle in } M \text{ is absolutely connected.} \]

It is easy to transform this principle into a statement concerning the incidence correspondence associated with the system \( M \), i.e., the correspondence
in which to every cycle of \( M \) correspond all the points which belong to that cycle. The transformed statement can be itself incorporated into the following more general connectedness theorem on algebraic correspondences:

**Theorem.** Let \( T/k \) be an irreducible algebraic correspondence between two varieties \( V \) and \( V' \) and let \( (P, P') \) be a general point pair of \( T/k \). We make the following assumptions:

1. \( T^{-1} \) is rational [i.e., \( k(P) \subset k(P') \)] and semi-regular.
2. The field \( k(P) \) is maximally algebraic in \( k(P') \). Let \( W/k \) be any connected subvariety of \( V \). Then if \( V \) is analytically irreducible (in particular, locally normal) at each point of \( W \), the total transform of \( W \) under \( T \) is a \( k \)-connected subvariety of \( V' \).

This last theorem covers a good deal more ground than does the principle of degeneration. The incidence correspondence of an algebraic system has the special property (not shared by arbitrary correspondences) that it has no fundamental points: to every point of the representative variety \( V \) of the system \( M \) there corresponds, on the carrier variety \( V' \), a variety of the same fixed dimension \( r \), \( r \) = the dimension of the cycles of the system \( M \). If we deal, however, with an arbitrary correspondence \( T \) and if a point \( Q \) of \( V \) happens to be a fundamental point of \( T \), then the variety \( T\{Q\} \), whose connectedness is being claimed, may very well have dimension higher than that of the total transform of the general point of \( V/k \). The connectedness of that variety \( T\{Q\} \) is, in that case, not at all trivial even in the classical case (and—to our knowledge—has never been proved in the classical case, even for birational transformations \( T \)).

It is in the proof of this theorem that the holomorphic functions come directly into play. With the given subvariety \( W \) of \( V \) we associate "functions" on \( V \) which "are defined and holomorphic" along \( W \). These functions are, by definition, certain specified elements of the direct product of the completions of the local rings of \( V \) at the various points of \( W \). They are those elements of this direct product which can be represented by a finite number of sequences of elements of the function field of \( V \), in such a manner that (1) each sequence converges uniformly on some open subset \( \Gamma \) of \( W \) and (2) the sets \( \Gamma \) cover \( W \). These functions form a ring which we denote by \( o^* \). As a first tangible evidence of the nonartificiality of this new concept, we have the following:

**Connectedness criterion.** If \( V \) is analytically irreducible at each point of \( W \), then \( W/k \) is connected if and only if the ring \( o^* \) of holomorphic functions along \( W \) is an integral domain.

With this theorem, we are still very far from the proof of the connectedness theorem for algebraic correspondences. The high point, and also the most difficult part, of the whole theory is still to come. It is represented by a theorem of
invariance of rings of holomorphic functions under rational transformations. It is noteworthy that while the proof of the principle of degeneration in the classical case is essentially a simple exercise in topology, our proof of this principle in the abstract case can be given only after a long and difficult journey. The end of this journey is as follows:

Let $T$ be an algebraic correspondence between two irreducible varieties $V/k$ and $V'/k$ such that $T^{-1}$ is a rational transformation, semi-regular at each point of $V'$. Let $W'$ be the total transform $T[W]$ of $W$, where $W$ is any subvariety of $V$. We have the ring $o_{W'}^*$ of functions on $V'$, defined and holomorphic along $W'$. We first show that there always exists a natural isomorphism $H_{w, w^*}$ of $o_w^*$ into $o_{W'}^*$. The fundamental theorem of invariance asserts the following:

**Theorem.** Let $(P, P')$ be a general point pair of $T/k$. If (a) $k(P)$ is maximally algebraic in $k(P')$ and (b) if $V$ is locally normal at each point of $W$, then $H_{w, w^*}$ is an isomorphism of $o_w^*$ onto $o_{W'}^*$. [Note that condition (a) is automatically satisfied if $T$ is a birational transformation.]

This theorem, together with the above connectedness criterion, gives immediately the connectedness theorem for algebraic correspondences.

There is a number of very difficult problems suggested by the theory of holomorphic functions and which are still open even in the classical case. One of them is to prove that:

I. The ring $o_{W'}^*$ is Noetherian.

The elements of $o_{W'}^*$ which belong to the field of rational functions on $V$ form themselves a ring, denoted by $o_W$. This ring is the intersection of all the local rings of the points of $W$. The second problem is to prove the following:

II. The ring $o_W$ is Noetherian.

The non units in $o_W$ form an ideal, say $m$. Whether or not $o_W$ is Noetherian, one may consider the completion of $o_W$ with respect to the powers of this ideal. The next problem is to prove the following:

III. If $W$ is connected, then $o_{W'}^*$ is the completion of $o_W$.

A special case of III is the following conjecture: if $o_W$ consists only of constants, then $o_{W'} = o_{W'}^*$. The existence of nonconstant holomorphic functions on $V$, defined along $W$, is closely connected with the existence of a rational transformation $T$ of $V$ into some other variety $V'$, such that $W$ is the total transform of a point of $V'$. It is obvious that the latter implies the former, but I have no proof of the converse. At any rate, if $W$ can be transformed into a point $Q'$ of some variety $V'$ by a rational transformation, then it is easily seen that $o_{W'} = o_{Q'}^*$, and from the theorem of invariance of rings of holomorphic functions it follows that $o_{W'}^* = o_{Q'}^*$. Hence in this case the conjectures I, II, III are true.

6. Transcendental theory of specializations. My report is very incomplete as it stands, but it would be glaringly incomplete if I had not said anything about the role of valuation theory in abstract algebraic geometry. The ordinary theory
of specializations applies to finite sets of quantities; it is a finite theory of specializations. This theory does not do everything that continuity does in classical geometry, for it contains nothing that corresponds to the notion of a branch, whether algebraic or transcendental. It does not tell us anything about the different modes of approach to a point on a given variety. For this reason, the finer differential aspects of the local geometry of a variety, in particular the analysis of the neighborhood of a singular point, are outside the province of the finite theory of specializations. What was needed here was a theory which deals with the simultaneous specialization of all the rational functions on a given variety, therefore a transcendental theory of specializations. Valuation theory meets precisely this requirement. It is to be observed that it was precisely the general valuation theory, as developed by Krull, i.e., the theory of valuations having arbitrary value groups (nondiscrete as well as discrete, non-Archimedean value groups as well as Archimedean), that turned out to be the necessary tool for the solution of such a concrete algebro-geometric problem as the local uniformization of algebraic varieties and for partial progress in the problem of the resolution of singularities. Nothing less than the general valuation theory would have served that purpose, and that is so for the following two reasons:

(1) Without the general concept of a valuation it is not possible even to formulate the problem of local uniformization in pure algebraic terms. The nearest algebraic substitute for the neighborhood of a point \( Q \) of a variety \( V \) is the set of all modes of approach to the point \( Q \), therefore essentially, the set of all valuations of the function field of \( V \) which have center at \( Q \). Here, by the center of a valuation \( v \) we mean the point whose coordinates are the \( v \)-residues of the coordinates of the general point of the variety \( V \). And the only statement which can reproduce in algebraic terms, and without loss of power, that what has been a classical conjecture, namely that the complete neighborhood of the point \( Q \) can be represented by a finite number of power series expansions, is the statement that any valuation \( v \) of center \( Q \) can be uniformized with respect to \( V \). By this I mean that there exists a birational transform \( V' \) of \( V \) such that (a) the center of \( v \) on \( V' \) is a simple point \( Q' \) of \( V' \) and (b) the local ring of \( V \) at \( Q \) is contained in the local ring of \( V' \) at \( Q' \) (this second condition implies that every element of the function field of \( V \) which is holomorphic at \( Q \) is also holomorphic at \( Q' \)).

(2) The set of all valuations of the function field of \( V \) is a compact space in a suitable natural topology of that set. This space is called the Riemann surface of the function field of \( V \). Now it is the compactness of the Riemann surface that makes it possible to apply the theorem of local uniformization of abstract algebraic geometry (i.e., the uniformization of a single valuation) to the classical problem of local uniformization of the complete neighborhood of a point of a variety. No reasonable proper subset of the set of all valuations has that compactness property. Thus, the set of all discrete valuations is not compact (except in the case of curves, in which case all valuations are, of course, discrete).
and neither is the set of all algebraic valuations (represented by algebraic branches).

The problem of local uniformization has been settled, so far, only in the case of characteristic zero. The extension of the present proof to the case of non-zero characteristic will call for considerable algebraic skill and ingenuity. It is not a problem for geometers; it is a problem for algebraists with a feeling and intuition for all the unpleasant things that can happen in the case of nonzero characteristic. The essential difficulties of this problem are already apparent in the case of algebraic surfaces over an algebraically closed ground field. In this case I can prove the theorem of local uniformization for every valuation except those which are nondiscrete and have rational rank 1 (i.e., those whose value group is a dense set of rational numbers). The case which is particularly difficult is the one in which the value group of the valuation contains rational numbers having in the denominator arbitrarily high powers of the characteristic \( p \). In this case, already such a simple surface as \( z^p = f(x, y) \) becomes untractable.

Also the problem of the resolution of singularities is still unsolved; or—to put it into more cautious terms—no solution of this problem has ever come to my direct attention. For characteristic zero, this problem has been solved, so far, only for varieties of dimension \( \leq 3 \).

7. Concluding remarks. The Italian geometers have erected, on somewhat shaky foundations, a stupendous edifice: the theory of algebraic surfaces. It is the main object of modern algebraic geometry to strengthen, preserve, and further embellish this edifice, while at the same time building up also the theory of algebraic varieties of higher dimension. The bitter complaint that Poincaré has directed, in his time, against the modern theory of functions of a real variable cannot be deservedly directed against modern algebraic geometry. We are not intent on proving that our fathers were wrong. On the contrary, our whole purpose is to prove that our fathers were right.

The arithmetic trend in algebraic geometry is not in itself a radical departure from the past. This trend goes back to Dedekind and Weber who have developed, in their classical memoir, an arithmetic theory of fields of algebraic functions of one variable. Abstract algebraic geometry is a direct continuation of the work of Dedekind and Weber, except that our chief object is the study of fields of algebraic functions of more than one variable. The work of Dedekind and Weber has been greatly facilitated by the previous development of classical ideal theory. Similarly, modern algebraic geometry has become a reality partly because of the previous development of the general theory of ideals. But here the similarity ends. Classical ideal theory strikes at the very core of the theory of algebraic functions of one variable, and there is in fact a striking parallelism between this theory and the theory of algebraic numbers. On the other hand, the general theory of ideals strikes at most of the foundations of algebraic geometry and falls short of the deeper problems which we face in the post-foundational stage. Furthermore, there is nothing in modern commutative
algebra that can be regarded even remotely as a development parallel to the
theory of algebraic function fields of more than one variable. This theory is
after all itself a chapter of algebra, but it is a chapter about which modern
algebraists knew very little. All our knowledge here comes from geometry. For
all these reasons, it is undeniably true that the arithmetization of algebraic
gometry represents a substantial advance of algebra itself. In helping geom-
etry, modern algebra is helping itself above all. We maintain that abstract
algebraic geometry is one of the best things that happened to commutative
algebra in a long time.

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