Proceedings of the

International Congress of Mathematicians

Seoul 2014

SEOUL ICM 2014
INTERNATIONAL CONGRESS OF MATHEMATICIANS

Proceedings of the **International Congress of Mathematicians**

Seoul 2014

VOLUME III
Invited Lectures

Editors
Sun Young Jang
Young Rock Kim
Dae-Woong Lee
Ikkwon Yie

KM KYUNG MOON SA

Editors

Sun Young Jang, University of Ulsan
Young Rock Kim, Hankuk University of Foreign Studies
Dae-Woong Lee, Chonbuk National University
Ikkwon Yie, Inha University

Technical Editors

Young Rock Kim, The Korean TEX Society
Hyun Woo Kwon, The Korean TEX Society

Proceedings of the International Congress of Mathematicians
August 13–21, 2014, Seoul, Korea

Published by

KYUNG MOON SA Co. Ltd.
174, Wausan-ro Mapo-gu Seoul, Korea
Tel: +82-2-332-2004 Fax: +82-2-336-5193
E-mail: kyungmoon@kyungmoon.com
Homepage: www.kyungmoon.com

ISBN 978-89-6105-806-3
ISBN 978-89-6105-803-2 (set)

Printed in Korea

Contents

7. Lie Theory and Generalizations

8. Analysis and its Applications

9. Dynamical Systems and Ordinary Differential Equations

10. Partial Differential Equations

11. Mathematical Physics

7. Lie Theory and Generalizations

$\widehat{\mathcal{D}}$-modules on rigid analytic spaces

Konstantin Ardakov

Abstract. We give an overview of the theory of $\widehat{\mathcal{D}}$-modules on rigid analytic spaces and its applications to admissible locally analytic representations of p-adic Lie groups.

Mathematics Subject Classification (2010). Primary 14G22; Secondary 16S38, 22E50, 32C38.

Keywords. $\mathcal{D}$-modules, rigid analytic geometry, Beilinson-Bernstein localisation, locally analytic representations, p-adic Lie groups

1. $\widehat{\mathcal{D}}$-modules on rigid analytic spaces

1.1. Rigid analytic spaces. Let K be a field complete with respect to a non-archimedean norm. The ultrametric inequality $|x-y| \leqslant \max|x|,|y|$ implies that the "unit circle" $\{|z| = 1\}$ is *open* in the affine line over K, and the "closed unit disc" $\{|z| \leqslant 1\}$ is disconnected, being the disjoint union of the unit circle and the open unit disc. This is a basic feature of non-archimedean geometry: every K-analytic manifold is totally disconnected with respect to its natural topology.

In order to make the category of K-analytic manifolds more geometric, Tate invented *rigid analytic spaces* [35], by introducing a new (Grothendieck) topology on this category, with a basis given by *K-affinoid varieties* which are by definition the maximal ideal spectra of *K-affinoid algebras*. The *n^{th}-Tate algebra* is the algebra $K\langle x_1, \ldots, x_n\rangle$ of K-valued functions on the n-dimensional polydisc that can be globally defined by a single power series which converges on the entire polydisc, and a K-affinoid algebra is by definition any homomorphic image of a Tate algebra.

The theory of rigid analytic spaces has now reached maturity comparable to that of the theory of complex analytic manifolds, thanks to the works of Kiehl [20], Raynaud [25], Berkovich [4], Huber [18] and many others. It is now an indispensable part of modern arithmetic geometry, and has found many striking applications such as Tate's uniformisation of elliptic curves with bad reduction, and the proof of the Local Langlands conjecture for GL_n by Harris and Taylor.

1.2. Rigid analytic quantisation. We assume now that K is discretely valued, has characteristic zero and that its residue field has characteristic $p > 0$. Let K° denote its ring of integers and let $\pi \in K^\circ$ be a uniformiser. In a series of papers including [5–7], Berthelot introduced the sheaf of arithmetic differential operators $\widehat{\mathcal{D}^{(m)}_{\mathcal{X},\mathbb{Q}}}$ of level m on every smooth formal K°-scheme $\mathcal{X}$ in an attempt to better understand the p-adic cohomology of algebraic

Proceedings of the International Congress of Mathematicians, Seoul, 2014

varieties in characteristic p.

Let $\mathcal{X} = \widehat{\mathbb{A}^1}$ be the formal affine line over K°. One of the origins of this work was the observation that it is possible to obtain the ring of global sections $\Gamma(\mathcal{X}, \widehat{\mathcal{D}^{(0)}_{\mathcal{X},\mathbb{Q}}})$ by defining a non-commutative multiplication $*$ on the second Tate algebra $K\langle x, y\rangle$, which is completely determined by the relation

$$y * x - x * y = 1.$$

In other words, if t denotes a local coordinate on $\mathcal{X}$ and ∂_t is the corresponding vector field then $\Gamma(\mathcal{X}, \widehat{\mathcal{D}^{(0)}_{\mathcal{X},\mathbb{Q}}})$ is isomorphic to the *Tate-Weyl algebra* $K\langle t; \partial_t\rangle$ which can be defined by π-adically completing the usual Weyl algebra $A_1(K^\circ) = K^\circ[t; \partial_t]$ with coefficients in K°, and inverting π. In this way we view $\Gamma(\mathcal{X}, \widehat{\mathcal{D}^{(0)}_{\mathcal{X},\mathbb{Q}}})$ as a naive "rigid analytic quantisation" of the two-dimensional polydisc.

The aim of this paper is to sketch the construction of the algebra $\widehat{\mathcal{D}}(X)$ of infinite order differential operators on a smooth K-affinoid variety X, developed in joint work with S. J. Wadsley: proofs will appear elsewhere. Morally $\widehat{\mathcal{D}}(X)$ is a "rigid analytic quantisation" of the *entire* cotangent bundle T^*X.

1.3. Lie algebroids. Let $k \to R$ be a morphism of commutative rings. Recall [26] that a *k-R-Lie algebra* or a *Lie algebroid* is a pair (L, a) consisting of a k-Lie algebra and R-module L, together with an R-linear k-Lie algebra homomorphism a from L to the set of k-linear derivations $\mathrm{Der}_k R$ of R, such that $[v, rw] = r[v, w] + a(v)(r)w$ for all $v, w \in L$ and all $r \in R$. It is possible to form a unital associative k-algebra $U(L)$ called the *enveloping algebra* of (L, a) which is generated as a k-algebra by R and L, subject to appropriate natural relations. Enveloping algebras of Lie algebroids simultaneously generalise the ordinary enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ over a field k, and also the algebra $\mathcal{D}(X)$ of (crystalline) differential operators on a smooth affine algebraic variety X over k, since $\mathcal{T}(X) = \mathrm{Der}_k \mathcal{O}(X)$ is itself naturally a k-$\mathcal{O}(X)$-Lie algebra such that $U(\mathcal{T}(X)) = \mathcal{D}(X)$.

The ring $U(L)$ has a natural positive filtration with associated graded ring the symmetric R-algebra $\mathcal{S}(L)$ whenever L is a projective R-module; thus $U(L)$ is an algebraic quantisation of the underlying topological space $\mathrm{Spec}\, \mathcal{S}(L)$. In this way, the enveloping algebra $U(\mathfrak{g})$ can be viewed as an algebraic quantisation of $\mathfrak{g}^* = \mathrm{Spec}\, \mathcal{S}(\mathfrak{g})$, and $\mathcal{D}(X)$ as an algebraic quantisation of the cotangent bundle $T^*X = \mathrm{Spec}\, \mathcal{S}(\mathcal{T}(X))$.

1.4. Quantised rigid analytic cotangent bundles. Let $\mathcal{O}(X)$ be the algebra of rigid K-analytic functions on a smooth K-affinoid variety X, let $\mathcal{O}(X)^\circ$ be its subring of power-bounded elements, and let $\mathcal{T}(X)$ be the Lie algebra of continuous K-linear derivations of $\mathcal{O}(X)$.

Definition. *We say that an $\mathcal{O}(X)^\circ$-submodule $\mathcal{L}$ of $\mathcal{T}(X)$ is a* Lie lattice *if it is a sub K°-$\mathcal{O}(X)^\circ$-Lie algebra of $\mathcal{T}(X)$, is finitely generated as a module over $\mathcal{O}(X)^\circ$ and generates $\mathcal{T}(X)$ as a K-vector space. Let $\widehat{U(\mathcal{L})}$ be the π-adic completion of $U(\mathcal{L})$ and let $\widehat{U(\mathcal{L})_K} := \widehat{U(\mathcal{L})} \otimes_{K^\circ} K$. We define $\widehat{\mathcal{D}}(X)$ to be the inverse limit of the $\widehat{U(\mathcal{L})_K}$ where $\mathcal{L}$ runs over all possible Lie lattices in $\mathcal{T}(X)$.*

Every Lie lattice $\mathcal{L}$ gives rise to a tower

$$\widehat{U(\mathcal{L})_K} \leftarrow \widehat{U(\pi\mathcal{L})_K} \leftarrow \widehat{U(\pi^2\mathcal{L})_K} \leftarrow \cdots$$

of Noetherian Banach K-algebras, whose inverse limit is a Fréchet K-algebra in the sense of non-archimedean functional analysis [32]. Since any two Lie lattices in $\mathcal{T}(X)$ contain a π-power multiple of each other, the inverse limit of this tower is isomorphic to $\widehat{\mathcal{D}}(X)$, regardless of the choice of the Lie lattice $\mathcal{L}$.

Example. Let X denote the closed disc of radius 1 in the affine line over K, with local coordinate t. Then $\mathcal{O}(X)$ is the first Tate algebra $K\langle t\rangle$ and $\mathcal{O}(X)^\circ$ is the subalgebra $K^\circ\langle t\rangle := K\langle t\rangle \cap K^\circ[[t]]$. Let $\mathcal{L} = K^\circ\langle t\rangle\partial_t$, so that $\pi^n\mathcal{L}$ is a Lie lattice in $\mathcal{T}(X)$ for each $n \geqslant 0$ and

$$\widehat{U(\pi^n\mathcal{L})_K} \cong K\langle t; \pi^n\partial_t\rangle$$

is a deformation of the Tate-Weyl algebra over K. Thus

$$\widehat{\mathcal{D}}(X) = \bigcap_{n\geqslant 0} K\langle t; \pi^n\partial_t\rangle = \left\{ \sum_{i=0}^{\infty} a_i\partial_t^i \in K\langle t\rangle[[\partial_t]] : \lim_{i\to\infty} \frac{a_i}{\pi^{in}} = 0 \quad \text{for all} \quad n \geqslant 0 \right\}$$

is naturally in bijection with $\mathcal{O}(T^*X)$.

If $Y \hookrightarrow X$ is an open embedding of smooth K-affinoid varieties and $\mathcal{L}$ is a Lie lattice in $\mathcal{T}(X)$, then $\mathcal{O}(Y)^\circ \otimes_{\mathcal{O}(X)^\circ} \mathcal{L}$ need not be a Lie lattice in $\mathcal{T}(Y)$ in general. However, a sufficiently large π-power multiple of $\mathcal{O}(Y)^\circ \otimes_{\mathcal{O}(X)^\circ} \mathcal{L}$ *is* a Lie lattice in $\mathcal{T}(Y)$, and the functoriality of enveloping algebras of Lie algebroids induces a ring map $\widehat{\mathcal{D}}(X) \to \widehat{\mathcal{D}}(Y)$. We have the following non-commutative analogue of Tate's Acyclicity Theorem:

Theorem. *Let X be a smooth K-affinoid variety. Then $\widehat{\mathcal{D}}$ is a sheaf on X with vanishing higher cohomology.*

This construction extends naturally to a sheaf of K-algebras $\widehat{\mathcal{D}}$ on arbitrary smooth rigid analytic varieties over K.

1.5. Coadmissible $\widehat{\mathcal{D}}$-modules. Recall [33] that Schneider and Teitelbaum defined a *Fréchet-Stein* algebra to be the inverse limit of a countable inverse system of Noetherian K-Banach algebras $(A_n)_{n\in\mathbb{N}}$ with flat transition maps.

Theorem. *Let X be a smooth K-affinoid variety. Then the algebra $\widehat{\mathcal{D}}(X)$ is Fréchet-Stein.*

There is a well-behaved abelian category of *coadmissible A-modules* associated with any Fréchet-Stein algebra A, whose objects are inverse limits of compatible familes $(M_n)_{n\in\mathbb{N}}$ where each M_n is a finitely generated module over A_n. Let X be a smooth rigid K-analytic variety, and let $(X_j)_j$ be an admissible K-affinoid covering of X. It is possible to prove a precise non-commutative analogue of Kiehl's Theorem from [20], which allows us to glue the resulting categories of coadmissible $\widehat{\mathcal{D}}(X_j)$-modules in an appropriate way in order to obtain the category $\mathcal{C}_X$ of *coadmissible $\widehat{\mathcal{D}}$-modules* on X.

Every $\widehat{\mathcal{D}}$-module that is coherent as an $\mathcal{O}_X$-module is coadmissible in this sense. As in the classical theory [17] over $\mathbb{C}$, we may think of these $\widehat{\mathcal{D}}$-modules as rigid vector bundles

equipped with a flat connection, and thereby obtain a link between our $\widehat{\mathcal{D}}$-modules and the well-established theory of p-adic differential equations [19]. There is also a natural exact analytification functor from the category of coherent $\mathcal{D}$-modules on a smooth algebraic variety Y over K to $\mathcal{C}_{Y^{an}}$. For these reasons, we will regard $\mathcal{C}_X$ as an appropriate rigid analytic analogue of the category of coherent algebraic $\mathcal{D}$-modules.

1.6. Functoriality. In the classical setting [17], it is known that the inverse and direct image functors for $\mathcal{D}$-modules preserve $\mathcal{O}$-quasi-coherence, but need not in general preserve $\mathcal{D}$-coherence. Since our category of coadmissible $\widehat{\mathcal{D}}$-modules is modelled on the category of *coherent* algebraic $\mathcal{D}$-modules, and since it is well-known that there is no obvious well-behaved analogue of quasi-coherent $\mathcal{O}$-modules in rigid analytic geometry, it is unreasonable to expect to be able to define direct and inverse image functors in full generality in our current setting. However, given a morphism $f : X \to Y$ between smooth rigid analytic varieties, it is possible to define a *transfer bimodule* $\widehat{\mathcal{D}}_{X\to Y} := \mathcal{O}_X \widehat{\otimes}_{f^{-1}\mathcal{O}_Y} f^{-1}\widehat{\mathcal{D}}_Y$, and a direct image functor

$$\begin{array}{rccc} f_+ & : & \mathcal{C}_X^r & \to & \mathcal{C}_Y^r \\ & & \mathcal{M} & \mapsto & f_*\left(\mathcal{M}\widehat{\otimes}_{\widehat{\mathcal{D}}_X}\widehat{\mathcal{D}}_{X\to Y}\right) \end{array}$$

between the derived categories of coadmissible right $\widehat{\mathcal{D}}$-modules, at least in the case when f is *proper*. It would be interesting to investigate whether the classical inverse and direct image functors extend to our setting in a greater generality.

1.7. Dimension theory. Whenever A is an *Auslander-regular ring* [13], the functor $M \mapsto \mathrm{RHom}_A(M, A)$ induces an anti-equivalence between the derived categories of finitely generated left, and right, A-modules [36]. This allows us to associate with any finitely generated A-module M its *canonical dimension* $d(M)$, defined in terms of the vanishing of the Ext groups $\mathrm{Ext}^j_A(M, A)$.

When A is the ring of regular functions on a smooth affine variety X over a field, $d(M)$ is the Krull dimension of the support of the associated sheaf $\widetilde{M}$ on X.

Theorem. *Let X be the d-dimensional polydisc and let $\mathcal{L}$ be the free $\mathcal{O}(X)^\circ$-submodule of $\mathcal{T}(X)$ spanned by the standard vector fields. For every $n \geqslant 0$, the deformed Tate-Weyl algebra $\widehat{U(\pi^n\mathcal{L})_K}$ is an Auslander-regular ring of global dimension d.*

Proof. This version of Bernstein's Inequality for deformed Tate-Weyl algebras follows from [1, Theorem B]. □

Schneider and Teitelbaum observed in [32, §8] that if $A = \varprojlim A_n$ is a Fréchet-Stein algebra such that each A_n is Auslander-regular of the same global dimension, then the canonical dimension function extends naturally to the category of coadmissible A-modules. It follows from the above result that their formalism applies to our algebras $\widehat{\mathcal{D}}(X)$ whenever X is sufficiently small, and allows us to define the canonical dimension of a coadmissible $\widehat{\mathcal{D}}$-module on an arbitrary smooth rigid K-analytic variety.

Definition. *We say that a non-zero coadmissible $\widehat{\mathcal{D}}$-module is* holonomic *if its canonical dimension is zero.*

1.8. Support and Kashiwara's equivalence. The support of an abelian sheaf on a topological space is a fundamental invariant. Since our sheaves are defined on a space with a

Grothendieck topology, the usual definition of support in terms of stalks seems inferior to the alternative one given by

$$\operatorname{Supp}\mathcal{M} := X - \bigcup \{U \quad \text{admissible open in } X : \mathcal{M}_{|U} = 0\}.$$

It is natural to hope that $\operatorname{Supp}\mathcal{M}$ is an analytic subspace of X for every coadmissible $\widehat{\mathcal{D}}$-module $\mathcal{M}$. However, morally a coadmissible $\widehat{\mathcal{D}}$-module is a coherent sheaf on a rigid analytic quantisation of T^*X and the projection map $T^*X \to X$ isn't proper, so this hope is probably unreasonable. Nevertheless, it seems possible that there is an appropriately large subcategory of coadmissible $\widehat{\mathcal{D}}$-modules whose objects do have analytic support.

As there is no natural exhaustive ring filtration on the sheaf $\widehat{\mathcal{D}}$ due to the presence of completions, it is not clear at present how to define a good analogue of the *characteristic variety* for coadmissible $\widehat{\mathcal{D}}$-modules. Nevertheless it is conceivable that in the future it will be possible to do this by "microlocalising" coadmissible $\widehat{\mathcal{D}}$-modules to appropriate Lagrangian affinoid subspaces of T^*X, and thereby make more precise the words "rigid analytic quantisation". In any case, the notion of support defined above is sufficient for us to be able to formulate a rigid-analytic version of the fundamental *Kashiwara equivalence*:

Theorem. *Let $i : Y \to X$ be a closed immersion of smooth rigid analytic varieties. Then the functor i_+ induces an equivalence of abelian categories between $\mathcal{C}_Y$ and the full subcategory $\mathcal{C}_X^Y$ consisting of objects $\mathcal{M}$ in $\mathcal{C}_X$ with support contained in the image of Y.*

2. p-adic representations of p-adic Lie groups

2.1. Locally analytic representations. Let L be a finite extension of $\mathbb{Q}_p$, assume that our ground field K contains L and let G be a locally L-analytic group. In a series of papers including [29, 31–33], Schneider and Teitelbaum developed the theory of *admissible locally analytic G-representations* in locally convex K-vector spaces. This theory has found applications to several areas, including p-adic automorphic forms [22], p-adic interpolation [15], non-commutative Iwasawa theory [30] and the p-adic local Langlands programme [3, 10–12].

By definition, the *locally analytic distribution algebra of G over K* is the strong dual $D(G,K)$ of the vector space of locally analytic K-valued functions on G. It may be viewed as a certain K-Fréchet space completion of the group ring $K[G]$.

When the group G is compact, Schneider and Teitelbaum showed that $D(G,K)$ is a Fréchet-Stein algebra, so the notion of coadmissible $D(G,K)$-module makes sense. A locally analytic representation V of an arbitrary locally L-analytic group G is *admissible* if its strong dual is coadmissible as a module over the distribution algebra $D(H,K)$ of every compact open subgroup H of G.

One of the most basic problems in this theory is to gain a better understanding of the *irreducible* admissible locally analytic representations of G, or equivalently, the simple coadmissible modules over the distribution algebra $D(G,K)$.

2.2. Arens-Michael envelopes. There is a natural embedding of the Lie algebra $\mathfrak{g}$ of G into $D(G,K)$, which extends to an embedding of K-algebras $U(\mathfrak{g}_K) \hookrightarrow D(G,K)$, where $\mathfrak{g}_K := K \otimes_L \mathfrak{g}$. It follows from the work of Kohlhaase [21] that the closure of the image

consists of the K-valued locally analytic distributions on G which are supported at the identity in a suitable sense, and is isomorphic to the Hausdorff completion $\widehat{U(\mathfrak{g}_K)}$ of $U(\mathfrak{g}_K)$ with respect to all submultiplicative seminorms on $U(\mathfrak{g}_K)$. Following Schmidt [27], we call this completion the *Arens-Michael envelope* of $U(\mathfrak{g}_K)$.

If $\{x_1, \ldots, x_d\}$ is a K-basis for $\mathfrak{g}_K$, then $\widehat{U(\mathfrak{g}_K)}$ can be identified with the vector space of power series in the x_i converging everywhere on K^d:

$$\widehat{U(\mathfrak{g}_K)} = \left\{ \sum_{\alpha \in \mathbb{N}^d} \lambda_\alpha \mathbf{x}^\alpha \in K[[x_1, \ldots, x_d]] : \sup_{\alpha \in \mathbb{N}^d} |\lambda_\alpha| r^{-|\alpha|} < \infty \quad \text{for all} \quad r > 0 \right\}.$$

This allows us to view $\widehat{U(\mathfrak{g}_K)}$ as a "rigid analytic quantisation" of $\mathfrak{g}_K^*$.

2.3. Infinitesimal central characters. Assume from now on that G is an open subgroup of the group of L-rational points of a split semisimple L-algebraic group $\mathbf{G}$. Let $\mathfrak{g}$ be the Lie algebra of G. The classical "Harish-Chandra" centre $Z(\mathfrak{g}_K)$ of $U(\mathfrak{g}_K)$ remains central in $D(G, K)$, and Kohlhaase showed that the Arens-Michael envelope of $Z(\mathfrak{g}_K)$ is in fact the centre of $D(G, K)$ whenever the centre of G is trivial.

Theorem. *Let M be a simple coadmissible $\widehat{U(\mathfrak{g}_K)}$-module. Then there exists a K-algebra homomorphism $\theta_M : Z(\mathfrak{g}_K) \to \overline{K}$ such that $z \cdot m = \theta_M(z)m$ for all $z \in Z(\mathfrak{g}_K)$ and $m \in M$. Thus M has an* infinitesimal central character.

This result follows from our analogue of Quillen's Lemma [1, Theorem D] for affinoid enveloping algebras. Dospinescu and Schraen have extended this Theorem to simple coadmissible $D(G, K)$-modules in [14].

2.4. Beilinson-Bernstein Localisation. It follows from Theorem 2.3 that in the quest for simple coadmissible $\widehat{U(\mathfrak{g}_K)}$-modules, it will be sufficient to study the central quotients

$$\widehat{\mathcal{U}^\theta} := \widehat{U(\mathfrak{g}_K)}/\langle \ker \theta \rangle$$

for every central character $\theta : Z(\mathfrak{g}_K) \to \overline{K}$ in turn. It is well-known that a good way to understand the uncompleted algebras $U(\mathfrak{g}_K)/\langle \ker \theta \rangle$ is through *geometric representation theory* [2], which interprets them as rings of globally defined twisted differential operators on the flag variety $\mathbf{G}/\mathbf{B}$ associated with $\mathfrak{g}_K$.

Theorem. *Let $(\mathbf{G}/\mathbf{B})^{\mathrm{an}}$ be the rigid analytic flag variety. Let $\mathfrak{t}_K$ be a Cartan subalgebra of $\mathfrak{g}$ and let $\lambda \in \mathfrak{t}_K^*$ be a dominant regular weight. Then there is an equivalence of abelian categories*

$$\left\{ \begin{array}{c} coadmissible \\ \widehat{\mathcal{U}^{\lambda\phi}}\text{–}modules \end{array} \right\} \cong \left\{ \begin{array}{c} coadmissible \\ \widehat{\mathcal{D}^\lambda}\text{–}modules\ on\ (\mathbf{G}/\mathbf{B})^{\mathrm{an}} \end{array} \right\}$$

where $\phi : Z(\mathfrak{g}_K) \to S(\mathfrak{t}_K)$ is the Harish-Chandra homomorphism.

Here $\widehat{\mathcal{D}^\lambda}$ denotes a λ-twisted version of the ring $\widehat{\mathcal{D}}$ from §1.4. This rigid analytic analogue of the Beilinson-Bernstein Localisation Theorem has several precursors, including [8, Theorem 3.2], [23, Théorème 2.1] and [1, Theorem C].

2.5. Canonical dimension estimates. Schneider and Teitelbaum's dimension theory from [32, §8] applies not only to our algebras $\widehat{\mathcal{D}}(X)$ as explained in §1.7 above, but also to the Arens-Michael envelopes $\widehat{U(\mathfrak{g}_K)}$ and the distribution algebras $D(G,K)$ whenever G is compact locally $\mathbb{Q}_p$-analytic group. The canonical dimension of a coadmissible $D(G,K)$-module M is zero precisely when M is finite dimensional as a K-vector space.

Using the folklore observation [9] that the main mechanism behind the Beilinson-Bernstein Localisation Theorem is a quantisation of the Springer resolution, we obtain the following analogue of Bernstein's Inequality for $\widehat{U(\mathfrak{g}_K)}$.

Theorem. *Suppose that p is a very good prime for $\mathbf{G}$. Let r be the half the smallest possible dimension of a non-zero $\mathbf{G}(K)$-orbit in $\mathfrak{g}_K^*$ and let M be a coadmissible $\widehat{U(\mathfrak{g}_K)}$-module. Then either $d(M)=0$ or $d(M)\geqslant r$.*

We refer the reader to [1, §6.8, §9.9] for the meaning of the words "very good prime", and the precise values that the invariant r takes. Roughly speaking, r is the square root of the dimension of G: for example if $\mathbf{G}=SL_n$ then $r=n-1$. Theorem 2.5 is an analogue of Smith's Theorem for classical enveloping algebras of complex semisimple Lie algebras [34], and follows easily from the corresponding statement for semisimple affinoid enveloping algebras [1, Theorem 9.10]. A similar estimate holds for semisimple Iwasawa algebras [1, Theorem A], and semisimple locally analytic distribution algebras [28, Theorem 9.9].

2.6. Equivariant $\widehat{\mathcal{D}}$-modules. At the time of writing, the main applications of our methods to the theory of locally analytic representations have been the dimension estimates explained above. However, we believe that there is significant scope for other applications. Using $\widehat{\mathcal{D}}$-modules it should be possible to construct irreducible coadmissible $D(G,K)$-modules geometrically, and to better understand the admissible representations arising in the p-adic local Langlands programme for $\mathrm{GL}_2(\mathbb{Q}_p)$ and for other p-adic Lie groups.

There have been several attempts to prove a version of the Beilinson-Bernstein Localisation Theorem for locally analytic distribution algebras, including [28] and [24]. We expect that it will be possible in the future to show that the abelian category of admissible locally analytic representations of G with dominant regular infinitesimal central character $\lambda\phi$ is anti-equivalent to the category of *coadmissible G-equivariant $\widehat{\mathcal{D}^\lambda}$-modules* on the rigid analytic flag variety.

References

[1] Ardakov, K. and Wadsley, S. J., *On irreducible representations of compact p-adic analytic groups*, Ann. Math., **178** (2013), 453–557.

[2] Beilinson, A. and Bernstein, J., *Localisation of* $\mathfrak{g}$*-modules*, C. R. Acad. Sci. Paris. Sér. I Math., **292**(1) (1981), 15–18.

[3] Berger, L., *La correspondance de Langlands locale p-adique pour* $\mathrm{GL}_2(\mathbb{Q}_p)$, Astérisque, **339**: Exp. No. 1017 du Séminaire Bourbaki (2011), 157–180.

[4] Berkovich V., *Spectral theory and analytic geometry over non-Archimedean fields*,

Mathematical Surveys and Monographs 39, American Mathematical Society, Providence, RI, 1990.

[5] Berthelot, P., *Cohomologie rigide et théorie des $\mathscr{D}$-modules*, Lecture Notes in Mathematics, **1454**, Springer, Berlin, 1990.

[6] ———, *$\mathscr{D}$-modules arithmétiques I. Opérateurs différentiels de niveau fini*, Ann. Sci. École Norm. Sup. (4), **29**(2) (1996), 185–272.

[7] ———, *Introduction à la théorie arithmétique des $\mathscr{D}$-modules*, Astérisque, **279**: Cohomologies p-adiques et applications arithmétiques, II. (2002), 1–80.

[8] Bezrukavnikov, R., Mirković, I., and Rumynin, D., *Localization of modules for a semisimple Lie algebra in prime characteristic*, Ann. Math., **167**(3) (2008), 945–991.

[9] Bezrukavnikov, R., *Noncommutative counterparts of the Springer resolution*, Eur. Math. Soc., International Congress of Mathematicians, II, Zürich, 2006, 1119–1144.

[10] Breuil, C., *Vers le socle localement analytique pour* GL_n*, I.*, http://www.math.u-psud.fr / breuil/PUBLICATIONS/socleI.pdf.

[11] Breuil, C., *The emerging p-adic Langlands programme*, Proceedings of the International Congress of Mathematicians, II., Hindustan Book Agency, New Delhi, 2010.

[12] Colmez, P., *Représentations de* $\mathrm{GL}_2(\mathbb{Q}_p)$ *et* (ϕ, Γ)*-modules*, Astérisque, **330** (2010), 281–509.

[13] Clark, J., *Auslander-Gorenstein rings for beginners*, Trends Math., Birkhäuser Boston Inc., Boston, MA, 2001.

[14] Dospinescu, G. and Schraen, B., *Endomorphism algebras of admissible p-adic representations of p-adic Lie groups*, Representation Theory, **17** (2013), 237–246.

[15] Emerton, M., *On the interpolation of systems of eigenvalues attached to automorphic Hecke eigenforms*, Invent. Math., **164**(1) (2006), 1–84.

[16] Harris, M. and Taylor, R., *The geometry and cohomology of some simple Shimura varieties*. Annals of Mathematics Studies 151, Princeton University Press, Princeton, NJ, 2001. With an appendix by Vladimir G. Berkovich.

[17] Hotta, R., Takeuchi, K., and Tanisaki, T., *D-modules, perverse sheaves, and representation theory*. Progress in Mathematics 236, Birkhäuser Boston Inc., Boston, MA, 2008. Translated from the 1995 Japanese edition by Takeuchi.

[18] Huber, R., *A generalisation of formal schemes and rigid analytic varieties*, Math. Z., **217**(4) (1994), 513–551.

[19] Kedlaya, K. S., *p-adic differential equations*, Cambridge Studies in Advanced Mathematics 125, Cambridge University Press, Cambridge, 2010.

[20] Kiehl, R., *Theorem A und Theorem B in der nichtarchimedischen Funktionentheorie*, Invent. Math., **2** (1967), 256–273.

[21] Kohlhaase, J., *Invariant distributions on p-adic analytic groups*, Duke Math. J., **137**(1) (2007), 19–62.

[22] Loeffler, *D. Overconvergent algebraic automorphic forms*, Proc. Lond. Math. Soc. (3), **102**(2) (2011), 193–228.

[23] Noot-Huyghe, C., *Un théorème de Beilinson-Bernstein pour les $\mathscr{D}$-modules arithmétiques*, Bull. Soc. Math., France, **137**(2) (2009), 159–183.

[24] Patel, D., Schmidt, T., and Strauch, M., *Locally analytic representations and sheaves on the Bruhat-Tits building*, http://arxiv.org/abs/1201.3646.

[25] Raynaud, M., *Géométrie analytique rigide d'aprés Tate, Kiehl, $\cdots$*, Bull. Soc. Math., France, Mém., Table Ronde d'Analyse non archimédienne, France, Paris, 1974.

[26] Rinehart, G., *Differential forms on general commutative algebras*, Trans. Amer. Math. Soc., **108** (1963), 195–222.

[27] Schmidt, T., *Stable flatness of nonarchimedean hyperenveloping algebras*, J. Algebra, **323**(3) (2010), 757–765.

[28] ———, *On locally analytic Beilinson-Bernstein localisation and the canonical dimension*, Math. Z., **275**(3-4) (2013), 793–833.

[29] Schneider, P. and Teitelbaum, J., *$U(\mathfrak{g})$-finite locally analytic representations*, Representation Theory, **5** (2001), 111–128, With an appendix by Dipendra Prasad.

[30] ———, *Banach space representations and Iwasawa theory*, Israel J. Math., **127**, (2002), 359–380.

[31] ———, *Locally analytic distributions and p-adic representation theory, with applications to* GL_2, J. Amer. Math. Soc., **15**(2) (2002), 443–468.

[32] ———, *Algebras of p-adic distributions and admissible representations*, Invent. Math., **153**(1) (2003), 145–196.

[33] ———, *Duality for admissible locally analytic representations*, Representation Theory, **9** (2005), 297–326.

[34] Smith, S. P., *Krull dimension of factor rings of the enveloping algebra of a semisimple Lie algebra*, Math. Proc. Camb. Phil. Soc., **93**(3) (1983), 459–466.

[35] Tate, J., *Rigid analytic spaces*, Invent. Math., **12** (1971), 258–289.

[36] Yekutieli, A. and Zhang, J. J., *Rings with Auslander dualizing complexes*, J. Algebra, **213**(1) (1999), 1–51.

Mathematical Institute, University of Oxford, Oxford OX2 6GG
E-mail: ardakov@maths.ox.ac.uk

Recurrence on the space of lattices

Yves Benoist

Abstract. This is an introduction to recurrence properties on finite volume homogeneous spaces based on examples.

Mathematics Subject Classification (2010). Primary 22E40; Secondary 37C85, 60J05.

Keywords. Lie groups, discrete subgroups, homogeneous dynamics, Markov chains.

1. Introduction

Extending a recurrence theorem due to Eskin and Margulis in [9], We proved with Jean-François Quint in [4] a recurrence theorem for random walks on finite volume homogeneous spaces. This text is an introduction to these two recurrence theorems.

We will explain in this preprint the main ideas of the proof of both recurrence theorems by working on the first non-trivial examples. We will not seek for generality, but we hope that these examples will help the reader to understand the meaning of these theorems. For a complete account on these theorems the reader is refered to [4] and [9].

2. Markov-Feller chains

We first define Markov-Feller operators and their recurrence properties.

A Markov chain on a space X is a mathematical model which describes the evolution of a stochastic process $(x_t)_{t\in\mathbb{N}}$, for which the position x_{t+1} at time $t+1$ is chosen randomly according to a law P_{x_t} which depends only on the position x_t at time t. In this paper we will deal only with Markov-Feller chains on second countable locally compact spaces X, i.e. with Markov chains for which the law P_{x_t} on X depends continuously on the point x_t.

In a more formal way, let X be a second countable locally compact space. A *Markov-Feller chain* on X is a continuous map $x \to P_x$ from X to the space $\mathcal{P}(X)$ of Borel probability measures on X. As usual, this space $\mathcal{P}(X)$ is endowed with the $*$-weak topology. We also denote by P the induced *Markov-Feller operator* on the Banach space $\mathcal{C}_b(X)$ of continuous bounded functions on X. It is given, for f in $\mathcal{C}_b(X)$ and x in X, by $Pf(x) = \int_X f(y)\,\mathrm{d}P_x(y)$.

Iterating n times this Markov chain, one gets a Markov chain $x \to P_x^n$. This probability P_x^n is the law of x_n when you know only the position of the chain at time zero $x_0 = x$.

Proceedings of the International Congress of Mathematicians, Seoul, 2014

This Markov chain $x \to P_x^n$ is defined inductively by $P_x^1 = P_x$ and, for $n \geq 1$, $P_x^{n+1} = \int_X P_y \, \mathrm{d}P_x^n(y)$. Its associated Markov operator on $\mathcal{C}_b(X)$ is nothing but the n^{th} power P^n.

Here are two very strong recurrence property of P.

Definition 2.1. We say that P is recurrent on X if, for every $\varepsilon > 0$ and x in X, one can find a compact set $M \subset X$ and an integer n_0 such that, for all $n \geq n_0$, one has $P_x^n(M) \geq 1 - \varepsilon$.

This means that there is no escape of mass for the laws of the Markov-Feller chain, i.e. any $*$-weak limit of a subsequence of P_x^n will be a probability measure.

Definition 2.2. We say that P is uniformly recurrent on X if, for every $\varepsilon > 0$, one can find a compact $M \subset X$ such that, for all x in X, one can find an integer n_0 such that, for all $n \geq n_0$, one has $P_x^n(M) \geq 1 - \varepsilon$.

This means that the compact set M can be chosen independantly of the starting point x.

Most of the Markov chains we will study will be obtained in the following way. Let G be a second countable locally compact group acting continuously on X, and μ be a Borel probability measure on G. The Markov-Feller chain on X will be the corresponding random walk on X, i.e. the transition probability will be $x \to P_{\mu,x} := \mu * \delta_x$. In other words, the corresponding Markov-Feller operator P_μ is given by, for all f in $\mathcal{C}_b(X)$ and x in X, $P_\mu f(x) = \int_G f(gx) \, \mathrm{d}\mu(g)$.

3. Finite volume homogeneous spaces

We introduce now the random walk on finite volume homogeneous spaces and state precisely the two recurrence theorems we want to explain.

3.1. Recurrence on G/Λ.

The reader non familiar with Lie groups may skip this general section. Indeed later on we will mainly focus on examples.

Let G be a connected real algebraic Lie group, let Λ be a lattice in G i.e. Λ is a discrete subgroup of finite covolume in G and $X := G/\Lambda$. Let $\mu \in \mathcal{P}(G)$ be a probability measure on G, with a finite exponential moment, $\int_G \|g\|^\delta \, \mathrm{d}\mu(g) < \infty$, for some $\delta > 0$. Let Γ_μ be the closed subgroup generated by the support of μ and H_μ be the Zariski closure of Γ_μ. We will assume that H_μ is semisimple. We will denote by H_μ^{nc} the smallest algebraic cocompact normal subgroup of H_μ. In [9] Eskin and Margulis proved the following:

Theorem 3.1. (Eskin-Margulis) *Assume that μ has exponential moment, that H_μ is semisimple and that the centralizer of H_μ^{nc} in G is trivial. Then X is uniformly P_μ-recurrent.*

They conjectured in [9, 2.5] the following statement which we proved in [4].

Theorem 3.2. (Benoist-Quint) *Assume that μ has exponential moments and that H_μ is semisimple. Then X is P_μ-recurrent.*

Here is a reformulation of Theorem 3.2.

Corollary 3.3. *Same assumptions as in Theorem 3.2. Let x be in X. Any weak limit ν_∞ of the sequence $\nu_n := \mu^{*n} * \delta_x$ in the space of finite measures on X, is a probability measure, i.e. $\nu_\infty(X) = 1$.*

Eskin-Margulis recurrence theorem 3.1 is used in [1] as the starting point for the classifications of both the μ-stationary probability measures on X and the Γ_μ-invariant closed subsets of X when G is a simple group and $H_\mu = G$. Benoist-Quint recurrence theorem 3.2 is used in [2] and [3] to extend these classifications to any Lie group G as soon as H_μ is semisimple with no compact factor. We recommend the survey [5] for an introduction to this classification theorem.

Here is a straightforward corollary of Theorem 3.2

Corollary 3.4. *Let Γ be a discrete subgroup of G whose Zariski closure is semisimple. Then any discrete Γ-orbit in G/Λ is finite.*

Proof of Corollary 3.4. By the recurrence property such a Γ-orbit supports a stationary probability measure ν i.e. a measure satisfying $\mu * \nu = \nu$. By the maximum principle, all the points on this Γ-orbit have same mass for ν. Hence this orbit is finite. □

For the sake of simplicity, we always assume from now on that μ has compact support and that H_μ has no compact factor.

3.2. The space of unimodular lattices in $\mathbb{R}^d$.

The main example of finite volume homogeneous space $X = G/\Lambda$ is the space $X_d = \mathrm{SL}(d,\mathbb{R})/\mathrm{SL}(d,\mathbb{Z})$. In this case, the compact subsets are described by the Mahler compactness criterion below.

The space X_d is also the space of unimodular lattices Δ of $\mathbb{R}^d$, i.e. the set of discrete subgroups of $\mathbb{R}^d$ spanned by a basis $v_1, \dots, v_d$ of $\mathbb{R}^d$ of determinant 1.

For $0 \leq i \leq d$, we define the i^{th}-systole function α_i on X_d by

$$\alpha_i(x) = \min\{\|v\| \mid v \in \Lambda^i x \text{ non-zero pure tensor}\}. \tag{3.1}$$

The minimum is taken among tensor v that can be written as $v = v_1 \wedge \cdots \wedge v_i$ with $v_1, \dots, v_i$ linearly independant elements of the lattice x. For instance $\alpha_1(x)$ is the length of the shortest non-zero vector in the lattice $x \subset \mathbb{R}^d$. By convention we set $\alpha_0 \equiv \alpha_d \equiv 1$. These systole functions are continuous. Their relevance lies in the following criterion.

Lemma 3.5 (Mahler compactness criterion). *For $0 < i < d$, the systole functions α_i^{-1} are proper.*

We recall that a real valued function f is said to be proper if the inverse image of a bounded set is relatively compact. When $i = 1$, Lemma 3.5 means that, a sequence x_n in X_d goes to infinity if and only if there exists a sequence of non-zero vectors $v_n \in x_n$ converging to 0.

4. The contraction properties

We give in this section sufficient conditions for the recurrence and for the uniform recurrence of a Markov-Feller operator. These conditions called **CH** and **UCH** are easy to check since they involve only one iteration of the Markov chain.

Let X be a second countable locally compact space and P a Markov-Feller operator on X. We will say that P satisfies the contraction hypothesis if

CH for every compact L of X, there exists a Borel function $f = f_L : X \to [0, \infty]$ such that,

(i) f takes finite values on L,
(ii) for every $M < \infty$, $f^{-1}([0, M])$ is relatively compact in X,
(iii) there exists constants $a < 1$, $b > 0$ such that $Pf \leq af + b$.

Note that f is not assumed to be finite nor continuous.

This **CH** means that there exist on X functions f which have a very strong P-subharmonicity property: the Markov operator contracts f up to an additive constant.

We will say that P satisfies the uniform contraction hypothesis if

UCH There exists a proper function $f : X \to [0, \infty[$ such that $Pf \leq af + b$, where $a < 1$ and $b > 0$.

This **UCH** means that the function f in **CH** can be chosen to be everywhere finite. This **UCH** is a variation of a condition due to Foster that one can find in [10], [13] and [9]. This **UCH** is shown in [13] to be related to the existence of an exponential moment for the first return time in some bounded sets of X.

Lemma 4.1. *Let X be a second countable locally compact space and P a Markov-Feller operator on X.*

a) *Assume that P satisfies the contraction hypothesis* **CH** *on X, then P is recurrent on X.*

b) *Assume that P satisfies the uniform contraction hypothesis* **UCH** *on X, then P is uniformly recurrent on X.*

Proof. a) Let x be a point in X and $f = f_x$ be the function given by the hypothesis **CH** for the compact set $L = \{x\}$. Choose for M the closure of the set

$$\{y \in X \mid f(y) \leq \tfrac{2B}{\varepsilon}\}$$

so that the indicator function of the complementary set M^c satisfies $\mathbf{1}_{M^c} \leq \frac{\varepsilon}{2B} f$.

According to the hypothesis **CH**, one has, for every $n \geq 1$

$$P^n f \leq a^n f + b(1 + \cdots + a^{n-1}) \leq a^n f + B$$

with $B = \frac{b}{1-a}$. One then has the inequalities, for all x in X,

$$P^n_x(M^c) = P^n(\mathbf{1}_{M^c})(x) \leq \frac{\varepsilon}{2B} P^n f(x) \leq \frac{\varepsilon a^n}{2B} f(x) + \frac{\varepsilon}{2} \leq \varepsilon$$

as soon as n is sufficiently large so that $f(x) \leq \frac{B}{a^n}$.

b) Same proof with a function f which does not depend on the point x. □

5. Countable spaces

In this section we give basic examples of Markov operators on countable spaces and describe their recurrence properties.

The first example does not satisfy **UCH**.

Example 5.1. [Random walk on groups] Let G be a discrete infinite group acting on itself by left multiplication, let μ be a probability measure on G whose support spans G, then P_μ is not recurrent on G.

Proof. There are no ergodic stationary probability measure ν on G. Indeed, the set of element g for which $\nu(g)$ is maximum is finite and G-invariant. □

Remark 5.2. There is a classical notion of recurrence for a Markov chain that we will call here 0-*recurrence*. It says that, for all neighborhood U of the starting point, almost all trajectories of the Markov chain comes back in U. When $G = \mathbb{Z}$ the above Markov chain P_μ is not recurrent on G eventhough it is 0-recurrent.

The second example is very simple but it gives a fairly good picture of what is a Markov chain satisfying **UCH**.

Example 5.3 (Markov chain satisfying UCH). Consider the Markov chain $x \to P_x$ on $X = \mathbb{N}$ given by $P_x = \frac{1}{3}\delta_{x+1} + \frac{2}{3}\delta_{x-1}$ when $x > 0$ and $P_x = \delta_{x+1}$ when $x = 0$. This Markov chain satisfies the uniform contraction hypothesis **UCH**. In particular it is uniformly recurrent.

Proof. It satisfies **UCH** with the function $f : x \to 2^{x/2}$. Indeed, one has the inequality $Pf \leq \frac{2\sqrt{2}}{3} f + 1$. □

The next example enlights the difference between **UCH** and **CH**.

Example 5.4 (Markov chain satisfying CH but not UCH). The trivial Markov chain on $X = \mathbb{N}$ given by the transition probabilities $P_x = \delta_x$ satisfies the contraction hypothesis **CH**. In particular P is recurrent. However P is not uniformly recurrent.

Proof. It satisfies **CH** with the functions $f_n : x \to 1$ when $x \leq n$ and $f_n : x \to \infty$ otherwise. Indeed, one has the inequality $Pf_n \leq \frac{1}{2} f_n + 1$. □

A Markov chain P on a countable set X is said to be *transitive* if for all x, y in X, there exists $n \geq 1$ such that $P_x^n(y) > 0$. The following example tells us very roughly that, except for Example 5.4, the conditions **UCH** and **CH** are equivalent.

Example 5.5 (CH + T implies UCH). Let P be a transitive Markov chain on a countable set X satisfying the contraction hypothesis **CH**. Then P satisfies also the uniform contraction hypothesis **UCH**.

Proof. Since P satisfies **CH**, there exists a stationary probability measure ν on X. Since P is transitive on X, this stationary probability measure has full support on X. By [6, Prop. 1.8], ν is the unique stationary probability measure on X. Hence, for all x in X, one has

$$\lim_{n\to\infty} \tfrac{1}{n} \textstyle\sum_{k\leq n} P_x^k = \nu.$$

For any $\varepsilon > 0$, one can find a finite set F such that $\nu(F^c) \leq \varepsilon/4$. In particular, for all x in X there exists $n_1 > 0$ such that

$$P_x^{n_1}(F^c) \leq \varepsilon/2.$$

Since P satisfies **CH**, there exists a finite set $M \subset X$ and $n_2 \geq 0$ such that, for all $n \geq n_2$, for all y in F, one has $P_y^n(M^c) \leq \varepsilon/2$. Then, for all $n \geq n_1 + n_2$, one has $P_x^n(M^c) \leq \varepsilon$. Hence P satisfies **UCH**. □

6. The uniform contraction hypothesis UCH

In this section we sketch the proof of Margulis-Eskin recurrence theorem. We begin by simpler examples to enlight one by one the ideas entering the proof.

6.1. Linear random walk.

The first idea is a uniform contraction property for the linear random walk on vector spaces which is nothing but a reformulation of the positivity of the first Lyapounov exponent.

Let H be a real algebraic semisimple Lie group with no compact factor. Let μ be a Borel probability measure on H which is Zariski dense, i.e. whose support spans a Zariski dense subgroup of H. For sake of simplicity we will assume from now on that this support is compact. Let V be a real finite dimensional algebraic representation of H. We set V^H for the set of fixed points of H in V.

The following functions on V are contracted by the Markov operators P_μ on V. They will be the building blocks for the construction of f.

Lemma 6.1 ([9, Lemma 4.2]). *Let V be a real algebraic representation of H such that $V^H = \{0\}$. Let φ be the function on V given by $\varphi(v) = \|v\|$. Then there exists $\delta > 0$, $a_0 < 1$ and $n_0 \geq 1$ such that*

$$P_\mu^{n_0}\varphi^{-\delta} \leq a_0\varphi^{-\delta} \tag{6.1}$$

Proof. This Lemma 6.1 is proven in [9, Lemma 4.2]. We can assume that V is irreducible. The proof relies on Furstenberg theorem on the positivity of the first Lyapounov exponent of μ which tells us that, uniformly for v in $V \smallsetminus 0$, the limit $\lambda_1 = \lim_{n\to\infty} \int_H \log \frac{\|gv\|}{\|v\|} \, \mathrm{d}\mu^{*n}(g)$ exists and is positive. One then write the asymptotic expansion up to order 2 of $e^{-\delta \log(\|hv\|/\|v\|)}$ and computes its image by P_μ^n. □

Since it is harmless to replace μ by the convolution power μ^{*n_0}, we will always assume implicitely that $n_0 = 1$.

6.2. The pointed torus.

Before dealing with the spaces $X = G/\Lambda$, we explain here on a simpler example how Lemma 6.1 is used to prove the uniform contraction hypothesis **UCH**.

Proposition 6.2. *Let μ be a probability measure on $\mathrm{SL}(d, \mathbb{Z})$ with finite support. Assume that H_μ is semisimple with no compact factor and has no non-zero invariant vectors on $\mathbb{R}^d$. Then the Markov operator P_μ on the pointed torus $\mathbb{T}^d \smallsetminus 0$ satisfies* **UCH**.

Proof. We choose for function f on $\mathbb{T}^d \smallsetminus 0$, a small negative power of the distance to 0, i.e. $f(x) = d(x,0)^{-\delta}$ with δ small enough.

For x in a small neighborhood U of 0, the random walk is linear hence by Lemma 6.1, one has $P_\mu f(x) \leq a_0 f(x)$, for some constant $a_0 < 1$.

For x in the compact set U^c, $P_\mu f$ is bounded by a constant $b > 0$.

In both cases, one has $P_\mu f \leq a_0 f + b$. □

6.3. $H = \mathbf{SL(2,\mathbb{R})}$ and $X = \mathbf{SL(2,\mathbb{R})/SL(2,\mathbb{Z})}$.

We can now give the proof of Margulis-Eskin recurrence theorem in the simplest case.

Proposition 6.3. *Let μ be a Zariski dense compactly supported probability measure on $\mathrm{SL}(2,\mathbb{R})$. The Markov operator P_μ on $X = \mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$ satisfies* **UCH**.

Proof. We follow the same strategy. We choose for function f on X, a small negative power of the systole, i.e. $f = \alpha_1^{-\delta}$ with δ small enough.

We want to bound $P_\mu f$ by $a\, f + b$. The difficulty is that in general an average of maximum is not always bounded by the maximum of the average. We fix a constant $C > 0$ such that, for all g in the support of μ, one has $\|g\| \leq C$ and $\|g^{-1}\| \leq C$. Let v be a vector of x such that $\alpha_1(x) = \|v\|$. We distinguish two cases.

First case. If all non-collinear vectors w in x satisfy $\|w\| \geq C^2\|v\|$. Then one has $\|gv\| \leq \|gw\|$, and hence $\alpha_1(gx) = \|gv\|$. Using Lemma 6.1, one gets, with $a_0 < 1$,

$$P_\mu \alpha_1^{-\delta}(x) = P_\mu \varphi^{-\delta}(v) \leq a_0 \varphi^{-\delta}(v) = a_0 \alpha_1^{-\delta}(x)\,. \tag{6.2}$$

Second case. If there exists a non-collinear vector w in x with $\|w\| \leq C^2\|v\|$. Then we use the inequality

$$\|v \wedge w\| \leq \|v\|\,\|w\| \tag{6.3}$$

and the fact that, since x has covolume 1, the left-hand side is bounded below by 1. We deduce that $\alpha_1(x) \geq C^{-1}$. Hence, by Mahler criterion, x belongs to a compact subset of X. The continuous function $P_\mu f$ is bounded on this compact set by a constant $b > 0$. In both cases, one has $P_\mu f \leq a_0 f + b$. □

6.4. $H = \mathbf{SL(3,\mathbb{R})}$ and $X = \mathbf{SL(3,\mathbb{R})/SL(3,\mathbb{Z})}$.

The main new idea needed to prove Margulis-Eskin recurrence theorem in the second simplest case, is the use of all the systole functions α_i.

Proposition 6.4. *Let μ be a Zariski dense compactly supported probability measure on $\mathrm{SL}(3,\mathbb{R})$. The Markov operator P_μ on $X = \mathrm{SL}(3,\mathbb{R})/\mathrm{SL}(3,\mathbb{Z})$ satisfies* **UCH**.

Proof. We follow the same strategy as for Proposition 6.3, but we will use negative powers of both systole functions. For $i = 1$ and $i = 2$, we introduce the functions f_i on X given by $f_i(x) = \alpha_i(x)^{-\delta}$ with δ small enough.

We fix a constant $C > 0$ such that, for all g in the support of μ, one has $\|g\| \leq C$ and $\|g^{-1}\| \leq C$. Let v be a vector of x such that $\alpha_1(x) = \|v\|$. We still distinguish two cases.

First case. If all non-collinear vectors w in x satisfy $\|w\| \geq C^2\|v\|$. Then the same calculation (6.2), gives the bound $P_\mu f_1(x) \leq a_0 f_1(x)$.

Second case. If there exists a non collinear vector w in x with $\|w\| \leq C^2\|v\|$. Then we use the same inequality

$$\|v \wedge w\| \leq \|v\| \, \|w\| \tag{6.4}$$

to deduce $\alpha_1(x) \geq C^{-1}\alpha_2(x)^{\frac{1}{2}}$, and then $\alpha_1(gx) \geq C^{-2}\alpha_2(x)^{\frac{1}{2}}$. One gets the bound $P_\mu f_1(x) \leq C^{2\delta} f_2^{\frac{1}{2}}(x)$. In both cases, one has

$$P_\mu f_1 \leq a_0 f_1 + C^{2\delta} f_2^{\frac{1}{2}} \,, \tag{6.5}$$

for some constants $a_0 < 1$. This is not exactly what we wanted. That is why, we use the companion inequality obtained by using the systole α_2 of the dual lattice

$$P_\mu f_2 \leq a_0 f_2 + C^{2\delta} f_1^{\frac{1}{2}} \,. \tag{6.6}$$

Note that for every $\varepsilon_0 > 0$ and $z > 0$, one has $z^{\frac{1}{2}} \leq \varepsilon_0 \, z + \varepsilon_0^{-1}$. Setting $f = f_1 + f_2$, one deduces from (6.5) and (6.6), the upper bound

$$P_\mu f \leq a f + b \,,$$

for the constants $a = a_0 + \varepsilon_0 \, C^{2\delta}$ and $b = 2\,\varepsilon_0^{-1} C^{2\delta}$. If ε_0 is small enough one has $a < 1$ as required □

6.5. $H = \mathbf{SL}(d, \mathbb{R})$ and $X = \mathbf{SL}(d, \mathbb{R})/\mathbf{SL}(d, \mathbb{Z})$.

> For larger d, the main new idea for proving Margulis-Eskin recurrence theorem is Inequality (6.7) which allows us to compare the various systole functions α_i.

Proposition 6.5. *Let μ be a Zariski dense compactly supported probability measure on $\mathrm{SL}(d, \mathbb{R})$. The Markov operator P_μ on $X = \mathrm{SL}(d, \mathbb{R})/\mathrm{SL}(d, \mathbb{Z})$ satisfies* **UCH**.

The new key point will be to replace Inequality (6.4) by the following key inequality. We recall that an element u in $\Lambda^r \mathbb{R}^d$ is a pure tensor, if one can write $u = u_1 \wedge \cdots \wedge u_r$ with all u_i in $\mathbb{R}^d$.

Lemma 6.6. *For all pure tensors $u \in \Lambda^r \mathbb{R}^d$, $v \in \Lambda^s \mathbb{R}^d$ and $w \in \Lambda^t \mathbb{R}^d$, one has*

$$\|u\| \, \|u \wedge v \wedge w\| \leq \|u \wedge v\| \, \|u \wedge w\| \,. \tag{6.7}$$

Proof of Lemma 6.6. Set $\langle u \rangle$ for the vector subspaces spanned by the u_i's. One can reduce to the case where the subspaces $\langle u \rangle$, $\langle v \rangle$ and $\langle w \rangle$ are orthogonal. Then we only have to check the easy inequality $\|v \wedge w\| \leq \|v\| \, \|w\|$. □

Proof of Proposition 6.5. We follow the same strategy as for Proposition 6.4, but we will use negative powers of all the systole functions. For $0 \leq i \leq d$, we introduce the functions $f_i = \alpha_i^{-\delta}$ on X with δ small enough. We fix a constant $C > 0$ such that, for all g in the support of μ and all $i \leq d$, one has $\|\Lambda^i g\| \leq C$ and $\|\Lambda^i g^{-1}\| \leq C$.

Fix i with $0 < i < d$. Using the key inequality (6.7) with $r = i - j$ and $s = t = j$ with $0 < j \leq \min(i, d - i)$, instead of using Inequality (6.4), one replace the bounds (6.5) and (6.6) by the following bound.

$$P_\mu f_i \leq a_0 f_i + C^{2\delta} \textstyle\sum_{j>0} f_{i-j}^{\frac{1}{2}} f_{i+j}^{\frac{1}{2}}\,, \tag{6.8}$$

for some constants $a_0 < 1$.

Setting

$$f = \sum_{0<i<d} \psi_i \;\text{ where }\; \psi_i = \varepsilon_0^{i(d-i)} f_i\,,$$

with ε_0 very small, one deduces the upper bounds

$$\begin{aligned} P_\mu \psi_i &\leq a_0 \psi_i + \varepsilon_0\, C^{2\delta} \sum_{j>0} \psi_{i-j}^{\frac{1}{2}} \psi_{i+j}^{\frac{1}{2}} \\ &\leq a_0 \psi_i + \varepsilon_0\, C^{2\delta} \sum_{0\leq k\leq d} \psi_k\,, \end{aligned}$$

and hence

$$P_\mu f \leq a f + b\,,$$

for the constant $a = a_0 + d\,\varepsilon_0\, C^{2\delta}$ and for $b = 2\,\varepsilon_0\, C^{2\delta}$. If ε_0 is small enough one has $a < 1$ as required □

6.6. H irreducible on $\mathbb{R}^d$ and $X = \mathrm{SL}(d,\mathbb{R})/\mathrm{SL}(d,\mathbb{Z})$.

This case is not more difficult than the previous one.

Proposition 6.7. *Let μ be a compactly supported probability measure on $\mathrm{SL}(d,\mathbb{R})$ such that H_μ is a semisimple group with no compact factors which acts irreducibly on $\mathbb{R}^d$. Then the Markov operator P_μ on $X = \mathrm{SL}(d,\mathbb{R})/\mathrm{SL}(d,\mathbb{Z})$ satisfies* **UCH**.

In this case, the vector space $V = \Lambda^i \mathbb{R}^d$ is the sum $V = V_+ \oplus V_0$ with V_0 the set of H_μ-invariant vectors and V_+ the H_μ-invariant supplementary subspace. We write $v = v_+ + v_0$ for the corresponding decomposition of a vector v in V. The new feature is that this subspace V_0 might be non-trivial. This is harmless because of the following lemma.

We will write $f \ll g$ for $f \leq C\, g$ where C is a constant.

Lemma 6.8. *Keep these notations, in particular, H_μ is irreducible on $\mathbb{R}^d$. For $0 < i < d$ and all pure tensor v in $\Lambda^i \mathbb{R}^d$, one has $\|v\| \ll \|v_+\|$.*

Proof of Lemma 6.8. This follows from a compacity argument, since by the irreducibility assumption, the space V_0 does not contain non-zero pure tensors. □

Proof of Proposition 6.7. The proof is exactly the same as for Proposition 6.5. We just notice that, by Lemmas 6.1 and 6.8, the function $\varphi : v \mapsto \|v\|$ on $V = \Lambda^i \mathbb{R}^d$ still satisfies Inequality (6.1) on the set of pure tensors of $\Lambda^i \mathbb{R}^d$. □

A proof of a more general case of Eskin-Margulis recurrence theorem will be given in section 7.5.

7. The contraction hypothesis CH

In this section, we want to explain the proof of Benoist-Quint recurrence theorem.

7.1. $H = \mathbf{SL}(2,\mathbb{R})$ and $X = \mathbf{SL}(3,\mathbb{R})/\mathbf{SL}(3,\mathbb{Z})$.

We begin by the simplest case. The main new idea is a modification of the systole function in which one replaces the lattice x by its intersection with an ε_0-neighborhood of the expanding space.

Proposition 7.1. *Let μ be a Zariski dense compactly supported probability measure on $\mathrm{SL}(2,\mathbb{R})$. The Markov operator P_μ on $X = \mathrm{SL}(3,\mathbb{R})/\mathrm{SL}(3,\mathbb{Z})$ satisfies* **CH**.

Remark 7.2. We consider the group $H = \mathrm{SL}(2,\mathbb{R})$ as a subgroup of $G = \mathrm{SL}(3,\mathbb{R})$ fixing the last vector e_3 of the standard basis e_1, e_2, e_3 of $\mathbb{R}^3$. Since the centralizer of H in G is non trivial, Margulis-Eskin recurrence theorem does not apply to this case. Indeed P_μ does not satisfy **UCH**, because, by Mahler criterion, the closed H-invariant subsets

$$Y_\varepsilon := \{x \in X \mid \varepsilon\, e_3 \in x\}$$

are going away from any compact subsets of X when $\varepsilon \searrow 0$.

The vector space $V = \mathbb{R}^3$ is the sum $V = V_+ \oplus V_0$ with $V_+ = \mathbb{R}^2$ and $V_0 = \mathbb{R}$. We still write $v = v_+ + v_0$ for the corresponding decomposition of a vector v in V. Same for $V^* = \Lambda^2 V$. A new key point will be to replace Inequality (6.4) by the following inequality (7.1).

Lemma 7.3. *For every v, w in $\mathbb{R}^3$, one has*

$$\|(v \wedge w)_+\| \leq \|v_+\|\,\|w_0\| + \|v_0\|\,\|w_+\| \tag{7.1}$$

Proof of Lemma 7.3. One has $(v \wedge w)_+ = v_+ \wedge w_0 + v_0 \wedge w_+$. □

Proof of Proposition 7.1. We follow the same strategy as for Proposition 6.4. Since the positivity of the Lyapounov exponent occurs only in the V_+ direction and since the projection of a lattice in V_+ might be dense, we have to introduce the following modification of the systole functions. We fix $\varepsilon_0 > 0$ small, and we set, for x in X,

$$\alpha_{\varepsilon_0,1}(x) = \min\{\|v_+\| \mid v \in x \smallsetminus \{0\}\ ,\ \|v_0\| < \varepsilon_0\}. \tag{7.2}$$

The minimum is taken among all non zero vectors v of x belonging to the ε_0-neighborhood of the plane V_+. The new feature is that this quantity $\alpha_{\varepsilon_0,1}(x)$ is not always positive, indeed

$$\alpha_{\varepsilon_0,1}(x) = 0 \Longleftrightarrow x \in Y_\varepsilon \ \text{ for some } \ \varepsilon < \varepsilon_0.$$

Similarly using the dual lattice $x^* = \Lambda^2 x$ in the dual space $V^* = V_+^* \oplus V_0^*$, we set

$$\alpha_{\varepsilon_0,2}(x) = \min\{\|v_+\| \mid v \in x^* \smallsetminus \{0\}\ ,\ \|v_0\| < \varepsilon_0\}. \tag{7.3}$$

We introduce the functions $f_{\varepsilon_0,i} = \alpha_{\varepsilon_0,i}^{-\delta}$ with δ small enough. We fix a constant $C > 0$ such that, for all g in the support of μ, one has $\|g\| \leq C$ and $\|g^{-1}\| \leq C$. Let v be a vector of x such that $\|v_0\| < \varepsilon_0$ and $\alpha_{\varepsilon_0,1}(x) = \|v_+\|$.

First case. If all the non collinear vector $w = w_+ + w_0$ in x with $\|w_0\| < \varepsilon_0$ satisfy $\|w_+\| \geq C^2\|v_+\|$. The same arguments as in (6.2), gives the bound $P_\mu f_{\varepsilon_0,1}(x) \leq a_0 f_{\varepsilon_0,1}(x)$ with $a_0 < 1$.

Second case. There exists a non collinear vector w with $\|w_0\| < \varepsilon_0$ satisfying $\|w_+\| \leq C^2\|v_+\|$.

In case $\|v_+\| < \varepsilon_0$, we have $\|(v \wedge w)_0\| < \varepsilon_0$, and we use Inequality (7.1) to deduce $2\varepsilon_0 C^2 \alpha_{\varepsilon_0,1}(x) \geq \alpha_{\varepsilon_0,2}(x)$ and get the bound $P_\mu f_{\varepsilon_0,1}(x) \leq (2\varepsilon_0 C^3)^\delta f_{\varepsilon_0,2(x)}$.

In case $\|v_+\| \geq \varepsilon_0$, one has the bound $P_\mu f_{\varepsilon_0,1}(x) \leq \varepsilon_0^{-\delta} C^\delta$. In all these three cases, one has

$$P_\mu f_{\varepsilon_0,1} \leq a_0 f_{\varepsilon_0,1} + (2\varepsilon_0 C^3)^\delta f_{\varepsilon_0,2} + \varepsilon_0^{-\delta} C^\delta \,, \tag{7.4}$$

for some constant $a_0 < 1$. Similarly, one has

$$P_\mu f_{\varepsilon_0,2} \leq a_0 f_{\varepsilon_0,2} + (2\varepsilon_0 C^3)^\delta f_{\varepsilon_0,1} + \varepsilon_0^{-\delta} C^\delta \,. \tag{7.5}$$

Setting $f_{\varepsilon_0} = f_{\varepsilon_0,1} + f_{\varepsilon_0,2}$, one deduces then from (7.4) and (7.5), the upper bound

$$P_\mu f_{\varepsilon_0} \leq a\, f_{\varepsilon_0} + b \,,$$

for the constants $a = a_0 + (2\varepsilon_0\, C^3)^\delta$ and $b = 2\,\varepsilon_0^{-\delta} C^\delta$. If ε_0 is small enough one has $a < 1$ as required. □

7.2. $H = \mathbf{SL}(d_1, \mathbb{R}) \times \mathbf{SL}(d_2, \mathbb{R})$ and $X = \mathbf{SL}(d, \mathbb{R})/\mathbf{SL}(d, \mathbb{Z})$.

In this case the main new idea is to replace the norm by a function φ_{ε_0} which takes into account suitable powers of the norm in the irreducible subrepresentations of H_μ.

Proposition 7.4. *Let $d = d_1 + d_2$. Let μ be a Zariski dense compactly supported probability measure on $\mathrm{SL}(d_1, \mathbb{R}) \times \mathrm{SL}(d_2, \mathbb{R})$. Then the Markov operator P_μ on $X = \mathrm{SL}(d,R)/\mathrm{SL}(d,Z)$ satisfies* **CH**.

We will need a stronger inequality generalizing both (6.7) and (7.1).

Let $\mathbb{R}^d := \mathbb{R}^{d_1} \oplus \mathbb{R}^{d_2}$ be the associated orthogonal decomposition. For any couple $\lambda = (\lambda_1, \lambda_2) \in \mathbb{N}^2$, we denote by $u \to u_\lambda$ the projector of $\Lambda^* \mathbb{R}^d$ on the component $\Lambda^{\lambda_1}\mathbb{R}^{d_1} \otimes \Lambda^{\lambda_2}\mathbb{R}^{d_2}$. We endow $\mathbb{N}^2$ with the partial order

$$\lambda \leq \mu \Longleftrightarrow (\lambda_1 \leq \mu_1 \ \text{ and } \ \lambda_2 \leq \mu_2). \tag{7.6}$$

For any λ, μ in $\mathbb{N}^2$ we denote by $m_{\lambda,\mu}$ the minimum and $M_{\lambda,\mu}$ the maximum of λ and μ, that is $m_{\lambda,\mu} = (\min(\lambda_1, \mu_1), \min(\lambda_2, \mu_2))$ and similarly for the maximum. We denote by

$$R(\lambda, \mu) := \{\nu \in \mathbb{N}^2 \mid m_{\lambda,\mu} \leq \nu \leq M_{\lambda,\mu}\}$$

the "rectangle" between $m_{\lambda,\mu}$ and $M_{\lambda,\mu}$, and by R the rectangle

$$R := \{\nu \in \mathbb{N}^2 \mid \nu \leq (d_1, d_2)\}.$$

Lemma 7.5. (Mother inequality for $SL \times SL$) *For any pure tensors u, v, w in $\Lambda^*(\mathbb{R}^{d_1} \oplus \mathbb{R}^{d_2})$, and λ, μ in R, one has*

$$\|u_\lambda\|\, \|(u \wedge v \wedge w)_\mu\| \ll \max_{\substack{\nu,\rho \in R(\lambda,\mu) \\ \nu+\rho = \lambda+\mu}} \|(u \wedge v)_\nu\|\, \|(u \wedge w)_\rho\|. \tag{7.7}$$

The only proof of Lemma 7.5 that I know relies on representation theory. We will explain this proof in Section 7.3.

Example 7.6. For u, v, w vectors in $\mathbb{R}^d = \mathbb{R}^{d_1} \oplus \mathbb{R}^{d_2}$, one has

$$\begin{aligned}
&\|u_{1,0}\| \, \|(u\wedge v\wedge w)_{3,0}\| \ll \|(u\wedge v)_{2,0}\| \, \|(u\wedge w)_{2,0}\|,\\
&\|u_{1,0}\| \, \|(u\wedge v\wedge w)_{2,1}\| \ll \|(u\wedge v)_{2,0}\|\|(u\wedge w)_{1,1}\|+\|(u\wedge v)_{1,1}\|\|(u\wedge w)_{2,0}\|,\\
&\|u_{1,0}\| \, \|(u\wedge v\wedge w)_{1,2}\| \ll \|(u\wedge v)_{1,1}\|\|(u\wedge w)_{1,1}\|,\\
&\|u_{1,0}\| \, \|(u\wedge v\wedge w)_{0,3}\| \ll \|(u\wedge v)_{0,2}\|\|(u\wedge w)_{1,1}\|+\|(u\wedge v)_{1,1}\|\|(u\wedge w)_{0,2}\|.
\end{aligned}$$

Among these inequalities, the most difficulty inequality is the third one since the terms

$$\|(u \wedge v)_{0,2}\| \, \|(u \wedge w)_{2,0}\| + \|(u \wedge v)_{2,0}\| \, \|(u \wedge w)_{0,2}\|$$

do not occur on the right hand side.

For $\lambda \in R$ we set

$$|\lambda| := (d_1 - \lambda_1)\lambda_1 + (d_2 - \lambda_2)\lambda_2.$$

Let $\varepsilon_0 > 0$. For v in $\Lambda^i E$, with $0 < i < d$, we define

$$\varphi_{\varepsilon_0}(v) = \max_{\lambda \in R \smallsetminus 0} \varepsilon_0^{\frac{-(d-i)i}{|\lambda|}} \|v_\lambda\|^{\frac{1}{|\lambda|}}. \tag{7.8}$$

Note that this function is the inverse of the function denoted φ_{ε_0} in [4].

Lemma 7.7. *There exists $\delta > 0$, $a_0 < 1$ and $n_0 \geq 1$ such that,*

$$P_\mu^{n_0} \varphi_{\varepsilon_0}^{-\delta} \leq a_0 \varphi_{\varepsilon_0}^{-\delta} \quad \text{for any } \varepsilon_0 > 0. \tag{7.9}$$

Proof of Lemma 7.7. This follows from Lemma 6.1. □

Proof of Proposition 7.4. The proof is the same as for Proposition 7.1, replacing Inequality (6.1) by (7.9) and Inequality (7.1) by (7.7). We define for x in X

$$\alpha_{\varepsilon_0}(x) = \min\{\varphi_{\varepsilon_0}(v) \mid v \in \Lambda^\bullet x \smallsetminus 0 \ , \ \text{pure tensor with } \|v_0\| < \varepsilon_0\},$$

where the minimum is taken over all the non-zero pure tensor v in some $\Lambda^i x$ for which $\|v_0\| < \varepsilon_0$. We also introduce the function on X

$$f_{\varepsilon_0}(x) = \alpha_{\varepsilon_0}(x)^{-\delta}.$$

If δ and ε_0 are small enough, this function f_{ε_0} satisfies

$$P_\mu f_{\varepsilon_0} \leq a \, f_{\varepsilon_0} + b$$

for some constants $a < 1$ and $b > 0$. Moreover, for x in X, one has the equivalence : $f_{\varepsilon_0}(x) = \infty$ if and only if, for some i, $\Lambda^i x$ contains an H-invariant pure tensor v with $\|v\| < \varepsilon_0^{(d-i)i}$.

This proves that P_μ satisfies **CH** on X. □

7.3. Mother inequality.

In this section, we sketch the proof of Inequality (7.7). We will see that it is a special case of the Mother Inequality (7.11) based on Representation Theory.

Let $H \subset \mathrm{SL}(\mathbb{R}^d)$ be a semisimple algebraic subgroup, $A \subset H$ be a maximal split subtorus of H, $\Sigma = \Sigma(A, H)$ be the set of (restricted) roots, i.e. Σ is the set of non-zero weights of A in the Lie algebra $\mathfrak{h}$ of H. We choose a system $\Sigma^+ \subset \Sigma$ of positive roots. Let P be the set of algebraic characters of A. We endow P with the partial order given, for λ, μ in P, by

$$\lambda \leq \mu \Longleftrightarrow \mu - \lambda \text{ is a sum of positive roots.} \tag{7.10}$$

For any real algebraic irreducible representation of H, the set of weights of A in this representation has a unique maximal element λ called the (restricted) highest weight of the representation. Let P^+ be the set of all these highest weights. For any algebraic representation of H in a real finite dimensional vector space V, for λ in P^+, we denote by $v \mapsto v_\lambda$ the H-equivariant projection on the sum of all the irreducible subrepresentations of V whose highest weight is equal to λ.

Lemma 7.8 (Mother inequality). *Let $H \subset \mathrm{SL}(\mathbb{R}^d)$ be a semisimple algebraic subgroup. For pure tensors u, v, w in $\Lambda^*\mathbb{R}^d$ and λ, μ in P^+, one has*

$$\|u_\lambda\| \, \|(u \wedge v \wedge w)_\mu\| \ll \max_{\substack{\nu,\rho \in P^+ \\ \nu+\rho \geq \lambda+\mu}} \|(u \wedge v)_\nu\| \, \|(u \wedge w)_\rho\|. \tag{7.11}$$

Proof of Lemma 7.8 $\Rightarrow$ Lemma 7.5. Let $H = \mathrm{SL}(d_1, \mathbb{R}) \times \mathrm{SL}(d_2, \mathbb{R})$, $d = d_1 + d_2$. We choose the Lie algebra $\mathfrak{a}$ to be the set of diagonal matrices in $\mathfrak{h}$, and we choose the positive roots of $\mathfrak{h}$ to be the linear forms $e_i^* - e_j^*$ with either $1 \leq i < j \leq d_1$ or $d_1 < i < j \leq d$. We can embed the rectangle R as a subset of the set P^+ of dominant weights. Indeed, for λ in R, the representation of H in $\Lambda^{\lambda_1}\mathbb{R}^{d_1} \otimes \Lambda^{\lambda_2}\mathbb{R}^{d_2}$ is irreducible with highest weight

$$\widetilde{\lambda} = e_1^* + \cdots + e_{\lambda_1}^* + e_{d_1+1}^* + \cdots + e_{d_1+\lambda_2}^*.$$

One can describe the restriction to the subset $\widetilde{R}+\widetilde{R} \subset P^+$ of the partial order (7.10). Indeed, one has the equivalence, for λ, μ, ν, ρ in R,

$$\widetilde{\nu} + \widetilde{\rho} \geq \widetilde{\lambda} + \widetilde{\mu} \Longleftrightarrow (\ \nu + \rho = \lambda + \mu \text{ and } \min(\lambda, \mu) \leq \nu \leq \max(\lambda, \mu)\ ).$$

In the left-hand side, the inequality is defined by (7.10) while, in the right-hand side, it is defined by (7.6). This proves that the bound (7.11) can be reformulated as the bound (7.7). □

Proof of Lemma 7.8. Follows directly from the next two lemmas. □

Lemma 7.9. *Let H be a real algebraic reductive group, V be a real algebraic representation of H. For λ, μ in P^+ and v, w in V, one has*

$$\|v_\lambda\| \, \|w_\mu\| \ll \|(v \otimes w)_{\lambda+\mu}\| \, .$$

Proof. This bound follows by a compacity argument, once one has noticed that Equality $(v \otimes w)_{\lambda+\mu} = 0$ implies $v_\lambda \otimes w_\mu = 0$. □

Lemma 7.10. *Let $V = \mathbb{R}^d$ and $r, s, t \geq 0$. There exists a linear map*

$$\Psi : \Lambda^{r+s}V \otimes \Lambda^{r+t}V \to \Lambda^r V \otimes \Lambda^{r+s+t}V \text{ such that}$$
$$(u \wedge v) \otimes (u \wedge w) \mapsto u \otimes (u \wedge v \wedge w) ,$$

for all pure tensors $u \in \Lambda^r V$, $v \in \Lambda^s V$ and $w \in \Lambda^t V$.

This map Ψ is unique and is $\mathrm{GL}(V)$ equivariant.

Proof. This exercise in exterior algebra is left to the reader. See [4]. □

7.4. Benoist-Quint recurrence theorem.

We show how Theorem 3.2 can be deduced from the previous ideas.

We will only deal with the following case which, thanks to Margulis Arithmeticity Theorem (see [12]), is the most important one.

Proposition 7.11. *Let $G \subset \mathrm{SL}(d, \mathbb{R})$ be a semisimple algebraic subgroup defined over $\mathbb{Q}$ and $\Lambda = G \cap \mathrm{SL}(d, \mathbb{Z})$. Let μ be a Zariski dense compactly supported probability measure on a semisimple subgroup H with no compact factors. Then the Markov operator P_μ on $X = G/\Lambda$ satisfies* **CH**.

Proof. We recall that the quotient $X = G/\Lambda$ is closed in $X_d = \mathrm{SL}(d, \mathbb{R})/\mathrm{SL}(d, \mathbb{Z})$ (see [7]). Hence, we can assume that $G = \mathrm{SL}(d, \mathbb{R})$ and $X = X_d$. We keep the notation of Section 7.3. We choose an element H_0 in the interior of the Weyl chamber, and set, for λ in P^+, $|\lambda| = \lambda(H_0)$. Let $\varepsilon_0 > 0$. Exactly as in Formula (7.8), for v in $\Lambda^i \mathbb{R}^d$ with $0 < i < d$, we define

$$\varphi_{\varepsilon_0}(v) = \max_{\lambda \in P^+ \smallsetminus 0} \varepsilon_0^{\frac{-(d-i)i}{|\lambda|}} \|v_\lambda\|^{\frac{1}{|\lambda|}}$$

so that Lemma 7.7 is still true with this function φ_{ε_0}. As in the proof of Proposition 7.4, we define, for x in X,

$$\alpha_{\varepsilon_0}(x) = \min\{\varphi_{\varepsilon_0}(v) \mid v \in \Lambda^\bullet x \smallsetminus 0 \text{ , pure tensor with } \|v_0\| < \varepsilon_0\}.$$

and check the condition **CH** with the same functions $f_{\varepsilon_0} = \alpha_{\varepsilon_0}^{-\delta}$ provided that δ and ε_0 are small enough. □

7.5. Eskin-Margulis recurrence theorem.

We show how Theorem 3.1 can be deduced from Theorem 3.2.

We will again only deal with the most important case.

Proposition 7.12. *Let $G \subset \mathrm{SL}(d, \mathbb{R})$ be a semisimple algebraic subgroup defined over $\mathbb{Q}$ and $\Lambda = G \cap \mathrm{SL}(d, \mathbb{Z})$. Let μ be a Zariski dense compactly supported probability measure on a semisimple subgroup H with no compact factors and with trivial centralizer in G. Then the Markov operator P_μ on $X = G/\Lambda$ satisfies* **UCH**.

Proof. We consider the function f_{ε_0} of Section 7.4 restricted to the finite volume G-orbit $G/\Lambda \subset \mathrm{SL}(d, \mathbb{R})/\mathrm{SL}(d, \mathbb{Z})$. We only have to check that, for ε_0 small enough, f_{ε_0} is everywhere finite on G/Λ.

Assume by contradiction that this is not the case, then there exists a sequence of H-invariant non-zero vectors $v_n \in \Lambda^i\mathbb{R}^d$ such that $\|v_n\| \searrow 0$ and $g_n \in G$ such that $g_n v_n$ belongs to the lattice $\Lambda^i\mathbb{Z}^d$. As a consequence, there exists n_0 such that, for $n \geq n_0$, every G-invariant polynomial F on $\Lambda^i\mathbb{R}^d$ with $F(0) = 0$ satisfies also $F(v_n) = 0$. This means that v_n is an unstable vector. By Kempf Theorem in [11], the stabilizer of an unstable vector is a parabolic subgroup $P \neq G$. Hence the semisimple group H is included in P. As a consequence H has a non-trivial centralizer. Contradiction. □

We conclude this survey by an open question: it is very likely that Theorems 3.1 and 3.2 are still true without any moment assumption on μ.

References

[1] Y. Benoist and J.-F. Quint, *Mesures stationnaires et fermés invariants des espaces homogènes*, Ann. of Math., **174** (2011), 1111–1162.

[2] ———, *Stationary measures and invariant subsets of homogeneous spaces (II)*, Journ. Am. Math. Soc., **26** (2013), 659–734.

[3] ———, *Stationary measures and invariant subsets of homogeneous spaces (III)*, *Ann. of Math.*, **178** (2013), 1017–1059.

[4] ———, *Random walks on finite volume homogeneous spaces*, Invent. Math. **187** (2012), 37–59.

[5] ———, *Introduction to random walks on homogeneous spaces* Japan. J. Math., **7** (2012), 135–166.

[6] ———, *Random walks on reductive groups*, preprint, 2013.

[7] A. Borel, *Introduction aux groupes arithmétiques*, Hermann, 1969.

[8] E. Breuillard, *Local limit theorems and equidistribution of random walks on the Heisenberg group* Geom. Funct. Anal., **15** (2005), 35–82.

[9] A. Eskin and G. Margulis, *Recurrence properties of random walks on finite volume homogeneous manifolds*, in Random walks and geometry, W. de Gruiter (2004), 431–444.

[10] F. Foster, *On the stochastic matrices associated with certain queuing processes*, Ann. Math. Stat., **24** (1953), 355–360.

[11] G. Kempf, *Instability in invariant theory*, Ann. of Math., **108** (1978), 299–316.

[12] G. Margulis, *Discrete subgroups of semisimple Lie groups*, Springer, 1991.

[13] S. Meyn, R. Tweedie, *Markov chain and stochastic stability*, Spinger, 1993.

CNRS - Université Paris-Sud, 91405 Orsay France

E-mail: yves.benoist@math.u-psud.fr

Diophantine geometry and uniform growth of finite and infinite groups

Emmanuel Breuillard

Abstract. We survey a number of recent results regarding the geometry and spectra of finite and infinite groups. In particular we discuss the uniform Tits alternative for infinite linear groups highlighting the inputs from diophantine geometry and the consequences for finite groups.

Mathematics Subject Classification (2010). Primary 22E40; Secondary 11G50.

Keywords. Diophantine geometry, heights, small points, Lehmer conjecture, exponential growth, amenability, Tits alternative, spectral gaps, approximate groups.

1. Uniform growth for infinite groups and Lehmer's conjecture

Let Γ be a finitely generated group and S a finite symmetric (i.e. $S = S^{-1}$) generating set containing the identity. The study of the *growth of* Γ is the study of the number of elements in the n-th fold product set $S^n = S \cdot \cdots \cdot S \subset \Gamma$ as a function of n. The set S^n is also the ball of radius n in the Cayley graph $Cay(\Gamma, S)$ of Γ relative to the generating set S, namely the graph with vertex set Γ in which two group elements x, y are linked by an (undirected) edge if $x = ys$ for some $s \in S \setminus \{1\}$. A simple way to quantify the growth of Γ with respect to S is to introduce the *exponential growth rate*

$$\rho_S := \lim_{n \to +\infty} |S^n|^{1/n} \tag{1.1}$$

The limit exists by sub-multiplicativity $|S^{n+m}| \leqslant |S^n| \cdot |S^m|$. Note that $\rho_S \leqslant |S^n|^{1/n}$ for each $n \geqslant 1$. The group Γ is said to be of exponential growth if $\rho_S > 1$. While ρ_S typically depends on S, the property that it is strictly bigger than 1 is easily seen to be independent of the choice of generating set S. Similarly one says that Γ has polynomial growth if there are constants $C, d > 0$ independent of n such that $|S^n| \leqslant Cn^d$ for all $n \geqslant 1$.

The growth of groups has been widely studied since the 1950's and the initial works of Svarc [83] and Milnor [70] who noticed that fundamental groups of negatively curved compact manifolds have exponential growth. See [42, 44] and [68] for thorough recent expository texts. We begin with a quick historical review of some important developments regarding group growth:

- Milnor and Wolf [91] proved that nilpotent groups have polynomial growth and that solvable groups have either exponential growth or are virtually nilpotent (i.e. contain a nilpotent subgroup of finite index).

Proceedings of the International Congress of Mathematicians, Seoul, 2014

- Tits showed that linear groups, i.e. subgroups of $GL_n(K)$ over a (commutative) field K have exponential growth unless they are virtually nilpotent, a consequence of his famous alternative: any linear group either contains a non-abelian free group, or is virtually solvable [85].
- Gromov [46] famously proved that every finitely generated group with polynomial growth is virtually nilpotent.
- Grigorchuk [40], answering by the negative a question of Milnor, gave the first example of a group with *intermediate growth*, i.e. whose growth is neither polynomial nor exponential: the so-called *Girgorchuk group* (see [43]). Recently Bartholdi and Erschler [4], using ingenious variants of Grigorchuk's construction, built for each $\alpha \in (.77, 1)$ groups for which $e^{c_1 n^\alpha} \leqslant |S^n| \leqslant e^{c_2 n^\alpha}$ for some constants $c_1, c_2 > 0$. Interesting groups with oscillating behaviors also exist (see [4, 25, 56]).
- Kleiner [57] gave a new proof of Gromov's theorem using harmonic functions and arguments closely related to the work of Colding and Minicozzi [27] in differential geometry. These arguments were pushed further by Shalom and Tao [82] to show that if $|S^n| \leqslant n^{\varepsilon(\log\log n)^\varepsilon}$ for some small absolute constant $\varepsilon > 0$, then the group is virtually nilpotent.
- The Grigorchuk *gap conjecture* asserts that if $|S^n| \leqslant e^{n^\alpha}$ for some $\alpha < \frac{1}{2}$, then the group has polynomial growth and hence is virtually nilpotent ([41, 42]).

A finitely generated group is said to have *uniform exponential growth* if

$$\inf_S \rho_S > 1,$$

where S varies among all (finite symmetric) generating subsets of the group. Gromov [47, Remark 5.2.] asked in the early eighties whether every group with exponential growth has uniform exponential growth. The answer is no. The first example was given more than a decade later by J.S. Wilson [90]. He built a group Γ containing a non-abelian free subgroup, and hence having exponential growth, and subsets $S_n := \{1, a_n^{\pm 1}, b_n^{\pm 1}\}$ generating Γ such that $\rho_{S_n} \to 1$ as $n \to +\infty$. Wilson's group is a subgroup of the group of automorphisms of a rooted tree (as is Grigorchuk's group by the way). It is known however that hyperbolic groups [59], solvable groups [75], linear groups in characteristic zero [33] or positive characteristic [14] have uniform exponential growth when they have exponential growth.

Although non virtually nilpotent linear groups have uniform exponential growth, the exponential growth rate ρ_S can be arbitrarily close to 1 when S and the group are allowed to vary. This fact, observed by Grigorchuk and de la Harpe in [45], can be seen as a consequence of the existence of the Grigorchuk group of intermediate growth. Indeed consider the Grigorchuk group G, generated by the usual four generators a, b, c, d (see e.g. [42, p 21]) and list the relations of G as reduced words in four letters of non-decreasing length $(w_n)_{n\geqslant 1}$. Then $G = \langle a, b, c, d | w_1, w_2, \dots, w_n, \dots \rangle$ is a presentation of G. Truncate this presentation after the n-th relator: we get this way a finitely presented group G_n. Clearly G_n surjects onto G and converges to G in the topology of marked groups: this means in particular that a ball $B_G(1, R)$ of radius R centered at the identity in G will be in bijection with the same ball $B_{G_n}(1, R)$ in G_n provided n is large enough and R is fixed. Consequently:

$$\rho_{S,G_n} \leqslant |B_{G_n}(1, R)|^{1/R} = |B_G(1, R)|^{1/R} = e^{\varepsilon(R)},$$

where $\varepsilon(R)$ tends to 0 as R tends to infinity, because G has sub-exponential growth. Grigorchuk and de la Harpe [6, 45] establish that each G_n has a quotient Γ_n containing the direct product of a finite number (increasing with n) of copies of a non-abelian free group as a subgroup of finite index. In particular $\rho_{S,G_n} \geqslant \rho_{S,\Gamma_n} > 1$. Moreover the Γ_n are clearly linear, since they contain a linear group of finite index. In conclusion:

Fact 1. *There are linear groups of exponential growth $\Gamma_n \leqslant GL_{d_n}(\mathbb{Z})$ each generated by a set S_n of 4 matrices and their inverses such that ρ_{S_n} tends to 1 as n tends to infinity.*

As far as we know it is an open problem to show that such a phenomenon of slow exponential growth arises as well in the class of all Gromov hyperbolic groups. However we conjecture that this cannot happen for linear groups of bounded dimension:

Conjecture 1.1 (Growth conjecture). *Given $d \in \mathbb{N}$, there is $\varepsilon(d) > 0$ such that for every field K and every finite subset $S \subset GL_d(K)$, either $\rho_S = 1$ and $\langle S \rangle$ is virtually nilpotent, or*

$$\rho_S > 1 + \varepsilon(d).$$

The examples of Grigorchuk and de la Harpe described above imply that $\varepsilon(d)$ must tend to 0 as d tends to infinity. Besides, their examples contain a direct product of a large number of copies of the free group, hence cannot be linear in bounded dimension, i.e. $d_n \to +\infty$.

In [12] we observed the following:

Fact 2. *The Growth conjecture implies the Lehmer conjecture.*

Let us recall the Lehmer conjecture. Given an algebraic number x with minimal polynomial $\pi_x = a_d X^d + \cdots + a_1 X + a_0 \in \mathbb{Z}[X]$, write $\pi_x = a_d \prod_1^d (X - x_i)$ and define the Mahler measure of π_x as

$$M(\pi_x) := |a_d| \prod_{|x_i| \geqslant 1} |x_i|$$

Lehmer's conjecture (initially stated as a problem) asserts that $M(\pi_x)$ ought to be bounded away from 1 unless it is equal to 1. Kronecker's theorem tells us that $M(\pi_x) = 1$ if and only if x is a root of unity (i.e. π_x is a cyclotomic polynomial). Hence Lehmer's conjecture is the statement that there is some absolute $\varepsilon > 0$ such that

$$M(\pi_x) > 1 + \varepsilon,$$

for every algebraic number $x \in \overline{\mathbb{Q}}$, which is not a root of unity. The smallest known Mahler measure is the Mahler measure of the so-called Lehmer polynomial $X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1$, which is approximately 1.17628. Somewhat surprisingly this number coincides with the growth rate of the $(2, 3, 7)$ triangle group $\langle s, t, u | s^2 = t^2 = u^2 = 1, (st)^2 = (tu)^3 = (us)^7 = 1 \rangle$ which is also the discrete subgroup of isometries of the hyperbolic plane of smallest possible co-volume (see [35]).

Fact 2 above can be easily seen by considering the following set of matrices:

$$S_x := \{ \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}^{\pm 1}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{\pm 1}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \}. \tag{1.2}$$

A simple calculation (see [12, §7]), involving estimating the number of points of height at most n in the ring $\mathbb{Z}[x]$, $x \in \mathbb{C}$, shows that

$$M(\pi_x) \geqslant \rho_{S_x} \tag{1.3}$$

where ρ_{S_x} is the rate of exponential growth of S_x. There is no equality in general in (1.3) because $M(\pi_x)$ can be large, while $\rho_{S_x} \leqslant 3$ since there are only two generators and their inverses. But there is equality in some cases, for example when x is a Salem number in the interval $[1, 2]$. It can easily be shown that $\langle S_x \rangle$ is virtually nilpotent if and only if x is a root of unity. If x is transcendental then $\langle S_x \rangle$ is isomorphic to the wreath product $\mathbb{Z}wr.\mathbb{Z}$ and thus ρ_{S_x} is bounded away from 1. If x is not an algebraic unit or has a Galois conjugate $\sigma(x)$ such that $|\sigma(x)| > 1 + \eta$, then it is straightforward to establish a lower bound on ρ_{S_x} of the form $1 + \varepsilon(\eta)$, where $\varepsilon > 0$ depends only on η (see [12]). However there are algebraic numbers all of whose conjugates are close to the unit circle.

Note that for every possible choice of x, the group $\langle S_x \rangle$ in this example is solvable of derived length 2. Somewhat surprisingly, it turns out that the solvable case is the most difficult one. Indeed we have shown the following:

Theorem 1.2 (Growth gap [16, 17]). *There is $\varepsilon = \varepsilon(d) > 0$ such that given any field K and any finite symmetric set S containing 1 in $GL_d(K)$ and generating a non virtually solvable subgroup,*

$$\rho_S > 1 + \varepsilon.$$

Again we see that $\varepsilon(d)$ must tend to 0 as d tends to infinity, because the examples given above of Grigorchuk and de la Harpe contain a free subgroup, hence are not virtually solvable.

Theorem 1.2 is deduced from a more general statement, the uniform Tits alternative, which we discuss further below. At the heart of its proof lies some diophantine geometry and the behavior of large Galois orbits of algebraic numbers of small height. Earlier results in this direction, in particular Eskin-Mozes-Oh [33] and its strengthening by Gelander and the author [14], were focusing on proving a uniform lower bound on ρ_S where the matrix entries of S were constrained within a certain fixed finitely generating ring (this is in particular the situation when S varies among the generating sets of a fixed finitely generated subgroup). The main novelty of Theorem 1.2 is the uniformity in the field of definition of the subgroup $\langle S \rangle$. Of course, this is where the interplay with number theory comes in.

It is not the first time that a connection between the exponential growth rate of groups and some properties of algebraic numbers is made. For example Cannon [26] observed that the exponential growth rate ρ_S of the fundamental group of a closed surface of genus $g \geqslant 2$ in its standard presentation as $\langle a_1, \dots, a_g, b_1, \dots, b_g | \prod_{i=1}^{g} [a_i, b_i] = 1 \rangle$ is a Salem number, i.e. an algebraic number having only one conjugate outside of the closed unit disc and at least one on the unit circle. More generally it is known [36, Chp 9.] that Gromov hyperbolic groups such as fundamental groups of closed hyperbolic manifolds admit a rational growth series, and thus the associated growth rates are algebraic numbers. See also the nice survey [35].

We end this section with some suggestions for further research. It can be easily seen, thanks to Theorem 1.2 that the growth conjecture reduces to the case of $GL_2(\mathbb{C})$ and even to the subgroups $\langle S_x \rangle$ considered above. In light of this it would be interesting to determine whether the converse to Fact 2 above holds, i.e. whether the Growth and the Lehmer conjecture are equivalent. This seems highly plausible and very likely related to Bernoulli

convolutions. Another interesting problem would be to verify that Theorem 1.2 extends to sub-semi-groups.

2. Uniform Tits alternative and uniform spectral gap estimates

The growth gap theorem (Theorem 1.2) above is a direct consequence of the following uniform Tits alternative:

Theorem 2.1 (Uniform Tits alternative [16, 17]). *Given $d \in \mathbb{N}$, there is $N = N(d) \in \mathbb{N}$ such that if K is a field and S is a finite subset of $GL_d(K)$ with $S = S^{-1}$ and $1 \in S$, then*

- *either $\langle S \rangle$ is virtually solvable,*
- *or S^N contains two generators of a non-abelian free subgroup.*

That Theorem 1.2 follows from this is clear, because $\rho_S \geqslant \rho_{S^N}^{1/N} \geqslant 3^{1/N}$, where we have the last inequality, because S^N contains a pair $\{a, b\}$ generating a free subgroup.

Recall that the Tits alternative ([48, 85] first conjectured by Bass and Serre) asserts that every finitely generated linear group admits a non-abelian free subgroup unless it is virtually solvable. It is an alternative, because the two cases are mutually exclusive: non-abelian free subgroups do not contain solvable subgroups of finite index.

The proof of J. Tits uses the dynamics of powers of linear transformations on projective space and the so-called *ping-pong lemma*, well-known to hyperbolic geometers since Fricke and Klein. See [13, 48] for expositions. Theorem 2.1 is thus a strengthening of the Tits alternative, in which the generating pair for the free subgroup is shown to arise already in a ball of universally bounded radius in the Cayley graph of the linear group $\langle S \rangle$.

Theorem 2.1 improves on an earlier result of Gelander and the author [14] in which the bound N was proven to be uniform as S varies among the generating sets of a fixed linear group. That was the same kind of uniformity as was obtained by Eskin-Mozes-Oh [33] for the rate of growth (as remarked in the situation of Theorem 1.2 above): it assumes that the matrix entries of S lie in a fixed finitely generated ring. The key point in Theorem 2.1 is the uniformity of N in the field K. This new uniformity is intimately linked to number theory and as we will see below to properties of a certain height (in the sense of Diophantine geometry) on the representation variety of the free group in GL_d over $\overline{\mathbb{Q}}$. It is also key to proving uniform spectral gap and diameter estimates for finite quotients such as $SL_d(\mathbb{F}_p)$ as shown in work of Gamburd and the author [21].

The Tits alternative [85] implies that finitely generated non virtually solvable linear groups are non-amenable, because they contain a free subgroup. In a similar way, Theorem 2.1 shows that this non-amenability is uniform when the generating set varies. The non-amenability of a group can be quantified in terms of so-called Kazhdan constants

$$\kappa(S, \pi) := \inf_{f \in \mathcal{H}_\pi, ||f||=1} \max_{s \in S} \{||\pi(s)f - f||\} \tag{2.1}$$

with respect to a set S and a unitary representation π with Hilbert space $\mathcal{H}_\pi$. A discrete group Γ is said to be *non-amenable* if

$$\kappa(S, \lambda_\Gamma) > 0$$

for some (hence all) finite subset S of Γ, where λ_Γ is the left regular representation of Γ, i.e. the unitary representation with Hilbert space $\ell^2(\Gamma)$ defined by

$$\lambda_\Gamma(g)f(x) = f(g^{-1}x).$$

We can now state some spectral corollaries of Theorem 2.1.

Corollary 2.2 (uniform non-amenability). *There is $\varepsilon = \varepsilon(d) > 0$ such that if K is a field and $S \subset GL_d(K)$ a finite subset, either $\langle S\rangle$ is virtually solvable, or $\kappa(S, \lambda_{\langle S\rangle}) > \varepsilon$.*

That a uniform Tits alternative would imply such a spectral bound was observed by Shalom in [81, Theorem 8.4] in the context of hyperbolic groups. We now discuss further similar spectral bounds, all inspired by [81] and [72]. A discrete group Γ is said to have Kazhdan property (T) if there is a finite subset S and a uniform $\varepsilon = \varepsilon_S > 0$ such that $\kappa(S, \pi) > \varepsilon$ for every unitary representation π of Γ without invariant vectors. Usually there is no uniform lower bound on ε_S independent of the choice of S among generating subsets of a fixed group with property (T) (although that remains an open problem from $\mathrm{SL}_3(\mathbb{Z})$). Gelander and Zuk [34] showed that no such lower bound exists in the case when Γ has a non-discrete image in a connected topological group.

However if we restrict the set of representations to those coming from the ambient group, one can sometimes obtain a uniform lower bound. Indeed Theorem 2.1 implies, via the well-known *tensor power trick* (see [29, 71]):

Corollary 2.3 (Uniform Kazhdan constant). *Given $d \in \mathbb{N}$ there is $\varepsilon = \varepsilon(d) > 0$ such that the following holds. If G is a real Lie group with $\dim(G) \leqslant d$ and π is a unitary representation of G which is strongly L^p, then for every finite subset $S \subset G$ generating a non-virtually solvable discrete subgroup*

$$\kappa(S, \pi) > \frac{\varepsilon}{\sqrt{p}}.$$

Recall that a unitary representation π is called strongly L^p if there is a dense subspace of vectors $\xi \in \mathcal{H}_\pi$ whose matrix coefficients $g \mapsto \langle \pi(g)\xi, \xi\rangle$ belong to $L^p(G)$ for each ξ. M. Cowling [28] proved that for every simple Lie group with property (T) (e.g. $G = SL_n(\mathbb{R})$, $n \geqslant 3$) there is some $p_0 > 0$ such that every unitary representation of G without non zero invariant vectors is strongly L^{p_0}. See [63, 74] for the value of $p_0(G)$. Hence in this case $\kappa(S, \pi)$ can be bounded from below independently of π.

We conclude this section with a natural suggestion for further research. That is to give good bounds on $\varepsilon(d)$ and $N(d)$ from Theorems 1.2 and 2.1. The proof of the uniform Tits alternative given in [15–17] is effective, except at one point (the constant in [15, Lemma 2.1(b)] and Lemma 3.9 below). However even this constant can be made effective although with a relatively poor bound. At any case it would be interesting to work out an explicit lower bound on $\varepsilon(d)$ in terms of d only and compare it to the upper bound given by the examples of Grigorchuk and de la Harpe described in the previous section.

3. Heights on character varieties of semi-simple groups

3.1. A reformulation in terms of first order logic. The uniform Tits alternative (Theorem 2.1 above) for subgroups of GL_d is uniform over all fields: the length of the two words

giving rise to generators of a free subgroup is universally bounded in terms of d only. Fields of different characteristic have to be dealt with independently, but it turns out that if the field has characteristic $p > 0$, then the uniformity in Theorem 2.1 is much easier to establish, is uniform in p, and requires no significantly new ingredient than what was already known from the previous uniformity result by Gelander and the author [14]. So in what follows we will focus on the zero characteristic case. It turns out that proving that the bound $N(d)$ holds uniformly over all fields of characteristic zero is equivalent to proving it for the field of algebraic numbers $\overline{\mathbb{Q}}$. This can be seen in a number of ways. For example by specialization. One other way is to view the statement of Theorem 2.1 as a countable union of statements expressible in first order logic. Let us be more precise.

To begin with, the condition on the set S viewed as a k-tuple of elements in GL_d for $\langle S\rangle$ to be virtually solvable is an algebraic condition: it defines a certain closed algebraic subvariety of $(GL_d)^k$, which we denote by $\mathcal{V}_{sol}$. In fact virtually solvable subgroups of GL_d in characteristic zero admit a subgroup of bounded index (i.e. $< c(d)$) which is conjugate to a subgroup of the upper-triangular matrices. Hence $\langle S\rangle$ is virtually solvable if and only if a certain number of words with letters in S and bounded length have a common fixed point in the flag variety.

On the other hand, to say that no two words w_1, w_2 of length at most $N(d)$ can ever generate a free subgroup is equivalent to say that there is an integer n, such that for all possible choices of w_1, w_2 among words of length at most $N(d)$ with letters in S, one can always find a non-trivial word of length at most n in the free group, such that $w(w_1(S), w_2(S)) = 1$. Clearly this is a finite set of algebraic conditions on S viewed as a k-tuple in $(GL_d)^k$. Call this subvariety $\mathcal{W}_n$.

Theorem 2.1 is the statement that for each integer n, $\mathcal{W}_n$ is contained in $\mathcal{V}_{sol}$. This implication is in itself a statement of first order logic, because the algebraic varieties involved are defined over $\mathbb{Q}$. In particular if it holds for some algebraically closed field of characteristic zero, it holds for all of them, because any two algebraically closed field with the same characteristic have the same first order theory.

The discussion regarding $\mathcal{V}_{sol}$ shows that there is an integer n_0 depending on d only such that $\mathcal{V}_{sol}(\mathbb{C}) \subset \mathcal{W}_{n_0}(\mathbb{C})$. Now Theorem 2.1 tells us that this is an equality. So finally Theorem 2.1 can be reformulated as the statement that

$$\mathcal{V}_{sol} = \mathcal{W}_n$$

for each integer $n \geqslant n_0(d)$.

Note in passing that Theorem 2.1 cannot be deduced automatically by logical compactness from the original Tits alternative. The reason is that the condition that two matrices generate a free subgroup is not expressible in first order logic: it is a countable union of first order logic statements. Indeed it is very hard in general to understand the locus of tuples, say in $(GL_d)^k$, which generate a free subgroup. Not much can be said on this set even in the case of GL_2.

Anyways, reducing (the characteristic zero case of) Theorem 2.1 to the field of algebraic numbers $\overline{\mathbb{Q}}$ allows to introduce the theory of heights and take advantage of known results in Diophantine geometry, in particular regarding the action of the Galois group. For this purpose, we introduced in [15] a certain conjugation invariant normalized height $\widehat{h}$ on $(GL_d)^k$, to be discussed below.

As we saw in Section 1, the uniform growth of linear groups is closely related to the properties of algebraic numbers of high degree and small height, in particular to the Lehmer

conjecture. There the Lehmer conjecture was the obstacle to prove uniform growth. This situation can be reversed in the non solvable case, by first establishing a strong analogue of the Lehmer conjecture for this normalized height $\widehat{h}$. Let us first set up some notation.

3.2. A normalized height on reductive groups. In this paragraph we discuss the Height gap theorem (Theorem 3.6 below), which is the key ingredient for the uniformity in the field in the proof of the uniform Tits alternative.

Let $\mathbb{G}$ be a connected reductive algebraic group defined over a number field K (such as $\mathbb{G} = GL_d$). Let (ρ, W) be a faithful linear representation of $\mathbb{G}$. Let V_K be the set of places of K, i.e. equivalence classes of absolute values on K. Associated to each $v \in V_K$ is a local field K_v, the completion of K with respect to v, and an absolute value $|\cdot|_v$ defined on an algebraic closure $\overline{\mathbb{Q}}_v$ of K_v. Picking a basis of W, we can define a norm $||\cdot||_v$ on $W_{K_v} := W \otimes K_v$ for each $v \in V_K$ to be equal to

- the Euclidean norm $\sqrt{\sum_i |x_i|_v^2}$ if v is archimedean (i.e. $K_v = \mathbb{R}$ or $\mathbb{C}$),
- the sup norm $\max |x_i|_v$ if v is non archimedean.

Let S denote as before a finite subset of $\mathbb{G}(K)$. Set

$$h_\rho(S) := \frac{1}{[K:\mathbb{Q}]} \sum_{v \in V_K} n_v \log^+ ||\rho(S)||_v,$$

where n_v is the degree of the local extension $[K_v : \mathbb{Q}_v]$, $\log^+$ is short for $\max\{0, \log\}$, and

$$||\rho(S)||_v := \max\{||\rho(s)||_v, s \in S\},$$

where $||\rho(s)||_v$ is the operator norm of the endomorphism $\rho(s)$ of W_{K_v} associated to the norm $||\cdot||_v$.

Definition 3.3 (Normalized height). *We set*

$$\widehat{h}_\rho(S) := \lim_{n \to +\infty} \frac{1}{n} h_\rho(S^n),$$

where $S^n = S \cdot \cdots \cdot S$ is the n-fold product set.

While $h_\rho(S)$ depends on the particular choice of basis used to defined the Euclidean and sup-norm on W_{K_v}, the normalized height $\widehat{h}(S)$ does not depend on this choice.

The definition of the normalized height is modeled on the definition of the Néron-Tate height in the theory of abelian varieties. Here the normalization encodes the way the powers S^n grow in each valuation. In particular this height carries some important information on the subgroup $\langle S \rangle$ generated by S. Heights on subgroups of matrices generated by one element have been studied by Talamanca in [89]. Our height is a natural extension of Talamanca's height to the case when S has more than one element.

The limit in the definition of $\widehat{h}_\rho$ exists because of sub-additivity. Indeed it is straightforward to check that $h_\rho(S^{n+m}) \leqslant h_\rho(S^n) + h_\rho(S^m)$ for all integers $n, m \geqslant 1$.

Moreover the height h_ρ and normalized height $\widehat{h}_\rho$ do not depend on the choice of the number field K. Namely if we replace K by any finite extension K' of K, so that S is again defined over K', then the value of the heights for K and for K' are the same.

Example 3.4. Consider the set $S = S_x$, for $x \in \overline{\mathbb{Q}}^{\times}$, from (1.2) in Section 1 and let ρ be the natural 2-dimensional representation of GL_2. Then

$$\widehat{h}_\rho(S_x) = h(x),$$

where $h(x)$ is the classical absolute Weil height (see [11] for background) of the algebraic number x, namely

$$\begin{aligned} h(x) &:= \frac{1}{[\mathbb{Q}(x):\mathbb{Q}]} \sum_{v \in V_{\mathbb{Q}(x)}} n_v \log^+ |x|_v \\ &= \frac{1}{D}(\log |a_D| + \sum_{y \in Gal(\overline{\mathbb{Q}}|\mathbb{Q})\cdot x} \log^+ |y|) \\ &= \frac{1}{D} \log M(\pi_x), \end{aligned}$$

where $M(\pi_x)$ is as before the Mahler measure of the minimal polynomial $\pi_x := a_D X^D + \cdots + a_1 X + a_0 \in \mathbb{Z}[X]$ of x. ◇

Example 3.5. Let $S = \{g\} \in \mathbb{G}(K)$ a singleton in $\mathbb{G} = GL_d$ with the natural d-dimensional representation ρ. Let $\lambda_1, \ldots, \lambda_d$ the eigenvalues of g. Then

$$\widehat{h}_\rho(S) = \frac{1}{[K:\mathbb{Q}]} \sum_{v \in V_K} n_v \log^+ \max_i \{|\lambda_i|_v\}.$$

In particular

$$\frac{1}{d}\widehat{h}_\rho(S) \leqslant \max_i \{h(\lambda_i)\} \leqslant \widehat{h}_\rho(S).$$

◇

To get a better understanding of this height, let us record now its main properties. Below S is a finite subset of $\mathbb{G}(\overline{\mathbb{Q}})$.

Properties of $\widehat{h}_\rho(S)$:

(i) (linearity in powers) $\forall n \in \mathbb{N}$, $\widehat{h}_\rho(S^n) = n\widehat{h}_\rho(S)$,

(ii) (conjugation invariance) $\forall g \in \mathbb{G}(\overline{\mathbb{Q}})$, $\widehat{h}_\rho(gSg^{-1}) = \widehat{h}_\rho(S)$,

(iii) (height zero points) $\widehat{h}_\rho(S) = 0$ if and only if $\langle S \rangle$ is virtually unipotent.

(iv) (change of representation) given two faithful linear representations ρ_1, ρ_2 of $\mathbb{G}$ there are constants $C_1, C_2 > 0$ such that

$$C_1 \widehat{h}_{\rho_2}(S) \leqslant \widehat{h}_{\rho_1}(S) \leqslant C_2 \widehat{h}_{\rho_2}(S)$$

for all finite subsets $S \subset \mathbb{G}(\overline{\mathbb{Q}})$,

(v) (comparison between h and $\widehat{h}$) There is $C = C(\mathbb{G}, \rho) > 0$ such that for every finite subset $S \subset \mathbb{G}(\overline{\mathbb{Q}})$ generating a Zariski-dense subgroup of $\mathbb{G}$ assumed semisimple, one can find $g \in \mathbb{G}(\overline{\mathbb{Q}})$ such that

$$\widehat{h}_\rho(S) \leqslant h_\rho(gSg^{-1}) \leqslant C\widehat{h}_\rho(S).$$

By virtually unipotent in item (iii) we mean that $\langle S\rangle$ has a subgroup of finite index, which can be conjugated inside a unipotent subgroup of $\mathbb{G}$ (i.e. there is a basis of W where all matrices in this subgroup are upper-triangular with all eigenvalues equal to 1).

Item (v) suggests that the other natural way to build a conjugation invariant height on $\mathbb{G}$ leads in fact to a comparable quantity, at least if S is not degenerate. There is at least one more natural way to define a conjugation invariant height function on k-tuples of $\mathbb{G}$. One may consider the stable quotient $\mathbb{G}^k /\!\!/ \mathbb{G}$ in the sense of Geometric Invariant Theory, where the quotient is via the diagonal action of $\mathbb{G}$ on $\mathbb{G}^k$ by coordinate-wise conjugation. This algebraic variety, whose coordinate ring is the ring of invariants $\mathbb{C}[\mathbb{G}^k]^{\mathbb{G}}$, is also called the variety of $\mathbb{G}$-characters of the free group F_k on k letters. Then one may simply consider a height on this variety defined using the usual Weil height machine [51, 61]. This height will be comparable, up to additive and multiplicative constants, to our height $\widehat{h}(S)$.

One particularly nice way to parametrize $\mathbb{G}^k /\!\!/ \mathbb{G}$ is to consider the traces of short words in the k-tuple. Fricke and Klein showed in the 19th century that $\mathrm{tr}(a), \mathrm{tr}(b)$ and $\mathrm{tr}(ab)$ are coordinates on the character variety of the free group F_2 on $\mathrm{SL}_2(\mathbb{C})$, namely away from some singular locus, these 3 values determine the conjugacy class of the pair (a, b). More recently Procesi [76] extended this to $GL_d(\mathbb{C})$, showing that the coordinate ring of $\mathbb{G}^k /\!\!/ \mathbb{G}$, when $\mathbb{G} = GL_d(\mathbb{C})$ is generated by the traces $\mathrm{tr}(w(g_1, \dots, g_k))$, where w ranges through all (positive) words in k letters whose length is bounded by a bound depending only on d.

Given an embedding $\rho : \mathbb{G} \to GL_d$, traces of words are no longer enough to tell apart non $\mathbb{G}$-conjugate tuples, but the induced natural morphism $\mathbb{G}^k /\!\!/ \mathbb{G} \to GL_d^k /\!\!/ GL_d$, is a finite morphism: indeed if two generic k-tuples in $\mathbb{G}$ are conjugate in GL_d, they must be conjugate by an element of the normalizer of $\mathbb{G}$ in GL_d, because they generate a Zariski-dense subgroup of $\mathbb{G}$. This normalizer, when acting on $\mathbb{G}$, contains the inner automorphisms as a subgroup of finite index, see [87].

A consequence of Corollary 3.8 below is that there is $N \in \mathbb{N}$ and $C > 0$ depending only on the embedding ρ, such that for every finite subset $S \subset \mathbb{G}(\overline{\mathbb{Q}})$,

$$\frac{1}{C|S|^C}\widehat{h}_\rho(S) - C \leqslant \max_{|w|\leqslant N} h(\mathrm{tr}(\rho(w(S))) \leqslant C\widehat{h}_\rho(S)$$

So with this parametrization of $\mathbb{G}^k /\!\!/ \mathbb{G}$, $k = |S|$, we obtain a height function, which is comparable to our normalized height $\widehat{h}_\rho(S)$.

Our main theorem regarding $\widehat{h}_\rho$ is the following. It can be seen as an analogue for reductive groups of the Lehmer conjecture:

Theorem 3.6 (Height gap [15]). *There is $\varepsilon = \varepsilon(\mathbb{G}, \rho) > 0$ such that*

$$\widehat{h}_\rho(S) > \varepsilon$$

for every finite subset $S \subset \mathbb{G}(\overline{\mathbb{Q}})$ such that the subgroup $\langle S\rangle$ generated by S is not virtually solvable.

By way of contrast, one can see that the Lehmer conjecture itself is equivalent to the existence of some $\varepsilon_\rho > 0$ such that

$$\widehat{h}_\rho(S) > \frac{\varepsilon_\rho}{[K : \mathbb{Q}]} \tag{3.1}$$

for every number field K and all finite subsets $S \subset \mathbb{G}(K)$ generating a non-virtually unipotent subgroup, that is a subgroup whose elements have only roots of unity as eigenvalues (for those subsets $\widehat{h}_\rho(S) = 0$ by property (iii) above).

For example, if $\mathbb{G}$ is the multiplicative group $\mathbb{G}_m$, and $S := \{x\}$ a singleton, then $\widehat{h}(S) = h(x)$, the Weil height of x. Theorem 3.6 does not apply to this situation, because all subgroups of $\mathbb{G}_m$ are abelian, hence solvable.

Similarly the first example given above with $S_x \subset GL_2$ also shows that no such uniform lower bound can be expected when $\langle S\rangle$ is virtually solvable. There $\widehat{h}_\rho(S_x)$ was exactly the Weil height $h(x)$ of the algebraic number x, and hence could be very small (e.g. take $x = 2^{1/n}$). So the theorem claims in fact a uniform lower bound on the height, rather than on the height times the degree as the Lehmer conjecture asks. Uniform lower bounds on heights are related to the so-called *Bogomolov property* in Diophantine geometry. An algebraic extension F of $\mathbb{Q}$ of infinite degree is said to have the Bogomolov property if there is a uniform $\varepsilon > 0$ such that all elements of F with Weil height at most ε are in fact roots of unity (hence have zero Weil height). See [1, 11] for recent results about this property.

The proof of Theorem 3.6 makes use of some important facts borrowed from Diophantine geometry. Most importantly, Zhang's theorem [93], [11, Thm 4.2.] and Bilu's theorem [8], [11, Thm 4.3.1]. Zhang's theorem says that the points of very small height that lie on a proper algebraic subvariety of an algebraic torus $(\overline{\mathbb{Q}}^\times)^r$, must in fact lie in a finite union of even smaller dimensional subsets, unless the subvariety itself is a translate of a subtorus. In particular torsion points (and even points of small height!) cannot be Zariski-dense in a subvariety unless this subvariety is very special... For example, there are only finitely many points of height $< 1/100$ on the line $x + y = 1$, see [92] for optimal bounds. These lines of thought form by now a well-established branch of Diophantine geometry, encompassing such far-reaching statements as the Manin-Mumford conjecture, the Andre-Oort conjecture, etc.

This idea carries the key to the proof of the Height gap of Theorem 3.6, because too small a $\widehat{h}_\rho(S)$ would yield too many points of small height and contradict the Zariski-density of the group generated by S (after reduction to the case with the Zariski-closure of $\langle S\rangle$ is semisimple).

Zhang's theorem, as well as many other results in Diophantine geometry (starting with Szpiro-Ullmo-Zhang [84]) can be established via equidistribution methods. The prototype of these results is Bilu's theorem, according to which the Galois orbit of any algebraic number whose height is close to zero but non zero is almost equidistributed on the unit circle with its Lebesgue measure. In fact Zhang's theorem can be deduced from Bilu's, see [8]. Although we make use of Bilu's theorem in the proof of Theorem 3.6, it would be interesting to come up with a more direct argument proving an equidistribution result for the Galois orbit of S in the character variety $\mathbb{G}^k /\!\!/ \mathbb{G}$, before reaching a contradiction:

Problem. Give a proof of Theorem 3.6 via equidistribution.

3.7. Large eigenvalues. As the reader would have guessed by now, there is a relationship between our normalized height $\widehat{h}_\rho(S)$ and the Weil height of the eigenvalues of subgroup elements in $\langle S\rangle$. For example it is clear that if λ is an eigenvalue of $\rho(g)$ for some $g \in S^n$, then $|\lambda|_v \leqslant ||\rho(g)||_v \leqslant n||\rho(S)||_v$ for all $v \in V_K$ and so $h(\lambda) \leqslant n\widehat{h}_\rho(S)$. An important consequence of Theorem 3.6 and the analysis done in its proof is the following converse:

Corollary 3.8 (finding large eigenvalues). *There is a constant $C > 0$ depending only on $\mathbb{G}$*

and ρ such that if $S \subset \mathbb{G}(\overline{\mathbb{Q}})$ is a finite set, then there is a positive integer $k \leqslant C$ and an element $g \in S^k$ such that for some eigenvalue λ of $\rho(g)$.

$$h(\lambda) \geqslant \frac{1}{|S|^C} \widehat{h}_\rho(S)$$

In particular, there is a uniform $N_1 = N_1(d) > 0$ such that if two elements $a, b \in GL_d(\overline{\mathbb{Q}})$ generate a non-virtually solvable subgroup, then there is an element g expressible as a word of length at most N_1 with letters in a and b and an eigenvalue λ of g with Weil height $h(\lambda) \geqslant 1$.

In the proof of the uniform Tits alternative, a crucial step consists in finding in S^N, for some bounded N, an element with a large eigenvalue. The above corollary does just that. The largeness of the eigenvalue is measured in terms of its height.

An important fact regarding the joint spectral radius of a bounded set of matrices is encapsulated in Lemma 3.9 below. It is used many times in order to produce a large eigenvalue, both in the proof of Theorem 3.6 and of Theorem 2.1.

Let K be a local field, that is either $\mathbb{R}$, $\mathbb{C}$, or a finite extension of the p-adic numbers, or (in characteristic p) of the field of Laurent series $\mathbb{F}_p((t))$. We choose an absolute value $|\cdot|$ on K and extend it (such an extension is unique) to an algebraic closure of K. Let S be a bounded subset of $d \times d$ matrices in $M_d(K)$, endowed with the operator norm $||\cdot||$ induced by the choice of some norm on K^d. Define the spectral radius of S to be

$$R(S) = \lim_{n \to +\infty} ||S^n||^{1/n},$$

where $||S^n||$ denotes the maximum of the (operator) norms of the elements in S^n. We also let $\Lambda(S)$ be the maximal eigenvalue:

$$\Lambda(S) := \max_{s \in S} \{|\lambda|; \lambda \in Spec\{s\}\},$$

where $Spec(s)$ is the set of eigenvalues of s. If $S = \{s\}$ is a singleton, the well-known spectral radius formula tells us that

$$R(\{s\}) = \Lambda(\{s\}).$$

Obviously $\Lambda(S) \leqslant R(S)$, for all S, but general $\Lambda(S) < R(S)$ if there are two or more matrices in S. However, somewhat surprisingly, we have the following converse inequality:

Lemma 3.9 (Spectral radius formula for several matrices). *There is $c = c(d) > 0$ such that for every bounded subset $S \subset M_d(K)$, there is an integer $k \leqslant d^2$ such that*

$$\Lambda(S^k)^{1/k} \geqslant c(d) \cdot R(S),$$

moreover $c(d) = 1$ if K is non-archimedean (i.e. K not $\mathbb{R}$ or $\mathbb{C}$).

In particular, if K is non-archimedean (for example a p-adic field), then

$$R(S) = \max_{k \leqslant d^2} \Lambda(S^k)^{1/k}.$$

This information is crucial, because it says that in order to find an element in a small power of S with a large eigenvalue, it is enough to have a good lower bound on $R(S)$. And since

$$\widehat{h}_\rho(S) = \frac{1}{[K:\mathbb{Q}]} \sum_{v \in V_K} n_v \log^+ R_{K_v}(S),$$

with the obvious notation, we understand now why the Height gap theorem 3.6 above is precisely what is needed in order to find an element with large eigenvalue.

We note in passing that this lemma produces an element in S^k, $k \leqslant d^2$, whose individual powers are already responsible for most of the growth of the full power set S^n. This feature is reminiscent of one of the key claims in the Gleason-Yamabe solution to Hilbert's fifth problem on the structure of locally compact groups. See the so-called Gleason-Yamabe lemmas [55, Thm II.13], [38, Lemma 5.4].

Problem. Find a good lower bound on $c(d)$ in terms of d.

3.10. A group theoretic consequence. The Grigorchuk group is one of the simplest example of a finitely generated infinite periodic group. Periodic means that every element has finite order. According to a classical theorem of Schur (see e.g. [30, 80]) every finitely generated periodic linear group is finite. A simple consequence of the Height gap theorem is that one can always quickly get out of torsion elements unless one belongs to a finite subgroup, namely we have a quantitative version of Schur's theorem ([17, Cor 3.6]):

Corollary 3.11 (Escaping torsion elements). *Let K be a field and S a finite subset of $GL_d(K)$. If the subgroup generated by S is infinite, then one can find a word of length at most $N_1(d)$ with letters in $S \cup S^{-1}$, which has infinite order. Here $N_1(d)$ is independent of S and K.*

To see the connection with the Height gap theorem, note that (in characteristic zero) unless $\langle S \rangle$ is virtually unipotent, $\widehat{h}_\rho(S) > 0$ and hence there is $g \in S^k$ and an eigenvalue λ of g such that $h(\lambda) > 0$ by Corollary 3.8, where k is bounded in terms of d only. In particular this g has infinite order. We also remark proving escape from elements of bounded torsion is easier. It follows from the Eskin-Mozes-Oh escape from subvarieties lemma, see [33, Lemma 3.2], or [22, Lemma 3.11], according to which for any proper subvariety $\mathcal{V}$ of the Zariski closure of $\langle S \rangle$, one can find a word of bounded length with letters in S lying outside the subvariety. However the length of the word can only be bounded by a function of the degree of $\mathcal{V}$, and in the case of $\mathcal{V} = \{g; g^e = 1\}$, this degree increases with e.

It is also worth pointing out that, as in the Growth gap theorem (Theorem 1.2), the constant $N_1(d)$ from the above statement must tend to infinity. The same examples of Grigorchuk and de la Harpe can be used to show, and this was done by Bartholdi and de Cornulier in [3], that for each n, there is a 3-generated infinite linear group all of whose elements lying in the word ball of radius n are of finite order.

3.12. Does the spectral gap imply a height gap? As we already mentioned, the uniform Tits alternative (Theorem 2.1) can be derived (see [16, 17]) from the Height gap theorem and the ping-pong techniques introduced by Tits in his original paper [85]. The uniform spectral gap estimate for non-amenable linear groups, i.e. Corollary 2.2 is a simple consequence of Theorem 2.1. It is interesting to wonder whether one can also go backwards and prove the Height gap theorem assuming the uniform spectral gap for non-amenable linear groups. As it turns out, the uniform spectral gap is not quite enough to get the Height gap theorem as stated in Theorem 3.6. But one can recover the weak form of it discussed after the statement of Theorem 3.6 in (3.1), as follows.

Claim 3.13. *For a finite (symmetric) subset S of $\mathbb{G}(K)$, K any number field,*

$$\widehat{h}_\rho(S) \geqslant \frac{1}{C[K:\mathbb{Q}]} \log(r_S^{-1}),$$

where $C > 0$ is a constant depending only of $\mathbb{G}$ and ρ, and $r_S = ||\lambda_{\langle S\rangle}(\mu_S)||$ is the spectral radius of μ_S on the regular representation of $\langle S\rangle$.

To see the claim, set $\Phi_C(g) := e^{-C[K:\mathbb{Q}]h_\rho(g)}$ for some $C > 0$ and view it as a function defined on the group of adèles $G := \mathbb{G}(\mathbb{A}_K)$. We may assume that $\mathbb{G} = SL_d$ and ρ is the identity embedding. If C is large enough (depending on d only) a simple volume computation shows that Φ_C belongs to $L^2(G)$. Now, $h(xy) \leqslant h(x) + h(y)$, so $\Phi_C(xy) \geqslant \Phi_C(x)\Phi_C(y)$, from which one obtains $\forall g \in G$,

$$\langle \lambda_G(g)\Phi_C, \Phi_C\rangle_{L^2(G)} \geqslant \Phi_C(g^{-1})\langle \Phi_C, \Phi_C\rangle_{L^2(G)},$$

hence integrating over $\mu_S^{(n)}$, the n-th fold convolution power of the uniform probability measure supported on S,

$$\begin{aligned} ||\lambda_G(\mu_S^{(n)})|| &\geqslant \mathbb{E}_{\mu_S^{(n)}} \Phi_C(g^{-1}) = \mathbb{E}_{\mu_S^{(n)}} \Phi_C(g) \\ &\geqslant e^{-C[K:\mathbb{Q}]\mathbb{E}_{\mu_S^{(n)}}(h(g))} \geqslant e^{-C[K:\mathbb{Q}]h(S^n)}, \end{aligned}$$

where we applied the Jensen inequality to go from the first to the second line. The claim then follows by taking the n-th root and letting n tend to infinity. Note that $||\lambda_G(\mu_S))|| = r_S$, because $\langle S\rangle$ is a discrete subgroup of G ([5, F.1.11]). This argument is inspired from those in [72] and [88].

4. Uniform spectral gap and uniform diameter bounds for finite groups of Lie type

We now present some applications of the Height gap theorem to diameter bounds and spectral gaps for finite groups of Lie type.

4.1. Diameter bounds. Let G be a finite group and S a symmetric generating subset of G. The diameter $diam_S(G)$ is the least integer n such that every element of G can be written as a product of at most n elements from S. There is an extensive literature on diameter bounds for finite groups, from abelian groups to simple groups, including the Rubik's cube group, etc. One of the most celebrated conjecture is Babai's conjecture:

Conjecture 4.2 (Babai's conjecture). *For every finite simple group and every symmetric generating set S*

$$diam_S(G) \leqslant C(\log|G|)^C,$$

where $C > 0$ is an absolute constant (independent of G and S).

For example this conjecture is widely open in the special case of alternating groups $G = A_n$, the best bounds to date are due to Helfgott and Seress [50] and are in $\exp((\log\log|G|)^{O(1)})$ for these groups.

However Babai's conjecture was recently shown to hold for finite simple groups of Lie type and bounded rank. Such simple groups can also be described as those admitting a non trivial linear representation of bounded degree (over some, possibly finite, field).

Theorem 4.3 (Case of groups of Lie type). *If K is a field and $G \leqslant GL_d(K)$ a finite simple subgroup generated by a finite subset S, then*

$$diam_S(G) \leqslant C(\log |G|)^C,$$

where $C = C(d) > 0$ is a constant depending on d but otherwise independent of S, G and K.

Most approaches towards Babai's conjecture use the classification of finite simple groups. We will see in the next section a diameter bound, weaker than the one claimed by Babai's conjecture, but whose proof is independent of the classification.

The bound in the above theorem is a direct consequence of the following result applied repeatedly to the powers of a fixed generating set.

Theorem 4.4 (Product theorem). *If K is a field and $G \leqslant GL_d(K)$ a finite simple subgroup generated by a finite subset S, then*

$$|SSS| \geqslant \min\{|S|^{1+\varepsilon}, |G|\}$$

where $\varepsilon = \varepsilon(d) > 0$ is a constant depending on d but otherwise independent of S, G and K.

These results are due to Pyber and Szabó [77, 78] and, independently, to Green, Tao and the author [22], following work of Helfgott [49], who first solved Babai's conjecture for the family of groups $SL_2(\mathbb{F}_p)$ and $SL_3(\mathbb{F}_p)$, p prime. While Helfgott's arguments used clever ad hoc matrix computations coupled with techniques from additive combinatorics (the sum-product theorem), the proof of the general case is mainly based on algebraic geometry over finite fields. It can largely be seen as a derivative of the techniques introduced by Larsen and Pink [62] in their classification of finite subgroups of algebraic groups, and used in this context by Hrushovski [53]. See [19, 20, 78] for a discussion of these arguments.

While this represents a significant advance compared to what was known prior to these developments, the polylogarithmic bound of Theorem 4.3 is most likely not optimal. Indeed we conjecture:

Conjecture 4.5 (Logarithmic diameter). *If K is a field and $G \leqslant GL_d(K)$ a finite simple subgroup generated by a finite symmetric set S, then*

$$diam_S(G) \leqslant C \log |G|,$$

where $C = C(d) > 0$ is a constant depending on d but otherwise independent of S, G and K.

The product theorem (Theorem 4.4 above) falls short of proving any exponential growth at any early stage, because iterating n times the bound $|SSS| \geqslant |S|^{1+\varepsilon}$ gives only subexponential growth in $\exp(Cn^\alpha)$ for some $\alpha < 1$. In particular it is not optimal for small n and not sufficient to get logarithmic diameter.

As it turns out, the uniform Tits alternative can be used precisely for this purpose of establishing exponential growth at an early stage. Coupled with Theorem 4.4 above, used at

a later stage, it can say something towards this conjecture. Indeed the fact that the uniform Tits alternative holds over $\overline{\mathbb{Q}}$ allows for its reinterpretation in terms of a series of equality between *a priori* unrelated algebraic subvarieties as we pointed out in Paragraph 3.1. The equality between two algebraic varieties defined over $\mathbb{Z}$ implies their equality modulo p for every large enough prime. This will mean that unless the group G generated by S has a large solvable subgroup (of index bounded in terms of d only), one will be able to find two short words (of length L bounded in terms of d only) with letters in S admitting no relation of length $\leqslant \ell(p)$ for some function $\ell(p)$ tending to $+\infty$ as p gets large. In particular this will give at least $2^{\ell(p)}$ elements in $S^{L\ell(p)} \subset G$.

The question is how large can $\ell(p)$ be. Applying standard bounds on the *effective nullstellensatz* (e.g. those in [69]) one sees that $\ell(p)$ can be taken as large as $c(\log p)^\alpha$, where α is some positive exponent strictly less than one. This falls short of reaching the range where the product theorem can be applied successfully to get the desired logarithmic bound, as one would need to be able to have $\alpha = 1$. However one can instead play with several primes in order to turn the nullstellensatz bounds to one's advantage and obtain:

Theorem 4.6 (Uniform growth at almost all primes). *There is a constant $A = A(d) \geqslant 1$ such that, for every $\varepsilon > 0$, except perhaps for a (small yet possibly infinite) set of primes $\mathcal{P}_{\varepsilon-bad}$ satisfying $|\mathcal{P}_{\varepsilon-bad} \cap [1, X]| \leqslant X^\varepsilon$, for all $X \geqslant 1$, every symmetric subset $S \subset GL_d(\overline{\mathbb{F}_p})$ satisfies $|S^{\log p}| \geqslant p^{\varepsilon/A}$, unless the subgroup generated by S has a solvable subgroup of index at most A.*

Conjecturally the set of bad primes ought to be empty and uniform growth should take place at all primes. Recall that according to the prime number theorem, there are roughly $X/\log X$ primes less than X. So we see that uniform exponential growth does indeed take place at most primes. However we cannot say for which primes it does.

The proof of the above statement, already as outlined above, follows the same lines as the main argument in the paper by Gamburd and the author [21], which we will discuss in the next paragraph. With this we obtain:

Corollary 4.7 (diameter of perfect subgroups of $GL_d(\mathbb{F}_p)$). *Given $\varepsilon > 0$, if p is a prime not in $\mathcal{P}_{\varepsilon-bad}$, then for every perfect subgroup $G \leqslant GL_d(\mathbb{F}_p)$ generated by elements of order p*

$$\max_S diam_S G \leqslant \frac{C}{\varepsilon} \log p,$$

Here $C = C(d) > 0$ is a constant independent of p.

While this improves (at least for good primes) on the polylogarithmic bound given by Pyber and Szabó in Theorem 8 of [77], the proof uses their analog of the product theorem above for perfect groups combined with Theorem 4.6. Finite simple subgroups $GL_d(\mathbb{F}_p)$ are perfect and generated by their elements of order p (unless their order is prime to p, in which case they are bounded in size by a function of d only). In particular, we can reformulate this consequence in the following way:

Corollary 4.8 (Logarithmic diameter for almost all primes). *Conjecture 4.5 holds for simple subgroups of $GL_d(\mathbb{F}_p)$ for a density one set of primes.*

What if $G \leqslant GL_d(\mathbb{F}_p)$ is an arbitrary subgroup, not necessarily perfect or generated by elements of order p ? Well, in that case we can describe what happens regarding the

diameter by studying the subgroup G_p of G generated of elements of order p. Clearly this is a characteristic subgroup of G. Moreover G/G_p has order prime to p, and the Frattini argument coupled with Jordan's theorem[1] says that G has a subgroup whose index is bounded in terms of d only of the form AG_p, where A is abelian. Now the last term in the derived series of G_p is a perfect normal group to which Corollary 4.7 applies, while moding out by this normal subgroup gives a solvable quotient of bounded derived length. This allows to estimate the diameter of G. Diameters of abelian and solvable groups are typically much larger than those of simple or perfect groups. See e.g. [58] for a recent paper on the diameter of abelian groups.

Problem (Function field analogues). Theorem 4.6 and its corollaries above say something interesting only when p is large. For a fixed p, it would be interesting to derive a statement of a similar flavor (from the characteristic p case of the uniform Tits alternative) for the quotient fields $\mathbb{F}_p[X]/(\pi)$, where π varies among the irreducible polynomials of $\mathbb{F}_p[X]$.

We finally note that it is plausible that an even stronger phenomenon than the logarithmic diameter bounds of Corollary 4.7 holds for finite simple groups of bounded rank. Namely it is likely that

$$diam_S(G) \leqslant C\frac{\log|G|}{\log|S|}$$

for every generating subset S. This means that we take into account the size of the generating set, whereas the previous statement does not distinguish between $|S|$ large or small. This kind of bound would be optimal.

Related statements occur in the work of Liebeck, Shalev, Nikolov [64] and others, where one allows to take arbitrary conjugates of S to compute the diameter (or rather *width*) of G. For example it is known, thanks to recent work of Gill, Pyber, Short and Szabó [37], that every element in a finite simple group of Lie type of rank at most d can be written as a product of only $C(d)\frac{\log|G|}{\log|S|}$ conjugates of elements from any subset $S \subset G$ (of size $\geqslant 2$). We refer the reader to the recent survey by Martin Liebeck [65] for many beautiful recent results in this direction.

We finally quote another result, proved by Green, Guralnick and the author [24], which provides further evidence towards Conjecture 4.5.

Theorem 4.9 (Logarithmic diameter for almost all generating sets). *If K is a field and $G \leqslant GL_d(K)$ a finite simple subgroup, then*

$$diam_{\{a^{\pm 1},b^{\pm 1}\}}(G) \leqslant C\log|G|,$$

for all but a proportion $\leqslant 1/|G|^\delta$ of all pairs $\{a,b\} \subset G$, where $C,\delta > 0$ are certain constants depending on d but otherwise independent of S, G and K.

4.10. Uniform spectral gap bounds, Ellenberg's property $\widehat{\tau}$. In what follows, we will say that the Cayley graph of a finite group G with finite symmetric generating set S is an ε-expander if

$$\kappa(S, \ell_0^2(G)) > \varepsilon,$$

[1]Usually Jordan's theorem [54] is cited as a theorem about subgroups of $GL_d(\mathbb{C})$, but Jordan's original proof, unlike the more often quoted geometric argument due to Frobenius (see [30]), assumes only that all subgroups elements are semisimple and nothing on the field.

where $\ell^2_0(G)$ is the regular representation of G on functions with zero average, and $\kappa(S, \ell^2_0(G))$ is the Kazhdan constant defined in (2.1). We refer the reader to the books [66, 79] and surveys [52, 60] for the background on expanders. Let us only mention that expander graphs with N vertices have logarithmic diameter (i.e. $O(\log N)$) and that the simple random walk on them equidistributes after logarithmically many steps.

As it turns out, the Height gap theorem and the uniform Tits alternative can also be used to prove uniform spectral gap estimates for Cayley graphs of finite simple groups of Lie type. A very general method, due to Bourgain and Gamburd, allows to establish spectral gaps for Cayley graphs of finite groups. We refer the reader to [20, 24, 60] for an exposition of this method. An important requirement for the method to work is to be able to assert that the probability that the simple random walk on the Cayley graph of G hits any given subgroup decays exponentially fast at an initial stage (say for $c \log |G|$ steps). When $G = \mathrm{SL}_2(\mathbb{F}_p)$ and the Cayley graph has girth $\geqslant c \log p$, then this easily achieved as did Bourgain and Gamburd in their seminal paper [9]. Using the uniform Tits alternative, one can claim that this happens without any girth condition, at least at almost all primes p. We have:

Theorem 4.11 (Breuillard-Gamburd [21]). *For every $\delta > 0$ there is $\varepsilon > 0$ such that, given any $X > 1$, for all but at most X^δ primes $p \leqslant X$, all Cayley graphs of $SL_2(\mathbb{F}_p)$ are ε-expanders.*

We record here the following folklore conjecture, which implies Conjecture 4.5 above:

Conjecture 4.12. *Given $d \geqslant 1$, there is $\varepsilon > 0$ such that all Cayley graphs of all finite simple subgroups of GL_d over some field are ε-expanders.*

Although this conjecture seems out of reach at the moment even for $SL_2(\mathbb{F}_p)$'s and the entire family of primes p, the following looks more approachable:
Problem: Generalize Theorem 4.11 to higher rank finite simple groups of Lie type.

A related question is that of the spectral gap for finite subsets of compact groups. Following Jordan Ellenberg in [32] we will say that a topologically finitely generated compact group G has *property $\widehat{\tau}$* if for every finite subset S of G generating a dense subgroup

$$\kappa(S, L^2_0(G)) > 0,$$

where $L^2_0(G)$ is the regular representation of G on square integrable functions with zero average on G. The Kazhdan constant $\kappa(S, L^2_0(G))$ was defined above in (2.1).

The terminology echoes Lubotzky's property (τ), which is a property of a finitely generated group: property (τ) for a finitely generated group Γ means that for some (hence all) finite generating subsets S of Γ, one has

$$\kappa(S, L^2_0(\widehat{\Gamma})) > 0,$$

where $\widehat{\Gamma}$ is the profinite completion of Γ. We refer the reader to the forthcoming book [67] as well as [19] for more on property (τ).

Note that every finitely generated group with Kazhdan's property (T) has Lubotzky's property (τ). And there are also many other examples. However it surprisingly difficult to even exhibit one example of an infinite compact group G with Ellenberg's property $(\widehat{\tau})$. Using Theorem 4.11 it is possible to prove that certain infinite products $\prod_1^\infty \mathrm{SL}_2(\mathbb{F}_{p_i})$ for a sparse increasing sequence of primes have property $\widehat{\tau}$, thus giving the first examples of infinite compact groups with this property.

However difficult it appears to produce examples of compact groups with property $(\widehat{\tau})$, it is conjectured that this property is quite common and should hold in particular for all semisimple compact real Lie groups and the adèles groups $SL_n(\widehat{\mathbb{Z}})$. However it is not even known for $SU(2)$ and $SL_2(\mathbb{Z}_p)$ and these cases already appear to be very difficult (see [79, p.58] and [10]). See also Varju's paper [86, Corollary 4] for the current state of the art regarding spectral gap bounds for $SL_n(\widehat{Z})$ and other compact groups. For a connection between property $(\widehat{\tau})$ for certain Galois groups and Bogomolov's property for field extension, we refer the reader to Ellenberg's article [32] and to [31].

Acknowledgements. The author was partially supported by ERC Grant GADA 208091 and ANR-11-BS01-013, ANR-12-BS01-0011, ANR-13-BS01-0006. I am very grateful to all my co-authors, present and past, for sharing their time and ideas with me, making my experience in research in mathematics so rewarding personally.

References

[1] Amoroso, F., David, S., and Zannier, U., *On fields with Property* (B), Proc. Amer. Math. Soc., **142** (2014), no. 6, 1893–1910.

[2] Babai, L., Seress, *Á., On the diameter of Cayley graphs of the symmetric group*, J. Combin. Theory Ser. A, **49** (1) (1988), 175–179.

[3] Bartholdi, L. and de Cornulier, Y., *Infinite groups with large balls of torsion elements and small entropy*, Arch. Math., (Basel) **87** (2006), no. 2, 104–112.

[4] L. Bartholdi and A. Erschler, *Groups of given intermediate word growth*, preprint, 2011, arXiv:1110.3650.

[5] B. Bekka, P. de la Harpe, and A. Valette, *Kazhdan's property* (T), New Mathematical Monographs, 11, Cambridge University Press, Cambridge, 2008, xiv+472.

[6] Benli, M. G., Grigorchuk, R., and de la Harpe, P., *Amenable groups without finitely presented amenable covers*, Bull. Math. Sci., **3** (2013), no. 1, 73–131.

[7] Benjamini, I., Finucane, H., and Tessera, R., *On the scaling limit of finite vertex transitive graphs with large diameter*, preprint, arXiv:1203.5624.

[8] Y. Bilu, *Limit distribution of small points on algebraic tori*, Duke Math. J., **89** (1997), no. 3, 465–476.

[9] J. Bourgain and A. Gamburd, *Uniform expansion bounds for Cayley graphs of* $\mathrm{SL}_2(\mathbb{F}_p)$, Ann. of Math., **167** (2008), no. 2, 625–642.

[10] ———, *On the spectral gap for finitely-generated subgroups of* $SU(2)$, Invent. Math., **171** (2008), no. 1, 83-121.

[11] Bombieri, E. and Gubler, W., *Heights in Diophantine geometry*, New Mathematical Monographs, 4. Cambridge University Press, Cambridge, 2006. xvi+652.

[12] E. Breuillard, *On uniform exponential growth for solvable groups*, Pure Appl. Math. Q., **3** (2007), no. 4, part 1, 949–967.

[13] E. Breuillard and T. Gelander, *On dense free subgroups of Lie groups*, J. of Algebra, **261** (2003), no. 2, 448–467.

[14] ———, *Uniform independence for linear groups*, Invent. Math., **173** (2008), no. 2, 225–263.

[15] E. Breuillard, *A Height Gap Theorem for finite subsets of* $\mathrm{GL}_d(\overline{\mathbb{Q}})$ *and non amenable subgroups*, Ann. of Math., (2) **174** (2011), no. 2, 1057–1110.

[16] ———, *Heights on* SL_2 *and free subgroups*, in Geometry, rigidity, and group actions, 455–493, Chicago Lectures in Math., Univ. Chicago Press, Chicago, IL, 2011.

[17] ———, *A strong Tits alternative*, preprint, arXiv:0804.1395.

[18] ———, *Expander graphs, property* (τ) *and approximate groups*, PCMI 12 Geometric Group Theory, Lecture notes of a IAS-PCMI summer school held in Park City Utah in July 2012.

[19] ———, *A brief introduction to approximate groups* in Thin Groups and Superstrong Approximation, MSRI publication Breuillard and Oh eds., Cambridge Univ. Press, 2014.

[20] ———, *Approximate groups and super-strong approximation*, in Group St. Andrews, 2013, to appear.

[21] E. Breuillard, A. Gamburd, *Strong uniform expansion in* $\mathrm{SL}(2,p)$, Geom. Funct. Anal., **20** (2010), no. 5, 1201–1209.

[22] Breuillard, E., Green, B., and Tao, T., *Approximate subgroups of linear groups*, Geom. Funct. Anal., 21 (2011), no. 4, 774–819.

[23] ———, *Small soubling in groups*, to appear in the Erdös centennial volume.

[24] Breuillard, E., Green, B., Guralnick, R., and Tao, T., *Expansion in finite simple groups of Lie type*, to appear in J. Eur. Math. Soc.

[25] Brieussel, J., *Behaviors of entropy on finitely generated groups*, Ann. Probab., **41** (2013), no. 6, 4116–4161.

[26] Cannon, J. W. and Wagreich, Ph., *Growth functions of surface groups*, Math. Ann., **293** (1992), no. 2, 239–257.

[27] Colding, T. and Minicozzi, W. II, *Harmonic functions on manifolds*, Ann. of Math., (2) 146 (1997), no. 3, 725–747.

[28] Cowling, M., *Sur les coefficients des reprÃl'sentations unitaires des groupes de Lie simples*, in Analyse harmonique sur les groupes de Lie, (Sém., Nancy-Strasbourg 1976-1978), II, pp. 132–178, Lecture Notes in Math., **739**, Springer, Berlin, 1979.

[29] Cowling, M., Haagerup, U., and Howe, R. E., *Almost* L^2 *matrix coefficients*, J. Reine Angew. Math., **387** (1988), 97–110.

[30] C.W. Curtis and I. Reiner, *Representation Theory of Finite Groups and Associative Algebras*, Interscience, New York, 1962.

[31] Ellenberg, J., Hall, Ch., and Kowalski, E., *Expander graphs, gonality, and variation of Galois representations*, Duke Math. J., **161** (2012), no. 7, 1233–1275.

[32] Ellenberg, J., *Super strong approximation for monodromy groups*, in Thin Groups and Superstrong Approximation, MSRI publication Breuillard and Oh eds., Cambridge Univ. Press, 2014.

[33] Eskin, A., Mozes, S., and Oh, H., *On uniform exponential growth for linear groups*, Invent. Math., **160** (2005), no. 1, 1–30.

[34] Gelander, T. and Zuk, A., *Dependence of Kazhdan constants on generating subsets*, Israel J. Math., **129** (2002), 93–98.

[35] Ghate, E. and Hironaka, E., *The arithmetic and geometry of Salem numbers*, Bull. Amer. Math. Soc., (N.S.) **38** (2001), no. 3, 293–314.

[36] Ghys, E. and de la Harpe, P., *Sur les groupes hyperboliques d'après Mikhael Gromov (Bern, 1988)*, Progr. Math., **83**, Birkhäuser Boston, Boston, MA, 1990.

[37] Gill, N., Pyber, L., Short, I., and Szabó, E., *On the product decomposition conjecture for finite simple groups*, Groups Geom. Dyn., **7** (2013), no. 4, 867–882.

[38] Goldbring, I. and Van den Dries, L., *Notes on Hilbert 5-th problem*.

[39] Green, B., *Approximate algebraic structure*, ICM Seoul 2014 Proceedings.

[40] R. I. Grigorchuk, *Degrees of growth of finitely generated groups and the theory of invariant means*, Math. USSR-Izv., **25** (1985), no. 2, 259–300.

[41] _____, *On the Gap Conjecture concerning group growth*, Bull. Math. Sci., **4** (2014), no. 1, 113–128.

[42] _____, *Milnor's Problem on the Growth of Groups and its Consequences*, preprint, arXiv:1111.0512.

[43] R. I. Grigorchuk and I. Pak, *Groups of intermediate growth: an introduction*, Enseign. Math., (2) 54 (2008), no. 3-4, 251–272.

[44] Grigorchuk, R., de la Harpe, P., *On problems related to growth, entropy, and spectrum in group theory*, J. Dynam. Control Systems, **3** (1997), no. 1, 51–89.

[45] _____, *Limit behaviour of exponential growth rates for finitely generated groups*, Essays on geometry and related topics, Vol. 1, 2, 351–70, Monogr. Enseign. Math., **38**, Enseignement Math., Geneva, 2001.

[46] Gromov, M., *Groups of polynomial growth and expanding maps*, Inst. Hautes Études Sci. Publ. Math., No. 53 (1981), 53–73.

[47] _____, *Metric structures for Riemannian and non-Riemannian spaces*, based on original 1981 French original, Modern BirkhÃd'user Classics, BirkhÃd'user, 2007, xx+585.

[48] P. de la Harpe, *Free Groups in Linear Groups*, L'Enseignement Mathématique, **29** (1983), 129–144.

[49] H. Helfgott, *Growth and generation in $SL_2(\mathbb{Z}/p\mathbb{Z})$*, Ann. of Math., **167** (2008), 601–623.

[50] Helfgott, H. and Seress, Á., On the diameter of permutation groups, *Ann. of Math.* (2) 179 (2014), no. 2, 611–658.

[51] Hindry, M. and Silverman, J., *Diophantine geometry, an introduction*, Graduate Texts in Mathematics, 201, Springer-Verlag, New York, 2000.

[52] S. Hoory, N. Linial, and A. Wigderson, *Expander graphs and their applications*, Bull. Amer. Math. Soc. (N.S.), **43** (2006), no. 4, 439–561.

[53] Hrushovski, E., *Stable group theory and approximate subgroups*, J. Amer. Math. Soc., **25** (2012), no. 1, 189–243.

[54] Jordan, C., *Mémoire sur les équations différentielles linéaires à intégrale algébrique*, J. Reine Angew. Math., **84** (1878), 89–215.

[55] Kaplansky, I., *Lie algebras and locally compact groups*, The University of Chicago Press, Chicago, Ill.-London, 1971, xi+148.

[56] Kassabov, M. and Pak, I., *Groups of oscillating intermediate growth*, Ann. of Math., (2) 177 (2013), no. 3, 1113–1145.

[57] Kleiner, Br., *A new proof of Gromov's theorem on groups of polynomial growth*, J. Amer. Math. Soc., **23** (2010), no. 3, 815–829.

[58] Klopsch, B. and Lev, V. F., *Generating abelian groups by addition only*, Forum Math., **21** (2009), no. 1, 23–41.

[59] Koubi, M., *Croissance uniforme dans les groupes hyperboliques*, Ann. Inst. Fourier, 48(5):1441–1453, 1998.

[60] Kowalski, E., *lectures notes on expanders*, http://www.math.ethz.ch/ kowalski/ expanders.html.

[61] Lang, S., *Diophantine geometry*, Interscience Tracts in Pure and Applied Mathematics, No. 11, Interscience Publishers, New York-London, 1962, x+170.

[62] M. Larsen and R. Pink, *Finite subgroups of algebraic groups*, J. Amer. Math. Soc., 24 (2011), no. 4, 1105–1158.

[63] Li, J.S., *The minimal decay of matrix coefficients for classical groups*, in Harmonic Analysis in China, Math. Appl. 327, Kluwer, Dordrecht, 1995, 146–169.

[64] Liebeck, M. W., Nikolov, N., and Shalev, A., *A conjecture on product decompositions in simple groups*, Groups Geom. Dyn. , **4** (4):799–812, (2010).

[65] Liebeck, M., *Probabilistic and asymptotic aspects of finite simple groups*, in Probabilistic group theory, combinatorics, and computing, 1–34, Lecture Notes in Math., 2070, Springer, London, 2013.

[66] Lubotzky, A., *Discrete groups, expanding graphs and invariant measures*, Progress in Mathematics, 125. BirkhÃd'user Verlag, Basel, 1994, xii+195.

[67] Lubotzky, A. and Zuk, A., *Property* (τ), book in preparation.

[68] Mann, A., *How groups grow*, London Mathematical Society Lecture Note Series, 395. Cambridge University Press, 2012, x+199.

[69] Masser, D. and Wüstholz, G., *Fields of large transcendence degree generated by values of elliptic functions*, Invent. Math., **72** (1983), no. 3, 407–464.

[70] Milnor, J., *A note on curvature and fundamental group*, J. Differential Geometry, 2:1–7, 1968.

[71] Moore, C. C., *Exponential decay of correlation coefficients for geodesic flows*, in Group representations, ergodic theory, operator algebras, and mathematical physics, (Berkeley, Calif., 1984), 163–181, MSRI Publ., 6, Springer, 1987.

[72] Nevo, A., *The spectral theory of amenable actions and invariants of discrete groups*, Geom. Dedicata, 100 (2003), 187–218.

[73] Nori, M. V., *On subgroups of* $GL_n(\mathbb{F}_p)$, Invent. math., **88** (1987), 257–275.

[74] Oh, H., *Uniform pointwise bounds for matrix coefficients of unitary representations and applications to Kazhdan constants*, Duke Math. J., **113** (2002), no. 1, 133–192.

[75] Osin, D., *The entropy of solvable groups*, Ergodic Theory Dynam. Systems, 23(3):907–918, 2003.

[76] Procesi, C., *The invariant theory of* $n \times n$ *matrices*, Advances in Math., **19** (1976), no. 3, 306–381.

[77] L. Pyber and E. Szabó, *Growth in finite simple groups of Lie type*, preprint, 2010, arXiv:1001.4556.

[78] L. Pyber and E. Szabó, *Growth in linear groups* in Thin Groups and Superstrong Approximation, MSRI publication Breuillard and Oh eds., Cambridge Univ. Press, 2014.

[79] Sarnak, P., *Some applications of modular forms.*, Cambridge Tracts in Mathematics, **99**, Cambridge University Press, 1990, x+111.

[80] Schur I., *Uber Gruppen periodischer Substitutionen*, Sitzber. Preuss. Akad. Wiss., 1911, 619–627.

[81] Shalom, Y., *Explicit Kazhdan constants for representations of semisimple and arithmetic groups*, Ann. Inst. Fourier, (Grenoble) **50** (2000), no. 3, 833–863.

[82] Shalom, Y. and Tao, T., *A finitary version of Gromov's polynomial growth theorem*, Geom. Funct. Anal., **20** (2010), no. 6, 1502–1547.

[83] Svarc, A., *A volume invariant of covering*, Dokl. Akad. Nauj SSSR, **105** (1955), 32–34.

[84] Szpiro, L., Ullmo, E., and Zhang, S., *Équirépartition des petits points*, Invent. Math., **127** (1997), no. 2, 337–347.

[85] Tits, J., *Free subgroups in linear groups*, J. Algebra, **20** (1972), 250–270.

[86] Varjù, P., *Random walks in compact groups*, preprint, arXiv:1209.1745.

[87] Vinberg, E. B., *On invariants of a set of matrices*, J. Lie Theory, **6** (1996), no. 2, 249–269.

[88] Virtser, A., *On the simplicity of the spectrum of characteristic Lyapunov exponents of the product of random matrices*, Theory Probabl. Appl., **28** (1983), no. 1, 122–135.

[89] Talamanca, V., *A Gelfand-Beurling type formula for heights on endomorphism rings*, J. Number Theory, **83** (2000), no. 1, 91–105.

[90] Wilson, J. S., *On exponential growth and uniformly exponential growth for groups*, Invent. Math., 155(2):287–303, 2004.

[91] Wolf, J., *Growth of finitely generated solvable groups and curvature of Riemanniann manifolds*, J. Differential Geometry, **2**:421–446, 1968.

[92] Zagier, D., *Algebraic numbers close to both* 0 *and* 1, Math. Comp., **61** (1993), no. 203, 485–491.

[93] Zhang, Sh., *Small points and adelic metrics*, J. Algebraic Geom., **4** (1995), no. 2, 281–300.

Laboratoire de Mathématiques, Bâtiment 425, Université Paris Sud 11, 91405 Orsay FRANCE
E-mail: emmanuel.breuillard@math.u-psud.fr

Schur-Weyl duality and categorification

Jonathan Brundan

Abstract. In some joint work with Kleshchev in 2008, we discovered a *higher level analog of Schur-Weyl duality*, relating parabolic category $\mathcal{O}$ for the general linear Lie algebra to certain cyclotomic Hecke algebras. Meanwhile Rouquier and others were developing a general axiomatic approach to the study of *categorical actions of Lie algebras*. In this survey, we recall aspects of these two theories, then explain some related recent developments due to Losev and Webster involving *tensor product categorifications*.

Mathematics Subject Classification (2010). Primary 17B10; Secondary 16G99, 18D99.

Keywords. Schur-Weyl duality, highest weight categories, categorification.

1. Introduction

The double centralizer property. To set the scene in this introduction, we are going to briefly recall two classic topics in representation theory, namely:

- *Schur-Weyl duality* relating the representation theory of the general linear and symmetric groups;
- *Soergel's functor* relating the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ for a semisimple Lie algebra to modules over coinvariant algebras.

First though we formulate some abstract *double centralizer property*. This applies to both of the above situations, as well as to the generalizations to be discussed in subsequent sections. (For the reader not familiar with the notion of a highest weight category, we will recall its meaning shortly.)

Theorem 1.1. *Let $\mathcal{M}$ be a highest weight category with a finite weight poset. Assume that the injective hulls of all its standard objects are projective. Let T be a* prinjective generator *for $\mathcal{M}$, that is, a prinjective (= both projective and injective) object such that every indecomposable prinjective object is isomorphic to a summand of T. Let $C := \mathrm{End}_{\mathcal{M}}(T)^{\mathrm{op}}$ and C-mod denote the category of finite dimensional left C-modules. Then the quotient functor*

$$\mathbb{V} := \mathrm{Hom}_{\mathcal{M}}(T, -) : \mathcal{M} \to C\text{-mod}$$

is fully faithful on projectives.

Proof. Note to start with that if $0 \to P \to Q \to R \to 0$ is a short exact sequence in $\mathcal{M}$ with P projective and Q prinjective, then R has a Δ-flag; see [46, Proposition 2.3]. Now let P

Proceedings of the International Congress of Mathematicians, Seoul, 2014

be any projective object in $\mathcal{M}$. Since P has a Δ-flag, the assumption on $\mathcal{M}$ implies that its injective hull Q is prinjective. Then we see that the cokernel of the inclusion $P \hookrightarrow Q$ has a Δ-flag, so its injective hull R is prinjective too. This proves the existence of an exact sequence $0 \to P \to Q \to R$ such that Q and R are direct sums of summands of T. Now argue as in [51, Corollary 1.7]. □

Remark 1.2. There is also a version of Theorem 1.1 for highest weight categories with infinite weight posets. For this, one needs to replace T by a family $\{T_d\}_{d \in D}$ of prinjective objects of $\mathcal{M}$ such that every indecomposable prinjective is a summand of at least one and at most finitely many of the T_d's. Then the algebra C becomes the vector space $\bigoplus_{d,d' \in D} \mathrm{Hom}_{\mathcal{M}}(T_d, T_{d'})$ viewed as a locally unital algebra via the opposite of composition, and C-mod means the category of locally unital finite dimensional left C-modules.

Schur-Weyl duality. The first situation in which the double centralizer property arises involves the representation theory of the general linear group $GL_n(\Bbbk)$ and the symmetric group S_d over an algebraically closed field $\Bbbk$. From our categorical perspective, this is only really interesting in the case that $\Bbbk$ is of positive characteristic. For simplicity we assume that $n \geq d$; the case $n < d$ is more subtle and is discussed in [39] from a similar point of view to this introduction.

Let $\mathcal{R}$ denote the category of *polynomial representations of degree* d for the algebraic group $G := GL_n(\Bbbk)$, that is, the Serre subcategory of its category of rational representations generated by the dth tensor power $V^{\otimes d}$ of the natural G-module V. It is a highest weight category with weight poset Λ being the set of all partitions of d, partially ordered by the usual dominance ordering. This poset indexes a set $\{L(\lambda)\}_{\lambda \in \Lambda}$ of representatives for the isomorphism classes of irreducible objects of $\mathcal{R}$; explicitly, one takes $L(\lambda)$ to be the irreducible highest weight module for G of highest weight λ. For $\lambda \in \Lambda$, we also have the projective cover $P(\lambda)$ of $L(\lambda)$ in $\mathcal{R}$, and the standard module

$$\Delta(\lambda) := P(\lambda) \Big/ \sum_{\substack{\mu \in \Lambda \text{ with } \mu \not\leq \lambda \\ f \in \mathrm{Hom}_G(P(\mu), \mathrm{rad}\, P(\lambda))}} \mathrm{im}\, f.$$

In this context $\Delta(\lambda)$ turns out to be isomorphic to the *Weyl module* whose character is given by the Schur polynomial s_λ. The statement that $\mathcal{R}$ is a highest weight category means that each $P(\lambda)$ has a finite filtration with top section $\Delta(\lambda)$ and other sections of the form $\Delta(\mu)$ for $\mu > \lambda$.

The symmetric group S_d acts on the right on the tensor space $T := V^{\otimes d}$ by permuting tensors, and this action induces an isomorphism between the group algebra $\Bbbk S_d$ and the endomorphism algebra $C := \mathrm{End}_G(T)^{\mathrm{op}}$. The module T is a projective module in $\mathcal{R}$, as more generally is the *divided power*

$$\Gamma^\mu(V) := \{v \in V^{\otimes d} \mid v \cdot g = v \text{ for all } g \in S_\mu\}$$

for any n-part composition $\mu \vDash_n d$, where S_μ denotes the parabolic subgroup $S_{\mu_1} \times \cdots \times S_{\mu_n}$ of S_d. As T is self-dual with respect to the natural duality on $\mathcal{R}$, it is also injective. Then it is a special feature of this situation that all of the standard modules $\Delta(\lambda)$ embed into T. Hence the injective hull of each standard module is a summand of T, so projective, and moreover T is a prinjective generator for $\mathcal{R}$. This verifies the hypotheses of Theorem 1.1. We deduce that the functor

$$\mathbb{V} := \mathrm{Hom}_G(T, -) : \mathcal{R} \to \Bbbk S_d\text{-mod}$$

is fully faithful on projectives.

In Green's monograph [30], this quotient functor $\mathbb{V}$ is called the *Schur functor*, and it is used in a systematic way to recover the representation theory of the symmetric group from that of the general linear group, thereby reversing the flow of information compared to Schur's classical work over $\mathbb{C}$. Green's approach also emphasizes the role of the *Schur algebra* $S(n,d)$, which is a certain finite dimensional algebra whose module category is equivalent to $\mathcal{R}$. From the perspective of the double centralizer property, the Schur algebra can be *defined* as the endomorphism algebra

$$S(n,d) := \mathrm{End}_{\Bbbk S_d}(T),$$

and then the equivalence $\mathcal{R} \xrightarrow{\sim} S(n,d)\text{-mod}$ can be seen as follows. Let

$$P := \bigoplus_{\mu \vDash_n d} \Gamma^\mu(V).$$

This is a projective generator for $\mathcal{R}$, hence the functor $\mathrm{Hom}_G(P,-)$ defines an equivalence of categories $\mathcal{R} \xrightarrow{\sim} A\text{-mod}$, where $A := \mathrm{End}_G(P)^{\mathrm{op}}$. Setting $Y := \mathbb{V}P$, it remains to observe that

$$A = \mathrm{End}_G(P)^{\mathrm{op}} \cong \mathrm{End}_{\Bbbk S_d}(Y)^{\mathrm{op}} \cong \mathrm{End}_{\Bbbk S_d}(Y^*) \cong \mathrm{End}_{\Bbbk S_d}(T) = S(n,d).$$

The non-trivial first isomorphism here is defined by applying the functor $\mathbb{V}$; the fact that it is an isomorphism follows from (indeed, is equivalent to) the double centralizer property. The second isomorphism is just taking linear duals to turn left modules into right modules. The final isomorphism follows on checking that $Y^* \cong \bigoplus_{\mu \vDash_n d} \Bbbk(S_\mu \backslash S_d) \cong T$ as right $\Bbbk S_d$-modules, i.e. both Y^* and T are direct sums of the same permutation modules.

Soergel's functor. Everything in the remainder of the article will be defined over the ground field $\mathbb{C}$. Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra. Fix a Borel subalgebra $\mathfrak{b}$ containing a Cartan subalgebra $\mathfrak{t}$. Let $\mathcal{O}_0$ be the principal block of the BGG category $\mathcal{O}$ attached to this data. Thus $\mathcal{O}_0$ consists of all finitely generated $\mathfrak{g}$-modules which are locally finite over $\mathfrak{b}$, semisimple over $\mathfrak{t}$, and which have the same generalized central character as the trivial module. The irreducible modules in $\mathcal{O}_0$ are parametrized naturally by the Weyl group W of $\mathfrak{g}$. We denote them by $\{L(w)\}_{w \in W}$; explicitly, $L(w)$ is the irreducible highest weight module of highest weight $w\rho - \rho$ where ρ is the half-sum of the positive roots.

The category $\mathcal{O}_0$ is a highest weight category with weight poset W partially ordered by the opposite of the usual Bruhat order, i.e. the longest element $w_0 \in W$ is minimal. Its standard modules $\Delta(w)$ defined according to the general recipe explained above are better known as *Verma modules*, and may be denoted instead by $M(w)$. It is well known that the socle of each Verma module in $\mathcal{O}_0$ is isomorphic to $L(w_0)$. Moreover the only indecomposable projective module $P(w)$ that is also injective is the *antidominant projective* $P(w_0)$. This puts us in the situation of Theorem 1.1 with $T := P(w_0)$.

In [50], Soergel proved that the algebra $C := \mathrm{End}_{\mathfrak{g}}(T)^{\mathrm{op}}$ is canonically isomorphic to the *coinvariant algebra*, that is, the quotient of $S(\mathfrak{t})$ by the ideal generated by all homogeneous W-invariant polynomials of strictly positive degree. Equivalently, by a classical theorem of Borel, C is the cohomology algebra $H^*(G/B,\mathbb{C})$ of the flag variety associated to $\mathfrak{g}$. Soergel also showed that the functor

$$\mathbb{V} := \mathrm{Hom}_{\mathfrak{g}}(T,-) : \mathcal{O}_0 \to C\text{-mod}$$

is fully faithful on projectives, as asserted by Theorem 1.1. Moreover, all of the *Soergel modules* $Q(w) := \mathbb{V}P(w)$ admit unique (up to automorphism) gradings making them into self-dual graded modules over the naturally graded algebra C. Hence, letting $P := \bigoplus_{w \in W} P(w)$ and $Q := \bigoplus_{w \in W} Q(w)$ graded in this way, we get induced a grading on the endomorphism algebra

$$A := \mathrm{End}_{\mathfrak{g}}(P)^{\mathrm{op}} \cong \mathrm{End}_C(Q)^{\mathrm{op}},$$

where the isomorphism comes from the double centralizer property.

In fact, as shown by Beilinson, Ginzburg and Soergel in [4], the graded algebra A is a Koszul algebra. Since P is a projective generator for $\mathcal{O}_0$, the category $\mathcal{O}_0$ is equivalent to the category A-mod. This means that $\mathcal{O}_0$ has a natural graded lift, namely, the category A-grmod of finite dimensional *graded* left A-modules. This graded category is related intimately to the Iwahori-Hecke algebra associated to the Weyl group of $\mathfrak{g}$ and the Kazhdan-Lusztig conjecture.

Organization of the article. The rest of the article is an attempt to explain some generalizations of the above examples. The first of these, discussed in section 2, is the *Schur-Weyl duality for higher levels* introduced in [13]. This is built around a double centralizer property as above in which the category $\mathcal{M}$ is a sum of blocks of parabolic category $\mathcal{O}$ for the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$, and the endomorphism algebra C of a suitably chosen prinjective generator T is some degenerate cyclotomic Hecke algebra. A key feature of this example is that the category $\mathcal{M}$ admits a categorical $\mathfrak{sl}_{\mathbb{Z}}$-action in the general sense of Chuang and Rouquier [24, 47].

In fact the category $\mathcal{M}$ fits into the axiomatic framework of *tensor product categorifications* introduced recently by Losev and Webster [44]. They show that all tensor product categorifications of integrable highest weight modules satisfy a double centralizer property in which the algebra C is some cyclotomic quiver Hecke algebra. From this they are able to deduce a striking uniqueness theorem. We sketch these results in section 3. When applied to our category $\mathcal{M}$, the Losev-Webster uniqueness theorem implies the equivalence of $\mathcal{M}$ with various other categories which have appeared elsewhere in the literature. Some examples are discussed in section 4, together with some further generalizations and possible future directions.

2. Schur-Weyl duality for higher levels

Categorical actions. Throughout, we let $I \subseteq \mathbb{Z}$ be some non-empty interval, $I_+ := I \cup (I+1)$, and $\mathfrak{sl}_I$ be the (complex) special linear Lie algebra of $I_+ \times I_+$ matrices. It is generated by its matrix units $e_i := e_{i,i+1}$ and $f_i := e_{i+1,i}$ for each $i \in I$. Also let V_I be the natural $\mathfrak{sl}_I$-module of column vectors with standard basis $\{v_i\}_{i \in I_+}$. We denote the weight of v_i with respect to the Cartan subalgebra $\mathfrak{t}_I$ of $\mathfrak{sl}_I$ consisting of diagonal matrices by $\varepsilon_i \in \mathfrak{t}_I^*$.

The following is essentially [47, Definition 5.32].

Definition 2.1. Let $\mathcal{C}$ be a $\mathbb{C}$-linear abelian category such that all its objects have finite length and there are enough projectives. A *categorical $\mathfrak{sl}_I$-action* on $\mathcal{C}$ is the data of an endofunctor F, a right adjoint E to F (with a specified adjunction), and natural transformations $x \in \mathrm{End}(F)$ and $s \in \mathrm{End}(F^2)$ satisfying the axioms (SL1)–(SL4) formulated below. For the first axiom, we let F_i be the subfunctor of F defined by the generalized i-eigenspace of x, i.e.

$F_iM = \sum_{k\geq 0} \ker(x_M - i)^k$ for each $M \in \mathcal{C}$.

(SL1) We have that $F = \bigoplus_{i\in I} F_i$, i.e. $FM = \bigoplus_{i\in I} F_iM$ for each $M \in \mathcal{C}$.

(SL2) For $d \geq 0$ the endomorphisms $x_j := F^{d-j}xF^{j-1}$ and $s_k := F^{d-k-1}sF^{k-1}$ of F^d satisfy the relations of the degenerate affine Hecke algebra H_d, i.e. the x_j's commute like in the polynomial algebra $\mathbb{C}[x_1, \dots, x_d]$, the s_j's satisfy the Coxeter relations of the simple transpositions in the symmetric group S_d, $s_jx_{j+1} = x_js_j + 1$, and $s_jx_k = x_ks_j$ for $k \neq j, j+1$.

(SL3) The functor F is isomorphic to a right adjoint of E.

For the final axiom, we let $c : \mathrm{id} \to EF$ and $d : FE \to \mathrm{id}$ be the unit and counit of the given adjunction, respectively. The endomorphisms x and s of F and F^2 induce endomorphisms x' and s' of E and E^2 too:

$$x' : E \overset{cE}{\to} EFE \overset{ExE}{\to} EFE \overset{Ed}{\to} E,$$
$$s' : E^2 \overset{cE^2}{\to} EFE^2 \overset{EcFE^2}{\to} E^2F^2E^2 \overset{E^2sE^2}{\to} E^2F^2E^2 \overset{E^2FdE}{\to} E^2FE \overset{E^2d}{\to} E^2.$$

Let E_i be the subfunctor of E defined by the generalized i-eigenspace of $x' \in \mathrm{End}(E)$. The axioms so far imply that $E = \bigoplus_{i\in I} E_i$ and moreover F_i and E_i are biadjoint, so they are both exact and send projectives to projectives.

(SL4) Let $K_0(\mathcal{C})$ be the split Grothendieck group of the category of projectives in $\mathcal{C}$. The endomorphisms f_i and e_i of $[\mathcal{C}] := \mathbb{C}\otimes_{\mathbb{Z}} K_0(\mathcal{C})$ induced by F_i and E_i, respectively, make $[\mathcal{C}]$ into an integrable representation of $\mathfrak{sl}_I$. Moreover the classes of the indecomposable projective objects are weight vectors.

There is also a much more general notion of a categorical action of an arbitrary Kac-Moody algebra $\mathfrak{g}$ on a category $\mathcal{C}$, which was introduced independently by Rouquier [47] and Khovanov and Lauda [38]. We will refer to this general notion in later discussion, but are not going to repeat its definition in full here. It involves a certain 2-category $\mathcal{U}(\mathfrak{g})$ defined in [47] by generators and relations, which is closely related to the diagrammatic category introduced in [38]. In particular, the degenerate affine Hecke algebra appearing in our axiom (SL2) gets replaced by the *quiver Hecke algebra* (or *Khovanov-Lauda-Rouquier algebra*) associated to $\mathfrak{g}$. The equivalence of the general definition with the special version stated above depends on the isomorphism theorem between affine Hecke algebras and quiver Hecke algebras in type A_n from [47, Proposition 3.15] (see also [14]), as well as on [47, Theorem 5.27].

Definition 2.2. Given two categorical $\mathfrak{sl}_I$-actions on categories $\mathcal{C}_1$ and $\mathcal{C}_2$, a functor $\mathbb{G} : \mathcal{C}_1 \to \mathcal{C}_2$ is *strongly equivariant* if there exists an isomorphism of functors $\zeta : \mathbb{G}\circ F \xrightarrow{\sim} F\circ\mathbb{G}$ with

$$x\mathbb{G}\circ\zeta = \zeta\circ\mathbb{G}x \qquad \text{in } \mathrm{Hom}(\mathbb{G}\circ F, F\circ\mathbb{G}),$$
$$s\mathbb{G}\circ F\zeta\circ\zeta F = F\zeta\circ\zeta F\circ\mathbb{G}s \qquad \text{in } \mathrm{Hom}(\mathbb{G}\circ F^2, F^2\circ\mathbb{G}).$$

A *strongly equivariant equivalence* is a strongly equivariant functor $\mathbb{G} : \mathcal{C}_1 \to \mathcal{C}_2$ that is also an equivalence of categories. It is then automatic that $[\mathcal{C}_1] \cong [\mathcal{C}_2]$ as $\mathfrak{sl}_I$-modules.

First example of a categorical action. In this subsection, we explain our favorite example of a categorical action; for this the interval I will be $\mathbb{Z}$. Let $\mathfrak{g} := \mathfrak{gl}_n(\mathbb{C})$, $\mathfrak{t}$ be the Cartan subalgebra consisting of diagonal matrices, and $\mathfrak{b}$ be the Borel subalgebra of upper triangular matrices. Inside $\mathfrak{t}^*$, we have the standard coordinate functions $\delta_1, \dots, \delta_n$, where δ_i picks out the ith diagonal entry of a diagonal matrix. Let $(-,-)$ be the symmetric bilinear form on $\mathfrak{t}^*$ defined from $(\delta_i, \delta_j) := \delta_{i,j}$. Also set

$$\rho := -\delta_2 - 2\delta_3 - \cdots - (n-1)\delta_n.$$

We identify the set $\mathfrak{t}^*_\mathbb{Z} := \mathbb{Z}\delta_1 \oplus \cdots \oplus \mathbb{Z}\delta_n$ of integral weights with $\mathbb{Z}^n$, so that $\lambda \in \mathfrak{t}^*_\mathbb{Z}$ is identified with the n-tuple $(\lambda_1, \dots, \lambda_n)$ defined from $\lambda_i := (\lambda + \rho, \delta_i)$. The *Bruhat order* $\leq$ on $\mathfrak{t}^*_\mathbb{Z}$ is the partial order generated by the basic relation that $\lambda < \mu$ if μ is obtained from the n-tuple λ by switching some pair of entries λ_i and λ_j for $i < j$ with $\lambda_i < \lambda_j$.

Let $\mathcal{O}$ be the BGG category of all finitely generated $\mathfrak{g}$-modules M that are locally finite over $\mathfrak{b}$ and satisfy

$$M = \bigoplus_{\lambda \in \mathfrak{t}^*_\mathbb{Z}} M_\lambda,$$

where M_λ denotes the λ-weight space with respect to $\mathfrak{t}$. The irreducible modules in $\mathcal{O}$ are the modules $\{L(\lambda)\}_{\lambda \in \mathfrak{t}^*_\mathbb{Z}}$, where $L(\lambda)$ is the irreducible highest weight module of highest weight λ. The category $\mathcal{O}$ is a highest weight category with weight poset $(\mathfrak{t}^*_\mathbb{Z}, \leq)$. Its standard modules are the Verma modules $M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$.

In any highest weight category $\mathcal{C}$, projective objects have finite Δ-flags. Hence there is a map $K_0(\mathcal{C}) \hookrightarrow K_0(\mathcal{C}^\Delta)$, the right hand side denoting the Grothendieck group of the exact subcategory of $\mathcal{C}$ consisting of objects with a Δ-flag. In category $\mathcal{O}$, all the chains in the partial order $\leq$ are all finite, in which case this map is actually an isomorphism. This means that the classes $[M(\lambda)]$ of the Verma modules can be interpreted as elements of $[\mathcal{O}]$.

Now we define a categorical $\mathfrak{sl}_\mathbb{Z}$-action on $\mathcal{O}$. For the endofunctors F and E, we take the functors $F := - \otimes U$ and $E := - \otimes U^\vee$, where U is the natural $\mathfrak{g}$-module of column vectors and $U^\vee$ is its dual. These are both left and right adjoint to each other in a canonical way. For the natural transformation $x \in \mathrm{End}(F)$, we let $x_M : M \otimes U \to M \otimes U$ be the endomorphism defined by the action of the Casimir tensor $\Omega := \sum_{i,j=1}^n e_{i,j} \otimes e_{j,i}$. For $s \in \mathrm{End}(F^2)$, we let $s_M : M \otimes U \otimes U \to M \otimes U \otimes U$ be the map $m \otimes u \otimes v \mapsto m \otimes v \otimes u$. The axioms (SL1)–(SL4) are checked in [24, §7.4]. The hardest one is (SL4); for this one shows that the map

$$[\mathcal{O}] \to V_\mathbb{Z}^{\otimes n}, \qquad [M(\lambda)] \mapsto v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_n}$$

is an isomorphism of $\mathfrak{sl}_\mathbb{Z}$-modules. This follows from a slightly stronger statement, namely, that $F_i M(\lambda)$ has a Δ-flag with sections $M(\lambda + \delta_j)$ for all $j = 1, \dots, n$ such that $\lambda_j = i$; similarly $E_i M(\lambda)$ has a Δ-flag with sections $M(\lambda - \delta_j)$ for all $j = 1, \dots, n$ such that $\lambda_j = i + 1$. On passing to the Grothendieck group, these two descriptions match the actions of f_i and e_i on the monomial $v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_n} \in V_\mathbb{Z}^{\otimes n}$.

We have just identified $[\mathcal{O}]$ with $V_\mathbb{Z}^{\otimes n}$ so that the classes of the Verma modules correspond to the monomials. The classes of the indecomposable projective modules give another natural basis for $[\mathcal{O}]$. Under our identification it is known that this basis corresponds to *Lusztig's canonical basis* for $V_\mathbb{Z}^{\otimes n}$. We skip the precise definition of the canonical basis here; it arises by applying Lusztig's general construction of tensor product of based modules from [45, Ch. 27] to the n-fold tensor product of the q-analog of the module $V_\mathbb{Z}$ for quantized enveloping

algebra $U_q\mathfrak{sl}_\mathbb{Z}$ (then specializing at $q = 1$). In fact the statement just made is an equivalent formulation of the Kazhdan-Lusztig conjecture for the Lie algebra $\mathfrak{g}$; see e.g. [12, Theorem 4.5] where the dual statement is explained.

Second example of a categorical action. Now take I to be an arbitrary non-empty interval and fix also a composition $\nu = (\nu_1, \dots, \nu_l)$ of integers with $1 \leq \nu_i \leq |I|$ for each i. Our next example of a categorical $\mathfrak{sl}_I$-action is going to have complexified Grothendieck group isomorphic to

$$\bigwedge^\nu V_I := \bigwedge^{\nu_1} V_I \otimes \cdots \otimes \bigwedge^{\nu_l} V_I.$$

Let us introduce some notation for the obvious monomial basis of this module: set

$$\Lambda := \left\{ \lambda = (\lambda_{i,j})_{1\leq i\leq l, 1\leq j\leq \nu_i} \;\middle|\; \lambda_{i,j} \in I_+, \lambda_{i,1} > \cdots > \lambda_{i,\nu_i} \text{ for each } i \right\},$$
$$v_\lambda := (v_{\lambda_{1,1}} \wedge \cdots \wedge v_{\lambda_{1,\nu_1}}) \otimes \cdots \otimes (v_{\lambda_{l,1}} \wedge \cdots \wedge v_{\lambda_{l,\nu_l}}) \in \bigwedge^\nu V_I.$$

Then $\{v_\lambda \mid \lambda \in \Lambda\}$ is a basis for $\bigwedge^\nu V_I$. Each of the modules $\bigwedge^{\nu_i} V_I$ is minuscule, so all of its weight spaces are one-dimensional. Hence the map

$$\Lambda \to (\mathfrak{t}_I^*)^{\oplus l}, \qquad \lambda \mapsto (|\lambda_1|, \dots, |\lambda_l|) \text{ where } |\lambda_i| := \sum_{j=1}^{\nu_i} \varepsilon_{\lambda_{i,j}}$$

is injective. Let $\leq$ be the usual dominance ordering on $\mathfrak{t}_I^*$ defined from $\lambda \leq \mu$ if $\mu - \lambda$ is a sum of simple roots $\varepsilon_i - \varepsilon_{i+1}$ ($i \in I$). Then define a partial order $\leq$ on Λ by $\lambda \leq \mu$ if and only if $|\lambda_1| + \cdots + |\lambda_i| \geq |\mu_1| + \cdots + |\mu_i|$ for each $i = 1, \dots, l$, with equality in case $i = l$. We refer to this as the *reverse dominance ordering*.

We identify the set Λ with a subset of the set $\mathfrak{t}_\mathbb{Z}^*$ of integral weights from the previous subsection so that $\lambda \in \Lambda$ corresponds to the weight $\lambda \in \mathfrak{t}_\mathbb{Z}^*$ with

$$(\lambda + \rho, \delta_{\nu_1+\cdots+\nu_{i-1}+j}) = \lambda_{i,j}$$

for each i, j, i.e. it is the tuple $(\lambda_{1,1}, \dots, \lambda_{1,\nu_1}, \dots, \lambda_{l,1}, \dots, \lambda_{l,\nu_l}) \in \mathbb{Z}^n$. Then let $\mathcal{M}$ be the Serre subcategory of the category $\mathcal{O}$ from the previous subsection generated by the modules $\{L(\lambda)\}_{\lambda\in\Lambda}$. In fact $\mathcal{M}$ is a sum of blocks of the parabolic category $\mathcal{O}$ associated to the standard parabolic subalgebra with Levi factor $\mathfrak{gl}_{\nu_1}(\mathbb{C}) \oplus \cdots \oplus \mathfrak{gl}_{\nu_l}(\mathbb{C})$. As is well known, $\mathcal{M}$ is again a highest weight category with weight poset Λ partially ordered by the reverse dominance ordering $\leq$ introduced in the previous subsection; this order is just the restriction of the Bruhat order $\leq$ on $\mathfrak{t}_\mathbb{Z}^*$ to Λ. The standard module $\Delta(\lambda) \in \mathcal{M}$ is the parabolic Verma module of highest weight λ. The functors F_i and E_i for $i \in I$ restrict to well-defined endofunctors of $\mathcal{M}$. Hence we can define a categorical $\mathfrak{sl}_I$-action on $\mathcal{M}$ with $F := \bigoplus_{i\in I} F_i$, $E := \bigoplus_{i\in I} E_i$, and x and s being the restrictions of the ones on $\mathcal{O}$. For (SL4), one checks that the map

$$[\mathcal{M}] \to \bigwedge^\nu V_\mathbb{Z}, \qquad [\Delta(\lambda)] \mapsto v_\lambda$$

is an isomorphism of $\mathfrak{sl}_I$-modules.

The natural inclusion $\mathcal{M} \hookrightarrow \mathcal{O}$ has a left adjoint $\pi : \mathcal{O} \to \mathcal{M}$, defined by taking the largest quotient that belongs to $\mathcal{M}$. As it is left adjoint to an exact functor, π sends projectives to projectives. In fact it is even the case that the restriction of π to $\mathcal{O}^\Delta$ is exact,

with $\pi(M(\lambda)) \cong \Delta(\lambda)$ if $\lambda \in \Lambda$ and $\pi(M(\lambda)) = 0$ otherwise. This means that there is a commuting diagram of linear maps

$$\begin{array}{ccc} [\mathcal{O}] & \xrightarrow{[\pi]} & [\mathcal{M}] \\ \downarrow & & \downarrow \\ V_{\mathbb{Z}}^{\otimes n} & \longrightarrow & \bigwedge^{\nu} V_I, \end{array}$$

where the vertical maps are the isomorphisms introduced above, and the bottom map is an obvious surjection. The functor π sends the projective cover of $L(\lambda)$ in $\mathcal{O}$ to its projective cover $P(\lambda)$ in $\mathcal{M}$ if $\lambda \in \Lambda$, or to zero otherwise. This parallels the effect of the bottom map in the above commuting diagram on the canonical bases of $V_{\mathbb{Z}}^{\otimes n}$ and $\bigwedge^{\nu} V_I$. Hence the basis $\{[P(\lambda)]\}_{\lambda \in \Lambda}$ for $[\mathcal{M}]$ corresponds to the canonical basis $\{b_\lambda\}_{\lambda \in \Lambda}$ of the based module $\bigwedge^{\nu} V_I$.

Higher level Schur-Weyl duality. We continue with the notation of the previous subsection, assuming in addition that the interval I is *finite*. Set $o := \min(I) - 1$ and $\varpi_i := \sum_{I \ni j \leq i} \varepsilon_j \in \mathfrak{t}_I^*$. The module $\bigwedge^{\nu} V_I$ has a unique highest weight in the dominance ordering, namely, the weight

$$\varpi := \sum_{i=1}^{l} \varpi_{o+\nu_i}.$$

We let $\kappa \in \Lambda$ be the unique element satisfying $|\kappa_1| + \cdots + |\kappa_l| = \varpi$, so that v_κ spans the highest weight space of $\bigwedge^{\nu} V_I$. The $\mathfrak{sl}_I$-submodule of $\bigwedge^{\nu} V_I$ generated by v_κ is a copy of the irreducible highest weight module $V(\varpi)$. Let

$$\iota : V(\varpi) \hookrightarrow \bigwedge^{\nu} V_I$$

be the inclusion. Higher level Schur-Weyl duality categorifies this homomorphism.

Recall that $\bigwedge^{\nu} V_I$ is a based module with canonical basis $\{b_\lambda\}_{\lambda \in \Lambda}$ corresponding to the indecomposable projectives $\{P(\lambda)\}_{\lambda \in \Lambda}$ in $\mathcal{M}$. By the general theory of based modules, there is a subset $\Lambda^\circ \subset \Lambda$ such that $\{b_\lambda\}_{\lambda \in \Lambda^\circ}$ is the canonical basis of the irreducible submodule $V(\varpi)$. The best way to describe this set Λ° combinatorially is to note that the set Λ that labels the basis of our based module $\bigwedge^{\nu} V_I$ comes equipped with an explicit crystal structure defined via Kashiwara's tensor product rule; then Λ° is the connected component of this crystal generated by κ.

The representation theoretic significance of Λ° was first noticed in [13]: it is exactly the set of weights that index the indecomposable projective modules $P(\lambda) \in \mathcal{M}$ that are also injective. Moreover the hypotheses of Theorem 1.1 are all satisfied in the present situation. For the prinjective generator T, we take

$$T := \bigoplus_{d \geq 0} F^d L(\kappa) \in \mathcal{M}.$$

Setting $C := \mathrm{End}_{\mathfrak{g}}(T)^{\mathrm{op}}$, Theorem 1.1 implies that the the functor

$$\mathbb{V} : \mathcal{M} \to C\text{-mod}$$

is fully faithful on projectives. For this to be good for anything, we of course need to identify the algebra C explicitly.

To state the main result, let H_d^f be the quotient of the degenerate affine Hecke algebra H_d by the two-sided ideal generated by $f := \prod_{i=1}^{l}(x_1 - (o + \nu_i))$. This finite dimensional algebra is known as a degenerate *cyclotomic Hecke algebra*. It contains a system of mutually orthogonal idempotents $\{1_{\boldsymbol{i}}\}_{\boldsymbol{i}\in\mathbb{C}^d}$ indexed by words $\boldsymbol{i} = i_1 \dots i_d \in \mathbb{C}^d$; these are defined so that $1_{\boldsymbol{i}}$ projects any H_d^f-module onto the generalized i_j-eigenspace of x_j for all j. Then let H_d^ω be the algebra $eH_d^f e$ where e is the central idempotent $\sum_{\boldsymbol{i}\in I^d} 1_{\boldsymbol{i}} \in H_d^f$.

Theorem 2.3 (Brundan-Kleshchev). *There is a well-defined right action of H_d^ω on $F^d L(\kappa)$ defined so that each of its generators x_j and s_k act via the natural transformations from (SL2). This action induces an isomorphism $H_d^\omega \xrightarrow{\sim} \mathrm{End}_{\mathfrak{g}}(F^d L(\kappa))^{\mathrm{op}}$. Hence $C \cong \bigoplus_{d\geq 0} H_d^\omega$.*

Remark 2.4. Theorem 2.3 was first proved in [13] under the assumption that $\nu_1 \geq \dots \geq \nu_l$; this restriction was removed in [8]. The original proof goes via finite W-algebras and a result of Vust establishing some generalization of classical Schur-Weyl duality for centralizers in the general linear Lie algebra. Vust's result itself is quite non-trivial; its proof was completed in [40] by an invariant theory argument depending on the normality of closures of conjugacy classes of matrices. As discussed further in Remark 3.5, Losev and Webster have subsequently found a completely different proof of Theorem 2.3 based on the uniqueness of minimal categorifications of integrable highest weight modules established in [47].

By works of Ariki [1] and Grojnowski [31], the category

$$C\text{-mod} = \bigoplus_{d\geq 0} H_d^\omega\text{-mod}$$

admits a categorical $\mathfrak{sl}_I$-action making it into a minimal categorification of the irreducible $\mathfrak{sl}_I$-module $V(\omega)$. The appropriate functors F and E are the induction and restriction functors going between H_d^ω-mod and H_{d+1}^ω-mod. The functor $\mathbb{V} : \mathcal{M} \to C\text{-mod}$ is then strongly equivariant in the sense of Definition 2.2.

The left adjoint to the quotient functor $\mathbb{V}$ sends the indecomposable projectives in C-mod to the ones in $\mathcal{M}$ indexed by the set Λ°. It induces a linear map $[C\text{-mod}] \hookrightarrow [\mathcal{M}]$ which corresponds exactly to the inclusion $\iota : V(\omega) \hookrightarrow \bigwedge^\nu V_I$ mentioned already above. Thus the classes of the indecomposable projectives in $[C\text{-mod}]$ coincide with the canonical basis $\{b_\lambda\}_{\lambda\in\Lambda^\circ}$ for $V(\omega)$. There are also certain *Specht modules* $\{S(\lambda)\}_{\lambda\in\Lambda}$ which have an intrinsic definition in terms of H_d^ω. In fact, as show in [13], the Specht module $S(\lambda)$ is the image of the parabolic Verma module $\Delta(\lambda)$ under the quotient functor $\mathbb{V}$. Then one can deduce almost everything known about the representation theory of the degenerate cyclotomic Hecke algebras H_d^ω from that of $\mathcal{M}$. This is done systematically in [15], leading to another proof of *Ariki's categorification theorem* from [1] for a generic parameter. This argument is similar to the way that Green recovers the representation theory of the symmetric group from the general linear group in [30].

The point of the double centralizer property is that it gives a way to recover the category $\mathcal{M}$ (up to equivalence) from the algebra C and knowledge of the *Young modules* $Y(\lambda) := \mathbb{V}P(\lambda)$ for each $\lambda \in \Lambda$. Indeed if Y is any *Young generator* for C-mod, that is, a direct sum of Young modules with each occuring at least once, then the double centralizer property shows that $\mathcal{M}$ is equivalent to the category A-mod where $A := \mathrm{End}_C(Y)^{\mathrm{op}}$. One application of

this is given in [13]: it is shown there that a particular Young generator Y may be obtained by taking a direct sum of all of the so-called *permutation modules* introduced by Dipper, James and Mathas in [25] (or rather, their degenerate analogs). For this choice, the algebra A is the *cyclotomic Schur algebra*, i.e. the degenerate version of the algebra introduced in [25]. Hence by the double centralizer property the category $\mathcal{M}$ is equivalent to the category of finite dimensional modules over the cyclotomic Schur algebra. This argument is similar to the proof of the equivalence of the categories $\mathcal{R}$ and $S(n,d)$-mod from the introduction.

3. Tensor product categorifications

Discussion of the definition. In this section we are going to focus on some results of Losev and Webster from [44]. These put the Schur-Weyl duality for higher levels discussed above into a general axiomatic framework. We begin by formulating their definition in a very special case, namely, for tensor products of minuscule representations of $\mathfrak{sl}_I$ for a finite interval $I \subset \mathbb{Z}$. The following is exactly like in [16, Definition 2.9]. Note also that the category $\mathcal{M}$ defined in the previous section is an example, thus establishing the existence of such structures.

Definition 3.1. Let $\nu = (\nu_1, \dots, \nu_l)$ be a composition of n and $I \subset \mathbb{Z}$ be a finite interval. A *tensor product categorification* of $\bigwedge^\nu V_I$ means a highest weight category $\mathcal{M}$ together with an endofunctor F of $\mathcal{M}$, a right adjoint E to F (with specified adjunction), and natural transformations $x \in \mathrm{End}(F)$ and $s \in \mathrm{End}(F^2)$ satisfying the axioms (SL1)–(SL3) from Definition 2.1 and the axioms (TP1)–(TP2) below.

(TP1) The weight poset Λ is the set of tuples $(\lambda_1, \dots, \lambda_l) \in (\mathfrak{t}_I^*)^{\oplus l}$ such that each λ_i is a weight of $\bigwedge^{\nu_i} V_I$, ordered by the *reverse dominance ordering* $\lambda \leq \mu$ if and only if $\lambda_1 + \dots + \lambda_i \geq \mu_1 + \dots + \mu_i$ for each i with equality when $i = l$.

(TP2) The exact functors F_i and E_i send objects with Δ-flags to objects with Δ-flags. Moreover the linear isomorphism $[\mathcal{M}] \xrightarrow{\sim} \bigwedge^\nu V_I, [\Delta(\lambda)] \mapsto v_\lambda$ intertwines the endomorphisms f_i and e_i of $[\mathcal{M}]$ induced by F_i and E_i with the endomorphisms of $\bigwedge^\nu V_I$ arising from the actions of the Chevalley generators f_i and e_i of $\mathfrak{sl}_I$.

In [44], Losev and Webster have introduced a substantially more general notion of tensor product categorication of $V(\nu_1) \otimes \dots \otimes V(\nu_l)$ for arbitrary integrable highest weight modules $V(\nu_1), \dots, V(\nu_l)$ for an arbitrary Kac-Moody algebra $\mathfrak{g}$. We are going to explain their definition somewhat informally. To start with, since they work with an arbitrary $\mathfrak{g}$ rather than $\mathfrak{sl}_I$, the axioms (SL1)–(SL3) are replaced by the corresponding axioms for a categorical $\mathfrak{g}$-action, i.e. the natural transformations defining the degenerate affine Hecke algebra action on F^d are replaced by natural transformations defining a quiver Hecke algebra action.

A more significant issue is that the modules $V(\nu_i)$ are no longer assumed to be minuscule, so their weight spaces are not all one-dimensional. We still have a natural poset Ξ, namely, the set of l-tuples $\xi = (\xi_1, \dots, \xi_l)$ such that ξ_i is a weight of the $\mathfrak{g}$-module $V(\nu_i)$, ordered by the reverse dominance ordering as above. The theory of based modules also produces a couple of natural bases for $V(\nu_1) \otimes \dots \otimes V(\nu_l)$, both indexed by the set Λ that is the Cartesian product of the underlying highest weight crystals: the *monomial basis* $\{v_\lambda\}_{\lambda \in \Lambda}$ arising from the naive tensor product of the canonical bases in each $V(\nu_i)$, and the *canonical basis* $\{b_\lambda\}_{\lambda \in \Lambda}$ for $V(\nu_1) \otimes \dots \otimes V(\nu_l)$ itself defined via Lusztig's general construction from

[45, Ch. 27]. However now there is only a surjection

$$\rho : \Lambda \twoheadrightarrow \Xi,$$

rather than the isomorphism that we exploited in the previous section. This is a shadow of the problem at a categorical level: the category $\mathcal{M}$ is no longer going to be a highest weight category. Rather, it is the following weakening of the notion of highest weight category introduced by Losev and Webster (building on an earlier notion of standardly stratified algebra studied by a number of authors in the literature).

Definition 3.2. Let Ξ be an interval-finite poset and $\rho : \Lambda \twoheadrightarrow \Xi$ be a surjective function with finite fibers. A *standardly stratified category* of this type is a $\mathbb{C}$-linear abelian category $\mathcal{M}$ together with a given set of representatives $\{L(\lambda)\}_{\lambda\in\Lambda}$ for its irreducible objects, satisfying the axioms (SS1)–(SS3) below.

(SS1) All objects of $\mathcal{M}$ are of finite length, there are enough projectives and injectives, and $\mathrm{End}_{\mathcal{M}}(L(\lambda)) \cong \mathbb{C}$ for each λ.

For $\xi \in \Xi$, let $\mathcal{M}_{\leq\xi}$ be the Serre subcategory of $\mathcal{M}$ generated by $\{L(\lambda)\}_{\lambda\in\Lambda,\rho(\lambda)\leq\xi}$. Define $\mathcal{M}_{<\xi}$ similarly, and let $\pi_\xi : \mathcal{M}_{\leq\xi} \to \mathcal{M}_\xi$ be the quotient of $\mathcal{M}_{\leq\xi}$ by $\mathcal{M}_{<\xi}$. The *associated graded category* is $\mathrm{gr}\,\mathcal{M} := \bigoplus_{\xi\in\Xi} \mathcal{M}_\xi$. The *standardization functor* is $\Delta := \bigoplus_{\xi\in\Xi} \Delta_\xi : \mathrm{gr}\,\mathcal{M} \to \mathcal{M}$ where $\Delta_\xi : \mathcal{M}_\xi \to \mathcal{M}_{\leq\xi}$ is some choice of a left adjoint to π_ξ.

(SS2) The standardization functor is exact.

Let $P(\lambda)$ be the projective cover of $L(\lambda)$ in $\mathcal{M}$ and $\Delta(\lambda)$ be the projective cover of $L(\lambda)$ in $\mathcal{M}_{\leq\rho(\lambda)}$. In other words, $\Delta(\lambda)$ is the largest quotient of $P(\lambda)$ that belongs to $\mathcal{M}_{\leq\rho(\lambda)}$:

$$\Delta(\lambda) = P(\lambda) \Big/ \sum_{\substack{\mu\in\Lambda \text{ with } \rho(\mu)\not\leq\rho(\lambda)\\ f\in\mathrm{Hom}_G(P(\mu),P(\lambda))}} \mathrm{im}\, f.$$

(SS3) Each $P(\lambda)$ admits a finite Δ-flag with $\Delta(\lambda)$ at the top and lower sections of the form $\Delta(\mu)$ for $\mu \in \Lambda$ with $\rho(\mu) > \rho(\lambda)$.

Now we can complete our sketch of what it means for $\mathcal{M}$ to be a tensor product categorification of $V(\nu_1) \otimes \cdots \otimes V(\nu_l)$. Of course it should be standardly stratified with $\rho : \Lambda \twoheadrightarrow \Xi$ as defined just before Definition 3.2. Moreover $\mathrm{gr}\,\mathcal{M}$ should admit a categorical action of $\mathfrak{g}^{\oplus l}$ making it into a minimal categorification of the irreducible $\mathfrak{g}^{\oplus l}$-module $V(\nu_1) \otimes \cdots \otimes V(\nu_l)$; we denote the functors F_i and E_i for the action of the jth copy of $\mathfrak{g}$ here by ${}_jF_i$ and ${}_jE_i$, respectively. By the general theory of categorifications of integrable highest weight modules, the isomorphism classes of indecomposable projectives in $\mathrm{gr}\,\mathcal{M}$ are canonically labelled by the $\mathfrak{g}^{\oplus l}$-crystal Λ; this is explained in [41] or follows from the theory of perfect bases from [5]. Then there are two axioms which give some compatibility between $\mathrm{gr}\,\mathcal{M}$ and $\mathcal{M}$:

(TP1′) The indecomposable projective object of $\mathrm{gr}\,\mathcal{M}$ labelled by $\lambda \in \Lambda$ is isomorphic to the one arising by taking the image of $\Delta(\lambda)$ under the functor $\pi_{\rho(\lambda)}$.

(TP2′) For each $M \in \mathrm{gr}\,\mathcal{M}$, the object $F_i\Delta(M)$ (resp. $E_i\Delta(M)$) admits a filtration with sections $\Delta({}_jF_iM)$ (resp. $\Delta({}_jE_iM)$) for $j = 1, \dots, l$.

(This formulation of the definition looks slightly different but is equivalent to the one in [44].)

The problem of *existence* of such general tensor product categorifications was addressed already in earlier work of Webster [60]. In this, he introduced certain *tensor product algebras*, which can naturally be viewed as generalizations of cyclotomic quotients of quiver Hecke algebras. Then he uses the category of finite dimensional modules over these algebras to construct arbitrary tensor product categorifications.

The Losev-Webster uniqueness theorem. Having sketched the definition of tensor product categorification, we can now paraphrase the main result established in [44] as follows. Recall Definition 2.2 (which has an analog for arbitrary $\mathfrak{g}$).

Theorem 3.3 (Losev-Webster). *Let $V(\nu_1), \dots, V(\nu_l)$ be integrable highest weight for some Kac-Moody algebra $\mathfrak{g}$. Any tensor product categorification of $V(\nu_1) \otimes \cdots \otimes V(\nu_l)$ is unique up to strongly equivariant equivalence.*

Here we restate this in a special case:

Corollary 3.4. *Let $I \subset \mathbb{Z}$ be a finite interval. Any tensor product categorification of the $\mathfrak{sl}_I$-module $\bigwedge^\nu V_I$ in the sense of Definition 3.1 is strongly equivariantly equivalent to the category $\mathcal{M}$ constructed from parabolic category $\mathcal{O}$ in the previous section.*

In order to emphasize the similarity between the present situation and the Schur-Weyl duality for higher levels from the previous section, let us say a few words about the strategy behind the proof of Theorem 3.3. As we mentioned earlier, the tensor product $V(\nu_1) \otimes \cdots \otimes V(\nu_l)$ has a canonical basis $\{b_\lambda\}_{\lambda \in \Lambda}$ parametrized by the Cartesian product Λ of the underlying highest weight crystals. Kashiwara's tensor product rule gives the set Λ a canonical structure of $\mathfrak{g}$-crystal. (Building on earlier arguments from [42], Losev and Webster even give an interpretation of this crystal structure in terms of tensor product categorifications, which is the key to the proof of property (P1) stated in the next paragraph.) Let $\kappa \in \Lambda$ be the label of the highest weight vector of $V(\nu_1) \otimes \cdots \otimes V(\nu_l)$ of weight $\omega := \nu_1 + \cdots + \nu_l$, and let Λ° be the connected component of the $\mathfrak{g}$-crystal Λ generated by κ. Then the vectors $\{b_\lambda\}_{\lambda \in \Lambda^\circ}$ span a $\mathfrak{g}$-submodule of $V(\nu_1) \otimes \cdots \otimes V(\nu_l)$ isomorphic to $V(\omega)$.

Now let $\mathcal{M}$ be a tensor product categorification of $V(\nu_1) \otimes \cdots \otimes V(\nu_l)$. Losev and Webster show:

(P1) For $\lambda \in \Lambda$, the projective object $P(\lambda)$ is injective if and only if $\lambda \in \Lambda^\circ$. Moreover $T := \bigoplus_{d \geq 0} F^d L(\kappa)$ is a prinjective generator for $\mathcal{M}$.

(P2) The algebra $C := \mathrm{End}_{\mathcal{M}}(T)^{\mathrm{op}}$ is the direct sum $\bigoplus_{d \geq 0} H_d^\omega$ of the cyclotomic quiver Hecke algebras attached to $\mathfrak{g}$ and the dominant weight ω.

(P3) The double centralizer property holds, i.e. $\mathbb{V} := \mathrm{Hom}_{\mathcal{M}}(T, -) : \mathcal{M} \to C\text{-mod}$ is fully faithful on projectives.

(P4) For $\lambda \in \Lambda$ the isomorphism type of the C-module $Y(\lambda) := \mathbb{V}P(\lambda)$ is independent of the particular choice of the tensor product categorification $\mathcal{M}$.

(These statements make sense as written only in the case that $\mathfrak{g}$ is of finite type; for the general case one needs to modify them in an analogous way to Remark 1.2, taking $T_d := F^d L(\kappa)$ for $d \in \mathbb{N}$.) By the double centralizer property (P3), the category $\mathcal{M}$ can be recovered from the algebra C and its modules $\{Y(\lambda)\}_{\lambda \in \Lambda}$. Hence (P2) and (P4) establish the uniqueness of $\mathcal{M}$ up to equivalence. The strong equivariance follows by some further considerations in a similar vein.

Remark 3.5. Here we sketch the Losev-Webster proof of (P2); note in view of [14] that this generalizes Theorem 2.3 above. By (P1), the left adjoint to the quotient functor $\mathbb{V}$ induces an embedding $[C\text{-mod}] \hookrightarrow [\mathcal{M}]$ which corresponds to the inclusion $V(\omega) \hookrightarrow V(\nu_1) \otimes \cdots \otimes V(\nu_l)$ at the level of Grothendieck groups. Thus the category of projectives in C-mod is equivalent to the bottom section of Rouquier's canonical filtration of the category of projectives in $\mathcal{M}$ from [47, Theorem 5.8]. (Indeed, Rouquier's filtration parallels Lusztig's canonical filtration of the based module $V(\nu_1) \otimes \cdots \otimes V(\nu_l)$.) Since $\mathrm{End}_{\mathcal{M}}(L(\kappa)) \cong \mathbb{C}$, it follows that this category is a realization of the minimal categorication of $V(\omega)$. But it is also known that the cyclotomic quiver Hecke algebras H_d^ω give such a realization with $F := H_{d+1}^\omega \otimes_{H_d^\omega} -$ (e.g. see [34]); in this setting it is obvious that $\mathrm{End}(F^d L(\kappa))^{\mathrm{op}} \cong H_d^\omega$. The property (P2) is now clear from the uniqueness of minimal categorifications established in [47].

Graded lifts. The following theorem should by now come as no surprise. Note this builds essentially on the base case $l = 1$, where the identification of the canonical basis with the basis arising from indecomposable projectives in the minimal categorification of an integrable highest weight module was established already by Rouquier [48] and Varagnolo and Vasserot [55] (for symmetric Cartan matrices over $\mathbb{C}$ only).

Theorem 3.6 (Webster). *Let $V(\nu_1), \dots, V(\nu_l)$ be integrable highest weight modules for some Kac-Moody algebra $\mathfrak{g}$ with a* symmetric *Cartan matrix. Let $\mathcal{M}$ be a tensor product categorification of $V(\nu_1) \otimes \cdots \otimes V(\nu_l)$ (over the ground field $\mathbb{C}$). Identify $[\mathcal{M}]$ with $V(\nu_1) \otimes \cdots \otimes V(\nu_l)$ so that each $[\Delta(\lambda)]$ is identified with the monomial v_λ (= the tensor product of canonical basis vectors in each $V(\nu_i)$). Then each $[P(\lambda)]$ coincides with the canonical basis vector b_λ.*

This is proved in [58] using the realization of $\mathcal{M}$ given by the tensor product algebras. In the special case that $\mathfrak{g}$ is of finite type A and each ν_i is minuscule, it follows already from Corollary 3.4 and the Kazhdan-Lusztig conjecture for the general linear Lie algebra, as we discussed already in the previous section; see also [16, Corollary 5.29] for a more direct argument in this case along the lines of [58].

The basic idea of Webster's proof of Theorem 3.6 involves the construction of certain *graded lifts* of tensor product categorifications. For Webster, these are no problem since his tensor product algebras are naturally graded. One can also understand them along similar lines to Soergel's definition of the graded lift of category $\mathcal{O}_0$ sketched in the introduction. This depends on the existence of a grading on C arising from the natural grading on quiver Hecke algebras. The images $Y(\lambda) := \mathbb{V}P(\lambda)$ all turn out to admit a unique grading (up to automorphism) with respect to which they are graded-self-dual modules over the graded algebra C. Then, setting $Y := \bigoplus_{\lambda \in \Lambda} Y(\lambda)$, the graded lift of $\mathcal{M}$ arises from the category A-grmod where $A := \mathrm{End}_C(Y)^{\mathrm{op}}$.

The point then is that the graded category A-grmod admits a categorical action of the quantized enveloping algebra $U_q(\mathfrak{g})$; see e.g. [16, Definition 5.5] where this definition is spelled out in the special case that $\mathfrak{g} = \mathfrak{sl}_I$. The grading shift functor makes its Grothendieck group into a $\mathbb{Z}[q, q^{-1}]$-module, hence tensoring over $\mathbb{Z}[q, q^{-1}]$ with $\mathbb{Q}(q)$, we obtain a $\mathbb{Q}(q)$-vector space. The categorical action of $U_q(\mathfrak{g})$ on A-grmod makes this into a $U_q(\mathfrak{g})$-module isomorphic to the q-analog of $V(\nu_1) \otimes \cdots \otimes V(\nu_l)$. The next step is to show that the category of graded projectives in A-grmod admits a duality which corresponds to Lusztig's bar involution on the q-analog of $V(\nu_1) \otimes \cdots \otimes V(\nu_l)$. This machinery reduces the proof

of Theorem 3.6 to verifying that the grading on the algebra A is positive (with A_0 being semisimple); for Webster this is the property that the graded lift of $\mathcal{M}$ is *mixed*. Finally that is established by appealing to some geometric construction involving quiver varieties; see [58, Theorem 6.8].

In the special case that $\mathfrak{g} = \mathfrak{sl}_I$ for a finite interval I and all the weights ν_i are minuscule, the grading on the algebra A defined in the previous paragraph makes it into a *Koszul algebra* (hence it is mixed). The proof of this can actually be deduced from the Koszulity of blocks of parabolic category $\mathcal{O}$ established already by Beilinson, Ginzburg and Soergel [4] (also Backelin [2] for the singular-singular case). This is explained by Webster in [60], and independently by Hu and Mathas in [33]; see also [59] which suggests an entirely different approach to see the Koszulity via quiver varieties.

4. Some applications and future directions

Level two examples arising from Khovanov's arc algebra. There is a completely explicit description of the basic algebra underlying all tensor product categorifications in the sense of Definition 3.1 in which the level l is two. This arises from Khovanov's arc algebra, which was introduced originally in [36, 37] in the course of his work on categorification of the Jones polynomial. We give a very brief sketch here.

We fix an interval $I \subseteq \mathbb{Z}$ and let $I_+ := I \cup (I+1)$ as before. A *weight diagram* means a number line with vertices at each of the integers in the set I_+ labelled by one of the symbols $\vee, \wedge, \circ$ or $\times$; on excluding finitely many of the vertices, we require that no $\vee$ should appear to the left of an $\wedge$. Assume that we are given some set Λ of weight diagrams that is closed under the following swaps of labels of any pair of adjacent vertices:

$$\circ\times \leftrightarrow \vee\wedge \leftrightarrow \wedge\vee \leftrightarrow \times\circ, \quad \circ\vee \leftrightarrow \vee\circ, \quad \circ\wedge \leftrightarrow \wedge\circ, \quad \times\vee \leftrightarrow \vee\times, \quad \times\wedge \leftrightarrow \wedge\times .$$

To the set Λ, we associated in [18] a certain positively graded basic algebra K_Λ, which is some generalization of Khovanov's arc algebra. In [19] we defined some endofunctors of K_Λ-mod defined by some explicit bimodules (generalizing Khovanov's geometric bimodules from [37]), and used these to prove directly that K_Λ is Koszul. Then in [20, (3.11)] we used certain of these functors to define biadjoint endofunctors F_i and E_i of K_Λ-mod for each $i \in I$. Finally in [20, (5.3)–(5.4)] we defined natural transformations $F_i \to F_i$ and $F_i \circ F_j \to F_j \circ F_i$ which, when suitably signed, satisfy the quiver Hecke algebra relations. Thus we obtain all of the data needed for a categorical $\mathfrak{sl}_I$-action on K_Λ-mod. (In [20] we considered only the case that I is finite but the constructions there apply in any case.)

Now we specialize to the case that the set I is finite and that Λ consists of all weight diagrams such that exactly m of the labels are either $\vee$ or $\times$, and exactly n of the labels are either $\wedge$ or $\times$. Then it is straightforward to check that K_Λ-mod is actually a tensor product categorification of $\bigwedge^m V_I \otimes \bigwedge^n V_I$ in the sense of Definition 3.1. Applying Corollary 3.4, we deduce that K_Λ-mod is equivalent to the parabolic category $\mathcal{O}$ denoted $\mathcal{M}$ in section 2 for $\nu = (m, n)$. In this way, one quite easily reproves an equivalence of categories established originally in [20, 52].

The super Kazhdan-Lusztig conjecture. In Definition 3.1, we assumed that the interval I was finite. The case $I = \mathbb{Z}$ is also interesting. Note for this that the Lie algebra $\mathfrak{sl}_\mathbb{Z}$ has four natural families of minuscule representations: the integrable highest weight modules $V(\omega_m)$

indexed by the fundamental dominant weights ω_m, the integrable lowest weight modules $V(-\omega_m)$, the exterior powers $\bigwedge^n V_{\mathbb{Z}}$ where $V_{\mathbb{Z}}$ is the natural $\mathfrak{sl}_{\mathbb{Z}}$-module, and the exterior powers $\bigwedge^n W_{\mathbb{Z}}$ where $W_{\mathbb{Z}}$ is dual to $V_{\mathbb{Z}}$.

In [16] we proved a version of the Losev-Webster uniqueness theorem for $\mathfrak{sl}_{\mathbb{Z}}$-tensor product categorifications involving tensor products of the exterior powers $\bigwedge^n V_{\mathbb{Z}}$ and $\bigwedge^n W_{\mathbb{Z}}$ (which are neither highest nor lowest weight). To set this up formally, one needs to modify Definition 3.1 slightly since the poset Λ defined exactly as in (TP1) need no longer have finite chains; the fix is to replace the Grothendieck group $[\mathcal{M}]$ in (TP2) with $[\mathcal{M}^{\Delta}]$. In all cases, projectives have finite Δ-flags, so that $[\mathcal{M}]$ still embeds naturally into $[\mathcal{M}^{\Delta}]$. We also proved that any such tensor product categorification admits a unique (up to equivalence) graded lift, and this graded lift is Koszul. Both of these results were deduced ultimately as applications of Corollary 3.4 and the known Koszulity of the graded lifts for finite intervals.

The main example of such $\mathfrak{sl}_{\mathbb{Z}}$-tensor product categorifications comes by considering parabolic category $\mathcal{O}$ for the general linear Lie superalgebra. Using this one can define a category $\mathcal{M}$ admitting a categorical $\mathfrak{sl}_{\mathbb{Z}}$-action in a very similar way to the second example from section 2. This leads to a construction of tensor product categorifications of any number of the modules of the form $\bigwedge^n V_{\mathbb{Z}}$ or $\bigwedge^n W_{\mathbb{Z}}$, with tensor factors appearing in any order. In particular the (integral part of) full category $\mathcal{O}$ for $\mathfrak{gl}_{m|n}(\mathbb{Z})$ relative to the standard Borel gives a tensor product categorification of $V_{\mathbb{Z}}^{\otimes m} \otimes W_{\mathbb{Z}}^{\otimes n}$, while its category of (finite dimensional) integrable representations gives a tensor product categorification of $\bigwedge^m V_{\mathbb{Z}} \otimes \bigwedge^n W_{\mathbb{Z}}$. The results from the previous paragraph imply at once that these categories all admit Koszul graded lifts. Moreover the super Kazhdan-Lusztig conjecture formulated originally in [6] and first proved by Cheng, Lam and Wang in [23] falls out easily from Corollary 3.4. See also [9] for a recent survey.

The existence of a Koszul graded lift of the category of integrable representations of $\mathfrak{gl}_{m|n}(\mathbb{C})$ had been proved earlier in [21]. In fact there is an explicit construction of this category in terms of Khovanov's arc algebra from the previous subsection: one just applies the results sketched there to $I = \mathbb{Z}$ with Λ consisting of all weights in which exactly m vertices are labelled $\times$ or $\vee$ and exactly n vertices are labelled $\circ$ or $\vee$. This produces another tensor product categorification of $\bigwedge^m V_{\mathbb{Z}} \otimes \bigwedge^n W_{\mathbb{Z}}$. Then the uniqueness of such tensor product categorifications implies that this is strongly equivariantly equivalent to the category of representations of $\mathfrak{gl}_{m|n}(\mathbb{C})$. In this way, one can recover the main theorem of [21].

Lowest tensored highest weight modules. There is one more interesting family of examples coming from Khovanov's arc algebra. Take the interval I to be $\mathbb{Z}$. Fix also integers $m, n \in \mathbb{Z}$. Let Λ be the set of all bipartitions $\lambda = (\lambda^\vee, \lambda^\wedge)$. We identify bipartition $\lambda \in \Lambda$ with the weight diagram having label $\vee$ at vertices $n+1-\lambda^\vee_1, n+2-\lambda^\vee_2, n+3-\lambda^\vee_3, \dots$ and label $\wedge$ at vertices $m+\lambda^\wedge_1, m-1+\lambda^\wedge_2, m-2+\lambda^\wedge_3, \dots$ (both $\vee$ and $\wedge$ means $\times$, neither means $\circ$). Then as above we get associated an arc algebra K_Λ and the data of a categorical $\mathfrak{sl}_{\mathbb{Z}}$-action on a suitable category of K_Λ-modules. This turns out to be a tensor product categorification of $V(-\omega_n) \otimes V(\omega_m)$, i.e. lowest weight tensored highest weight minuscule representations of $\mathfrak{sl}_{\mathbb{Z}}$. Actually, there is some further loss of finiteness here: although finitely generated projective modules still have finite Δ-flags, the standard modules in this category have infinite length in general. This means that one needs to modify Definition 3.1 again, allowing certain direct limits of highest weight categories.

There is another naturally occurring example of such a tensor product categorification of $V(-\omega_n) \otimes V(\omega_m)$. This is given by Deligne's category $\underline{\mathrm{Rep}}(GL_\delta)$ where $\delta := m - n$. By

definition, Deligne's catgory is the Karoubification of the *oriented Brauer category* $\mathcal{OB}(\delta)$ as defined in [10]. As conjectured in [22], this categorification is expected to be strongly equivariantly equivalent to one arising from the arc algebra K_Λ from the previous paragraph.

In [58], Webster has also introduced categorifications of integrable lowest tensored highest weight representations associated to arbitrary Kac-Moody algebras. These arise as certain cyclotomic quotients of the 2-Kac-Moody algebra $\mathcal{U}(\mathfrak{g})$. Yet more examples, which should of course be closely related to special cases of Webster's categories, arise from the cyclotomic oriented Brauer categories $\mathcal{OB}^{f,f'}$ defined in [10] and studied further in [17]. These are attached to a pair f, f' of monic polynomials of degree ℓ and produce $\mathfrak{sl}_\mathbb{Z}$-tensor product categorifications of the form $V(-\omega') \otimes V(\omega)$ where ω and ω' are level ℓ dominant weights defined from f and f', respectively.

Other sorts of categorical actions. We end by listing several recent works which hint at the existence of various undeveloped (or at least underdeveloped) parallel theories of categorical actions.

In [35], Kang, Kashiwara and Tsuchioka have introduced *quiver Hecke superalgebras*, and proved some isomorphism theorems relating them to the affine Sergeev superalgebras and affine Hecke-Clifford superalgebras which arose in [11, 53]. The quiver Hecke superalgebra for the trivial quiver with one (odd) vertex is closely related to the spin Hecke algebra of [57] and the odd nil-Hecke algebra of [28]. There is slowly emerging a parallel theory of super categorical actions based around these algebras. It seems reasonable to expect that there should be a version of Rouquier's canonical filtration in this setting, and results like the uniqueness of minimal categorifications and more generally of tensor product categorifications. An interesting example comes from the category $\mathcal{O}$ for the Lie superalgebra $\mathfrak{q}_n(\mathbb{C})$; we hope this new point of view will one day shed light on the Kazhdan-Lusztig conjecture for $\mathfrak{q}_n(\mathbb{C})$ formulated in [7].

In [3], some new canonical bases have been defined which are related to category $\mathcal{O}$ for the symplectic and orthogonal Lie algebras and the orthosymplectic Lie superalgebras. This points towards another twisted theory of categorification, in which the role of degenerate affine Hecke algebras is played by the generalized Wenzl (VW) algebra as suggested in [27]. In [26], Ehrig and Stroppel have also introduced some twisted version of the Khovanov arc algebra which should control the "level two" part of this picture.

There is also a completely different sort of twisted quiver Hecke algebras related to affine Hecke algebras of types B and C. These were introduced by Varagnolo and Vasserot in [56], who used them to prove the Lascoux-Leclerc-Thibon-type conjecture formulated in [29].

Finally we mention very briefly another very rich example of a categorical action. This arises from the category $\mathcal{O}$ in the sense of [32] attached to the rational Cherednik algebras associated to the complex reflection groups $S_n \ltimes (\mathbb{Z}/\ell\mathbb{Z})^n$. In [54], Varagnolo and Vasserot conjectured this category to be equivalent to a truncation of parabolic category $\mathcal{O}$ for the affine general linear Lie algebra. Independent proofs of this conjecture have recently been given by Losev [43] and Rouquier, Shan, Varagnolo and Vasserot [49]. Losev's proof makes essential use of the theory of categorical actions, which he extends to something he calls a *Schur categorification*.

Acknowledgements. Research supported in part by NSF grant no. DMS-1161094. Thanks to Ben Webster for suggesting the formulation of Theorem 1.1 to me.

References

[1] S. Ariki, *On the decomposition numbers of the Hecke algebra of $G(m, 1, n)$*, J. Math. Kyoto Univ., **36** (1996), 789–808.

[2] E. Backelin, *Koszul duality for parabolic and singular category $\mathcal{O}$*, Represent. Theory, **3** (1999), 139–152.

[3] H. Bao and W. Wang, *A new approach to Kazhdan-Lusztig theory of type B via quantum symmetric pairs*, arXiv:1310.0103.

[4] A. Beilinson, V. Ginzburg and W. Soergel, *Koszul duality patterns in representation theory*, J. Amer. Math. Soc., **9** (1996), 473–527.

[5] A. Berenstein and D. Kazhdan, *Geometric and unipotent crystals II: from unipotent bicrystals to crystal bases*, Contemp. Math., **433** (2007), 13–88.

[6] J. Brundan, *Kazhdan-Lusztig polynomials and character formulae for the Lie superalgebra $\mathfrak{gl}(m|n)$*, J. Amer. Math. Soc., **16** (2003), 185–231.

[7] ———, *Kazhdan-Lusztig polynomials and character formulae for the Lie superalgebra $\mathfrak{q}(n)$*, Advances Math., **182** (2004), 28–77.

[8] ———, *Centers of degenerate cyclotomic Hecke algebras and parabolic category $\mathcal{O}$*, Represent. Theory, **12** (2008), 236–259.

[9] ———, *Representations of the general linear Lie superalgebra in the BGG category $\mathcal{O}$*, to appear in the retrospective volumes of the West Coast Lie Theory Seminar.

[10] J. Brundan, J. Comes, D. Nash and A. Reynolds, *A basis theorem for the affine oriented Brauer category and its cyclotomic quotients*, preprint, arXiv:1404.6574.

[11] J. Brundan and A. Kleshchev, *Hecke-Clifford superalgebras, crystals of type $A^{(2)}_{2l}$ and modular branching rules for $\widehat{S}_n$*, Represent. Theory, **5** (2001), 317–403.

[12] ———, *Representations of shifted Yangians and finite W-algebras*, Mem. Amer. Math. Soc., **196** (2008), 107.

[13] ———, *Schur-Weyl duality for higher levels*, Selecta Math., **14** (2008), 1–57.

[14] ———, *Blocks of cyclotomic Hecke algebras and Khovanov-Lauda algebras*, Invent. Math., **178** (2009), 451–484.

[15] ———, *The degenerate analogue of Ariki's categorification theorem*, Math. Z., **266** (2010), 877–919.

[16] J. Brundan, I. Losev, and B. Webster, *Tensor product categorifications and the super Kazhdan-Lusztig conjecture*, arXiv:1310.0349.

[17] J. Brundan and A. Reynolds, *Representations of the cyclotomic oriented Brauer category*, in preparation.

[18] J. Brundan and C. Stroppel, *Highest weight categories arising from Khovanov's diagram algebra I: cellularity*, Mosc. Math. J., **11** (2011), 685–722.

[19] ———, *Highest weight categories arising from Khovanov's diagram algebra II: Koszulity*, Transform. Groups, **15** (2010), 1–45.

[20] J. Brundan and C. Stroppel, *Highest weight categories arising from Khovanov's diagram algebra III: category $\mathcal{O}$*, Represent. Theory, **15** (2011), 170–243.

[21] ———, *Highest weight categories arising from Khovanov's diagram algebra IV: the general linear supergroup*, J. Eur. Math. Soc., **14** (2012), 373–419.

[22] ———, *Gradings on walled Brauer algebras and Khovanov's arc algebra*, Advances Math., **231** (2012), 709–773.

[23] S.-J. Cheng, N. Lam, and W. Wang, *Brundan-Kazhdan-Lusztig conjecture for general linear Lie superalgebras*, arXiv:1203.0092.

[24] J. Chuang and R. Rouquier, *Derived equivalences for symmetric groups and $\mathfrak{sl}_2$-categorification*, Annals Math., **167** (2008), 245–298.

[25] R. Dipper, G. James and A. Mathas, *Cyclotomic q-Schur algebras*, Math. Z., **229** (1999), 385–416.

[26] M. Ehrig and C. Stroppel, *Diagrams for perverse sheaves on isotropic Grassmannians and the supergroup $SOSp(m|2n)$*, arXiv:1306.4043.

[27] ———, *Nazarov-Wenzl algebras, coideal subalgebras and categorified skew Howe duality*, arXiv:1310.1972.

[28] A. Ellis, M. Khovanov, and A. Lauda, *The odd nilHecke algebra and its diagrammatics*, Int. Math. Res. Notices, **4** (2014), 991–1062.

[29] N. Enomoto and M. Kashiwara, *Symmetric crystals and affine Hecke algebras of type B*, Proc. Japan Acad. Ser. A Math. Sci. **82**, (2006), 131–136.

[30] J. A. Green, *Polynomial Representations of GL_n*, Lecture Notes in Math., vol. 830, Springer-Verlag, 1980.

[31] I. Grojnowski, *Affine $\widehat{\mathfrak{sl}}_p$ controls the modular representation theory of the symmetric group and related Hecke algebras*, arXiv:math/9907129.

[32] V. Ginzburg, N. Guay, E. Opdam, and R. Rouquier, *On the category $\mathcal{O}$ for rational Cherednik algebras*, Invent. Math., **154** (2003), 617–651.

[33] J. Hu and A. Mathas, *Quiver Schur algebras I: linear quivers*, arXiv:1110.1699.

[34] S.-J. Kang and M. Kashiwara, *Categorification of highest weight modules via Khovanov-Lauda-Rouquier algebras*, Invent. Math., **190** (2012), 699–742.

[35] S.-J. Kang, M. Kashiwara, and S. Tsuchioka, *Quiver Hecke superalgebras*, arXiv:1107.1039.

[36] M. Khovanov, *A categorification of the Jones polynomial*, Duke Math. J., **101** (2000), 359–426.

[37] ———, *A functor-valued invariant of tangles*, Alg. Geom. Topology, **2** (2002), 665–741.

[38] M. Khovanov and A. Lauda, *A diagrammatic approach to categorification of quantum groups III*, Quantum Topology, **1** (2010), 1–92.

[39] S. Koenig, H. Slungård, and C. Xi, *Double centralizer properties, dominant dimension and tilting modules*, J. Algebra, **240** (2001), 393–412.

[40] H. Kraft and C. Procesi, *Closures of conjugacy classes of matrices are normal*, Invent. Math., **53** (1979), 227–247.

[41] A. Lauda and M. Vazirani, *Crystals from categorified quantum groups*, Advances Math., **228** (2011), 803–861.

[42] I. Losev, *Highest weight $\mathfrak{sl}_2$-categorifications I: crystals*, Math. Z., **274** (2013), 1231–1247.

[43] ———, *Proof of Varagnolo-Vasserot conjecture on cyclotomic categories $\mathcal{O}$*, arXiv: 1305.4894.

[44] I. Losev and B. Webster, *On uniqueness of tensor products of irreducible categorifications*, arXiv:1303.4617.

[45] G. Lusztig, *Introduction to Quantum Groups*, Birkhäuser, 1993.

[46] V. Mazorchuk and C. Stroppel, *Projective-injective modules, Serre functors and symmetric algebras*, J. Reine Angew. Math., **616** (2008), 131–165.

[47] R. Rouquier, *2-Kac-Moody algebras*, arXiv:0812.5023.

[48] ———, *Quiver Hecke algebras and 2-Lie algebras*, Algebra Colloq., **19** (2012), 359–410.

[49] R. Rouquier, P. Shan, M. Varagnolo, and E. Vasserot, *Categorification and cyclotomic rational double affine Hecke algebras*, arXiv:1305.4456.

[50] W. Soergel, *Kategorie $\mathcal{O}$, perverse Garben und Moduln über den Koinvarianten zur Weylgruppe*, J. Amer. Math. Soc., **3** (1990), 421–445.

[51] C. Stroppel, *TQFT with corners and tilting functors in the Kac-Moody case*, arXiv: math/0605103.

[52] ———, *Parabolic category $\mathcal{O}$, perverse sheaves on Grassmannians, Springer fibres and Khovanov homology*, Compositio Math., **145** (2009), 954–992.

[53] S. Tsuchioka, *Hecke-Clifford superalgebras and crystals of type $D_l^{(2)}$*, Publ. Res. Inst. Math. Sci. **46** (2010), 423–471.

[54] M. Varagnolo and E. Vasserot, *Cyclotomic double affine Hecke algebras and affine parabolic category $\mathcal{O}$*, Advances Math., **225** (2010), 1523–1588.

[55] ———, *Canonical bases and KLR-algebras*, J. Reine Angew. Math., **659** (2011), 67–100.

[56] ———, *Canonical bases and affine Hecke algebras of type B*, Invent. Math., **185** (2011), 593–693.

[57] W. Wang, *Spin Hecke algebras of finite and affine types*, Advances Math., **212** (2007), 723–748.

[58] B. Webster, *Canonical bases and higher representation theory*, arXiv:1209.0051.

[59] ———, *Tensor product algebras in type A are Koszul*, arXiv:1209.2777.

[60] ———, *Knot invariants and higher representation theory*, arXiv:1309.3796.

Department of Mathematics, University of Oregon, Eugene, OR 97403, USA
E-mail: brundan@uoregon.edu

Boundaries, rigidity of representations, and Lyapunov exponents

Uri Bader and Alex Furman

Abstract. In this paper we discuss some connections between measurable dynamics and rigidity aspects of group representations and group actions. A new ergodic feature of familiar group boundaries is introduced, and is used to obtain rigidity results for group representations and to prove simplicity of Lyapunov exponents for some dynamical systems.

Mathematics Subject Classification (2010). Primary 37A; Secondary 22E.

Keywords. Boundary theory, isometric ergodicity, characteristic maps, superrigidity, Lyapunov exponents.

1. Introduction

Boundary theory is a broad term referring to constructions of auxiliary spaces that are used to analyze asymptotic properties of spaces and groups, to study representations and group actions, and for other applications. The topics discussed in this paper revolve around rigidity phenomena, inspired by Margulis' superrigidity, and are then connected to the problem of simplicity of the Lyapunov exponents in classical dynamics. Much of the work on which this paper is based is yet unpublished. So rather than aiming at outmost generality, we chose to illustrate the main ideas by presenting key results and to include sketches of their proofs. Results about representations have natural cocycle versions; and while we focus here on real Lie groups much of the work can be extended to algebraic groups over more general fields.

Notations. The abbreviation for locally compact secondly countable group is *lcsc*. We shall use symbols G, H, S, and even Γ to denote lcsc groups; with Γ being often discrete countable group, and G, H mostly used for real Lie groups, or (real points of) algebraic groups over $\mathbb{R}$.

By an action $\Gamma \curvearrowright X$ of a group Γ on a set X we mean a map $\Gamma \times X \to X$, $(g, x) \mapsto g.x$, so that $e.x = x$ and $gh.x = g.(h.x)$ for every $g, h \in \Gamma$, $x \in X$. If Γ is a lcsc group, a *Borel* Γ*-space* X is a standard Borel space X with a Γ-action for which $\Gamma \times X \to X$ is a Borel map. A *Lebesgue* Γ*-space* is a Borel Γ-space X with a Borel probability measure m on X that is quasi-invariant under every $g \in \Gamma$, i.e. $g_*m \sim m$ for all $g \in \Gamma$. A Lebesgue Γ-space (X, m) is *ergodic* if the only measurable $E \subset X$ with $m(g^{-1}E \triangle E) = 0$ for every $g \in \Gamma$ satisfies $m(E) = 0$ or $m(E) = 1$. The notion of a Lebesgue Γ-space depends only on the measure class $[m]$ of m; a reference to m will often be omitted from the notation for Lebesgue Γ-spaces. If a Lebesgue Γ-space X has a probability measure m that is actually

Proceedings of International Congress of Mathematicians, 2014, Seoul

Γ-invariant, i.e. $m(g^{-1}E) = m(E)$ for every measurable $E \subset X$ and $g \in \Gamma$, we will say that the action $\Gamma \curvearrowright (X, m)$ is *probability measure preserving* (p.m.p.). If X is a topological space (in particular a compact metrizable space) and the action map $\Gamma \times X \to X$ is continuous, we say that X is a *topological* Γ*-space* (a *compact* Γ*-space*). A topological Γ-space X is *proper* if for every compact subset $Q \subset X$ the set $\{g \in \Gamma \mid gQ \cap Q \neq \emptyset\}$ is precompact in Γ. We shall use the notation X^Γ for the set of Γ-fixed points in X.

Let Γ be a lcsc group, X a Lebesgue Γ-space, and V a Borel Γ-space. A measurable Γ-*equivariant function* is a Borel function $f : X \to V$ such that for every $g \in \Gamma$, $f(g.x) = g.f(x)$ for a.e. $x \in X$. We denote by $\mathrm{Map}_\Gamma(X, V)$ the space of equivalence classes of Γ-equivariant functions, where functions that agree m-a.e. are identified. We shall use the term Γ-map to describe such a class $\phi \in \mathrm{Map}_\Gamma(X, V)$. By a Γ-map $p : X \to Y$ between Lebesgue Γ-spaces we mean a Γ-map for which $p[\mu] = [\nu]$, where $[\mu]$ denotes the Γ-invariant measure class on X and $[\nu]$ the one on Y. Given a Lebesgue Γ-space X we denote by $\mathrm{Aut}_\Gamma(X)$ the group of invertible self Γ-maps $X \to X$, i.e. measure class preserving bijections $X \to X$, defined up to null sets, that commute with Γ.

2. Boundaries

In this section we introduce a version of the concept of a boundary, or rather boundary pair (Definition 2.3), and discuss the basic properties of this notion. A more detailed discussion will appear in [4]. In our context a Γ-boundary is a Lebesgue Γ-space, and as such it may have many realizations on topological Γ-spaces. Furthermore, even as a Lebesgue space a Γ-boundary may not be unique.

Isometric ergodicity. A Lebesgue Γ-space (X, m) is *isometrically ergodic* if for any isometric action $\Gamma \to \mathrm{Iso}(M, d)$ on a separable metric space (M, d), any Γ-map $f : X \to M$ is essentially constant; in which case its essential value is a Γ-fixed point. In short,

$$\mathrm{Map}_\Gamma(X, M) = \mathrm{Map}(X, M^\Gamma).$$

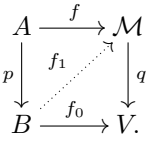
Isometric ergodicity implies ergodicity, by taking the two point space $M = \{0, 1\}$ with the trivial Γ-action. Isometric ergodicity is a natural strengthening of *ergodicity with unitary coefficients*, introduced by Burger and Monod [14], where one considers only Hilbert spaces M with unitary Γ-representations. For p.m.p. actions $\Gamma \curvearrowright (X, m)$ both notions are equivalent to weak-mixing (cf. [23]). However, here we shall be mostly interested in Lebesgue Γ-spaces that have no invariant probability measure in the relevant measure class.

Next we introduce a *relative* notion of isometric ergodicity, or equivalently, isometric ergodicity of Γ-maps between Lebesgue Γ-spaces. We first define a relative notion of a metric space. Given a Borel map $q : \mathcal{M} \to V$ between standard Borel spaces, a metric on q is a Borel function $d : \mathcal{M} \times_V \mathcal{M} \to [0, \infty)$ whose restriction d_v to each fiber $M_v = q^{-1}(\{v\})$ is a separable metric. A *fiber-wise isometric* Γ*-action* on such $\mathcal{M}$ consists of q-compatible actions $\Gamma \curvearrowright \mathcal{M}$, $\Gamma \curvearrowright V$, so that the maps between the fibers $g : M_v \to M_{g.v}$ are isometries, i.e.

$$d_{g.v}(g.x, g.y) = d_v(x, y) \qquad (x, y \in M_v,\ v \in V,\ g \in \Gamma).$$

Definition 2.1. A map $p : A \to B$ between Lebesgue Γ-spaces is *relatively isometrically ergodic* if for every fiber-wise isometric Γ-action on $\mathcal{M} \to V$ as above, and for any q-

compatible Γ-maps $f : A \to \mathcal{M}$, $f_0 : B \to V$, there is a compatible Γ-map $f_1 : B \to \mathcal{M}$ making the following diagram commutative:



Note that isometric ergodicity relative to the trivial action on a point is just the (absolute) isometric ergodicity. Let us list without proofs some basic properties of the notion of relatively isometrically ergodic maps. Some of them are reminiscent of properties of relatively weakly mixing extensions in the context of p.m.p. actions. In fact, for p.m.p. actions, or more generally relatively p.m.p. maps between Lebesgue Γ-spaces, relative isometric ergodicity is equivalent to relative weak mixing. However this remark will play no role in the sequel.

Proposition 2.2.

(i) *The property of relative isometric ergodicity is closed under composition of Γ-maps.*

(ii) *If $A \to B \to C$ are Γ-maps, and $A \to C$ is relatively isometrically ergodic, then so is $B \to C$, but $A \to B$ need not be relatively isometrically ergodic.*

(iii) *For Lebesgue Γ-spaces A and B, if the projection $A \times B \to B$ is relatively isometrically ergodic then A is (absolutely) isometrically ergodic. This is an "if and only if" in case B is a p.m.p action, but not in general.*

(iv) *If Γ is a lattice in a lcsc group G, and $p : A \to B$ is a relatively isometrically ergodic G-maps between G-spaces, then $p : A \to B$ remains relatively isometrically ergodic if viewed as a Γ-map between Γ-spaces.*

Boundary pairs. Recall the notion of an amenable action, or an amenable Lebesgue Γ-space in the sense of Zimmer [36]. We shall use the fact that if B is an amenable Lebesgue Γ-space, then given a metrizable compact Γ-space M, the set $\mathrm{Map}_\Gamma(B, \mathrm{Prob}(M))$ is non-empty, i.e. there exist Γ-map $\phi : B \to \mathrm{Prob}(M)$. More generally, given an affine Γ-action on a convex weak-* compact set $Q \subset E^*$, where E is a separable Banach space, there exists a Γ-map $\phi \in \mathrm{Map}_\Gamma(B, Q)$.

Definition 2.3. Let Γ be a lcsc group. A pair (B_-, B_+) of Lebesgue Γ-spaces forms a *boundary pair* if the actions $\Gamma \curvearrowright B_-$ and $\Gamma \curvearrowright B_+$ are amenable, and the projections

$$B_- \times B_+ \longrightarrow B_-, \qquad B_- \times B_+ \longrightarrow B_+$$

are relatively isometrically ergodic. A Lebesgue Γ-space B for which (B, B) is a boundary pair will be called a Γ-*boundary*.

Remarks 2.4.

(1) If (B_-, B_+) is a boundary pair for Γ, then $B_- \times B_+$ is isometrically ergodic. This follows by applying Propositions 2.2.(iii) and (i) to maps

$$B_- \times B_+ \longrightarrow B_-, \qquad B_- \times B_+ \longrightarrow B_+ \longrightarrow \{*\}.$$

Therefore a Γ-boundary in the sense of Definition 2.3, is also doubly ergodic with unitary coefficients, i.e. is a *strong* Γ-*boundary* in the sense of Burger-Monod [14].

(2) Every lcsc group Γ admits boundary/ies in the above sense, see Theorem 2.7 below.

(3) Being a boundary is inherited by lattices: for any lcsc group G any G-boundary B is also a Γ-boundary for any lattice $\Gamma < G$.

(4) Let B_1 be a G_1-boundary, and B_2 be a G_2-boundary for some lcsc groups G_1, G_2. Then $B = B_1 \times B_2$ is a $G_1 \times G_2$-boundary.

Some Examples. Most examples of boundaries that are used in rigidity theory turn out to have the properties stated in Definition 2.3. Let us outline the proofs in two basic cases.

Theorem 2.5. *Let G be a connected semi-simple Lie group, $P < G$ be a minimal parabolic subgroup. Then $B = G/P$ with the Lebesgue measure class is a G-boundary, and is a Γ-boundary for any lattice $\Gamma < G$.*

Proof. Since P is amenable, $G \curvearrowright G/P$ is an amenable action (cf. [36]). So it remains to show that the projection $G/P \times G/P \longrightarrow G/P$ is relatively isometrically ergodic. Typical (here from the measurable point of view) pairs g_1P, g_2P intersect along a coset of the centralizer $A' = \mathcal{Z}_G(A)$ of a maximal split torus $A < P$. So as a Lebesgue G-space $G/P \times G/P$ is the same as G/A', and the projection corresponds to the map $gA' \mapsto gP$.

The following is a version of Mautner's Lemma.

Lemma 2.6. *The P-space P/A' is isometrically ergodic.*

Proof. There is a natural correspondence between P-equivariant maps $P/A' \to M$ from the transitive P-action on P/A', and the P-orbits of A'-fixed points $x_0 \in M$. Mautner's phenomenon in this context, is the statement that in an isometric action $P \to \mathrm{Iso}(M, d)$ any A'-fixed point x_0 is fixed also by all elements $u \in P$ for which one can find a sequence $a_n \in A'$ with $a_n^{-1} u a_n \to e$. Indeed, using continuity of the homomorphism $P \to \mathrm{Iso}(M, d)$, for an A'-fixed x_0 we have

$$d(u.x_0, x_0) = d(ua_n.x_0, a_n.x_0) = d(a_n^{-1}ua_n^{-1}.x_0, x_0) \to d(x_0, x_0) = 0.$$

There x_0 is fixed by any such u. The Lemma is proven because the minimal parabolic $P < G$ is generated by A' and elements u as above, so x_0 is P-fixed, and the corresponding map is constant. □

The relative isometric ergodicity for the transitive G-actions $\pi : G/A' \to G/P$ follows formally from the isometric ergodicity of a.e. stabilizer $\mathrm{Stab}_G(gP)$ on its fiber $\pi^{-1}(\{gP\})$, but these are isomorphic to the action $P \curvearrowright P/A'$ which is isometrically ergodic by Mautner's Lemma 2.6. This proves that G/P is a G-boundary. This property is inherited by any lattice $\Gamma < G$, so $\Gamma \curvearrowright G/P$ is also a boundary action. □

For products of groups $G = G_1 \times \cdots \times G_n$ one can use the product of G_i-boundaries of the factors $B = B_1 \times \cdots \times B_n$ to obtain a G-boundary (Remark 2.4.(4)). Thus this result can be extended to products of semi-simple groups over various fields.

Let us now show that any lcsc group Γ has Γ-boundaries. Specifically we shall show that the Furstenberg-Poisson boundary for a generating spread-out random walk on Γ forms a boundary pair in the sense of Definition 2.3. This strengthens the result of Kaimanovich [28] showing ergodicity with unitary coefficients for $\Gamma \curvearrowright B \times \check{B}$ below.

Theorem 2.7 (Furstenberg-Poisson boundaries). *Let Γ be a lcsc group and μ be a spread-out generating measure on Γ. Denote by (B,ν) and $(\check{B},\check{\nu})$ the Furstenberg-Poisson boundaries for (Γ,μ) and $(\Gamma,\check{\mu})$. Then $(\check{B},B)$ is a boundary pair for Γ and for any of its lattices. Taking a symmetric spread-out generating μ, the Furstenberg-Poisson boundary $B=\check{B}$ is a Γ-boundary.*

Proof. Amenability of the actions $\Gamma \curvearrowright B$, $\Gamma \curvearrowright \check{B}$ is well known (Zimmer [36]), so it remains to prove relative isometric ergodicity. It suffices to treat one of the projections, say

$$\mathrm{pr}_B : B \times \check{B} \longrightarrow B, \qquad \mathrm{pr}_B(x,y) = x.$$

We shall do so by establishing the following stronger property, whose proof uses a combination of Martingale Convergence Theorem for the $\check{\mu}$-random walk (see (2.1) below), and Poincaré recurrence for a non-invertible p.m.p. skew-product (see (2.2)).

Lemma 2.8. *Given a positive $\nu\times\check{\nu}$-measure subset $E\subset B\times\check{B}$ and $\epsilon>0$ there is $g\in\Gamma$ and a positive ν-measure subset $C\subset \mathrm{pr}_B(E)\cap \mathrm{pr}_B(g^{-1}E)$ so that for $x\in C$*

$$\check{\nu}(g(E_x)) > 1-\epsilon$$

where $E_x=\{y\in\check{B}\mid (x,y)\in E\}$.

Proof. Denote by $(\Omega,\mu^{\mathbb{N}})$ the infinite product space $(\Gamma,\mu)^{\mathbb{N}}$. The Furstenberg-Poisson boundary (B,ν) of (Γ,μ) can be viewed as a quotient $\mathbf{bnd} : (\Omega,\mu^{\mathbb{N}}) \to (B,\nu)$, where $\mathbf{bnd}(\omega)$ is the limit of the paths of the μ-random walk

$$\mathbf{bnd}(\omega) = \lim_{n\to\infty} \pi_n(\omega), \qquad \pi_n(\omega)=\omega_1\omega_2\cdots\omega_n$$

with the convergence being understood as convergence of values of bounded μ-harmonic functions (cf. [17, 28, 29]). The $\check{\mu}$-boundary $(\check{B},\check{\nu})$ can also be viewed as a quotient of $(\Omega,\mu^{\mathbb{N}})$, using

$$\check{\mathbf{bnd}}(\omega) = \lim_{n\to\infty} \check{\pi}_n(\omega), \qquad \check{\pi}_n(\omega)=\omega_1^{-1}\omega_2^{-1}\cdots\omega_n^{-1}.$$

By the Furstenberg-Poisson formula, every measurable set $D\subset\check{B}$ defines a bounded $\check{\mu}$-harmonic function $h_D:\Gamma\to[0,1]$ by

$$h_D(g) = \int_{\check{B}} 1_D(y)\, dg_*\check{\nu}(y) = \check{\nu}(g^{-1}D).$$

Furthermore, by the Martingale Convergence Theorem, for $\mu^{\mathbb{N}}$-a.e. $\omega\in\Omega$ we have

$$h_D(\check{\pi}_n(\omega)) \to 1_D(\check{\mathbf{bnd}}(\omega)). \tag{2.1}$$

In particular, the set $\Omega_D=\{\omega\in\Omega\mid h_D(\check{\pi}_n(\omega))=\nu(\omega_n\cdots\omega_1 D)\to 1\}$ satisfies $\mu^{\mathbb{N}}(\Omega_D)=\check{\nu}(D)$. Given $E\subset B\times\check{B}$ with $\nu\times\check{\nu}(E)>0$ and $\epsilon>0$, consider the measurable sets

$$E^* = \{(\omega,x)\in\Omega\times B\mid \omega\in\Omega_{E_x}\},$$
$$E^*_N = \{(\omega,x)\in E^*\mid \forall n\geq N,\ \ \check{\nu}(\omega_n\cdots\omega_1 E_x)>1-\epsilon\}.$$

We have

$$\mu^{\mathbb{N}} \times \nu(E^*) = \int_B \mu^{\mathbb{N}}(\Omega_{E_x})\, d\nu(x) = \int_B \check{\nu}(E_x)\, d\nu(x) = \nu \times \check{\nu}(E) > 0.$$

Since E^*_N increase to E^*, we can find N large enough so that $\nu \times \check{\nu}(E^*_N) > 0$.

The fact that ν is μ-stationary implies that the following skew-product transformation

$$S : (\omega_1, \omega_2, \dots, x) \mapsto (\omega_2, \omega_3, \dots, \omega_1.x) \qquad \text{on} \qquad \Omega \times B \tag{2.2}$$

preserves the probability measure $\mu^{\mathbb{N}} \times \nu$. Therefore, Poincaré recurrence implies that we can find

$$n > N \qquad \text{so that} \qquad \mu^{\mathbb{N}} \times \nu(S^{-n}(E^*_N) \cap E^*_N) > 0.$$

Denote $F = S^{-n}(E^*_N) \cap E^*_N$ and let $F_\omega = \{x \in B \mid (\omega, x) \in F\}$. By Fubini, there is a positive $\mu^{\mathbb{N}}$-measure set of ω, for which $\nu(F_\omega) > 0$. Fix such an ω and set

$$g = \check{\pi}_n(\omega)^{-1} = \omega_n \cdots \omega_1, \qquad C = F_\omega.$$

Then $C \subset \mathrm{pr}_B(E^*_N) \subset \mathrm{pr}_B(E)$ and for every $x \in C$ one has $\check{\nu}(gE_x) > 1 - \epsilon$. □

Let us now complete the proof of the Theorem by showing how the property described in the Lemma implies relative isometric ergodicity. First consider an arbitrary Borel probability measure β on a metric space (M, d), and for a small radius $\rho > 0$ define $w(m, \rho) = \beta(\mathrm{Ball}(m, \rho))$. We point out that for β-a.e. $m \in M$ one has $w(m, \rho) > 0$ (this is easier to see for separable spaces). Note also that β is a Dirac mass δ_m iff for every $\epsilon > 0$ there exists $m' \in M$ with $w(m', \epsilon) > 1 - \epsilon$.

Now consider a fiber-wise isometric Γ-action on some $q : \mathcal{M} \to V$ and pair of compatible Γ-maps $f : B \times \check{B} \to \mathcal{M}$, $f_0 : B \to V$. For ν-a.e. $x \in B$, the pushforward of $\check{\nu}$ by $f(x, -)$ is a probability measure β_x on the fiber $q^{-1}(\{f_0(x)\})$ that we shall denote (M_x, d_x). To construct the required map $f_1 : B \to \mathcal{M}$ we will show that a.e. β_x is a Dirac measure and define f_1 by $\beta_x = \delta_{f_1(x)}$. Assuming this is not the case, there exists $\epsilon > 0$ and a positive measure set $A \subset B$ so that the function

$$w_x(y, \rho) = \beta_x\left(\mathrm{Ball}_{d_x}(f(x, y), \rho)\right)$$

satisfies $w_x(y, \epsilon) < 1 - \epsilon$ for all $(x, y) \in A \times \check{B}$. Since $\check{\nu} \times \nu$-a.e. $w_x(y, \epsilon) > 0$, there exists a measurable map $A \to \check{B}$, $x \mapsto y_x$, with $w_x(y_x, \epsilon) > 0$. Then the set

$$E = \{(x, z) \in A \times \check{B} \mid z \in \mathrm{Ball}_{d_x}(f(x, y_x), \epsilon)\}$$

has positive measure, and by Lemma 2.8, there is $C \subset \mathrm{pr}_B(E) = A$ and $g \in \Gamma$ so that for $x \in C$ one has $g.x \in C \subset A$ and

$$1 - \epsilon < \check{\nu}(g\mathrm{Ball}_{d_x}(f(x, y_x), \epsilon)) = \check{\nu}(\mathrm{Ball}_{d_{g.x}}(f(g.x, g.y_x), \epsilon)) = w_{g.x}(g.y_x, \epsilon).$$

This contradiction completes the proof that $(B, \check{B})$ is a Γ-boundary pair. □

Let us add a purely geometric example.

Example 2.9. Let M be a compact Riemannian manifold of negative curvature, $\partial\tilde{M}$ the boundary of the universal cover $\tilde{M}$ of M, and let ν_o^{PS} be the Patterson-Sullivan measure relative to some $o \in \tilde{M}$. Then $\partial\tilde{M}$ with the Patterson-Sullivan class is a Γ-boundary for the fundamental group $\Gamma = \pi_1(M)$.

The relative isometric ergodicity in this context can be shown using an analogue of Lemma 2.8, whose proof in this case would use Poincaré recurrence of the geodesic flow on the unit tangent bundle SM with Bowen-Margulis-Sullivan measure, combined with Lebesgue differentiation instead of Martingale convergence used in the preceding proof. However, both Example 2.9 and Theorem 2.7, can also be established in a different way using Theorem 5.6 below. The proof of the latter is inspired by Kaimanovich [28].

3. Characteristic maps

One of the applications of boundaries is a construction of characteristic maps (a.k.a. boundary maps) associated to representations of the group, or to cocycles of ergodic p.m.p. actions of the group. In particular, characteristic maps play a key role in higher rank superrigidity (see [18; 33, Chapters V, VI; 38]). In this section we shall illustrate the use of relative isometric ergodicity by deducing special properties of characteristic maps in three settings: for convergence actions, actions on the circle, and linear representations over $\mathbb{R}$. All the results have natural analogues in the context of measurable cocycles over ergodic p.m.p. actions, but we shall not state these results here.

Convergence actions. Let G be a lcsc group and M be a compact G-space. For $n \geq 2$ we denote by $M^{(n)}$ the subset of M^n consisting of distinct n-tuples, that is

$$M^{(n)} = \{(m_i) \in M^n \mid m_i \neq m_j \text{ if } i \neq j \in \{1, \dots, n\}\}.$$

The G-action $G \curvearrowright M$ is a *convergence action*, if the diagonal G-action on $M^{(3)}$ is proper. To avoid trivial examples we assume G is not compact and $\mathrm{card}(M) > 2$ (in which case $\mathrm{card}(M) = 2^{\aleph_0}$). Subgroups of G that stabilize a point, or an unordered pair of points in M, are called *elementary*.

Examples of convergence actions include (but not restricted to) non-elementary groups of isometries of proper δ-hyperbolic spaces acting on their Gromov boundary. This includes Gromov-hyperbolic groups and their non-elementary subgroups, relatively hyperbolic groups and other examples. In the case of relatively hyperbolic groups peripheral subgroups are elementary.

Remark 3.1. Any convergence action $G \curvearrowright M$ has a unique minimal G-invariant closed subset $L(G) \subset M$. Given a closed subgroup $H < G$, both $H \curvearrowright M$ and $H \curvearrowright L(H) \subset M$ are convergence actions. So given a group Γ and a homomorphism $\rho : \Gamma \to G$ with unbounded and non-elementary image in a group with a convergence action $G \curvearrowright M$, upon replacing G by $\overline{\rho(\Gamma)}$ and M by $L(\overline{\rho(\Gamma)})$, we may assume $\rho(\Gamma)$ to be dense in G and $G \curvearrowright M$ to be a minimal convergence action. To avoid trivial situations, one assumes that $\rho(\Gamma)$ is non-elementary and not precompact in G.

Theorem 3.2. *Let Γ be an lcsc group, (B_+, B_-) a boundary pair for Γ, $G \curvearrowright M$ a convergence action, and $\rho : \Gamma \to G$ a homomorphism with $\rho(\Gamma)$ non-elementary and not precompact in G. Then there exist Γ-maps $\phi_+ : B_+ \to M$, $\phi_- : B_- \to M$ such that the image of*

$$\phi_{\bowtie} = \phi_+ \times \phi_- : B_- \times B_+ \to M^2$$

is essentially contained in $M^{(2)}$ and

(i) $\mathrm{Map}_\Gamma(B_-, \mathrm{Prob}(M)) = \{\delta \circ \phi_-\}, \quad \mathrm{Map}_\Gamma(B_+, \mathrm{Prob}(M)) = \{\delta \circ \phi_+\}.$

(ii) $\mathrm{Map}_\Gamma(B_- \times B_+, M) = \{\phi_- \circ \mathrm{pr}_-, \phi_+ \circ \mathrm{pr}_+\},$

(iii) $\mathrm{Map}_\Gamma(B_- \times B_+, M^{(2)}) = \{\phi_\bowtie,\ \tau \circ \phi_\bowtie\}$ *where* $\tau(m, m') = (m', m)$.

Sketch of the proof. Let $G \curvearrowright \Sigma$ be a proper action of G on some locally compact separable space (e.g. $\Sigma = M^{(3)}$). We claim that

$$\mathrm{Map}_\Gamma(B_- \times B_+, \Sigma) = \emptyset.$$

Indeed, properness implies that the quotient Σ/G is Hausdorff and the stabilizers $K_s = \mathrm{Stab}_G(s)$, $s \in \Sigma$, are compact subgroups. Ergodicity of $B_- \times B_+$ implies that any Γ-map $\Psi : B_- \times B_+ \to \Sigma$ essentially ranges into a single G-orbit $G.s_0 \cong G/K_{s_0}$. It is easy to see that there exists a compact subgroup $K < G$ such that $K_{s_0} < K$ and G/K carries a G-invariant metric (e.g, K is the stabilizer of a K_{s_0}-invariant positive function in $L^2(G)$). We obtain a Γ-invariant map $B_- \times B_+ \to G/K$, which is constant by isometric ergodicity, and we conclude that $\rho(\Gamma)$ is contained in a conjugate of K. Thus the existence of such Ψ contradicts the assumption that $\rho(\Gamma)$ is not precompact.

By amenability of $\Gamma \curvearrowright B_\pm$ we may choose Γ-maps $\Phi_\pm \in \mathrm{Map}_\Gamma(B_\pm, \mathrm{Prob}(M))$. Consider the function $\Psi \in \mathrm{Map}_\Gamma(B_- \times B_+, \mathrm{Prob}(M^3))$ defined by

$$\Psi : (x, y) \mapsto \Phi_-(x) \times \Phi_+(y) \times \frac{\Phi_-(x) + \Phi_+(y)}{2} \in \mathrm{Prob}(M^3).$$

Since $G \curvearrowright M^{(3)}$ is proper, the action $G \curvearrowright \mathrm{Prob}(M^{(3)})$ is also proper, and using the above argument with $\Sigma = \mathrm{Prob}(M^{(3)})$, it follows that Ψ is supported on the big diagonal $\Delta_{12} \cup \Delta_{23} \cup \Delta_{31}$, where $\Delta_{ij} = \{(m_1, m_2, m_3) \in M^3 \mid m_i = m_j\}$. This implies that for a.e. (x, y) the measure $\Psi(x, y)$ is atomic with at most two atoms, and consequently that $\Phi_-(x)$ and $\Phi_+(y)$ must be Dirac measures. We define ϕ_- by $\Phi_-(x) = \delta_{\phi_-(x)}$, and ϕ_+ by $\Phi_+(y) = \delta_{\phi_+(y)}$. We also conclude that the essential image of any Γ-map $B_+ \to \mathrm{Prob}(M)$ consists of δ-measure. It follows that such a map is unique: indeed, given $\Phi'_- : B_- \to \mathrm{Prob}(M)$ we may also consider the map

$$x \mapsto \frac{\Phi_-(x) + \Phi'_-(x)}{2}$$

and conclude that a.e $\Phi'_-(x) = \Phi_-(x)$. A similar argument applies to give uniqueness of $y \mapsto \delta_{\phi+(y)}$ as an element of $\mathrm{Map}_\Gamma(B_+, \mathrm{Prob}(M))$. This proves (i).

Given $\psi \in \mathrm{Map}_\Gamma(B_- \times B_+, M)$, consider $\Psi = \psi \times (\phi \circ \mathrm{pr}_-) \times (\phi \circ \mathrm{pr}_+)$ as a Γ-map $B_- \times B_+ \to M^3$. Since $\Gamma \curvearrowright M^{(3)}$ is a proper action, it follows that Ψ takes values in $M^3 \setminus M^{(3)}$, and more specifically in Δ_{12} or in Δ_{13} (because Δ_{23} is impossible), which gives (ii), while (iii) easily follows from (ii). □

Note that this proof used only the amenability of B_-, B_+ and isometric ergodicity of their product, $B_- \times B_+$, but did not rely on the relative isometric ergodicity of the projections.

Actions on the circle. Consider an action of some group Γ on a circle S^1 by homeomorphisms. Up to passing to an index two subgroup, we may assume the action to be orientation preserving, and obtain a homomorphism $\Gamma \to \mathrm{Homeo}_+(S^1)$. Hereafter we shall assume that the action has no finite orbits. It is well known that in such case Γ has a unique minimal set $K \subset S^1$, and either $K = S^1$, or K is a Cantor set. In the latter case, $S^1 \setminus K$ is a countable

dense set of open arcs; collapsing these arcs one obtains a degree one map $h : S^1 \to S^1$ that intertwines the given Γ-action with a minimal Γ-action on the circle.

Given a minimal Γ-action on the circle the following dichotomy holds (see Margulis [34], Ghys [22]): either $\rho(\Gamma)$ is equicontinuous, in which case it is conjugate into rotation group $\mathrm{SO}(2)$, or the centralizer Z of $\rho(\Gamma)$ in $\mathrm{Homeo}_+(S^1)$ is a finite cyclic group, and the $|Z|$-to-1 cover $g : S^1 \to S^1/Z$ intertwines the given minimal Γ-action with a *minimal and strongly proximal*[1] one, where the latter term can be taken to mean that for any proper closed arc $J \subsetneq S^1$ and any non-empty open arc $U \neq \emptyset$ there is $g \in \Gamma$ with $gJ \subset U$. To sum up, any group action with only infinite orbits is semi-conjugate either to rotations, or to a minimal and strongly proximal action. We shall focus on the latter class of actions.

Theorem 3.3. *Let $\Gamma \to \mathrm{Homeo}_+(S^1)$ be a minimal and strongly proximal action on the circle, and let (B_-, B_+) be a boundary pair for Γ. Then there exist Γ-maps $\phi_+ : B_+ \to S^1$, $\phi_- : B_- \to S^1$ such that the image of*

$$\phi_{\bowtie} = \phi_- \times \phi_+ : B_- \times B_+ \to (S^1)^2$$

is essentially contained in the space of distinct pairs $(S^1)^{(2)}$, and

(i) $\mathrm{Map}_\Gamma(B_-, \mathrm{Prob}(S^1)) = \{\delta \circ \phi_-\}, \quad \mathrm{Map}_\Gamma(B_+, \mathrm{Prob}(S^1)) = \{\delta \circ \phi_+\}.$

(ii) $\mathrm{Map}_\Gamma(B_- \times B_+, S^1)$ *has a canonical cyclic order.*

Sketch of the proof. Following Ghys [22] we note that

$$d(\mu_1, \mu_2) = \max\left\{|\mu_1(J) - \mu_2(J)| \,:\, J \subset S^1 \text{ is an arc}\right\},$$

is a $\mathrm{Homeo}_+(S^1)$-invariant metric on the subspace $\mathrm{Prob}_c(S^1)$ of all continuous (i.e. atomless) probability measures on S^1. This shows that there are no Γ-maps from any isometrically ergodic Γ-space A to $\mathrm{Prob}_c(S^1)$, because $\mathrm{Prob}_c(S^1)^\Gamma = \emptyset$ under the assumption of minimality and strongly proximality of $\rho(\Gamma)$.

By amenability there exist $\Phi_\pm \in \mathrm{Map}_\Gamma(B_\pm, \mathrm{Prob}(S^1))$, and the above argument shows that they take values in atomic measures. We claim that $\Phi_\pm = \delta_{\phi_\pm}$ for some unique $\phi_\pm \in \mathrm{Map}_\Gamma(B_\pm, S^1)$. Indeed, fix $w > 0$ and let $A_-(x) = \{a \in S^1 \mid \Phi_-(x)(\{a\}) > w\}$ denote the set of atoms of $\Phi_-(x)$ of weight $\geq w$, and define $A_+(y)$ similarly. Then, for $w > 0$ small enough, $x \mapsto A_-(x)$ is a Γ-equivariant assignment of non-empty finite subsets of S^1; and by ergodicity the cardinality of $A_-(x)$ is a.e. constant $k \in \mathbb{N}$. Similarly for $A_+(y)$.

Let us say that (x, y) is a *good pair* if $A_-(x)$ is *unlinked* with $A_+(y)$, i.e. they belong to disjoint arcs. Since the set of good pairs is Γ-invariant, it is either null or conull in $B \times \check{B}$ by ergodicity. Choose proper closed arcs $I, J \subsetneq S^1$ so that $E = \{x \in B \mid A_-(x) \subset J\}$ has $\nu(E) > 0$, and $F = \{y \mid A_+(y) \subset I\}$ has $\check{\nu}(F) > 0$. By minimality and strong proximality, there exists $g \in \Gamma$ with $gJ \cap I = \emptyset$. Then $gE \times F$ is a positive measure set of good pairs. Hence for a.e. (x, y) the sets $A_-(x)$ and $A_+(y)$ are unlinked.

For a.e. fixed $x \in B$, the complement $S^1 \setminus A_-(x) = \bigsqcup_{i=1}^k U_i(x)$ is a disjoint union of k open arcs $U_i(x)$, where the enumeration is cyclic and $x \mapsto U_1(x)$ can be assumed to be measurable. The Γ-action cyclically permutes these intervals: $\rho(g)U_i(x) = U_{\pi_x(i)}(g.x)$ by some

[1] In general, a Γ-action $\Gamma \curvearrowright M$ on a compact metrizable M is minimal and strongly proximal if for every $\nu \in \mathrm{Prob}(M)$ the closure $\overline{\Gamma.\nu} \subset \mathrm{Prob}(M)$ contains $\delta_M = \{\delta_x \mid x \in M\}$.

$\pi_x \in \mathrm{Sym}_k$. The fact that $A_-(x)$ is unlinked from a.e. $A_+(y)$ means that $A_+(y) \in U_{i(x,y)}(x)$ for some measurable $i : B_- \times B_+ \to \{1, \dots, k\}$, while relative isometric ergodicity of $B_- \times B_+ \to B_-$ implies that $i(x,y) = i(x)$ is essentially independent of $y \in B_+$. Thus for a.e. $x \in B_-$, the closure $J(x) = \overline{U_{i(x)}(x)}$ contains a.e. $A_+(y)$. By minimality of the Γ-action on S^1, it follows that $J(x) = S^1$ and $k = 1$. As $w > 0$ was arbitrary, it follows that $\Phi_-(x) = \delta_{\phi_-(x)}$ for some $\phi_- \in \mathrm{Map}_\Gamma(B_-, S^1)$. Similarly we get $\Phi_+ = \delta \circ \phi_+$ for a unique $\phi_+ \in \mathrm{Map}_\Gamma(B_+, S^1)$.

For a.e. $(x,y) \in B_- \times B_+$ the given orientation of S^1 defines a cyclic order on every triple in $\mathrm{Map}_\Gamma(B_- \times B_+, S^1)$ by evaluation. This order is Γ-invariant, and is therefore a.e. constant by ergodicity. This shows (ii). □

It is possible that $\mathrm{Map}_\Gamma(B_- \times B_+, S^1) = \{\phi_- \circ \mathrm{pr}_-, \phi_+ \circ \mathrm{pr}_+\}$; but short of proving this we will rely on the cyclic order (ii) that would suffice for our arguments. We note also that the concept of relative isometric ergodicity allows to improve the argument from [11] that was based only on double ergodicity. See also [12, §2] for a different argument that gives the above result in the special case of B being a Furstenberg-Poisson boundary.

Linear representations. Let G be a connected, center-free, simple, non-compact, real Lie group, $P < G$ a minimal parabolic subgroup, $A < P$ a maximal split torus, and $A' = \mathcal{Z}_G(A)$ its centralizer. Since $A' < P$ one has a natural G-equivariant projection $\mathrm{pr}_1 : G/A' \to G/P$. The Weyl group of G can be defined as $\mathcal{N}_G(A)/\mathcal{Z}_G(A) = \mathcal{N}_G(A')/A' = \mathrm{Aut}_G(G/A')$; and can also be used to parameterize $\mathrm{Map}_G(G/A', G/P)$. If $\mathrm{w}_{\mathrm{long}} \in \mathrm{Weyl}_G$ denotes the *long element* of this Coxeter group, then $\mathrm{pr}_2 = \mathrm{pr}_1 \circ \mathrm{w}_{\mathrm{long}} : G/A' \to G/P$ is the opposite projection, so that

$$\mathrm{pr}_1 \times \mathrm{pr}_2 : G/A' \to G/P \times G/P$$

is an embedding, whose image is the *big* G-orbit. For $G = \mathrm{PSL}_d(\mathbb{R})$, $A < P$ are the diagonal and the upper triangular subgroups, G/A' is the space of d-tuples $(\ell_1, \dots, \ell_d)$ of 1-dimensional subspaces that span $\mathbb{R}^d$, $\mathrm{Weyl}_G \cong \mathrm{Sym}_d$ acts by permutations, $\mathrm{w}_{\mathrm{long}}$ is the order reversing involution $j \mapsto (n+1-j)$, G/P is the space of flags $(E_1, \dots, E_d)$ consisting of nested vector subspaces $E_j < E_{j+1}$ with $\dim(E_j) = j$, and

$$\mathrm{pr}_1 : (\ell_1, \dots, \ell_d) \mapsto (\ell_1, \ell_1 \oplus \ell_2, \dots, \ell_1 \oplus \dots \oplus \ell_d = \mathbb{R}^d),$$
$$\mathrm{pr}_2 : (\ell_1, \dots, \ell_d) \mapsto (\ell_d, \ell_{d-1} \oplus \ell_d, \dots, \ell_1 \oplus \dots \oplus \ell_d = \mathbb{R}^d).$$

The image of $\mathrm{pr}_1 \times \mathrm{pr}_2$ consists of pairs of flags that are in a general position.

If $\mathrm{rank}_{\mathbb{R}}(G) = 1$, then one can identify G/A' with the space of oriented but unparameterized geodesic lines in the symmetric space X of G, G/P with sphere at infinity $\partial_\infty X$, $G/A' \to G/P$ associating the limit at $-\infty$ of the geodesic, and $\mathrm{Weyl}_G \cong \mathbb{Z}/2\mathbb{Z}$ acting by flipping the orientation/ endpoints of the geodesics. The image of G/A' in $G/P \times G/P$ consists of all distinct pairs. In this case $G \curvearrowright G/P$ is a convergence action. Zariski dense subgroups are non-elementary and not precompact in G (conversely, a non-elementary non-precompact subgroup in a rank one G is Zariski dense in a possibly smaller rank one subgroup $G' < G$). Hence the following result in the special case of $\mathrm{rank}_{\mathbb{R}}(G) = 1$ can also be deduced from Theorem 3.2.

Theorem 3.4. *Let Γ be an lcsc group, (B_+, B_-) a boundary pair for Γ, G a non-compact connected simple Lie group, and $\rho : \Gamma \to G$ a homomorphism. Assume that $\rho(\Gamma)$ is*

Zariski dense in G. Then there exist Γ-maps $\phi_- : B_- \to G/P$, $\phi_+ : B_+ \to G/P$ and $\phi_\bowtie : B_- \times B_+ \to G/A'$ such that

$$\mathrm{pr}_1 \circ \phi_\bowtie(x,y) = \phi_-(x), \qquad \mathrm{pr}_2 \circ \phi_\bowtie(x,y) = \phi_+(y) \qquad (x \in B_-,\, y \in B_+),$$

and

(i) $\mathrm{Map}_\Gamma(B_-, \mathrm{Prob}(G/P)) = \{\delta \circ \phi_-\}, \quad \mathrm{Map}_\Gamma(B_+, \mathrm{Prob}(G/P)) = \{\delta \circ \phi_+\},$

(ii) $\mathrm{Map}_\Gamma(B_- \times B_+, G/P) = \{\mathrm{pr}_1 \circ w \circ \phi_\bowtie \mid w \in \mathrm{Weyl}_G\},$

(iii) $\mathrm{Map}_\Gamma(B_- \times B_+, G/A') = \{w \circ \phi_\bowtie \mid w \in \mathrm{Weyl}_G\}.$

As we have mentioned the above theorem is aimed at higher rank target groups G, where Weyl_G has more than just $\{e, \mathrm{w}_{\mathrm{long}}\}$. In the forthcoming paper [4] we prove a general version of Theorem 3.4, which is valid for algebraic groups G defined over an arbitrary local field (in fact over any spherically complete field). The proof uses the formalism of representations of ergodic actions, developed in our recent paper [7]. There we show that for every ergodic Lebesgue Γ-space X there exists an algebraic subgroup $H < G$ and $\phi \in \mathrm{Map}_\Gamma(X, G/H)$ having the following universal property: for every G-variety V and $\psi \in \mathrm{Map}_\Gamma(X, V)$ there exists a G-algebraic morphism $\pi : G/H \to V$ so that $\psi = \pi \circ \phi$ a.e. on X (this is closely related to Zimmer's notion of algebraic hull, see also [13]). We apply this result to our setting and let $\phi_+ : B_+ \to G/H_+$, $\phi_- : B_- \to G/H_-$ and $\phi_0 : B_+ \times B_- \to G/H_0$ be the corresponding universal Γ-maps. Theorem 3.4 follows easily once we show that $H_+ = H_- = P$ and $H_0 = A'$ up to conjugations. It is precisely this point, where relative isometric ergodicity of $\mathrm{pr}_\pm : B_- \times B_+ \to B_\pm$ is used.

Sketch of the proof. We first explain that the amenability of B_+ implies that H_+ is amenable. Indeed, there exists a boundary map $B_+ \to \mathrm{Prob}(G/P)$ and the ergodicity of B_+ implies that its image is essentially contained in a unique G-orbit, as the G-orbits on $\mathrm{Prob}(G/P)$ are locally closed [38]. We get a map $B_+ \to G.\mu \simeq G/G_\mu$ for some $\mu \in \mathrm{Prob}(G/P)$. The stabilizer in G of any point of $\mathrm{Prob}(G/P)$ is amenable and algebraic (we work over $\mathbb{R}$). In particular, $V = G/G_\mu$ is algebraic, and by the universal property of H_+ there exists a G-map $G/H_+ \to G/G_\mu$. Thus, up to conjugation, $H_+ < G_\mu$. In particular, H_+ is amenable. Similarly H_- is amenable.

Considering the composed map $\phi : B_- \times B_+ \to B_+ \to G/H_+ = V$ and using the universal property of $\phi_0 : B_- \times B_+ \to G/H_0$, we get a G-map $\pi : G/H_0 \to G/H_+$ such that $\pi \circ \phi_0 = \phi_+ \circ \mathrm{pr}_+$. We assume, as we may (by conjugating), that $H_0 < H_+$. Denoting by R_+ the unipotent radical of H_+, we obtain the containment $H_0 < H_0 R_+ < H_+$, and the corresponding G-maps $G/H_0 \to G/H_0R_+ \to G/H_+$. We get the following commutative diagram

$$\begin{array}{ccc} B_- \times B_+ & \longrightarrow & G/H_0R_+ \\ {\scriptstyle \mathrm{pr}_+}\downarrow & \overset{\psi}{\nearrow} & \downarrow{\scriptstyle q} \\ B_+ & \xrightarrow{\phi_+} & G/H_+ \end{array}$$

in which the existence of the map ψ is guaranteed by the isometric ergodicity of pr_+. Indeed, q is fiber-wise Γ-isometric as its fibers are H_+/R_+-homogeneous spaces and therefore carry an H_+/R_+-invariant metric, as the latter group is compact by abelian (since it is reductive and amenable). By the universal property of ϕ_+, we conclude that q is an isomorphism. We

therefore obtain $H_+ = H_0 R_+$. Similarly, denoting by R_- the unipotent radical of H_- we obtain $H_- = H_0 R_-$.

The composed $\phi_+ \times \phi_- : B_+ \times B_- \to G/H_0 \to G/H_+ \times G/H_-$ is $\Gamma \times \Gamma$-equivariant, hence the Zariski closure of its essential image is $\rho(\Gamma) \times \rho(\Gamma)$-invariant. Since $\rho(\Gamma)$ is Zariski dense in G, it follows that the image of G/H_0 is Zariski dense in $G/H_+ \times G/H_-$. Equivalently, the set $R_+ H_0 R_-$ is Zariski dense in G. At this point the proof reduces to the following algebraic group theoretic lemma.

Lemma 3.5. *Let G be a reductive group, $H_0 < H_+, H_- < G$ algebraic subgroups. Assume that $H_+ = H_0 R_+$, $H_- = H_0 R_-$ and $R_+ H_0 R_-$ is Zariski dense in G, where R_+ and R_- are the unipotent radicals of H_+ and H_- correspondingly. Then H_+ and H_- are opposite parabolics in G, and H_0 is their intersection.*

Finally, by the amenability of H_+, H_- these parabolics must be minimal in G, H_0 conjugate to A', and the result follows. □

4. Applications to rigidity

Let us now demonstrate how boundary theory can be used to obtain restrictions on linear representations, convergence actions, and actions on the circle. These results are inspired by the celebrated Margulis' superrigidity [32, 33], and the developments that followed, including [15, 21, 38]. Our aim is to illustrate the techniques rather than to obtain most general results, in particular we do not state the cocycle versions of the results that can be obtained by similar methods.

Convergence action of a lattice in a product. Consider a homomorphism $\rho : \Gamma \to G$ where $G \curvearrowright M$ is a convergence action. In view of Remark 3.1, we may assume $\rho(\Gamma)$ is dense in G and $G \curvearrowright M$ is a minimal convergence action, $\mathrm{card}(M) = 2^{\aleph_0}$ and G is non-compact.

Theorem 4.1. *Let $S = S_1 \times \cdots \times S_n$ be a product of lcsc groups, $\Gamma < S$ a lattice, such that $\mathrm{pr}_i(\Gamma)$ is dense in S_i for each $i \in \{1, \ldots, n\}$. Assume $G \curvearrowright M$ is a minimal convergence action, G is not compact, $\mathrm{card}(M) > 2$, and $\rho : \Gamma \to G$ is a continuous homomorphism with a dense image. Then for some $i \in \{1, \ldots, n\}$ there exists a continuous homomorphism $\bar{\rho} : S_i \to G$ such that $\rho = \bar{\rho} \circ \mathrm{pr}_i$.*

Sketch of the proof for $n = 2$. Choose a boundary B_i for each S_i, for example using Theorem 2.7, and set $B = B_1 \times B_2$. Then B is an S-boundary and a Γ-boundary (Remark 2.4). By Theorem 3.2 we have a unique Γ-map $\phi : B \to M$. Consider the map

$$\Phi : B \times B = B_1 \times B_2 \times B_1 \times B_2 \longrightarrow M^2, \quad (x, y, x', y') \mapsto (\phi(x, y), \phi(x, y')).$$

By Theorem 3.2(iii) we have three cases: $\Phi(B \times B)$ is contained in the diagonal $\Delta \subset M^2$, $\Phi = \phi_{\bowtie}$, or $\Phi = \tau \circ \phi_{\bowtie}$, where $\phi_{\bowtie} = \phi \times \phi$ and $\tau(m, m') = (m', m)$. In the first case we see that $\phi(x, y)$ is independent of $y \in B_2$, and therefore descends to a Γ-map $B_1 \to M$. In the second case, ϕ is independent of $x \in B_1$, and descends to $B_2 \to M$. The third case gives that ϕ is independent of both parameters, thus its essential image is a Γ-fixed point in M. This is incompatible with $\rho(\Gamma)$ being non-elementary. We conclude that for some $i \in \{1, 2\}$, $\phi : B_1 \times B_2 \to M$ factors through B_i. We shall apply the following general lemma, letting $X = B_i$, and $\Lambda = \mathrm{pr}_i(\Gamma)$, which is dense in $T = S_i$.

Lemma 4.2. *Let T be a lcsc group, $\Lambda < T$ a dense subgroup, M a compact metrizable space, $G < \mathrm{Homeo}(M)$ a closed subgroup and $\rho : \Lambda \to G$ a homomorphism. Assume that there exists a Lebesgue T-space (X, μ), a Λ-map $\phi : X \to M$ so that the G-action on M with $\eta = \phi_* \mu \in \mathrm{Prob}(M)$ satisfies the following condition:*

$(*)$ *a sequence $\{g_n\}$ in G satisfies $g_n \to e$ in G if (and only if)*

$$\int_M (h \circ g_n - h) \cdot k \, d\eta \to 0 \qquad (h, k \in C(M)) \tag{4.1}$$

Then $\rho : \Lambda \to G$ extends to a continuous homomorphism $\bar{\rho} : T \to G$.

Furthermore, the combination of the following two conditions implies the condition $(*)$ defined above:

$(*)_1$ $(g_n)_* \eta \to \eta$ *in weak-* topology* $\implies$ $\{g_n\}$ *is bounded in G,*

$(*)_2$ $\forall g \in G \setminus \{e\}$, $\exists k, h \in C(M)$ *so that* $\int_M (h \circ g - h) \cdot k \, d\eta \neq 0$.

Proof. Since Λ is dense in T, existence of a continuous extension $\bar{\rho} : T \to G$ is equivalent to showing that $g_n = \rho(\lambda_n) \to e$ in G for every sequence $\{\lambda_n\}$ in Λ with $\lambda_n \to e$ in T. The T-action by pre-composition on $L^\infty(X, \mu)$, equipped with the weak-* topology from $L^1(X, \mu)$, is continuous. Take $\Lambda \ni \lambda_n \to e$ in T, functions $h, k \in C(M)$, and define $\tilde{h} \in L^\infty(X, \mu)$, $\tilde{k} \in L^1(X, \mu)$ by $\tilde{h} = h \circ \phi$, $\tilde{k} = k \circ \phi$. Then

$$\int_M (h \circ \rho(\lambda_n) - h) \cdot k \, d\eta = \int_X (\tilde{h} \circ \lambda_n - \tilde{h}) \cdot \tilde{k} \, d\mu \to 0.$$

Hence (4.1) detects the convergence $\rho(\lambda_n) \to e$ in G.

To see that $(*)_1 + (*)_2 \implies (*)$, note that for any sequence $\{g_n\}$ in G with (4.1), there is weak-* convergence $(g_n)_* \eta \to \eta$ by taking $k = 1$ and varying $h \in C(M)$. Thus $\{g_n\}$ is precompact in G by $(*)_1$. Condition $(*)_2$ implies that e is the only possible limit point for $\{g_n\}$. This completes the proof of the Lemma. □

To complete the proof of Theorem 4.1 we check conditions $(*)_1$, $(*)_2$. In our context $\mathrm{supp}(\eta) = M$, because $\mathrm{supp}(\eta)$ is a $\rho(\Gamma)$-invariant closed subset of M, while $\rho(\Gamma)$ is dense in G and $G \curvearrowright M$ is minimal. This implies $(*)_2$. For $(*)_1$ observe that it follows from the convergence property of $G \curvearrowright M$ that if $g_i \to \infty$ in G and $(g_i)_* \eta \to \xi \in \mathrm{Prob}(M)$, then ξ is supported on one or two points, while $\mathrm{supp}(\eta) = M$ is a continuum. Therefore the conditions of Lemma are satisfied, and we get a continuous extension $\bar{\rho} : S_i \to G$ as claimed. □

Weyl groups. The argument showing that the map $\phi \in \mathrm{Map}_\Gamma(B, M)$ factors through one of the boundaries B_i in the proof of Theorem 4.1, might appear to be ad-hoc. But in fact, it can be made conceptual as follows. Given a group Γ and a choice of a Γ-boundary, define the associated *generalized Weyl group* to be

$$\mathrm{W}_{\Gamma, B} = \mathrm{Aut}_\Gamma(B \times B),$$

the group of measure class preserving automorphisms of $B \times B$ that commute with Γ. For non-amenable Γ, a Γ-boundary cannot be trivial, so $\mathrm{W}_{\Gamma, B}$ always contains the non-trivial involution $w_{\mathrm{flip}} : (x, y) \mapsto (y, x)$.

Example 4.3. For a boundary which is a product of Γ-spaces, $B = \prod_{i\in I} B_i$, the generalized Weyl group contains a subgroup isomorphic to $\prod_{i\in I} \mathbb{Z}/2\mathbb{Z}$ obtained by flipping factors of $B \times B \simeq \prod_{i\in I}(B_i \times B_i)$.

Given a Borel Γ-space V, $\mathrm{W}_{\Gamma,B}$ acts on $\mathrm{Map}_\Gamma(B\times B, V)$ by precompositions. For any Γ-map $\phi : B \to V$ we obtain a subgroup of $\mathrm{W}_{\Gamma,B}$ - the stabilizer of $\phi\circ \mathrm{pr}_+ \in \mathrm{Map}_\Gamma(B\times B, V)$ under this action. The subgroups obtained this way are called *special subgroups*.

It is easy to check that the special subgroups of $\prod_{i\in I} \mathbb{Z}/2\mathbb{Z}$ in Example 4.3 are the subgroups of the form $\prod_{i\in J} \mathbb{Z}/2\mathbb{Z}$ for $J \subset I$. In the setting of convergence actions, Theorem 3.2(ii) shows that the action of $\mathrm{W}_{\Gamma,B}$ on $\mathrm{Map}_\Gamma(B \times B, M)$ must factor trough a group of order two. The kernel of this action is clearly a special subgroup, it is the stabilizer of $\phi \circ \mathrm{pr}_+$, and one deduces that $\phi : \prod B_i \to M$ factors through $B_i \to M$ for some $i \in I$. Invoking now Lemma 4.2, one obtains this way an alternative proof of Theorem 4.1.

Considering now a lattice in a product of groups acting on a circle, we may apply a similar strategy. By Theorem 3.3(ii) the action of $\mathrm{W}_{\Gamma,B}$ on $\mathrm{Map}_\Gamma(B \times B, S^1)$ factors through a cyclic subgroup. Considering again the subgroup $\prod_{i\in I} \mathbb{Z}/2\mathbb{Z}$ and its special subgroups, we conclude that $\phi : \prod B_i \to S^1$ factors through $B_i \to S^1$ for some $i \in I$. The extension Lemma 4.2 applies to $G = \mathrm{Homeo}_+(S^1)$, and one deduces the following.

Theorem 4.4. *Let Γ be an irreducible lattice in a product $S = S_1 \times \cdots \times S_n$ of lcsc groups as in Theorem 4.1. Let $\rho : \Gamma \to \mathrm{Homeo}_+(S^1)$ be a minimal strongly proximal action on the circle. Then, ρ extends to a continuous homomorphism that factors through some S_i, namely $\bar\rho_i : S_i \to \mathrm{Homeo}_+(S^1)$ so that $\rho = \bar\rho_i \circ \mathrm{pr}_i$. Moreover, if $\bar\rho_i(S_i)$ is non-discrete, then it could be conjugated to $\mathrm{PSL}_2(\mathbb{R}) < \mathrm{Homeo}_+(S^1)$, so Γ may be assumed to act via fractional linear transformations.*

The addendum about $\mathrm{PSL}_2(\mathbb{R})$ follows from the general fact that a lcsc group acting minimally and strongly proximally on the circle is either discrete, or could be conjugated into $\mathrm{PSL}_2(\mathbb{R})$. Let us also remark, that under some mild assumptions (e.g. Γ is finitely generated and projects injectively to the S_i-factors) one can prove that if $\rho(S)$ is non-discrete then, up to finite index and a compact factor, $\Gamma < S$ is an arithmetic lattice in a finite product of a real and possible p-adic algebraic groups, one of which is $\mathrm{PSL}_2(\mathbb{R})$, and Γ acts on the circle through this factor [10].

Next consider a connected simple Lie group S with $\mathrm{rank}_{\mathbb{R}}(S) \geq 2$. Let $B = S/Q$ where $Q < S$ is a minimal parabolic. It is an S-boundary by Theorem 2.5. As a Lebesgue S-space, $B \times B = S/Q \times S/Q \cong S/A'$, where A' is the centralizer of a maximal split torus $A < Q$. The generalized Weyl group $\mathrm{W}_{S,S/Q}$, consisting of automorphisms of S/A' as a Lebesgue S-space, is easily seen to coincide with the classical Weyl group $\mathrm{Weyl}_S = \mathcal{N}_S(A)/\mathcal{Z}_S(A) = \mathcal{N}_S(A')/A'$:

$$\mathrm{W}_{S,S/Q} \cong \mathrm{Weyl}_S\,.$$

Let $\Gamma < S$ be a lattice. Then $B = S/Q$ is also a Γ-boundary and $\mathrm{W}_{\Gamma,S/Q}$ contains Weyl_S. This inclusion is an isomorphism ([9]) and the three notions of special subgroups: of $\mathrm{W}_{\Gamma,S/Q}$, of $\mathrm{W}_{S,S/Q}$, and of Weyl_S seen as a Coxeter group, all coincide [6]. Since S is assumed to be simple, the Coxeter group Weyl_S is irreducible. It is not hard to see that if W is an irreducible Coxeter group and $W' < W$ a proper special subgroup then the action of W on the coset space W/W' is faithful: W' contains no nontrivial subgroup which is normal in W. It follows that for any Borel Γ-space V with $V^\Gamma = \emptyset$, the action of $\mathrm{W}_{\Gamma,S/Q}$ on the orbit of $\phi \circ \mathrm{pr}_+ \mathrm{Map}_\Gamma(S/Q \times S/Q, V)$ for $\phi \in \mathrm{Map}_\Gamma(S/Q, V)$ is faithful. This allows to deduce the following result of Ghys [21].

Theorem 4.5. *Let Γ be a lattice in a connected simple Lie group S with $\mathrm{rank}_{\mathbb{R}}(S) \geq 2$. Then any Γ-action on the circle has a finite orbit.*

Indeed, assuming Γ has an action on the circle with only infinite orbits, one could find a minimal such action by applying a semiconjugation. Since Γ cannot act minimally by rotations (because $\Gamma/[\Gamma,\Gamma]$ is finite), it would also have a minimal strongly proximal action. Theorem 3.3(ii) then guarantees that the action of $\mathrm{Weyl}_S \simeq \mathrm{W}_{\Gamma,S/Q}$ on $\mathrm{Map}_\Gamma(S/Q \times S/Q, S^1)$ factors through a cyclic quotient, contradicting its faithfulness because Weyl_G is not cyclic for a higher rank G.

Similarly, we have the following result that might be seen as a generalization of the special case of Margulis superrigidity stating that any homomorphism from a higher rank lattice Γ into a rank one group has precompact image.

Theorem 4.6. *Let Γ be a lattice in a connected simple Lie group S with $\mathrm{rank}_{\mathbb{R}}(S) \geq 2$. Then for any homomorphism $\rho : \Gamma \to G$ where $G \curvearrowright M$ is a non-trivial convergence action, $\rho(\Gamma)$ is elementary or precompact in G.*

In [6] the basic idea of the last result is developed further for a class of target groups that includes mapping class groups, automorphism groups of finite dimensional CAT(0) cubical complexes etc.

Taking the target group G to be a connected simple real Lie group one can use Theorem 3.4 to obtain the following.

Theorem 4.7. *Let Γ be a lcsc group, B a Γ-boundary, G a connected, non-compact, simple real Lie group, and $\rho : \Gamma \to G$ a homomorphism with Zariski dense image. Then there exists a homomorphism*

$$\pi : \mathrm{W}_{\Gamma,B} \longrightarrow \mathrm{Weyl}_G, \qquad \pi(w_{\mathrm{flip}}) = \mathrm{w}_{\mathrm{long}},$$

satisfying that the preimage of a special subgroup of Weyl_G is a special subgroup of $\mathrm{W}_{\Gamma,B}$. Furthermore, there is a map $\phi_{\bowtie} \in \mathrm{Map}_\Gamma(B \times B, G/A')$ satisfying

$$\phi_{\bowtie} \circ w = \pi(w) \circ \phi_{\bowtie} \qquad (w \in \mathrm{W}_{\Gamma,B}),$$

and $\phi \in \mathrm{Map}_\Gamma(B, G/P)$ such that $\phi(x) = \mathrm{pr}_1 \circ \phi_{\bowtie}(x,y)$, $\phi(y) = \mathrm{pr}_2 \circ \phi_{\bowtie}(x,y)$.

We remark that there is a natural notion of a preorder relation on $\mathrm{W}_{\Gamma,B}$, generalizing the classical Bruhat order on Coxeter groups, and one can show that the map π considered here is order preserving. We will not elaborate on this here.

Note that the theorem above could be applied in particular to a lattice Γ in a simple Lie group S. Then one deduces some cases of Margulis superrigidity, e.g. it follows that a lattice in $\mathrm{SL}_n(\mathbb{R})$ cannot have an unbounded representation in $\mathrm{SL}_m(\mathbb{R})$ if $n > m$. However, a more efficient approach to superrigidity phenomena with algebraic targets, one that avoids boundary theory almost completely, is proposed in [7].

Commensurator superrigidity. Finally, let us show how existence and uniqueness of characteristic maps can be used to prove results analogous to Margulis' commensurator superrigidity [33] (see also [13]). Let S be a lcsc group, and $\Gamma < S$ be a lattice. Recall that the commensurator of Γ in S is the subgroup of S given by

$$\mathrm{Commen}_S(\Gamma) = \left\{ s \in S \mid \Gamma \cap \Gamma^s \text{ has finite index in } \Gamma, \text{ and in } \Gamma^s \right\},$$

where $\Gamma^s = \{g^s = sgs^{-1} \mid g \in \Gamma\}$ denotes conjugation.

Theorem 4.8. *Let S be a lcsc group, $\Gamma < S$ a lattice, Λ a dense subgroup in S such that $\Gamma < \Lambda < \mathrm{Commen}_S(\Gamma)$.*

(i) *Let $G \curvearrowright M$ be a minimal convergence action and $\rho : \Lambda \to G$ be a continuous homomorphism with a dense image. Then ρ extends to a continuous homomorphism $\bar{\rho} : S \to G$.*

(ii) *Let $\rho : \Lambda \to \mathrm{Homeo}_+(S^1)$ be such that $\Lambda \curvearrowright S^1$ acts minimally and strongly proximally. Then ρ extends to a continuous homomorphism $\bar{\rho} : S \to \mathrm{Homeo}_+(S^1)$, whose image is either discrete or is conjugate to $\mathrm{PSL}_2(\mathbb{R}) < \mathrm{Homeo}_+(S^1)$.*

(iii) *Let G be a connected, simple, center-free, non-compact, real Lie group, and let $\rho : \Lambda \to G$ be a homomorphism with Zariski dense image. Then ρ extends to a continuous homomorphism $\bar{\rho} : S \to G$.*

In view of Remark 3.1, the assumption of density of $\rho(\Lambda)$ in G and minimality of $G \curvearrowright M$ is not restrictive. Minimality and strong proximality can also be assumed for the circle case, see the discussion preceding Theorem 3.3. Of course, case (iii) is a special case of the original Margulis commensurator superrigidity, that was used to give a criterion for arithmeticity of lattices in semi-simple Lie groups. To this end one needs to consider also algebraic target groups over $\mathbb{C}$, and over $\mathbb{Q}_p$ where p is a prime (cf. [13]). We include it here just to emphasize the analogy with the other cases.

Sketch of the proof. The non-degeneracy assumptions on $\rho(\Lambda)$ are already satisfied by $\rho(\Gamma)$. For example in (i), the set $L(\rho(\Gamma))$ being stable under replacing Γ by finite index subgroups is necessarily $\rho(\Lambda)$-invariant. As $\rho(\Lambda)$ is dense in G and $G \curvearrowright M$ is minimal, it follows that $L(\rho(\Gamma)) = M$. Similarly in (ii), one shows that already Γ acts minimally and strongly proximally on S^1. In (iii) $\rho(\Gamma)$ is Zariski dense, because the identity component of its Zariski closure is normalized by $\rho(\Lambda)$ and hence by all of G, but the latter is simple.

Choose an S-boundary B, say using Theorem 2.7. Then B is a boundary for Γ, and for any finite index subgroup $\Gamma' < \Gamma$ (Remark 2.4). Consider the compact G-space Q where: in (i) $Q = M$, in (ii) $Q = S^1$, in (iii) $Q = G/P$. Then using the above properties of $\rho(\Gamma)$ we obtain a characteristic Γ-map

$$\phi \in \mathrm{Map}_\Gamma(B, Q)$$

by applying just the first claim in Theorems 3.2, 3.3, 3.4 in cases (i), (ii), (iii), respectively. For any fixed $\lambda \in \Lambda$ the group $\Gamma' = \Gamma \cap \Gamma^{\lambda^{-1}}$ has finite index in Γ, and therefore is also a lattice in S. The measurable map $\psi : B \to Q$ defined by $\psi(x) = \rho(\lambda)^{-1}\phi(\lambda.x)$ is Γ'-equivariant, because for $g \in \Gamma'$ one has $g^\lambda \in \Gamma$, and so

$$\psi(g.x) = \rho(\lambda)^{-1}\phi(\lambda g.x) = \rho(\lambda)^{-1}\phi(g^\lambda \lambda.x) = \rho(\lambda^{-1}g^\lambda)\phi(\lambda.x) = \rho(g)\psi(x).$$

So both ϕ and ψ are in $\mathrm{Map}_{\Gamma'}(B, Q)$, which means that $\phi = \psi$. Thus

$$\phi(\lambda.x) = \rho(\lambda).\phi(x)$$

for a.e. $x \in B$, and this holds for every $\lambda \in \Lambda$. Hence the map $\phi : B \to Q$ is Λ-equivariant. This allows to show that $\rho : \Lambda \to G$ extends to a continuous $\bar{\rho} : S \to G$ using, for example, the extension Lemma 4.2, once conditions $(*)_1$, $(*)_2$ have been verified. For convergence groups this was done in the proof of Theorem 4.1.

Similar arguments apply to the case (ii) of the circle. One shows using minimality and strong proximality of Γ-action on S^1 that $\eta = \phi_*\nu$ is a full support continuous measure.

It easy to see that if $(g_i)_*\eta \to \xi$ for some $g_i \to \infty$ in $\mathrm{Homeo}_+(S^1)$ then ξ has atoms and cannot be η. This proves $(*)_1$, while $(*)_2$ follows from the fact that η has full support.

In case (iii) of $G \curvearrowright G/P$, the measure η is *proper* meaning that $\eta(V) = 0$ for any proper algebraic subspace $V \subset G/P$. Then $(*)_1$ follows from Furstenberg's lemma about quasi-projective-transformations ([19]) and $(*)_2$ is a consequence of properness of η.

Therefore $\rho : \Lambda \to G$ extends to a continuous homomorphism $\bar{\rho} : S \to G$ in all three cases. In the circle case (ii), there is an additional fact: a lcsc group, e.g. $\bar{\rho}(S)$, with a minimal and strongly proximal action on the circle is either discrete or could be conjugated to $\mathrm{PSL}_2(\mathbb{R})$. □

5. An application to Lyapunov exponents

In this section we shall apply boundary theory – Theorem 3.4 – to obtain results about Lyapunov exponents for some matrix valued functions on a class of p.m.p. systems. In his first proof of superrigidity Margulis used non-vanishing of Lyapunov exponents for certain matrix valued functions to construct characteristic maps; our approach ([5]) follows a converse direction.

The Multiplicative Ergodic Theorem of Oseledets [35] (see also Kaimanovich [27], Karlsson-Margulis [30]) describes the asymptotic behavior of products of matrix valued functions along orbits of a p.m.p. system. More precisely, let (X, m, T) be an invertible, ergodic, p.m.p. system, and $F : X \to \mathrm{SL}_d(\mathbb{R})$ a measurable map with

$$\int_X \log \|F(x)\| \, dm(x) < +\infty. \tag{5.1}$$

Multiplying F along T-orbits one obtains a measurable cocycle $\mathbb{Z} \times X \to \mathrm{SL}_d(\mathbb{R})$

$$F_n(x) = \begin{cases} F(T^{n-1}x) \cdots F(Tx)F(x) & \text{if} \quad n \geq 1, \\ I & \text{if} \quad n = 0, \\ F(T^n x)^{-1} \cdots F(T^{-1}x)^{-1} & \text{if} \quad n < 0. \end{cases} \tag{5.2}$$

The cocycle equation being $F_{k+n}(x) = F_k(T^n x)F_n(x)$ for $k, n \in \mathbb{Z}$. The Multiplicative Ergodic Theorem asserts that there exist: a partition $d = d_1 + \cdots + d_s$, constants

$$\gamma_1 > \gamma_2 > \cdots > \gamma_s \qquad \text{with} \qquad d_1\gamma_1 + \cdots + d_s\gamma_s = 0$$

and a measurable equivariant[2] splitting into vector subspaces

$$\mathbb{R}^d = L_1(x) \oplus L_2(x) \oplus \cdots \oplus L_s(x), \qquad \dim L_j(x) = d_j,$$

so that for m-a.e. $x \in X$ for every $v \in L_j(x) \setminus \{0\}$ one has

$$\lim_{n\to\infty} \frac{1}{n} \log \|F_n(x)v\| = \gamma_j, \qquad \lim_{n\to\infty} \frac{1}{n} \log \|F_{-n}(x)v\| = -\gamma_j.$$

Rewriting $\gamma_1 > \gamma_2 > \cdots > \gamma_s$ with their multiplicities we get the *Lyapunov exponents*

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d, \qquad \lambda_1 + \cdots + \lambda_d = 0,$$

[2] satisfying m-a.e. $L_j(T^n x) = F_n(x)L_j(x)$ for $1 \leq j \leq s$.

that can be recorded as $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_d)$; we refer to this element $\Lambda \in \mathfrak{sl}_d(\mathbb{R})$ as the *Lyapunov spectrum* of F on (X, m, T). The Multiplicative Ergodic Theorem can be restated as the assertion that a.e. sequence $\{F_n(x)\}_{n\in\mathbb{Z}}$ follows with a sub-linear deviation the sequence $\{U_x \exp(n\Lambda)V_x^{-1}\}_{n\in\mathbb{Z}}$ for some $U_x, V_x \in \mathrm{O}_d(\mathbb{R})$. More generally, given a simple real Lie group G and an integrable[3] measurable map $F : X \to G$, we define the *Lyapunov spectrum* to be the element in the positive Weyl chamber

$$\Lambda \in \mathfrak{a}_+$$

of the Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{g} = \mathrm{Lie}(G)$, so that $u_x \exp(n\Lambda)v_x^{-1}$ represent the asymptotic behavior of m-a.e. sequences $F_n(x) \in G$, $n \in \mathbb{Z}$.

The spectrum Λ of an integrable $F : X \to G$ over an ergodic invertible system (X, m, T) is *non-degenerate* if $\Lambda \neq 0$, and is called *simple* if Λ is a regular element in $\mathfrak{a}_+$. In the basic case $G = \mathrm{SL}_d(\mathbb{R})$, non-degeneracy of the spectrum corresponds to $\lambda_1 > 0$, and simplicity to the strict inequalities

$$\lambda_1 > \lambda_2 > \cdots > \lambda_d$$

in which case $s = d$ and $d_j = \dim L_j(x) = 1$ for all $1 \leq j \leq d$.

In general, there is no explicit formula for Λ (or even λ_1) in terms of $F : X \to G$ on (X, m, T), and the dependence of the Lyapunov exponents on F and (X, m, T) is mostly mysterious. The best studied situation is that of Random Walks, where (X, m) is the invertible Bernoulli shift $(G^{\mathbb{Z}}, \mu^{\mathbb{Z}})$ with $(Tx)_i = x_{i+1}$ and $F(x) = x_1$. From the fundamental work of Furstenberg [17], Guivarc'h-Raugi [25, 26], and Gol'dsheid-Margulis [24], it is known that if $\mathrm{supp}(\mu)$ generates a Zariski dense subgroup in G, then the Lyapunov spectrum is simple. More recently, Avila and Viana [1–3] gave sufficient conditions for simplicity of the Lyapunov for certain classes of systems that allowed them to prove simplicity of the Lyapunov spectrum of Kontsevich-Zorich cocycle. Here we shall describe an approach ([5]) that allows to prove simplicity of the Lyapunov spectrum using boundary theory.

Simplicity of the Lyapunov spectrum. Let (X, m, T) be an ergodic, invertible, p.m.p. system and Γ be some auxiliary group, that we assume to be countable discrete for clarity of presentation, and let

$$f : X \to \Gamma$$

be a measurable map. It generates a measurable cocycle $\mathbb{Z} \times X \to \Gamma$, denoted $f_n(x)$, similarly to (5.2). Let $\widetilde{m}$ denote the (infinite) measure on the space $\Gamma^{\mathbb{Z}}$ obtained by pushing forward the product $m \times c_\Gamma$ of m with the counting measure c_Γ on Γ by the map

$$X \times \Gamma \longrightarrow \Gamma^{\mathbb{Z}}, \qquad (x, g) \mapsto (f_n(x)g^{-1})_{n\in\mathbb{Z}}.$$

This measure describes the distribution of paths (g_i) of a stochastic walk with (not necessarily independent) increments $f(T^n x) = g_{i+1}g_i^{-1}$ that starts from an arbitrary initial value g_0. Let us write $\widetilde{X}$ for the space $\Gamma^{\mathbb{Z}}$ with the measure $\widetilde{m}$ or another measure in its measure class. The measure $\widetilde{m}$ (and its class) are preserved by the commuting actions of $\mathbb{Z}$ and Γ:

$$n : (g_i) \mapsto (g_{i+n}), \qquad g : (g_i) \mapsto (g_i g^{-1}) \qquad (n \in \mathbb{Z},\ g \in \Gamma).$$

Consider the future tail equivalence relation $\sim_+$ on $\widetilde{X}$ defined by $(g_i) \sim_+ (g'_i)$ if for some $k \in \mathbb{Z}$ one has $g_{i+k} = g'_i$ for all $i \geq i_0$. Let $B_+ = \widetilde{X}/\!\!/ \sim_+$ denote the space of $\sim_+$-ergodic

[3] (5.1) holds for some/any embedding $G < \mathrm{SL}_d(\mathbb{R})$

components. To make this more precise, one may replace $\widetilde{m}$ by an equivalent probability measure $\widetilde{m}_1$, push down $\widetilde{m}_1$ by the projection $\Gamma^{\mathbb{Z}} \to \Gamma^{\mathbb{N}}$, and take the ergodic components for the semi-group $\mathbb{N}$ acting by the shift. Then B_+ is a Lebesgue Γ-space which is a quotient of $\widetilde{X}$. Similarly, one defines the past tail equivalence relation $\sim_-$ and the corresponding Γ-quotient $\widetilde{X} \to B_-$. We shall say that the quotients $p_- : \widetilde{X} \to B_-$ and $p_+ : \widetilde{X} \to B_+$ are *weakly independent*, denoted $B_- \perp B_+$, if

$$(p_- \times p_+)[\widetilde{m}] \;=\; p_-[\widetilde{m}] \times p_+[\widetilde{m}], \tag{5.3}$$

where $[\widetilde{m}]$ denote the measure class of $\widetilde{m}$, and the equality is of measure classes.

Example 5.1. Let μ be a generating probability measure on a (countable) group Γ, (X, m, T) be the Bernoulli system $(\Gamma^{\mathbb{Z}}, \mu^{\mathbb{Z}})$ with the shift $T : (x_i) \mapsto (x_{i+1})$, and $f : X \to \Gamma$ given by $f(x) = x_1$. Then $\widetilde{X}$ is the space of paths for random walks and B_+ and B_- are the Furstenberg-Poisson boundaries for μ and $\check{\mu}$ respectively. They are weakly independent $B_+ \perp B_-$. Note that the assumption that μ is generating is essential here, for if μ is supported on a proper subgroup $\Gamma_0 < \Gamma$ then the non-trivial Γ-space Γ/Γ_0 is a common quotient of $\widetilde{X}$, B_+, B_-, and $B_+ \not\perp B_-$.

Example 5.2. Let M be a closed Riemannian manifold of negative curvature, $\Gamma = \pi_1(M)$ the fundamental group, $X = SM$ unit tangent bundle, T the time one geodesic flow on X, and m be Lebesgue-Liouville measure, or Bowen-Margulis measure, or any other Gibbs measure. We define a cocycle $f_n : X \to \Gamma$, $n \in \mathbb{Z}$, by

$$\widetilde{T}^n(\sigma(x)) = f_n(x).\sigma(T^n x)$$

where $\widetilde{T}^t$ is the geodesic flow on the unit tangent bundle $\widetilde{SM}$ of the universal cover $\widetilde{M}$, and $\sigma : SM \to \widetilde{SM}$ is the section of the covering map $\pi : \widetilde{SM} \to SM$ corresponding to a measurable choice of a fundamental domain, say a Dirichlet domain. The B_- and B_+ are then realized on the geometric boundary $\partial\widetilde{M}$ and the measure classes represent those of stable/unstable foliations. One has weak independence $B_- \perp B_+$ as a consequence of the local product structure of the conditional measures on stable/unstable leaves, and the mixing condition.

Theorem 5.3. *Let (X, m, T) be an invertible ergodic p.m.p. system, $f : X \to \Gamma$ a measurable map, so that $B_- \perp B_+$ in the above sense. Then*

(i) *Let G be a connected, non-compact, center free, simple, real Lie group and $\rho : \Gamma \to G$ a representation with Zariski dense image. Then the map*

$$F : X \xrightarrow{f} \Gamma \xrightarrow{\rho} G$$

has simple Lyapunov spectrum over (X, m, T), provided it is integrable.

(ii) *Let $\Gamma \curvearrowright (Z, \zeta)$ be an ergodic p.m.p. action, and $\rho : \Gamma \times Z \to G$ a Zariski dense cocycle into G as above. Then the skew-product*

$$(X \times Z, m \times \zeta, T_f), \qquad T_f : (x, z) \mapsto (Tx, f(x).z) \tag{5.4}$$

is ergodic, and the map $F : X \times Z \to G$, $F(x, z) = \rho(f(x), z)$, has simple Lyapunov spectrum provided it is integrable.

Non-degeneracy of the Lyapunov spectrum ($\lambda_1 > 0$) remains valid if Zariski density condition on ρ is replaced by the weaker condition that the algebraic hull of ρ is non-amenable.

Let us note some consequences of this result. In the random walk setting (Example 5.1) we recover the results of Guivarc'h-Raugi, Gol'dsheid-Margulis on simplicity of the Lyapunov spectrum for Zariski dense random walk on a simple Lie group G by applying part (i) of the theorem to $X = G^{\mathbb{Z}}$, $m = \mu^{\mathbb{Z}}$ with the shift. The addendum about non-degeneracy of the spectrum is precisely Furstenberg's condition for $\lambda_1 > 0$. Part (ii) gives already a new result:

Corollary 5.4. *Let Γ be a (countable) group, $\Gamma \curvearrowright (Z, \zeta)$ an ergodic p.m.p. action, $\rho : \Gamma \times Z \to G$ a Zariski dense cocycle. Let μ be a generating probability measure on Γ with $\log \|\rho(g, z)\| \in L^1(\mu \times \zeta)$, let $X = \Gamma^{\mathbb{Z}}$, $m = \mu^{\mathbb{Z}}$, $T : (x_i) \mapsto (x_{i+1})$, and $\overline{T} : (x, z) \mapsto (Tx, x_1.z)$. Then the cocycle $F_n : X \times Z \to G$ given by*

$$F_n(x, z) = \rho(x_n \cdots x_1, z)$$

has a simple Lyapunov spectrum. If ρ is only assumed to have non-amenable algebraic hull, then the spectrum is non-degenerate.

The result about non-degeneracy of the Lyapunov spectrum in this setting is due to Ledrappier [31].

Corollary 5.5. *Let M be a compact negatively curved manifold, T^t the geodesic flow on the unit tangent bundle $X = SM$ to M, m a Gibbs measure on X, and $f_n : X \to \Gamma = \pi_1(M)$ a cocycle as in Example 5.2. Then*

(i) *Given a Zariski dense representation $\rho : \Gamma \to G$ in a simple Lie group, the Lyapunov spectrum of $F = \rho \circ f$ is simple.*

(ii) *Given any ergodic p.m.p. action $\Gamma \curvearrowright (Z, \zeta)$ the skew-product $X \times_f Z$ is ergodic and if $\rho : \Gamma \times G$ is a Zariski dense cocycle with $\log \|\rho(g, -)\| \in L^1(Z)$, $g \in \Gamma$, then the Lyapunov spectrum of $F_n(x, z) = \rho(f_n(x), z)$ is simple.*

Part (ii) for the case where M is a constant curvature surface, can be restated as asserting that for an ergodic p.m.p. action $\mathrm{SL}_2(\mathbb{R}) \curvearrowright (X, m)$ and Zariski dense integrable cocycle $\rho : \mathrm{PSL}_2(\mathbb{R}) \times X \to G$ the restriction to the diagonal subgroup $F_n(x) = \rho(g^n, x)$ for any hyperbolic $g \in \mathrm{PSL}_2(\mathbb{R})$, has simple Lyapunov spectrum. This was recently obtained by Eskin-Matheus [16].

Theorem 5.3 can also be used to prove simplicity of the Lyapunov spectrum for the Kontsevich-Zorich cocycle (in the main stratum).

Outline of the proof. The main observation is that the setting of $f : X \to \Gamma$ and condition $B_- \perp B_+$, described above, allow one to use boundary theory.

Theorem 5.6. *Let (X, m, T) and $f : X \to \Gamma$ be as above, and assume that $B_- \perp B_+$. Then $\widetilde{X} /\!\!/ \mathbb{Z}$ is isometrically ergodic, projections $\widetilde{X} /\!\!/ \mathbb{Z} \to B_\pm$ are relatively isometrically ergodic, and (B_-, B_+) is a boundary pair for Γ.*

We shall not describe the proof of this result here, but remark that amenability of $B_\pm$ follows from amenability of $\mathbb{N}$ (as in Zimmer's [36]), and other statements reduce to relative isometric ergodicity of the maps $\widetilde{X} /\!\!/ \mathbb{Z} \to B_\pm$. The proof of this key property is motivated by Kaimanovich [28].

Observation 5.7. Let V be a Borel Γ-space, and $\mathrm{Map}_f(X,V)$ denote the space of all *f-equivariant maps*, i.e. measurable maps $\phi : X \to V$ satisfying m-a.e. $\phi(Tx) = f(x).\phi(x)$. Then there exists a natural bijection between f-equivariant maps and $\mathbb{Z}$-invariant Γ-equivariant maps $\widetilde{X} \to V$, which gives a bijection

$$\mathrm{Map}_f(X,V) \;\cong\; \mathrm{Map}_\Gamma(\widetilde{X}/\!/\mathbb{Z}, V).$$

This observation gives the following fact, that was included in the statement of Theorem 5.3.(ii).

Corollary 5.8 (of Theorem 5.6). *Let (X,m,T) and $f : X \to \Gamma$ be such that $B_- \perp B_+$. Then for any ergodic p.m.p. action $\Gamma \curvearrowright (Z,\zeta)$ the skew-product (5.4) is ergodic.*

Note that for the random walk setting (Example 5.1) this can be deduced from Kakutani's random ergodic theorem, and for the geodesic flow setting (Example 5.2) with M being *locally symmetric*, it follows from Moore's ergodicity. However it is new for geodesic flow on general negatively curved manifolds, and potentially in other situations.

Proof. The claim is that T_f-invariant functions $F \in L^2(X \times Z, m \times \zeta)$ are a.e. constant. Such an F can be viewed as a measurable f-equivariant map $X \to L^2(Z,\zeta)$, $x \mapsto F(x,-)$. By 5.7 it corresponds to a Γ-map $\Phi : \widetilde{X}/\!/\mathbb{Z} \to L^2(Z,\zeta)$. Since $\widetilde{X}/\!/\mathbb{Z}$ is isometrically ergodic (Theorem 5.6), Φ is constant $\phi_0 \in L^2(Z,\zeta)^\Gamma$. As $\Gamma \curvearrowright (Z,\zeta)$ is ergodic, ϕ_0 is ζ-a.e. a constant c_0, and F is $m \times \zeta$-a.e. constant $F(x,z) = c_0$. □

Let us outline the proof of Theorem 5.3. We focus on part (i) that refers to the simplicity of the Lyapunov spectrum of $F = \rho \circ f : X \to G$ where $\rho : \Gamma \to G$ is a Zarsiki dense representation. The proof of part (ii) that refers to cocycles follows the same outline.

By Theorem 5.6, the pair (B_-, B_+) constructed from (X,m,T) and $f : X \to \Gamma$ is a boundary pair for Γ. Therefore from Theorem 3.4 there exist Γ-maps

$$\phi_- : B_- \longrightarrow G/P, \qquad \phi_+ : B_- \longrightarrow G/P, \qquad \phi_\bowtie : B_- \times B_+ \longrightarrow G/A'$$

so that $\phi_- = \mathrm{pr}_1 \circ \phi_\bowtie$ and $\phi_- = \mathrm{pr}_2 \circ \phi_\bowtie$, where G/A' is viewed as a subset

$$G/A' \subset G/P \times G/P.$$

For $n \in \mathbb{Z}$ denote by $\mathcal{F}_{\geq n} = \sigma(f \circ T^n, f \circ T^{n+1}, \dots)$ the σ-algebra generated by the maps $f \circ T^k : X \to \Gamma$, $k \geq n$. Similarly define $\mathcal{F}_{<n} = \sigma(f \circ T^{n-1}, f \circ T^{n-2}, \dots)$. Then $\mathcal{F}_{\geq n} \subset \mathcal{F}_{\geq n-1}$ and $\mathcal{F}_{<n} \supset \mathcal{F}_{<n-1}$.

Proposition 5.9. *There exists a map $\nu_- : X \to \mathrm{Prob}(G/P)$ with the following properties:*

(i) *The map $x \mapsto \nu_-(x)$ is $\mathcal{F}_{\geq 0}$-measurable and satisfies*

$$\nu_-(x) = \mathbb{E}\left(F(T^{-1}x)_*\nu_-(T^{-1}x) \mid \mathcal{F}_{\geq 0}\right).$$

(ii) *For m-a.e. $x \in X$ there is weak-* convergence to Dirac measure*

$$\delta_{\psi_-(x)} = \lim_{n\to\infty} F(T^{-1}x)F(T^{-2}x)\cdots F(T^{-n}x)_*\nu_-(T^{-n}x),$$

where ψ_- is an F-equivariant map $X \to G/P$.

(iii) *For m-a.e. $x \in X$ the measure $\nu_-(x)$ is proper, i.e. gives zero mass to proper algebraic subspaces $W \subsetneq G/P$.*

There is a $\mathcal{F}_{<0}$-measurable map $\nu^+ : X \to \mathrm{Prob}(G/P)$ and $\psi_+ \in \mathrm{Map}_F(X, G/P)$ with similar properties with respect to T^{-1}. Moreover, there exists

$$\psi_{\bowtie} \in \mathrm{Map}_F(X, G/A'), \qquad \text{so that} \qquad \psi_- = \mathrm{pr}_1 \circ \psi_{\bowtie}, \qquad \psi_+ = \mathrm{pr}_2 \circ \psi_{\bowtie}$$

where $\mathrm{pr}_i : G/A' \to G/P$ are the projections.

Sketch of the proof. The map $\psi_- \in \mathrm{Map}_F(X, G/P)$ is defined by applying the correspondence from 5.7 to the pull-back of $\phi_- \in \mathrm{Map}_\Gamma(B_-, G/P)$ via the quotient $\widetilde{X} /\!\!/ \mathbb{Z} \to B_-$. Define ν_- to be the conditional expectation (average) of the Dirac measures $\delta_{\psi_-(x)}$

$$\nu_-(x) = \mathbb{E}\left(\delta_{\psi_-(x)} \mid \mathcal{F}_{\geq 0}\right).$$

Property (i) then follows from this definition, and (ii) follows by applying Martingale Convergence Theorem.

We shall not give here the proof of property (iii), but point out that it uses the $B_- \perp B_+$ assumption as a well as Zariski density of ρ. □

The following well known lemma allows one to prove quantitative results (linear growth of ergodic sums) from qualitative information (consistent growth of ergodic sums).

Lemma 5.10. *Let (X, m, T) be an ergodic p.m.p. system, and $h \in L^1(X, m)$ such that $h(x) + h(Tx) + \cdots + h(T^n x) \to +\infty$ for m-a.e. $x \in X$. Then $\int h\, dm > 0$.*

Contraction of measures on G/P can indicate growth.

Lemma 5.11. *Let $Q \subset \mathrm{Prob}(G/P)$ be a compact set of proper measures, $\{\nu_n\}$ a sequence in Q, and let $\{a_n\}$ be a sequence in the Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{g} = \mathrm{Lie}(G)$, so that*

$$\exp(a_n)_* \nu_n \longrightarrow \delta_{eP}.$$

Then for any positive root, $\chi : \mathfrak{a} \to \mathbb{R}$ one has $\chi(a_n) \to \infty$.

Combining these two Lemmas, one may deduce simplicity of the spectrum in the following very special situation: assume that

- an integrable $F : X \to G$ takes values in the Cartan subgroup $A = \exp(\mathfrak{a})$, so we can write $F(x) = \exp(a(x))$ for an appropriate function $a : X \to \mathfrak{a}$,
- some map $\nu : X \to \mathrm{Prob}(G/P)$, taking values in a compact $Q \subset \mathrm{Prob}(G/P)$ consisting of proper measures, satisfies weak-* convergence

$$F(T^{-1}x)F(T^{-2}x)\cdots F(T^{-n}x)_* \nu_-(T^{-n}x) \longrightarrow \delta_{eP}.$$

Then one has $\Lambda = \int a\, dm$, and since $\chi(\Lambda) = \int \chi(a(x))\, dm(x) > 0$ for every positive root χ, the spectrum Λ is simple.

Returning to the general case described in Proposition 5.9, one can use the maps $\psi_{\bowtie} : X \to G/A'$ and $\psi_- : X \to G/P$ to find a measurable $c : X \to G$ so that

$$c(Tx)F(x)c(x)^{-1} \in A = \exp(\mathfrak{a}), \qquad \text{and} \qquad c(x)\psi_-(x) = eP,$$

while all $\nu(x) = c(x)\nu_-(x)$ still remain proper measures. To arrive at the special situation described above we need to control integrability of the A-valued $c(Tx)F(x)c(x)^{-1}$ and to ensure uniform properness for $\nu(x)$. This can be achieved by passing to an *induced system* (in the sense of Kakutani) as follows. There exist compact sets $C \subset G$ and $Q \subset \mathrm{Prob}(G/P)$ where Q consists of proper measures only, so that the set

$$X^* = \{x \in X \mid c(x) \in C,\ \nu(x) \in Q\}$$

has $m(X^*) > 0$. Let m^* be the normalized restriction $m^* = m(X^*)^{-1} \cdot m|_{X^*}$, denote the first return time to X^* by $n(x) = \inf\{n \geq 1 \mid T^n x \in X^*\}$, and define

$$T^*x = T^{n(x)}x, \qquad F^*(x) = c(T^{n(x)}x)F_{n(x)}(x)c(x)^{-1}.$$

From the ergodic theorem $\int n(x)\, dm^*(x) = m(X^*)^{-1}$, and it follows that the Lyapunov spectra, Λ of F on (X, m, T) and Λ^* of F^* on (X^*, m^*, T^*), are positively proportional

$$\Lambda^* = \frac{1}{m(X^*)} \cdot \Lambda \in \mathfrak{a}^+.$$

But F^* on (X^*, m^*, T^*) satisfies the condition of the special case above, hence Λ^* is simple, and therefore so is the original $\Lambda \in \mathfrak{a}^+$.

Finally the addendum about non-degeneracy of the Lyapunov spectrum when $\rho(\Gamma)$ is just assumed to be non-amenable, follows from the simplicity criterion by considering the Levi decomposition of the Zariski closure of $\rho(\Gamma)$. This completes the outline of the proof of Theorem 5.3.

Acknowledgements. U.B was supported in part by the ERC grant 306706. A.F. was supported in part by the NSF grants DMS 1207803.

References

[1] A. Avila and M. Viana, *Extremal Lyapunov exponents: an invariance principle and applications*, Invent. Math., **181** (2010), no. 1, 115–189.

[2] ———, *Simplicity of Lyapunov spectra: a sufficient criterion*, Port. Math. (N.S.), **64** (2007), no. 3, 311–376.

[3] ———, *Simplicity of Lyapunov spectra: proof of the Zorich-Kontsevich conjecture*, Acta Math., **198** (2007), no. 1, 1–56.

[4] U. Bader and A. Furman, *Stronger boundary pairs and characteristic maps*. in preparation.

[5] ———, *Boundary theory and simplicity of the Lyapunov spectrum*. in preparation.

[6] ———, *Superrigidity via Weyl groups: hyperbolic-like targets*. preprint.

[7] ———, *Algebraic representations of ergodic actions and super-rigidity*, available at arXiv:1311.3696. preprint.

[8] ———, *Boundaries, Weyl groups, and superrigidity*, Electron. Res. Announc. Math. Sci., **19** (2012), 41–48.

[9] U. Bader, A. Furman, A. Gorodnik, and B. Weiss, *Rigidity of group actions on homogeneous spaces, III*, available at arxiv:1201.5367.

[10] U. Bader, A. Furman, and R. Sauer, *Lattice envelopes for some countable groups*. preprint.

[11] U. Bader, A. Furman, and A. Shaker, *Superrigidity via Weyl groups: actions on the circle*, available at arXiv:math/0605276.

[12] M. Burger, *An extension criterion for lattice actions on the circle*, Geometry, rigidity, and group actions, Chicago Lectures in Math., Univ. Chicago Press, Chicago, IL, 2011, pp. 3–31.

[13] M. Burger and N. A'Campo, *Réseaux arithmétiques et commensurateur d'après G. A. Margulis*, Invent. Math., **116** (1994), no. 1-3, 1–25 (French).

[14] M. Burger and N. Monod, *Continuous bounded cohomology and applications to rigidity theory*, Geom. Funct. Anal, **12** (2002), no. 2, 219–280.

[15] M. Burger and S. Mozes, CAT*(-1)-spaces, divergence groups and their commensurators*, J. Amer. Math. Soc., **9** (1996), no. 1, 57–93.

[16] A. Eskin and C. Matheus, *Semisimplicity of the Lyapunov spectrum for irreducible cocycles* (2013), available at arXiv:1309.0160[math.DS].

[17] H. Furstenberg, *A Poisson formula for semi-simple Lie groups*, Ann. of Math. (2) **77** (1963), 335–386.

[18] ———, *Rigidity and cocycles for ergodic actions of semisimple Lie groups (after G. A. Margulis and R. Zimmer)*, Bourbaki Seminar, Vol. 1979/80, Lecture Notes in Math., vol. 842, Springer, Berlin, 1981, pp. 273–292.

[19] ———, *A note on Borel's density theorem*, Proc. Amer. Math. Soc. **55** (1976), no. 1, 209–212.

[20] T. Gelander, A. Karlsson, and G. A. Margulis, *Superrigidity, generalized harmonic maps and uniformly convex spaces*, Geom. Funct. Anal., **17** (2008), no. 5, 1524–1550.

[21] É. Ghys, *Actions de réseaux sur le cercle*, Invent. Math., **137** (1999), no. 1, 199–231 (French).

[22] ———, *Groups acting on the circle*, Enseign. Math. (2), **47** (2001), no. 3-4, 329–407.

[23] E. Glasner and B. Weiss, *Weak mixing properties for nonsingular actions* (2013), available at arXiv:1308.0159[math.DS].

[24] I. Ya. Gol′dsheĭd and G. A. Margulis, *Lyapunov exponents of a product of random matrices*, Uspekhi Mat. Nauk, **44** (1989), no. 5(269), 13–60 (Russian); English transl., Russian Math. Surveys **44** (1989), no. 5, 11–71.

[25] Y. Guivarc'h and A. Raugi, *Products of random matrices: convergence theorems*, Random matrices and their applications (Brunswick, Maine, 1984), Contemp. Math., vol. 50, Amer. Math. Soc., Providence, RI, 1986, pp. 31–54.

[26] ———, *Frontière de Furstenberg, propriétés de contraction et théorèmes de convergence*, Z. Wahrsch. Verw. Gebiete, **69** (1985), no. 2, 187–242 (French).

[27] V. A. Kaimanovich, *Lyapunov exponents, symmetric spaces and a multiplicative ergodic theorem for semisimple Lie groups*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **164** (1987), no. Differentsialnaya Geom. Gruppy Li i Mekh. IX, 29–46, 196–197 (Russian, with English summary); English transl., J. Soviet Math., **47** (1989), no. 2, 2387–2398.

[28] ———, *Double ergodicity of the Poisson boundary and applications to bounded cohomology*, Geom. Funct. Anal., **13** (2003), no. 4, 852–861.

[29] V. A. Kaimanovich and A. M. Vershik, *Random walks on discrete groups: boundary and entropy*, Ann. Probab., **11** (1983), no. 3, 457–490.

[30] A. Karlsson and G. A. Margulis, *A multiplicative ergodic theorem and nonpositively curved spaces*, Comm. Math. Phys., **208** (1999), no. 1, 107–123.

[31] F. Ledrappier, *Positivity of the exponent for stationary sequences of matrices*, Lyapunov exponents (Bremen, 1984), Lecture Notes in Math., vol. 1186, Springer, Berlin, 1986, pp. 56–73.

[32] G. A. Margulis, *Discrete groups of motions of manifolds of nonpositive curvature*, Proceedings of the International Congress of Mathematicians (Vancouver, B.C., 1974), Vol. 2, Canad. Math. Congress, Montreal, Que., 1975, pp. 21–34 (Russian).

[33] ———, *Discrete subgroups of semisimple Lie groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 17, Springer-Verlag, Berlin, 1991.

[34] ———, *Free subgroups of the homeomorphism group of the circle*, C. R. Acad. Sci. Paris Sér. I Math., **331** (2000), no. 9, 669–674 (English, with English and French summaries).

[35] V. I. Oseledec, *A multiplicative ergodic theorem. Characteristic Ljapunov, exponents of dynamical systems*, Trudy Moskov. Mat. Obšč., **19** (1968), 179–210 (Russian).

[36] R. J. Zimmer, *Amenable ergodic group actions and an application to Poisson boundaries of random walks*, J. Functional Analysis, **27** (1978), no. 3, 350–372.

[37] ———, *Strong rigidity for ergodic actions of semisimple Lie groups*, Ann. of Math. (2), **112** (1980), no. 3, 511–529.

[38] ———, *Ergodic theory and semisimple groups*, Monographs in Mathematics, vol. 81, Birkhäuser Verlag, Basel, 1984.

711, Amado Mathematics building, Technion, Haifa, Israel 32000
E-mail: uri.bader@gmail.com

Department of Mathematics, Statistics, and Computer Science University of Illinois at Chicago 322 Science and Engineering Offices (M/C 249) 851 S. Morgan Street Chicago, IL 60607-7045, USA
E-mail: furman@uic.edu

Modular representation theory of symmetric groups

Alexander Kleshchev

Abstract. We review some recent advances in modular representation theory of symmetric groups and related Hecke algebras. We discuss connections with Khovanov-Lauda-Rouquier algebras and gradings on the blocks of the group algebras $F\Sigma_n$, which these connections reveal; graded categorification and connections with quantum groups and crystal bases; modular branching rules and the Mullineaux map; graded cellular structure and graded Specht modules; cuspidal systems for affine KLR algebras and imaginary Schur-Weyl duality, which connects representation theory of these algebras to the usual Schur algebras of smaller rank.

Mathematics Subject Classification (2010). Primary 20C30; Secondary 20C08, 17B37.

Keywords. Symmetric group, modular representation, Hecke algebra, quantum group.

1. Introduction

The classical problem of understanding representation theory of symmetric groups is especially difficult in positive characteristic. For example, there is no effective algorithm for computing the dimensions of irreducible modules in that case. In this expository paper, we review some results on *modular* representation theory of symmetric groups. Let F be a filed of characteristic $p > 0$ and Σ_n be the symmetric group on n letters.

We discuss connections with Khovanov-Lauda-Rouquier algebras and gradings on the blocks of $F\Sigma_n$ which these connections reveal, graded categorification and connections with quantum groups and crystal bases, modular branching rules and the Mullineaux map, graded cellular structure and graded Specht modules, cuspidal systems for affine KLR algebras and imaginary Schur-Weyl duality which connects representation theory of these algebras to the usual Schur algebras of smaller rank.

2. Graded isomorphism theorem

2.1. Basic notation. Let $I := \mathbb{Z}/p\mathbb{Z}$ be identified with the simple subfield of F. We associate to p the affine *Cartan matrix* $\mathtt{C} = (\mathtt{c}_{ij})_{i,j\in I}$ of type $\mathtt{A}^{(1)}_{\mathtt{p}-1}$ so that $\mathtt{c}_{ii} = 2, \mathtt{c}_{ij} = -1$ if $|i - j| = 1$ and $p \neq 2$, $\mathtt{c}_{0,1} = \mathtt{c}_{1,0} = -2$ if $p = 2$ and $\mathtt{c}_{ij} = 0$ otherwise.

The group algebra $F\Sigma_n$ will be denoted by H_n. We have the simple transpositions $s_r := (r, r + 1) \in \Sigma_n$ for $r = 1, \dots n - 1$, and the *Murphy elements*:

$$x_t := (1, t) + (2, t) + \cdots + (t - 1, t) \in H_n \qquad (1 \leq t \leq n).$$

Proceedings of the International Congress of Mathematicians, Seoul, 2014

The Murphy elements commute.

Let V be a finite dimensional H_n-module. The eigenvalues of the Murphy elements in V belong to I, see for example [27, Lemma 7.1.2]. Given a *word* $\boldsymbol{i} = (i_1, \dots, i_n) \in I^n$, we define the $\boldsymbol{i}$*-word space* of V as follows:

$$V_{\boldsymbol{i}} = \{v \in V \mid (x_r - i_r)^N v = 0 \text{ for } N \gg 0 \text{ and } r = 1, \dots, n\}.$$

We have a *word space decomposition* $V = \bigoplus_{\boldsymbol{i} \in I^n} V_{\boldsymbol{i}}$. Using the word space decomposition of the left regular H_n-module, we get a system of orthogonal idempotents $\{1_{\boldsymbol{i}} \mid \boldsymbol{i} \in I^n\}$ in the group algebra H_n (some of which are zero) such that $\sum_{\boldsymbol{i} \in I^n} 1_{\boldsymbol{i}} = 1$, and $1_{\boldsymbol{i}} V = V_{\boldsymbol{i}}$ for all $\boldsymbol{i} \in I^n$ and all finite dimensional H_n-modules V.

The symmetric group Σ_n acts on I^n by place permutations, and let I^n/Σ_n be the set of orbits. Fix an orbit $\alpha \in I^n/\Sigma_n$. Define $1_\alpha := \sum_{\boldsymbol{i} \in \alpha} 1_{\boldsymbol{i}} \in H_n$. It is easy to check that 1_α is a central idempotent in H_n. In fact, by [34] or [3, Theorem 1], 1_α is either zero or a primitive central idempotent in H_n. Hence the algebra

$$H_\alpha := 1_\alpha H_n$$

is either zero or a single *block* of the group algebra H_n. If $x \in H_n$ we denote $1_\alpha x \in H_\alpha$ again by x.

2.2. Graded presentation. Define special elements of H_α as follows:

$$y_t := \sum_{\boldsymbol{i} \in \alpha} (x_t - i_r) 1_{\boldsymbol{i}} \qquad (1 \le t \le n), \tag{2.1}$$

$$\psi_r := \sum_{\boldsymbol{i} \in \alpha} (s_r + P_r(\boldsymbol{i})) Q_r(\boldsymbol{i})^{-1} 1_{\boldsymbol{i}} \qquad (1 \le r < n), \tag{2.2}$$

where $P_r(\boldsymbol{i})$ and $Q_r(\boldsymbol{i})^{-1}$ are certain explicit polynomials in $F[y_r, y_{r+1}]$ defined in [6]. This gives us the following elements of H_α:

$$\{1_{\boldsymbol{i}} \mid \boldsymbol{i} \in \alpha\} \cup \{y_1, \dots, y_n\} \cup \{\psi_1, \dots, \psi_{n-1}\}. \tag{2.3}$$

Finally, choose signs $\varepsilon_{ij} \in \pm 1$ for all $i, j \in I$ with $|i - j| = 1$ so that $\varepsilon_{ij}\varepsilon_{ji} = -1$, and define the polynomials in $F[u, v]$:

$$Q_{ij}(u, v) := \begin{cases} 0 & \text{if } i = j; \\ 1 & \text{if } \mathtt{c}_{ij} = 0; \\ \varepsilon_{ij}(u^{-\mathtt{c}_{ij}} - v^{-\mathtt{c}_{ji}}) & \text{if } \mathtt{c}_{ij} < 0. \end{cases} \tag{2.4}$$

The following result was first proved in [6, Theorem 1.1], see also [38]:

Theorem 2.1. *The algebra H_α is generated by the elements (2.3) subject* only *to the following relations (for all admissible $\boldsymbol{i} = (i_1, \dots), \boldsymbol{j}, r, t$):*

$$1_{\boldsymbol{i}} 1_{\boldsymbol{j}} = \delta_{\boldsymbol{i},\boldsymbol{j}} 1_{\boldsymbol{i}}, \tag{2.5}$$

$$\textstyle\sum_{\boldsymbol{i} \in \alpha} 1_{\boldsymbol{i}} = 1; \tag{2.6}$$

$$y_r 1_{\boldsymbol{i}} = 1_{\boldsymbol{i}} y_r; \tag{2.7}$$

$$y_r y_t = y_t y_r; \tag{2.8}$$

$$\psi_r 1_{\boldsymbol{i}} = 1_{s_r \boldsymbol{i}} \psi_r; \tag{2.9}$$

$$(y_t \psi_r - \psi_r y_{s_r(t)}) 1_{\boldsymbol{i}} = \delta_{i_r, i_{r+1}} (\delta_{t,r+1} - \delta_{t,r}) 1_{\boldsymbol{i}}; \tag{2.10}$$

$$\psi_r^2 1_{\boldsymbol{i}} = Q_{i_r, i_{r+1}}(y_r, y_{r+1}) 1_{\boldsymbol{i}}; \tag{2.11}$$

$$\psi_r \psi_t = \psi_t \psi_r \qquad (|r-t| > 1); \tag{2.12}$$

$$(\psi_{r+1}\psi_r\psi_{r+1} - \psi_r\psi_{r+1}\psi_r) 1_{\boldsymbol{i}} = \delta_{i_r, i_{r+2}} \frac{Q_{i_r, i_{r+1}}(y_{r+2}, y_{r+1}) - Q_{i_r, i_{r+1}}(y_r, y_{r+1})}{y_{r+2} - y_r} 1_{\boldsymbol{i}}; \tag{2.13}$$

$$y_1^{\delta_{i_1,0}} 1_{\boldsymbol{i}} = 0. \tag{2.14}$$

We note that the ratio in (2.13) is always a polynomial in y's. Theorem 2.1 is saying that blocks of groups algebras of symmetric groups are isomorphic to certain *cyclotomic Khovanov-Lauda-Rouquier (KLR) algebras* [20, 21, 38, 39]. We return to this in Section 7. The presentation of H_α given in Theorem 2.1 allows us to define a *grading* on H_α by setting:

$$\deg(1_{\boldsymbol{i}}) := 0, \quad \deg(y_r 1_{\boldsymbol{i}}) := 2, \quad \deg(\psi_r 1_{\boldsymbol{i}}) := -\mathtt{c}_{i_r, i_{r+1}}.$$

From now on, we will always consider *graded* H_α-modules, unless otherwise stated. The irreducible ungraded H_α-modules are gradable in a unique way up to isomorphism and degree shift, so considering graded modules does divert our attention from the main goal of understanding irreducible H_n-modules.

2.3. Basics of graded algebra. For any graded algebra H, we denote by $H\text{-mod}$ the abelian category of all finitely generated graded H-modules, with morphisms $\hom_H(\cdot,\cdot)$ being *degree-preserving* module homomorphisms. Denote by $H\text{-proj}$ the full subcategory of finitely generated projective graded H-modules. Set

$$\mathcal{L} := \mathbb{Z}[q, q^{-1}].$$

The Grothendieck group $[H\text{-mod}]$ is a $\mathcal{L}$-module via $q^m[M] := [q^m M]$, where $q^m M$ denotes the module obtained by shifting the grading in M up by m, i.e. $(q^m M)_n := M_{n-m}$. For $n \in \mathbb{Z}$, let $\mathrm{Hom}_H(M,N)_n := \hom_H(q^n M, N)$, and set

$$\mathrm{Hom}_H(M,N) := \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_H(M,N)_n.$$

If M is finitely generated, forgetting all gradings, $\mathrm{Hom}_H(M,N)$ is the usual Hom.

For graded H-modules M and N we write $M \cong N$ to mean that M and N are isomorphic as graded modules and $M \simeq N$ to mean that $M \cong q^d N$ for some $d \in \mathbb{Z}$. For a finite dimensional graded vector space $V = \oplus_{d \in \mathbb{Z}} V_n$, its *graded dimension* is

$$\dim_q V := \sum_{d \in \mathbb{Z}} (\dim V_d) q^d \in \mathcal{L}.$$

Given $M, L \in H\text{-mod}$ with L irreducible, we write $[M : L]_q$ for the corresponding *graded composition multiplicity*, i.e. $[M : L]_q := \sum_{n \in \mathbb{Z}} a_d q^d$, where a_d is the multiplicity of $q^d L$ in a graded composition series of M.

Define the *formal character* of $M \in H_\alpha\text{-mod}$ as the formal sum

$$\mathrm{ch}_q M := \sum_{\boldsymbol{i} \in \alpha} (\dim_q M_{\boldsymbol{i}}) \boldsymbol{i}.$$

Consider the antiautomorphism

$$H_\alpha \to H_\alpha,\ h \mapsto h', \tag{2.15}$$

which is identity on the generators (2.3). If $M = \bigoplus_{d\in\mathbb{Z}} M_d \in H_\alpha\text{-mod}$, then the *graded dual* $M^\circledast$ is the graded H_α-module such that $(M^\circledast)_d := M^*_{-d}$, for all $d \in \mathbb{Z}$, and the H_α-action is given by $(xf)(m) = f(x'm)$, for all $f \in M^\circledast, m \in M, x \in H_\alpha$.

For every irreducible H_α-module L, there is a unique choice of the grading shift so that $L^\circledast \cong L$ [20, Section 3.2]. We always choose the shifts for irreducible H_α-modules in this way.

2.4. The KLR algebra A_α. We denote by A_α the algebra given by the generators (2.3) and the relations (2.5)–(2.13). This is the *KLR algebra* corresponding to $\mathtt{C}$. It is an infinite dimensional graded algebra with the natural surjection

$$A_\alpha \twoheadrightarrow H_\alpha. \tag{2.16}$$

All H_α-modules will be considered as A_α-modules via the functor of inflation $\mathrm{infl} : H_\alpha\text{-mod} \to A_\alpha\text{-mod}$. This functor has a left adjoint $\mathrm{pr} : A_\alpha\text{-mod} \to H_\alpha\text{-mod},\ M \mapsto H_\alpha \otimes_{A_\alpha} M$. The definition of the graded duality '$\circledast$' and the formal character ch_q for H_α-modules extends to A_α-modules.

In the important special case $\alpha = n\alpha_i$, the algebra $A_{n\alpha_i}$ is the usual affine nilHecke algebra. It has a representation on the polynomial space $P_n = F[x_1, \dots, x_n]$ with each y_t acting as multiplication by x_t and each ψ_r acting as the divided difference operator $f \mapsto ({}^{s_r}f - f)/(y_r - y_{r+1})$. The module P_n is graded so that $\deg(x_r) = 2$. We shift the degrees and define

$$P(i^{(n)}) := q^{-n(n-1)/2} P_n. \tag{2.17}$$

3. Branching and categorification

3.1. More notation. Following [19, §1.1], we have a realization of the Cartan matrix $\mathtt{C}$. In particular we have the simple roots $\{\alpha_i \mid i \in I\}$, the fundamental dominant weights $\{\Lambda_i \mid i \in I\}$, and the form $(\cdot,\cdot)$ such that

$$(\alpha_i, \alpha_j) = \mathtt{c}_{i,j} \quad \text{and} \quad (\Lambda_i, \alpha_j) = \delta_{i,j} \qquad (i, j \in I).$$

Denote $Q_+ := \bigoplus_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i$ and $P := \bigoplus_{i\in I} \mathbb{Z}\Lambda_i$. For $\alpha = \sum_{i\in I} n_i\alpha_i \in Q_+$, we write $\mathrm{ht}(\alpha) := \sum_{i\in I} n_i$. We have a bijection

$$I^n/\Sigma_n \xrightarrow{\sim} \{\alpha \in Q_+ \mid \mathrm{ht}(\alpha) = n\},\ \Sigma_n \cdot (i_1, \dots, i_n) \mapsto \alpha_{i_1} + \dots + \alpha_{i_n},$$

and from now on we *identify the two sets*. So we have the algebras H_α for $\alpha \in Q_+$.

3.2. Induction and restriction functors. We want to study the induction and restriction functors between H_n-modules and H_{n-1}-modules. In particular, we are interested to know as much as possible about restrictions of irreducible H_n-modules to H_{n-1}, since this could help us to understand irreducible H_n-modules by induction.

It makes sense to refine induction and restriction to blocks. For any $\alpha \in Q_+$ of height n and $i \in I$, there is an obvious graded algebra homomorphism $H_\alpha \to H_{\alpha+\alpha_i}$. It maps the identity element of H_α to the idempotent

$$1_{\alpha,\alpha_i} = \sum_{\boldsymbol{i} \in \alpha+\alpha_i,\ i_{n+1}=i} 1_{\boldsymbol{i}} \in H_{\alpha+\alpha_i}.$$

Now, define the functors

$$e_i := 1_{\alpha,\alpha_i} H_{\alpha+\alpha_i} \otimes_{H_{\alpha+\alpha_i}} - : H_{\alpha+\alpha_i}\text{-mod} \to H_\alpha\text{-mod},$$
$$f_i := H_{\alpha+\alpha_i} 1_{\alpha,\alpha_i} \otimes_{H_\alpha} - : H_\alpha\text{-mod} \to H_{\alpha+\alpha_i}\text{-mod}.$$

For $M \in H_\alpha$-mod, define

$$\varepsilon_i(M) := \max\{k \mid e_i^k \neq 0\}, \quad \varphi_i(M) := \max\{k \mid f_i^k \neq 0\}.$$

3.3. First branching rules. Let L be an irreducible H_α-module. It has been first proved in [23] (in the ungraded setting) that $e_i L$ is either zero or it has a simple socle and head isomorphic to each other, and similarly for $f_i L$, see also [16] for a more conceptual proof and a generalization. In particular, $e_i L$ and $f_i L$ are either zero or *indecomposable*, which is far from obvious. Let

$$\tilde{e}_i L := \operatorname{soc} e_i L, \quad \tilde{f}_i L := \operatorname{soc} f_i L.$$

This defines maps

$$\tilde{e}_i, \tilde{f}_i : B \to B \sqcup \{0\},$$

where B is the set of irreducible H_α-modules for all $\alpha \in Q_+$ up to isomorphism and degree shift. Recall that all algebras and modules are graded, moreover the irreducible modules are graded *canonically* so that they are gradedly self-dual. This applies in particular to the irreducible modules $\tilde{e}_i L$, $\tilde{f}_i L$.

For $n \in \mathbb{Z}$, denote the corresponding *quantum integer*

$$[n]_q := (q^n - q^{-n})/(q - q^{-1}).$$

Then one can refine the results on the socle as follows [7, Theorem 4.12]:

Theorem 3.1. *Let $\alpha \in Q_+$, $i \in I$ and L be an irreducible H_α-module. Then:*

(i) $(e_i L)^{\circledast} \cong e_i L$ *and* $(f_i L)^{\circledast} \cong f_i L$.

(ii) $e_i L$ *and* $f_i L$ *are indecomposable or zero. Moreover:*

$$\operatorname{soc} e_i L \cong q^{\varepsilon_i(L)-1} \tilde{e}_i L, \qquad \operatorname{head} e_i L \cong q^{1-\varepsilon_i(L)} \tilde{e}_i L,$$
$$\operatorname{soc} f_i L \cong q^{\varphi_i(L)-1} \tilde{f}_i L, \qquad \operatorname{head} f_i L \cong q^{1-\varphi_i(L)} \tilde{f}_i L.$$

(iii) $[e_i L : \tilde{e}_i L]_q = [\varepsilon_i(\lambda)]_q$ *and* $[f_i L : \tilde{f}_i L]_q = [\varphi_i(\lambda)]_q$.

(iv) $\varepsilon_i(\tilde{e}_i L) = \varepsilon_i(L) - 1$ *and* $\varepsilon_i(N) < \varepsilon_i(L) - 1$ *for any other composition factor N of $e_i L$;* $\varphi_i(\tilde{f}_i L) = \varphi_i(L) - 1$ *and* $\varphi_i(K) < \varphi_i(L) - 1$ *for any other composition factor K of $f_i L$.*

(v) $\mathrm{End}_{H_{\alpha-\alpha_i}}(e_iL) \cong F[x]/(x^{\varepsilon_i(L)})$, *the truncated polynomial algebra with the variable* x *of degree* 2, *and* $\mathrm{End}_{H_{\alpha+\alpha_i}}(f_iL) \cong F[x]/(x^{\varphi_i(L)})$.

(vi) e_iL *is irreducible if and only if* $\varepsilon_i(L) = 1$.

Ungraded versions of these results were first obtained in [22, 23, 25, 26]. For more branching rules see §5.4.

3.4. Crystal operators. Let us return to the set B and the operations $\tilde{e}_i, \tilde{f}_i : B \to B \sqcup \{0\}$. Every elements of $[L] \in B$ is an isomorphism class of an irreducible H_α-module L for some $\alpha \in Q_+$. This allows us to define a function

$$\mathrm{wt} : B \to P,\ [L] \mapsto \Lambda_0 - \alpha.$$

Moreover, for all $i \in I$, we have functions $\varepsilon_i, \varphi_i : B \to \mathbb{Z}_{\geq 0}$.

Let $\mathfrak{g}$ be the affine Kac-Moody Lie algebra corresponding to the Cartan matrix $\mathtt{C}$, i.e. $\mathfrak{g} = \widehat{\mathfrak{sl}}_p(\mathbb{C})$, see [19, Section 7].

Theorem 3.2. *The tuple* $(B, \varepsilon_i, \varphi_i, \tilde{e}_i, \tilde{f}_i, \mathrm{wt})$ *is the Kashiwara's crystal associated to the irreducible* $\mathfrak{g}$*-module* $V(\Lambda_0)$ *with highest weight* Λ_0.

This theorem has been first proved by Lascoux, Leclerc and Thibon [32] by comparing the branching rules from [22] and [23] with the explicit combinatorial description of the crystal obtained in [36]. A more conceptual proof was found in [15], see also [27]. The remaining results of this section can be considered as steps towards an 'explanation' of the theorem, the 'real explanation' coming perhaps from [39] and [42].

3.5. Divided powers. We define divided power analogues of the functors e_i, f_i. In order to do this, we exploit the algebras A_α. There is an obvious embedding $A_{\alpha,\beta} := A_\alpha \otimes A_\beta \to A_{\alpha+\beta}$ mapping $1 \otimes 1 \mapsto 1_{\alpha,\beta} := \sum_{\boldsymbol{i}\in\alpha,\,\boldsymbol{j}\in\beta} 1_{\boldsymbol{ij}}$. Consider the functors

$$\mathrm{Ind}_{\alpha,\beta}^{\alpha+\beta} := A_{\alpha+\beta}1_{\alpha,\beta} \otimes_{A_{\alpha,\beta}} - : A_{\alpha,\beta}\text{-mod} \to A_{\alpha+\beta}\text{-mod}, \tag{3.1}$$

$$\mathrm{Res}_{\alpha,\beta}^{\alpha+\beta} := 1_{\alpha,\beta}A_{\alpha+\beta} \otimes_{A_{\alpha+\beta}} - : A_{\alpha+\beta}\text{-mod} \to A_{\alpha,\beta}\text{-mod}\,. \tag{3.2}$$

Let $i \in I$ and $n \geq 1$. Recalling the $A_{n\alpha_i}$-module $P(i^{(n)})$ from (2.17), set

$$\theta_i^{(n)} := \mathrm{Ind}_{\alpha,n\alpha_i}^{\alpha+n\alpha_i}(- \boxtimes P(i^{(n)})) : A_\alpha\text{-mod} \to A_{\alpha+n\alpha_i}\text{-mod},$$
$$(\theta_i^*)^{(n)} := \mathrm{Hom}_{A'_{n\alpha_i}}(P(i^{(n)}), -) : A_{\alpha+n\alpha_i}\text{-mod} \to A_\alpha\text{-mod},$$

where $A'_{n\alpha_i} := 1 \otimes A_{n\alpha_i} \subseteq A_{\alpha,n\alpha_i}$. Define

$$e_i^{(n)} := \mathrm{pr} \circ (\theta_i^*)^{(n)} \circ \mathrm{infl} : H_{\alpha+n\alpha_i}\text{-mod} \to H_\alpha\text{-mod},$$
$$f_i^{(n)} := q^{n^2-n(\Lambda_0-\alpha,\alpha_i)}\mathrm{pr} \circ \theta_i^{(n)} \circ \mathrm{infl} : H_\alpha\text{-mod} \to H_{\alpha+n\alpha_i}\text{-mod}\,.$$

By [7, Lemma 4.8], $e_i^n \cong [n]_q^! e_i^{(n)}$, $f_i^n \cong [n]_q^! f_i^{(n)}$, where $[n]_q^! := [1]_q \dots [n]_q$, the functors $e_i^{(n)}, f_i^{(n)}$ are exact, and send finite dimensional (resp. projective) modules to finite dimensional (resp. projective) modules. Finally, we need the degree shift functors

$$k_i^n : H_\alpha\text{-mod} \to H_\alpha\text{-mod},\ M \mapsto q^{n(\Lambda-\alpha,\alpha_i)}M \qquad (n \in \mathbb{Z}).$$

Consider the (locally unital) algebra $H := \oplus_{\alpha\in Q_+} H_\alpha$, the categories

$$H\text{-mod} := \bigoplus_{\alpha\in Q_+} H_\alpha\text{-mod} \quad \text{and} \quad H\text{-proj} := \bigoplus_{\alpha\in Q_+} H_\alpha\text{-proj},$$

and the Grothendieck groups $[H\text{-mod}] = \bigoplus_{\alpha\in Q_+}[H_\alpha\text{-mod}]$ with $\mathcal{L}$-basis B, and $[H\text{-proj}] = \bigoplus_{\alpha\in Q_+}[H_\alpha\text{-proj}]$. Let

$$\langle .,. \rangle : H\text{-proj} \times H\text{-mod} \to \mathcal{L}, \quad \langle [P], [M] \rangle := \dim_q \ \mathrm{Hom}_H(P, M), \tag{3.3}$$

be the *Cartan pairing*. The pairing is *sesquilinear*, i.e. anti-linear in the first argument and linear in the second. We have a similar form $\langle .,. \rangle : H\text{-proj} \times H\text{-proj} \to \mathcal{L}$.

The exact functors $e_i^{(n)}, f_i^{(n)}$ and $k_i^{\pm 1}$ induce $\mathcal{L}$-linear endomorphisms $E_i^{(n)}, F_i^{(n)}$ and $K_i^{\pm 1}$, respectively of $[H\text{-mod}]$ and $[H\text{-proj}]$.

3.6. LLT categorification. On the other hand, let $U_q(\mathfrak{g})$ be the quantized enveloping algebra of $\mathfrak{g}$ over $\mathbb{Q}(q)$ with Chevalley generators $E_i^{(n)}, F_i^{(n)}, K_i^{\pm 1}$ for $i \in I$. Let $V(\Lambda_0)$ be the irreducible $U_q(\mathfrak{g})$-module with highest weight Λ_0 and a fixed highest weight vector v_+. The module $V(\Lambda_0)$ has a unique compatible bar-involution $- : V(\Lambda_0) \to V(\Lambda_0)$ such that $\overline{v_+} = v_+$.

The *Shapovalov form* $\langle .,. \rangle$ is the unique sesquilinear $\mathbb{Q}(q)$-valued form on $V(\Lambda_0)$ such that $\langle v_\Lambda, v_\Lambda \rangle = 1$ and $\langle uv, w \rangle = \langle v, \tau(u)w \rangle$ for all $u \in U_q(\mathfrak{g})$ and $v, w \in V(\Lambda_0)$, where τ is anti-linear anti-automorphism defined by $\tau(K_i) = K_i^{-1}$, $\tau(E_i) = qF_iK_i^{-1}$, $\tau(F_i) = q^{-1}K_iE_i$.

Let $U_q(\mathfrak{g})_\mathcal{L}$ be the Lusztig's $\mathcal{L}$-form, i.e. the $\mathcal{L}$-subalgebra of $U_q(\mathfrak{g})$ generated by the quantum divided powers $E_i^{(n)} := E_i^n/[n]_q^!$ and $F_i^{(n)} := F_i^n/[n]_q^!$ for all $i \in I$ and $n \geq 1$. Let $V(\Lambda_0)_\mathcal{L} := U_q(\mathfrak{g})_\mathcal{L} \cdot v_+$ be the standard $\mathcal{L}$-form of $V(\Lambda_0)$, and $V(\Lambda_0)^*_\mathcal{L} = \{v \in V(\Lambda_0) \mid \langle v, w \rangle \in \mathcal{L} \text{ for all } w \in V(\Lambda_0)_\mathcal{L}\}$ be the costandard $\mathcal{L}$-form.

The following is the graded version [7, Theorem 4.18] of the categorification theorems proved by Lascoux-Leclerc-Thibon [32], Ariki [1], and Grojnowsky [15]:

Theorem 3.3. *The linear operators E_i, F_i and K_i on the Grothendieck group*

$$[H\text{-proj}]_{\mathbb{Q}(q)} := [H\text{-proj}] \otimes_\mathcal{L} \mathbb{Q}(q)$$

satisfy the defining relations of the Chevalley generators of $U_q(\mathfrak{g})$. So $[H\text{-proj}]_{\mathbb{Q}(q)}$ is a $U_q(\mathfrak{g})$-module. Moreover:

(i) *There is a unique isomorphism $\delta : V(\Lambda_0) \xrightarrow{\sim} [H\text{-proj}]_{\mathbb{Q}(q)}$ of $U_q(\mathfrak{g})$-modules, such that $\delta(v_+) = [\mathtt{triv}_{H_0}]$, where $\mathtt{triv}_{H_0} \in H_0\text{-proj}$ is the one-dimensional vector space F considered as a module over $H_0 \cong F$.*

(ii) *The restriction of δ to $V(\Lambda_0)_\mathcal{L}$ is an isomorphism $\delta : V(\Lambda_0)_\mathcal{L} \xrightarrow{\sim} [H\text{-proj}]$ of $U_q(\mathfrak{g})_\mathcal{L}$-modules, which intertwines $\circledast$ with the bar-involution on $V(\Lambda_0)_\mathcal{L}$ and induces the isomorphisms on weight spaces $V(\Lambda_0)_{\Lambda_0-\alpha,\mathcal{L}} \xrightarrow{\sim} [H_\alpha\text{-proj}]$ for all $\alpha \in Q_+$.*

(iii) *The isomorphism δ identifies the Shapovalov form on $V(\Lambda_0)_\mathcal{L}$ with the Cartan pairing on $[H\text{-proj}]$.*

(iv) *Let $\delta^* : [H\text{-mod}] \to V(\Lambda_0)^*_\mathcal{L}$ be the dual map:*

$$\delta^*([M])(v) := \langle \delta(v), [M] \rangle \qquad (v \in V(\Lambda_0)_\mathcal{L}).$$

Then δ^* is an isomorphism of $U_q(\mathfrak{g})_{\mathcal{L}}$-modules, which intertwines $\circledast$ with the bar-involution on $V(\Lambda_0)^*_{\mathcal{L}}$, and induces the isomorphisms $[H_\alpha\text{-mod}] \xrightarrow{\sim} V(\Lambda_0)^*_{\Lambda_0-\alpha,\mathcal{L}}$ for all $\alpha \in Q_+$.

(v) *The following diagram is commutative:*

$$\begin{array}{ccc} V(\Lambda_0)_{\mathcal{L}} & \xrightarrow[\delta]{\sim} & [H\text{-proj}] \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ V(\Lambda_0)^*_{\mathcal{L}} & \xleftarrow[\delta^*]{\sim} & [H\text{-mod}], \end{array}$$

where $a : V(\Lambda)_{\mathcal{L}} \hookrightarrow V(\Lambda)^*_{\mathcal{L}}$ is the canonical inclusion, and $b : [H\text{-proj}] \to [H\text{-mod}]$ is the $\mathcal{L}$-linear map induced by the natural inclusion of H-proj into H-mod. In particular, b is injective and becomes an isomorphism over $\mathbb{Q}(q)$.

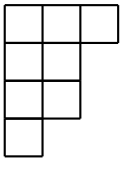
We complete this section with a special case of the Chuang-Rouquier result [10] on derived equivalence of the algebras H_α. Recall from [19] that the (affine) Weyl group W of $\mathfrak{g}$ acts on the weights of $V(\Lambda_0)$.

Theorem 3.4. *Let $\alpha, \beta \in Q_+$. Then the derived categories $D^b(H_\alpha\text{-mod})$ and $D^b(H_\beta\text{-mod})$ are equivalent if and only if the weights $\Lambda_0 - \alpha$ and $\Lambda_0 - \beta$ belong to the same W-orbit.*

The equivalence in the theorem is induced by a complex of functors, which is built out of the functors e_i and f_i using adjunctions, see [10, §6].

4. Combinatorics of partitions and homogeneous representations

4.1. Partitions and nodes. Let $\mathscr{P}_n$ be the set of all partitions of n and put $\mathscr{P} := \bigsqcup_{n\geq 0} \mathscr{P}_n$. If $\mu \in \mathscr{P}_n$, we write $n = |\mu|$. A partition $\mu = (\mu_1, \mu_2, \dots)$ is called *p-restricted* if $\mu_k - \mu_{k+1} < p$ for all $k = 1, 2, \dots$. Let $\mathscr{RP}_n$ be the set of all p-restricted partitions of n, and put $\mathscr{RP} := \bigsqcup_{n\geq 0} \mathscr{RP}_n$. The *Young diagram* of a partition $\mu = (\mu_1, \mu_2, \dots)$ is $\{(a, b) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \mid 1 \leq b \leq \mu_a\}$. The elements of this set are the *nodes of* μ. More generally, a *node* is any element of $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$. We identify partitions with their Young diagrams, so that a node (a, b) = box in row a and column b. For example,

$(3, 2, 2, 1) =$ 

To each node $A = (a, b)$ we associate its *residue*:

$$\operatorname{res} A := (b - a) \pmod p \in I.$$

An *i-node* is a node of residue i. Let $c_i(\mu)$ be the number of i-nodes of μ, and define the *content* of μ to be $\operatorname{cont}(\mu) = \sum_{i\in I} c_i(\mu)\alpha_i \in Q_+$. Denote

$$\mathscr{P}_\alpha := \{\mu \in \mathscr{P} \mid \operatorname{cont}(\mu) = \alpha\}, \quad \mathscr{RP}_\alpha := \mathscr{RP} \cap \mathscr{P}_\alpha \qquad (\alpha \in Q_+).$$

A node $A \in \mu$ is *removable (for μ)* if $\mu \setminus \{A\}$ is a partition. A node $B \notin \mu$ is an *addable node (for μ)* if $\mu \cup \{B\}$ is a partition. We denote $\mu_A := \mu \setminus \{A\},\ \mu^B := \mu \cup \{B\}$.

Let $i \in I$, and $A_1, \dots, A_n$ be the addable and removable i-nodes of μ ordered so that A_m is to the left of A_{m+1} for each $m = 1, \dots, n-1$. Consider the sequence $(\tau_1, \dots, \tau_n)$ where $\tau_r = +$ if A_r is addable and $-$ if A_r is removable. If there are $1 \le r < s \le n$ with $\tau_r = +$, $\tau_s = -$ and $\tau_{r+1} = \cdots = \tau_{s-1} = 0$ then replace τ_r and τ_s by 0. Keep doing this until left with a sequence $(\sigma_1, \dots, \sigma_n)$ in which no $+$ appears to the left of a $-$. This is the *reduced i-signature* of μ (it is well-defined).

If $(\sigma_1, \dots, \sigma_n)$ is the reduced i-signature of μ, we set

$$\varepsilon_i(\mu) := \#\{r = 1, \dots, n \mid \sigma_r = -\}, \quad \varphi_i(\mu) := \#\{r = 1, \dots, n \mid \sigma_r = +\}.$$

Let $\{r_1 > \cdots > r_{\varepsilon_i(\mu)}\} = \{r \mid \sigma_r = -\}$, and $\{a_1 < \cdots < a_{\varphi_i(\mu)}\} = \{a \mid \sigma_a = +\}$. If $\varepsilon_i(\mu) > 0$, set $\tilde{e}_i\mu := \mu_{A_{r_1}}$; otherwise set $\tilde{e}_i\mu := 0$. If $\varphi_i(\mu) > 0$, set $\tilde{f}_i\mu := \mu^{A_{a_1}}$; otherwise set $\tilde{f}_i\mu := 0$. The removable nodes $A_{r_1}, \dots, A_{r_{\varepsilon_i(\mu)}}$ of μ are called *i-normal*, and the addable nodes $A_{a_1}, \dots, A_{a_{\varphi_i(\mu)}}$ of μ are called *i-conormal*.

4.2. Tableaux. Let $\mu \in \mathscr{P}_n$. A *μ-tableau* $\mathtt{T}$ is an insertion of the integers $1, \dots, n$ into the nodes of μ, allowing no repeats. The *residue sequence* of $\mathtt{T}$ is

$$\boldsymbol{i}^{\mathtt{T}} = (i_1, \dots, i_n) \in I^n,$$

where i_r is the residue of the node occupied by r in $\mathtt{T}$ ($1 \le r \le n$). A μ-tableau $\mathtt{T}$ is *row-strict* (resp. *column-strict*) if its entries increase from left to right (resp. from top to bottom) along the rows (resp. columns) of μ. A μ-tableau $\mathtt{T}$ is *standard* if it is row- and column-strict. Let $\mathrm{St}(\mu)$ be the set of all standard μ-tableaux.

Let $\mathtt{T}^\mu$ be the *leading* μ-tableau, i.e. the tableau in which the numbers $1, 2, \dots, n$ appear in order from left to right along the successive rows, working from top row to bottom row. For example, if $\mu = (3, 2, 2, 1)$ then $\mathtt{T}^\mu$ is

1	2	3
4	5	
6	7	
8		

Set $\boldsymbol{i}^\mu := \boldsymbol{i}^{\mathtt{T}^\mu}$. The group Σ_n acts on the set of μ-tableaux by acting on the entries of the tableaux. For each μ-tableau $\mathtt{T}$, define $w^{\mathtt{T}} \in \Sigma_n$ from $w^{\mathtt{T}}\mathtt{T}^\mu = \mathtt{T}$.

Let $\le$ be the *Bruhat order* on Σ_d. Define the *Bruhat order* on the set of all μ-tableaux as follows: $\mathtt{S} \trianglelefteq \mathtt{T}$ if and only if $w^{\mathtt{S}} \le w^{\mathtt{T}}$. Then the leading μ-tableau $\mathtt{T}^\mu$ is the unique minimal element of $\mathrm{St}(\mu)$.

Let $\mu \in \mathscr{P}$, $i \in I$, and A be a removable i-node of μ. We set

$$d_A(\mu) = \#\left\{\begin{array}{c}\text{addable } i\text{-nodes of } \mu \\ \text{strictly to the left of } A\end{array}\right\} - \#\left\{\begin{array}{c}\text{removable } i\text{-nodes of } \mu \\ \text{strictly to the left of } A\end{array}\right\}.$$

Given $\mathtt{T} \in \mathrm{St}(\mu)$, the *degree* of $\mathtt{T}$ is defined in [8, §3.5] inductively as follows. If $n = 0$, we set $\deg(\mathtt{T}) := 0$. Otherwise, let A be the node occupied by n in $\mathtt{T}$. Let $\mathtt{T}_{<n} \in \mathrm{St}(\mu_A)$ be the tableau obtained by removing A and set

$$\deg(\mathtt{T}) := d_A(\mu) + \deg(\mathtt{T}_{<n}).$$

5. Branching and graded cellular structure

5.1. Crystal combinatorics and irreducible modules. Using the terminology introduced in Section 4, we can now state the following theorem of Misra and Miwa [36]:

Theorem 5.1. *For any partition μ, define* $\mathrm{wt}(\mu) := \Lambda_0 - \mathrm{cont}(\mu)$. *Then the tuple* $(\mathscr{RP}, \varepsilon_i, \varphi_i, \tilde{e}_i, \tilde{f}_i, \mathrm{wt})$ *is the Kashiwara's crystal associated to* $V(\Lambda_0)$.

Comparing this with Theorem 3.2, we deduce that there is a unique isomorphism of crystals

$$(\mathscr{RP}, \varepsilon_i, \varphi_i, \tilde{e}_i, \tilde{f}_i, \mathrm{wt}) \xrightarrow{\sim} (B, \varepsilon_i, \varphi_i, \tilde{e}_i, \tilde{f}_i, \mathrm{wt}). \tag{5.1}$$

Under this isomorphism, to every $\mu \in \mathscr{RP}_\alpha$, we associate the irreducible H_α-module D^μ, and

$$\{D^\mu \mid \mu \in \mathscr{RP}_\alpha\} \tag{5.2}$$

is a complete and irredundant set of irreducible H_α-modules up to isomorphism and degree shift.

On the other hand, there is another approach to the classification of irreducible H_α-modules, based on the theory of *Specht modules*, and which goes back to James [18]. In modern terms, this is the approach through *cell modules* [13]. The *graded cellular structure* of H_α, which we present here, has been worked out by Hu and Mathas [17].

5.2. Graded cellular structure. Fix $\alpha \in Q_+$ and $\mu \in \mathscr{P}_\alpha$. Recall the leading standard tableau $\mathtt{T}^\mu \in \mathrm{St}(\mu)$ and the corresponding residue sequence $\boldsymbol{i}^\mu$. For $k = 1, \dots, n$, let A_k be the box occupied with k in $\mathtt{T}^\mu$. Observe that A_k is a removable node for the partition μ^k, obtained from μ by removing $A_{k+1}, \dots, A_n$. Set $d_k(\mu) := d_{A_k}(\mu^k)$. Note that $\deg(T^\mu) = d_1(\mu) + \dots + d_n(\mu)$. Define

$$y^\mu := y_1^{d_1(\mu)} \dots y_n^{d_n(\mu)}.$$

Given $\mathtt{T} \in \mathrm{St}(\mu)$, recall the element $w^\mathtt{T} \in \Sigma_n$. Pick any reduced decomposition $w^\mathtt{T} = s_{m_1} \dots s_{m_l}$, and set

$$\psi^\mathtt{T} := \psi_{m_1} \dots \psi_{m_l} \in H_\alpha. \tag{5.3}$$

This element in general depends on the choice of the reduced decomposition. Finally, recalling (2.15), for any standard tableaux $\mathtt{S}, \mathtt{T} \in \mathrm{St}(\mu)$, we define

$$\psi^{\mathtt{S},\mathtt{T}} := \psi^\mathtt{S} y^\mu 1_{\boldsymbol{i}^\mu} (\psi^\mathtt{T})'.$$

It is easy to see that $\deg(\psi^{\mathtt{S},\mathtt{T}}) = \deg(\mathtt{S}) + \deg(\mathtt{T})$. The following theorem was proved by Hu and Mathas [17]:

Theorem 5.2. *Let $\alpha \in Q_+$. Then* $\{\psi^{\mathtt{S},\mathtt{T}} \mid \mu \in \mathscr{P}_\alpha,\ \mathtt{S}, \mathtt{T} \in \mathrm{St}(\mu)\}$ *is a graded cellular basis of H_α.*

The following immediate corollary was originally proved in [7, Theorem 4.20] by a different method:

Corollary 5.3. *Let $\alpha \in Q_+$ and $\boldsymbol{i}, \boldsymbol{j} \in \alpha$. Then*

$$\dim_q 1_{\boldsymbol{i}} H_\alpha 1_{\boldsymbol{j}} = \sum_{\mu \in \mathscr{P}_\alpha,\ \mathtt{S},\mathtt{T} \in \mathrm{St}(\mu),\ \boldsymbol{i}^\mathtt{S} = \boldsymbol{i},\ \boldsymbol{j}^\mathtt{T} = \boldsymbol{j}} q^{\deg(\mathtt{S}) + \deg(\mathtt{T})}.$$

In particular, $\dim_q H_\alpha = \sum_{\mu\in\mathscr{P}_\alpha,\ \mathtt{S},\mathtt{T}\in\mathrm{St}(\mu)} q^{\deg(\mathtt{S})+\deg(\mathtt{T})}$.

The graded version of the Graham-Lehrer theory [13] can be found in [17]. In particular, from a graded cellular basis we get graded cell modules $\{S^\mu \mid \mu \in \mathscr{P}_\alpha\}$. It is shown in [17] that these are just the *graded Specht modules* as constructed originally in [8]. On the other hand, it is noted in [8] that, if we forget the grading, then S^μ is the usual dual Specht module of [18].

Recall from (5.2) the simple H_α-modules D^μ defined using the crystal isomorphism (5.1).

Theorem 5.4. *If μ is p-restricted, then S^μ has a simple head isomorphic to D^μ.*

There might be more to this theorem than meets the eye. Firstly, it is the graded aspect: recall that $(D^\mu)^\circledast \cong D^\mu$, and the theorem claims that the natural map $S^\mu \twoheadrightarrow \operatorname{head} S^\mu \cong D^\mu$ is a homogeneous degree zero map. Secondly, it is known from James [18] that the head of S^μ is simple when μ is p-restricted, and this is more or less how James classifies the irreducible modules (actually our D^λ is isomorphic to James' $D^{\lambda'} \otimes \mathtt{sgn}$). But it is not at all clear that the James classification agrees with the classification (5.2) coming from the isomorphism of crystals (5.1). There are several ways of seeing this, none being trivial. One comes from the fact that the original branching rules are proved in [22] for the modules D^λ in James' classification, which allows us to identify them with the classification (5.2).

5.3. Mullineux map. Branching rules yield a simple solution to the Mullineux problem [37]. Tensoring with the sign representation yields a bijection

$$\mathscr{R}\mathscr{P}(\alpha) \xrightarrow{\sim} \mathscr{R}\mathscr{P}(-\alpha),\ \mu \mapsto \mathbf{M}(\mu),$$

where $\mathbf{M}(\mu)$ is defined from

$$D^{\mathbf{M}(\mu)} \cong D^\mu \otimes \mathtt{sgn} \qquad (\mu \in \mathscr{R}\mathscr{P}_\alpha).$$

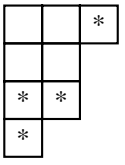
The problem is to describe $\mathbf{M}(\mu)$ explicitly in combinatorial terms. The following theorem, proved in [23], gives an answer to this question:

Theorem 5.5. *Let $\mu \in \mathscr{R}\mathscr{P}_n$, and $\emptyset$ be the empty partition. Pick a sequence $i_1, \dots, i_n \in I$ such that $\emptyset = \tilde{e}_{i_1} \dots \tilde{e}_{i_n}\mu$. Then $\mathbf{M}(\mu) = \tilde{f}_{-i_n} \dots \tilde{f}_{-i_1}\emptyset$.*

Thus a computation of $\mathbf{M}(\mu)$ is reduced to combinatorics of the crystal graph of Theorem 5.1 described in §4.1. However, there a faster algorithm originally conjectured by Mullineux [37]. The Mullineux Conjecture has been first proved in [12] and simpler proofs were later found in [2] and [9]. We now describe this algorithm or rather its more elegant version suggested by Xu [43].

The *rim* of μ is the set of all nodes $(i, j) \in \mu$ such that that $(i + 1, j + 1) \notin \mu$. The *$p$-rim* of μ is the union of the *p-segments*, which are defined as follows. The first p-segment of μ consists of the first p nodes of the rim, reading along the rim from bottom-left to top-right. The next p-segment is obtained by similar reading off the nearest p nodes of the rim, but *starting from the column immediately to the right of the rightmost node of the first p-segment*. And so on. All but the last p-segment contain exactly p nodes, while the last may contain less. In the following example $p = 3$, there are two p-segments, and the nodes of the p-rim are

marked with $*$'s.



Let let $J(\mu)$ be the partition obtained from μ by deleting every node in the p-rim that is at the rightmost end of a row of μ but that is not the pth node of a p-segment. Let $j(\mu) = |\mu| - |J(\mu)|$ be the total number of nodes deleted. Now, in Xu's reformulation [43], the result is as follows:

Theorem 5.6. *$\mathbf{M}(\mu)$ is the partition $\lambda = (\lambda_1, \lambda_2, \dots)$ with $\lambda_r = j(J^{r-1}(\mu))$.*

In the example above, we get $\mathbf{M}((3, 2^2, 1)) = (2, 1^6)$.

5.4. More branching rules. We complete this section with more result on branching.

Theorem 5.7. *Let $\alpha \in Q_+$, $i \in I$, $\mu \in \mathscr{R}\mathscr{P}_\alpha$, A be a removable node of μ such that μ_A is p-restricted, and B be an addable node for μ such that μ^B is p-restricted. Moreover, let $A_1, \dots, A_{\varepsilon_i(\mu)}$ be the i-normal nodes of μ counted from left to right, and $B_1, \dots, B_{\varphi_i(\mu)}$ be the i-conormal nodes for μ counted from right to left.*

(i) $\operatorname{Hom}_{H_{\alpha-\alpha_i}}(S^{\mu_A}, e_i D^\mu) \neq 0$ *if and only if A is i-normal for μ, in which case we have* $\dim_q \operatorname{Hom}_{H_{\alpha-\alpha_i}}(S^{\mu_{A_m}}, e_i D^\mu) = q^{m-1}$ *for all $m = 1, \dots, \varepsilon_i(\mu)$.*

(ii) $\operatorname{Hom}_{H_{\alpha+\alpha_i}}(S^{\mu^B}, f_i D^\mu) \neq 0$ *if and only if B is i-conormal for μ, in which case we have* $\dim_q \operatorname{Hom}_{H_{\alpha+\alpha_i}}(S^{\mu^{B_m}}, f_i D^\mu) = q^{m-1}$ *for all $m = 1, \dots, \varphi_i(\mu)$.*

(iii) *D^{μ_A} appears as a composition factor of $e_i D^\mu$ if and only if A is i-normal for μ, in which case we have $[e_i D^\mu : D^{\mu_{A_m}}]_q = [m]_q$ for all $m = 1, \dots, \varepsilon_i(\mu)$.*

(iv) *D^{μ^B} appears as a composition factor of $f_i D^\mu$ if and only if B is i-conormal for μ, in which case we have $[f_i D^\mu : D^{\mu^{B_m}}]_q = [m]_q$ for all $m = 1, \dots, \varphi_i(\mu)$.*

This is a graded version of the results [5, 22, 25, 26]. The graded version is deduced using the graded endomorphism algebra description of Theorem 3.1(v) and the related filtrations of $e_i D^\mu$ and $f_i D^\mu$ obtained in [5] and [26]. The following corollary follows immediately from Theorem 5.7 on tensoring with sign, cf. [25], and often provides us with some new non-trivial branching multiplicities:

Corollary 5.8. *Let $\alpha \in Q_+$, $i \in I$, $\mu \in \mathscr{R}\mathscr{P}_\alpha$, and let $A_1, \dots, A_{\varepsilon_i(\mu)}$ be the $(-i)$-normal nodes of $\mathbf{M}(\mu)$ labeled from left to right, and $B_1, \dots, B_{\varphi_i(\mu)}$ be the $(-i)$-conormal nodes for $\mathbf{M}(\mu)$ labeled from right to left. Then:*

(i) *$[e_i D^\mu : D^{\mathbf{M}(\mathbf{M}(\mu)_{A_m})}]_q = [m]_q$ for all $m = 1, \dots, \varepsilon_i(\mu)$ such that $\mathbf{M}(\mu)_{A_m}$ is p-restricted.*

(ii) *$[f_i D^\mu : D^{\mathbf{M}(\mathbf{M}(\mu)^{B_m})}]_q = [m]_q$ for all $m = 1, \dots, \varphi_i(\mu)$ such that $\mathbf{M}(\mu)^{B_m}$ is p-restricted.*

Let us consider the example where $p = 3$ and $\mu = (3^2, 2, 1^2)$. We draw the corresponding Young diagram with the residues of the boxes written in them.

0	1	2
2	0	1
1	2	
0		
2		

Note that both 2-removable boxes are normal, the removable 1-box is not, and there are no removable 0-boxes. So $e_0 D^\mu = e_1 D^\mu = 0$. As for the composition factors of $e_2 D^\mu$, Theorem 5.7 shows that $[e_2 D^\mu : D^{(3^2,2,1)}]_q = 1$ and $[e_2 D^\mu : D^{(3^2,1^3)}]_q = [2]_q = q + q^{-1}$. Moreover, since $\mathbf{M}(\mu) = (3^2, 1^4)$ has the leftmost normal 1-node $(6, 1)$ and $\mathbf{M}(3^2, 1^3) = (3, 2, 1^4)$, Corollary 5.8 yields another composition factor $D^{(3,2,1^4)}$ of multiplicity 1. It is easy to verify using decomposition matrices in [18] that in this example we have discovered all composition factors, i.e. $[\mathrm{res}^{\Sigma_{10}}_{\Sigma_9}] = [D^{(3^2,2,1)}] + (q + q^{-1})[D^{(3^2,1^3)}] + [D^{(3,2,1^4)}]$.

Unfortunately, this technique is not powerful enough to always yield all composition factors, see [25, Section 1]. So it leads only to a lower bound on the dimensions of irreducible H_n-modules. A family of irreducible modules for which this lower bound is equal to the actual dimension is described in the next section.

5.5. Homogeneous representations. Let $\alpha \in Q_+$. An irreducible H_α-module is called *homogeneous* if it is concentrated in degree zero. To describe the homogeneous representations, for a partition $\mu = (\mu_1 \geq \cdots \geq \mu_u > 0) \in \mathscr{R}\mathscr{P}_n$ consider the hook length $\chi(\mu) := \lambda_1 + u - \max\{t \mid \lambda_t = \lambda_1\}$. Then μ is called *homogeneous* if $\chi(\mu) \leq p$, cf. [24], where we worked with transposed partitions.

Let $\mu \in \mathscr{R}\mathscr{P}_n$ be a homogeneous partition. A tableaux $\mathtt{T} \in \mathrm{St}(\mu)$ is called *p-standard* if $a < b$ whenever a occupies a box (r, s) in $\mathtt{T}$ and b occupies a box (r', s') with $r > r'$, $s < s'$ and $r - r' + s' - s + 1 = p$. Let $\mathrm{St}^p(\mu)$ be the set of all p-standard μ-tableaux. The results of [24, 35] and [31] can be restated as follows:

Theorem 5.9. *Let $\mu \in \mathscr{R}\mathscr{P}_\alpha$. The irreducible H_α-module D^μ is homogeneous if and only if μ is a homogeneous partition. In this case, D^μ has a basis $\{v_\mathtt{T} \mid \mathtt{T} \in \mathrm{St}^p(\mu)\}$ with the action of the homogeneous generators of H_α given as follows:*

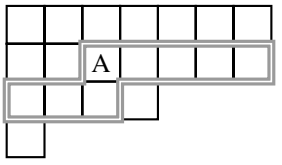
$$1_{\boldsymbol{i}} v_\mathtt{T} = \delta_{\boldsymbol{i},\boldsymbol{i}^\mathtt{T}} v_\mathtt{T},\ y_t v_\mathtt{T} = 0,\ \psi_r v_\mathtt{T} = \begin{cases} v_{s_r\mathtt{T}} & \text{if } s_r\mathtt{T} \in \mathrm{St}^p(\mu); \\ 0 & \text{otherwise.} \end{cases}$$

6. Presentations and bases of cell modules

6.1. Garnir tableaux. Let $\mu \in \mathscr{P}_n$. We now explain an explicit presentation of the graded Specht module S^μ obtained in [29]. First, we need more notation. Let $A = (r, s)$ be a node of μ. It is called a *Garnir node* if $(r + 1, s) \in \mu$, i.e. A is not at the bottom of its column. Then the *A-Garnir belt* $\mathbf{B}^A$ is

$$\mathbf{B}^A := \{\, (r, t) \in \mu \mid s \leq t \leq \mu_r \,\} \cup \{\, (r + 1, u) \in \mu \mid 1 \leq u \leq s \,\}.$$

For example, if $A = (2,3)$ then $\mathbf{B}^A$ for $\mu = (7,7,4,1)$ is highlighted below:

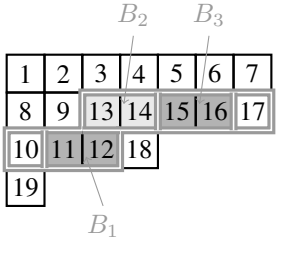


The *A-Garnir tableau* is the μ-tableaux $\mathtt{G}^A$ defined as follows. Let $u = \mathtt{T}^\mu(r,s)$ be the entry of the leading μ-tableau $\mathtt{T}^\mu$ which occupies the node $A = (r,s)$, and $v = \mathtt{T}^\mu(r+1,s)$. To get $\mathtt{G}^A$, insert the numbers $u, u+1, \dots, v$ into the nodes of the Garnir belt going from left bottom to top right, and the other numbers into the same positions as in $\mathtt{T}^\mu$. Continuing the previous example, $u = 10, v = 17$, and:

$$\mathtt{T}^\mu = \begin{array}{|c|c|c|c|c|c|c|}\hline 1&2&3&4&5&6&7\\\hline 8&9&10&11&12&13&14\\\hline 15&16&17&18\\\cline{1-4} 19\\\cline{1-1}\end{array}, \quad \mathtt{G}^A = \begin{array}{|c|c|c|c|c|c|c|}\hline 1&2&3&4&5&6&7\\\hline 8&9&13&14&15&16&17\\\hline 10&11&12&18\\\cline{1-4} 19\\\cline{1-1}\end{array}$$

Fix a Garnir node $A = (r,s)$ of μ. A *brick* is a set of p successive nodes in the same row $\{(t,u),(t,u+1),\dots,(t,u+p-1)\} \subseteq \mathbf{B}^A$ such that $\operatorname{res}(t,u) = \operatorname{res} A$. Note that $\mathbf{B}^A$ is a disjoint union of the bricks that it contains, together with less than p nodes at the end of row r which are not contained in a brick and less than p nodes at the beginning of row $r+1$ which are not contained in a brick.

Let k be the number of bricks in $\mathbf{B}^A$ (possibly zero). We label the bricks $B_1^A, B_2^A, \dots, B_k^A$ going from left to right along row $r+1$ and then from left to right along row r of $\mathtt{G}^A$. For example, the following picture shows the bricks in the $(2,3)$-Garnir belt of $\mu = (7,7,4,1)$ when $p = 2$:



We have $k = 3$, there are two bricks B_2, B_3 in row 2 and one brick B_1 in row 3. Finally, $(3,1)$ and $(2,7)$ are the nodes in the Garnir belt which are not contained in a brick.

Assume now that $k > 0$ and let d be the smallest entry in $\mathtt{G}^A$ which is contained in a brick in $\mathbf{B}^A$. In the example above, $n = 11$. Define

$$w_t^A = \prod_{a=d+tp-p}^{d+tp-1} (a, a+p) \in \Sigma_n \qquad (1 \le t < k). \tag{6.1}$$

Informally, w_t^A is a brick permutation swapping B_t and B_{t+1}. The elements $w_1^A, w_2^A, \dots, w_{k-1}^A$ are the Coxeter generators of the *brick permutation group*

$$\Sigma^A := \langle w_1^A, w_2^A, \dots, w_{k-1}^A \rangle \cong \Sigma_k.$$

By convention, Σ^A is the trivial group if $k = 0$.

Let Gar^A be the set of all row-strict μ-tableaux which are obtained from the Garnir tableau $\mathtt{G}^A$ by acting with the brick permutation group Σ^A on $\mathtt{G}^A$. Note that all of the tableaux in Gar^A, except for $\mathtt{G}^A$, are standard. Moreover, $\mathtt{G}^A$ is the maximal element of Gar^A, with respect to the Bruhat order, and there is a unique minimal tableaux $\mathtt{T}^A$ in Gar^A.

Let f be the number of bricks in row r of the Garnir belt $\mathbf{B}^A$, and let $\mathscr{D}^A$ be the set of minimal length left coset representations of $\Sigma_f \times \Sigma_{k-f}$ in $\Sigma^A \cong \Sigma_k$. By definition Σ^A is a subgroup of Σ_n, so $\mathscr{D}^A$ is a subset of Σ_n, and, in particular, its elements act on μ-tableaux. We have

$$\mathrm{Gar}^A = \{w\mathtt{T}^A \mid w \in \mathscr{D}^A\}. \tag{6.2}$$

Continuing the example above, $\mathtt{T}^A$ is the tableau

$$\mathtt{T}^A = \begin{array}{|c|c|c|c|c|c|c|}\hline 1&2&3&4&5&6&7\\\hline 8&9&11&12&13&14&17\\\hline 10&15&16&18\\\cline{1-4} 19\\\cline{1-1}\end{array}$$

and $\mathrm{Gar}^A = \{\mathtt{T}^A, w_2^A\mathtt{T}^A, \mathtt{G}^A = w_1^A w_2^A \mathtt{T}^A\}$.

6.2. Presenting Specht modules. Set $\tau_r^A := (\psi_{w_r^A} + 1)$. Any element $u \in \Sigma^A$ can be written as a reduced product $u = w_{r_1}^A \dots w_{r_a}^A$ of simple generators $w_1^A, \dots, w_{k-1}^A$ of Σ^A. Then define $\tau_u^A := \tau_{r_1}^A \dots \tau_{r_a}^A$.

Suppose that $\mu \in \mathscr{P}_\alpha$ and $A \in \mu$ is a Garnir node. The *Garnir element* is

$$g^A := \sum_{u \in \mathscr{D}^A} \tau_u^A \psi^{\mathtt{T}^A} 1_{\boldsymbol{i}^\mu} \in H_\alpha.$$

A special case of the main result of [29] is:

Theorem 6.1. *Let $\alpha \in Q_+$ be of height n and $\mu \in \mathscr{P}_\alpha$. The graded Specht module S^μ has a homogeneous vector z^μ of degree $\deg(\mathtt{T}^\mu)$ such that S^μ is generated as a graded H_α-module by z^μ subject only to the following relations:*

(i) $1_{\boldsymbol{j}} z^\mu = \delta_{\boldsymbol{j},\boldsymbol{i}^\mu} z^\mu$ *for all* $\boldsymbol{j} \in \alpha$;

(ii) $y_r z^\mu = 0$ *for all* $r = 1, \dots, n$;

(iii) $\psi_r z^\mu = 0$ *for all* $1 \le r < n$ *such that r and $r+1$ are in the same row of* $\mathtt{T}^\mu$;

(iv) (homogeneous Garnir relations) $g^A z^\mu = 0$ *for all Garnir nodes A in μ.*

In other words, the theorem says that $S^\mu = q^{\deg(\mathtt{T}^\mu)} H_\alpha / J_\alpha^\mu$, where J_α^μ is the homogeneous left ideal of H_α generated by the elements (i) $1_{\boldsymbol{j}} - \delta_{\boldsymbol{j},\boldsymbol{i}^\mu}$ for all $\boldsymbol{j} \in I^\alpha$; (ii) y_r for all $r = 1, \dots, n$; (iii) ψ_r for all $1 \le r < n$ such that r and $r+1$ are in the same row of $\mathtt{T}^\mu$; (iv) g^A for all Garnir nodes $A \in \mu$.

We refer the reader to [11] for further developments on this presentation. A homogeneous basis of S^μ can now be given as follows [8]:

Theorem 6.2. *Let $\alpha \in Q_+$, $\mu \in \mathscr{P}_\alpha$ and $z^\mu \in S^\mu$ be the element from Theorem 6.1. For $\mathtt{T} \in \mathrm{St}(\mu)$, define $v^\mathtt{T} := \psi^\mathtt{T} z^\mu$. Then $v^\mathtt{T}$ is a homogeneous vector of degree $\deg(v^\mathtt{T}) = \deg(\mathtt{T})$,*

and $\{v^{\mathtt{T}} \mid \mathtt{T} \in \mathrm{St}(\mu)\}$ *is a basis of* S^μ. *In particular,*

$$\dim_q S^\mu = \sum_{\mathtt{T} \in \mathrm{St}(\mu)} q^{\deg(\mathtt{T})}.$$

7. Representation theory of KLR algebras

We now return to the KLR algebra A_α, defined in §2.4. Graded H_α-modules inflate to graded A_α-modules via the natural surjection (2.16). In particular, the irreducible modules D^μ inflate to irreducible A_α-modules. However, representation theory of A_α is more rich and perhaps more natural than that of H_α. For example, A_α has some important infinite dimensional modules, which H_α 'cannot see'.

So it is possible that understanding irreducible A_α-modules is a 'more manageable' and more natural task than understanding irreducible H_α-modules. By the way, irreducible H_α-modules D^μ can be distinguished among all irreducible A_α-modules by the simple condition that all words $\boldsymbol{i} = (i_1, \dots, i_n)$ appearing in the formal character of D^μ have the property that $i_1 = 0 \neq i_2$, see [33, Proposition 2.4].

One of the interesting ideas, due to Turner [41] and others, is 'incorporating' representation theory of smaller symmetric groups or Schur algebras into representation theory of Σ_n. Curiously, this phenomenon is appearing very naturally in representation theory of A_α in the form of the so-called imaginary Schur-Weyl duality described below. The results of this section are mainly from [28] and [30]; a different approach is suggested in [40].

7.1. Convex preorders. Recall the Cartan matrix $\mathtt{C}$ and the simple roots α_i labeled by $i \in I$. Let $I' = I \setminus \{0\}$. As in [19], we have the affine *root system* Φ and the corresponding finite root subsystem $\Phi' = \Phi \cap \mathbb{Z}\text{-}\operatorname{span}(\alpha_i \mid i \in I')$. Denote by Φ'_+ and Φ_+ the sets of *positive* roots in Φ' and Φ, respectively. Then $\Phi_+ = \Phi_+^{\mathrm{im}} \sqcup \Phi_+^{\mathrm{re}}$, where $\Phi_+^{\mathrm{im}} = \{n\delta \mid n \in \mathbb{Z}_{>0}\}$ for the null-root $\delta := \sum_{i \in I} \alpha_i$.

A *convex preorder* on Φ_+ is a preorder $\preceq$ such that the following three conditions hold for all $\beta, \gamma \in \Phi_+$:

(1) $\beta \preceq \gamma$ or $\gamma \preceq \beta$;

(2) if $\beta \preceq \gamma$ and $\beta + \gamma \in \Phi_+$, then $\beta \preceq \beta + \gamma \preceq \gamma$;

(3) $\beta \preceq \gamma$ and $\gamma \preceq \beta$ if and only if β and γ are proportional.

We fix a convex preorder $\preceq$ on Φ_+ such that $\alpha_i \succ n\delta \succ \alpha_0$ for all $i \in I'$; this is always possible. (This additional assumption is for convenient only.) Then

$$\{\beta \in \Phi_+^{\mathrm{re}} \mid \beta \succ \delta\} = \{\beta + n\delta \mid \beta \in \Phi'_+,\ n \in \mathbb{Z}_{\geq 0}\},$$
$$\{\beta \in \Phi_+^{\mathrm{re}} \mid \beta \prec \delta\} = \{-\beta + n\delta \mid \beta \in \Phi'_+,\ n \in \mathbb{Z}_{>0}\}.$$

We have that $\beta \preceq \gamma$ and $\gamma \preceq \beta$ happens for $\beta \neq \gamma$ if and only if both β and γ are imaginary. We write $\beta \prec \gamma$ if $\beta \preceq \gamma$ but $\gamma \not\preceq \beta$. The following set is *totally ordered* with respect to $\preceq$:

$$\Psi := \Phi_+^{\mathrm{re}} \cup \{\delta\}.$$

Let $l := p - 1$. An *l-multipartition* of n is a tuple $\underline{\mu} = (\mu^{(1)}, \dots, \mu^{(l)})$ of partitions such that $|\mu^{(1)}| + \dots + |\mu^{(l)}| = n$. The set of all l-multipartitions of n is denoted by $\mathscr{P}_n^l$,

and $\mathscr{P}^l := \sqcup_{n\geq 0}\mathscr{P}^l_n$. A *root partition of* $\alpha \in Q_+$ is a pair $(M, \underline{\mu})$, where M is a tuple $(m_\rho)_{\rho\in\Psi}$ of non-negative integers such that $\sum_{\rho\in\Psi} m_\rho\rho = \alpha$, and $\underline{\mu}$ is an l-multipartition of m_δ. Clearly all but finitely many integers m_ρ are zero, so we can always choose a finite subset

$$\rho_1 \succ \cdots \succ \rho_s \succ \delta \succ \rho_{-t} \succ \cdots \succ \rho_{-1}$$

of Ψ such that $m_\rho = 0$ for ρ outside of this subset. Then, denoting $m_u := m_{\rho_u}$, we can write any root partition of α in the form

$$(M, \underline{\mu}) = (\rho_1^{m_1}, \dots, \rho_s^{m_s}, \underline{\mu}, \rho_{-t}^{m_{-t}}, \dots, \rho_{-1}^{m_{-1}}), \tag{7.1}$$

where all $m_u \in \mathbb{Z}_{\geq 0}$, $\underline{\mu} \in \mathscr{P}^l$, and $\sum_{u=1}^s m_u\rho_u + |\underline{\mu}|\delta + \sum_{u=1}^t m_{-u}\rho_{-u} = \alpha$. We write $\Pi(\alpha)$ for the set of all root partitions of α.

Denote by $\mathtt{Se}$ the set of all finitary tuples $M = (m_\rho)_{\rho\in\Psi} \in \mathbb{Z}_{\geq 0}^\Psi$ of non-negative integers, so that a root partition is a pair $(M, \underline{\mu})$ with $M \in \mathtt{Se}$ and $\underline{\mu} \in \mathscr{P}^l_{m_\delta}$. The left lexicographic order on $\mathtt{Se}$ is denoted $\leq_l$ and the right lexicographic order on $\mathtt{Se}$ is denoted $\leq_r$. We will use the following *bilexicographic* partial order on $\mathtt{Se}$:

$$M \leq N \qquad \text{if and only if} \qquad M \leq_l N \text{ and } M \geq_r N.$$

We will use the following partial order on the set $\Pi(\alpha)$ of root partitions of α:

$$(M, \underline{\mu}) \leq (N, \underline{\nu}) \quad \text{if and only if} \quad M < N \text{ or } M = N \text{ and } \underline{\mu} = \underline{\nu}.$$

7.2. Cuspidal systems. Recall from (3.1) and (3.2) the functors $\mathrm{Ind}_{\alpha,\beta}$ and $\mathrm{Res}_{\alpha,\beta}$. For $M \in A_\alpha\text{-mod}$ and $N \in A_\beta\text{-mod}$, denote

$$M \circ N := \mathrm{Ind}_{\alpha,\beta} M \boxtimes N.$$

We also write $M^{\circ n}$ for $M \circ \cdots \circ M$ (n times).

A *cuspidal system* (for a fixed convex preorder) is the following data:

- A *cuspidal* irreducible A_ρ-module L_ρ assigned to every $\rho \in \Phi_+^{\mathrm{re}}$, i.e. an irreducible A_ρ-module with the following property: if $\beta, \gamma \in Q_+$ are non-zero elements such that $\rho = \beta + \gamma$ and $\mathrm{Res}_{\beta,\gamma} L_\rho \neq 0$, then β is a sum of roots less than ρ and γ is a sum of roots greater than ρ.
- An irreducible *imaginary* $A_{n\delta}$-module $L(\underline{\mu})$ assigned to every $\underline{\mu} \in \mathscr{P}^l_n$, i.e. an irreducible $A_{n\delta}$-module with the following property: if $\beta, \gamma \in Q_+ \setminus \Phi_+^{\mathrm{im}}$ are non-zero elements such that $n\delta = \beta + \gamma$ and $\mathrm{Res}_{\beta,\gamma} L(\underline{\mu}) \neq 0$, then β is a sum of real roots less than δ and γ is a sum of real roots greater than δ. In addition, it is required that $L(\underline{\lambda}) \not\cong L(\underline{\mu})$ unless $\underline{\lambda} = \underline{\mu}$.

Given a root partition π as in (7.1), set $\mathtt{sh}(\pi) := \sum_{\rho\in\Phi_+^{\mathrm{re}}} m_\rho(m_\rho - 1)/2 \in \mathbb{Z}$, and define the corresponding *(proper) standard module*:

$$\Delta(\pi) := q^{\mathtt{sh}(\pi)} L_{\rho_1}^{\circ m_1} \circ \cdots \circ L_{\rho_s}^{\circ m_s} \circ L(\underline{\mu}) \circ L_{\rho_{-t}}^{\circ m_{-t}} \circ \cdots \circ L_{\rho_{-1}}^{m_{-1}}. \tag{7.2}$$

Theorem 7.1. *For any convex preorder there exists a cuspidal system, unique up to permutation of irreducible imaginary modules. Moreover:*

(i) *For every root partition π, the standard module $\Delta(\pi)$ has irreducible head; denote this irreducible module $L(\pi)$.*

(ii) *$\{L(\pi) \mid \pi \in \Pi(\alpha)\}$ is a complete and irredundant system of irreducible A_α-modules up to isomorphism and degree shift.*

(iii) *For every root partition π, we have $L(\pi)^\circledast \cong L(\pi)$.*

(iv) *For all root partitions $\pi, \sigma \in \Pi(\alpha)$, we have that $[\Delta(\pi) : L(\pi)]_q = 1$, and $[\Delta(\pi) : L(\sigma)]_q \neq 0$ implies $\sigma \leq \pi$.*

(v) *The induced module $L_\rho^{\circ n}$ is irreducible for all $\rho \in \Phi_+^{\mathrm{re}}$ and $n \in \mathbb{Z}_{>0}$.*

7.3. Minuscule representations and imaginary tensor spaces. Theorem 7.1 gives a 'rough classification' of irreducible A_α-modules. The main problem is that we did not give a canonical definition of individual irreducible imaginary modules $L(\underline{\mu})$. So far, we just know that the amount of such modules for $A_{n\delta}$ is equal to the number of l-multipartitions of n, and we have labeled them by such multipartitions in an arbitrary way.

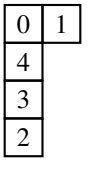
To address this problem, we begin with an explicit description of the *minuscule* representations—the irreducible imaginary A_δ-modules. These correspond to l-multipartitions of 1. There are of course exactly l such multipartitions, namely $\underline{\mu}(1), \dots, \underline{\mu}(l)$, where

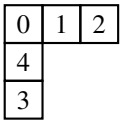
$$\underline{\mu}(i) := (\emptyset, \dots, \emptyset, (1), \emptyset, \dots, \emptyset)$$

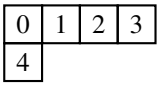
with the partition (1) in the ith position.

Let $i \in I' = \{1, 2, \dots, p-1\}$ (we identify the set I of residues modulo p with integers $0, 1, \dots, p-1$). Consider the hook partition $\chi^i = (i, 1^{p-i})$ for all $i \in I'$. For example, if $p = 5$, here are the corresponding Young diagram with the residues of the boxes written in them.

$\chi^1 =$ 

$\chi^2 =$ 

$\chi^3 =$ 

$\chi^4 =$ 

Note that the partitions χ^i for $i \in I'$ are homogenous in the sense of §5.5. In particular, we have the corresponding homogeneous irreducible H_δ-modules D^{χ^i} defined explicitly in Theorem 5.9. Define the A_δ-modules

$$L(\underline{\mu}(i)) := L_{\delta,i} := \operatorname{infl} D^{\chi^i} \qquad (i \in I').$$

For example, $L_{\delta,1}$ and $L_{\delta,p-1}$ are 1-dimensional with characters

$$\operatorname{ch}_q L_{\delta,1} = (0, p-1, p-2, \dots, 1), \quad \operatorname{ch}_q L_{\delta,p-1} = (0, 1, 2, \dots, p-1),$$

while for $p > 3$, the module $L_{\delta,p-2}$ is $(p-2)$-dimensional with character

$$\operatorname{ch}_q L_{\delta,p-2} = \textstyle\sum_{r=0}^{p-3} (0, 1, \dots, r, p-1, r+1, \dots, p-2).$$

Define the *imaginary tensor space of color $i \in I'$* to be the $A_{n\delta}$-module

$$M_{n,i} := L_{\delta,i}^{\circ n}.$$

Fix for now $i \in I'$ and suppress i from the indices, so that we have the imaginary tensor space $M_n = M_{n,i}$. The $A_{n\delta}$-module structure on M_n yields an algebra homomorphism $A_{n\delta} \to \mathrm{End}_F(M_n)$. Define the *imaginary Schur algebra* $\mathscr{S}_n$ as the image of $A_{n\delta}$ under this homomorphism, i.e. $\mathscr{S}_n = A_{n\delta}/\mathrm{Ann}_{A_{n\delta}}(M_n)$. Modules over $A_{n\delta}$ which factor through to $\mathscr{S}_n$ are called *imaginary modules* (of color i). It turns out that this notion agrees with the notion of an irreducible imaginary module in the sense of cuspidal systems.

Theorem 7.2. *Let $n \in \mathbb{Z}_{>0}$. Then:*

(i) *M_n is a projective $\mathscr{S}_n$-module.*

(ii) *The endomorphism algebra $\mathrm{End}_{A_{n\delta}}(M_n)^{\mathrm{op}} = \mathrm{End}_{\mathscr{S}_n}(M_n)^{\mathrm{op}}$ is isomorphic to the group algebra $F\Sigma_n$ of the symmetric group Σ_n (concentrated in degree zero). Thus M_n can be considered as a right $F\Sigma_n$-module.*

(iii) $\mathrm{End}_{F\Sigma_n}(M_n) = \mathscr{S}_n$.

In view of the theorem, we have an exact functor

$$\gamma_n : \mathscr{S}_n\text{-mod} \to F\Sigma_n\text{-mod}, \quad V \mapsto \mathrm{Hom}_{\mathscr{S}_n}(M_n, V).$$

Unless $p > n$ or $p = 0$, the $\mathscr{S}_n$-module M_n is not a projective generator, and γ_n is not an equivalence of categories. To fix this problem, we need to upgrade from the *imaginary Schur-Weyl duality* of Theorem 7.2 to an *imaginary Howe duality*.

7.4. Imaginary How and Ringel dualities. Let $\mathtt{x}_n := \sum_{g \in \Sigma_n} g$. Define the *imaginary exterior* and *imaginary divided* powers respectively as follows:

$$\Lambda_n := M_n \mathtt{x}_n, \quad Z_n := \{m \in M_n \mid mg - \mathtt{sgn}(g)m = 0 \text{ for all } g \in \Sigma_n\}.$$

For $h \in \mathbb{Z}_{>0}$, denote by $X(h,n)$ the set of all compositions of n with h parts:

$$X(h,n) := \{(n_1, \dots, n_h) \in \mathbb{Z}_{\geq 0}^h \mid n_1 + \dots + n_h = n\}.$$

The corresponding set of partitions is

$$X_+(h,n) := \{(n_1, \dots, n_h) \in X(h,n) \mid n_1 \geq \dots \geq n_h\}.$$

For a composition $\nu = (n_1, \dots, n_h) \in X(h,n)$, we define the functor of *imaginary induction*:

$$I_\nu^n := \mathrm{Ind}_{n_1\delta,\dots,n_h\delta} : A_{n_1\delta,\dots,n_h\delta}\text{-mod} \to A_{n\delta}\text{-mod}\,.$$

Given imaginary $A_{n_b\delta}$-modules V_b for $b = 1, \dots, h$, the module $I_\nu^n(V_1 \boxtimes \dots \boxtimes V_h)$ is also imaginary. Define

$$Z^\nu := I_\nu^n(Z_{n_1} \boxtimes \dots \boxtimes Z_{n_h}), \quad \Lambda^\nu := I_\nu^n(\Lambda_{n_1} \boxtimes \dots \boxtimes \Lambda_{n_h}).$$

Now, let $S_{h,n}$ be the classical Schur algebra, whose representations are the same as the degree n polynomial representations of the general linear group $GL_h(F)$ [14]. It is a finite dimensional quasi-hereditary algebra with irreducible, standard, costandard, and indecomposable tilting modules

$$L_h(\lambda),\ \Delta_h(\lambda),\ \nabla_h(\lambda),\ T_h(\lambda) \qquad (\lambda \in X_+(h,n)).$$

Theorem 7.3. *We have:*

(i) *For each $\nu \in X(h,n)$ the $\mathscr{S}_n$-module Z^ν is projective. Moreover, for any $h \geq n$, the module $Z := \bigoplus_{\nu \in X(h,n)} Z^\nu$ is a projective generator for $\mathscr{S}_n$.*

(ii) *The endomorphism algebra $\mathrm{End}_{\mathscr{S}_n}(Z)^{\mathrm{op}}$ is isomorphic to the classical Schur algebra $S_{h,n}$ concentrated in degree zero. Thus Z can be considered as a right $S_{h,n}$-module.*

(iii) $\mathrm{End}_{S_{h,n}}(Z) = \mathscr{S}_n$.

This theorem allows us to use Morita theory and define quasi-inverse equivalences of categories:

$$\alpha_{h,n} : \mathscr{S}_n\text{-mod} \to S_{h,n}\text{-mod}, \quad V \mapsto \mathrm{Hom}_{\mathscr{S}_n}(Z, V) \tag{7.3}$$

$$\beta_{h,n} : S_{h,n}\text{-mod} \to \mathscr{S}_n\text{-mod}, \quad W \mapsto Z \otimes_{S_{h,n}} W. \tag{7.4}$$

Let $\mu \in \mathscr{P}_n$ and $h \geq n$. We can also consider μ as an element of $X_+(h,n)$. Define the $\mathscr{S}_n$-modules:

$$L(\mu) := \beta_{h,n}(L_h(\mu)),\ \Delta(\mu) := \beta_{h,n}(\Delta_h(\mu)),$$
$$\nabla(\mu) := \beta_{h,n}(\nabla_h(\mu)),\ T(\mu) := \beta_{h,n}(T_h(\mu)).$$

These definitions turn out to be independent of the choice of $h \geq n$. An easy consequence of the theorem above is that the imaginary Schur algebra $\mathscr{S}_n$ is a finite dimensional quasi-hereditary algebra with irreducible, standard, costandard, and indecomposable tilting modules $L(\mu),\ \Delta(\mu),\ \nabla(\mu),\ T(\mu)$ labeled by $\mu \in X_+(h,n)$. In particular, inflating the irreducible modules $L(\mu)$ from $\mathscr{S}_n$ to $A_{n\delta}$, we get:

Theorem 7.4. *The irreducible imaginary $A_{n\delta}$-modules of color i are exactly the modules $\{L(\lambda) \mid \lambda \in \mathscr{P}_n\}$ (up to isomorphism).*

Moreover:

Theorem 7.5. *We have:*

(i) *Let $h \geq n$. The $\mathscr{S}_n$-module $\bigoplus_{\nu \in X(h,n)} \Lambda^\nu$ is a full tilting module.*

(ii) *We have isomorphisms of endomorphism algebras*

$$\mathrm{End}_{\mathscr{S}_n}\left(\bigoplus_{\nu \in X(h,n)} \Lambda^\nu\right)^{\mathrm{op}} \cong S_{h,n} \quad \textit{and} \quad \mathrm{End}_{S_{h,n}}\left(\bigoplus_{\nu \in X(h,n)} \Lambda^\nu\right) \cong \mathscr{S}_n.$$

The additional nice property of the constructed Morita equivalence is that imaginary induction commutes with tensor products:

Theorem 7.6. *Let $h \geq n$ and $\nu = (n_1, \dots, n_a) \in X(a,n)$. The following functors are isomorphic:*

$$I^n_\nu(\beta_{h,n_1} - \boxtimes \cdots \boxtimes \beta_{h,n_a} -) : S_{h,n_1}\text{-mod} \times \cdots \times S_{h,n_a}\text{-mod} \to \mathscr{S}_n\text{-mod},$$
$$\beta_{h,n}(- \otimes \cdots \otimes -) : S_{h,n_1}\text{-mod} \times \cdots \times S_{h,n_a}\text{-mod} \to \mathscr{S}_n\text{-mod}\,.$$

7.5. Gelfand-Graev character fragment and imaginary Jacobi-Trudi formula. We can say quite a bit about the characters of irreducible imaginary modules. An important role in the paper is played by an analogue of the *Gelfand-Graev representation*, cf. e.g. [4]. Let

$$\boldsymbol{i} = (i_1, \dots, i_p)$$

be any word appearing in the (explicitly known) formal character of L_δ. Define the corresponding *Gelfand-Graev words* $\boldsymbol{g}_{\boldsymbol{i}}^{(t)} := i_1^t i_2^t \dots i_p^t$ for all $t \in \mathbb{Z}_{>0}$. For any composition $\mu = (\mu_1, \dots, \mu_n) \in X(n,n)$ define

$$\boldsymbol{g}_{\boldsymbol{i}}^{\mu} := \boldsymbol{g}_{\boldsymbol{i}}^{(\mu_1)} \dots \boldsymbol{g}_{\boldsymbol{i}}^{(\mu_n)}$$

and $c_{\boldsymbol{i}}(\mu) := ([\mu_1]_q^! \dots [\mu_n]_q^!)^p \in \mathcal{L}$. If $V \in A_{n\delta}$-mod, it is known that

$$\dim_q V_{\boldsymbol{g}_{\boldsymbol{i}}^{\mu}} = c_{\boldsymbol{i}}(\mu) m_{\boldsymbol{i},\mu}(V)$$

for some $m_{\boldsymbol{i},\mu}(V) \in \mathcal{L}$. We are going to describe $m_{\boldsymbol{i},\mu}(V)$ for many important imaginary modules.

There are explicitly defined *Gelfand-Graev idempotents* $\gamma_{n,\boldsymbol{i}} \in A_{n\delta}$. The *Gelfand-Graev module* is the projective module $\Gamma_{n,\boldsymbol{i}} := q^{-pn(n-1)/2} A_{n\delta} \gamma_{n,\boldsymbol{i}}$.

Theorem 7.7. *For any $V \in A_{n\delta}$-mod and $\mu \in X(n,n)$, we have*

$$m_{\boldsymbol{i},\mu}(V) = \dim_q \operatorname{Hom}_{A_{n\delta}}(\Gamma_{\mu_1,\boldsymbol{i}} \circ \dots \circ \Gamma_{\mu_n,\boldsymbol{i}}, V).$$

Since we have equivalences of categories (7.3) and (7.4), every finite dimensional graded $\mathscr{S}_n$-module V can be written as $V = \beta_{n,n}(W)$ (up to degree shift). Then we can describe the *Gelfand-Graev fragment* of $\operatorname{ch}_q V$ as follows:

Theorem 7.8. *Let $\boldsymbol{i}$ be any word appearing in the formal character of L_δ, $\mu \in X(n,n)$, $W \in S_{h,n}$-mod, and $V = \beta_{n,n}(W) \in \mathscr{S}_n$-mod. Denote by W_μ the usual weight space of W. Then*

$$\dim_q V_{\boldsymbol{g}_{\boldsymbol{i}}^{\mu}} = c_{\boldsymbol{i}}(\mu) \dim W_\mu.$$

Note that the Gelfand-Graev fragment is described in terms of the formal characters of a 'smaller rank' Schur algebra.

The formal characters of the modules $\Delta(\mu)$ are important; for example in the case $p > n$ we have $\Delta(\mu) = L(\mu)$. An *imaginary Jacobi-Trudi formula* allows us to compute the formal characters of the modules $\Delta(\lambda)$ explicitly.

First of all, the characters of the modules $\Delta(1^n) = \Lambda_n = L(1^n)$ are well-understood: let $\boldsymbol{i} = (i_1, \dots, i_p)$ be a word appearing in L_δ. Then $\boldsymbol{i}^n$ is a word of $\Delta(1^n)$, and $\Delta(1^n)$ is the homogeneous irreducible module associated to the connected component of $\boldsymbol{i}^n$ in the word graph, see [31].

Let $\mu = (\mu_1, \dots, \mu_a) \in \mathscr{P}_n$. Denote by $\circ$ the quantum shuffle product, see e.g. [20, §2]. Then $\operatorname{ch}_q \Delta(1^k) \circ \operatorname{ch}_q \Delta(1^l) = \operatorname{ch}_q \Delta(1^l) \circ \operatorname{ch}_q \Delta(1^k)$ for all $k, l \in \mathbb{Z}_{>0}$. So we can use the quantum shuffle product to make sense of the following determinant as an element of $\mathscr{A}\langle I \rangle_{n\delta}$:

$$\mathbf{D}(\mu) := \det\left(\operatorname{ch}_q \Delta(1^{\mu_r - r + s})\right)_{1 \le r,s \le a}.$$

where $\mathrm{ch}_q\,\Delta(1^0)$ is interpreted as (multiplicative) identity, and $\mathrm{ch}_q\,\Delta(1^m)$ is interpreted as (multiplicative) zero if $m<0$. For example, for $\mu=(3,1,1)$:

$$\begin{aligned}\mathbf{D}((3,1,1)) =& \det\begin{pmatrix}\mathrm{ch}_q\,\Delta(1^3) & \mathrm{ch}_q\,\Delta(1^4) & \mathrm{ch}_q\,\Delta(1^5)\\ 1 & \mathrm{ch}_q\,\Delta(1) & \mathrm{ch}_q\,\Delta(1^2)\\ 0 & 1 & \mathrm{ch}_q\,\Delta(1)\end{pmatrix}\\ =&\mathrm{ch}_q\,\Delta(1^3)\circ\mathrm{ch}_q\,\Delta(1)\circ\mathrm{ch}_q\,\Delta(1)+\mathrm{ch}_q\,\Delta(1^5)\\ &-\mathrm{ch}_q\,\Delta(1^4)\circ\mathrm{ch}_q\,\Delta(1)-\mathrm{ch}_q\,\Delta(1^3)\circ\mathrm{ch}_q\,\Delta(1^2).\end{aligned}$$

Theorem 7.9. *Let μ^{tr} be the partition transpose to μ. Then $\mathrm{ch}_q\,\Delta(\mu)=\mathbf{D}(\mu^{\mathrm{tr}})$.*

For example, let $p=2$. Then $I'=\{1\}$. The $i=1$. In this case the character of L_δ is $(0,1)$, and the character of $\Delta(1^n)$ is $(0,1,0,1,\dots,0,1)$. So

$$D((1,1))=\det\begin{pmatrix}\boldsymbol{i}_\delta & \boldsymbol{i}_{2\delta}\\ 1 & \boldsymbol{i}_\delta\end{pmatrix}=\boldsymbol{i}_\delta\circ\boldsymbol{i}_\delta-\boldsymbol{i}_{2\delta}=(0101)+(q+q^{-1})^2(0011).$$

7.6. Classification of imaginary irreducible modules. In Theorem 7.4, we have classified the irreducible imaginary representations of $A_{n\delta}$ of color i. Since we now want to distinguish between the imaginary representations of different colors, we will use the notation $L_i(\mu)$ for these irreducible imaginary representations of color i corresponding to a partition μ.

Theorem 7.10. *For an l-multipartition $\underline{\lambda}=(\lambda^{(1)},\dots,\lambda^{(l)})$ of n, define*

$$L(\underline{\lambda}):=L_1(\lambda^{(1)})\circ\cdots\circ L_l(\lambda^{(l)}).$$

Then $\{L(\underline{\lambda})\mid\underline{\lambda}\in\mathscr{P}_n^l\}$ is a complete and irredundant system of imaginary irreducible $A_{n\delta}$-modules.

Acknowledgements. Research supported by the NSF grant no. DMS-1161094 and the Humboldt Foundation.

References

[1] S. Ariki, *On the decomposition numbers of the Hecke algebra of $G(m,1,n)$*, J. Math. Kyoto Univ., **36** (1996), 789–808.

[2] C. Bessenrodt and J. B. Olsson, *On residue symbols and the Mullineux conjecture*, J. Algebraic Combin., **7** (1998), 227–251.

[3] J. Brundan, *Centers of degenerate cyclotomic Hecke algebras and parabolic category $\mathcal{O}$*, Represent. Theory, **12** (2008), 236–259.

[4] J. Brundan, R. Dipper, and A. Kleshchev, *Quantum linear groups and representations of $GL_n(\mathbb{F}_q)$*, Mem. Amer. Math. Soc., **149** (2001), No 706, viii+112.

[5] J. Brundan and A. Kleshchev, *On translation functors for general linear and symmetric groups*, Proc. London Math. Soc. (3), **80** (2000), 75–106.

[6] ———, *Blocks of cyclotomic Hecke algebras and Khovanov-Lauda algebras*, Invent. Math., **178** (2009), 451–484.

[7] ———, *Graded decomposition numbers for cyclotomic Hecke algebras*, Adv. Math., **222** (2009), 1883–1942.

[8] J. Brundan, A. Kleshchev, and W. Wang, *Graded Specht modules*, J. reine angew. Math., **655** (2011), 61–87.

[9] J. Brundan and J. Kujawa, *A new proof of the Mullineux conjecture*, J. Algebraic Combin., **18** (2003), 13–39.

[10] J. Chuang and R. Rouquier, *Derived equivalences for symmetric groups and $\mathfrak{sl}_2$-categorification*, Ann. of Math. (2), **167** (2008), 245–298.

[11] M. Fayers, *Dyck tilings and the homogeneous Garnir relations for graded Specht modules*, arXiv:1309.6467.

[12] B. Ford and A. Kleshchev, *A proof of the Mullineux conjecture*, Math. Z., **226** (1997), 267–308.

[13] J. J. Graham and G. I. Lehrer, *Cellular algebras*, Invent. Math., **123** (1996), 1–34.

[14] J.A. Green, *Polynomial representations of GL_n*, Springer-Verlag, Berlin, 2007.

[15] I. Grojnowski, *Affine $\mathfrak{sl}_p$ controls the representation theory of the symmetric group and related Hecke algebras*, arXiv:math.RT/9907129.

[16] I. Grojnowski and M. Vazirani, *Strong multiplicity one theorems for affine Hecke algebras of type A*, Transform. Groups, **6** (2001), 143–155.

[17] J. Hu and A. Mathas, *Graded cellular bases for the cyclotomic Khovanov-Lauda-Rouquier algebras of type A*, Adv. Math., **225** (2010), 598–642.

[18] G. D. James, *The representation theory of the symmetric groups*, Lecture Notes in Mathematics 682, Springer, Berlin, 1978.

[19] V. G. Kac, *Infinite Dimensional Lie Algebras*, Cambridge University Press, Cambridge, 1990.

[20] M. Khovanov and A. Lauda, *A diagrammatic approach to categorification of quantum groups I.*, Represent. Theory, **13** (2009), 309–347.

[21] ———, *A diagrammatic approach to categorification of quantum groups II.*, Trans. Amer. Math. Soc., **363** (2011), 2685–2700.

[22] A. Kleshchev, *Branching rules for modular representations of symmetric groups II.*, J. reine angew. Math., **459** (1995), 163–212.

[23] ———, *Branching rules for modular representations of symmetric groups III: some corollaries and a problem of Mullineux*, J. London Math. Soc., **54** (1996), 25–38.

[24] ———, *Completely splittable representations of symmetric groups*, J. Algebra, **181** (1996), 584–592.

[25] ———, *On the decomposition numbers and branching coefficients for symmetric and special linear groups*, Proc. Lond. Math. Soc. (3), **75** (1997), 497–558.

[26] ———, *Branching rules for modular representations of symmetric groups, IV.*, J. Algebra, **201** (1998), 547–572.

[27] ———, *Linear and Projective Representations of Symmetric Groups*, Cambridge University Press, Cambridge, 2005.

[28] ———, *Cuspidal systems for affine Khovanov-Lauda-Rouquier algebras*, Math. Z., **276** (2014), 691–726.

[29] A. Kleshchev, A. Mathas, and A. Ram, *Universal graded Specht modules for cyclotomic Hecke algebras*, Proc. Lond. Math. Soc. (3), **105** (2012), 1245–1289.

[30] A. Kleshchev and R. Muth, *Imaginary Schur-Weyl duality*, arXiv:1312.6104.

[31] A. Kleshchev and A. Ram, *Homogeneous representations of Khovanov-Lauda algebras*, J. Eur. Math. Soc., **12** (2010), 1293–1306.

[32] A. Lascoux, B. Leclerc, and J.-Y. Thibon, *Hecke algebras at roots of unity and crystal bases of quantum affine algebras*, Commun. Math. Phys., **181** (1996), 205–263.

[33] A. Lauda and M. Vazirani, *Crystals from categorified quantum groups*, Adv. Math., **228** (2011), 803–861.

[34] S. Lyle and A. Mathas, *Blocks of cyclotomic Hecke algebras*, Adv. Math., **216** (2007), 854–878.

[35] O. Mathieu, *On the dimension of some modular irreducible representations of the symmetric group*, Lett. Math. Phys., **38** (1996), 23–32.

[36] K. C. Misra and T. Miwa, *Crystal base of the basic representation of* $U_q(\widehat{\mathfrak{sl}}_n)$, Comm. Math. Phys., **134** (1990), 79–88.

[37] G. Mullineux, *Bijections of* p*-regular partitions and* p*-modular irreducibles of the symmetric groups*, J. London Math. Soc. (2), **20** (1979), 60–66.

[38] R. Rouquier, *2-Kac-Moody algebras*, arXiv:0812.5023.

[39] ———, *Quiver Hecke algebras and* 2*-Lie algebras*, Algebra Colloq., **19** (2012), 359–410.

[40] P. Tingley and B. Webster, *Mirkovic-Vilonen polytopes and Khovanov-Lauda-Rouquier algebras*, arXiv:1210.6921.

[41] W. Turner, *Rock blocks*, Mem. Amer. Math. Soc., **202** (2009), no. 947, viii+102.

[42] M. Varagnolo and E. Vasserot, *Canonical bases and KLR-algebras*, J. Reine Angew. Math., **659** (2011), 67–100.

[43] M. Xu, *On Mullineux conjecture in the representation theory of symmetric groups*, Comm. Algebra, **25** (1997), 1797–1803.

Department of Mathematics, University of Oregon, Eugene, OR 97403, USA
E-mail: klesh@uoregon.edu

Multi-fusion categories of Harish-Chandra bimodules

Victor Ostrik

Abstract. We survey some results on tensor products of irreducible Harish-Chandra bimodules. It turns out that such tensor products are semisimple in suitable Serre quotient categories. We explain how to identify the resulting semisimple tensor categories and describe some applications to representation theory.

Mathematics Subject Classification (2010). Primary 17B35, 18D10; Secondary 22E47, 14F05.

Keywords. Harish-Chandra modules, tensor categories, finite W-algebras.

1. Introduction

The notion of tensor category is ubiquitous in representation theory. A classical example is the theory of Tannakian categories (see [17, 58]) which shows that a linear algebraic group can be recovered from the tensor category of its finite dimensional representations. The tensor product in a Tannakian category is commutative in a very strong sense. In this note we will be interested in tensor categories for which the tensor product is not assumed to be commutative. One reason for the relevance of such categories to representation theory is very simple: the category of bimodules over an arbitrary algebra A is a tensor category with tensor product given by tensoring over A and this tensor product is non-commutative in general.

A classical notion in the representation theory of a complex semisimple Lie algebra $\mathfrak{g}$ is that of *Harish-Chandra bimodules*. These objects were introduced by Harish-Chandra [24] in order to reduce some questions of continuous representation theory of complex semisimple groups (considered as real Lie groups) to pure algebra. A number of deep results on Harish-Chandra bimodules are known, see e.g. [7, 26, 61]. In this note we will be interested in just one aspect of the theory, namely, in the structure of the tensor category of Harish-Chandra bimodules. Some significant steps towards a complete description of this category were made in [61], however due to non-semisimplicity this description is necessarily quite complicated. It turns out that the classical notion of *associated variety* (see e.g. [27, 65]) provides us with a kind of filtration on this category; moreover one can define an "associated graded" category with respect to this filtration which is much simpler than the original category but still carries an important information about the category of Harish-Chandra bimodules. Thus one defines certain interesting semisimple subquotients of the tensor category Harish-Chandra bimodules associated with various nilpotent orbits in $\mathfrak{g}$ which we call *cell categories*, see Section 3.4. The idea of this definition can be traced back to the work of Joseph [29] and the name is

Proceedings of the International Congress of Mathematicians, Seoul, 2014

justified by the connection with the theory of Kazhdan-Lusztig cells [31]. A nice property of the cell categories is that they are *multi-fusion* in a sense of [21]. The known results from the theory of multi-fusion categories turned out to be powerful enough to identify these categories with some categories constructed from some finite groups. This is interesting in its own right but also gives a better understanding of some notions of representation theory such as Lusztig's quotients and Lusztig's subgroups.

One can hope to apply the ideas above in the following way. The Harish-Chandra bimodules act on various categories of $\mathfrak{g}-$modules via tensoring over the universal enveloping algebra of $\mathfrak{g}$. We can exploit this action restricted to the semisimple subquotients as above in order to obtain interesting information about such categories of $\mathfrak{g}-$modules. One example of such application is the theory of finite $W-$algebras where this strategy allowed to obtain the information on the number of finite dimensional simple modules, see Section 4. It was suggested by Bezrukavnikov that similar approach might work for Harish-Chandra modules, see Section 4.3.

The cell categories described above can be realized via truncated convolution of some perverse sheaves on the flag variety associated with $\mathfrak{g}$. One advantage of this description is a greater flexibility. For example we can replace the complex semisimple Lie algebra $\mathfrak{g}$ by a semisimple algebraic group G defined over a field of arbitrary characteristic. Moreover, using the tensor categorical construction of the *Drinfeld center* one connects the cell categories with the theory of *character sheaves* on G, see [6, 11, 45]. We describe briefly these developments in Section 5.

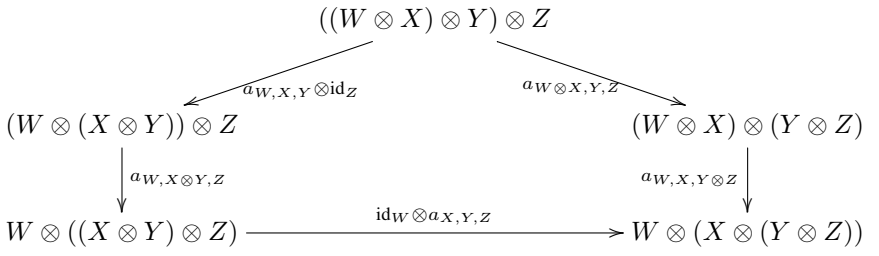
This paper is organized as follows. In Section 2 we review briefly some notions of the theory of tensor categories. In Section 3 we introduce the Harish-Chandra bimodules and define the cell categories. In Section 4 we explain how to use Whittaker modules and Premet's $W-$algebras in order to establish some basic properties of the cell categories. Conversely we show that the actions of the cell categories can be used in order to get an information about finite dimensional representations of $W-$algebras. Finally in Section 5 we describe the interaction of the cell categories and some classes of sheaves on algebraic varieties associated with $\mathfrak{g}$.

My understanding of the subject described in this note was deeply influenced by collaborations with Roman Bezrukavnikov, Pavel Etingof, Michael Finkelberg, Ivan Losev, and Dmitri Nikshych. It is my great pleasure to express my sincere gratitude and appreciation to them. Thanks are also due to Jonathan Brundan, Victor Ginzburg, and Ivan Losev for useful comments on a preliminary version of this paper.

2. Multi-fusion categories

2.1. Monoidal categories. For the purposes of this note a *monoidal category* is a quadruple $(\mathcal{C}, \otimes, a, \mathbf{1})$ where $\mathcal{C}$ is a category, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bifunctor (called *tensor product*), a is a natural isomorphism $a_{X,Y,Z} : (X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z)$ (so a is called *associativity constraint*), $\mathbf{1} \in \mathcal{C}$ is an object (called *unit object*) such that the following axioms hold:

1. Pentagon axiom: the following diagram commutes for all $W, X, Y, Z \in \mathcal{C}$:



2. Unit axiom: both functors $\mathbf{1}\otimes?$ and $?\otimes\mathbf{1}$ are isomorphic to the identity functor.

It is well known that this definition of monoidal category reduces to the traditional one (see e.g. [46]) if we fix an isomorphism $\mathbf{1}\otimes\mathbf{1}\simeq\mathbf{1}$.

Further one defines natural notions of tensor functors and tensor equivalences, see e.g. [46]. From a practical point of view tensor equivalent monoidal categories are indistinguishable.

A basic example of monoidal category is category of $R-$bimodules over a ring (with unity) R. In this case the tensor product is tensor product $\otimes_R$ over R, the unit object is R considered as a bimodule, and the associativity constraint is the obvious one. A closely related example is a category of endofunctors of a category with tensor product given by the composition. Also modules over a *commuative* ring form a monoidal category, most familiar example being the category of vector spaces over a field k.

Here is another more abstract example.

Example 2.1 ([59]). Let A be a group and let S be an abelian group, both written multiplicatively. We consider the category where the objects are elements of A and the morphisms are given by $\mathrm{Hom}(g,h)=\emptyset$ if $g\neq h$ and $\mathrm{Hom}(g,g)=S$ for any g. We have a bifunctor $g\otimes h=gh$ and $\alpha\otimes\beta=\alpha\beta$ for objects $g,h\in A$ and morphisms $\alpha,\beta\in S$. The associativity constraint amounts to a morphism $\omega_{g,h,k}\in\mathrm{Hom}(ghk,ghk)=S$ for any three elements $g,h,k\in A$. One verifies that the pentagon axiom reduces to the equation $\partial\omega=1$ which says that ω is a 3-cocycle on A with values in S. Moreover, any 2-cochain ψ determines a tensor structure on the identity functor between tensor categories with associativity constraints given by 3-cocycles ω and $\omega\cdot\partial\psi$. We see that monoidal structures on our category are parameterized by the cohomology group $H^3(A,S)$.

We explain now that nontrivial associativity constraints do appear in tensor categories of bimodules.

Example 2.2. Let R be an algebra over a field k with trivial center. Recall that $R-$bimodule M is *invertible* if there exists a $R-$bimodule N such that $M\otimes_R N\simeq N\otimes_R M\simeq R$. The invertible bimodules form a tensor category with respect to $\otimes_R$ (morphisms being the *isomorphisms* of bimodules). This category is equivalent to the category of type described in Example 2.1. The group of automorphisms of any object is $k^\times$, so the associativity constraint determines a class $\omega\in H^3(\mathrm{Pic}(R),k^\times)$ where $\mathrm{Pic}(R)$ is the group of isomorphism classes of invertible $R-$bimodules (this is the non-commutative *Picard group* of R). This class is often nontrivial. Indeed for any automorphism ϕ of R we can define invertible bimodule R_ϕ as follows: $R_\phi=R$ as a vector space and the action is given by $(a,b)\cdot c:=ac\phi(b)$. The bimodule R_ϕ is isomorphic to R if and only if ϕ is inner, so we get a well known embedding

$\mathrm{Out}(R) \subset \mathrm{Pic}(R)$. Now assume that ϕ is outer automorphism such that ϕ^2 is inner, that is $\phi^2(x) = gxg^{-1}$ for some invertible element $g \in R$ and all $x \in R$; thus ϕ generates subgroup $\mathbb{Z}/2\mathbb{Z} \subset \mathrm{Out}(R) \subset \mathrm{Pic}(R)$. It is easy to see that then $\phi(g) = \pm g$; we leave it to the reader to check that the restriction of the class ω to $\mathbb{Z}/2\mathbb{Z}$ is nontrivial if and only if $\phi(g) = -g$. Here is an example when this is the case:

$$R = \mathbb{C}\langle g, x, y\rangle/(xy - yx - 1, g^2 - 1, gx + xg, gy + yg), \;\; \phi(g) = -g, \phi(x) = -y, \phi(y) = x.$$

Remark 2.3. For a commutative algebra R one defines the category of invertible $R-$modules similarly to Example 2.2. Since we have canonically $M \otimes_R N \simeq N \otimes_R M$, this category has an additional structure of *symmetric tensor category*. This implies that the cohomology class representing the associativity constraint (see Example 2.1) is always trivial, see [19, Section 7].

A crucially important technical assumption on a monoidal category $\mathcal{C}$ is that of *rigidity*. We recall that for an object $X \in \mathcal{C}$ its *right dual* is an object $X^* \in \mathcal{C}$ together with *evaluation* and *coevaluation* morphisms $\mathrm{ev}_X : X^* \otimes X \to \mathbf{1}$ and $\mathrm{coev}_X : \mathbf{1} \to X \otimes X^*$ such that the composition

$$X \xrightarrow{\mathrm{coev}_X \otimes \mathrm{id}_X} (X \otimes X^*) \otimes X \xrightarrow{a_{X,X^*,X}} X \otimes (X^* \otimes X) \xrightarrow{\mathrm{id}_X \otimes \mathrm{ev}_X} X$$

equals the identity morphism and the composition

$$X^* \xrightarrow{\mathrm{id}_{X^*} \otimes \mathrm{coev}_X} X^* \otimes (X \otimes X^*) \xrightarrow{a^{-1}_{X^*,X,X^*}} (X^* \otimes X) \otimes X^* \xrightarrow{\mathrm{ev}_X \otimes \mathrm{id}_{X^*}} X^*$$

equals the identity morphism. Similarly, a *left dual* of X is ${}^*X \in \mathcal{C}$ such that X is right dual of ${}^*X$. A monoidal category $\mathcal{C}$ is *rigid* if any object of $\mathcal{C}$ has both left and right duals.

Example 2.4. A vector space considered as an object of the monoidal category of vector spaces has left or right dual if and only if it is finite dimensional. A bimodule over an algebra A has a right dual if and only if it is finitely generated projective when considered as a left $A-$module. The category considered in Example 2.1 is always rigid.

2.2. Semisimplicity. We will fix an algebraically closed field k of characteristic zero. We recall that a $k-$linear category $\mathcal{C}$ is called *semisimple* if there is a collection $\{L_i\}_{i\in J}$ of objects of $\mathcal{C}$ such that

(i) $\dim_k \mathrm{Hom}(L_i, L_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases}$

(ii) any object of $\mathcal{C}$ is isomorphic to a finite direct sum of the objects L_i and any direct sum (including the empty one) is contained in $\mathcal{C}$.

For example if $\mathcal{A}$ as a $k-$linear abelian category with finite dimensional spaces of morphisms the category of semisimple objects in $\mathcal{A}$ (that is the full subcategory consisting of direct sums of simple objects) is semisimple.

Clearly the isomorphism classes of the objects L_i are uniquely determined by the category $\mathcal{C}$. These objects are *simple objects* of $\mathcal{C}$ (note that zero object is not simple).

For a semisimple category $\mathcal{C}$ let $K(\mathcal{C})$ be its *Grothendieck group*; this is a free abelian group with basis $[L_i]_{i\in J}$. For any $M \in \mathcal{C}$ we have its *class*:

$$[M] := \sum_{i\in J} \dim_k \mathrm{Hom}(L_i, M)[L_i] \in K(\mathcal{C}).$$

We say that a semisimple category is *finite* if the isomorphism classes of simple objects form a finite set.

Definition 2.5. A *multi-fusion category* over k is a $k-$linear rigid monoidal category which is finite semisimple. A *fusion category* is a multi-fusion category such that the unit object is simple.

Let $\mathbf{1} = \oplus_{i\in I}\mathbf{1}_i$ be the decomposition of the unit object of a multi-fusion category into the sum of simple objects. One shows that the objects $\mathbf{1}_i$ are "orthogonal idempotents", that is $\mathbf{1}_i \otimes \mathbf{1}_j \simeq \delta_{ij}\mathbf{1}_i$. For any simple object $L \in \mathcal{C}$ there are unique $i, j \in I$ such that $\mathbf{1}_i \otimes L = L = L \otimes \mathbf{1}_j$. We say that a multi-fusion category $\mathcal{C}$ is *indecomposable* if for any pair $i, j \in I$ there exists a simple L with $L = \mathbf{1}_i \otimes L \otimes \mathbf{1}_j$. One shows that any multi-fusion category naturally decomposes into a unique direct sum of indecomposable ones.

The Grothendieck group of a multi-fusion category has a natural structure of a ring via $[M] \cdot [N] = [M \otimes N]$. The *Grothendieck ring* $K(\mathcal{C})$ of a multi-fusion category $\mathcal{C}$ together with its basis consisting of the classes of simple objects is a *based ring* in the sense of [39].

Remark 2.6. The Grothendieck ring $K(\mathcal{C})$ of a multi-fusion category $\mathcal{C}$ together with its basis determines the tensor product and the unit object in $\mathcal{C}$ uniquely up to isomorphism. Thus the only part of information describing $\mathcal{C}$ and not contained in $K(\mathcal{C})$ is the associativity constraint.

Example 2.7.

(i) Let A be a finite group. Consider a fusion category where simple objects L_g are labeled by $g \in A$ and $L_g \otimes L_h \simeq L_{gh}$. Similarly to Example 2.1 the possible associativity constraints in this category are classified by $H^3(A, k^\times)$. We will denote such a category with associativity constraint given by $\omega \in H^3(A, k^\times)$ by Vec_A^ω; we set $\mathrm{Vec}_A = \mathrm{Vec}_A^{\omega_0}$ where ω_0 is the neutral element of $H^3(A, k^\times)$. The Grothendieck ring $K(\mathrm{Vec}_A^\omega)$ is the group ring $\mathbb{Z}[A]$ with a basis $\{g\}_{g\in A}$.

(ii) Let R be a finite dimensional semisimple $k-$algebra, e.g. $R = k \oplus k$. Then the category of finite dimensional $R-$bimodules with tensor product $\otimes_R$ is a multi-fusion category. Its Grothendieck ring is the ring of matrices over $\mathbb{Z}$ with a basis consisting of matrix units.

(iii) Let A be a finite group and Y is a finite set on which A acts. Consider the category $\mathrm{Coh}_A(Y \times Y)$ of finite dimensional $A-$equivariant vector bundles on the set $Y \times Y$. This category has a natural *convolution tensor product* defined as follows. Let $p_{ij} : Y \times Y \times Y \to Y \times Y, i, j \in \{1, 2, 3\}$ be the various projections; then for $F_1, F_2 \in \mathrm{Coh}_A(Y \times Y)$ we set $F_1 * F_2 = p_{13*}(p_{12}^*(F_1) \otimes p_{23}^*(F_2))$ (here p_{ij*} and p_{ij}^* are the functors of direct and inverse image and $\otimes$ is the pointwise tensor product). Then the bifunctor $*$ has a natural associativity constraint and thus $\mathrm{Coh}_A(Y \times Y)$ is a multi-fusion category. In the special case of trivial A we get example (ii) above; in the case when Y consists of one point we get the category $\mathrm{Rep}(A)$ of representation of A; if the action of A on Y is free and transitive we get category Vec_A from (i).

2.3. Module categories and dual categories. Let $\mathcal{C}$ be a monoidal category and let $\mathcal{M}$ be a category. We say that $\mathcal{C}$ *acts* on $\mathcal{M}$ if we are give a tensor functor from $\mathcal{C}$ to the category of endofunctors of $\mathcal{M}$. Equivalently, we have a bifunctor $\mathcal{C} \times \mathcal{M} \to \mathcal{M}, (X, M) \mapsto X \otimes M$ endowed with the natural associativity isomorphism $(X \otimes Y) \otimes M \simeq X \otimes (Y \otimes M)$ such that

the counterpart of the pentagon axiom holds and the functor $\mathbf{1}\otimes ? : \mathcal{M} \to \mathcal{M}$ is isomorphic to the identity functor. Thus in such situation we often say that $\mathcal{M}$ is a *module category* over $\mathcal{C}$. Further one defines module functors between module categories and, in particular, the equivalences of module categories, see [52].

Convention. In the case when both categories $\mathcal{C}$ and $\mathcal{M}$ are $k-$linear we will consider only $k-$linear actions. If $\mathcal{C}$ is a multi-fusion category all module categories over $\mathcal{C}$ are assumed to be finite semisimple and non-zero.

Example 2.8.

(i) Let Y be a finite set with an action of a finite group A. Then the category $\mathcal{M} = \mathrm{Coh}(Y)$ of finite dimensional vector bundles on Y has an obvious structure of module category over Vec_A. It is easy to recover the $A-$set Y from $\mathcal{M}$: the set Y is just the set of isomorphism classes of simple objects in $\mathcal{M}$ and the action of A is recovered by considering the action of simple objects of Vec_A on simple objects of $\mathcal{M}$.

(ii) We can generalize the example above to the category Vec_A^ω from Example 2.7 (i). Pick a cocycle $\tilde{\omega}$ representing ω. Let $B \subset A$ be a subgroup and let ψ be a 2-cochain on B such that $\partial\psi = \tilde{\omega}|_B$. Then ψ determines an multiplication morphism $R_B \otimes R_B \to R_B$ where $R_B = \oplus_{g\in B} L_g$; moreover this morphism makes R_B into *associative algebra in the category* Vec_A^ω. Let $\mathcal{M} = \mathcal{M}(B,\psi)$ be the category of right R_B-modules in the category Vec_A^ω; then the left tensoring with object of Vec_A^ω makes $\mathcal{M}$ into module category over Vec_A^ω. Note that the simple objects of $\mathcal{M}$ are naturally labeled by the cosets A/B; moreover the action of simple objects of Vec_A^ω on simple objects of $\mathcal{M}$ is the same as the action of A on A/B. Thus we consider the module category $\mathcal{M}(B,\psi)$ as a cohomologically twisted version of the action of A on A/B. More generally one can consider a direct sum of module categories of the form $\mathcal{M}(B,\psi)$; this is a twisted version of the action of A on a finite set. We will use for such module categories the notation $\mathcal{M} = \mathrm{Coh}(Y)$ where it is understood that the "set" Y carries the cohomological information describing the module category $\mathcal{M}$ (thus Y is completely determined when a finite collection of pairs (B,ψ) as above is specified). We refer the reader to [13, 4.2] for the notion of "A-set of centrally extended points" which is a formalization of the cohomological data above in the special case when $\tilde{\omega}$ is trivial. Such notions are important since it is known that any module category over Vec_A^ω is equivalent as a module category to $\mathrm{Coh}(Y)$ where Y is such cohomologically twisted $A-$set, see [53, Example 2.1].

Let $\mathcal{M}$ be a module category over an indecomposable multi-fusion category $\mathcal{C}$. Then one defines the dual category $\mathcal{C}^*_\mathcal{M}$ to be the category of all endofunctors of $\mathcal{M}$ which commute with the action of $\mathcal{C}$, see [52, 4.2]. The category $\mathcal{C}^*_\mathcal{M}$ has a natural monoidal structure where the tensor product is given by the composition of functors. It is known that the category $\mathcal{C}^*_\mathcal{M}$ is again an indecomposable multi-fusion category, see [21, Theorem 2.18] (this result fails if k is allowed to have positive characteristic).

Example 2.9. Let $\mathcal{C} = \mathrm{Vec}_A$ and let $\mathcal{M}$ be as in Example 2.8 (i). In this case the category $\mathcal{C}^*_\mathcal{M}$ is precisely the category $\mathrm{Coh}_A(Y\times Y)$ from Example 2.7 (iii). Thus using module categories from Example 2.8 (ii) we get a cohomologically twisted version of the category $\mathrm{Coh}_A(Y\times Y)$. We will use similar notation $(\mathrm{Vec}_A^\omega)^*_\mathcal{M} = \mathrm{Coh}_{A,\omega}(Y\times Y)$ where it is understood that the $A-$set Y is cohomologically twisted as in Example 2.8 (ii). We note that the that direct

summands $\{\mathbf{1}_i\}_{i\in I}$ in the decomposition of the unit object $\mathbf{1} \in \mathrm{Coh}_A(Y \times Y)$ are precisely the projection functors from the module category $\mathcal{M}$ to its indecomposable direct summands; in particular the set I is in natural bijection with the set of such summands.

The category $\mathcal{C}^*_{\mathcal{M}}$ consists of endofunctors of $\mathcal{M}$; thus it acts in an obvious way on $\mathcal{M}$. The following result justifies the terminology:

Theorem 2.10 (see Remark 2.19 in [21]). *Let $\mathcal{C}$ be an indecomposable multi-fusion category and let $\mathcal{M}$ be a module category over $\mathcal{C}$. Then $\mathcal{C}^*_{\mathcal{M}}$ is also indecomposable multi-fusion and the natural functor $\mathcal{C} \to (\mathcal{C}^*_{\mathcal{M}})^*_{\mathcal{M}}$ is an equivalence of tensor categories.*

Let $F : \mathcal{C} \to \mathcal{D}$ be a tensor functor between indecomposable multi-fusion categories. We say that F is *injective* if it is fully faithful and *surjective* if it is dominant, that is any object of $\mathcal{D}$ is contained in $F(X)$ for suitable $X \in \mathcal{C}$. Now let $\mathcal{M}$ be a module category over $\mathcal{D}$. Then $\mathcal{M}$ can be considered as a module category over $\mathcal{C}$ and we have a natural dual tensor functor $F^* : \mathcal{D}^*_{\mathcal{M}} \to \mathcal{C}^*_{\mathcal{M}}$. It is shown in [21, 5.7] that this duality interchanges injective and surjective functors.

Example 2.11. Let $A \xrightarrow{f} \bar{A}$ be a surjective homomorphism of finite groups and let $\tilde{\omega}$ be a 3-cocycle representing class $\omega \in H^3(\bar{A}, k^\times)$ such that $f^*(\omega)$ is zero element of $H^3(A, k^\times)$. Then any 2-cochain ψ such that $\partial\psi = f^*(\tilde{\omega})$ defines a tensor structure on the functor $F : \mathrm{Vec}_A \to \mathrm{Vec}^{\omega}_{\bar{A}}$ sending L_g to $L_{f(g)}$. The functor F is surjective. Conversely, it is easy to see that any surjective tensor functor $\mathrm{Vec}_A \to \mathcal{C}$ where $\mathcal{C}$ is a fusion category is isomorphic to the one of this form.

It is easy to see that the category $\mathcal{C}^*_{\mathcal{M}}$ is fusion if and only if the module category $\mathcal{M}$ is not a nontrivial direct sum of module categories over $\mathcal{C}$, that is $\mathcal{M}$ is *indecomposable* over $\mathcal{C}$. We have the following consequence of the discussion above:

Corollary 2.12 (see Lemma 3.1 in [37]). *Let $\mathcal{M}$ be a module category over Vec_A and let $\mathcal{C}$ be a full multi-fusion subcategory of $(\mathrm{Vec}_A)^*_{\mathcal{M}}$ such that $\mathcal{M}$ is indecomposable over $\mathcal{C}$. Then there exists a surjective functor $F : \mathrm{Vec}_A \to \mathrm{Vec}^{\omega}_{\bar{A}}$ such that the action of Vec_A on $\mathcal{M}$ factors through F and such that $\mathcal{C} = (\mathrm{Vec}^{\omega}_{\bar{A}})^*_{\mathcal{M}} \subset (\mathrm{Vec}_A)^*_{\mathcal{M}}$.*

Proof. Let $G : \mathcal{C} \to (\mathrm{Vec}_A)^*_{\mathcal{M}}$ be the embedding functor; clearly it is injective. Then the dual functor $G^* : ((\mathrm{Vec}_A)^*_{\mathcal{M}})^*_{\mathcal{M}} \to \mathcal{C}^*_{\mathcal{M}}$ is surjective. By Theorem 2.10 we have $((\mathrm{Vec}_A)^*_{\mathcal{M}})^*_{\mathcal{M}} = \mathrm{Vec}_A$ and the category $\mathcal{C}^*_{\mathcal{M}}$ is fusion. By Example 2.11 the result follows. □

The module categories over a fixed indecomposable multi-fusion category $\mathcal{C}$ form a 2-category, where the morphisms are the module functors and 2-morphisms are the natural transformations of the module functors. This 2-category is semisimple in the following sense: for any module categories $\mathcal{M}_1$ and $\mathcal{M}_2$ the category of module functors $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$ from $\mathcal{M}_1$ to $\mathcal{M}_2$ is finite semisimple, see [21, Theorem 2.18]). It is clear that the composition of functors makes $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$ into a module category over $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_1) = \mathcal{C}^*_{\mathcal{M}_1}$. One shows that the 2-functor $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, ?)$ is a 2-equivalence of 2-categories of module categories over $\mathcal{C}$ and over $\mathcal{C}^*_{\mathcal{M}}$, see [53, Proposition 2.3] or [50]. For example this means that there is one to one correspondence between the module categories over $\mathrm{Coh}_{A,\omega}(Y \times Y)$ and over Vec^{ω}_A; moreover to compute the module functors between the module categories over $\mathrm{Coh}_{A,\omega}(Y \times Y)$ we can compute the module functors between the corresponding module categories over Vec^{ω}_A.

Example 2.13. Let $\mathcal{M}_1 = \mathcal{M}(B_1, \psi_1)$ and $\mathcal{M}_2 = \mathcal{M}(B_2, \psi_2)$ be the module categories over $\mathcal{C} = \mathrm{Vec}_A$ as in Example 2.8 (ii). Assume that ψ_1 and ψ_2 are both trivial. Then the category $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$ identifies with the category $\mathrm{Coh}_{B_1}(A/B_2)$ of B_1-equivariant vector bundles on the B_1-set A/B_2, see e.g. [53, Proposition 3.2].

In general it is difficult to find a number of simple objects in the category $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$. Here is a special case when this is possible to do. Let $\mathbf{1} = \oplus_{i\in I}\mathbf{1}_i$ be the decomposition of the unit object of $\mathcal{C}$ into simple summands. Let $\mathcal{C}\otimes\mathbf{1}_i$ be the full subcategory of $\mathcal{C}$ consisting of objects X such that $X\otimes\mathbf{1}_i \simeq X$. It is clear that $\mathcal{C}\otimes\mathbf{1}_i$ is stable under the left multiplications by objects from $\mathcal{C}$. Thus $\mathcal{C}\otimes\mathbf{1}_i$ is a module category over $\mathcal{C}$. Note that for any module category $\mathcal{M}$ over $\mathcal{C}$ the Grothendieck group $K(\mathcal{M})$ is naturally a module over the Grothendieck ring $K(\mathcal{C})$.

Lemma 2.14 (Lemma 3.4 in [37]). *Let $\mathcal{M}$ be a module category over a multi-fusion category $\mathcal{C}$. Then the number of simple objects in the category $\mathrm{Fun}_{\mathcal{C}}(\mathcal{C}\otimes\mathbf{1}_i, \mathcal{M})$ equals the dimension of $\mathrm{Hom}_{K(\mathcal{C})}(K(\mathcal{C}\otimes\mathbf{1}_i), K(\mathcal{M}))$.*

2.4. Drinfeld center. One of the most important constructions in the theory of tensor categories is the construction of *Drinfeld center*, see [30, 47]. One definition in the spirit of Section 2.3 is as follows. A monoidal category $\mathcal{C}$ acts on itself by left and right multiplications, so $\mathcal{C}$ is a *bimodule category* over itself. Then the Drinfeld center $\mathcal{Z}(\mathcal{C})$ of $\mathcal{C}$ is the category of endofunctors of $\mathcal{C}$ commuting with these actions. The composition makes $\mathcal{Z}(\mathcal{C})$ into a monoidal category, but we have more structure here: $\mathcal{Z}(\mathcal{C})$ is naturally a braided tensor category, see [30, 47]. It is easy to see ([53, 2.3]) that our definition is equivalent to the classical one: the objects of $\mathcal{Z}(\mathcal{C})$ are pairs (X, ϕ) where X is an object of $\mathcal{C}$ and ϕ is an isomorphism of functors $X\otimes ? \simeq ?\otimes X$ satisfying some natural conditions, see [30, 47, 51]. We have a natural *forgetful functor* $\mathcal{Z}(\mathcal{C})\to\mathcal{C}$ sending (X,ϕ) to X. The right adjoint of this functor (if it exists) is called the *induction functor*.

It is known that the Drinfeld center of an indecomposable multi-fusion category is a fusion category, see [21, Theorem 2.15] or [51]; in particular the induction functor exists in this case. Another important property is the Morita invariance: for a module category $\mathcal{M}$ we have a natural tensor equivalence $\mathcal{Z}(\mathcal{C}^*_{\mathcal{M}}) \simeq \mathcal{Z}(\mathcal{C})$, see [53, Corollary 2.6] or [51].

Example 2.15. Recall that $\mathrm{Coh}_{A,\omega}(Y\times Y)$ is $(\mathrm{Vec}^{\omega}_A)^*_{\mathcal{M}}$ for suitable $\mathcal{M}$. Thus we get a somewhat surprising result: $\mathcal{Z}(\mathrm{Coh}_{A,\omega}(Y\times Y))$ does not depend on Y and is equivalent to $\mathcal{Z}(\mathrm{Vec}^{\omega}_A)$.

3. Harish-Chandra bimodules

3.1. Basic definitions. Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ and let $Z(\mathfrak{g})\subset U(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. Recall that a *central character* is a homomorphism $\chi : Z(\mathfrak{g})\to\mathbb{C}$. For a central character χ we have two sided ideal $U(\mathfrak{g})\mathrm{Ker}\,(\chi)\subset U(\mathfrak{g})$ and we will set $U_\chi := U(\mathfrak{g})/U(\mathfrak{g})Ker(\chi)$.

Recall that for a $U(\mathfrak{g})-$bimodule M one defines an adjoint $\mathfrak{g}-$action by the formula $ad(x)m := xm - mx$; we will denote by M_{ad} the space M with this action of $\mathfrak{g}$.

Definition 3.1. A $U(\mathfrak{g})-$bimodule M is called $ad(\mathfrak{g})$-algebraic if M_{ad} can be decomposed into a direct sum of finite dimensional $\mathfrak{g}-$modules. We say that an $ad(\mathfrak{g})$-algebraic

$U(\mathfrak{g})-$bimodule M is *Harish-Chandra bimodule* if it is finitely generated as $U(\mathfrak{g})-$bimodule.

Remark 3.2. The definitions of Harish-Chandra bimodules in the literature (see e.g. [25, 26, 61]) differ slightly from each other with $ad(\mathfrak{g})-$algebraicity being the crucial part.

Example 3.3. Consider $U(\mathfrak{g})$ as $U(\mathfrak{g})-$bimodule. Then the Poincaré-Birkhoff-Witt (or PBW) filtration on $U(\mathfrak{g})$ is stable under the adjoint action. Hence $U(\mathfrak{g})$ is a Harish-Chandra bimodule. Since $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ is Noetherian any subquotient of Harish-Chandra bimodule is again Harish-Chandra bimodule. Hence I and $U(\mathfrak{g})/I$ are Harish-Chandra bimodules for any two sided ideal $I \subset U(\mathfrak{g})$. In particular U_χ is a Harish-Chandra bimodule.

Remark 3.4. Assume that $\mathfrak{g}$ is a complexification of the Lie algebra of a real semisimple Lie group $G_{\mathbb{R}}$ with a maximal compact subgroup K. It was shown by Harish-Chandra that the study of continuous representations of $G_{\mathbb{R}}$ to a large extent reduces to the study of so called $(\mathfrak{g}, K)-$modules (or Harish-Chandra modules). We recall that a $(\mathfrak{g}, K)-$module is a finitely generated $U(\mathfrak{g})-$module endowed with compatible locally finite action of K, see e.g. [65, 2.1(a)]; thus this is a purely algebraic object. The notion of Harish-Chandra bimodule is a special case of this when we take in the place of $G_{\mathbb{R}}$ a complex simply connected Lie group with Lie algebra $\mathfrak{g}$ considered as a real Lie group (so the complexified Lie algebra is isomorphic to $\mathfrak{g} \oplus \mathfrak{g}$).

The following well known result is of crucial importance for this note:

Lemma 3.5. *If M and N are Harish-Chandra bimodules, then so is $M \otimes_{U(\mathfrak{g})} N$.*

Proof. It is immediate from definitions that the canonical surjection $M \otimes N \to M \otimes_{U(\mathfrak{g})} N$ commutes with the adjoint action. Hence $M \otimes_{U(\mathfrak{g})} N$ is $ad(\mathfrak{g})-$algebraic.

Let $M_0 \subset M$ be a finite dimensional $ad(\mathfrak{g})-$invariant subspace of M generating M as $U(\mathfrak{g})-$bimodule. It is easy to see that M_0 generates M as left $U(\mathfrak{g})-$module and as right $U(\mathfrak{g})-$module. Let $N_0 \subset N$ be a similar subspace of N. Then the image of $M_0 \otimes N_0$ clearly generates $M \otimes_{U(\mathfrak{g})} N$. □

Let $\mathcal{H}$ denote the category of Harish-Chandra bimodules (where the morphisms are homomorphisms of bimodules). The tensor product over $U(\mathfrak{g})$ with the obvious associativity isomorphisms makes $\mathcal{H}$ a tensor category with the unit object $U(\mathfrak{g})$, see Example 3.3. However this category has some unpleasant properties: the endomorphism algebra of $U(\mathfrak{g})$ identifies with $Z(\mathfrak{g})$, so the Hom$-$spaces are infinite dimensional in general.

Remark 3.6. It is easy to see that for any K as in Remark 3.4 the tensor product $M \otimes_{U(\mathfrak{g})} N$ of a Harish-Chandra bimodule M and $(\mathfrak{g}, K)-$module N is again $(\mathfrak{g}, K)-$module. In other words, the category of Harish-Chandra bimodules acts naturally on the category of $(\mathfrak{g}, K)-$modules.

3.2. Irreducible Harish-Chandra bimodules. For a central character χ let $\mathcal{H}^\chi$ be the full subcategory of $\mathcal{H}$ consisting of bimodules M such that $M\mathrm{Ker}\,(\chi) = 0$ (in other words, the right action of $Z(\mathfrak{g})$ on M factorizes through χ). A very precise description of category $\mathcal{H}^\chi$ was given by Bernstein and Gelfand in [7]. This description is based on the category $\mathcal{O}$ of $\mathfrak{g}-$modules introduced by Bernstein, Gelfand and Gelfand in [8]. We refer the reader to [25] for the basic definitions and results on the category $\mathcal{O}$.

Recall that for any *weight* λ one defines the *Verma module* $M(\lambda) \in \mathcal{O}$. The center $Z(\mathfrak{g})$ acts on $M(\lambda)$ via central character χ_λ. It follows from Harish-Chandra's theorem (see e.g.

[25, 1.10]) that any central character arises in this way; moreover $\chi_\lambda = \chi_\mu$ if and only if there exists an element w of the *Weyl group* W such that $w(\lambda+\rho)-\rho = \mu$ where ρ is the sum of the fundamental weights. Thus for any central character χ there exists a *dominant* (see [25, 3.5]) weight λ such that $\chi = \chi_\lambda$. From now on we will restrict ourselves to the case of *regular integral* central characters χ (this means that $\chi = \chi_\lambda$ where λ is a highest weight of a finite dimensional representation of $\mathfrak{g}$). For example $Z(\mathfrak{g})$ acts on the trivial $\mathfrak{g}-$module via the regular integral character χ_0.

Theorem 3.7 (Theorem 5.9 in [7]). *Assume that the weight λ is regular, integral, and dominant. The functor $M \mapsto M \otimes_{U(\mathfrak{g})} M(\lambda)$ is an equivalence of the category ${}^{\chi}\mathcal{H}$ and the subcategory of $\mathcal{O}$ consisting of modules with integral weights.*

As a consequence we see that any object of the category ${}^{\chi}\mathcal{H}$ has finite length (this holds with no restrictions on χ, see e.g. [26, Satz 6.30]). Also the simple objects in the category ${}^{\chi}\mathcal{H}$ are labeled by the integral weights (see [7, Proposition 5.4] for the general case).

Now we consider the left action of $Z(\mathfrak{g})$. For two central characters χ_1 and χ_2 let ${}^{\chi_1}\mathcal{H}^{\chi_2}$ be the category of Harish-Chandra bimodules M such that $\mathrm{Ker}\,(\chi_1)M = M\mathrm{Ker}\,(\chi_2) = 0$. We also set $\mathcal{H}(\chi) := {}^{\chi}\mathcal{H}^{\chi}$. It is clear that for $M \in {}^{\chi_1}\mathcal{H}^{\chi_2}$ and $N \in {}^{\chi_3}\mathcal{H}^{\chi_4}$ we have $M \otimes_{U(\mathfrak{g})} N \in {}^{\chi_1}\mathcal{H}^{\chi_4}$ and $M \otimes_{U(\mathfrak{g})} N = 0$ unless $\chi_2 = \chi_3$. In particular, the category $\mathcal{H}(\chi)$ is a tensor category with unit object U_χ.

For a regular integral χ_2 Theorem 3.7 implies that the category ${}^{\chi_1}\mathcal{H}^{\chi_2}$ is nonzero if and only if χ_1 is integral. Moreover one shows using Theorem 3.7 that the categories $\mathcal{H}(\chi)$ are tensor equivalent for various regular integral χ. Furthermore, we have

Corollary 3.8. *Let $\chi = \chi_\lambda$ where λ is regular, integral, and dominant. The simple objects of $\mathcal{H}(\chi)$ are naturally labeled by the elements of W: for any $w \in W$ there exists $M_w \in \mathcal{H}(\chi)$ such that $M_w \otimes_{U(\mathfrak{g})} M(\lambda)$ is the irreducible $\mathfrak{g}-$module with highest weight $w(\lambda+\rho)-\rho$.*

3.3. Associated varieties. In this section we identify $\mathfrak{g}^*$ and $\mathfrak{g}$ via the Killing form. Let G be the complex connected adjoint algebraic group with the Lie algebra $\mathfrak{g}$. An element $x \in \mathfrak{g}$ is *nilpotent* if $ad(x) : \mathfrak{g} \to \mathfrak{g}$ is nilpotent. Let $\mathcal{N} \subset \mathfrak{g}$ be the *nilpotent cone*, that is the set of all nilpotent elements. Clearly $\mathcal{N}$ is a closed $G-$invariant subvariety of $\mathfrak{g}$. It is a classical fact (see [32]) that $\mathcal{N}$ is a finite union of $G-$orbits, $\mathcal{N} = \sqcup\mathbb{O}$. The $G-$orbits appearing in $\mathcal{N}$ are called *nilpotent orbits*. For a nilpotent orbit $\mathbb{O}$ let $\bar{\mathbb{O}} \subset \mathcal{N}$ be its Zariski closure; clearly $\bar{\mathbb{O}}$ is a union of nilpotent orbits.

The associated varieties (see e.g. [65]) provide a convenient measure of "size" of a Harish-Chandra bimodule. Let M be a Harish-Chandra bimodule. Then there exists a finite dimensional $ad(\mathfrak{g})-$invariant subspace $M_0 \subset M$ generating M as a left $U(g)-$module. Then the PBW filtration on $U(\mathfrak{g})$ induces a compatible filtration on M (note that this filtration is $ad(\mathfrak{g})-$invariant, so it is compatible with both left and right $U(\mathfrak{g})-$actions). The associated graded grM with respect to this filtration is a left module over $grU(\mathfrak{g}) = S(\mathfrak{g})$. Let $V(M)$ be the support of this module in $\mathfrak{g} \simeq \mathfrak{g}^* = Spec(S(\mathfrak{g}))$. The following properties of $V(M)$ are easy to verify (see e.g. [65]):

(1) $V(M)$ is independent of the choice of M_0;

(2) $V(M)$ is invariant under the adjoint action of G;

(3) for an exact sequence $0 \to M_1 \to N \to M_2 \to 0$ we have $V(N) = V(M_1) \cup V(M_2)$;

(4) $V(M_1 \otimes_{U(\mathfrak{g})} M_2) \subset V(M_1) \cap V(M_2)$;

(5) for $M \in \mathcal{H}^\chi$ we have $V(M) \subset \mathcal{N}$.

The following result of Joseph is fundamental:

Theorem 3.9 ([27], see also [14, 65]). *Assume that $M \in \mathcal{H}$ is irreducible. Then $V(M) = \bar{\mathbb{O}}$ for some nilpotent orbit $\mathbb{O}$.*

Assume that $M \in \mathcal{H}(\chi)$ where χ is regular integral. It follows from the results of [1, 2, 27] that $V(M) = \bar{\mathbb{O}}$ where $\mathbb{O}$ is *special* nilpotent orbit in the sense of Lusztig, (all the nilpotent orbits are special in type A; however this is not the case in other types); and all special nilpotent orbits can be obtained in this way.

The theory of associated varieties is closely related with the theory of Kazhdan-Lusztig cells, see [31]. Namely let us introduce the following equivalence relation on the Weyl group W: $u \sim v$ if $V(M_u) = V(M_v)$. Then W is partitioned into equivalence classes labeled by the special nilpotent orbits. It follows from the results of [1, 2, 27] that this partition coincides with the partition of W into *two sided cells* as defined in [31]. In particular the set of two sided cells is in natural bijection with the set of special nilpotent orbits.

3.4. Cell categories. Let χ be a regular integral central character and let $\mathbb{O}$ be a nilpotent orbit. We define full subcategories $\mathcal{H}(\chi)_{\leq \bar{\mathbb{O}}}$ and $\mathcal{H}(\chi)_{<\bar{\mathbb{O}}}$ as follows: $M \in \mathcal{H}(\chi)_{\leq \bar{\mathbb{O}}}$ (respectively, $M \in \mathcal{H}(\chi)_{<\bar{\mathbb{O}}}$) if and only if $\bar{V}(M) \subset \bar{\mathbb{O}}$ (respectively, $V(M) \subset \bar{\mathbb{O}} - \mathbb{O}$). It follows easily from the properties of associated varieties that $\mathcal{H}(\chi)_{\leq \bar{\mathbb{O}}}$ and $\mathcal{H}(\chi)_{<\bar{\mathbb{O}}}$ are Serre subcategories of $\mathcal{H}(\chi)$; also $\mathcal{H}(\chi)_{\leq \bar{\mathbb{O}}}$ is closed under the tensor product and the tensor product of bimodules from $\mathcal{H}(\chi)_{<\bar{\mathbb{O}}}$ and $\mathcal{H}(\chi)_{\leq \bar{\mathbb{O}}}$ is contained in $\mathcal{H}(\chi)_{<\bar{\mathbb{O}}}$.

We define $\tilde{\mathcal{H}}(\chi)_{\mathbb{O}}$ to be the Serre quotient category $\mathcal{H}(\chi)_{\leq \bar{\mathbb{O}}}/\mathcal{H}(\chi)_{<\bar{\mathbb{O}}}$ (note that the category $\tilde{\mathcal{H}}(\chi)_{\mathbb{O}}$ is nonzero if and only if the nilpotent orbit $\mathbb{O}$ is special). Let $\mathcal{H}(\chi)_{\mathbb{O}} \subset \tilde{\mathcal{H}}(\chi)_{\mathbb{O}}$ be the full subcategory consisting of the semisimple objects in $\tilde{\mathcal{H}}(\chi)_{\mathbb{O}}$. The tensor product $\otimes_{U(\mathfrak{g})}$ descends to a well defined tensor product functor $\otimes$ on the category $\tilde{\mathcal{H}}(\chi)_{\mathbb{O}}$ endowed with the associativity constraint.

Theorem 3.10 (see [11, 34, 37]). *The restriction of $\otimes$ to $\mathcal{H}(\chi)_{\mathbb{O}} \subset \tilde{\mathcal{H}}(\chi)_{\mathbb{O}}$ takes values in the subcategory $\mathcal{H}(\chi)_{\mathbb{O}}$. Moreover, the category $\mathcal{H}(\chi)_{\mathbb{O}}$ is an indecomposable multi-fusion category.*

We will explain some ideas of the proof of Theorem 3.10 in Section 4. In the same time we will give a precise description of the cell category $\mathcal{H}(\chi)_{\mathbb{O}}$ as a multi-fusion category. For now we will explain that the category $\mathcal{H}(\chi)_{\mathbb{O}}$ does contain the unit object. We recall (see e.g. [26, 1.9]) that a two sided ideal $I \subset U(\mathfrak{g})$ is *primitive* if it is the annihilator of an irreducible $\mathfrak{g}-$module. It follows from Schur's lema that for a primitive ideal I the intersection $I \cap Z(\mathfrak{g}) = \mathrm{Ker}\,(\chi)$ for some central character χ; let Pr_χ be the set of all such primitive ideals (this set is finite; it is explicitly known in all cases thanks to the deep work of Joseph [28]).

Let $I \in Pr_\chi$. It was proved by Joseph [27] that $V(U(\mathfrak{g})/I) = \bar{\mathbb{O}}$ for some special nilpotent orbit $\mathbb{O}$ (this result is closely related with Theorem 3.9, see [65, Corollary 4.7]). It is also known that for any ideal $I' \supset I, I' \neq I$ the dimension of $V(U(\mathfrak{g}))/I')$ is strictly smaller than the dimension of $V(U(\mathfrak{g}))/I)$, see [15, 3.6]. Therefore $U(\mathfrak{g})/I$ contains a unique simple sub-bimodule M_I; moreover $V(U(\mathfrak{g})/I) = V(M_I) = \bar{\mathbb{O}}$ and $V((U(\mathfrak{g})/I)/M_I) \subset \bar{\mathbb{O}} - \mathbb{O}$. In other words $U(\mathfrak{g})/I \simeq M_I$ in the category $\tilde{\mathcal{H}}(\chi)_{\mathbb{O}}$; in particular $U(\mathfrak{g})/I \in \mathcal{H}(\chi)_{\mathbb{O}}$. Also for two distinct $I, J \in Pr_\chi$ with $V(U(\mathfrak{g})/I) = V(U(\mathfrak{g})/J) = \bar{\mathbb{O}}$ we have $U(\mathfrak{g})/I \otimes_{U(\mathfrak{g})}$

$U(\mathfrak{g})/J = U(\mathfrak{g})/(I+J)$ whence $V(U(\mathfrak{g})/I \otimes_{U(\mathfrak{g})} U(\mathfrak{g})/J) \subset \bar{\mathbb{O}} - \mathbb{O}$. Equivalently $U(\mathfrak{g})/I \otimes_{U(\mathfrak{g})} U(\mathfrak{g})/J = M_I \otimes M_J = 0$ in the category $\mathcal{H}(\chi)_{\mathbb{O}}$.

It is well known that for a simple Harish-Chandra bimodule $M \in {}^{\chi_1}\mathcal{H}^{\chi_2}$ there exist $I \in Pr_{\chi_1}$ and $J \in Pr_{\chi_2}$ such that I is the annihilator of M considered as a left $U(\mathfrak{g})-$module and J is the annihilator of M considered as a right $U(\mathfrak{g})-$module; moreover $V(M) = V(U(\mathfrak{g})/I) = V(U(\mathfrak{g})/J)$, see e.g. [26, 7.7, 17.8]. Clearly $U(\mathfrak{g})/I \otimes_{U(\mathfrak{g})} M = M \otimes_{U(\mathfrak{g})} U(\mathfrak{g})/J = M$. Let $Pr_\chi(\bar{\mathbb{O}}) \subset Pr_\chi$ consists of I with $V(U(\mathfrak{g})/I) = \bar{\mathbb{O}}$. It follows from the above that

$$\mathbf{1} = \bigoplus_{I \in \mathrm{Pr}_\chi(\bar{\mathbb{O}})} U(\mathfrak{g})/I = \bigoplus_{I \in \mathrm{Pr}_\chi(\bar{\mathbb{O}})} M_I$$

is the unit object of $\mathcal{H}(\chi)_{\mathbb{O}}$. Again there is an important connection with the theory of Kazhdan-Lusztig cells [31]: it follows from the results of [1, 2, 38] that two elements $u, v \in W$ are in the same *left cell* if and only if there exists $\mathbf{1}_i$ such that $M_u \otimes \mathbf{1}_i \simeq M_u$ and $M_v \otimes \mathbf{1}_i \simeq M_v$. In particular the set Pr_χ is in bijection with the set of left cells in W.

4. Actions of cell categories

4.1. Whittaker modules. Let $e \in \mathfrak{g}$ be a nilpotent element. By the Jacobson-Morozov theorem we can pick $h, f \in \mathfrak{g}$ such that e, f, h is an sl_2-triple, that is $[h, e] = 2e, [h, f] = -2f, [e, f] = h$. Then $\mathfrak{g}$ decomposes into eigenspaces for $ad(h)$:

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i), \quad \mathfrak{g}(i) = \{x \in \mathfrak{g} | [h, x] = ix\}.$$

In particular $e \in \mathfrak{g}(2)$ and $f \in \mathfrak{g}(-2)$. Using the Killing form $(,)$ on $\mathfrak{g}$ one defines a skew-symmetric bilinear form $x, y \mapsto (e, [x, y])$ on the space $\mathfrak{g}(-1)$; it turns out that this form is non-degenerate. Pick a lagrangian subspace $l \subset \mathfrak{g}(-1)$ and set $\mathfrak{m} = \mathfrak{m}_l = l \oplus \bigoplus_{i \leq -2} \mathfrak{g}(i)$. Then $\xi(x) = (x, e)$ is a Lie algebra homomorphism $\mathfrak{m} \to \mathbb{C}$. Let $\mathfrak{m}_\xi$ be the Lie subalgebra of $U(\mathfrak{g})$ spanned by $x - \xi(x), x \in \mathfrak{m}$.

Definition 4.1 ([49]). We say that $\mathfrak{g}-$module is *Whittaker* if the action of $\mathfrak{m}_\xi$ on it is locally nilpotent.

Let Wh be the category of Whittaker modules (this is a full Serre subcategory of category of $\mathfrak{g}-$modules). We have a functor from Wh to vector spaces

$$M \mapsto \{m \in M | xm = \xi(x)m, \ \forall x \in \mathfrak{m}\}.$$

Let $U(\mathfrak{g}, e)$ be the algebra of endomorphisms of this functor; thus the functor above upgrades to a functor Sk : Wh $\to U(\mathfrak{g}, e) - mod$. An important result proved by Skryabin [60] (see also [22] and [33]) is that this functor is an equivalence of categories. Thus we call Sk the Skryabin equivalence.

Remark 4.2. The algebras $U(\mathfrak{g}, e)$ are *finite $W-$algebras* introduced by Premet [55]. We refer the reader to [35] for a nice survey of their properties.

A particularly important property of algebras $U(\mathfrak{g}, e)$ is that they do not depend on the choice of lagrangian subspace l (more precisely the algebras defined using different choices of l are canonically isomorphic), see [22]. In particular, the centralizer Q of e, h, f in G acts naturally on $U(\mathfrak{g}, e)$, see [35, 2.6].

4.2. Irreducible finite dimensional representations of finite $W-$algebras. Let $M \in \mathcal{H}$ and $N \in \mathrm{Wh}$. For $x \in \mathfrak{g}$ and $m \otimes n \in M \otimes_{U(\mathfrak{g})} N$ we have $x(m \otimes n) = ad(x)m \otimes n + m \otimes xn$. The subalgebra $\mathfrak{m}$ consists of nilpotent elements, so $ad(x)$ is locally nilpotent for any $x \in \mathfrak{m}$. Hence $M \otimes_{U(\mathfrak{g})} N \in \mathrm{Wh}$, in other words the tensor category $\mathcal{H}$ acts on the category Wh. Let ${}^{\chi}\mathrm{Wh}$ be the full subcategory of $M \in \mathrm{Wh}$ such that $Z(\mathfrak{g})$ acts on M through the central character χ. Clearly the action above restrict to an action of $\mathcal{H}(\chi)$ on ${}^{\chi}\mathrm{Wh}$.

We will be interested in the set $Y = Y(\chi)$ of isomorphism classes of irreducible modules M in ${}^{\chi}\mathrm{Wh}$ such that $\mathrm{Sk}(M)$ is finite dimensional. Since Sk is an equivalence, Y is also the set of irreducible finite dimensional representations of $W-$algebra $U(\mathfrak{g}, e)$. For any $M \in Y$ its annihilator is a primitive ideal of $U(\mathfrak{g})$; thus we get a map $\mathrm{Ann}_{\chi} : Y \to Pr_{\chi}$. It was proved by Premet [56] that any ideal I in the image of this map is contained in $Pr_{\chi}(\bar{\mathbb{O}})$ where $\mathbb{O} = Ge$ is the nilpotent orbit containing e. Moreover, it was conjectured by Premet and proved by Losev [33] (see also [23, 56, 57]) that any $I \in Pr_{\chi}(\bar{\mathbb{O}})$ is in the image of this map. Recall that the group Q acts on the algebra $U(\mathfrak{g}, e)$. Thus we get an action of Q on the set Y. One shows that the unit component $Q^0 \subset Q$ acts trivially, so we get an action of the component group $C(e) = Q/Q^0$ on Y (it is well known that the group $C(e)$ is isomorphic to the component group of the centralizer of e in G or, equivalently, $C(e)$ is equivariant fundamental group of the orbit $\mathbb{O}$). It was proved by Losev [34] that each fiber of the map Ann_{χ} is exactly one $C(e)-$orbit in Y. We will seek for a precise description of these orbits. Actually there is a little bit more information here. Let $M \in Y$ and let $Q_M \subset Q$ be its stabilizer in the group Q. Then Q_M acts projectively on $\mathrm{Sk}(M)$, so we have a cohomology class in $H^2(Q_M, \mathbb{C}^{\times})$ describing this action. The data of the set Y together with $Q-$action and 2-cocycles above (which should be compatible in an obvious way) can be described as the data of "$Q-$set of centrally extended points", see Example 2.8 (ii). Recall that in the special case when the group Q is finite, precisely the same data describe a structure of module category over Vec_Q on the category $\mathrm{Coh}(Y)$, see Example 2.8 (ii). In general, $\mathrm{Coh}(Y)$ acquires the structure of module category over Vec_A for any finite subgroup $A \subset Q$.

Let ${}^{\chi}\mathrm{Wh}^f \subset {}^{\chi}\mathrm{Wh}$ be the full subcategory consisting of semisimple N such that $\mathrm{Sk}(N)$ is finite dimensional. It was proved by Losev [34] that for $M \in \mathcal{H}(\chi)_{\leq \mathbb{O}}$ and $N \in {}^{\chi}\mathrm{Wh}^f$ we have $M \otimes_{U(\mathfrak{g})} N \in {}^{\chi}\mathrm{Wh}^f$. Moreover, $M \otimes_{U(\mathfrak{g})} N = 0$ for $M \in \mathcal{H}(\chi)_{<\mathbb{O}}$. Thus the category $\mathcal{H}(\chi)_{\mathbb{O}}$ acts on ${}^{\chi}\mathrm{Wh}^f$. On the other hand the group Q acts on the category ${}^{\chi}\mathrm{Wh}^f$ (or rather on the equivalent category of $U(\mathfrak{g}, e)-$modules) via twisting: an element $g \in Q$ sends a $U(\mathfrak{g}, e)-$module to itself with the action of $U(\mathfrak{g}, e)$ twisted by an automorphism g. One shows that these two actions commute. We pick a finite subgroup $A \subset Q$ which surjects on $C(e) = Q/Q^0$ and restrict the above action of Q to A. Then the category ${}^{\chi}\mathrm{Wh}^f$ is a module category over fusion category Vec_A (note that ${}^{\chi}\mathrm{Wh}^f \simeq \mathrm{Coh}(Y)$, and this is the same structure of the module category as in the previous paragraph). Since any $M \in \mathcal{H}(\chi)_{\mathbb{O}}$ produces a functor $M \otimes_{U(\mathfrak{g})} ? : {}^{\chi}\mathrm{Wh}^f \to {}^{\chi}\mathrm{Wh}^f$ commuting with the action of Vec_A we get a canonical tensor functor

$$\mathcal{H}(\chi)_{\mathbb{O}} \to \mathrm{Fun}_{\mathrm{Vec}_A}({}^{\chi}\mathrm{Wh}^f, {}^{\chi}\mathrm{Wh}^f) = (\mathrm{Vec}_A)^*_{{}^{\chi}\mathrm{Wh}^f} = \mathrm{Coh}_A(Y \times Y),$$

see Example 2.9.

Theorem 4.3. ([34, 37]) *The functor $\mathcal{H}(\chi)_{\mathbb{O}} \to Coh_A(Y \times Y)$ is fully faithful.*

Remark 4.4. An important tool in the proof of Theorem 4.3 is the notion of Harish-Chandra bimodules for $W-$algebras introduced by Ginzburg [23] and Losev [34]. It is possible to replace the group Q by the finite group A since the action of Q on $U(\mathfrak{g}, e)$ has the following

property: there is embedding of the Lie algebra $\mathfrak{q}$ of Q to $U(\mathfrak{g}, e)$ (considered as a Lie algebra) such that the differential of $Q-$action on $U(\mathfrak{g}, e)$ coincides with the adjoint action of $\mathfrak{q}$, see [34, 1.1(1)].

One consequence of Theorem 4.3 is the fact that the category $\mathcal{H}(\chi)_{\mathbb{O}}$ closed under the tensor product, see [34, Corollary 1.3.2]. This is a crucial step in the proof of Theorem 3.10. Moreover one shows that the module category ${}^{\chi}\mathrm{Wh}^f$ over $\mathcal{H}(\chi)_{\mathbb{O}}$ is indecomposable, see [37, Theorem 5.1]. Thus we can apply Corollary 2.12 and get the following

Corollary 4.5. *There is a quotient $\bar{A}$ of A and $\omega \in H^3(\bar{A}, \mathbb{C}^\times)$ such that the action of Vec_A on ${}^{\chi}Wh^f$ factors through tensor functor $Vec_A \to Vec^{\omega}_{\bar{A}}$ and the action on ${}^{\chi}Wh^f$ induces tensor equivalence $\mathcal{H}(\chi)_{\mathbb{O}} \simeq Coh_{A,\omega}(Y \times Y)$.*

It turns out that the quotient map $A \to \bar{A}$ always factorizes through $A \to Q \to Q/Q^0 = C(e)$. Thus $\bar{A}$ is naturally a quotient of the group $C(e)$. It was shown in [37] that $\bar{A}$ coincides with the *Lusztig's quotient* of $C(e)$ which was introduced by Lusztig [38, Section 13]. Also it was shown in [37] (see also [10]) that the class ω is trivial in almost all cases. However it is not trivial in the case case of nilpotent orbits corresponding to so called *exceptional* two sided cells, see [54].

It follows from the results in Section 5.2 below that the rational Grothendieck ring $K(\mathcal{H}(\chi)_{\mathbb{O}}) \otimes \mathbb{Q}$ is naturally a quotient of the group algebra $\mathbb{Q}[W]$. Thus for any module category $\mathcal{M}$ over $\mathcal{H}(\chi)_{\mathbb{O}}$ the rational Grothendieck group $K(\mathcal{M}) \otimes \mathbb{Q}$ is naturally $W-$module. In the special case $\mathcal{M} = \mathcal{H}(\chi)_{\mathbb{O}} \otimes \mathbf{1}_i$ we obtain the *constructible representations* of W, see [38, 5.29]; these representations are explicitly known. On the other hand let Spr be the *Springer representation* of $W \times C(e)$ (this is top rational cohomology of the *Springer fiber* associated with $e \in \mathbb{O}$ with the natural action of $C(e)$ and the action of W defined by Springer [63]). It was proved by Dodd [18] that there is $W \times C(e)-$equivariant embedding $K(\mathrm{Coh}(Y)) \otimes \mathbb{Q} \subset \mathrm{Spr}$. Using this result together with some results by Lusztig on Springer representation [44] and Lemma 2.14 the module category $\mathrm{Coh}(Y)$ over $\mathrm{Vec}^{\omega}_{\bar{A}}$ was explicitly determined in all cases in [37]. We recall that the indecomposable summands of $\mathrm{Coh}(Y)$ are naturally labeled by the simple summands $\mathbf{1}_i$ of $\mathbf{1} \in \mathcal{H}(\chi)_{\mathbb{O}}$, see Example 2.9. Moreover, each such summand is of the form $\mathcal{M}(B_i, \psi_i)$, see Example 2.8 (ii). It was shown in [37] that we have $C(e)-$equivariant isomorphism $\mathbb{Q}[\bar{A}/B_i] \simeq \mathrm{Hom}_W(K(\mathcal{H}(\chi)_{\mathbb{O}} \otimes \mathbf{1}_i) \otimes \mathbb{Q}, \mathrm{Spr}(\mathbb{O}))$; moreover this determines the subgroups $B_i \subset \bar{A}$ uniquely up to conjugacy. It turned out that the subgroups B_i precisely coincide with *Lusztig's subgroups* [39, Proposition 3.8] associated to various left cells in W (recall that the summands $\mathbf{1}_i$ are labeled by the left cells contained in the two sided cell corresponding to $\mathbb{O}$, see Section 3.4). Note that the map $\mathrm{Ann}_\chi : Y \to \mathrm{Pr}_\chi(\bar{\mathbb{O}})$ has the following interpretation: for any $M \in Y$ there is a unique $\mathbf{1}_i$ such that $\mathbf{1}_i \otimes M \simeq M$ and $\mathrm{Ann}_\chi(M)$ is precisely the primitive ideal I such that $\mathbf{1}_i = M_I$, see Section 3.4. This implies that the fiber of the map Ann_χ over $I \in \mathrm{Pr}_\chi(\mathbb{O})$ such that $\mathbf{1}_i = M_I$ is precisely $C(e)-$set $\bar{A}/B_i$.

Now let us assume that the two sided cell corresponding to the orbit $\mathbb{O}$ is not exceptional (so the class ω is trivial). It follows from the computations described above that there is one class $\psi \in H^2(\bar{A}, \mathbb{C}^\times)$ such that the classes ψ_i are just inverse images of ψ under the embeddings $B_i \subset \bar{A}$, see [37, Theorem 7.4]. Equivalently, the class describing the projective action of Q_M on $M \in Y$ is the inverse image of ψ under the map $Q_M \subset Q \to C(e) \to \bar{A}$. The recent results of Losev imply that the class ψ is always trivial. To prove this we can assume that $\mathfrak{g}$ is simple. The result certainly holds if $H^2(\bar{A}, \mathbb{C}^\times) = 0$. It follows from the classification of the nilpotent orbits that if $H^2(\bar{A}, \mathbb{C}^\times) \neq 0$ then either $\mathfrak{g}$ is classical or $\mathfrak{g}$ is

exceptional and $\bar{A}$ is the symmetric group on four or five letters. In both cases there exists a 1-dimensional $U(\mathfrak{g}, e)-$module fixed by the action of Q: for the classical $\mathfrak{g}$ this is [36, Theorem 1.2] and for the exceptional $\mathfrak{g}$ one can use the *generalized Miura transform* (see [35, 2.2]) since e must be even in this case. Thus we obtain the desired triviality of ψ since a projective 1-dimensional representation is equivalent to an actual representation.

Remark 4.6.

(i) There is a conjectural extension of the picture above to the case of non-integral central characters χ, see [37, 7.6] The computations suggest that in this case non-trivial 2-cocycles will arise quite often.

(ii) The results above give a description of the set Y (we note that for the Lie algebras of type A such a description is due to Brundan and Kleshchev [16]). An immediate next question is what are dimensions of the spaces $\mathrm{Sk}(M)$, $M \in Y$, or, equivalently, what are dimensions of the irreducible representations of $W-$algebras. A complete answer to this question is given in a recent paper [36]; remarkably in the same time some old questions about the *Goldie ranks* of the primitive ideals are resolved in *loc. cit.*

4.3. Harish-Chandra modules. It would be interesting to investigate whether the ideas above apply to the categories $\mathcal{H}^K$ of $(\mathfrak{g}, K)-$modules as in Remark 3.4. Recall that we have a *Cartan decomposition* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k}$ is the complexified Lie algebra of K. For a finitely generated $(\mathfrak{g}, K)-$module M one defines its associated variety $V(M) \subset \mathfrak{p}$, see [65]. Let ${}^{\chi}\mathcal{H}^K$ be the full subcategory of $\mathcal{H}^K$ consisting of modules M such that $Z(\mathfrak{g})$ acts on M via central character χ. Then for $M \in {}^{\chi}\mathcal{H}^K$ we have $V(M) \subset \mathfrak{p} \cap \mathcal{N}$, see [65, Corollary 5.13].

Clearly the category $\mathcal{H}(\chi)$ acts on ${}^{\chi}\mathcal{H}^K$ via $\otimes_{U(\mathfrak{g})}$. Let us fix a nilpotent orbit $\mathbb{O}$ and consider the Serre subcategories ${}^{\chi}\mathcal{H}^K_{\leq \mathbb{O}}$ and ${}^{\chi}\mathcal{H}^K_{<\mathbb{O}}$ consisting of $M \in {}^{\chi}\mathcal{H}^K$ with $V(M) \subset \mathfrak{p} \cap \bar{\mathbb{O}}$ and $V(M) \subset \mathfrak{p} \cap (\bar{\mathbb{O}} - \mathbb{O})$. We can form the quotient category ${}^{\chi}\tilde{\mathcal{H}}^K_{\mathbb{O}} = {}^{\chi}\mathcal{H}^K_{\leq \mathbb{O}}/{}^{\chi}\mathcal{H}^K_{<\mathbb{O}}$; then $\otimes_{U(\mathfrak{g})}$ gives us a bifunctor $\otimes : \mathcal{H}(\chi)_{\mathbb{O}} \times {}^{\chi}\tilde{\mathcal{H}}^K_{\mathbb{O}} \to {}^{\chi}\tilde{\mathcal{H}}^K_{\mathbb{O}}$. Let ${}^{\chi}\mathcal{H}^K_{\mathbb{O}} \subset {}^{\chi}\tilde{\mathcal{H}}^K_{\mathbb{O}}$ be the full subcategory of semisimple objects.

Conjecture 4.7. *For $M \in \mathcal{H}(\chi)_{\mathbb{O}}$ and $N \in {}^{\chi}\mathcal{H}^K_{\mathbb{O}}$ we have $M \otimes N \in {}^{\chi}\mathcal{H}^K_{\mathbb{O}}$.*

Conjecture 4.7 would imply that ${}^{\chi}\mathcal{H}^K_{\mathbb{O}}$ is a module category over $\mathcal{H}(\chi)_{\mathbb{O}}$. By Corollary 4.5 we have a tensor equivalence $\mathcal{H}(\chi)_{\mathbb{O}} = \mathrm{Coh}_{A,\omega}(Y \times Y)$ and by the results of Section 2.3 we have a classification of all indecomposable module categories over $\mathrm{Coh}_{A,\omega}(Y \times Y)$. It would be very interesting to decompose the category ${}^{\chi}\mathcal{H}^K_{\mathbb{O}}$ and to identify the indecomposable summands in terms of this classification.

5. Sheaves

Let F be an algebraically closed field of arbitrary characteristic. In this Section we will consider consider various classes of sheaves on algebraic varieties over F: $D-$modules (F is of characteristic zero), constructible sheaves in the classical topology ($F = \mathbb{C}$), and constructible $l-$adic sheaves (l is invertible in F). The corresponding categories of sheaves are $k-$linear where $k = F$, k is arbitrary of characteristic zero, and $k = \bar{\mathbb{Q}}_l$ respectively. Recall that the theories of such sheaves are parallel up to some extent. Thus we will not specify the kind of sheaves we deal with below unless this is necessary; the results are parallel in all three setups.

5.1. Convolution and Hecke algebra. Let G be a semisimple algebraic group over F of the same Dynkin type as $\mathfrak{g}$. Let $\mathcal{B}$ be the *flag variety* of G. We recall that $\mathcal{B}$ is a projective variety which is a homogeneous space for G; furthermore the *Bruhat decomposition* gives a canonical bijection between $G-$orbits on $\mathcal{B}\times\mathcal{B}$ and the Weyl group W. Let $D^b_G(\mathcal{B}\times\mathcal{B})$ be the suitable $G-$equivariant derived category of sheaves on $\mathcal{B}\times\mathcal{B}$, see e.g. [11, 2.2]. The category $D^b_G(\mathcal{B}\times\mathcal{B})$ contains a natural abelian subcategory $\mathcal{P}$ consisting of $D-$modules or perverse sheaves. The simple objects in the category $\mathcal{P}$ are the intersection cohomology complexes of closures of $G-$orbits on $\mathcal{B}\times\mathcal{B}$. This gives a natural bijection $w\to I_w$ between W and the isomorphism classes of simple objects in $\mathcal{P}$.

The category $D^b_G(\mathcal{B}\times\mathcal{B})$ has a natural monoidal structure with respect to tensor product given by *convolution*, see e.g. [11, 2.4] (this construction is parallel to Example 2.7 (iii)). It follows from the Decomposition Theorem [4] that the convolution $I_u * I_v$ is isomorphic to a direct sum of shifted I_w:

$$I_u * I_v \simeq \bigoplus_{w\in W, i\in\mathbb{Z}} n^w_{u,v}(i) I_w[i]$$

where the multiplicities $n^w_{u,v}(i)\in\mathbb{Z}_{\geq 0}$. Let $K(\mathcal{P})$ be the algebra over $\mathbb{Z}[t,t^{-1}]$ which encodes the multiplicities $n^w_{u,v}(i)$ above: the algebra has a basis c_w and

$$c_u\cdot c_v=\sum_{w\in W,i\in\mathbb{Z}} n^w_{u,v}(i)c_w t^i.$$

It is a classical result that the algebra $K(\mathcal{P})$ together with its basis $\{c_w\}$ identifies with the *Hecke algebra* together with the *Kazhdan-Lusztig basis*, see e.g. [64, 2.5]. In particular, the multiplicities $n^w_{u,v}(i)$ are computable in principle.

Lusztig defined (see [39]) the *asymptotic Hecke algebra* J in the following way: for $w\in W$ let $a(w)=\max\{i\in\mathbb{Z}|n^w_{u,v}(i)\neq 0 \text{ for some } u,v\in W\}$. Let J be a free $\mathbb{Z}-$module with basis $t_w, w\in W$ endowed with multiplication $t_u t_v=\sum_{w\in W} n^w_{u,v}(a(w))t_w$. It was shown by Lusztig that this multiplication is associative and has a unit. Moreover there is a canonical isomorphism of associative algebras $\mathbb{Q}[W]\simeq J\otimes\mathbb{Q}$, see [39, 3.2]. Furthermore, for any subset $T\subset W$ let J_T be the abelian subgroup of J spanned by $t_u, u\in T$. It is easy to see that there is a finest partition $W=\sqcup C$ such that the decomposition $J=\bigoplus_C J_C$ is a direct sum of algebras. This partition is known to coincide with partition of W into two sided Kazhdan-Lusztig cells, see [39, 3.1]. It is also known that the function a takes a constant value on any two sided cell C; we will denote this value by $a(C)$.

The constructions above was categorified in [43]. Namely, for any two sided cell C let $\mathcal{J}_C\subset\mathcal{P}$ be the full subcategory consisting of direct sums $I_w, w\in C$. The category $\mathcal{J}_C$ has a monoidal structure given by the *truncated convolution* $\bullet$, see [43]. For example

$$I_u\bullet I_v\simeq\bigoplus_{w\in C} n^w_{u,v}(a(C))I_w.$$

Hence the assignment $I_w\mapsto t_w$ induces isomorphism of based rings $K(\mathcal{J}_C)\simeq J_C$. It was shown in [10] (see also [45]) that the category $\mathcal{J}_C$ is rigid. As a consequence$\mathcal{J}_C$ is an indecomposable multi-fusion category.

5.2. $D-$modules and Harish-Chandra bimodules. In this section we assume that F is of characteristic zero. The Beilinson-Bernstein theorem [3] is a fundamental result in representation theory of $\mathfrak{g}$. It states that the category of $U(\mathfrak{g})-$modules with the trivial central

character $\chi = \chi_0$ is equivalent to the category of $D-$modules on $\mathcal{B}$. As a consequence one deduces that the category of Harish-Chandra bimodules $\mathcal{H}(\chi_0)$ is equivalent to the category of $D-$modules $\mathcal{P}$. However the Beilinson-Bernstein equivalence is not a tensor equivalence. Luckily it was shown in [5] that a composition of the Beilinson-Bernstein equivalence and a *long intertwining functor* has a natural structure of tensor functor. Using a suitable truncation of this functor the following result was shown in [11, Corollary 4.5(b)]:

Theorem 5.1. *Let C be the two-sided cell corresponding to a special nilpotent orbit $\mathbb{O}$ (see Section 3.3). Then there is a natural tensor equivalence $\mathcal{H}(\chi_0)_{\mathbb{O}} \simeq \mathcal{J}_C$.*

Theorem 5.1 gives a useful information on the Grothendieck ring $K(\mathcal{H}(\chi)_{\mathbb{O}})$ (we recall that for a regular integral character χ the category $\mathcal{H}(\chi)_{\mathbb{O}}$ is tensor equivalent to $\mathcal{H}(\chi_0)_{\mathbb{O}}$, see Section 3.2). In particular we get a homomorphism $\mathbb{Q}[W] \simeq J \otimes \mathbb{Q} \to J_C \otimes \mathbb{Q} = K(\mathcal{J}_C) \otimes \mathbb{Q} = K(\mathcal{H}(\chi_0)_{\mathbb{O}}) \otimes \mathbb{Q}$ alluded to in Section 4.2. In particular, we can consider $K(\mathcal{J}_C \otimes \mathbf{1}_i) \otimes \mathbb{Q}$ as $W-$modules; these $W-$ representations are precisely the constructible representations discussed in *loc. cit.*

Conversely Theorem 5.1 combined with Corollary 4.5 gives an explicit description of $\mathcal{J}_C$ as $\mathrm{Coh}_{\bar{A}}(Y \times Y)$ for the $D-$module version of the category $\mathcal{J}_C$. This implies similar description of $\mathcal{J}_C$ for other categories of sheaves under the assumption that the ground field F has characteristic 0. It was shown in [10] (see also [54] for the case of exceptional two sided cells) that the same description holds over a field F of arbitrary characteristic.

Remark 5.2. Theorem 5.1 was inspired by closely related results of Joseph [29]. Also a similar and related connection between Kazhdan-Lusztig cells and Harish-Chandra bimodules is contained in the work of Mazorchuk and Stroppel [48].

5.3. Drinfeld center and character sheaves. Lusztig introduced a very important class of *character sheaves* on the group G, see [40]. We recall the definition in the special case of *unipotent character sheaves*. Let

$$X = \{(b, b', g) \in \mathcal{B} \times \mathcal{B} \times G | gb = b'\}$$

we have two projections $f : X \to \mathcal{B} \times \mathcal{B}$, $f(b, b', g) = (b, b')$ and $\pi : X \to G$, $\pi(b, b', g) = g$. Note that group G acts on itself by conjugations and on X via $h \cdot (b, b', g) = (hb, hb', hgh^{-1})$ and both maps f, π are $G-$equivariant. Thus we have a functor $\Gamma : D^b_G(\mathcal{B} \times \mathcal{B}) \to D^b_G(G)$, $\Gamma = \pi_! f^*$. It follows from the Decomposition Theorem that $\Gamma(I_w), w \in W$ is isomorphic to a direct sum of shifted simple $G-$equivariant sheaves on G; a simple $G-$equivariant sheaf is called a unipotent character sheaf if it appears in such decomposition (possibly with some shift). Let $\mathcal{U}$ be the set of isomorphism classes of unipotent character sheaves on G; clearly this is a finite set.

One observes that the functor Γ above has formal properties similar to the induction functor from the monoidal category $D^b_G(\mathcal{B} \times \mathcal{B})$ to its Drinfeld center, see Section 2.4. Moreover, it is possible to identify a suitable version of the Drinfeld center of $G-$equivariant sheaves on $\mathcal{B} \times \mathcal{B}$ with suitable category of character sheaves. This was done in [11] using the abelian tensor category of Harish-Chandra bimodules and in [6] using suitable infinity categories. Furthermore applying a suitably truncated version of the same idea to the categories $\mathcal{J}_C$, the following result was proved in [11] (for the field F of characteristic zero) and in [45] (for the field F of arbitrary characteristic):

Theorem 5.3. *There is a partition $\mathcal{U} = \sqcup_C \mathcal{U}_C$ such that the Drinfeld center of the category $\mathcal{J}_C$ is naturally equivalent to the category of sheaves on G which are direct sums of objects from $\mathcal{U}_C$. In particular we have a bijection*

$$\mathcal{U}_C \leftrightarrow \{\textit{simple objects of the Drinfeld center of } \mathcal{J}_C\}.$$

The sets $\mathcal{U}_C$ were defined by Lusztig in [41, Section 16]. Recall that the category $\mathcal{J}_C$ is tensor equivalent to $\mathrm{Coh}_{\bar{A},\omega}(Y \times Y)$. Hence the Drinfeld center of $\mathcal{J}_C$ is equivalent to the Drinfeld center of $\mathrm{Vec}_{\bar{A}}^{\omega}$, see Example 2.15. The resulting bijection between $\mathcal{U}_C$ and simple objects of $\mathcal{Z}(\mathrm{Vec}_{\bar{A}}^{\omega})$ conjecturally coincides with Lusztig's one from [42, 17.8.3] which gives us a new approach to Lusztig's classification of character sheaves. On the other hand in [54] character sheaves were used in order to determine the associativity constraint in the categories $\mathcal{J}_C$ for exceptional cells C.

5.4. Some generalizations. Many constructions described in this paper extend to the case when the Weyl group W is replaced by an arbitrary Coxeter group. An important special case of the *affine Weyl groups* was considered in [9, 13] following conjectures made by Lusztig. In this case the counterparts of the cell categories are in one to one correspondence with all nilpotent orbits of $\mathfrak{g}$ and are of the form $\mathrm{Coh}_Q(Y \times Y)$ where the reductive group Q is the same as in Section 4.1 (note that the resulting categories are typically not multi-fusion categories since they have infinitely many simple objects). The set Y has a natural interpretation in terms of *non-restricted representations* of $\mathfrak{g}$ over fields of positive characteristic, see [12].

Using recent deep results by Elias and Williamson [20] on Soergel bimodules [62] Lusztig defined in [45, Section 10] the counterparts of the cell categories for an arbitrary Coxeter group W (note that these categories sometimes are not even tensor categories since they lack the unit object; however this is not very serious). It would be very interesting to identify the resulting categories. For example in the case of the dihedral group of order 10 one finds a cell category which contains a fusion subcategory with two simple objects $\mathbf{1}, X$ and the tensor product $X \otimes X = \mathbf{1} \oplus X$. This implies that the cell category is not of the form $\mathrm{Coh}_{A,\omega}(Y \times Y)$ in this case.

References

[1] Barbash, D. and Vogan, D., *Primitive ideals and orbital integrals in complex classical groups*, Math. Ann., **259** (1982), 153–199.

[2] Barbash, Vogan *Primitive ideals and orbital integrals in complex exceptional groups*, J. Algebra, **80** (1983), 350–382.

[3] Beilinson, A. and Bernstein, J., *Localization de $\mathfrak{g}-$modules*, C. R. Acad. Sci. Paris, **292** (1981), no. 1, 15–18.

[4] Beilinson, A., Bernstein, J., and Deligne, P., *Faisceaux pervers Asterisque*, **100**, 1982.

[5] Beilinson, A. and Ginzburg, V., *Wall-crossing functors and D-modules*, Represent. Theory, **3** (1999), 1–31.

[6] Ben Zvi, D. and Nadler, D., *The character theory of a complex group*, preprint, arXiv:0904.1247.

[7] Bernstein, J. and Gelfand, S., *Tensor products of finite and infinite dimensional representations of semisimple Lie algebras*, Compositio Math., **41** (1980), 245–285.

[8] Bernstein, J., Gelfand, I., and Gelfand, S., *A category of* $\mathfrak{g}-$*modules*, Funct. Anal. Appl., **10** (1976), 87–92.

[9] Bezrukavnikov, R., *On tensor categories attached to cells in affine Weyl groups, Representation theory of algebraic groups and quantum groups*, 69–91, Adv. Stud. Pure Math., 40, Math. Soc. Japan, Tokyo, 2004.

[10] Bezrukavnikov, R., Finkelberg, M., and Ostrik, V., *On tensor categories attached to cells in affine Weyl groups. III.*, Israel J. Math., **170** (2009), 207–234.

[11] _____, *Character D-modules via Drinfeld center of Harish-Chandra bimodules*, Invent. Math., **188** (2012), no. 3, 589–620.

[12] Bezrukavnikov, R. and Mirković, I., *Representations of semisimple Lie algebras in prime characteristic and the noncommutative Springer resolution*, Ann. of Math., **178** (2013), no. 3, 835–919.

[13] Bezrukavnikov, R. and Ostrik, V., *On tensor categories attached to cells in affine Weyl groups, II.*, Representation theory of algebraic groups and quantum groups, 101–119, Adv. Stud. Pure Math., 40, Math. Soc. Japan, Tokyo, 2004.

[14] Borho, W. and Brylinski, J., *Differential operators on homogeneous spaces. I. Irreducibility of the associated variety for annihilators of induced modules*, Invent. Math., **69** (1982), no. 3, 437–476.

[15] Borho, W. and Kraft, H., *Uber die Gelfand-Kirillov-Dimension*, Math. Ann., **220** (1976), 1–24.

[16] Brundan, J. and Kleshchev, A., *Representations of shifted Yangians and finite* $W-$*algebras*, Mem. Amer. Math. Soc., **196** (2008).

[17] Deligne, P., *Catégories tannakiennes*, The Grothendieck Festschrift, Vol. II, Progr. Math., **87**, Birkhäuser, Boston, MA, 1990, 111–195.

[18] Dodd, C., *Injectivity of a certain cycle map for finite dimensional W-algebras*, preprint, arXiv:1009.2456.

[19] Dugger, D., *Coherence for invertible objects and multigraded homotopy rings*, Algebr. Geom. Topol., **14** (2014), no. 2, 1055–1106.

[20] Elias, B. and Williamson, G., *The Hodge theory of Soergel bimodules*, to appear in Ann. of Math..

[21] Etingof, P., Nikshych, D., and Ostrik, V., *On fusion categories*, Ann. of Math. (2), **162** (2005), no. 2, 581–642.

[22] Gan, W. L. and Ginzburg, V., *Quantization of Slodowy slices*, IMRN, **5** (2002), 243–255.

[23] Ginzburg, V., *Harish–Chandra bimodules for quantized Slodowy slices*, Represent. Theory, **13** (2009), 236–271.

[24] Harish-Chandra, *Representations of semi-simple Lie groups I.*, Trans. Amer. Math. Soc., **75** (1953), 185–243.

[25] Humphreys, J. E., *Representations of semisimple Lie algebras in the BGG category $\mathcal{O}$*, Graduate texts in mathematics, **94**, Amer. Math. Soc., Providence, RI, 2008.

[26] Jantzen, J. C., *Einhüllende Algebren halbeinfacher Lie-Algebren*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, Berlin, 1983.

[27] Joseph, A., *On the associated variety of a primitive ideal*, J. Algebra, **93** (1985), 509–523.

[28] ———, *Goldie rank in the enveloping algebra of a semisimple Lie algebra, I.*, J. Algebra, **65** (1980), no. 2, 269–283.

[29] ———, *A sum rule for scale factors in the Goldie rank polynomials*, J. Algebra, **118**(2) (1988), 276–311.

[30] Joyal, A. and Street, R., *Tortile Yang-Baxter operators in tensor categories*, J. Pure Appl. Alg., **71** (1991), no. 1, 43–51.

[31] Kazhdan, D. and Lusztig, G., *Representations of Coxeter groups and Hecke algebras*, Inv. Math., **53** (1979), no. 2, 165–184.

[32] Kostant, B., *The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group*, Amer. J. Math., **81** (1959), 973–1032.

[33] Losev, I., *Quantized symplectic actions and $W-$algebras*, J. Amer. Math. Soc., **23** (2010), 35–59.

[34] ———, *Finite-dimensional representations of $W-$algebras*, Duke Math. J., **159** (2011), no. 1, 99–143.

[35] ———, *Finite $W-$algebras*, Proceedings of the International Congress of Mathematicians. Volume III, 1281–1307, Hindustan Book Agency, New Delhi, 2010.

[36] ———, *Dimensions of irreducible modules over W-algebras and Goldie ranks*, preprint, arXiv:1209.1083.

[37] Losev, I. and Ostrik, V., *Classification of finite dimensional irreducible modules over $W-$algebras*, to appear in *Compositio Math.*

[38] Lusztig, G., *Characters of reductive groups over a finite field*, Annals of Mathematics Studies, 107, Princeton University Press (1984).

[39] ———, *Leading coefficients of character values of Hecke algebras*, Proc. Symp. Pure Math. 47(2), Amer. Math. Soc., 1987, 235-262.

[40] ———, *Character sheaves, I.*, Adv. Math., **56** (1985), 193–237.

[41] ———, *Character sheaves, III.*, Adv. Math., **57** (1985), 266–315.

[42] ———, *Character sheaves, IV.*, Adv. Math., **59** (1986), 1–63.

[43] ———, *Cells in affine Weyl groups and tensor categories*, Adv. Math., **129** (1997), 85–98.

[44] ———, *Families and Springer's correspondence*, to appear in Pacific J. Math.

[45] ———, *Truncated convolution of character sheaves*, to appear Bull. Inst. Math. Acad. Sinica (N.S.).

[46] MacLane, S., *Categories for the working mathematician*, Graduate texts in mathematics, **5**, 2nd edition, Springer-Verlag, New York, 1998.

[47] Majid, S., *Representations, duals and quantum doubles of monoidal categories*, Rend. Circ. Mat. Palermo, **26** (1991), 197–206.

[48] Mazorchuk, V. and Stroppel, C., *Categorification of (induced) cell modules and the rough structure of generalized Verma modules*, Adv. Math., **219** (2008), 1363–1426.

[49] Moeglin, C., *Modèles de Whittaker et idéaux primitifs complètement premiers dans les algèbres enveloppantes I.*, C. R. Acad. Sci. Paris, **303** (1986), no. 17, 845–848.

[50] Müger, M., *From Subfactors to Categories and Topology I. Frobenius algebras in and Morita equivalence of tensor categories*, J. Pure Appl. Alg., **180** (2003), no. 1-2, 81–157.

[51] ———, *From Subfactors to Categories and Topology II. The quantum double of tensor categories and subfactors*, J. Pure Appl. Alg., **180** (2003), no. 1-2, 159–219.

[52] Ostrik, V., *Module categories, weak Hopf algebras and modular invariants*, Transf. Groups, **8** (2003), 177–206.

[53] ———, *Module categories over the Drinfeld double of a finite group*, IMRN, **27** (2003), 1507–1520.

[54] ———, *Tensor categories attached to exceptional cells in Weyl groups*, to appear in IMRN.

[55] Premet, A., *Special transverse slices and their enveloping algebras*, Adv. Math., **170** (2002), 1–55.

[56] ———, *Primitive ideals, non-restricted representations and finite $W-$algebras*, Mosc. Math. J., **7** (2007), 743–762.

[57] ———, *Commutative quotients of finite $W-$algebras*, Adv. Math., **225** (2010), 269–306.

[58] Saavedra Rivano, N., *Catégories Tannakiennes*, Lecture Notes in Math, **265**, Springer-Verlag, Berlin-New York, 1972.

[59] Sinh, H., *Gr-categories*, Université Paris VII doctoral thesis, 1975.

[60] Skryabin, S., *An appendix to [55].*

[61] Soergel, W., *The combinatorics of Harish-Chandra bimodules*, J. Reine Angew. Math., **429** (1992), 49–74.

[62] ———, *Kazhdan-Lusztig Polynome und unzerlegbare Bimoduln über Polynomringen*, J. Inst. Math. Jussieu, **6** (2007), 501–525.

[63] Springer, *Trigonometric sums, Green functions of finite groups and representations of Weyl groups*, Inv. Math., **36** (1976), 173–207.

[64] ———, *Quelques applications de la cohomologie d'intersection*, Bourbaki Seminar, vol. **1981/1982**, 249–273, Astérisque, 92–93, Soc. Math., France, Paris, 1982.

[65] Vogan, D. A., Jr, *Associated varieties and unipotent representations*, Harmonic analysis on reductive groups (Brunswick, ME, 1989), 315–388, Progr. Math., **101**, Birkhäuser Boston, Boston, MA, 1991.

Department of Mathematics, University of Oregon, Eugene, OR 97403, USA
E-mail: vostrik@uoregon.edu

On some recent developments in the theory of buildings

Bertrand Rémy

Abstract. Buildings are cell complexes with so remarkable symmetry properties that many groups from important families act on them. We present some examples of results in Lie theory and geometric group theory obtained thanks to these highly transitive actions. The chosen examples are related to classical and less classical (often non-linear) group-theoretic situations.

Mathematics Subject Classification (2010). 51E24, 20E42, 20E32, 20F65, 22E65, 14G22, 20F20.

Keywords. Algebraic, discrete, profinite group, rigidity, linearity, simplicity, building, Bruhat-Tits' theory, Kac-Moody theory.

1. Introduction

Buildings are cell complexes with distinguished subcomplexes, called apartments, requested to satisfy strong incidence properties. The notion was invented by J. Tits about 50 years ago and quickly became useful in many group-theoretic situations [75]. By their very definition, buildings are expected to have many symmetries, and this is indeed the case quite often. Buildings are relevant to Lie theory since the geometry of apartments is described by means of Coxeter groups: apartments are so to speak generalized tilings, where a usual (spherical, Euclidean or hyperbolic) reflection group may be replaced by a more general Coxeter group. One consequence of the existence of sufficiently large automorphism groups is the fact that many buildings admit group actions with very strong transitivity properties, leading to a better understanding of the groups under consideration.

The beginning of the development of the theory is closely related to the theory of algebraic groups, more precisely to Borel-Tits' theory of isotropic reductive groups over arbitrary fields and to Bruhat-Tits' theory of reductive groups over non-archimedean valued fields. In the former theory the involved buildings are spherical (i.e., the apartments are spherical tilings) and the group action reflects the existence, for the rational points of the algebraic group, of a strong combinatorial structure called Tits system (or BN-pair). Roughly speaking, such a structure formalizes the existence of a Bruhat decomposition indexed by a Coxeter group (called the Weyl group of the Tits system) and, among other things, leads to a uniform way of proving projective simplicity of rational points of classical groups. In the latter theory, the involved buildings are Euclidean (i.e., the apartments are Euclidean tilings) and the Weyl group of the Tits system is an affine Coxeter group. The group action on the building is a crucial tool to understand the subtle structure of the rational points of the alge-

Proceedings of the International Congress of Mathematicians, Seoul, 2014

braic group. For instance, by passing to cell stabilizers, Bruhat-Tits buildings parametrize remarkable compact open subgroups obtained from forms of the group over the valuation ring of the local ground field.

These two situations (spherical and Euclidean buildings), which are related to algebraic groups (via Borel-Tits and Bruhat-Tits' theory), will be called classical in the sequel of the report.

What is probably less well known is the fact that some buildings in which apartments are modeled on neither spherical nor affine tilings have recently led to interesting group-theoretic situations. One strong connection with geometric group theory is given by the existence, for any building, of a distance such that the resulting metric space is complete, contractible and non-positively curved in some suitable sense due to M. Gromov. In this case, the general theme is to study isometry groups of non-classical buildings by analogy with classical Lie-theoretic situations. In this analogy, buildings are seen as analogues of non-compact Riemannian symmetric spaces and their automorphism groups are seen as generalizations of semisimple Lie groups.

From that viewpoint, Kac-Moody theory is very useful even though it may not be so clear at first glance. This theory is usually presented as an infinite-dimensional generalization of semisimple Lie algebras, with applications in representation theory. It turns out that there exist some constructions of groups integrating (possibly in a loose sense) Kac-Moody Lie algebras. For these groups, nice structures from algebraic geometry are usually lost, but the combinatorial structures such as Tits systems stay available and can be translated in terms of strongly transitive group actions on (usually exotic) buildings. The notion of a ground field still makes sense and the possibility to work over a finite ground field leads to intriguing finitely generated groups or non-discrete locally profinite groups, according to the version of Kac-Moody groups chosen to begin with. These groups shall be compared to arithmetic groups and to non-archimedean Lie groups in positive characteristic, respectively.

Of course, not all exotic buildings with interesting isometry groups come from Kac-Moody theory. In dimension 2 for instance, products of arbitrary semi-homogeneous trees provide a much wider class; among groups acting properly discontinuously and cocompactly on these spaces, M. Burger and Sh. Mozes could exhibit the first simple torsion-free finitely presented groups. Still, one of the main ideas of this report is that Kac-Moody groups shall be investigated thoroughly since they are at good distance from the classical situation of Lie groups and their discrete subgroups. In some sense, it is a class of (discrete and profinite) groups which are new in the sense that striking new phenomena occur but on which we still have a very useful Lie-theoretic control (e.g. via infinite root systems).

The structure of this report is as follows. In the first section we recall some basic facts in building theory; we give some examples of results for both classical and non-classical buildings. The second section is dedicated to Euclidean buildings via two themes: compactifications of buildings (joint works with Y. Guivarc'h, and with A. Thuillier and A. Werner) and cohomology of arithmetic groups; we present two techniques of compactification, one of them using a promising relationship with non-archimedean analytic geometry. The third section deals with applications of Kac-Moody theory to the construction of interesting discrete, or non-discrete locally compact, groups (joint work with P.-E. Caprace); we explain for instance that these groups provide infinitely many quasi-isometry classes of finitely presented simple groups. It also mentions pro-p groups arising from Kac-Moody theory (joint work with I. Capdeboscq).

2. Building theory

In this section, we introduce the main subject matter of this report, namely the notion of a building. We briefly present two well-known families of buildings, that of spherical and of Euclidean buildings, and explain quickly how they are related to the theory of algebraic groups. We also mention other examples of buildings providing interesting spaces for geometric group theory.

2.1. Definition of a building. A general reference for buildings is [1]. In order to provide a definition, we first have to introduce the notion of a Coxeter complex.

- A *Coxeter group*, say W, is a group admitting a presentation: $W = \langle s \in S \mid (st)^{M_{s,t}} = 1\rangle$ where $M = [M_{s,t}]s, t \in S$ is a *Coxeter matrix* (i.e., symmetric with 1's on the diagonal and other entries in $\mathbf{N}_{\geqslant 2} \cup \{\infty\}$).
- For any Coxeter system (W, S) there is a natural simplicial complex Σ on the maximal simplices of which W acts simply transitively: Σ is called the *Coxeter complex* of (W, S).

Example 2.1. Let us go the other way round and start with a Euclidean or hyperbolic polytope whose dihedral angles are integral submultiples of π. Then, by a theorem of Poincaré's [53, IV.H.11], the group W generated by the reflections in the codimension 1 faces of the fundamental tile, is a discrete subgroup of the full isometry group of the ambient space. In fact, W is a Coxeter group and the tiling is a useful geometric realization of its Coxeter complex Σ.

The reason why we introduced Coxeter complexes is that they are so to speak generalized tilings on which the distinguished slices of a building are modeled. We freely use the previous notation W and Σ.

Definition 2.2. A building of type (W, S) is a cellular complex, covered by subcomplexes all isomorphic to Σ, called the *apartments*, such that:

(i) any two cells, called the *facets*, are contained in a suitable apartment;

(ii) given any two apartments A and A', there is a cellular isomorphism $A \simeq A'$ fixing $A \cap A'$.

The group W is called the *Weyl group* of the building. When W is a Euclidean reflection group [13, V §3], one says that the building is *affine* or, equivalently here, *Euclidean*.

Example 2.3. A tree all of whose vertices have valency $\geqslant 2$ (resp. a product of such trees) is a building with W equal to the infinite dihedral group D_∞ (resp. with W equal to $D_\infty \times D_\infty$).

The above examples of trees are elementary, but they are the only ones with infinite Weyl group which can be reasonably drawn. They are elementary examples but it is enough to consider them in order to see one difficulty in producing interesting group-theoretic situations from buildings. Indeed, take a tree T in which any two distinct vertices have distinct valencies. Then $\mathrm{Aut}(T) = \{1\}$, which shows that one has to make further assumptions on a building in order to obtain sufficiently transitive group actions.

Let us finish with some motivation of metric nature for the axioms. Assume that the Coxeter complex Σ of the Weyl group W admits an interesting W-invariant distance. This implies that each apartment carries a good metric structure, and one would like to show that this metric can be seen as being induced from a metric on the building itself. The first axiom precisely says that for any two arbitrary points in the building a distance can be computed (by choosing an apartment containing them) and the second axiom (up to some work to define suitable retractions onto apartments) can be used to show that the distance computed this way doesn't actually depend on the choice of the apartment containing the points. We will see in 2.3 that this fits very well with nice non-positive curvature properties for Coxeter complexes associated with infinite Weyl groups.

2.2. Spherical and Euclidean buildings. A building with a finite Weyl group is called *spherical*: this is because in that case the apartments are spherical tilings. The two families of affine and spherical buildings are very classical because they are closely related to the theory of algebraic groups.

First of all, it is well-known that if one starts with a reductive algebraic group, say G, over an arbitrary ground field, say k, then up to some isotropy condition on G with respect to k (namely, the existence of a non-central k-split torus) the group of rational points $G(k)$ admits a remarkable combinatorial structure called a *Tits system* (or also a BN-*pair*) [13, IV §2]. This is the main result of Borel-Tits' theory [10] and it can be reformulated as the fact that there exists a well-defined spherical building on which $G(k)$ acts *strongly transitively*, i.e. transitively on the inclusions of a chamber (i.e. a maximal facet) in an apartment.

Now, if the field k is endowed with a non-archimedean absolute value, under the above isotropy assumption (and further hypotheses on k when it is not locally compact), a similar statement says that there exists a well-defined Euclidean building on which $G(k)$ acts strongly transitively. This is one of the main results of Bruhat-Tits' theory but it doesn't exhaust the whole theory of reductive groups over valued fields [76] since one of the main tools (and objects of study at the same time) is given by forms of the group G over the valuation ring k° of k: see [19] for the building-theoretic part and [20] for the group scheme part of this deep theory.

In the spherical case, the theory of buildings may appear as a reformulation of some results proved by algebraic group-theoretic means. This is true for the statement formulated before, but quite not completely in the sense that the possibility to see the spherical building of a semisimple Lie group as the boundary at infinity of the associated symmetric space is a key step to prove Mostow's strong rigidity in differential geometry [55].

Moreover the structure of a Tits system with a finite Weyl group provides a uniform way to prove the projective simplicity of rational points of (suitable) simple isotropic algebraic groups, and reappeared recently in the theory of pseudo-reductive groups. The latter groups are algebraic groups whose study was initiated by J. Tits [81] and thoroughly made by B. Conrad, O. Gabber and G. Prasad [34]; a better understanding of these groups led to great progress in the theory of arbitrary algebraic groups in positive characteristic, with applications in number theory.

The theory of Euclidean buildings has a non-simplicial generalization which was already considered in Bruhat-Tits' work (it corresponds to the case when the valuation of the ground field k is not discrete). For geometric purposes, it was extended by B. Kleiner and B. Leeb in order to prove a strengthening of strong rigidity [48] stated by M. Gromov and called the rigidity of quasi-isometries. The non-simplicial buildings here are higher-dimensional

analogues of real trees. In Thurston's approach to Teichmüller theory, the latter trees appear as degenerations (technically speaking: asymptotic cones) of hyperbolic spaces; therefore it is quite natural to see group actions on these (so to speak, branching everywhere) Euclidean buildings appear at the boundary of some compactifications of representations spaces [57]. We will see in 3.2 that these buildings also appear naturally when combining Bruhat-Tits' theory and V. Berkovich's approach to non-archimedean analytic geometry.

The classification of spherical buildings, initially formulated in J. Tits' lecture notes [74], has been simplified and extended by J. Tits and R. Weiss in the book [82]. The classification of Euclidean buildings was done by J. Tits too [77]; as for Mostow rigidity, the proof is based on the fact that the boundary at infinity of an affine building is a spherical building. This classification was then completed by R. Weiss in the book [83]. Loosely speaking, in higher rank a spherical or a Euclidean building is related to some (possibly twisted, or even suitably generalized) algebraic group.

2.3. Some more buildings. During the last decades, some buildings of non-classical type (i.e. neither of spherical nor of Euclidean type) have become more and more interesting to study from various perspectives. For instance, the possibility to construct buildings in which the apartments are isomorphic to tilings in real hyperbolic spaces was the opportunity to obtain interesting contractible spaces of negative curvature for geometric group theory. These spaces led to important instances of Mostow rigidity [14] and quasi-isometric rigidity [15] in the setting of singular spaces.

Let us consider now the natural question of classifying buildings. As mentioned in 2.2, the classification of classical buildings is achieved in higher rank. Up to using the notion of a boundary at infinity [16, §II.8], it eventually amounts to classifying the spherical ones. In the classification of the latter buildings there are two key ingredients, namely the longest element in the (finite) Weyl group and a property, called the *Moufang property*, ensuring that the building has sufficiently many automorphisms [1, §7]. One important step consists in proving that a spherical building of rank $\geqslant 3$ automatically enjoys the Moufang property.

When dealing with buildings with infinite Weyl group, say W, the Moufang property often has to be taken as a hypothesis, and of course the existence of a longest element in W completely fails. In the attempt to classify non-affine buildings with infinite Weyl group, J. Tits had the idea to propose the hypothesis, as a substitute for the longest element in W, that the building under consideration admit a second *twin* building related to the previous one by a suitable opposition relation between the chambers [80]. The most important examples of Moufang twin buildings are provided by Kac-Moody groups as presented in 4.1, but there are other examples [2].

From the point of view of geometric group theory, an important reason why buildings sometimes play an interesting role as test spaces is probably the following result, due to G. Moussong and M. Davis [35].

Theorem 2.4. *Any building X admits a distance for which X is a complete, geodesic,* $\mathrm{CAT}(0)$*-space.*

The $\mathrm{CAT}(0)$-property is an important non-positive curvature property: roughly speaking, assuming that the space is geodesic (i.e. that any two points are always connected by a geodesic segment), it says that geodesic triangles are at least as thin as in the Euclidean plane; a $\mathrm{CAT}(0)$-space is automatically contractible. This property is fundamental in the sense that it is formulated in an elementary way but it has very deep consequences [16, Part II]. For

instance, it implies that an isometric group action with a bounded orbit (e.g. because the group is compact) has a fixed point: it is a generalization of the so-called Bruhat-Tits fixed point lemma. This result was initially used for Galois actions in a context of descent of the ground field for algebraic groups, but it has today a much broader spectrum of applications.

In view of 2.2, it is natural to see buildings with infinite Weyl groups as generalizations of Riemannian symmetric spaces. More generally, this can be done for all $\mathrm{CAT}(0)$-spaces, but it follows from remarkable papers by P.-E. Caprace and N. Monod that buildings (together with symmetric spaces) often play a prominent role in a metric space situation that might seem more general at first glance (see [27] for structure theory and [26] for discrete group actions). The main properties of semisimple Lie groups and of their discrete subgroups become therefore challenging questions for more general, sufficiently large, isometry groups of non-classical buildings with infinite Weyl groups. Among these questions, we have of course the problem of simplicity of isometry groups and the problem of rigidity of their natural actions (loosely speaking, a group action on a metric space is said to be *rigid* if there is no non-degenerate action of the group on reasonably different metric spaces). An additional question is, in some sense, a more basic one which detects to what extent the situation under consideration is new: it consists in deciding whether the isometry group of a metric space (or some subgroup of it) is linear or not, i.e. is a matrix group for some suitable dimension and field. There exist very useful sufficient conditions for linearity concerning groups acting on $\mathrm{CAT}(0)$ cell complexes [44], and some simplicity results for automorphism groups of exotic buildings [43]. In Section 4, the three questions of linearity, rigidity and simplicity are discussed for groups acting on Kac-Moody buildings.

3. Classical buildings

Let us go back to classical buildings for a while, and more precisely to Euclidean ones. The latter spaces are often presented as non-archimedean analogues of Riemannian symmetric spaces of the non-compact type associated to real semisimple Lie groups (of positive rank). This leads to natural questions, usually more precise than the questions mentioned in 2.3 (where the analogy is looser since it compares symmetric spaces and arbitrary buildings with infinite Weyl groups). This section discusses compactifications of Bruhat-Tits buildings and cohomology of arithmetic groups in positive characteristic. The first point will be the opportunity to mention a new approach to Bruhat-Tits' theory that uses non-archimedean analytic geometry in the sense of V. Berkovich.

3.1. Group-theoretic compactifications. There are many reasons to wish to compactify equivariantly symmetric spaces and Bruhat-Tits buildings associated to semisimple groups. Some of them are related to the computation of the cohomology of discrete subgroups of Lie groups, some other reasons are related to random walks on Lie groups and related geometries. We refer to the books [41] and [8] for more details and discuss here a partial compactification procedure that has the advantage to be generalized to arbitrary buildings.

The starting point of this procedure is the (maybe surprising at first glance) fact that for any locally group H, the set $\mathscr{S}_H$ of closed subgroups in H has a natural topology which is compact [12, §5]: it is called the *Chabauty topology* (hint: identify closed subgroups with homothety classes of measures on the ambient group satisfying suitable invariance properties for the action of their support). The idea to use this fact in order to compactify Riemannian

symmetric spaces (with underlying real Lie groups) is due to Y. Guivarc'h. It was generalized to the case of Bruhat-Tits buildings (with underlying non-archimedean Lie groups) in [42].

Let k be a locally compact local field, archimedean or not to begin with, and let G be a (simply connected) semisimple algebraic group over k. We let X be the symmetric space associated to $G(k)$ in the case when k is archimedean, or the Bruhat-Tits building of $G(k)$ if k is totally disconnected (2.2). In the first case, we have $X = G(k)/K$ where K is a maximal compact subgroup; in the second case, the $G(k)$-action on X admits any chamber as fundamental domain and the vertices in the closure of a given chamber parametrize the conjugacy classes of maximal compact subgroups (this follows from the Bruhat-Tits fixed point lemma of 2.3). It is a classical fact that the root system of a semisimple Lie group can be seen as a finite set of half-spaces in any maximal flat subspace A of X: see [4] in the real case; it is so by construction in the non-archimedean case, where A turns out to be an apartment [76]. Up to making a better choice in the second case, a maximal compact subgroup K in $G(k)$ always admits a fundamental domain given by a closed Weyl chamber in A, whose codimension 1 faces are called here *sector panels*; this is the geometric version of the Cartan decomposition of $G(k)$.

Now we restrict our attention to the case when k is non-archimedean and let $\{v_n\}_{n\geqslant 1}$ be a sequence of vertices in some closed Weyl chamber, say $\overline{\mathscr{Q}}$. By passing to stabilizers in $G(k)$ we obtain a sequence of maximal compact subgroups $\{K_{v_n}\}_{n\geqslant 1}$. If we further assume that for each sector panel Π of $\overline{\mathscr{Q}}$, the distance $d_X(v_n, \Pi)$ has a (possibly infinite) limit as $n \to +\infty$, then $\{K_{v_n}\}_{n\geqslant 1}$ converges in $\mathscr{S}_{G(k)}$. The limit group D is Zariski dense in some parabolic k-subgroup Q fixing a face of the chamber $\partial_\infty \overline{\mathscr{Q}}$ in the spherical building at infinity of X. Moreover D can be written as a semi-direct product $K \ltimes \mathscr{R}_u(Q)(k)$, where K is an explicit maximal compact subgroup of some reductive Levi factor of Q and $\mathscr{R}_u(Q)$ is the unipotent radical of Q. This convergence, proved by measure-theoretic means in the vein of ideas due to H. Furstenberg, is true in the archimedean case with vertices replaced by arbitrary points. It is the key fact to define a compact space $\overline{V}_X^{\mathrm{gp}}$ with a natural $G(k)$-action in any of the two cases.

Definition 3.1. The *group-theoretic compactification* of X is the closure of the set of maximal compact subgroups in $\mathscr{S}_{G(k)}$. In other words, it is the closure of the image of the $G(k)$-equivariant map $x \mapsto \mathrm{Stab}_{G(k)}(x)$ from X to $\mathscr{S}_{G(k)}$, which has to be restricted to the set V_X of vertices in X when X is a building (i.e., when k is ultrametric).

The next step then is to understand the geometry of $\overline{V}_X^{\mathrm{gp}}$ in Lie-theoretic terms. It turns out that, as in [72] for symmetric spaces, the group-theoretic compactification of the Bruhat-Tits building of the maximal semisimple quotient of each parabolic k-subgroup of G appears in the boundary [42, Theorem 16].

Theorem 3.2. *For any proper parabolic k-subgroup Q with radical $\mathscr{R}(Q)$, the group-theoretic compactification of the Bruhat-Tits building of $Q/\mathscr{R}(Q)$ lies in the boundary of $\overline{V}_X^{\mathrm{gp}}$. We let P be a minimal parabolic k-subgroup of G and we set $D_\varnothing = K \ltimes \mathscr{R}_u(P)(k)$, where K is the maximal compact subgroup of some reductive Levi factor of P. Then the conjugacy class of $D_\varnothing$ is $G(k)$-equivariantly homeomorphic to the maximal Furstenberg boundary $\mathscr{F}$ of $G(k)$, and it is the only closed $G(k)$-orbit in $\overline{V}_X^{\mathrm{gp}}$. In fact, for any closed subgroup $D \in \overline{V}_X^{\mathrm{gp}}$ there is a sequence $\{g_n\}_{n\geqslant 1}$ in $G(k)$ such that $\lim_{n\to+\infty} g_n D {g_n}^{-1}$ exists and belongs to $\mathscr{F}$.*

We have thus a description of a compactification of the set vertices of a Bruhat-Tits

building which looks like the description of the compactification of a moduli space, together with some basic statements on the dynamics of the group action on the boundary (the theory of Furstenberg boundaries is presented for instance in [51]).

Before explaining in 3.2 what can be done to compactify the full building X instead of V_X, let us finish by saying that $\overline{V}_X^{\mathrm{gp}}$ can be used to give a geometric classification (up to finite index) of remarkable closed subgroups in $G(k)$: the boundary of $\overline{V}_X^{\mathrm{gp}}$, seen as a subset of $\mathscr{S}_{G(k)}$, as well as the family of the normalizers of the groups in this boundary, can be characterized by means of dynamical notions (distality and amenability).

Remark 3.3. The results mentioned here are contained in [42] but many of them were generalized since then to arbitrary locally finite buildings by P.-E. Caprace and J. Lécureux [25]. Moreover J. Lécureux proved that the group action on the boundary is amenable, leading to positive answers to the Baum-Connes conjecture for interesting classes of groups [50].

3.2. Compactifications using analytic geometry. This subsection presents joint work with A. Thuillier and A. Werner.

There are two main problems with the compactification procedure described in 3.1. The first one is that $\overline{V}_X^{\mathrm{gp}}$ is only a compactification of the set of vertices in X. The second one is the fact that, if one has in mind the compactifications of symmetric spaces as defined by I. Satake [72] or by H. Furstenberg [40], the outcome should be a (finite) family of compact spaces. The group-theoretic compactification corresponds to the maximal Satake-Furstenberg one. The main idea in the papers [69] and [70], which allows one to overcome these two difficulties, is to combine Bruhat-Tits' theory of semisimple groups over valued fields and Berkovich's theory of analytic spaces over complete non-archimedean fields.

Berkovich geometry [6] is a version of analytic geometry over complete non-archimedean valued fields in which the spaces have nice local connectivity properties. This is surprising because local fields have a totally disconnected topology, but this good local behaviour is due to the fact that many points (of analytic nature) are added to the points given by algebraic considerations. In algebraic geometry the building blocks are algebraic spectra $\mathrm{Spec}(A)$ consisting of prime ideals of commutative rings A endowed with the Zariski topology, while in Berkovich geometry they are analytic spectra $\mathscr{M}(A)$ of Banach k-algebras, consisting of multiplicative bounded seminorms $A \to \mathbf{R}_+$. More precisely, let A be a Banach ring i.e., a commutative unit ring endowed with a Banach norm $\| \cdot \|_A$ that is submultiplicative. The *analytic spectrum* of A is the set $\mathscr{M}(A)$ of multiplicative seminorms $A \to \mathbf{R}_{\geqslant 0}$ which are bounded with respect to $\| \cdot \|_A$; this space is endowed with the coarsest topology making the evaluation maps $x \mapsto x(f)$ continuous ($f \in A$) and we henceforth use the notation $| f(x) |$ for $x(f)$. At last, to each variety V over k is attached a Berkovich analytic space over k, which is denoted by V^{an}. Loosely speaking, the good local connectivity properties of Berkovich analytic spaces come from the fact that the class of maps $x \mapsto f(x)$ is replaced by the wider class of maps $x \mapsto| f(x) |$. Recall that $\mathrm{Spec}(A)$ is in one-to-one correspondence with the set of equivalence classes of ring homomorphisms from A to an arbitrary field, where two maps are identified if they factorize through a common third map, and note that an algebraic map $x \mapsto f(x)$ can be composed with many absolute values coming from huge extensions of k.

If we go back to the compactification problem, we shall merely say that a crucial property is the fact that the attachment $V \mapsto V^{\mathrm{an}}$ is functorial and satisfies:

(i) if V is affine with coordinate ring $k[V]$, then V^{an} consists of all the multiplicative seminorms $k[V] \to \mathbf{R}_+$ extending the absolute value of k;

(ii) if V is projective, then V^{an} is compact.

Another key ingredient is a partially functorial behavior of the Bruhat-Tits building with respect to field extensions [66] combined with the possibility to work with any complete extension of k. In some sense, this implies the possibility to see any point (possibly in the relative interior of a cell) in X as a good vertex in the huger Bruhat-Tits building of G over some non-archimedean extension of k. By adapting faithfully flat descent in this context, one obtains the possibility to attach to each point $x \in X$ a Berkovich analytic subgroup G_x (defined over k as an analytic space), and the assignment $x \mapsto G_x$ is injective (in particular it takes distinct values for any two distinct points, even if they lie in the same cell). Finally, the following result [69, 2.1] is the main step to obtain an analytic filling of the group-theoretic compactification $\overline{V}_X^{\mathrm{gp}}$ of 3.1.

Theorem 3.4. *Let X be the building associated to a simply connected semisimple algebraic group G over a local field k.*

(i) *For any $x \in X$, there is an analytic subgroup G_x of G^{an} defined over k such that for any non-archimedean extension K/k, we have: $G_x(K) = \mathrm{Stab}_{G(K)}(x)$.*

(ii) *For any $x \in X$, there is a unique point $\vartheta(x) \in G^{\mathrm{an}}$ such that:*
$$G_x = \{g \in G^{\mathrm{an}} : |f(g)| \leqslant |f(\vartheta(x))| \text{ for any } f \in k[G]\}.$$

(iii) *The resulting map $x \mapsto \vartheta(x)$ is a $G(k)$-equivariant embedding of X into G^{an} with closed image.*

This result gives a map $X \to G^{\mathrm{an}}$ and then, in order to obtain equivariant compactifications of X, it suffices to compose it with analytifications of algebraic maps from G to proper varieties (e.g., the maps to flag varieties $G \to G/P$ where P is a parabolic k-subgroup of G). The desired compactifications are the closures of the image of X under these maps. When P varies over all the conjugacy classes of parabolic k-subgroups of G, one obtains all the expected analogues of the Satake-Furstenberg compactifications.

Remark 3.5. Together with the asymptotic cones [16, I.5] of symmetric spaces and Euclidean buildings alluded to in 2.3, the Bruhat-Tits buildings of G over non-archimedean extensions of k with dense valuations are other examples of non-simplicial Euclidean buildings that appear naturally.

The paper [69] also contains a Lie-theoretic description of the boundary structure of these compactifications and some extensions, from Bruhat-Tits' theory, of useful decompositions of the rational points $G(k)$. The paper [70] describes a variant of this compactification procedure which uses highest-weight theory and is closer in spirit to I. Satake's original ideas.

Remark 3.6. When G is split over k, the idea to combine Bruhat-Tits' theory and Berkovich geometry can be found already in [5, §5].

3.3. Cohomological and related questions. Non-compact Riemannian symmetric spaces and Bruhat-Tits buildings are contractible spaces acted upon properly by the Lie groups they are associated with. These actions are therefore very useful to compute or estimate the

cohomology of discrete subgroups of reductive Lie groups. Using this action and suitable compactifications, A. Borel and J.-P. Serre proved, among other things, that arithmetic and even S-arithmetic groups in characteristic 0 are of type F_∞ [9]. Recall that a group Γ is said to be *of type* F_m if it admits a free action on a contractible CW-complex whose m-skeleton has finitely many Γ-orbits; it is said to be *of type* F_∞ if it is of type F_n for any n. These conditions are related to other more algebraic finiteness properties stated in terms of resolutions [18, VIII]. The *finiteness length* of Γ is the largest m such that Γ is of type F_m, i.e. admits a classifying space with finite m-skeleton.

In the case when the global ground field leading to the arithmetic groups under consideration is not of characteristic 0, things get much more complicated for cohomology. For instance in characteristic $p > 0$, finite generation is not always true for arithmetic groups, and finitely generated lattices needn't be virtually torsion-free either. Still, combined efforts by K.-U. Bux, R. Köhl, S. Witzel and K. Wortman led to the following result.

Theorem 3.7. *Let K be a global function field, let S be a finite set of places of K and let $\mathcal{O}_S$ be the ring of S-integers in K. Let G be a connected, absolutely almost simple, K-isotropic K-group. For each $v \in S$ let r_v be the rank of G over the completion K_v of K with respect to v. Then the finiteness length of the S-arithmetic group $G(\mathcal{O}_S)$ is equal to $(\sum_{v\in S} r_v) - 1$.*

It was proved in [24] that $(\sum_{v\in S} r_v) - 1$ is an upper bound for the finiteness length of $G(\mathcal{O}_S)$ and equality was proved in [23]. The nice feature of many results in this vein is the mixture of classical techniques such as reduction theory in positive characteristic [45], K. Brown's criterion from algebraic topology [17] and the use of recent tools from geometric group theory such as singular Morse theory [7].

Note that, so far in this report, the fact that for a simple group G, the Bruhat-Tits building is a simplicial complex, has not been exploited yet (in general a Bruhat-Tits building is a polysimplicial complex). Examples of works where this geometric fact is used are given by the papers [59] and [60] which provide a key step towards an almost complete answer to the congruence subgroup problem. We will see in 4.3 that this can also be used to develop a singular version of Hodge theory in order to obtain some vanishing results for the cohomology of automorphism groups of exotic buildings.

Remark 3.8. In this section, most applications of Bruhat-Tits' theory that are presented (except [59] and [60]) mainly deal with the building-theoretic aspect of it and not with the delicate theory of forms of reductive algebraic groups over the valuation ring of the valued ground field. The volume formula proved by G. Prasad [58], which eventually leads to the classification of fake projective planes [61], is an example of a result that needs, among other things, Bruhat-Tits' theory at the latter level of subtlety.

4. Kac-Moody theory and exotic buildings

In this section, we are interested in families of non-classical buildings admitting sufficiently large groups of automorphisms, and being therefore good candidates for the comparison with symmetric spaces and Bruhat-Tits buildings associated to semisimple Lie groups. The main source of such buildings comes from an algebraic machinery which was not *a priori* designed for these purposes, namely Kac-Moody theory. We explain here why the analogy is indeed fruitful. In fact, Kac-Moody groups provide a good balance between persistence of classical

results from the theory of arithmetic groups and appearance of new phenomena. This is true in the framework of discrete groups, as well as in that of non-discrete locally compact groups. Moreover it is likely that this theory is also the source of many interesting profinite groups. As mentioned before, the three main questions organizing the study of Kac-Moody groups are those about linearity, rigidity and simplicity (but they are not the only ones).

4.1. Kac-Moody theory. Roughly speaking, Kac-Moody Lie algebras are infinite-dimensional generalizations of complex semisimple Lie algebras [47] and Kac-Moody groups integrate these Lie algebras over $\mathbf{Z}$, thus providing infinite-dimensional generalizations of Chevalley schemes [36]. Our goal in this section is to introduce the two versions of Kac-Moody groups, namely the minimal (possibly twisted) Kac-Moody groups and the complete ones; they are both presented and compared in J. Tits' Bourbaki talk [79].

Combinatorial Kac-Moody objects. The starting point to define all these objects is a *generalized Cartan matrix*; i.e. an integral matrix $A = [A_{s,t}]_{s,t\in S}$ satisfying: $A_{s,s} = 2$, $A_{s,t} \leqslant 0$ when $s \neq t$ and $A_{s,t} = 0 \Leftrightarrow A_{t,s} = 0$. It is more accurate to start with a *Kac-Moody root datum*, namely a 5-tuple $\mathcal{D} = \big(S, A, \Lambda, (c_s)_{s\in S}, (h_s)_{s\in S}\big)$, where A is a generalized Cartan matrix indexed by a finite set S and where Λ is a free $\mathbf{Z}$-module (with $\mathbf{Z}$-dual $\Lambda^\vee$); the elements c_s of Λ and h_s of $\Lambda^\vee$ are requested to satisfy $c_s(h_t) = A_{ts}$ for all $s, t \in S$. One defines then a complex Lie algebra $\mathfrak{g}_\mathcal{D}$ by a presentation generalizing Serre's presentation of finite-dimensional semisimple Lie algebras, involving $(h_s)_{s\in S}$ and the usual generators $(e_s)_{s\in S}$, $(f_s)_{s\in S}$ so that in particular $\mathbf{C}e_s \oplus \mathbf{C}h_s \oplus \mathbf{C}f_s \simeq \mathfrak{sl}_2(\mathbf{C})$.

Using the free abelian group $Q = \bigoplus_{s\in S} \mathbf{Z}\alpha_s$ on the symbols α_s, one defines a Q-gradation on $\mathfrak{g}_\mathcal{D}$ in which the degrees with non-trivial corresponding spaces belong to $Q^+ \cup Q^-$, where $Q^+ = \sum_{s\in S} \mathbf{N}\alpha_s$ and $Q^- = -Q^+$. The latter non-zero degrees are called *roots* and if $(c_s)_{s\in S}$ is free over $\mathbf{Z}$, they have the usual interpretation in terms of weight spaces. The *height* of a root $\alpha = \sum_{s\in S} n_s\alpha_s$ is the integer $\mathrm{ht}(\alpha) = \sum_{s\in S} n_s$. There is a natural action on the lattice Q by a Coxeter group W generated by involutions denoted again by $s \in S$; it is defined by setting $s.a_t = a_t - A_{st}a_s$. A root is called *real* if it is in the W-orbit of a *simple* root, i.e. some α_s; otherwise, it is said to be *imaginary*. The set of roots (resp. real roots, imaginary roots) is denoted by Δ (resp. Δ_{re}, Δ_{im}).

Minimal Kac-Moody groups. Using the divided powers $\frac{1}{n!}e_s^n$ and $\frac{1}{n!}f_s^n$ of the canonical generators e_s and f_s and of their Weyl group conjugates, J. Tits defined a certain $\mathbf{Z}$-form $\mathcal{U}_\mathcal{D}$ of the universal enveloping algebra $\mathcal{U}\mathfrak{g}_\mathcal{D}$. The ring $\mathcal{U}_\mathcal{D}$ has a filtration indexed by Q; some subrings as well as their completions with respect to some subsemigroups of Q are used to construct Kac-Moody groups. For the adjoint action on $\mathcal{U}\mathfrak{g}_\mathcal{D}$, the real root spaces have a locally nilpotent action which can be exponentiated to produce 1-parameter unipotent subgroups in the automorphism group of the $\mathbf{Z}$-form $\mathcal{U}_\mathcal{D}$ for suitable restrictions of parameters and elements in $\mathfrak{g}_\mathcal{D}$. By and large, the *minimal Kac-Moody group functor* $\mathfrak{G}_\mathcal{D}$ is an amalgamation of a split torus with character group Λ and of a quotient of the subgroup generated by these 1-parameter subgroups [78]. To each real root $\gamma \in \Delta_{\mathrm{re}}$ is attached a subgroup functor $\mathfrak{U}_\gamma$, which is isomorphic to the 1-dimensional additive group functor, but there is no subgroup associated to imaginary roots in minimal Kac-Moody groups. Non-split versions of minimal Kac-Moody groups can also constructed [63].

Example 4.1. The functor which sends a field k to the group $\mathrm{SL}_{n+1}(k[t, t^{-1}])$ is a minimal

Kac-Moody group functor of affine type $\widetilde{A}_n$.

Complete Kac-Moody groups. More generally, minimal Kac-Moody groups generalize groups of the form $\mathbf{G}(k[t,t^{-1}])$ where k is a field and $\mathbf{G}$ is a k-isotropic semisimple group. Accordingly, complete Kac-Moody groups generalize groups like $\mathbf{G}\big(k(\!(t)\!)\big)$. We present here a construction due to G. Rousseau [67] which provides group functors defined over $\mathbf{Z}$; the functors have a structure of ind-scheme generalizing constructions due to O. Mathieu [54] or Sh. Kumar [49] over the complex numbers.

For suitable affine group schemes over fields in characteristic 0, the algebra of invariant distributions [37, II §4 n°6] can be identified with the universal enveloping algebra of the Lie algebra of the group. Moreover $\mathbf{Z}$-forms of this algebra can be used to define, by duality, $\mathbf{Z}$-forms of the rings of regular functions: this eventually leads to group schemes over $\mathbf{Z}$ extending the initial groups over $\mathbf{C}$. For a Kac-Moody root datum $\mathcal{D}$, G. Rousseau associates to any closed set of roots Ψ, a pro-unipotent group scheme $\mathfrak{U}_\Psi^{\mathrm{ma}}$ defined over $\mathbf{Z}$ [67, 3.1]. With this approach, imaginary roots do lead to root groups (which seems to be a promising property for this version of Kac-Moody groups) and all the groups $\mathfrak{U}_\Psi^{\mathrm{ma}}$ have a filtration described thanks to the root system. At last, if k is a finite field of characteristic p, the groups $\mathfrak{U}_\Psi^{\mathrm{ma}}(k)$ are pro-p.

Remark 4.2. L. Carbone and H. Garland also defined a representation theoretic completion $\mathfrak{G}_{\mathcal{D}}^{\mathrm{cg}\lambda}(k)$ of $\mathfrak{G}_{\mathcal{D}}(k)$ for each dominant weight λ [33].

Connection with building theory. We can now go back to the main subject matter of this report, i.e. building theory. One crucial fact about minimal Kac-Moody groups is that any such group $\mathfrak{G}_{\mathcal{D}}(k)$ over some field k enjoys a combinatorial structure refining that of a Tits system [13, IV §2], and called a *twin* BN-*pair*. As a consequence, there is a pair $X_\pm$ of twin buildings as mentioned in 2.3 such that $\mathfrak{G}_{\mathcal{D}}(k)$ acts strongly transitively on each of them. The apartments are explicitly described thanks to the Weyl group W of $\mathcal{D}$ and the buildings $X_\pm$ are locally finite if and only if the ground field k is finite (if so, the full isometry groups $\mathrm{Iso}(X_\pm)$ are then locally compact for the compact open topology). Similarly, the complete group $\mathfrak{G}_{\mathcal{D}}^{\mathrm{ma}+}(k)$ has a natural strongly transitive action on a single building which is closely related to the twin buildings $X_\pm$ [67, Corollaire 3.18]. In the latter case, a chamber stabilizer is isomorphic to the semi-direct product of a finite-dimensional split torus and of the pro-unipotent group scheme $\mathfrak{U}_\Psi^{\mathrm{ma}}$ associated to $\Psi = \Delta^+$ (where Δ^+ is the set of all positive roots).

Remark 4.3. When k is finite, there is another more elementary completion $\mathfrak{G}_{\mathcal{D}}^{\mathrm{geom}}(k)$ obtained by taking the closure of the image of $\mathfrak{G}_{\mathcal{D}}(k)$ in the isometry group $\mathrm{Iso}(X_\pm)$ [68, 1.B].

4.2. Non-linearity, simplicity and rigidity. This subsection presents joint work with P.-E. Caprace.

We can now consider our three main questions: non-linearity, simplicity and rigidity, when dealing with minimal Kac-Moody groups over finite fields. The point is that these groups are, by definition, finitely generated groups which generalize arithmetic groups in positive characteristic like $\mathrm{SL}_{n+1}(\mathbf{F}_q[t,t^{-1}])$. Therefore the general theme is to try to answer the following question.

$(*)$ *To what extent is a finitely generated Kac-Moody group close to a discrete subgroup*

in a non-archimedean semisimple Lie group?

In what follows, Λ denotes a minimal Kac-Moody group $\mathfrak{G}_{\mathcal{D}}(\mathbf{F}_q)$ over some finite field of characteristic p.

Lattice property. The first result supporting the analogy of $(*)$ was proved independently in [32] and [62].

Theorem 4.4. *Assume that the Weyl group W of Λ is infinite and denote by $W(t) = \sum_{w\in W} t^{\ell(w)}$ its growth series. If $W(\frac{1}{q}) < \infty$, then the group Λ is a lattice of $X_+ \times X_-$; it is never cocompact.*

The statement means that the homogeneous space $\big(\mathrm{Isom}(X_+) \times \mathrm{Isom}(X_-)\big)/\Lambda$ carries an invariant measure of finite total volume. The proof relies on a simple measure-theoretic formula and an explicit description of a fundamental domain for the diagonal Λ-action on $X_+ \times X_-$. The fundamental domain is given by the product of a chamber and of a suitably chosen apartment of opposite sign. This can be seen by combinatorial arguments relevant to Tits systems (it is an analogue of the geometric interpretation of the Cartan decomposition mentioned in 3.1).

Normal subgroup property. The previous theorem suggests to try to prove the main results of the theory of discrete subgroups of Lie groups in the case of finitely generated Kac-Moody groups. A particularly well adapted part of this theory is G. Margulis' work on lattices in Lie groups [51] because many proofs there rely on measure-theoretic techniques (which can be more easily adapted to non-linear groups than arguments from pure algebraic group theory). One striking result in this field is a strong dichotomy called the normal subgroup property for higher-rank lattices. More precisely, one says that a group Γ has the *normal subgroup property* if for any $N \lhd \Gamma$ either N is finite and central in Γ, or N has finite index in Γ. Here is the result for Kac-Moody groups.

Theorem 4.5. *If the finitely generated Kac-Moody group Λ is a lattice of $X_+ \times X_-$, then it has the normal subgroup property.*

The proof is mainly a consequence of deep results due to Y. Shalom [73] and to U. Bader and Y. Shalom [3]. The idea is to follow Margulis' strategy: to sum up, we can assume that we are in a situation where $N \lhd \Lambda < \mathrm{Isom}(X_-) \times \mathrm{Isom}(X_+)$ for a center-free Λ; hence we have to prove that Λ/N is finite, i.e. is compact for the discrete topology! This apparently naive remark is a crucial trick because being compact here is equivalent to being amenable and having Kazhdan's property (T). Then the idea is to use a criterion due to Y. Shalom (resp. U. Bader and Y. Shalom) which says that in order to prove property (T) (resp. amenability) for the discrete quotient group Λ/N, it is enough to check it on the topological quotients $\overline{\mathrm{pr}_\pm(\Lambda)}/\overline{\mathrm{pr}_\pm(N)}$, where $\mathrm{pr}_\pm$ is the natural projection from $\mathrm{Isom}(X_-) \times \mathrm{Isom}(X_+)$ to $\mathrm{Isom}(X_\pm)$. Checking the latter points is easier because the involved topological groups have more structure: indeed, $\overline{\mathrm{pr}_\pm(\Lambda)}$ acts strongly transitively on the building $X_\pm$ since so does Λ. In fact, using Tits system arguments, one can see that the each topological quotient $\overline{\mathrm{pr}_\pm(\Lambda)}/\overline{\mathrm{pr}_\pm(N)}$ is compact. The paper [73] considers *cocompact* irreducible lattices in direct products, but the cocompactness assumption can relaxed to a weaker integrability condition involving an induction cocycle, which is checked in [65] thanks to combinatorial arguments.

Simplicity. The general strategy to prove simplicity of suitable (i.e. non-affine, irreducible) Kac-Moody lattices owes a lot to M. Burger and Sh. Mozes' seminal works [21] and [22]. Among other things, these papers prove the existence of finitely presented torsion free simple groups; these groups are constructed as lattices acting on products of two trees with a compact fundamental domain (in fact, the groups can be chosen to act transitively on the vertices of the square complex). The general idea is first to see the discrete groups under consideration as analogues of lattices in Lie groups in order to rule out infinite quotients, and then to exploit decisive differences with linear groups in order to rule out finite quotients too.

The first step, exploiting the analogy with lattices in Lie groups, is of course what was mentioned before in the Kac-Moody case. The point is to obtain the normal subgroup property without relying on any algebraic group structure on the ambient topological group. This structure is replaced by the fact that the latter topological group is the direct product of isometry groups of trees or buildings. The second step, where one has to stand by non-linear phenomena, is so far specific to each of the two situations: uniform lattices in products of trees in [22], or non-uniform lattices for products of (usually higher-dimensional) buildings in the Kac-Moody case. In the case of products of trees, this step relies on the possibility of obtaining some non-residual finiteness criteria involving transitivity conditions on the local actions (around each vertex) for the projection of the lattice on each of the two trees; this part was eventually improved by the possibility to embed explicitly well-known non-residually finite groups into suitable cocompact lattices of products of trees. In the Kac-Moody case, the arguments are relevant to Coxeter groups. This is where non-affineness of the Weyl group has to be exploited crucially: a strengthening of Tits' alternative for Coxeter groups implies that Coxeter complexes Σ of non-affine Coxeter groups contain lots of hyperbolic triples of roots (seen as half-spaces of Σ), i.e. with empty pairwise intersections. Combining this with a trick on infinite root systems and some defining relations for Kac-Moody groups leads to the following wide source of infinite finitely generated simple groups [29, §4].

Theorem 4.6. *Let Λ be a Kac-Moody group defined over the finite field $\mathbf{F}_q$. Assume that the Weyl group W is infinite and irreducible, and that $W(\frac{1}{q}) < \infty$. Then Λ is simple (modulo its finite center) whenever the buildings $X_\pm$ are not Euclidean and Λ is generated by its root subgroups.*

Remark 4.7. The assumption on generation by root groups is mild since the initial group Λ can be replaced by its finite index subgroup generated by the root groups, but the assumption excluding affine Weyl groups is crucial: indeed, groups of the form $\mathbf{G}(\mathbf{F}_q[t, t^{-1}])$, where $\mathbf{G}$ is a semisimple group over $\mathbf{F}_q$, are affine Kac-Moody groups and admit lots of (congruence) quotients.

Remark 4.8. Of course, the question of abstract simplicity for complete Kac-Moody groups makes sense too. Topological simplicity can be proved easily in this case by using Tits system arguments and the fact complete Kac-Moody groups over $\mathbf{F}_q$ are locally pro-p [64]. Using a beautiful mixture of dynamical and Lie-theoretic arguments, T. Marquis proved the (much better) abstract simplicity of the same groups [52].

Infinitely many quasi-isometry classes of simple groups. The wide choice of buildings admitting simple lattices is a very useful fact in geometric group theory. Recall that, after M. Gromov, it is natural to attach to each group Γ with finite symmetric generating set $S = S^{-1}$ its *Cayley graph*, i.e. the graph in which the vertices are the elements of Γ, which

are declared to be adjacent if and only if they differ from the right by an element of S. This is the starting point to see these groups as metric spaces. One important notion in this context is that of quasi-isometry between metric spaces, that is almost bi-Lipschitz equivalence except that additive constant are allowed. More precisely, two metric spaces (X, d_X) and (Y, d_Y) are said to be *quasi-isometric* to one another if there is a map $f : X \to Y$ such that there exist $C \geqslant 1$ and $D \geqslant 0$ satisfying for each $x, x' \in X$:

$$\frac{1}{C} \cdot d_X(x, x') - D \leqslant d_Y(f(x), f(x')) \leqslant C \cdot d_X(x, x') + D$$

and such that for any $y \in Y$ there exists $x \in X$ such that $d_Y(y, f(x)) \leqslant D$. The first condition says that f is a *quasi-isometric embedding* and the second condition is a coarse metric surjectivity assumption.

Now let G be a locally compact group admitting a finitely generated lattice Γ; then G admits a compact generating subset, say $\widehat{\Sigma}$. We denote by $d_{\widehat{\Sigma}}$ the word metric associated with $\widehat{\Sigma}$ and we fix a finite generating set Σ for Γ, leading to an associated word metric d_Σ. The lattice Γ is called *undistorted* in G if d_Σ is quasi-isometric to the restriction of $d_{\widehat{\Sigma}}$ to Γ. This amounts to saying that the inclusion of Γ in G is a quasi-isometric embedding from the metric space (Γ, d_Σ) to the metric space $(G, d_{\widehat{\Sigma}})$.

It is proved in [30] that any Kac-Moody lattice $\Lambda < \mathrm{Aut}(X_+) \times \mathrm{Aut}(X_-)$ is undistorted, and the most important consequence of this statement in geometric group theory is the following.

Theorem 4.9. *There exist infinitely many pairwise non-quasi-isometric finitely presented simple groups.*

Note that since any two trees are bi-Lipschitz equivalent, all uniform lattices of products of trees lie in the same quasi-isometry class.

4.3. Cohomological and related questions. It was mentioned in 3.3 that buildings, being simplicial complexes, are particularly well adapted to cohomology computation. Techniques from Hodge theory can be pushed quite far in this singular context [56]. Another approach, introduced by J. Dymara and T. Januszkiewicz, uses representation theoretic techniques as stated in [11] in the classical case, and leads to important results. The result below is a special case of [38, Theorem E].

Theorem 4.10. *Let Λ be a minimal Kac-Moody group over $\mathbf{F}_q$, defined by a generalized Cartan matrix A of size $n \times n$. Let $m < n$ be an integer such that all the principal submatrices of size $m \times m$ of A are Cartan matrices (i.e. are of finite type). Then for $1 \leqslant k \leqslant m-1$ and $q \gg 1$, the continuous cohomology groups $\mathrm{H}^k_{\mathrm{ct}}\big(\mathrm{Aut}(X_\pm), \rho\big)$ vanish for any unitary representation ρ.*

Degree 1 is of particular interest since $\mathrm{H}^1_{\mathrm{ct}}(G, \rho) = \{0\}$ for any unitary representation ρ is equivalent to Kazhdan's property (T) for G [46, Chap. 4]. When Λ is 2-spherical (i.e. when we have $m \geqslant 2$ above) Theorem 4.10 implies property (T) for the full automorphism groups $\mathrm{Aut}(X_\pm)$ with $q \gg 1$, hence for their product, and finally for any lattice in this product [51, III]. As a consequence, many Kac-Moody lattices have property (T) and this can be used to prove a super-rigidity result for isometric actions of higher-rank Kac-Moody groups on negatively curved metric spaces [29, §7].

Finite generation of maximal pro-p subgroups. Let us finish by mentioning another potential source of original results in a new group-theoretic framework. More precisely, if we now consider complete Kac-Moody groups as in 4.1 over finite fields, then we obtain locally pro-p groups (whatever the completion procedure, in fact). Moreover it follows from the Bruhat-Tits fixed point lemma that maximal pro-p subgroups in a given complete Kac-Moody group over $\mathbf{F}_q$ are all conjugate to one another [64]; they are finite index subgroups of chamber stabilizers for their natural action on the associated building.

Let $\mathfrak{G}_{\mathcal{D}}^{\mathrm{ma}+}(\mathbf{F}_q)$ be the algebraic completion of the minimal Kac-Moody group $\mathfrak{G}_{\mathcal{D}}(\mathbf{F}_q)$. Let A be the generalized Cartan matrix of the Kac-Moody root datum $\mathcal{D}$ defining the group functor $\mathfrak{G}_{\mathcal{D}}$. Let $U^{\mathrm{ma}+}$ be a pro-p Sylow subgroup of $\mathfrak{G}_{\mathcal{D}}^{\mathrm{ma}+}(\mathbf{F}_q)$. The following theorem [28, Theorem 2.2] shows that the maximal pro-p subgroups in complete Kac-Moody groups over finite fields of characteristic p have an interesting behavior, which still deserves deeper investigation.

Theorem 4.11. *Assume that the characteristic p of $\mathbf{F}_q$ is greater than the absolute value of any off-diagonal coefficient of the generalized Cartan matrix A. Then $U^{\mathrm{ma}+}$ is finitely generated as a pro-p group.*

Remark 4.12. An argument initially due to L. Carbone, M. Ershov and G. Ritter [31], combining a Frattini sugbroup argument and a Tits system argument, implies the projective simplicity of complete Kac-Moody groups over finite fields for many types A. It can be generalized to all types but still leads to a weaker result than Marquis's theorem [52] because of the assumption on the size of p with respect to the coefficients of A.

The connection with cohomology is as follows: under the assumptions of the theorem, it can be proved that the following more precise statements hold.

(i) The Frattini subgroup $\Phi(U^{\mathrm{ma}+})$ of $U^{\mathrm{ma}+}$ is equal to the abstract derived group $\big[U^{\mathrm{ma}+}, U^{\mathrm{ma}+}\big]$.

(ii) We have: $\Phi(U^{\mathrm{ma}+}) = \overline{\langle U_\gamma : \gamma \text{ non-simple positive real root}\rangle}$.

(iii) We have also: $\mathrm{H}_1(U^{\mathrm{ma}+}, \mathbf{Z}/p\mathbf{Z}) \simeq (\mathbf{Z}/p\mathbf{Z})^{\mathrm{size}(A)\cdot[\mathbf{F}_q:\mathbf{Z}/p\mathbf{Z}]}$.

The connection between the Frattini subgroup and homology is that we have $\mathrm{H}_1(V, \mathbf{Z}/p\mathbf{Z}) \cong V/\Phi(V)$ for any pro-p group V [71, Lemma 6.8.6]; moreover $\dim_{\mathbf{Z}/p\mathbf{Z}} \mathrm{H}_1(V, \mathbf{Z}/p\mathbf{Z})$ is the minimal size of a topologically generating set for V. The latter point suggests to compute higher homology groups for pro-p Sylow subgroups of complete locally compact Kac-Moody groups. The next interesting result would be to be able to decide under which conditions these groups are finitely presentable as pro-p groups. This is related to $\mathrm{H}_2(U^{\mathrm{ma}+}, \mathbf{Z}/p\mathbf{Z})$.

Of course the question of simplicity doesn't make sense for pro-p groups, but discussing linearity of these pro-p Sylow subgroups definitely makes sense. One hope would be to disprove linearity for as many examples as possible. There are only partial results in this direction so far [28, §4].

Remark 4.13. By studying full pro-p completions of suitably chosen subgroups of minimal Kac-Moody group over finite fields, M. Ershov could exhibit some examples of Golod-Shafarevich groups with property (T), which leads to the existence of infinite torsion residually finite non-amenable groups [39].

Acknowledgements. The author expresses his deep gratitude to all the co-authors mentioned in this report. This work was supported by the GDSous/GSG project (ANR-12-BS01-0003) and by the labex MILYON (ANR-10-LABX-0070) of Université de Lyon, within the Investissements d'Avenir program (ANR-11-IDEX-0007) operated by the French National Research Agency (ANR)

References

[1] Peter Abramenko and Kenneth S. Brown, *Buildings*, Graduate Texts in Mathematics, vol. 248, Springer, New York, 2008, Theory and applications, MR 2439729 (2009g:20055).

[2] Peter Abramenko and Bertrand Rémy, *Commensurators of some non-uniform tree lattices and Moufang twin trees*, Essays in geometric group theory, Ramanujan Math. Soc. Lect. Notes Ser., vol. 9, Ramanujan Math. Soc., Mysore, 2009, pp. 79–104. MR 2605356 (2011f:20111).

[3] Uri Bader and Yehuda Shalom, *Factor and normal subgroup theorems for lattices in products of groups*, Invent. Math., **163** (2006), no. 2, 415–454. MR 2207022 (2006m:22017).

[4] Werner Ballmann, Mikhael Gromov, and Viktor Schroeder, *Manifolds of nonpositive curvature*, Progress in Mathematics, vol. 61, Birkhäuser Boston, Inc., Boston, MA, 1985, MR 823981 (87h:53050).

[5] Vladimir G. Berkovich, *Spectral theory and analytic geometry over non-Archimedean fields*, Mathematical Surveys and Monographs, vol. 33, American Mathematical Society, Providence, RI, 1990. MR 1070709 (91k:32038).

[6] ———, *p-adic analytic spaces*, Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998), no. Extra Vol. II, 1998, pp. 141–151 (electronic), MR 1648064 (99h:14026).

[7] Mladen Bestvina and Noel Brady, *Morse theory and finiteness properties of groups*, Invent. Math. **129** (1997), no. 3, 445–470. MR 1465330 (98i:20039).

[8] Armand Borel and Lizhen Ji, *Compactifications of symmetric and locally symmetric spaces*, Mathematics: Theory & Applications, Birkhäuser Boston, Inc., Boston, MA, 2006, MR 2189882 (2007d:22030).

[9] Armand Borel and Jean-Pierre Serre, *Cohomologie d'immeubles et de groupes S-arithmétiques*, Topology **15** (1976), no. 3, 211–232, MR 0447474 (56 #5786).

[10] Armand Borel and Jacques Tits, *Groupes réductifs*, Inst. Hautes Études Sci. Publ. Math., (1965), no. 27, 55–150, MR 0207712 (34 #7527).

[11] Armand Borel and Nolan Wallach, *Continuous cohomology, discrete subgroups, and representations of reductive groups*, second ed., Mathematical Surveys and Monographs, vol. 67, American Mathematical Society, Providence, RI, 2000, MR 1721403 (2000j:22015).

[12] Nicolas Bourbaki, *Éléments de mathématique. Intégration VII-VIII*, Actualités Scientifiques et Industrielles, No. 1306, Hermann, Paris, 1963, MR 0179291 (31 #3539).

[13] ———, *Éléments de mathématique. Groupes et algèbres de Lie IV-VI*, Actualités Scientifiques et Industrielles, No. 1337, Hermann, Paris, 1968, MR 0240238 (39 #1590).

[14] Marc Bourdon, *Immeubles hyperboliques, dimension conforme et rigidité de Mostow*, Geom. Funct. Anal., **7** (1997), no. 2, 245–268, MR 1445387 (98c:20056).

[15] Marc Bourdon and Hervé Pajot, *Rigidity of quasi-isometries for some hyperbolic buildings*, Comment. Math. Helv., **75** (2000), no. 4, 701–736, MR 1789183 (2003a:30027).

[16] Martin R. Bridson and André Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften, vol. 319, Springer-Verlag, Berlin, 1999, MR 1744486 (2000k:53038).

[17] Kenneth S. Brown, *Finiteness properties of groups*, Proceedings of the Northwestern conference on cohomology of groups (Evanston, Ill., 1985), vol. 44, 1987, pp. 45–75, MR 885095 (88m:20110).

[18] ———, *Cohomology of groups*, Graduate Texts in Mathematics, vol. 87, Springer-Verlag, New York, 1994, Corrected reprint of the 1982 original., MR 1324339 (96a:20072).

[19] François Bruhat and Jacques Tits, *Groupes réductifs sur un corps local*, Inst. Hautes Études Sci. Publ. Math., (1972), no. 41, 5–251, MR 0327923 (48 #6265).

[20] ———, *Groupes réductifs sur un corps local. II. Schémas en groupes. Existence d'une donnée radicielle valuée*, Inst. Hautes Études Sci. Publ. Math., (1984), no. 60, 197–376. MR 756316 (86c:20042).

[21] Marc Burger and Shahar Mozes, *Groups acting on trees: from local to global structure*, Inst. Hautes Études Sci. Publ. Math., (2000), no. 92, 113–150 (2001), MR 1839488 (2002i:20041).

[22] ———, *Lattices in product of trees*, Inst. Hautes Études Sci. Publ. Math., (2000), no. 92, 151–194 (2001), MR 1839489 (2002i:20042).

[23] Kai-Uwe Bux, Ralf Köhl, and Stefan Witzel, *Higher finiteness properties of reductive arithmetic groups in positive characteristic: the rank theorem*, Ann. of Math., (2) **177** (2013), no. 1, 311–366, MR 2999042.

[24] Kai-Uwe Bux and Kevin Wortman, *Finiteness properties of arithmetic groups over function fields*, Invent. Math., **167** (2007), no. 2, 355–378, MR 2270455 (2007k:11082).

[25] Pierre-Emmanuel Caprace and Jean Lécureux, *Combinatorial and group-theoretic compactifications of buildings*, Ann. Inst. Fourier (Grenoble), **61** (2011), no. 2, 619–672, MR 2895068.

[26] Pierre-Emmanuel Caprace and Nicolas Monod, *Isometry groups of non-positively curved spaces: discrete subgroups*, J. Topol., **2** (2009), no. 4, 701–746, MR 2574741 (2011i:53052).

[27] ———, *Isometry groups of non-positively curved spaces: structure theory*, J. Topol., **2** (2009), no. 4, 661–700, MR 2574740 (2011i:53051).

[28] Inna Capdeboscq and Bertrand Rémy, *On some pro-p groups from infinite-dimensional Lie theory*, to appear in Mathematische Zeitschrift, 2014.

[29] Pierre-Emmanuel Caprace and Bertrand Rémy, *Simplicity and superrigidity of twin building lattices*, Invent. Math., **176** (2009), no. 1, 169–221, MR 2485882 (2010d:20056).

[30] ———, *Non-distortion of twin building lattices*, Geom. Dedicata, **147** (2010), 397–408, MR 2660586 (2011e:20038).

[31] Lisa Carbone, Mikhail Ershov, and Gordon Ritter, *Abstract simplicity of complete Kac-Moody groups over finite fields*, J. Pure Appl. Algebra, **212** (2008), no. 10, 2147–2162, MR 2418160 (2009d:20067).

[32] Lisa Carbone and Howard Garland, *Lattices in Kac-Moody groups*, Math. Res. Lett., **6** (1999), no. 3-4, 439–447, MR 1713142 (2000k:22026).

[33] ———, *Existence of lattices in Kac-Moody groups over finite fields*, Commun. Contemp. Math., **5** (2003), no. 5, 813–867, MR 2017720 (2004m:17031).

[34] Brian Conrad, Ofer Gabber, and Gopal Prasad, *Pseudo-reductive groups*, New Mathematical Monographs, vol. 17, Cambridge University Press, Cambridge, 2010, MR 2723571 (2011k:20093).

[35] Michael W. Davis, *Buildings are* $\mathrm{CAT}(0)$, Geometry and cohomology in group theory (Durham, 1994), London Math. Soc. Lecture Note Ser., vol. 252, Cambridge Univ. Press, Cambridge, 1998, pp. 108–123, MR 1709955 (2000i:20068).

[36] Michel Demazure, *Schémas en groupes réductifs*, Bull. Soc. Math., France, **93** (1965), 369–413, MR 0197467 (33 #5632).

[37] Michel Demazure and Pierre Gabriel, *Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs*, Masson & Cie, Éditeur, Paris; North-Holland Publishing Co., Amsterdam, 1970, Avec un appendice ıt Corps de classes local par Michiel Hazewinkel, MR 0302656 (46 #1800).

[38] Jan Dymara and Tadeusz Januszkiewicz, *Cohomology of buildings and their automorphism groups*, Invent. Math., **150** (2002), no. 3, 579–627, MR 1946553 (2003j:20052).

[39] Mikhail Ershov, *Golod-Shafarevich groups with property* (T) *and Kac-Moody groups*, Duke Math. J., **145** (2008), no. 2, 309–339, MR 2449949.

[40] Hillel Furstenberg, *A Poisson formula for semi-simple Lie groups*, Ann. of Math. (2), **77** (1963), 335–386, MR 0146298 (26 #3820).

[41] Yves Guivarc'h, Lizhen Ji, and J. C. Taylor, *Compactifications of symmetric spaces*, Progress in Mathematics, vol. 156, Birkhäuser Boston, Inc., Boston, MA, 1998, MR 1633171 (2000c:31006).

[42] Yves Guivarc'h and Bertrand Rémy, *Group-theoretic compactification of Bruhat-Tits buildings*, Ann. Sci. École Norm. Sup. (4), **39** (2006), no. 6, 871–920, MR 2316977 (2008f:20056).

[43] Frédéric Haglund and Frédéric Paulin, *Simplicité de groupes d'automorphismes d'espaces à courbure négative*, The Epstein birthday schrift, Geom. Topol. Monogr., vol. 1, Geom. Topol. Publ., Coventry, 1998, pp. 181–248 (electronic), MR 1668359 (2000b:20034).

[44] Frédéric Haglund and Daniel T. Wise, *Special cube complexes*, Geom. Funct. Anal., **17** (2008), no. 5, 1551–1620, MR 2377497 (2009a:20061).

[45] Günter Harder, *Minkowskische Reduktionstheorie über Funktionenkörpern*, Invent. Math., **7** (1969), 33–54, MR 0284441 (44 #1667).

[46] Pierre de la Harpe and Alain Valette, *La propriété* (T) *de Kazhdan pour les groupes localement compacts (avec un appendice de Marc Burger)*, Astérisque (1989), no. 175, 158, MR 1023471 (90m:22001).

[47] Victor G. Kac, *Infinite-dimensional Lie algebras*, third ed., Cambridge University Press, Cambridge, 1990, MR 1104219 (92k:17038).

[48] Bruce Kleiner and Bernhard Leeb, *Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings*, Inst. Hautes Études Sci. Publ. Math., (1997), no. 86, 115–197, (1998), MR 1608566 (98m:53068).

[49] Shrawan Kumar, *Kac-Moody groups, their flag varieties and representation theory*, Progress in Mathematics, vol. 204, Birkhäuser Boston, Inc., Boston, MA, 2002, MR 1923198 (2003k:22022).

[50] Jean Lécureux, *Amenability of actions on the boundary of a building*, Int. Math. Res. Not. IMRN, no. 17, 3265–3302, (2010), MR 1608566 (98m:53068).

[51] Grigori A. Margulis, *Discrete subgroups of semisimple Lie groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 17, Springer-Verlag, Berlin, 1991, MR 2680274 (2011k:22022).

[52] Timothée Marquis, *Abstract simplicity of locally compact Kac-Moody groups*, to appear in Compositio Mathematica, 2014.

[53] Bernard Maskit, *Kleinian groups*, Grundlehren der Mathematischen Wissenschaften, vol. 287, Springer-Verlag, Berlin, 1988, MR 959135 (90a:30132).

[54] Olivier Mathieu, *Construction d'un groupe de Kac-Moody et applications*, Compositio Math., **69** (1989), no. 1, 37–60, MR 986812 (90f:17012).

[55] George D. Mostow, *Strong rigidity of locally symmetric spaces*, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1973, Annals of Mathematics Studies, No. 78, MR 0385004 (52 #5874).

[56] Pierre Pansu, *Formules de Matsushima, de Garland et propriété (T) pour des groupes agissant sur des espaces symétriques ou des immeubles*, Bull. Soc. Math., France **126** (1998), no. 1, 107–139, MR 1651383 (2000d:53067).

[57] Anne Parreau, *Compactification d'espaces de représentations de groupes de type fini*, Math. Z., **272** (2012), no. 1-2, 51–86, MR 2968214.

[58] Gopal Prasad, *Volumes of S-arithmetic quotients of semi-simple groups*, Inst. Hautes Études Sci. Publ. Math., (1989), no. 69, 91–117, With an appendix by Moshe Jarden and the author., MR 1019962 (91c:22023).

[59] Gopal Prasad and M. S. Raghunathan, *Topological central extensions of semisimple groups over local fields*, Ann. of Math. (2), **119** (1984), no. 1, 143–201, MR 736564 (86e:20051a).

[60] ———, *Topological central extensions of semisimple groups over local fields. II*, Ann. of Math. (2), **119** (1984), no. 2, 203–268. MR 740894 (86e:20051b).

[61] Gopal Prasad and Sai-Kee Yeung, *Fake projective planes*, Invent. Math., **168** (2007), no. 2, 321–370, MR 2289867 (2008h:14038).

[62] Bertrand Rémy, *Construction de réseaux en théorie de Kac-Moody*, C. R. Acad. Sci. Paris Sér. I Math., **329** (1999), no. 6, 475–478, MR 1715140 (2001d:20028).

[63] ———, *Groupes de Kac-Moody déployés et presque déployés*, Astérisque (2002), no. 277, viii+348, MR 1909671 (2003d:20036).

[64] ———, *Topological simplicity, commensurator super-rigidity and non-linearities of Kac-Moody groups*, Geom. Funct. Anal., **14** (2004), no. 4, 810–852, With an appendix by P. Bonvin., MR 2084981 (2005g:22024).

[65] ———, *Integrability of induction cocycles for Kac-Moody groups*, Math. Ann., **333** (2005), no. 1, 29–43, MR 2169827 (2006k:22018).

[66] Guy Rousseau, *Immeubles des groupes réductifs sur les corps locaux*, U.E.R. Mathématique, Université Paris XI, Orsay, 1977, Thèse de doctorat, Publications Mathématiques d'Orsay, No. 221-77.68, MR 0491992 (58 #11158).

[67] ———, *Groupes de Kac-Moody déployés sur un corps local, II., Masures ordonnées*, preprint, 2012.

[68] Bertrand Rémy and Mark Ronan, *Topological groups of Kac-Moody type, right-angled twinnings and their lattices*, Comment. Math. Helv., **81** (2006), no. 1, 191–219, MR 2208804 (2007b:20063).

[69] Bertrand Rémy, Amaury Thuillier, and Annette Werner, *Bruhat-Tits theory from Berkovich's point of view. I., Realizations and compactifications of buildings*, Ann. Sci. Éc. Norm. Supér. (4), **43** (2010), no. 3, 461–554, MR 2667022 (2011j:20075).

[70] ———, *Bruhat-Tits theory from Berkovich's point of view. II Satake compactifications of buildings*, J. Inst. Math. Jussieu, **11** (2012), no. 2, 421–465, MR 2905310.

[71] Luis Ribes and Pavel Zalesskii, *Profinite groups*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, vol. 40, Springer-Verlag, Berlin, 2010, MR 2599132 (2011a:20058).

[72] Ichirô Satake, *On representations and compactifications of symmetric Riemannian spaces*, Ann. of Math. (2), **71** (1960), 77–110, MR 0118775 (22 #9546).

[73] Yehuda Shalom, *Rigidity of commensurators and irreducible lattices*, Invent. Math., **141** (2000), no. 1, 1–54, MR 1767270 (2001k:22022).

[74] Jacques Tits, *Buildings of spherical type and finite BN-pairs*, Lecture Notes in Mathematics, Vol. 386, Springer-Verlag, Berlin-New York, 1974, MR 0470099 (57 #9866).

[75] ———, *On buildings and their applications*, Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), Vol. 1, Canad. Math. Congress, Montreal, Que., 1975, pp. 209–220, MR 0439945 (55 #12826).

[76] ———, *Reductive groups over local fields*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 29–69, MR 546588 (80h:20064).

[77] ———, *Immeubles de type affine*, Buildings and the geometry of diagrams (Como, 1984), Lecture Notes in Math., vol. 1181, Springer, Berlin, 1986, pp. 159–190, MR 843391 (87h:20077).

[78] ———, *Uniqueness and presentation of Kac-Moody groups over fields*, J. Algebra, **105** (1987), no. 2, 542–573, MR 873684 (89b:17020).

[79] ———, *Groupes associés aux algèbres de Kac-Moody*, Astérisque (1989), no. 177-178, Exp. No. 700, 7–31, Séminaire Bourbaki, Vol. 1988/89, MR 1040566 (91c:22034).

[80] ———, *Twin buildings and groups of Kac-Moody type*, Groups, combinatorics & geometry (Durham, 1990), London Math. Soc. Lecture Note Ser., vol. 165, Cambridge Univ. Press, Cambridge, 1992, pp. 249–286, MR 1200265 (94d:20030).

[81] ———, *Résumés des cours au Collège de France (1973-2000)*, Documents mathématiques, no. 12, Soc. Math., de France, 2013.

[82] Jacques Tits and Richard M. Weiss, *Moufang polygons*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2002, MR 1938841 (2003m:51008).

[83] Richard M. Weiss, *The structure of affine buildings*, Annals of Mathematics Studies, vol. 168, Princeton University Press, Princeton, NJ, 2009, MR 2468338 (2009m:51022).

Université Lyon 1, CNRS - UMR 5208, Institut Camille Jordan, 43 boulevard du 11 novembre 1918, F-69622 Villeurbanne cedex, FRANCE

E-mail: remy@math.univ-lyon1.fr

Some qualitative properties of branching multiplicities

Nicolas Ressayre

Abstract. Let G be a connected reductive subgroup of a complex connected reductive group $\hat{G}$. We consider the multiplicities $c_{G,\hat{G}}$ as a function from the set of pairs of dominant weights to the set of integers. We recall that this function is piecewise quasipolynomial. Its support is a finitely generated semigroup ; we describe an irredundant list of inequalities determining the cone generated. The relation with the projection of coadjoint orbits for the Lie algebras of the compact forms of G and $\hat{G}$ is also recalled.

We also consider the multiplicities for the fusion products for G. More precisely, we explain how the small quantum cohomology rings of homogeneous spaces G/P allow to parametrize an irredundant set of inequalities determining the multiplicative eigenvalue problem for the compact form K of G.

Mathematics Subject Classification (2010). 14L24, 14N15, 14N35, 22C99.

Keywords. Branching multiplicities, additive and multiplicative eigenvalue problems, (quantum) Schubert calculus.

1. Introduction

Let G be a connected reductive subgroup of a complex connected reductive group $\hat{G}$. The branching problem consists in

decomposing irreducible representations of $\hat{G}$ as sum of irreducible G-modules.

Fix maximal tori $T \subset \hat{T}$ and Borel subgroups $B \supset T$ and $\hat{B} \supset \hat{T}$ of G and $\hat{G}$. Let $X(T)$ denote the group of characters of T and let $X(T)^+$ denote the set of dominant characters. For $\nu \in X(T)^+$, $V_G(\nu)$ denotes the irreducible representation of highest weight ν. Similarly we use notation $X(\hat{T})$, $X(\hat{T})^+$, $V_{\hat{G}}(\hat{\nu})$ relatively to $\hat{G}$. For any G-module V, the subspace of G-fixed vectors is denoted by V^G. For $\nu \in X(T)^+$ and $\hat{\nu} \in X(\hat{T})^+$, set

$$c_{G,\hat{G}}(\nu,\hat{\nu}) = \dim(V_G(\nu) \otimes V_{\hat{G}}(\hat{\nu}))^G. \tag{1.1}$$

The branching problem is equivalent to the knowledge of these coefficients since

$$V_{\hat{G}}(\hat{\nu}) = \sum_{\nu \in X(T)^+} c_{G\hat{G}}(\nu,\hat{\nu}) V_G(\nu)^*, \tag{1.2}$$

where $V_G(\nu)^*$ is the dual of $V_G(\nu)$.

Proceedings of the International Congress of Mathematicians, Seoul, 2014

For G diagonally embedded in $\hat{G} = G \times G$, $V_{\hat{G}}(\hat{\nu})$ is the tensor product of two irreducible representations of G and the coefficient $c_{G,\hat{G}}(\nu, \hat{\nu})$ are the multiplicities for the decomposition of $V_{\hat{G}}(\hat{\nu})$ as a sum of irreducible G-modules. If $G = GL_n(\mathbb{C})$, $X(T)^+$ identifies with the set of non-increasing sequences $\nu = (\nu_1 \geq \cdots \geq \nu_n)$ of n integers and the coefficients are the Littlewood-Richardson coefficients $c^{\nu}_{\lambda\mu}$. For $\hat{G} = GL_n(\mathbb{C})$ and $G = \hat{T}$, the multiplicities $c_{G,\hat{G}}(\nu, \hat{\nu})$ are the Kostka coefficients given by the Weyl character formula. In this note, we recall some qualitive results on the function

$$\begin{array}{rccc} c_{G\hat{G}} : & X(T)^+ \times X(\hat{T})^+ & \longrightarrow & \mathbb{Z}_{\geq 0} \\ & (\nu, \hat{\nu}) & \longmapsto & c_{G\hat{G}}(\nu, \hat{\nu}). \end{array}$$

More precisely:

(i) There exists a polyhedral convex cone $\mathbb{Q}_{\geq 0}\mathrm{LR}(G, \hat{G})$ outside which the multiplicities are zero. A multiplicity corresponding to a point in this cone can also vanish, but it becomes nonzero after scalar multiplication.

(ii) The cone $\mathbb{Q}_{\geq 0}\mathrm{LR}(G, \hat{G})$ is the support of some fan called GIT-fan and the function $c_{G\hat{G}}$ is quasipolynomial on each cone of the fan.

(iii) In Section 3, we give an explicit description of the irredundant list of linear inequalities of the cone $\mathbb{Q}_{\geq 0}\mathrm{LR}(G, \hat{G})$. These inequalities are parametrized by pairs of Schubert classes for some G and $\hat{G}$ homogeneous spaces that satisfy some cohomological condition. This condition is expressed by the Belkale-Kumar product.

(iv) We also present a description for coefficients on the boundary of the cone $\mathbb{Q}_{\geq 0}\mathrm{LR}(G, \hat{G})$: they are equal to similar coefficients for Levi subgroups of G and $\hat{G}$.

(v) The support $\mathrm{LR}(G, \hat{G})$ of the function $c_{G\hat{G}}$ is a finitely generated semigroup generating a group $\mathbb{Z}\mathrm{LR}(G, \hat{G})$. We present some results comparing $\mathrm{LR}(G, \hat{G})$ with the set $\mathbb{Z}\mathrm{LR}(G, \hat{G}) \cap \mathbb{Q}_{\geq 0}\mathrm{LR}(G, \hat{G})$ of integral points in the cone. This problem, called the question of saturation is far from being completely solved and we present some conjectures.

(vi) We present the PRV conjecture and its recent generalizations. These statement allow to produce easily points in the semigroup $\mathrm{LR}(G, \hat{G})$.

Consider the case of the tensor product decomposition, that is when G is diagonally embedded in $\hat{G} = G \times G$. Let K a maximal compact subgroup of G and $\mathfrak{k}$ its Lie algebra. It turns out that the cone $\mathbb{Q}_{\geq 0}\mathrm{LR}(G, \hat{G})$ identifies with the set of triples $(\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3)$ of adjoint K-orbits in $\mathfrak{k}$ such that the set $\mathcal{O}_1 + \mathcal{O}_2 + \mathcal{O}_3$ of sums contains 0. This spectral problem admits a multiplicative analogue: describe the set of triples $(\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3)$ of conjugacy classes in K such that $\mathcal{O}_1 \cdot \mathcal{O}_2 \cdot \mathcal{O}_3$ contains the unit e of K. By a result of Meinrenken-Woodward, these triples are described by a convex polytope. Here, we describe recent results that give explicitly the minimal list of linear inequalities characterizing this polytope. These inequalities are parametrized by triples of Schubert classes satisfying some conditions expressed using the quantum cohomology. If the additive case is related to tensor product decomposition, the multiplication problem is equivalent to the description of the asymptotic support of the fusion product. In this context, the question of saturation can also be asked: the only known result due to Belkale is in type A.

2. Quasipolynomiality and GIT-fan

2.1. A brief review on Geometric Invariant Theory. Consider a complex irreducible projective variety X acting on by a reductive group G. The set of G-linearized line bundles on X is a group (for the tenor product) denoted by $\mathrm{Pic}^G(X)$. For $\mathcal{L} \in \mathrm{Pic}^G(X)$, let $\mathrm{H}^0(X,\mathcal{L})$ denote the G-module of regular sections of $\mathcal{L}$ and let $\mathrm{H}^0(X,\mathcal{L})^G$ denote the subspace of G-invariant sections. We consider the following set of *semi-stable points* for $\mathcal{L}$:

$$X^{\mathrm{ss}}(\mathcal{L}) = \left\{x \in X : \exists n > 0 \text{ and } \tau \in \mathrm{H}^0(X,\mathcal{L}^{\otimes n})^G \text{ such that } \tau(x) \neq 0\right\}. \tag{2.1}$$

Note that this definition of $X^{\mathrm{ss}}(\mathcal{L})$ is NOT standard. Indeed, it is usually agreed that the open subset defined by the non vanishing of τ is affine. Our definition coincides with the usual one if $\mathcal{L}$ is ample. A line bundle $\mathcal{L}$ over X is said to be *semi-ample* if a positive power of $\mathcal{L}$ is base point free.

2.2. The GIT-fan. To simplify, we assume that the rank of the Picard group of X, and hence that of $\mathrm{Pic}^G(X)$ are finite. Otherwise one has to consider the Neron-Severi group (see [22]). Since $X^{\mathrm{ss}}(\mathcal{L}) = X^{\mathrm{ss}}(\mathcal{L}^{\otimes n})$ for any $n > 0$, the definition of $X^{\mathrm{ss}}(\mathcal{L})$ extends to the case when $\mathcal{L} \in \mathrm{Pic}^G(X)_{\mathbb{Q}}$. Following [22], we say that two points $\mathcal{L}_1$ and $\mathcal{L}_2$ in $\mathrm{Pic}^G(X)_{\mathbb{Q}}$ are *GIT-equivalent* if and only if $X^{\mathrm{ss}}(\mathcal{L}_1) = X^{\mathrm{ss}}(\mathcal{L}_2)$. A point $\mathcal{L} \in \mathrm{Pic}^G(X)_{\mathbb{Q}}$ is said to be *G-effective* if $X^{\mathrm{ss}}(\mathcal{L})$ is not empty. Let $\mathcal{C}^+(X)$ denote the cone of $\mathrm{Pic}^G(X)_{\mathbb{Q}}$ generated by the points $\mathcal{L} \otimes 1$ where $\mathcal{L} \in \mathrm{Pic}^G(X)$ is semi-ample. Let $\mathcal{C}^{G,+}(X)$ denote the cone of $\mathrm{Pic}^G(X)_{\mathbb{Q}}$ generated by the points $\mathcal{L} \otimes 1$ where $\mathcal{L} \in \mathrm{Pic}^G(X)$ is semi-ample and G-effective. Then $\mathcal{C}^+(X)$ and $\mathcal{C}^{G,+}(X)$ are convex. A sub-cone of $\mathcal{C}^+(X)$ is said to be *polyhedral in* $\mathcal{C}^+(X)$ if it is the intersection of $\mathcal{C}^+(X)$ with finitely many rational and closed half spaces of $\mathrm{Pic}^G(X)_{\mathbb{Q}}$. The geometry of the GIT-classes is described by the following result.

Theorem 2.1. *The cone $\mathcal{C}^{G,+}(X)$ is polyhedral in $\mathcal{C}^+(X)$. There are finitely many GIT-classes. Each GIT-class is the relative interior of some convex cone polyhedral in $\mathcal{C}^+(X)$. The closures of the GIT-classes in $\mathcal{C}^+(X)$ form a fan, called the GIT-fan.*

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two GIT-classes and fix $\mathcal{L}_1 \in \mathcal{C}_1$ and $\mathcal{L}_2 \in \mathcal{C}_2$. Then $X^{\mathrm{ss}}(\mathcal{L}_1)$ is contained in $X^{\mathrm{ss}}(\mathcal{L}_2)$ if and only if $\overline{\mathcal{C}_1}$ contains $\mathcal{C}_2$ if and only if $\overline{\mathcal{C}_2}$ is a face of $\overline{\mathcal{C}_1}$.

The points $\mathcal{L} \otimes 1$ for ample $\mathcal{L} \in \mathrm{Pic}^G(X)$ generate an open convex cone $\mathcal{C}^{++}(X)$ in $\mathcal{C}^+(X)$. Theorem 2.1 when $\mathcal{C}^+(X)$ is replaced by $\mathcal{C}^{++}(X)$ is proved in [59] following [22, 70]. The proofs in [59] can be applied without changing to get Theorem 2.1.

2.3. Application to branching coefficients. We now explain the geometric interpretation of the branching coefficients allowed by Borel-Weil theorem. Let $X = G/B \times \hat{G}/\hat{B}$. For any pair $(\nu,\hat{\nu})$ in $X(T) \times X(\hat{T})$, there exists a unique $(G \times \hat{G})$-linearized line bundle $\mathcal{L}(\nu,\hat{\nu})$ on X such that $T \times \hat{T}$ acts on the base-point of X with weight $-(\nu,\hat{\nu})$. Then $\mathcal{L}(\nu,\hat{\nu})$ is semi-ample if and only if ν and $\hat{\nu}$ are dominant. In this case, $H^0(X,\mathcal{L}(\nu,\hat{\nu}))$ is a $G \times \hat{G}$-module isomorphic to $V_G(\nu)^* \otimes V_{\hat{G}}(\hat{\nu})^*$. In particular $c_{G,\hat{G}}(\nu,\hat{\nu})$ is the dimension of $H^0(X,\mathcal{L}(\nu,\hat{\nu}))^G$ where G acts diagonally.

Consider the morphism $\theta : X(T \times \hat{T})_{\mathbb{Q}} \longrightarrow \mathrm{Pic}^G(X)_{\mathbb{Q}}$ that maps $(\nu,\hat{\nu})$ on $\mathcal{L}(\nu,\hat{\nu})$ endowed with the diagonal G-action. The pullback $\theta^{-1}(\mathcal{C}^+(X))$ is the cone $X(T \times \hat{T})^+_{\mathbb{Q}}$ generated by dominant weights, and $\theta^{-1}(\mathcal{C}^{G,+}(X))$ is

$$\mathbb{Q}_{\geq 0}\mathrm{LR}(G,\hat{G}) := \{\nu,\hat{\nu}) \in X(T \times \hat{T})^+_{\mathbb{Q}} : \exists n > 0 \quad c_{G,\hat{G}}(n\nu, n\hat{\nu}) \neq 0\}.$$

The pullback in $\mathbb{Q}_{\geq 0}\mathrm{LR}(G,\hat{G})$ of a GIT-class in $\mathrm{Pic}^G(X)_{\mathbb{Q}}$ is called a GIT-class.

Quasipolynomiality. Let $\mathfrak{g}$ and $\hat{\mathfrak{g}}$ denote the Lie algebras of G and $\hat{G}$. It turns out that $\mathbb{Q}_{\geq 0}\mathrm{LR}(G,\hat{G})$ has nonempty interior in $X(T\times\hat{T})_{\mathbb{Q}}$ if and only if no ideal of $\mathfrak{g}$ is an ideal of $\hat{\mathfrak{g}}$. Under this assumption, the GIT-classes $\mathcal{C}$ of nonempty interior in $X(T\times\hat{T})^+_{\mathbb{Q}}$ are called the *GIT-chambers*. Their closures $\overline{\mathcal{C}}$ are the maximal cones of the GIT-fan.

Theorem 2.2. *Let $\mathcal{C}$ be a GIT-chamber. There exists a cofinite lattice $\Lambda_{\mathcal{C}}$ of $X(T\times\hat{T})$ and a collection of polynomial functions f_l indexed by $l\in X(T\times\hat{T})/\Lambda_{\mathcal{C}}$ such that*

$$c_{G,\hat{G}}(\nu,\hat{\nu}) = f_l(\nu,\hat{\nu}),$$

for any $(\nu,\hat{\nu})\in\overline{\mathcal{C}}\cap X(T\times\hat{T})$ congruent to l modulo $\Lambda_{\mathcal{C}}$.

Meinrenken-Sjamaar proved Theorem 2.2 in [49] using symplectic geometry. A proof using the Riemann-Roch theorem for singular varieties can be found in [40]. Examples of GIT-fans and quasipolynomial functions can be found in [1, 19, 38, 40, 56]. Another example is given in Section 3.4.

3. The cone $\mathbb{Q}_{\geq 0}\mathrm{LR}(G,\hat{G})$

Theorems 2.1 or 5.1 imply that $\mathbb{Q}_{\geq 0}\mathrm{LR}(G,\hat{G})$ is a closed convex polyhedral cone in $X(T\times\hat{T})_{\mathbb{Q}}$. The aim of this section is to describe this cone by an explicit list of inequalities.

3.1. Spectral interpretation. Let K and $\hat{K}$ be two maximal compact subgroups of G and $\hat{G}$ such that $K\subset\hat{K}$. Let $\mathfrak{k}$ and $\hat{\mathfrak{k}}$ denote the Lie algebras of K and $\hat{K}$. Consider the projection $p:\hat{\mathfrak{k}}\longrightarrow\mathfrak{k}$ orthogonal for the Cartan-Killing form of $\hat{\mathfrak{k}}$. We are interested in the projections of adjoint orbits of $\hat{\mathfrak{k}}$.

Up to changing T, we may assume that $H=K\cap T$ is a Cartan subgroup of K. Consider the Lie algebra $\mathfrak{h}$ of H. Any root α of (G,T) induces (by derivation) a linear form (still denoted by α) on the Lie algebra $\mathrm{Lie}(T)$ of T. The Lie algebra $\mathrm{Lie}(H)$ of H identifies with the real Lie sub-algebra of $\xi\in\mathrm{Lie}(T)$ such that $\alpha(\xi)\in\sqrt{-1}\mathbb{R}$ for any root α. Consider the group $X_*(T)$ of one parameter subgroups of T and its paring $\langle\cdot,\cdot\rangle$ with $X(T)$. The dominant chamber in $X_*(T)_{\mathbb{R}}:=X_*(T)\otimes\mathbb{R}$ is

$$X_*(T)^+_{\mathbb{R}} = \{\lambda\in X_*(T)_{\mathbb{R}} : \langle\lambda,\alpha\rangle\geq 0 \quad \text{for any simple root } \alpha\}.$$

By derivation, $X_*(T)$ identifies with a sub-lattice of $\mathrm{Lie}(T)$, and hence, $X_*(T)_{\mathbb{R}}$ identifies with $\sqrt{-1}\,\mathrm{Lie}(H)$. Any adjoint K-orbit in $\mathfrak{k}$ contains a unique element belonging to $\sqrt{-1}X_*(T)^+_{\mathbb{R}}$; for any $\lambda\in X_*(T)^+_{\mathbb{R}}$, we denote by $\mathcal{O}_\lambda$ the adjoint K-orbit containing $\sqrt{-1}\lambda$. Similarly, we define the adjoint $\hat{K}$-orbit $\mathcal{O}_{\hat{\lambda}}$ for any $\hat{\lambda}\in X_*(\hat{T})^+_{\mathbb{R}}$. Let w_0 be the longest element of the Weyl group of G. Set

$$\mathcal{C}(K,\hat{K}) = \{(\lambda,\hat{\lambda})\in X_*(T)^+_{\mathbb{R}}\times X_*(\hat{T})^+_{\mathbb{R}} : \mathcal{O}_{-w_0\lambda}\subset p(\mathcal{O}_{\hat{\lambda}})\}.$$

The Kirwan convexity theorem (see [32]) in symplectic geometry shows that $\mathcal{C}(K,\hat{K})$ is a closed polyhedral cone.

Fix a W-invariant scalar product $(\cdot,\cdot)_T$ on $X_*(T)_{\mathbb{R}}$. Then, for $\lambda \in X_*(T)_{\mathbb{R}}$, $(\lambda,\cdot)_T$ is a linear form on $X_*(T)_{\mathbb{R}}$ and it corresponds to a point in $X^*(T)_{\mathbb{R}}$ for the pairing $\langle\cdot,\cdot\rangle$. Similarly $(\cdot,\cdot)_{\hat{T}}$ is fixed on $X_*(\hat{T})_{\mathbb{Q}}$.

Theorem 3.1. *Let* $(\lambda,\hat{\lambda}) \in X_*(T\times\hat{T})^+_{\mathbb{Q}}$. *Then* $(\lambda,\hat{\lambda}) \in \mathcal{C}(K,\hat{K})$ *if and only if* $((\lambda,\cdot)_T, (\hat{\lambda},\cdot)_{\hat{T}}) \in \mathbb{Q}_{\geq 0}\mathrm{LR}(G,\hat{G})$.

As it was pointed out by Guillemin-Sternberg [24], Heckman's work [26] implies Theorem 3.1. This result is also a consequence of Kempf-Ness' theorem [29].

Example 3.2. If G is diagonally embedded in $\hat{G} = G\times G$ then the branching problem is the problem of decomposition of tensor products of 2 irreducible representations of G. The cone $\mathbb{Q}_{\geq 0}\mathrm{LR}(G,\hat{G})$ is denoted by $\mathbb{Q}_{\geq 0}\mathrm{LR}(G^3)$. The cone $\mathcal{C}(K,\hat{K}) =: \mathcal{C}(K^3)$ identifies with the set of triples $(\mathcal{O}_1,\mathcal{O}_2,\mathcal{O}_3)$ of adjoint orbits in $\mathfrak{k}$ such that $\mathcal{O}_1+\mathcal{O}_2+\mathcal{O}_3$ contains 0. A good survey on this case is [39].

3.2. Belkale-Kumar Schubert calculus. It is known since A. Klyachko [34] that the inequalities that characterize the cone $\mathbb{Q}_{\geq 0}\mathrm{LR}(G,\hat{G})$ are related to the cohomology of flag varieties, that is to Schubert calculus. In 2006, Belkale-Kumar [7] defined a new product on the cohomology groups of flag varieties that is useful to parametrize irredundantly the inequalities of $\mathbb{Q}_{\geq 0}\mathrm{LR}(G,\hat{G})$.

Let P be a parabolic subgroup of G containing B. Let W and W_P denote respectively the Weyl groups of G and P. The Weyl group W is generated by the simple reflections s_α indexed by the simple roots α. The corresponding length function is denoted by l. Let W^P be the set of minimal length representative in the cosets of W/W_P. For any $w \in W^P$, let X_w be the corresponding Schubert variety (that is, the closure of BwP/P) and let $\sigma_w \in H^{2(\dim(G/P)-l(w))}(G/P,\mathbb{C})$ be its cohomology class. The structure coefficients $c^{w_3}_{w_1w_2}$ of the cup product are written as

$$\sigma_{w_1}.\sigma_{w_2} = \sum_{w_3\in W^P} c^{w_3}_{w_1w_2}\sigma_{w_3}, \qquad \forall w_1,w_2 \in W^P. \tag{3.1}$$

Let L be the Levi subgroup of P containing T and let Z be the neutral component of the center of L. Under the action of Z, the tangent space $\mathcal{T}_{P/P}G/P$ of G/P at the base point P/P decomposes as

$$\mathcal{T}_{P/P}G/P = \bigoplus_{\chi\in X(Z)} \mathcal{T}^\chi, \tag{3.2}$$

where Z acts on $\mathcal{T}^\chi$ with weight χ. By [2], each $\mathcal{T}^\chi$ is an irreducible L-module. For any $w \in W^P$, the tangent space $\mathcal{T}_w := \mathcal{T}_{P/P}w^{-1}X_w$ of the variety $w^{-1}X_w$ at the smooth point P/P also decomposes

$$\mathcal{T}_w = \bigoplus_{\chi\in X(Z)} \mathcal{T}^\chi_w, \tag{3.3}$$

where $\mathcal{T}^\chi_w = \mathcal{T}^\chi \cap \mathcal{T}_w$. The weights of T in $\mathcal{T}_w$ are the opposite of the elements of the inversion set

$$\Phi(w) = \{\alpha\in\Phi^+ : w\alpha \in -\Phi^+\},$$

where Φ^+ is the set of positive roots of G relatively to B. It is contained in the set $\Phi(G/P)$ of positive roots that are not roots of L. If $w \in W$ then $w \in W^P$ if and only if $\Phi(w)$ is contained

in $\Phi(G/P)$. For $\chi \in X(Z)$, denote $\Phi(w,\chi)$ the set of $\alpha \in \Phi(w)$ whose the restriction to Z is $-\chi$. Similarly, define $\Phi(G/P,\chi)$. Since σ_w has degree $2(\sharp\,\Phi(G/P) - \sharp\,\Phi(w))$ in the graded algebra $H^*(G/P)$, if $c_{w_1w_2}^{w_3} \neq 0$ then

$$\sharp\,\Phi(w_1) + \sharp\,\Phi(w_2) = \sharp\,\Phi(G/P) + \sharp\,\Phi(w_3), \tag{3.4}$$

that is

$$\sum_{\chi\in X(Z)} \left(\sharp\,\Phi(w_1,\chi) + \sharp\,\Phi(w_2,\chi) = \right) = \sum_{\chi\in X(Z)} \left(\sharp\,\Phi(G/P) + \sharp\,\Phi(w_3) \right). \tag{3.5}$$

The Belkale-Kumar product requires the equality (3.5) to hold term by term. More precisely, the structure constants $\tilde{c}_{w_1w_2}^{w_3}$ of the Belkale-Kumar product $\odot$,

$$\sigma_{w_1} \odot \sigma_{w_2} = \sum_{w_3\in W^P} \tilde{c}_{w_1w_2}^{w_3} \sigma_{w_3} \tag{3.6}$$

can be defined as follows (see [65, Proposition 2.4]):

$$\tilde{c}_{w_1w_2}^{w_3} = \begin{cases} c_{w_1w_2}^{w_3} & \text{if } \forall \chi \in X(Z) \quad \sharp\,\Phi(w_1,\chi) + \sharp\,\Phi(w_2,\chi) = \sharp\,\Phi(G/P) + \sharp\,\Phi(w_3), \\ 0 & \text{otherwise.} \end{cases} \tag{3.7}$$

Theorem 3.3 ([7]). *The product $\odot$ on $H^*(G/P,\mathbb{C})$ is commutative, associative and satisfies Poincaré duality.*

Denote by $\mathfrak{p}$ the Lie algebra of P and consider the convex cone C in $X(Z)_{\mathbb{Q}}$ generated by the weights of Z acting on $\mathfrak{p}$. It is a closed strictly convex polyhedral cone in $X(Z)_{\mathbb{Q}}$. Consider the partial order $\succcurlyeq$ on $X(Z)_{\mathbb{Q}}$ defined by $\alpha \succcurlyeq \beta$ if and only if $\alpha - \beta$ belongs to C. Then

$$\mathcal{T}^{\succcurlyeq\alpha} := \oplus_{\beta\succcurlyeq\alpha} \mathcal{T}^{\beta} \tag{3.8}$$

is P-stable. Observe that the tangent bundle $\mathcal{T}G/P$ of G/P identifies with the fiber product $G \times_P \mathcal{T}_{P/P}G/P$. Since $\mathcal{T}^{\succcurlyeq\alpha}$ is P-stable, it induces a G-homogeneous sub-bundle $\mathcal{T}^{\succcurlyeq\alpha}G/P$ of the tangent bundle $\mathcal{T}G/P$. This family of G-sub-bundles is decreasing: if $\alpha \succcurlyeq \beta$ then $\mathcal{T}^{\succcurlyeq\alpha}G/P$ is a sub-bundle of $\mathcal{T}^{\succcurlyeq\beta}G/P$. It is also integrable in the sense that

$$[\mathcal{T}^{\succcurlyeq\alpha}G/P, \mathcal{T}^{\succcurlyeq\beta}G/P] \subset \mathcal{T}^{\succcurlyeq\alpha+\beta}G/P. \tag{3.9}$$

This allows us to define a filtration ("Ăă la Hodge") of the De Rham complex and hence of the algebra $H^*(G/P,\mathbb{C})$ indexed by the group $X(Z)$. We consider the associated graded algebra.

Theorem 3.4 ([57]). *The $X(Z)$-graded algebra $GrH^*(G/P,\mathbb{C})$ associated to the $X(Z)$-filtration is isomorphic to the Belkale-Kumar algebra $(H^*(G/P,\mathbb{C}),\odot)$.*

Sketch of proof. The key point to prove the isomorphism is that the subspaces of the filtration of $H^*(G/P,\mathbb{C})$ are spanned by the Schubert classes $(\sigma_w)_{w\in W^P}$ that it contains. To obtain this result, we use Kostant's harmonic forms [37]. □

3.3. A description of $\mathbb{Q}_{\geq 0}\mathbf{LR}(G,\hat{G})$. Let λ be a one parameter subgroup of T. The set of $g \in G$ such that $\lim_{t\to 0}\lambda(t)g\lambda(t^{-1})$ exists in G is a parabolic subgroup of G denoted by $P(\lambda)$. Since λ is also a one parameter subgroup of $\hat{G}$ we have a parabolic subgroup $\hat{P}(\lambda)$ and an embedding $\phi_\lambda : G/P(\lambda) \longrightarrow \hat{G}/\hat{P}(\lambda)$. The comorphism ϕ_λ^* in cohomology induces a morphism (see [65]) $\phi_\lambda^\odot : (H^*(\hat{G}/\hat{P}(\lambda),\mathbb{C}),\odot) \longrightarrow (H^*(G/P(\lambda),\mathbb{C}),\odot)$.

A description of $\mathbb{Q}_{\geq 0}\mathbf{LR}(G,\hat{G})$. An indivisible dominant $\lambda \in X_*(T)$ is said to be *special* if the set of weights χ of T acting on $\hat{\mathfrak{g}}/\mathfrak{g}$ such that $\langle\chi,\lambda\rangle = 0$ spans an hyperplane of $X(T)_\mathbb{Q}$.

Theorem 3.5. *Assume that no ideal of $\mathfrak{g}$ is an ideal of $\hat{\mathfrak{g}}$. Let $(\nu,\hat{\nu}) \in X(T\times\hat{T})^+_\mathbb{Q}$. Then $(\nu,\hat{\nu})$ belongs to $\mathbb{Q}_{\geq 0}\mathrm{LR}(G,\hat{G})$ if and only if*

$$\langle w\lambda,\nu\rangle + \langle\hat{w}\lambda,\hat{\nu}\rangle \leq 0 \tag{3.10}$$

for any special $\lambda \in X_*(T)$, for any $w \in W^{P(\lambda)}$ and $\hat{w} \in \hat{W}^{\hat{P}(\lambda)}$ such that

$$\phi_\lambda^\odot(\sigma_{\hat{w}}) \odot \sigma_w = \sigma_e. \tag{3.11}$$

Moreover, this system of inequalities is irredundant.

Sketch of proof. Consider the action of G on $X = G/B\times\hat{G}/\hat{B}$. As explained in Section 2.3, it remains to determine $\mathcal{C}^{G,+}(X)$.

Let $\mathcal{L}$ be a semi-ample G-linearized line bundle on X. Let x be a point in X and $\lambda : \mathbb{C}^* \longrightarrow G$ be a one parameter subgroup. Since X is complete, $\lim_{t\to 0}\lambda(t)x$ exists; let z denote this limit. The image of λ fixes z, and hence, acts via λ on the fiber $\mathcal{L}_z$. There exists an integer denoted by $\mu^\mathcal{L}(x,\lambda)$ such that for all $t \in \mathbb{C}^*$ and $\tilde{z} \in \mathcal{L}_z$ we have:

$$\lambda(t).\tilde{z} = t^{-\mu^\mathcal{L}(x,\lambda)}\tilde{z}.$$

The integers $\mu^\mathcal{L}(x,\lambda)$ are used in [53] to give a numerical criterion (namely the Hilbert-Mumford criterion) for stability with respect to an ample $\mathcal{L}$. With definition (2.1) of semi-stability, the Hilbert-Mumford theorem admits the following direct generalization.

Lemma 3.6 (See *e.g.* [60, Lemma 2]). *Recall that $\mathcal{L}$ is semi-ample. Then, x is semi-stable for $\mathcal{L}$ if and only if $\mu^\mathcal{L}(x,\lambda) \leq 0$ for any one parameter subgroup λ of G.*

Assume now that λ is a dominant one parameter subgroup of T. Fix an irreducible component C of the fixed point set X^λ of λ in X. Denote by C^+ the Białinicky-Birula cell of points $x \in X$ such that $\lim_{t\to 0}\lambda(t)x \in C$. Let $\mathcal{L} \in \mathcal{C}^+(X)$. The integer $\mu^\mathcal{L}(x,\lambda)$ does not depend on x in C^+; let $\mu^\mathcal{L}(C,\lambda)$ denote this integer. A key observation is the following lemma.

Lemma 3.7. *If $G.C^+$ is dense in X and $X^{\mathrm{ss}}(\mathcal{L})$ is not empty then $\mu^\mathcal{L}(C,\lambda) \leq 0$.*

Proof. Since $X^{\mathrm{ss}}(\mathcal{L})$ is open, it intersects $G.C^+$. Since $X^{\mathrm{ss}}(\mathcal{L})$ is G-stable, it intersects C^+. Let $x \in C^+ \cap X^{\mathrm{ss}}(\mathcal{L})$. By Lemma 3.6, $\mu^\mathcal{L}(C,\lambda) = \mu^\mathcal{L}(x,\lambda) \leq 0$. □

The subvariety C^+ is $P(\lambda)$-stable and one can form the fibered product $G\times_{P(\lambda)}C^+$. Consider the morphism

$$\begin{array}{rccc}\eta: & G\times_{P(\lambda)}C^+ & \longrightarrow & X\\ & [g:x] & \longmapsto & gx.\end{array}$$

There exists $(w,\hat{w})\in W^{P(\lambda)}\times\hat{W}^{\hat{P}(\lambda)}$ such that $C^+=P(\lambda)w^{-1}B/B\times\hat{P}(\lambda)w^{-1}\hat{B}/\hat{B}$. Using Kleiman's transversality theorem, one can prove that η is dominant if and only if

$$\phi_\lambda^*(\sigma_{\hat{w}}).\sigma_w\neq 0. \tag{3.12}$$

Since

$$\mu^{\mathcal{L}(\nu,\hat{\nu})}(C,\lambda)=\langle w\lambda,\nu\rangle+\langle\hat{w}\lambda,\hat{\nu}\rangle,$$

this proves that inequalities (3.10) in the theorem are satisfied by points in $\mathbb{Q}_{\geq 0}\mathrm{LR}(G,\hat{G})$.

If we believe the theorem, we just obtained a redundant family of inequalities. We now explain how to prove that the inequalities corresponding to pairs $(w,\hat{w})$ satisfying condition (3.11) are sufficient to characterize the cone. We check that η is birational if and only if $\phi_\lambda^*(\sigma_{\hat{w}}).\sigma_w=\sigma_e$. One can prove that the stronger condition (3.11) is equivalent to the existence of an open G-stable subset Ω in X that intersects C such that the restriction of η to $G\times_{P(\lambda)}(C^+\cap\Omega)$ is an isomorphism onto Ω. In this case, the pair (C,λ) is said to be *well covering*. The point is that, given $\mathcal{L}\in\mathcal{C}^+(X)-\mathcal{C}^{G,+}(X)$, Kempf's theorey of instabily allows to construct a well covering pair (C,λ) such that $\mu^{\mathcal{L}}(C,\lambda)>0$. Note that Ω can be choosen to be the open Hesselink strata (see [33]).

Fix a well covering pair (C,λ), where $\lambda\in X_*(T)$ is dominant and indivisible. Let $\mathcal{F}(C,\lambda)$ denote the set of $\mathcal{L}\in\mathcal{C}^{G,+}(X)$ such that $\mu^{\mathcal{L}}(C,\lambda)=0$; it is a face of $\mathcal{C}^{G,+}(X)$.

Let $\mathcal{L}\in\mathcal{F}(C,\lambda)$ and $x\in X^{\mathrm{ss}}(\mathcal{L})\cap C^+$. Let $n>0$ and $\sigma\in H^0(X,\mathcal{L}^{\otimes n})^G$ such that $\sigma(x)\neq 0$. Set $z=\lim_{t\to 0}\lambda(t)x$. From $\mu^{\mathcal{L}}(x,\lambda)=0$, one can deduce that $\sigma(z)=\lim_{t\to 0}\lambda(t)\sigma(x)$ belongs to the pointed fiber $\mathcal{L}_z-\{z\}$. In particular z is semi-stable. It follows that the neutral component G_z° of the isotropy of z acts trivialy on $\mathcal{L}_z$. In λ is not special, this implies that $\mathcal{L}$ belongs to a codimension 2 linear subspace of $\mathrm{Pic}^G(X)_{\mathbb{Q}}$. In particular, $\mathcal{F}(C)$ has not codimension one and the inequality $\mu^{\mathcal{L}}(C,\lambda)$ can be removed.

The last step is to prove the irredundancy. Fix a well covering pair (C,λ) with some special λ. We have to prove that $\mathcal{F}(C,\lambda)$ has codimension one. Consider the restriction morphism

$$\rho:\mathrm{Pic}^G(X)_{\mathbb{Q}}\longrightarrow\mathrm{Pic}^{G^\lambda}(C)_{\mathbb{Q}}.$$

An explicit computation shows that ρ is surjective. Moreover, by induction, the dimension of $\mathcal{C}^{G^\lambda,+}(C)$ is equal to $\dim(\mathrm{Pic}^G(X)_{\mathbb{Q}})-1$. Let $\mathcal{M}\in\mathrm{Pic}^{G^\lambda}(C)$ such that $H^0(C,\mathcal{M})^{G^\lambda}\neq 0$. It would be sufficient to prove that there exists $\mathcal{L}\in\mathrm{Pic}^G(X)$ such that $H^0(X,\mathcal{L})^G\neq 0$ and $\rho(\mathcal{L})=\mathcal{M}$. This is not true directly but it is true after a little modification of $\mathcal{M}$. Let $\mathcal{L}\in\mathrm{Pic}^G(X)$ such that $\rho(\mathcal{L})=\mathcal{M}$. Fix a nonzero regular G^λ-invariant section τ of $\mathcal{M}$. Let η and $\Omega\subset X$ be as above. Let $E_1,\dots,E_k$ be the codimension one irreducible components of $X-\Omega$. Using the inverse of η one can prove that $\mathcal{L}_{|\Omega}$ admits a nonzero G-invariant section σ. Then, σ does not necessarily extend to a section of $\mathcal{L}$ on X; but it certainly extends to a section of $\mathcal{L}\otimes d\mathcal{O}(\sum_i E_i)$ for d big enough. Since no E_i contains C, $\mathcal{L}\otimes d\mathcal{O}(\sum_i E_i)$ belongs to $\mathcal{F}(C,\lambda)$.

By this method, one can produce a family of points in $\mathcal{F}(C,\lambda)$ that generates a codimension one cone. The irredundancy follows. □

3.4. The case of $\mathbb{Q}_{\geq 0}\mathrm{LR}(G^3)$. We assume that G is semi-simple and simply connected and we consider the cone $\mathbb{Q}_{\geq 0}\mathrm{LR}(G^3)$. The set of non-trivial weights of T acting on $\hat{\mathfrak{g}}/\mathfrak{g}$ is Φ. There are $\mathrm{rk}(G)$ special one parameter subgroups of T; for any simple root α exactly one λ_α is proportional to the fundamental coweight $\varpi_{\alpha^\vee}$. The parabolic subgroup $P(\lambda_\alpha)$ is the maximal parabolic subgroup P_α containing B associated to α.

Theorem 3.8. *Let $(\nu_1, \nu_2, \nu_3) \in (X^*(T)^+_{\mathbb{Q}})^3$. Then $(\nu_1, \nu_2, \nu_3) \in \mathbb{Q}_{\geq 0}\mathrm{LR}(G^3)$ if and only if*

$$\sum_{i=1}^{3} \langle w_i \varpi_{\alpha^\vee}, \nu_i \rangle \leq 0 \tag{3.13}$$

for any simple root α and any $w_1, w_2, w_3 \in W^{P_\alpha}$ such that

$$\sigma_{w_1} \odot \sigma_{w_2} \odot \sigma_{w_3} = \sigma_e. \tag{3.14}$$

Moreover, this list of inequalities is irredundant.

Theorem 3.8 without the irredundancy is the main result of [7].

The Horn case. Let $\mathcal{H}_n(\mathbb{C})$ be the space of Hermitian matrices of size n. For $A \in \mathcal{H}_n(\mathbb{C})$, we denote by $\lambda(A) = (\lambda_1 \geq \cdots \geq \lambda_n) \in \mathbb{R}^n$ the spectrum of A. For $G = \mathrm{GL}_n(\mathbb{C})$ diagonally embedded in $\hat{G} = G \times G$, Section 3.1 implies that the cone $\mathbb{R}_{\geq 0}\mathrm{LR}(G, \hat{G})$ identifies with

$$\mathrm{Horn}_{\mathbb{R}}(n) = \left\{ (\lambda(A), \lambda(B), \lambda(C)) \in \mathbb{R}^{3n} \ : \ \begin{array}{l} A, B, C \in \mathcal{H}_n(\mathbb{C}) \text{ s.t.} \\ A + B + C = 0 \end{array} \right\}.$$

After an easy reduction, we can apply Theorem 3.8 (with $\mathrm{SL}_n(\mathbb{C}) \subset \mathrm{SL}_n(\mathbb{C}) \times \mathrm{SL}_n(\mathbb{C})$) to obtain a description of $\mathrm{Horn}_{\mathbb{R}}(n)$. The homogeneous spaces G/P_α are the Grassmannian variety $\mathbb{G}(r, n)$ of r-dimensional subspaces of $\mathbb{C}^n$. The Schubert classes $(\sigma_I)_{I \in \mathcal{S}(r,n)}$ of $\mathbb{G}(r, n)$ are indexed by the set $\mathcal{S}(r, n)$ of subsets of $\{1, \cdots, n\}$ with r elements. Since $\mathbb{G}(r, n)$ is cominuscule, the product $\odot$ coincinde with the ordinary one.

Theorem 3.9. *Let (λ, μ, ν) be a triple of non-increasing sequences of n real numbers. Then $(\lambda, \mu, \nu) \in \mathrm{Horn}_{\mathbb{R}}(n)$ if and only if*

$$\sum_i \lambda_i + \sum_j \mu_j + \sum_k \nu_k = 0 \tag{3.15}$$

and

$$\sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j + \sum_{k \in K} \nu_k \leq 0, \tag{3.16}$$

for any $r \in \{1, \cdots, n-1\}$, for any $I, J, K \in \mathcal{S}(r, n)$ such that

$$\sigma_I . \sigma_J . \sigma_K = \sigma_e. \tag{3.17}$$

Moreover, this list of inequalities is irredundant.

Example 3.10 ($\mathbb{Q}_{\geq 0}\mathrm{LR}(\mathrm{SL}_3(\mathbb{C})^3)$). The symmetric group S_3 acts on $\mathbb{Q}_{\geq 0}\mathrm{LR}(G^3)$ by permuting the three copies of G. Since $(V_1 \otimes V_2 \otimes V_3)^G$ and $(V_1^* \otimes V_2^* \otimes V_3^*)^G$ have the same dimension the group $\mathbb{Z}/2\mathbb{Z}$ also acts on $\mathbb{Q}_{\geq 0}\mathrm{LR}(G^3)$. Finally, we get an action of

$S_3 \times \mathbb{Z}/2\mathbb{Z}$. For $\mathrm{SL}_3(\mathbb{C})$, we use the base of fundamental weights to identify $X(T)^+$ with $\mathbb{Z}^2_{\geq 0}$ and $\mathbb{Q}_{\geq 0}\mathrm{LR}(\mathrm{SL}_3(\mathbb{C})^3)$ with a cone in $\mathbb{Z}^6$. Let (ν_1, ν_2, ν_3) be three dominant weights corresponding to $(x_1, x_2, y_1, y_2, z_1, z_2) \in \mathbb{Z}^6_{\geq 0}$. If $x_1 + y_1 + z_1 + 2(x_2 + y_2 + z_2)$ is not a multiple of 3, then $c_{\mathrm{SL}_3}(\nu_1, \nu_2, \nu_3) = 0$ (to check this, one can consider the action of the center of $\mathrm{SL}_3(\mathbb{C})$). Assume now that 3 divides $x_1 + y_1 + z_1 + 2(x_2 + y_2 + z_2)$ and set $\delta = \frac{1}{3}(x_1 + y_1 - z_2 + 2(x_2 + y_2 - z_1)$. Then $c_{\mathrm{SL}_3}(\nu_1, \nu_2, \nu_3)$ is equal to the Littlewood-Richardson coefficient $c^{z_1+z_2+\delta \geq x_2 \delta \geq \delta}_{x_1+x_2 \geq x_2,\, y_1+y_2 \geq y_2}$.

There are two Grassmannians homogeneous under $\mathrm{SL}_3(\mathbb{C})$, $\mathbb{P}^2$ and its dual. In $\mathbb{P}^2$, the identity $[\mathbb{P}^1].[\mathbb{P}^1].[\mathbb{P}^2] = [\mathrm{pt}]$ gives the inequality $x_1 - x_2 + y_1 - y_2 + z_1 + 2z_2 \geq 0$ and the relation $[\mathbb{P}^2].[\mathbb{P}^2].[\mathrm{pt}] = [\mathrm{pt}]$ gives the inequality $x_1 + 2x_2 + y_1 + 2y_2 - 2z_1 - z_2 \geq 0$. Using the action of $S_3 \times \mathbb{Z}/2\mathbb{Z}$, we get 12 inequalities. Adding the inequalities of dominance, we get that $\mathbb{Q}_{\geq 0}\mathrm{LR}(\mathrm{SL}_3(\mathbb{C})^3)$ has 18 facets.

Consider the point $b = (1, 1, 1, 1, 1, 1) \in \mathbb{Q}_{\geq 0}\mathrm{LR}(\mathrm{SL}_3(\mathbb{C})^3)$. The cones generated by b and one of the 18 facets of $\mathbb{Q}_{\geq 0}\mathrm{LR}(\mathrm{SL}_3(\mathbb{C})^3)$ are the maximal cones of the GIT-fans. Assuming that 3 divides $x_1 + y_1 + z_1 + 2(x_2 + y_2 + z_2)$, the Littlewood-Richardson coefficient $c^{z_1+z_2+\delta \geq x_2 \delta \geq \delta}_{x_1+x_2 \geq x_2,\, y_1+y_2 \geq y_2}$ is given on the cones corresponding to the inequalities $x_1 \geq 0$, $x_1 - x_2 + y_1 - y_2 + z_1 + 2z_2 \geq 0$ and $x_1 + 2x_2 + y_1 + 2y_2 - 2z_1 - z_2 \geq 0$ respectively by the polynomials $1 + x_1$, $1 + \frac{1}{3}(x_1 - x_2 + y_1 - y_2 + z_1 + 2z_2)$ and $1 + \frac{1}{3}(x_1 + 2x_2 + y_1 + 2y_2 - 2z_1 - z_2)$.

Theorem 3.9 has a rich and long story starting with H. Weyl [71] who proved, in 1912, inequalities (3.16) for $G/P_\alpha = \mathbb{P}^{n-1}$. In 1998, A. Klyachko [34] made an important step proving the theorem is true (without the irredundancy) if condition (3.17) is replaced by

$$\sigma_I.\sigma_J.\sigma_K = d\sigma_e, \quad \text{for some positive integer } d. \tag{3.18}$$

In 2000, Belkale [4] proved that Klyachko's condition (3.18) can be replaced by condition (3.17). The irredundancy was first proved by Knutson-Tao-Woodward in [36] using the Honeycomb model for Littlewood-Richardson coefficients.

Let us now explain Horn's contribution. For $I = \{i_1 < \cdots < i_r\} \in \mathcal{S}(r, n)$, set $\tau^I = (i_r - r, \ldots, i_1 - 1)$ and $I^\vee = \{n + 1 - i_r < \cdots < n + 1 - i_1\}$. In 1962, Horn conjectured that Theorem 3.9 is true if one replaces condition (3.17) by

$$(\tau^I, \tau^J, \tau^K - (n - r)^r) \in \mathrm{Horn}(r), \tag{3.19}$$

where $(n - r)^r = (n - r, \ldots, n - r)$ in $\mathbb{R}^r$. By the classical Lesieur's result (see [43]), $\sigma_I.\sigma_J.\sigma_K = d\sigma_e$ is equivalent to $c^{\tau^{K^\vee}}_{\tau^I \tau^J} = d$. In particular, Klyachko's condition is equivalent to

$$c^{\tau^{K^\vee}}_{\tau^I \tau^J} > 0, \tag{3.20}$$

whereas condition (3.19) is equivalent to

$$\exists k > 0 \qquad c^{k\tau^{K^\vee}}_{k\tau^I\, k\tau^J} > 0. \tag{3.21}$$

The equivalence between conditions (3.20) and (3.21) is called saturation (see Section 5) and was first proved by Knutson-Tao in [35].

Horn's conjecture has the advantage to be inductive and elementary (without cohomology or representation theory). Theorem 3.9 has the advantage to give the minimal list of inequalities. In [63], we get the two advantages by giving an inductive algorithm to decide if a given Littlewood-Richardson coefficient is equal to one or not.

3.5. Some inequalities for nonzero Kronecker coefficients.. If $\alpha = (\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_e \geq 0)$ is a partition, we set $|\alpha| = \sum_i \alpha_i$ in such a way α is a partition of $|\alpha|$. Consider the symmetric group $\mathcal{S}_n$ on n letters. The irreducible representations of $\mathcal{S}_n$ are parametrized by the partitions of n, see *e.g.* [45, I. 7] . Let $[\alpha]$ denote the representation of $\mathcal{S}_{|\alpha|}$ corresponding to α. The Kronecker coefficients $g_{\alpha\,\beta\,\gamma}$, depending on three partitions α, β, and γ of the same integer n, are defined by

$$[\alpha] \otimes [\beta] = \sum_{\gamma} g_{\alpha\,\beta\,\gamma}[\gamma]. \tag{3.22}$$

The length $l(\alpha)$ of the partition α is the number of nonzero parts α_i.

Theorem 3.11 (see [64]). *Let e and f be two positive integers and $j \in \{2, \ldots, f+1\}$. Let α, β, and γ be three partitions of the same integer n such that*

$$l(\alpha) \leq e+1, \;\; l(\beta) \leq f+1, \;\; \textit{and} \;\; l(\gamma) \leq e+f+1. \tag{3.23}$$

Let $0 < r < e$, $0 < s < f$, $I \in \mathcal{S}(r, e)$, $J \in \mathcal{S}(s, f)$ and $K \in \mathcal{S}(r+s, e+f)$ such that

$$c_{\tau^I \tau^J}^{\tau^K} = 1. \tag{3.24}$$

If $g_{\alpha\,\beta\,\gamma} \neq 0$ then

$$n + \sum_{i \in I} \alpha_{i+1} - \alpha_1 + \sum_{j \in J} \beta_{j+1} - \beta_1 \geq \sum_{k \in K} \gamma_{k+1} - \gamma_1. \tag{3.25}$$

Sketch of proof. Given a complex vector space V and a partition α such that $l(\alpha) \leq \dim(V)$, let $S^\alpha V$ denote the irreducible $\mathrm{GL}(V)$-representation of highest weight α. Fix two complex vector spaces V_1, V_2 of dimension $e+1$ and $f+1$. The Schur-Weyl duality implies that

$$S^\gamma(V_1 \otimes V_2) = \bigoplus_{\alpha\beta} g_{\alpha\beta\gamma} S^\alpha V_1 \otimes S^\beta V_2.$$

We consider the action de $G = \mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$ on the product X of the manifolds of complete flags in V_1 and V_2 and some partial flag manifold on $V_1 \otimes V_2$. The coefficient $g_{\alpha\,\beta\,\gamma}$ is the dimension of the space of G-invariant sections of some line bundle on X. Then, we use techniques similar to those used to prove Theorem 3.5. □

3.6. Relations between cones $\mathbb{Q}_{\geq 0}\mathbf{LR}(G^3)$ for various G. In this section, G is assumed to be simple, simply connected and of simply-laced type. Consider an automorphism σ of the Dynkin diagram of G. It induces an automorphism, still denoted by σ, of G that stabilizes a maximal torus T, a Borel subgroup B and a compact form K of G. The fixed point set G^σ is a simple group with maximal torus T^σ, Borel subgroup B^σ and compact form K^σ. The inclusion $T^\sigma \subset T$ induces an immersion $X_*(T^\sigma)_{\mathbb{R}} \subset X_*(T)_{\mathbb{R}}$ satisfying $X_*(T^\sigma)^+_{\mathbb{R}} = X_*(T^\sigma)_{\mathbb{R}} \cap X_*(T)^+_{\mathbb{R}}$.

Theorem 3.12. *We have*

$$\mathcal{C}((K^\sigma)^3) = \mathcal{C}(K^3) \cap X_*(T^\sigma)^3_{\mathbb{R}}.$$

Sketch of proof. The proof uses Theorem 3.5 and compares conditions (3.11) in G and G^σ-homogeneous spaces. Beyond this general principle, the proof is case by case according to the following complete list:

(i) $(\mathrm{SL}_{2n}(\mathbb{C}), \mathrm{Sp}_{2n}(\mathbb{C}))$, $n \geq 2$;
(ii) $(\mathrm{SL}_{2n+1}(\mathbb{C}), \mathrm{SO}_{2n+1}(\mathbb{C}))$, $n \geq 2$;
(iii) $(\mathrm{Spin}_{2n}(\mathbb{C}), \mathrm{Spin}_{2n-1}(\mathbb{C}))$, $n \geq 4$;
(iv) $(\mathrm{Spin}_8(\mathbb{C}), G_2)$;
(v) (E_6, F_4).

The two first cases was proved in [8] (see also [68] for a simplification in some key argument). Case (iii) is proved in Braley's thesis [12]. The two reamaning cases are proved in Lee's thesis [42]. □

Remark 3.13. As a consequence of Theorem 3.12, it is proved in [63] that condition (3.11) in Theorem 3.5 in the cohomology of symplectic and odd orthogonal Grassmannians are equivalent to similar conditions for ordinary Grassmannians.

4. Reduction for coefficients on the boundary

In this section, we are intersted in the coefficients $c_{G,\hat{G}}(\nu, \hat{\nu})$ when $(\nu, \hat{\nu})$ belongs to the boundary of $\mathrm{LR}(G, \hat{G})$. Indeed, such multiplicities are equal to analogous numbers for Levi subgroups of G and $\hat{G}$. The results could be obtained by applying results of type "quantification commutes with reduction" in symplectic geometry (see [25]). Our proof is more direct. Indeed, it remains to prove that two spaces have the same dimension: we find an explicit and natural isomorphism.

Theorem 4.1. *Let $X = G/P \times \hat{G}/\hat{P}$ be a flag manifold for the group $G \times \hat{G}$. Let λ be a one-parameter subgroup of G and C be an irreducible component of the fixed point set X^λ of λ in X. Let G^λ be the centralizer of the image of λ in G. We assume that (C, λ) is a well covering pair . Let $\mathcal{L}$ be a G-linearized line bundle on X generated by its global sections such that λ acts trivially on the restriction $\mathcal{L}_{|C}$. Then the restriction map induces an isomorphism*

$$H^0(X, \mathcal{L})^G \longrightarrow H^0(C, \mathcal{L}_{|C})^{G^\lambda},$$

between the spaces of invariant sections of $\mathcal{L}$ and $\mathcal{L}_{|C}$.

Sketch of proof. Consider the closure $\overline{C^+}$ of the Białynicki-Birula cell C^+. The morphism

$$\begin{array}{rccc} \overline{\eta}: & G \times_{P(\lambda)} \overline{C^+} & \longrightarrow & X \\ & [g : x] & \longmapsto & gx \end{array}$$

that is proper and birational, induces a G-equivariant isomorphism

$$H^0(X, \mathcal{L}) \simeq H^0(G \times_{P(\lambda)} \overline{C^+}, \overline{\eta}^*(\mathcal{L})).$$

In particular

$$H^0(X, \mathcal{L})^G \simeq H^0(G \times_{P(\lambda)} \overline{C^+}, \overline{\eta}^*(\mathcal{L}))^G \simeq H^0(\overline{C^+}, \mathcal{L}_{|\overline{C^+}})^{P(\lambda)}.$$

On the other hand, since λ acts trivially on $\mathcal{L}_{|C}$, [60, Lemma 5] proves that

$$H^0(C^+, \mathcal{L}_{|C^+})^{P(\lambda)} \simeq H^0(C, \mathcal{L}_{|C})^{G^\lambda}. \tag{4.1}$$

Then we have to prove that

$$H^0(\overline{C^+}, \mathcal{L}_{|\overline{C^+}})^{P(\lambda)} \simeq H^0(C^+, \mathcal{L}_{|C^+})^{P(\lambda)};$$

that is, that any regular $P(\lambda)$-invariant section σ of $\mathcal{L}$ on C^+ extends to $\overline{C^+}$. Using the SL_2-theory, one can checks that such a section has no pole along the divisors of $\overline{C^+} - C$. We conclude that such a section extends to $\overline{C^+}$ by normality of $\overline{C^+}$ that is a Schubert variety. □

Let $\mathcal{F}$ be a face of $\mathbb{Q}_{\geq 0}\mathrm{LR}(G, \hat{G})$. Assume that $\mathcal{F}$ is regular, that is that it contains pairs $(\nu, \hat{\nu})$ of regular dominant weights. If S is a torus in G and H is a subgroup of G containing S, H^S denotes the centralizer of S in H. By [61], the regular face $\mathcal{F}$ corresponds to a pair $(S, \hat{w})$ where S is a subtorus of T and $\hat{w} \in \hat{W}$ such that

$$\hat{G}^S \cap \hat{w}\hat{B}\hat{w}^{-1} = \hat{B}^S, \tag{4.2}$$

and the span of $\mathcal{F}$ is the set of pairs $(\nu, \hat{\nu}) \in (X(T) \times X(\hat{T})) \otimes \mathbb{Q}$ such that

$$\nu_{|S} + \hat{w}\hat{\nu}_{|S} = 0 \in X(S) \otimes \mathbb{Q}. \tag{4.3}$$

Corollary 4.2. *Let $(\nu, \hat{\nu}) \in X(T)^+ \times X(\hat{T})^+$ be a pair of dominant weights. Assume that $(\nu, \hat{\nu})$ belongs to the span of $\mathcal{F}$ (equivalently that it satisfies condition* (4.3)*). Then*

$$c_{\nu\,\hat{\nu}}(G, \hat{G}) = c_{\nu\,\hat{w}\hat{\nu}}(G^S, \hat{G}^S).$$

Sketch of proof. The corollary is obtained by applying Theorem 4.1 with $X = G/B \times \hat{G}/\hat{B}$ and $C = G^S B/B \times \hat{G}^S \hat{w}\hat{B}/\hat{B}$. □

Several particular cases of Theorems 4.1 and its corollary was known before. If $G = T$ is a maximal torus of $G = \mathrm{GL}_n(\mathbb{C})$, our theorem is equivalent to [30, Theorem 5.8]. If $\hat{G} = G \times G$ (or more generally $\hat{G} = G^s$ for some integer $s \geq 2$) and G is diagonally embedded in $\hat{G}$ then $c_{\nu\,\hat{\nu}}(G, \hat{G})$ (resp. $c_{\nu\,\hat{w}\hat{\nu}}(G^S, \hat{G}^S)$) are tensor product multiplicities for the group G (resp. G^S). This case was recently proved independently by Derksen and Weyman in [21, Theorem 7.4] and King, Tollu and Toumazet in [31, Theorem 1.4] if $G = GL_n(\mathbb{C})$ and for any reductive group by Roth in [66]. If ν is regular then Theorem 4.1 can be obtained applying [14, Theorem 3] and [60]. Similar reductions can be found in [13, 46, 50].

Remark 4.3. In Section 3.4, we seen that the multiplicities corresponding to the points in the boundary of the cone $\mathbb{Q}_{\geq 0}\mathrm{LR}(\mathrm{SL}_3(\mathbb{C})^3)$ are equal to one. This agrees with Corollary 4.2, since the tensor product of two irreducible $\mathrm{SL}_2(\mathbb{C})$-modules is multiplicity free.

5. The question of saturation

5.1. The branching semigroup. Consider the set $\mathrm{LR}(G,\hat{G})$ of pairs $(\nu,\hat{\nu})$ of dominant weights such that $c_{G,\hat{G}}(\nu,\hat{\nu})\neq 0$.

Theorem 5.1 (Brion-Knop (see [23])). *The set* $\mathrm{LR}(G,\hat{G})$ *is a finitely generated semigroup in* $X(T)^+\times X(\hat{T})^+$.

Proof. Start with Frobenius' decomposition of $\mathbb{C}[\hat{G}]$ as a $\hat{G}\times\hat{G}$-module:

$$\mathbb{C}[\hat{G}]=\bigoplus_{\hat{\nu}\in X(\hat{T})^+} V_{\hat{G}}(\hat{\nu})\otimes V_{\hat{G}}(\hat{\nu})^*.$$

Let U and $\hat{U}^-$ denote the unipotent radicals of B and $\hat{B}^-$. Consider the algebra

$$\mathbb{C}[\hat{G}]^{U\times\hat{U}^-}=\bigoplus_{\hat{\nu}\in X(\hat{T})^+} V_{\hat{G}}(\hat{\nu})^U\otimes(V_{\hat{G}}(\hat{\nu})^*)^{\hat{U}^-}.$$

Observe that $\hat{T}$ acts on the line $(V_{\hat{G}}(\hat{\nu})^*)^{\hat{U}^-}$ by the weight $-\hat{\nu}$ and that $V_G(\nu)$ is a submodule of $V_{\hat{G}}(\hat{\nu})$ if and only if $V_{\hat{G}}(\hat{\nu})^U$ contains a T-eigenvector of weight ν. Then $(\nu,\hat{\nu})$ belongs to $\mathrm{LR}(G,\hat{G})$ if and only if $(\nu,-\hat{\nu})$ is a weight of $T\times\hat{T}$ acting on the algebra $\mathbb{C}[\hat{G}]^{U\times\hat{U}^-}$. This implies that $LR(G,\hat{G})$ is a semigroup.

To prove that $\mathrm{LR}(G,\hat{G})$ is finitely generated it is sufficient to prove that $\mathbb{C}[\hat{G}]^{U\times\hat{U}^-}$ is. Recall that $\mathbb{C}[G]^U$ is finitely generated. But

$$\mathbb{C}[\hat{G}]^{U\times\hat{U}^-}=(\mathbb{C}[G]^U\otimes\mathbb{C}[\hat{G}]^{\hat{U}^-})^G.$$

Since $\mathbb{C}[G]^U$ and $\mathbb{C}[\hat{G}]^{\hat{U}^-}$ are finitely generated and G is reductive the Hilbert theorem implies that $\mathbb{C}[\hat{G}]^{U\times\hat{U}^-}$ is finitely generated. □

The subgroup of $X(T\times\hat{T})$ generated by $\mathrm{LR}(G,\hat{G})$ is denoted by $\mathbb{Z}\mathrm{LR}(G,\hat{G})$. We already described the cone $\mathbb{Q}_{\geq 0}\mathrm{LR}(G,\hat{G})$. The following statement describes the group $\mathbb{Z}\mathrm{LR}(G,\hat{G})$.

Proposition 5.2 (see [17, 55]). *Assume that no ideal of* $\mathfrak{g}$ *is an ideal of* $\hat{\mathfrak{g}}$. *Let* $\hat{Z}$ *denote the center of* $\hat{G}$.

Then the group $\mathbb{Z}\mathrm{LR}(G,\hat{G})$ *is the set of pairs* $(\nu,\hat{\nu})\in X(T\times\hat{T})$ *such that*

$$\nu(t).\hat{\nu}(t)=1$$

for any $t\in\hat{Z}\cap G$.

The semigroup is said to be saturated if it can be recovered from the knowledge of the cone$\mathbb{Q}_{\geq 0}\mathrm{LR}(G,\hat{G})$ and the group $\mathbb{Z}\mathrm{LR}(G,\hat{G})$:

Definition 5.3. The semigroup $\mathrm{LR}(G,\hat{G})$ is said to be *saturated* if

$$\mathrm{LR}(G,\hat{G})=\mathbb{Q}_{\geq 0}\mathrm{LR}(G,\hat{G})\cap\mathbb{Z}\mathrm{LR}(G,\hat{G}). \tag{5.1}$$

5.2. The case of tensor product decomposition. In this section, we review results on the saturation of $\mathrm{LR}(G^3)$ for G is simple and simply connected. Observe that $\mathbb{Z}\mathrm{LR}(G,\hat{G})$ is the set $(\nu_1,\nu_2,\nu_3) \in X^*(T)^3$ such that $\nu_1+\nu_2+\nu_3$ belongs to the root lattice Λ_R.

Theorem 5.4 (Knutson-Tao). *The semigroup* $\mathrm{LR}(G^3)$ *is saturated for* $G = \mathrm{SL}_n(\mathbb{C})$.

The first proof [35] of Theorem 5.4 due to Knutson and Tao uses a combinatorial model for Littlewood-Richardson coefficients called honeycombs. Belkale reproved [5] this theorem using intersection theory. Derksen and Weyman reproved [20] this result using representations of quivers and Kapovich and Millson obtained a proof [28] using the geometry of Bruhat-Tits buildings.

The best known uniform generalization of Theorem 5.4 to any simple group G is

Theorem 5.5 (Kapovich-Millson [28]). *Let* ν_1, ν_2, *and* ν_3 *be three dominant weights of the simple group* G. *Let* k *be the square of the least common multiple of the coefficients of the highest root of* G *written in terms of simple roots.*

If $(V_G(N\nu_1)\otimes V_G(N\nu_2)\otimes V_G(N\nu_3))^G \neq \{0\}$ *for some positive integer* N *and* $\nu_1+\nu_2+\nu_3 \in \Lambda_R$, *then* $(V_G(k\nu_1)\otimes V_G(k\nu_2)\otimes V_G(k\nu_3))^G \neq \{0\}$.

Observe that for $G = \mathrm{SL}_n(\mathbb{C})$, $k = 1$. Belkale and Kumar [9] and Sam [67] obtained better constants than k for classical groups.

Two important conjectures in the topic are still open. The first one asserts that tensor product decompositions for simply-laced groups satisfy the saturation property. It was checked by explicit computations for $G = \mathrm{Spin}_8(\mathbb{C})$ in [27]. Observe that, for G of type E_8, the constant k in Theorem 5.5 is equal to $3\,600$. The second conjecture asserts that Theorem 5.5 is satisfied with $k = 1$ for any G if the weights ν_i are regular.

5.3. Other examples. If $G = \hat{T}$, $(\nu,\hat{\nu})$ belongs to the group $\mathbb{Z}\mathrm{LR}(G,\hat{G})$ if and only if $\hat{\nu}-\nu$ belongs to the root lattice $\hat{\Lambda}_R$. The multiplicities $c_{G,\hat{G}}(\nu,\hat{\nu})$ are given by the character of $V_{\hat{G}}(\hat{\nu})$. The saturation is well known in this case.

Others examples of semigroups have been determined explicitly in [55]. A consequence of these computations is the following list of examples of saturated semigroups.

Theorem 5.6. *For* $(G,\hat{G}) = (\mathrm{Spin}_{2n-1}(\mathbb{C}),\mathrm{Spin}_{2n}(\mathbb{C}))$, $(\mathrm{SL}_3(\mathbb{C}),G_2)$, $(G_2,\mathrm{Spin}_7(\mathbb{C}))$, $(\mathrm{Spin}_9(\mathbb{C}),F_4)$, (F_4,E_6), $(\mathrm{Sp}_4(\mathbb{C}),\mathrm{SL}_4(\mathbb{C}))$, $(\mathrm{Sp}_6(\mathbb{C}),\mathrm{SL}_6(\mathbb{C}))$, $(\mathrm{Sp}_8(\mathbb{C}),\mathrm{SL}_8(\mathbb{C}))$ *and* $(\mathrm{Sp}_{10}(\mathbb{C}),\mathrm{SL}_{10}(\mathbb{C}))$ *the semigroup* $\mathrm{LR}(G,\hat{G})$ *is saturated.*

6. PRV

6.1. The classical case. Recall that $X(T)^+$ is a fundamental domain for the action of W on $X(T)$; for any $\nu \in X(T)$, we denote by $\overline{\nu}$ the unique dominant element in the orbit $W.\nu$. Parthasarathy-Ranga Rao-Varadarajan conjectured in the sixties [54] (a weaker version of) the following

PRV conjecture. Let $V_G(\mu)$ and $V_G(\nu)$ be two irreducible G-modules with highest weights μ and ν respectively. Then, for any $w \in W$, the irreducible G-module $V_G(\overline{\mu+w\nu})$ with extremal weight $\mu+w\nu$, occurs with multiplicity at least one in $V_G(\mu)\otimes V_G(\nu)$.

This conjecture was proved independently by S. Kumar in [41] and O. Mathieu in [47].

6.2. A double generalization. The homogeneous space $\hat{G}/G$ is said to be *spherical* if it contains an open $\hat{B}$-orbit. It is said to be *spherical of minimal rank* if there exists a T-fixed point in $\hat{G}/\hat{B}$ whose the G-orbit is open. This condition is very strong and was classified in [62]. The pairs $(G, \hat{G})$ such that $\hat{G}/G$ is spherical of minimal rank reduces to the following list $(G, G\times G)$, $(\mathrm{Sp}_{2n}(\mathbb{C}), \mathrm{SL}_{2n}(\mathbb{C}))$, $(\mathrm{Spin}_{2n-1}(\mathbb{C}), \mathrm{Spin}_{2n}(\mathbb{C}))$, $(G_2, \mathrm{Spin}_7(\mathbb{C}))$ and (F_4, E_6).

Let $\rho : X(\hat{T}) \longrightarrow X(T)$ be the restriction morphism. Let Δ (resp $\hat{\Delta}$) denote the set of simple roots of G and $\hat{G}$. By [62, Lemma 4.6], $\rho(\hat{\Delta}) = \Delta$. Moreover, for any $\alpha \in \Delta$, we have the following alternative:

(i) there exists a unique $\hat{\alpha}_0 \in \hat{\Delta}$ such that $\rho(\hat{\alpha}_0) = \alpha$; or

(ii) there exist exactly two simple roots $\hat{\alpha}_1$ and $\hat{\alpha}_2$ in $\hat{\Delta}$ such that $\rho(\hat{\alpha}_1) = \rho(\hat{\alpha}_2) = \alpha$.

The set of simple roots satisfying the second assertion is denoted by Δ_2. For $\alpha \in \Delta_2$, let $\hat{W}_\alpha$ denote the subgroup of $\hat{W}$ generated by $s_{\hat{\alpha}_1}$ and $s_{\hat{\alpha}_2}$. Then $\hat{W}_\alpha$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$.

Theorem 6.1 ([51, 52]). *Fix two connected reductive groups $G \subset \hat{G}$ such that $\hat{G}/G$ is spherical of minimal rank. Let $\alpha \in \Delta_2$ and $\hat{w} \in \hat{W}$.*

Let $\hat{\nu}$ be a dominant weight of $\hat{G}$. Let $\hat{w}_1 \in \hat{W}_\alpha \hat{w}$ be such that $\langle \rho(\hat{w}_1\hat{\nu}), \alpha\rangle$ is maximal.

Fix an integer k such that $0 \le k \le \min(\langle \rho(s_{\hat{\alpha}_1}\hat{w}_1\hat{\nu}), \hat{\alpha}_1^\vee\rangle, \langle \rho(s_{\hat{\alpha}_2}\hat{w}_1\hat{\nu}), \hat{\alpha}_2^\vee\rangle)$. Then, if $\nu = \rho(\hat{w}_1\hat{\nu}) - k\alpha$ is dominant, the irreducible G-module $V_G(\nu)$ occurs with multiplicity at least one in $V_{\hat{G}}(\hat{\nu})$.

For $\hat{G} = G\times G$ and k extremal in the interval, Theorem 6.1 implies the PRV conjecture.

Sketch of proof. Denote the $\hat{G}$-linearized line bundle $\mathcal{L}_{\hat{\nu}}$ on $\hat{G}/\hat{B}$ such that $H^0(\hat{G}/\hat{B}, \mathcal{L}_{\hat{\nu}})$ is isomorphic to $V_{\hat{G}}(\hat{\nu})^*$. Set $\hat{v} = s_{\hat{\alpha}_1}\hat{w} \in \hat{W}$, $X^\circ(\hat{v}) = G.\hat{v}\hat{B}/\hat{B}$. Let $X(\hat{v})$ denote the closure of $X^\circ(\hat{v})$. By [15, Corollary 8], the restriction map

$$H^0(\hat{G}/\hat{B}, \mathcal{L}_{\hat{\nu}}) \longrightarrow H^0(X(\hat{v}), \mathcal{L}_{\hat{\nu}})$$

is surjective. Then, it is sufficient to prove the existence of a section $\sigma \in H^0(X(\hat{v}), \mathcal{L}_{\hat{\nu}})$ that is an eigenvector of weight $-\nu$ for B^-.

We first construct a section $\tau \in H^0(X(\hat{v}), \mathcal{L}_{\hat{\nu}}^{\otimes n})$ that is an eigenvector of weight $-n\nu$ for B^-, for some positive integer n. Consider $X = G/B^- \times \hat{G}/\hat{B}$, the neutral component $S \subset T$ of the kernel of α, the centralizer G^S of S in G, the fixed point set X^S of S in X and the irreducible component C of X^S containing $(B^-/B^-, \hat{v}\hat{B}/\hat{B})$. Let $\mathcal{L}^\nu \otimes \mathcal{L}_{\hat{\nu}}$ be the $(G\times\hat{G})$-linearized line bundle on X such that $H^0(X, \mathcal{L}^\nu \otimes \mathcal{L}_{\hat{\nu}}) \simeq V_G(\nu) \otimes V_{\hat{G}}(\hat{\nu})^*$. Actually, G^S is isomorphic to $(\mathrm{P})\,\mathrm{SL}_2(\mathbb{C})$ and C is isomorphic to $(\mathbb{P}^1)^3$. We can deduce that C contains points semi-stable for the action of G^S relatively to $\mathcal{L}^\nu \otimes \mathcal{L}_{\hat{\nu}}$. Then, a Luna's theorem (see [44, Corollary 2 and Remark 1] shows that C contains points semi-stable for the action of G. The existence of τ and n follows.

Secondly, we prove that $\dim(H^0(X^\circ(\hat{v}), \mathcal{L}_{\hat{\nu}})^{(B^-)_{-\nu}}) = 1$. Observe that the stabilizer $G_{\hat{v}\hat{B}/hB}$ contains T. Then, Forbenius' theorem allows to embed $H^0(X^\circ(\hat{v}), \mathcal{L}_{\hat{\nu}})^{(B^-)_{-\nu}}$ in $V_G(\hat{\nu})^{(T)_{\rho(\hat{v}\hat{\nu})}}$. Since $\rho(\hat{v}\hat{\nu})$ belongs to $\nu + \mathbb{Z}\alpha$ this space has dimension at most one (it is, once again, a consequence of the SL_2-theory). From the existence of τ, we deduce that $H^0(X^\circ(\hat{v}), \mathcal{L}_{\hat{\nu}})^{(B^-)_{-\nu}}$ and $V_G(\hat{\nu})^{(T)_{\rho(\hat{v}\hat{\nu})}}$ have dimension one.

Finally, fix a nonzero element $\sigma \in H^0(X^\circ(\hat{v}), \mathcal{L}_{\hat{\nu}})^{(B^-)_{-\nu}}$. By unicity $\sigma^{\otimes n}$ and the restriction $\tau_{|X^\circ(\hat{v})}$ must coincide (up to scalar multiplication). In particular, $\sigma^{\otimes n}$ extends to a section of $\mathcal{L}_{\hat{\nu}}^{\otimes n}$ on $X(\hat{v})$. Since $X(\hat{v})$ is normal (see [16, Theorem 1]), σ itself extends to a section of $\mathcal{L}_{\hat{\nu}}$ on $X(\hat{v})$. This ends the proof. □

It is natural to ask if the $\mathrm{LR}(\mathrm{Sp}_{2n}(\mathbb{C}), \mathrm{SL}_{2n}(\mathbb{C}))$ is saturated for any n.

7. The multiplicative Horn problem

In this section, we assume that G is simple and simply-connected.

7.1. The Meinrenken-Woodward polytope. Recall from Section 3.1 that $\mathcal{C}(K^3)$ identifies with the set of triples $(\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3)$ of adjoint orbits in $\mathfrak{k}$ such that $\mathcal{O}_1 + \mathcal{O}_2 + \mathcal{O}_3$ contains 0. We now consider a multiplicative analogous of $\mathcal{C}(K^3)$. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two conjugacy classes in K. Then the product $\mathcal{O}_1 \cdot \mathcal{O}_2 = \{k_1 k_2 : k_1 \in \mathcal{O}_1 \text{ and } k_2 \in \mathcal{O}_2\}$ is stable by conjugacy.

$$\text{What conjugacy classes are contained in } \mathcal{O}_1 \cdot \mathcal{O}_2? \qquad (\dagger)$$

Let θ be the longest root of G. The fundamental alcove in $X_*(T)_{\mathbb{R}}$ is

$$\mathcal{A}_* = \{\lambda \in X_*(T)_{\mathbb{R}} : \begin{cases} \langle \lambda, \alpha \rangle \geq 0 & \forall \alpha \in \Delta \\ \langle \lambda, \theta \rangle \leq 1 \end{cases} \}.$$

Consider the exponential map

$$\begin{array}{rccc} \exp : & \mathrm{Lie}(H) & \longrightarrow & H \\ & \mu & \longmapsto & \exp(\mu). \end{array}$$

Any conjugacy class in K contains a unique element of the form $\exp(\sqrt{-1}\lambda)$ for some $\lambda \in \mathcal{A}_*$ (see e.g. [11, Chapter IX. §5]); we denote by $\mathcal{O}_\lambda^K$ the conjugacy class containing $\exp(\sqrt{-1}\lambda)$.

To answer the question $(\dagger)$ we want to describe the set

$$\mathcal{P}_K = \{(\lambda_1, \lambda_2, \lambda_3) \in \mathcal{A}_*^3 : \mathcal{O}_{\lambda_1}^K \cdot \mathcal{O}_{\lambda_2}^K \cdot \mathcal{O}_{\lambda_3}^K \ni e\},$$

where e is the unit element of K. According to the convexity theorem proved by Meinrenken-Woodward [48], $\mathcal{P}_K$ is a convex polytope of nonempty interior in $\mathcal{A}$. The aim of this section is to describe the minimal list of inequalities that characterize $\mathcal{P}_K$.

7.2. The fusion product. Let $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}((z)) \oplus \mathbb{C}c$ be the affine Lie algebra with c central in $\tilde{\mathfrak{g}}$ and

$$[x \otimes f, y \otimes g] = [x, y] \otimes fg + (x, y) Res_{z=0}(gdf)c,$$

for $x, y \in \mathfrak{g}$ and $f, g \in \mathbb{C}((z))$. Set $\tilde{\mathfrak{g}}_+ = \mathfrak{g} \otimes z\mathbb{C}[[z]]$. Fix a positive integer and set $X(T)_\ell^+ = \{\nu \in X(T)^+ : \langle \nu, \theta^\vee \rangle \leq \ell\}$. For any $\nu \in X(T)_\ell^+$, there exists a unique $\tilde{\mathfrak{g}}$-module $V_{\tilde{\mathfrak{g}}}(\nu, \ell)$ such that c acts on by multiplication by ℓ and the subspace of $V_{\tilde{\mathfrak{g}}}(\nu, \ell)$ annihilated by $\tilde{\mathfrak{g}}_+$ is isomorphic to $V_{\mathfrak{g}}(\nu)$ as a $\mathfrak{g}$-module.

Consider now the projective line $\mathbb{P}^1$ with four distinct marked points $\{0, p_1, p_2, p_3\}$. Set $U = \mathbb{P}^1 - \{0\}$ and consider the ring $\mathcal{O}(U)$ of regular functions on U identified with $\mathbb{C}[z^{-1}]$. Then $\mathfrak{g} \otimes \mathcal{O}(U)$ is a sub-algebra of $\tilde{\mathfrak{g}}$ and acts on $V_{\tilde{\mathfrak{g}}}(0,\ell)$. For each point p_i, consider the evaluation map $ev_i : \mathcal{O}(U) \longrightarrow \mathbb{C}$ at p_i and the associated morphism (still denoted ev_i) from $\mathfrak{g} \otimes \mathcal{O}(U)$ to $\mathfrak{g}$. Fix three weights $\nu_1, \nu_2, \nu_3 \in X(T)^+_\ell$. Consider the action of $\mathfrak{g} \otimes \mathcal{O}(U)$ on $V_{\tilde{\mathfrak{g}}}(0,\ell) \otimes V_{\mathfrak{g}}(\nu_1) \otimes V_{\mathfrak{g}}(\nu_2) \otimes V_{\mathfrak{g}}(\nu_3)$ given by

$$\begin{aligned}\xi.(v_0 \otimes v_1 \otimes v_2 \otimes v_3) = \quad & (\xi v_0) \otimes v_1 \otimes v_2 \otimes v_3 + v_0 \otimes (ev_1(\xi)v_1) \otimes v_2 \otimes v_3 \\ & + v_0 \otimes v_1 \otimes (ev_2(\xi)v_2) \otimes v_3 \\ & + v_0 \otimes v_1 \otimes v_2 \otimes (ev_3(\xi)v_3).\end{aligned}$$

The *space of conformal blocks* $V^\dagger_{\mathbb{P}^1}(\nu_1,\nu_2,\nu_3)$ can be defined (see [3, Corollary 3.5]) as the space of $\mathfrak{g} \otimes \mathcal{O}(U)$-invariant linear forms on $V_{\tilde{\mathfrak{g}}}(0,\ell) \otimes V_{\mathfrak{g}}(\nu_1) \otimes V_{\mathfrak{g}}(\nu_2) \otimes V_{\mathfrak{g}}(\nu_3)$. It is finite dimensional and the *fusion multiplicities* are

$$N^\ell(\nu_1,\nu_2,\nu_3) = \dim(V^\dagger_{\mathbb{P}^1}(\nu_1,\nu_2,\nu_3)).$$

The fusion product $\circledast_\ell$ on $\oplus_{\nu \in X(T)^+_\ell} \mathbb{Z}V(\nu)$ is defined by

$$V(\nu_1) \circledast_\ell V(\nu_2) = \sum_{\nu_3 \in X(T)^+_\ell} N^\ell(\nu_1,\nu_2,\nu_3) V(-w_0\nu_3),$$

for any $\nu_1, \nu_2 \in X(T)^+_\ell$. The product $\circledast_\ell$ is associative and commutative (see e.g. [3]).

The fundamental alcove in $X^*(T)_{\mathbb{Q}}$ is

$$\mathcal{A}^*_{\mathbb{Q}} = \{\lambda \in X^*(T)_{\mathbb{Q}} : \begin{cases} \langle \lambda, \alpha^\vee \rangle \geq 0 & \forall \alpha \in \Delta \\ \langle \lambda, \theta^\vee \rangle \leq 1 \end{cases} \}.$$

The multiplicative analogous to $\mathbb{Q}_{\geq 0}\mathrm{LR}(G^3)$ is

$$\mathcal{P}_\circledast = \{(\nu_1,\nu_2,\nu_3) \in (\mathcal{A}^*_{\mathbb{Q}})^3 : \begin{array}{c} \ell\nu_1, \ell\nu_2, \ell\nu_3 \in X(T)^+_\ell \text{ and } N^\ell(\nu_1,\nu_2,\nu_3) \neq 0, \\ \text{for some positive } \ell. \end{array} \}.$$

Theorem 7.1 (see [69]). *Let $(\nu_1,\nu_2,\nu_3) \in (\mathcal{A}^*_{\mathbb{Q}})^3$. Then $(\nu_1,\nu_2,\nu_3) \in \mathcal{P}_\circledast$ if and only if $((\nu_1,\cdot)_T, (\nu_2,\cdot)_T), (\nu_3,\cdot)_T) \in \mathcal{P}_K$.*

7.3. Relation with $\mathbb{Q}_{\geq 0}\mathbf{LR}(G^3)$. The fusion multiplicities are related to the tensor product multiplicities by

$$\lim_{\ell \to \infty} N^\ell(\nu_1,\nu_2,\nu_3) = c_{G^3}(\nu_1,\nu_2,\nu_3). \tag{7.1}$$

A trivial consequence for the cones is the following.

Proposition 7.2. *The cone in $X^*(T^3)_{\mathbb{R}}$ generated by $\mathcal{P}_\circledast$ is $\mathbb{Q}_{\geq 0}\mathrm{LR}(G^3)$.*

7.4. Quantum Belkale-Kumar Schubert calculus. Fix a simple root α and consider the associated maximal parabolic subgroup P_α containing B. Let L_α be its Levi-subgroup containing T. The Picard group $\mathrm{Pic}(G/P_\alpha)$ identifies with $H^2(G/P_\alpha,\mathbb{Z}) = \mathbb{Z}\sigma_{s_\alpha}$. We denote by $\sigma^*_{s_\alpha}$ the element of $\mathrm{Hom}(H^2(G/P_\alpha,\mathbb{Z}),\mathbb{Z})$ mapping σ_{s_α} to 1.

Let $\gamma : \mathbb{P}^1 \longrightarrow G/P_\alpha$ be a curve. Identifying the group $\mathrm{Pic}(\mathbb{P}^1)$ to $\mathbb{Z}$ (by mapping ample line bundles on positive integers), the pullback of line bundles induces an element of $\mathrm{Hom}(H^2(G/P_\alpha,\mathbb{Z}),\mathbb{Z})$ called the degree of γ and denoted by $\underline{d}(\gamma)$. By construction $\underline{d}(\gamma) \in \mathbb{Z}_{\geq 0}\sigma^*_{s_\alpha}$.

Let ρ and ρ^{L_α} denote the half sum of positive roots of G and L_α respectively. Set

$$n_\alpha = \langle \alpha^\vee, 2(\rho - \rho^{L_\alpha})\rangle. \tag{7.2}$$

Fix a nonnegative integer d. Let $\overline{M}_{0,3}(G/P,\underline{d})$ be the moduli space of stable maps of degree $d\sigma^*_{s_\alpha}$ with 3 marked points into G/P_α. It is a projective variety and

$$\dim(\overline{M}_{0,3}(G/P,\underline{d})) = \dim(G/P_\alpha) + dn_\alpha.$$

It comes equipped with 3 evaluation maps $ev_i : \overline{M}_{0,3}(G/P,\underline{d}) \longrightarrow G/P_\alpha$. The Gromov-Witten invariant associated to three Schubert classes (corresponding to $w_i \in W^{P_\alpha}$) and a degree $\underline{d} = d\sigma^*_{s_\alpha}$ is then the intersection number

$$GW(w_1,w_2,w_3;d) = \int_{\overline{M}_{0,3}(G/P,\underline{d})} ev_1^*(\sigma_{w_1})\cdot ev_2^*(\sigma_{w_2})\cdot ev_3^*(\sigma_{w_3}).$$

Introduce a variable q and consider the group

$$\begin{aligned} QH^*(G/P_\alpha,\mathbb{Z}) &:= H^*(G/P,\mathbb{Z})\otimes\mathbb{Z}[q] \\ &= \oplus_{w\in W^{P_\alpha}} \mathbb{Z}[q]\sigma_w. \end{aligned}$$

The $\mathbb{Z}[q]$-linear quantum product $\star$ on $QH^*(G/P_\alpha,\mathbb{Z})$ is defined by, for any $w_1, w_2 \in W^{P_\alpha}$,

$$\sigma_{w_1}\star\sigma_{w_2} = \sum GW(w_1,w_2,w_3;d)q^d\sigma^\vee_{w_3},$$

where the sum runs over $w_3 \in W^{P_\alpha}$ and over nonnegative integers d.

The grading on $H^*(G/P_\alpha,\mathbb{Z})$ extends to the quantum setting by setting $\deg(q) = 2n_\alpha$. In particular $GW(w_1,w_2,w_3;\underline{d}) \neq 0$ implies

$$l(w_1) + l(w_2) + l(w_3) + dn_\alpha = 2\dim(G/P_\alpha). \tag{7.3}$$

Condition (7.3) can be rewritten like

$$\sharp\,\Phi(w_1) + \sharp\,\Phi(w_2) + \sharp\,\Phi(w_3) + dn_\alpha = 2\,\sharp\,\Phi(G/P_\alpha). \tag{7.4}$$

Set $h = d\alpha^\vee$. Since $2(\rho - \rho^{L_\alpha}) = \sum_{\beta\in\Phi(G/P)}\beta$, condition (7.4) can be rewritten like

$$\sum_{\chi\in X^*(Z)}\left(\sum_{i=1}^{3}\sharp\,\Phi(w_i,\chi) + \sum_{\beta\in\Phi(G/P,\chi)}\langle h,\beta\rangle\right) = 2\sum_{\chi\in X^*(Z)}\sharp\,\Phi(G/P,\chi). \tag{7.5}$$

The Belkale-Kumar quantum product requires the equality (7.5) to hold term by term. More precisely, set

$$\sigma_{w_1}\circledast\sigma_{w_2} = \sum_{\substack{w_3\in W^P \\ d\in\mathbb{Z}_{\geq 0}}} \widetilde{GW}(w_1,w_2,w_3;d)q^d\sigma^\vee_{w_3}, \tag{7.6}$$

where $\widetilde{GW}(w_1,w_2,w_3;d) = GW(w_1,w_2,w_3;d)$ if

$$\forall \chi \in X(Z) \qquad \sum_i \sharp\,\Phi(w_i,\chi) + \sum_{\beta\in\Phi(G/P_\alpha,\chi)} d\langle\alpha^\vee,\beta\rangle = 2\,\sharp\,\Phi(G/P,\chi), \tag{7.7}$$

and $\widetilde{GW}(w_1,w_2,w_3;d) = 0$ otherwise.

Theorem 7.3 (Belkale-Kumar [10]). *The product $\circledast$ is associative.*

7.5. Descriptions of $\mathcal{P}_K$. Let ϖ_α denote the fundamental weight associated to α. To any $(w_1,w_2,w_3) \in (W^{P_\alpha})^3$ and any $d \in \mathbb{Z}_{\geq 0}$, we associate the following linear inequality on points $(\lambda_1,\lambda_2,\lambda_3)$ in $X_*(T)_{\mathbb{R}}$:

$$\mathcal{I}_\alpha(w_1,w_2,w_3;d) \qquad \langle w_1\varpi_\alpha,\lambda_1\rangle + \langle w_2\varpi_\alpha,\lambda_2\rangle + \langle w_3\varpi_\alpha,\lambda_3\rangle \leq d.$$

Here, comes a first description of $\mathcal{P}_K$.

Theorem 7.4 (Teleman-Woodward, see [69]). *Let $(\lambda_1,\lambda_2,\lambda_3) \in \mathcal{A}^3_*$. Then $(\lambda_1,\lambda_2,\lambda_3) \in \mathcal{P}_K$ if and only if inequality $\mathcal{I}_\alpha(w_1,w_2,w_3;d)$ is fulfilled for any simple root α, any nonnegative integer d and any $w_1,w_2,w_3 \in W^{P_\alpha}$ such that*

$$GW(w_1,w_2,w_3;d) = 1 \tag{7.8}$$

in G/P_α.

Recently, Theorem 7.4 was improved as follows.

Theorem 7.5 (Belkale-Kumar [10], R. [58]). *Let $(\lambda_1,\lambda_2,\lambda_3) \in \mathcal{A}^3_*$. Then $(\lambda_1,\lambda_2,\lambda_3) \in \mathcal{P}_K$ if and only if inequality $\mathcal{I}_\alpha(w_1,w_2,w_3;d)$ is fulfilled for any simple root α, any nonnegative integer d and any (w_1,w_2,w_3) such that, in $QH^*(G/P_\alpha)$,*

$$\widetilde{GW}(w_1,w_2,w_3;d) = 1. \tag{7.9}$$

Now, Theorem 7.5 is optimal.

Theorem 7.6 (Belkale-Kumar [10]). *The list of inequalities given by Theorem 7.5 is irredundant.*

Remark 7.7. For G classical, condition (7.9) can be checked using works of Bertram, Buch, Kresch and Tamvakis or the software qcalc [18]. Explicit lists of inequalities can be downloaded on the homepage of the author. For example, the polype $\mathcal{P}_{\mathrm{Sp}_{12}}$ has 43 136 facets and 20 839 vertices.

In this multiplicative context, the question of saturation can be asked for the fusion product multiplicities by analogy with Section 5. The only known result is in type A:

Theorem 7.8 ([6]). *Let $G = \mathrm{SL}_n(\mathbb{C})$, ℓ be a positive integer and ν_1, ν_2 and ν_3 be three dominant weights in $X(T)^+_\ell$. We assume that $\nu_1+\nu_2+\nu_3$ belongs to the root lattice Λ_R.*

Then $N^\ell(\nu_1,\nu_2,\nu_3) \neq 0$ if and only if there exists a positive integer k such that $N^{k\ell}(k\nu_1,k\nu_2,k\nu_3) \neq 0$.

References

[1] Ivan V. Arzhantsev and Jürgen Hausen, *Geometric invariant theory via Cox rings*, J. Pure Appl. Algebra **213** (2009), no. 1, 154–172.

[2] H. Azad, M. Barry, and G. Seitz, *On the structure of parabolic subgroups*, Com. in Algebra **18** (1990), no. 2, 551–562.

[3] Arnaud Beauville, *Conformal blocks, fusion rules and the Verlinde formula*, Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993) (Ramat Gan), Israel Math. Conf. Proc., vol. 9, Bar-Ilan Univ., 1996, pp. 75–96.

[4] Prakash Belkale, *Local systems on $\mathbb{P}^1 - S$ for S a finite set*, Compositio Math. **129** (2001), no. 1, 67–86.

[5] ———, *Geometric proofs of Horn and saturation conjectures*, J. Algebraic Geom. **15** (2006), no. 1, 133–173.

[6] ———, *Quantum generalization of the Horn conjecture*, J. Amer. Math. Soc. **21** (2008), no. 2, 365–408.

[7] Prakash Belkale and Shrawan Kumar, *Eigenvalue problem and a new product in cohomology of flag varieties*, Invent. Math. **166** (2006), no. 1, 185–228.

[8] ———, *Eigencone, saturation and horn problems for symplectic and odd orthogonal groups*, J. Algebraic Geom. **19** (2010), no. 2, 199–242.

[9] ———, *Eigencone, saturation and Horn problems for symplectic and odd orthogonal groups*, J. Algebraic Geom. **19** (2010), no. 2, 199–242.

[10] ———, *The multiplicative eigenvalue problem and deformed quantum cohomology*, arXiv:1310.3191, October 2013.

[11] Nicolas Bourbaki, *Lie groups and Lie algebras. Chapters 7–9*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 2005, Translated from the 1975 and 1982 French originals by Andrew Pressley.

[12] Emily Braley, *Eigencone Problems for Odd and Even Orthogonal Groups*, ProQuest LLC, Ann Arbor, MI, 2012, Thesis (Ph.D.)–The University of North Carolina at Chapel Hill.

[13] Michel Brion, *Stable properties of plethysm: on two conjectures of Foulkes*, Manuscripta Math. **80** (1993), no. 4, 347–371.

[14] ———, *On the general faces of the moment polytope*, Internat. Math. Res. Notices (1999), no. 4, 185–201.

[15] ———, *On orbit closures of spherical subgroups in flag varieties*, Comment. Math. Helv. **76** (2001), no. 2, 263–299.

[16] ———, *Multiplicity-free subvarieties of flag varieties*, Commutative algebra (Grenoble/Lyon, 2001), Contemp. Math., vol. 331, Amer. Math. Soc., Providence, RI, 2003, pp. 13–23.

[17] ———, *Restriction de représentations et projections d'orbites coadjointes [d'après Belkale, Kumar et Ressayre]*, Astérisque (2012), no. 339, Exp. No. 1043, x, 30, Séminaire Bourbaki. Vol. 2011/2012.

[18] Anders Buch, *Quantum calculator — a software maple package*, Available at www.math.rutgers.edu/~asbuch/qcalc.

[19] Leandro Cagliero and Paulo Tirao, *A closed formula for weight multiplicities of representations of* $\mathrm{Sp}_2(\mathbb{C})$, Manuscripta Math. **115** (2004), no. 4, 417–426.

[20] Harm Derksen and Jerzy Weyman, *Semi-invariants of quivers and saturation for Littlewood-Richardson coefficients*, J. Amer. Math. Soc. **13** (2000), no. 3, 467–479.

[21] Harm Derksen and Jerzy Weyman, *The combinatorics of quiver representations*, Ann. Inst. Fourier **61** (2011), no. 3, 1061–1131.

[22] Igor V. Dolgachev and Yi Hu, *Variation of geometric invariant theory quotients*, Inst. Hautes Études Sci. Publ. Math. **87** (1998), 5–56, With an appendix by Nicolas Ressayre.

[23] Alexander G. Élashvili, *Invariant algebras*, Lie groups, their discrete subgroups, and invariant theory, Adv. Soviet Math., vol. 8, Amer. Math. Soc., Providence, RI, 1992, pp. 57–64.

[24] V. Guillemin and S. Sternberg, *Convexity properties of the moment mapping*, Invent. Math. **67** (1982), no. 3, 491–513.

[25] ———, *Geometric quantization and multiplicities of group representations*, Invent. Math. **67** (1982), no. 3, 515–538.

[26] G. J. Heckman, *Projections of orbits and asymptotic behavior of multiplicities for compact connected Lie groups*, Invent. Math. **67** (1982), no. 2, 333–356.

[27] Michael Kapovich, Shrawan Kumar, and John J. Millson, *The eigencone and saturation for Spin(8)*, Pure Appl. Math. Q. **5** (2009), no. 2, Special Issue: In honor of Friedrich Herzebruch. Part 1, 755–780.

[28] Michael Kapovich and John J. Millson, *A path model for geodesics in Euclidean buildings and its applications to representation theory*, Groups Geom. Dyn. **2** (2008), no. 3, 405–480.

[29] George Kempf and Linda Ness, *The length of vectors in representation spaces*, Algebraic geometry (Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978), Lecture Notes in Math., vol. 732, Springer, Berlin, 1979, pp. 233–243.

[30] Ronald C. King, Christophe Tollu, and Frédéric Toumazet, *The hive model and the factorisation of Kostka coefficients*, Sém. Lothar. Combin. **54A** (2005/07), Art. B54Ah, 22 pp. (electronic).

[31] ———, *Factorisation of Littlewood-Richardson coefficients*, J. Combin. Theory Ser. A **116** (2009), no. 2, 314–333.

[32] Frances Kirwan, *Convexity properties of the moment mapping. III*, Invent. Math. **77** (1984), no. 3, 547–552.

[33] Frances Clare Kirwan, *Cohomology of quotients in symplectic and algebraic geometry*, Mathematical Notes, vol. 31, Princeton University Press, Princeton, NJ, 1984.

[34] Alexander A. Klyachko, *Stable bundles, representation theory and Hermitian operators*, Selecta Math. (N.S.) **4** (1998), no. 3, 419–445.

[35] Allen Knutson and Terence Tao, *The honeycomb model of* $\mathrm{GL}_n(\mathbb{C})$ *tensor products. I. Proof of the saturation conjecture*, J. Amer. Math. Soc. **12** (1999), no. 4, 1055–1090.

[36] Allen Knutson, Terence Tao, and Christopher Woodward, *The honeycomb model of* $\mathrm{GL}_n(\mathbb{C})$ *tensor products. II. Puzzles determine facets of the Littlewood-Richardson cone*, J. Amer. Math. Soc. **17** (2004), no. 1, 19–48.

[37] Bertram Kostant, *Lie algebra cohomology and the generalized Borel-Weil theorem*, Ann. of Math. (2) **74** (1961), 329–387.

[38] P. Yu. Kotenkova, *GIT equivalence and diagonal actions*, Mat. Zametki **90** (2011), no. 2, 269–279.

[39] S. Kumar, *Additive Eigenvalue Problem (a survey), (With appendix by M. Kapovich)*, ArXiv e-prints (2013), 103.

[40] S. Kumar and D. Prasad, *Dimension of zero weight space: An algebro-geometric approach*, ArXiv e-prints (2013), 1–19.

[41] Shrawan Kumar, *Proof of the Parthasarathy-Ranga Rao-Varadarajan conjecture*, Invent. Math. **93** (1988), no. 1, 117–130.

[42] Brandyn Lee, *A Comparison of Eigencones Under Certain Diagram Automorphisms*, ProQuest LLC, Ann Arbor, MI, 2012, Thesis (Ph.D.)–The University of North Carolina at Chapel Hill.

[43] Léonce Lesieur, *Les problèmes d'intersection sur une variété de Grassmann*, C. R. Acad. Sci. Paris **225** (1947), 916–917.

[44] Domingo Luna, *Adhérences d'orbite et invariants*, Invent. Math. **29** (1975), no. 3, 231–238.

[45] Ian Grant Macdonald, *Symmetric functions and Hall polynomials*, second ed., Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1995, With contributions by A. Zelevinsky, Oxford Science Publications.

[46] Laurent Manivel, *Applications de Gauss et pléthysme*, Ann. Inst. Fourier (Grenoble) **47** (1997), no. 3, 715–773.

[47] Olivier Mathieu, *Construction d'un groupe de Kac-Moody et applications*, Compositio Math. **69** (1989), no. 1, 37–60.

[48] E. Meinrenken and C. Woodward, *Hamiltonian loop group actions and Verlinde factorization*, J. Differential Geom. **50** (1998), no. 3, 417–469.

[49] Eckhard Meinrenken and Reyer Sjamaar, *Singular reduction and quantization*, Topology **38** (1999), no. 4, 699–762.

[50] Pierre-Louis Montagard, *Une nouvelle propriété de stabilité du pléthysme*, Comment. Math. Helv. **71** (1996), no. 3, 475–505.

[51] Pierre-Louis Montagard, Boris Pasquier, and Nicolas Ressayre, *Generalizations of the PRV conjecture, II*, Preprint (2011), 1–17.

[52] _____, *Two generalisations of the PRV conjecture*, Compositio Math. **147** (2011), no. 4, 1321—1336.

[53] David Mumford, John Fogarty, and Frances Kirwan, *Geometric invariant theory*, 3d ed., Springer Verlag, New York, 1994.

[54] K. R. Parthasarathy, R. Ranga Rao, and V. S. Varadarajan, *Representations of complex semi-simple Lie groups and Lie algebras*, Ann. of Math. (2) **85** (1967), 383–429.

[55] B. Pasquier and N. Ressayre, *The saturation property for branching rules - examples*, Experimental Mathematics **22** (2013), no. 3, 299–312.

[56] Etienne Rassart, *A polynomiality property for Littlewood-Richardson coefficients*, J. Combin. Theory Ser. A **107** (2004), no. 2, 161–179.

[57] N. Ressayre, *Distribution on homogeneous spaces and belkale-kumar's product*, e-prints (2013), 1–43.

[58] _____, *On the quantum horn problem*, ArXiv e-prints (2013), 1–34.

[59] Nicolas Ressayre, *The GIT-equivalence for G-line bundles*, Geom. Dedicata **81** (2000), no. 1-3, 295–324.

[60] _____, *Geometric invariant theory and generalized eigenvalue problem*, Invent. Math. **180** (2010), 389–441.

[61] _____, *Geometric invariant theory and generalized eigenvalue problem II*, Ann. Inst. Fourier (to appear) (2010), 1–25.

[62] _____, *Spherical homogeneous spaces of minimal rank*, Adv. Math. **224** (2010), no. 5, 1784–1800.

[63] _____, *A cohomology free description of eigencones in type A, B and C*, Internat. Math. Res. Notices (to appear) (2011), 1–35.

[64] _____, *Horn inequalities for nonzero kronecker coefficients*, ArXiV e-prints (2012), 1–22.

[65] Nicolas Ressayre and Edward Richmond, *Branching Schubert calculus and the Belkale-Kumar product on cohomology*, Proc. Amer. Math. Soc. **139** (2011), 835–848.

[66] Mike Roth, *Reduction rules for Littlewood-Richardson coefficients*, Int. Math. Res. Not. IMRN (2011), no. 18, 4105–4134.

[67] Steven V Sam, *Symmetric quivers, invariant theory, and saturation theorems for the classical groups*, Adv. in Math. **229** (2012), no. 2, 1104–1135.

[68] Frank Sottile, *General isotropic flags are general (for grassmannian schubert calculus)*, J. Algebraic Geom. **1** (2010), no. 1, 1–3.

[69] Constantin Teleman and Christopher Woodward, *Parabolic bundles, products of conjugacy classes and Gromov-Witten invariants*, Ann. Inst. Fourier **53** (2003), no. 3, 713–748.

[70] Michael Thaddeus, *Geometric invariant theory and flips*, J. of Amer. Math. Soc. **9** (1996), no. 3, 691–723.

[71] Hermann Weyl, *Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung)*, Math. Ann. **71** (1912), no. 4, 441–479.

Université Lyon I, 43 boulevard du 11 novembre 1918, 69095 Villeurbanne, France
E-mail: ressayre@math.univ-lyon1.fr

Double affine Hecke algebras and Hecke algebras associated with quivers

To Maria and Gracco

Michela Varagnolo and Eric Vasserot

Abstract. This is a short survey of some geometrical and categorical approaches to the representation theory of several algebras related to Hecke algebras, including cyclotomic Hecke algebras, double affine Hecke algebras and quiver-Hecke algebras.

Mathematics Subject Classification (2010). 06B15.

Keywords. Hecke algebras, Quiver-Hecke algebras, categorical representations.

1. Introduction

Affine Hecke algebras are very important in representation theory and have been studied extensively over the past few decades, along with their degenerate version introduced by Drinfeld and Lusztig. About twenty years ago, Cherednik introduced the notion of double affine Hecke algebra, abbreviated as DAHA, which he used to prove the Macdonald's constant term conjecture for Macdonald polynomials. This algebra also admits degenerate versions, the rational one, which is also called Cherednik algebra, having been introduced by Etingof and Ginzburg in 2002.

A rational DAHA is defined for any complex reflection group W. Its representation theory yields a new approach to the representation theory of the Hecke algebra of W. Remarkably, this representation theory is also similar to the representation theory of semi-simple Lie algebras. In particular, it admits a highest weight category which is analogous to the BGG category O. Highest weight representations are infinite dimensional in general, but they admit a character. An important question is to determine the characters of simple modules.

One of the most important family of rational DAHA's is the cyclotomic one. One reason is that their representation theory is closely related to the representation theory of cyclotomic Hecke algebras, which are relevant in group theory. Another reason is that their highest weight category is closely related to the representation theory of affine Kac-Moody algebras. This was one important motivation for the development of categorical representations (in representation theory).

Categorical representations of Kac-Moody algebras is a relatively young subject that arises in Representation theory and in Knot theory. The first formal definition appeared in a paper of Chuang and Rouquier. The general case was treated independently by Rouquier

Proceedings of the International Congress of Mathematicians, Seoul, 2014

and by Khovanov and Lauda. The ideas leading to categorical representations were around for some two decades. One of the most remarkable application is the work of Ariki, inspired by a conjecture of Lascoux-Leclerc-Thibon, on cyclotomic Hecke algebras. It was observed there that the module category of cyclotomic Hecke algebras has endofunctors that on the level of the Grothendieck group give actions of Kac-Moody Lie algebras of (affine) type A.

This structure appears at several other places in Representation theory, such as the representations of symmetric groups, of the general linear groups or of Lie algebras of type A. An important fact is that the endofunctors come equipped with some natural transformations which satisfy the relations of a new algebra called quiver-Hecke algebra.

Our aim is not to give a general introduction to the subject. There are a lot of them available in the literature, both on DAHA's and on categorical representations. We'll simply focus on some recent results concerning the representation theory of these algebras.

2. Double affine Hecke algebras

2.1. Rational double affine Hecke algebras.

2.1.1. Definition. A *complex reflection group* W is a group acting on a finite dimensional complex vector space $\mathfrak{h}$ that is generated by complex reflections, i.e., non-trivial elements that fix a complex hyperplane in $\mathfrak{h}$ pointwise.

Given a complex reflection group W, let S be its set of complex reflections. For each $s \in S$ let $\alpha_s \in \mathfrak{h}^*$ be a generator of $\mathrm{Im}(s|_{\mathfrak{h}^*} - 1)$, and $\alpha_s^\vee$ be the generator of $\mathrm{Im}(s|_{\mathfrak{h}} - 1)$ such that $(\alpha_s, \alpha_s^\vee) = 2$. Let $c : S \to \mathbb{C}$, $s \mapsto c_s$ be a W-invariant function.

Definition 2.1 ([18]). The *rational DAHA*, abbreviated RDAHA, associated with $W, \mathfrak{h}$ and c is the quotient $H_c(W)$ of the algebra $\mathbb{C}W \ltimes T(\mathfrak{h} \oplus \mathfrak{h}^*)$ by the ideal generated by the relations $[x, x'] = [y, y'] = 0$ and $[y, x] = (y, x) - \sum_{s \in S} c_s\,(y, \alpha_s)(\alpha_s^\vee, x)$, for all $x, x' \in \mathfrak{h}^*$ and $y, y' \in \mathfrak{h}$.

The algebra $H_c(W)$ may as well be defined as the subalgebra of $\mathrm{End}_{\mathbb{C}}(\mathbb{C}[\mathfrak{h}])$ generated by the action of $w \in W$, the multiplication by all elements of $\mathfrak{h}^* \subset \mathbb{C}[\mathfrak{h}]$, and the *Dunkl-Opdam operators* $\partial_y + \sum_{s \in S} c_s\,(y, \alpha_s)\,\alpha_s^{-1}(s - 1)$ where $y \in \mathfrak{h}$. The (faithful) representation of $H_c(W)$ on $\mathbb{C}[\mathfrak{h}]$ is called the *polynomial representation*.

2.1.2. The highest weight category $\mathcal{O}_c(W)$. The algebra $H_c(W)$ contains commutative subalgebras $\mathbb{C}[\mathfrak{h}]$ and $\mathbb{C}[\mathfrak{h}^*]$. We define the category $\mathcal{O}_c(W)$ to be the category of $H_c(W)$-modules which are finitely generated over $\mathbb{C}[\mathfrak{h}]$ and locally nilpotent under the action of $\mathfrak{h}$. It is discussed in details in [21]. This is an analogue of the BGG category $\mathcal{O}$ for semisimple Lie algebras.

The algebra $H_c(W)$ admits a *triangular decomposition*. More precisely, the multiplication yields an isomorphism $H_c(W) \simeq \mathbb{C}[\mathfrak{h}] \otimes \mathbb{C}[W] \otimes \mathbb{C}[\mathfrak{h}^*]$. The most important objects in the category $\mathcal{O}_c(W)$ are the *standard modules* $\Delta_c(\tau) = \mathrm{Ind}_{W \otimes \mathbb{C}[\mathfrak{h}^*]}^{H_c(W)} \tau$, where τ is an irreducible representation of W with the zero action of $\mathfrak{h}$, and their irreducible quotients $L_c(\tau)$.

It is easy to see that $\mathcal{O}_c(W)$ contains all finite dimensional modules and that the standard module $\Delta_c(\mathrm{triv})$ is isomorphic to the polynomial representation, where triv is the trivial one-dimensional representation of W.

Definition 2.2. A *highest weight category* is a pair $(\mathcal{C}, \Lambda)$ where $\mathcal{C}$ artinian abelian category with enough projectives and injectives, such that the endomorphism algebra of the irreducible objects are one dimensional, and Λ is an interval-finite poset indexing a set of pairwise non-isomorphic irreducible object $\{L(\lambda)\,;\,\lambda \in \Lambda\}$ of $\mathcal{C}$.

Further, the following axioms hold. Let $P(\lambda)$ be the projective cover of $L(\lambda)$ in $\mathcal{C}$. Define the *standard object* $\Delta(\lambda)$ to be the largest quotient of $P(\lambda)$ such that $[\Delta(\lambda) : L(\mu)] = \delta_{\lambda,\mu}$ for $\mu \not< \lambda$. Then $P(\lambda)$ has a finite filtration with top section isomorphic to $\Delta(\lambda)$ and other sections of the form $\Delta(\mu)$ with $\mu > \lambda$.

The BGG category $\mathcal{O}$ for semisimple Lie algebras is an highest weight category. The category $\mathcal{O}_c(W)$ is also an an highest weight category.

Let $\mathcal{H}_t(W)$ be the *Hecke algebra* of W at the parameter $t = \exp(2\pi ic)$, see [7] for a definition. According to [21], there is a functor $KZ_c : \mathcal{O}_c(W) \to \mathcal{H}_t(W)$-mod, which is a *quotient functor* in the general sense of Gabriel. This functor has many good properties. In particular, by [41], this functor determines the highest weight category $\mathcal{O}_c(W)$ up to an equivalence.

More precisely, let R be a commutative local $\mathbb{C}$-algebra which is a domain and let $\mathcal{C}$ be a highest weight category over R. Let H be a finite projective R-algebra. An R-linear functor $F : \mathcal{C} \to H$-mod is a *highest weight cover* if it is a quotient functor which is fully faithful on projective modules. It is a *d-faithful highest weight cover* if it is a quotient functor which induces an isomorphism $\mathrm{Ext}^i_{\mathcal{C}}(M,N) \to \mathrm{Ext}^i_H(FM,FN)$ for all $i \leqslant d$ and all $M, N \in \mathcal{C}$ admitting a finite filtration whose sections are standard modules.

For any R-algebra R' and any R-linear category $\mathcal{C}$, let $\mathcal{C} \otimes_R R'$ be the R'-linear category with the same objects as $\mathcal{C}$ and with $\mathrm{Hom}_{\mathcal{C}\otimes_R R'}(M,N) = \mathrm{Hom}_{\mathcal{C}}(M,N) \otimes_R R'$ for each objects M, N.

Now, let K be the fraction field of R. We have the following.

Theorem 2.3 ([40]). *Assume that the K-algebra $H \otimes_R K$ is split semisimple and that $F_i : \mathcal{C}_i \to H$-mod is a 1-faithful highest weight cover for $i = 1, 2$. Then the category $\mathcal{C}_i \otimes_R K$ is semisimple and the functor $F_i \otimes_R K$ induces a bijection $\mathrm{Irr}(\mathcal{C}_i \otimes_R K) \simeq \mathrm{Irr}(H \otimes_R K)$. Let $\leqslant_i$ be the partial orders on $\mathrm{Irr}(H \otimes_R K)$ induced by the poset of $\mathcal{C}_i$. If $\leqslant_1$ is a refinement of $\leqslant_2$, then there is an equivalence of highest weight categories $\mathcal{C}_1 \simeq \mathcal{C}_2$.*

2.1.3. Support of modules in $\mathcal{O}_c(W)$. The functor KZ_c is not generally a category equivalence, since the restriction from $\mathfrak{h}$ to $\mathfrak{h}_{reg}$ kills any object of $\mathcal{O}_c(W)$ supported on $\mathfrak{h} \setminus \mathfrak{h}_{reg}$, the union of all reflecting hyperplanes of W. The support of an irreducible object is always a W-orbit of an intersection of reflecting hyperplanes by [20]. So it has, up to conjugacy, a parabolic subgroup W' attached to it by taking the stabilizer of a generic point in the intersection of these hyperplanes. Despite there usually being no non-trivial homomorphism $H_c(W') \to H_c(W)$, Bezrukavnikov and Etingof have constructed in [2] an induction functor and a restriction functor between the categories $\mathcal{O}_c(W)$ and $\mathcal{O}_c(W')$, for each $x \in \mathfrak{h}$ with stabilizer W'. Up to isomorphism, these functors are independent of the choice of the element x. Therefore, it is important to know the support of representations in $\mathcal{O}_c(W)$.

A module is supported at 0 if and only if it is finite dimensional. The values of the parameter c for which the module $L_c(\mathrm{triv})$ is finite dimensional has been determined in [52] by geometric methods (with some restrictions on W and c), see Section 2.2.4 below. More generally, the support of $L_c(\mathrm{triv})$ has been completely determined by Etingof in [16], using the Macdonald-Mehta integral for Weyl groups.

Example 2.4. The complex reflection groups have been classified in [57]. One infinite family appears, labelled $G(d,p,n)$, where d,e,n are positive integers such that p divides d. The subfamily $G(d,1,n)$ takes an important place. We have $G(d,1,n) = S_n \ltimes (\mathbb{Z}/d)^n$, the *wreath product* of the symmetric group S_n and the cyclic group $\mathbb{Z}/d$. We'll abbreviate $H_c(d,n) = H_c(G(d,1,n))$ and $\mathcal{O}_c(d,n) = \mathcal{O}_c(G(d,1,n))$. The algebra's $H_c(d,n)$ are called the *cyclotomic RDAHA*, and abbreviated CRDAHA. The Hecke algebra $\mathcal{H}_t(G(d,1,n))$ at the parameter $t = \exp(2\pi ic)$ associated with $H_c(d,n)$ is an important algebra in representation theory. It is called the *cyclotomic Hecke algebra*. We'll write $\mathcal{H}_t(d,n) = \mathcal{H}_t(G(d,1,n))$, hence the KZ-functor is a functor $KZ_c : \mathcal{O}_c(d,n) \to \mathcal{H}_t(d,n)$-mod.

To each tuple of integers $e, s_1, \dots, s_d$ with $e \geq 0$, one associates the *level d Fock space of multicharge* $s = (s_1, \dots, s_d)$. It is a semisimple $\widehat{\mathfrak{sl}}_e$-module $F(s)$ defined in a combinatorial way and equipped with a (dual) canonical basis, defined also in a combinatorial manner, see [50] and Section 4.2. The dimension of the support of all simple object in $\mathcal{O}_c(d,n)$ has been characterized in [48] via the representation theory of $F(s)$, using categorical representations, answering positively to a conjecture of Etingof in [17]. See Section 4.2.

2.2. Affine and double affine Hecke algebras.

2.2.1. Cartan data and braid groups. A *Cartan datum* consists of a finite-rank free abelian group X whose dual lattice is denoted $X^\vee$, a finite set of vectors $\Phi = \{\alpha_1, \dots, \alpha_n\} \subset X$ called *simple roots* and a finite set of vectors $\Phi^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\} \subset X^\vee$ called *simple coroots*. Set $I = \{1, \dots, n\}$. The $I \times I$ matrix A with entries $a_{ij} = (\alpha_j, \alpha_i^\vee)$ is assumed to be a *generalized Cartan matrix*.

Let $\alpha \in X$ and $\alpha^\vee \in X^\vee$ satisfy $(\alpha, \alpha^\vee) = 2$. The linear automorphism $s_{\alpha,\alpha^\vee}(\lambda) = \lambda - (\lambda, \alpha^\vee)\,\alpha$ of X is a reflection. If $\alpha^\vee$ is implicitly associated to α we write s_α for both $s_{\alpha,\alpha^\vee}$ and $s_{\alpha^\vee,\alpha}$. When $\alpha = \alpha_i$ and $\alpha^\vee = \alpha_i^\vee$ are a simple root and the corresponding coroot, we write $s_i = s_{\alpha_i}$. The s_i are called the *simple reflections*.

We'll assume that the Cartan datum is *non-degenerate*, i.e., the simple roots are linearly independent. The *Weyl group* W is the group of automorphisms of X (and of $X^\vee$) generated by the simple reflections s_i. The sets of *roots* and *coroots* are $R = \bigcup_i W(\alpha_i)$, $R^\vee = \bigcup_i W(\alpha_i^\vee)$. The *root* and *coroot lattices* are $Q = \mathbb{Z}\Phi \subset X$ and $Q^\vee = \mathbb{Z}\Phi^\vee \subset X^\vee$. The set of *positive roots* is $R_+ = R \cap Q_+$, where $Q_+ = \mathbb{N}\Phi$. For each element $\alpha = \sum_{i\in I} a_i\,\alpha_i$ in Q_+, let $|\alpha| = \sum_{i\in I} a_i$ be the *height* of α. The *dominant weights* are the elements of the cone $X_+ = \{\lambda \in X\,;\, (\lambda, \alpha_i^\vee) \geqslant 0 \text{ for all } i\}$.

The Cartan datum is *finite* if W is a finite group, or equivalently, R is a finite set. The finite Cartan data classify connected reductive algebraic groups G over any algebraically closed field. Then X is the character group $X^*(T)$ of a maximal torus T in G, called the *weight lattice* of G, and $X^\vee$ is the group $X_*(T)$ of one-parameter subgroups of T, called the *coweight lattice* of G. An element $\omega_i \in \mathbb{R} \otimes X$ is called a i-th *fundamental weights* if we have $(\omega_i, \alpha_j^\vee) = \delta_{i,j}$ for all j.

The Cartan datum is *affine* if its Cartan matrix A is singular, and for every proper subset $J \subset I$, the Cartan datum $(X, (\alpha_i)_{i\in J}, X^\vee, (\alpha_i^\vee)_{i\in J})$ is finite. This definition implies that the nullspace of A is one-dimensional. Since X is non-degenerate, then $\{\lambda \in Q\,;\, (\lambda, \alpha_i^\vee) = 0 \text{ for all } i\}$ is a sublattice of rank 1. It has a unique generator $\delta \in Q_+$, called the *nullroot*. The affine Cartan matrices are classified in [23] and [37].

The Weyl group W is a *Coxeter group* with defining relations $s_i^2 = 1$ and $s_j s_j s_i \cdots = s_j s_i s_j \cdots$ (m_{ij} factors on each side) where if $a_{ij}a_{ji} = 0, 1, 2, 3$ then $m_{ij} = 2, 3, 4, 6$ re-

spectively, and if $a_{ij}a_{ji} \geqslant 4$ there is no relation between s_i, s_j. The *length* $l(w)$ of $w \in W$ is the minimal l such that $w = s_{i_1} \cdots s_{i_l}$. Such an expression is called a *reduced factorization*. The *braid group* $B(W)$ is the group with generators T_i and the braid relations $T_jT_jT_i \cdots = T_jT_iT_j \cdots$ (m_{ij} factors on each side). If $w = s_{i_1} \cdots s_{i_l}$ is a reduced factorization, we set $T_w = T_{i_1} \ldots T_{i_l}$. There is a canonical homomorphism $B(W) \to W$, $T_i \mapsto s_i$.

The *affine Weyl group* is the semidirect product $W \ltimes X$. We use multiplicative notation for the group X, denoting $\lambda \in X$ by x^λ. So $W \ltimes X$ is generated by its subgroups W and X with the additional relations $s_i x^\lambda s_i = x^{s_i(\lambda)}$. For any finite Cartan datum X with Weyl group W and set of simple roots $\{\alpha_i\}$, there is an affine Cartan datum with weight lattice $\widetilde{X} = X \oplus \mathbb{Z}\,\delta$, Weyl group $\widetilde{W} = W \ltimes Q^\vee$ and set of simple roots $\widetilde{\Phi} = \{\tilde{\alpha}_0, \ldots, \tilde{\alpha}_n\}$. Here, we set $\tilde{\alpha}_0 = \delta - \theta$ where θ the highest root in R, and $\tilde{\alpha}_i = \alpha_i$ if $i \neq 0$. Let $\tilde{\alpha}_0^\vee, \ldots, \tilde{\alpha}_n^\vee$ be the affine simple coroots in the dual lattice $\widetilde{X}^\vee$. The canonical pairing $\widetilde{X} \times \widetilde{X}^\vee \to \mathbb{Z}$ is such that $(\delta, \widetilde{\alpha}_i^\vee) = 0$ for all i. There is also an affine Cartan datum with weight lattice $X_{\mathrm{aff}} = \widetilde{X} \oplus \mathbb{Z}\,\tilde{\omega}_0$, Weyl group $\widetilde{W}$ and set of simple roots $\widetilde{\Phi}$ such that $(\tilde{\omega}_0, \tilde{\alpha}_i^\vee) = \delta_{i,0}$.

2.2.2. Affine and double affine Hecke algebras. Consider a non-degenerate Cartan datum with weight lattice X, Weyl group W and root system R. To simplify, we'll assume that $\alpha_i^\vee \notin 2X^\vee$ for each i. Fix a commutative ground ring A and a W-invariant function $t : R \to A^\times$. We abbreviate $t_i = t_{\alpha_i}$.

Definition 2.5. The *affine Hecke algebra* $\mathcal{H}_t(W, X)$ is the A-algebra generated by elements T_i satisfying the braid relations of $B(W)$, the quadratic relations $(T_i - t_i)(T_i + t_i^{-1}) = 0$, and elements x^λ, with $\lambda \in X$, satisfying the relations of the group algebra AX and the relation

$$T_i x^\lambda - x^{s_i(\lambda)} T_i = (t_i - t_i^{-1})(x^\lambda - x^{s_i(\lambda)})(1 - x^{\alpha_i})^{-1}.$$

The subalgebra of $\mathcal{H}_t(W, X)$ generated by the elements T_i is isomorphic to the ordinary Hecke algebra $\mathcal{H}_t(W)$. The induced representation $\mathrm{Ind}_{\mathcal{H}_t(W)}^{\mathcal{H}_t(W,X)}(\mathrm{triv})$ is called the *polynomial representation*.

Definition 2.6. Let W be finite. The *double affine Hecke algebra* associated with X is the A-algebra $\mathcal{H}_t(\widetilde{W}, \widetilde{X})$.

2.2.3. Geometric realization of double affine Hecke algebras. Let G be a *universal Chevalley group*, i.e., G is a connected, simple and simply connected algebraic group over $\mathbb{C}$. Let $(X, R, X^\vee, R^\vee)$ be the *root datum* of G. Consider the corresponding affine Cartan data with Weyl group $\widetilde{W}$ and weight lattices $\widetilde{X}$ or X_{aff}. The affine Hecke algebra $\mathcal{H}_t(\widetilde{W}, X_{\mathrm{aff}})$ associated with X_{aff} contains $\mathcal{H}_t(\widetilde{W}, \widetilde{X})$ as a subalgebra, and we have a semidirect decomposition $\mathcal{H}_t(\widetilde{W}, X_{\mathrm{aff}}) = A[x^{\pm\omega_0}] \ltimes \mathcal{H}_t(\widetilde{W}, \widetilde{X})$. Thus, the representation theory of $\mathcal{H}_t(\widetilde{W}, \widetilde{X})$ may be deduced from the representation theory of $\mathcal{H}_t(\widetilde{W}, X_{\mathrm{aff}})$ by Clifford theory. The element $q = x^\delta$ in $\mathcal{H}_t(\widetilde{W}, X_{\mathrm{aff}})$ is central.

Set $\mathbf{F} = \mathbb{C}((\varpi))$ and $\mathbf{O} = \mathbb{C}[[\varpi]]$. Let $G(\mathbf{F})$ be the *loop group* of G (this is an infinite-dimensional group ind-scheme whose set of $\mathbb{C}$-points is equal to the set of $\mathbf{F}$-points of G). Since G is simply connected, the isomorphism classes of central extensions of $G(\mathbf{F})$ by $\mathbb{G}_m$ are naturally in bijection with the W-invariant even, negative-definite symmetric bilinear forms $X^\vee \times X^\vee \to \mathbb{Z}$, see e.g., [39]. Let $\widetilde{G}$ be the central extension associated with the minimal such pairing. The multiplicative group $\mathbb{G}_m$ acts naturally on $G(\mathbf{F})$ by 'rotation of

the loop' and this action lifts to $\widetilde{G}$. We denote the corresponding semi-direct product by $G_{\rm aff}$. The weight lattice of $G_{\rm aff}$ is $X_{\rm aff}$, the weight lattice of $\widetilde{G}$ is $\widetilde{X}$.

The affine flag manifold $\mathcal{B}$ is an ind-scheme equal to the fpqc quotient $G_{\rm aff}/I$, where $I \subset G$ is the *Iwahori subgroup*. The set of $\mathbb{C}$-points of $\mathcal{B}$ is canonically identified with the set of all conjugates of the Lie algebra $\mathfrak{i}$ of I, under the adjoint action of $G_{\rm aff}$ on its Lie algebra. For each $\mathfrak{b} \in \mathcal{B}$, let $\mathfrak{b}_{\rm nil}$ denote its pro-nilpotent radical. Set $\mathcal{N} = \{(x, \mathfrak{b}) \in \mathfrak{i}_{\rm nil} \times \mathcal{B}\,;\, x \in \mathfrak{b}_{\rm nil}\}$, an ind-coherent ind-scheme, see [53]. The ind-scheme $\mathcal{N}$ admits a natural action of $G_{\rm aff} \times \mathbb{G}_m$, where $\mathbb{G}_m$ acts by dilatations on $\mathfrak{i}_{\rm nil}$.

Let $K^{I\times\mathbb{G}_m}(\mathcal{N})$ be the Grothendieck group of the abelian category of $I\times\mathbb{G}_m$-equivariant coherent sheaves on $\mathcal{N}$. From now on, we assume that the function t on R is constant, i.e., the Hecke algebra depends on a single parameter t. Set $A = \mathbb{Z}[t^{-1}, t]$. Using correspondences on $\mathcal{N}$ we prove the following, see [19, 53].

Theorem 2.7. *There is an A-algebra structure on $K^{I\times\mathbb{G}_m}(\mathcal{N})$, and $\mathcal{H}_t(\widetilde{W}, X_{\rm aff})$ is isomorphic to $K^{I\times\mathbb{G}_m}(\mathcal{N})$ as A-algebras.*

Consider the tori $\widetilde{T} = {\rm Spec}(\mathbb{C}\widetilde{X})$ and $T_{\rm aff} = {\rm Spec}(\mathbb{C}X_{\rm aff})$ in $\widetilde{G}$ and $G_{\rm aff}$. A character $\chi : A[X_{\rm aff}] \to \mathbb{C}$ is a triple (s, τ, ζ) where $\zeta = \chi(t)$, $\tau = \chi(q)$ and s is an element of $\widetilde{T}$. The pair (s, τ) can be viewed as an element of the group $T_{\rm aff}$. It acts on the ind-scheme $\mathcal{B}$ by left multiplication. Let $\mathcal{B}^{s,\tau}$ be the fixed points subset.

For each $x \in \mathfrak{g}_{\rm aff}$, the *affine Springer fiber* $\mathcal{B}_x$ is the ind-scheme $\mathcal{B}_x = \{\mathfrak{b} \in \mathcal{B}\,;\, x \in \mathfrak{b}_{\rm nil}\}$. Set $\mathcal{B}_x^{s,\tau} = \mathcal{B}^{s,\tau} \cap \mathcal{B}_x$

Let $G(s, \tau, x) \subset G_{\rm aff}$ be the subgroup of elements commuting with x and (s, τ), and let $A(s, \tau, x)$ be the group of connected components of $G(s, \tau, x)$. An element of $\widetilde{G}$ is called *semisimple* if it is conjugate to an element of $\widetilde{T}$. Let $\Sigma_{\tau,\zeta}$ be the set of triples (s, x, π) where $s \in \widetilde{G}$ is semisimple, $x \in \mathfrak{g}_{\rm aff}$ is *topologically nilpotent* in the sense of [32] with ${\rm ad}_{(s,\tau)}(x) = \zeta^{-1}x$, and π is an irreducible representation of $A(s, \tau, x)$ which is a constituent of the natural representation of $A(s, \tau, x)$ in $H_*(\mathcal{B}_x^{s,\tau}, \mathbb{C})$. Two triples in $\Sigma_{\tau,\zeta}$ are *equivalent* if they are conjugated by an element of $G_{\rm aff}$.

The A-algebra $\mathcal{H}_t(\widetilde{W}, X_{\rm aff})$ has a triangular decomposition $\mathcal{H}_t(\widetilde{W}, X_{\rm aff}) \simeq AX_{\rm aff} \otimes_A \mathcal{H}_t(W) \otimes_A AX_{\rm aff}^\vee$. Let $\mathcal{O}_{\tau,\zeta}(\widetilde{W}, X_{\rm aff})$ be the category of all finitely generated modules over $\mathbb{C}$ which are locally finite over $\mathbb{C}X_{\rm aff}$ and such that q, t act by scalar multiplication by τ, ζ. Using the theorem above, we get the following.

Theorem 2.8 ([53]). *Assume that τ is not a root of 1 and that $\tau^k \neq \zeta^{2m}$ for each $k, m > 0$. The isomorphism classes of simple objects in $\mathcal{O}_{\tau,\zeta}(\widetilde{W}, X_{\rm aff})$ are in bijection with the equivalence classes of triples (s, x, π) in $\Sigma_{\tau,\zeta}$.*

Remark 2.9.

(a) Theorem 2.7 is an affine version of the Kazhdan-Lusztig classification of the simple modules of affine Hecke algebras in [31], see also Ginzburg's proof in [10].

(b) Let $\tau^{\mathbb{Z}} \subset \mathbb{C}^\times$ be the subgroup generated by τ. By [5], there is a bijection from the set of all $\tau^{\mathbb{Z}} \ltimes G(\mathbf{F})$-conjugacy classes in $G(\mathbf{F})$ containing a point in $G(\mathbf{O})$ onto the set $\mathcal{M}(G)$ of isomorphism classes of *topologically trivial semistable principal G-bundles* over the elliptic curve $E = \mathbb{C}^\times/\tau^{\mathbb{Z}}$. We deduce that the set of equivalence classes in $\Sigma_{\tau,\zeta}$ can be described in terms of isomorphism classes of *Higgs bundles* over E, see [4] for details.

(c) The theorem above admits a *global version* which yields a representation of an analogue of the double affine Hecke algebra in the cohomology groups of some fibers of the Hitchin map associated with a smooth projective curve C, see [60]. These fibers are closed subschemes of the moduli space of parabolic Higgs bundles over C. If $C = \mathbb{P}^1$, equipped with its natural $\mathbb{G}_m$-action, the algebra above is closely related to the graded version of $\mathcal{H}_t(\widetilde{W}, X_{\mathrm{aff}})$ introduced by Cherednik, see Section 2.2.4 below.

2.2.4. Application to finite dimensional representations. Fix a Cartan datum with weight lattice X, Weyl group W, root system R and set of simple roots $\Phi = \{\alpha_i\}$. Fix a commutative ground ring A. We'll assume that $\alpha_i^\vee \notin 2X^\vee$ for each i. Let $\kappa : R \to A^\times$ be a W-invariant function. We abbreviate $\kappa_i = \kappa_{\alpha_i}$.

The affine Hecke algebra $\mathcal{H}_t(W, X)$ admits a *graded version*, which is the A-algebra $\mathcal{H}'_\kappa(W, X)$ generated by elements σ_w, with $\in W$, satisfying the relations of the group algebra AW and elements ξ^λ, with $\lambda \in X$, satisfying the relations of the group algebra AX and the relation $\sigma_i \xi^\lambda - \xi^{s_i(\lambda)}\sigma_i = \kappa_i(\alpha_i^\vee, \lambda)$.

The induced representation $\mathrm{Ind}_{AW}^{\mathcal{H}'_\kappa(W,X)}(\mathrm{triv})$ is called the *polynomial representation*. It is faithful, which permits to view $\mathcal{H}'_\kappa(W, X)$ as a subalgebra of the semi-direct product $AW \ltimes D(T)_{\mathrm{rat}}$, where $D(T)_{\mathrm{rat}}$ is the ring of differential operators with rational coefficients on the torus T associated with the lattice X, see [34].

Assume that the Cartan datum is of finite type. By [34] the irreducible representations of the affine Hecke algebra $\mathcal{H}_t(W, X)$ may be described in terms of irreducible representations of graded affine Hecke algebras associated with root subsystems of R. Similarly, by [52] the irreducible representations of the double affine Hecke algebra $\mathcal{H}_t(\widetilde{W}, \widetilde{X})$ may be described in terms of irreducible representations of some graded double affine Hecke algebras.

Assume also that the Cartan datum is associated with a universal Chevalley group G. According to Cherednik, there is an exact fully faithful functor which embeds the category of finite dimensional $H_c(W)$-modules into the category of finite dimensional $\mathcal{H}'_\kappa(\widetilde{W}, \widetilde{X})$ for a good choice of the parameters, see e.g., [52, sec. 2.3]. By [2, sec. 5.4], all finite dimensional representations of $\mathcal{H}'_\kappa(\widetilde{W}, \widetilde{X})$ may indeed be described in terms of representations of rational DAHA's associated with root subsystems of maximal rank via a version of the Borel-de Siebenthal algorithm.

Using this, it is proved in [52] that $H_c(W)$ acts on the homology groups $H_*(\mathcal{B}_x^{s,\tau}, \mathbb{C})$, whenever the affine Springer fiber $\mathcal{B}_x$ has a finite dimensional cohomology, yielding a classification of all finite dimensional modules which are quotient of the polynomial representation.

3. Quiver-Hecke algebras

3.1. Quantum groups.

3.1.1. Definition. Fix a non-degenerate Cartan datum $(X, \Phi, X^\vee, \Phi^\vee)$ with a *symmetrizable* generalized Cartan matrix $A = (a_{i,j})_{i,j\in I}$, i.e., there exist non-zero integers d_i such that $d_i\, a_{ij} = d_j\, a_{ji}$ for all i, j. The integers d_i are unique up to an overall common factor. They can be assumed positive. Then d_i is the *length* of the root α_i. Note that the generalized Cartan matrix of finite and affine type are all symmetrizable.

Assume that for each $i \in I$, there exists $\omega_i \in X$, a fundamental weight, such that $(\omega_i, \alpha_j^\vee) = \delta_{i,j}$ for all $j \in I$. Let q be an indeterminate and set $q_i = q^{d_i}$. For $m, n \in \mathbb{N}$ we set $[n]_i = (q_i^n - q_i^{-n})/(q_i - q_i^{-1})$, $[n]_i! = \prod_{k=1}^n [k]!$ and $\left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right]_i = [m]_i!/[m-n]_i![n]_i!$.

Definition 3.1. The *quantum group* associated with $(X, \Phi, X^\vee, \Phi^\vee)$ is the associative algebra U_q over $\mathbb{Q}(q)$ with 1 generated by e_i, f_i, $i \in I$, and l_h, $h \in X^\vee$, satisfying the following relations

$$
\begin{aligned}
&l_0 = 1,\ l_h l_{h'} = l_{h+h'},\\
&l_h e_i l_{-h} = q^{(h,\alpha_i)} e_i,\ l_h f_i l_{-h} = q^{-(h,\alpha_i)} f_i,\\
&e_i f_j - f_j e_i = \delta_{i,j}(k_i - k_i^{-1})(q_i - q_i^{-1}) \text{ where } k_i = l_{d_i \alpha_i^\vee},\\
&\textstyle\sum_{r=0}^{1-a_{i,j}} \left[\begin{smallmatrix} 1-a_{ij} \\ r \end{smallmatrix}\right]_i e_i^{1-a_{ij}-r} e_j e_i^r = 0 \text{ if } i \neq j,\\
&\textstyle\sum_{r=0}^{1-a_{i,j}} \left[\begin{smallmatrix} 1-a_{ij} \\ r \end{smallmatrix}\right]_i e_i^{1-a_{ij}-r} e_j e_i^r = 0 \text{ if } i \neq j.
\end{aligned}
$$

Let U_q^+, U_q^- be the subalgebra of U_q generated by e_i's, f_i's respectively, and let U_q^0 be the subalgebra of U_q generated by l_h with $h \in X^\vee$. Then we have a triangular decomposition $U_q = U_q^- \otimes U_q^0 \otimes U_q^+$, and the *weight space decomposition* $U_q^- = \bigoplus_{\alpha \in Q_+} U_{q,\alpha}^-$ where $U_{q,\alpha}^- = \{x \in U_q^- \,;\, l_h x\, l_{-h} = q^{-(h,\alpha)} x \text{ for any } h \in X^\vee\}$.

Fort each $\lambda \in X$ there exists a unique irreducible *highest weight module* $L_q(\lambda)$ *with highest weight* λ, i.e., a U_q-module $L_q(\lambda) = M$ with a *weight space decomposition* $M = \bigoplus_{\mu \in X} M_\mu$, where $M_\mu = \{v \in M \,;\, l_h v = q^{(h,\mu)} v \text{ for all } h \in X^\vee\}$, such that there is a non-zero vector $v_\lambda \in M_\lambda$ with $e_i v_\lambda = 0$ for all $i \in I$ and $M = U_q\, v_\lambda$.

Let $A = \mathbb{Z}[q, q^{-1}]$ and set $e_i^{(n)} = e_i^n/[n]_i!$, $f_i^{(n)} = f_i^n/[n]_i!$ for all $n \in \mathbb{N}$. We define the *A-form* U_A to be the A-subalgebra of U_A generated by $e_i^{(n)}$, $f_i^{(n)}$, l_h with $i \in I$, $n \in \mathbb{N}$ and $h \in X^\vee$. We define the *A-form* $L_A(\lambda)$ to be the A-submodule of $L_q(\lambda)$ given by $L_A(\lambda) = U_A v_\lambda$.

According to Lusztig and Kashiwara, see [35], [28], the quantum group U_q^- admits a *canonical basis*, which is an A-basis $\mathbf{B}$ of the A-module $U_A^- = U_A \cap U_q^-$ such that, for each integrable dominant weight $\lambda \in X_+$ the set $\{b\, v_\lambda \,;\, b\, v_\lambda \neq 0\}$ is an A-basis of $L_A(\lambda)$.

3.2. Quiver-Hecke algebras.

3.2.1. Definition. Fix a symmetrizable generalized Cartan matrix $A = (a_{i,j})_{i,j \in I}$ and a commutative graded ring $\mathrm{k} = \bigoplus_{n \in \mathbb{Z}} \mathrm{k}_n$ such that k_0 is a field and $\mathrm{k}_n = 0$ if $n < 0$. Let $c_{i,j,p,q} \in \mathrm{k}$ be of degree $-2d_i(a_{ij} + p) - 2d_j q$. Assume that $c_{i,j,-a_{i,j},0}$ is invertible. For $i, j \in I$ let $Q_{i,j} \in \mathrm{k}[u, v]$ be such that $Q_{ij}(u, v) = Q_{ji}(v, u)$, $Q_{ij}(u, v) = 0$ if $i = j$ and $Q_{ij}(u, v) = \sum_{p,q \geqslant 0} c_{i,j,p,q}\, u^p v^q$ if $i \neq j$.

Definition 3.2 ([32, 42]). The *quiver-Hecke algebra* of degree $n \geqslant 0$ associated with A and $(Q_{i,j})_{i,j \in I}$ is the associative algebra $R(n)$ over k generated by $e(\mathbf{i})$, x_k, σ_l with $\mathbf{i} \in I^n$, $k \in [1, n]$, $l \in [1, n)$ satisfying the following defining relations

$$
\begin{aligned}
&e(\mathbf{i})\, e(\mathbf{i}') = \delta_{\mathbf{i},\mathbf{i}'}\, e(\mathbf{i}),\ \textstyle\sum_{\mathbf{i}} e(\mathbf{i}) = 1,\\
&x_k\, x_l = x_l\, x_k,\ x_k\, e(\mathbf{i}) = e(\mathbf{i})\, x_k,\\
&\sigma_l\, e(\mathbf{i}) = e(s_l \mathbf{i})\, \sigma_l,\ \ \sigma_k\, \sigma_l = \sigma_l\, \sigma_k \ \text{ if } |k - l| > 1,\\
&\sigma_l^2\, e(\mathbf{i}) = Q_{i_l, i_{l+1}}(x_l, x_{l+1})\, e(\mathbf{i}),
\end{aligned}
$$

$$(\sigma_k x_l - x_{s_k(l)}\sigma_k)\, e(\mathbf{i}) = \begin{cases} -e(\mathbf{i}) & \text{if } l = k,\ i_k = i_{k+1}, \\ e(\mathbf{i}) & \text{if } l = k+1,\ i_k = i_{k+1}, \\ 0 & \text{otherwise,} \end{cases}$$

$$(\sigma_{k+1}\,\sigma_k\,\sigma_{k+1} - \sigma_k\,\sigma_{k+1}\,\sigma_k)\, e(\mathbf{i}) = \begin{cases} a_k(\mathbf{i})\, e(\mathbf{i}) & \text{if } i_k = i_{k+2}, \\ 0 & \text{otherwise,} \end{cases}$$

where $a_k(\mathbf{i}) = (Q_{i_k,i_{k+1}}(x_k, x_{k+1}) - Q_{i_{k+2},i_{k+1}}(x_{k+2}, x_{k+1}))/(x_k - x_{k+2})$. The algebra $R(n)$ admits a $\mathbb{Z}$-grading given by $\deg e(\mathbf{i}) = 0$, $\deg x_k e(\mathbf{i}) = 2d_{i_k}$ and $\deg \sigma_l e(\mathbf{i}) = -d_{i_k}\, a_{i_k,i_{k+1}}$.

Now, fix a non-degenerate Cartan datum $(X, \Phi, X^\vee, \Phi^\vee)$ with generalized Cartan matrix A. Fix a dominant integral weight $\lambda \in X_+$. Given $i \in I$, set $s = (\lambda, \alpha_i^\vee)$ and fix a monic polynomial $a_i^\lambda(u) = \sum_{r=0}^{s} c_{i,r}\, u^{s-r}$ in $\mathrm{k}[u]$ of degree s such that the element $c_{i,r} \in \mathrm{k}$ has the degree $2rd_i$.

Definition 3.3. The *cyclotomic quiver-Hecke algebra* of degree $n \geqslant 0$ associated with $R(n)$, the weight $\lambda \in X_+$ and the polynomials a_i^λ is the quotient $R^\lambda(n)$ of the $\mathbb{Z}$-graded algebra $R(n)$ by the homogeneous two-sided ideal generated by the elements $a_{i_1}^\lambda(x_1)e(\mathbf{i})$ for all $\mathbf{i} \in I^n$.

Let $\mathrm{proj}(R(n))$, $\mathrm{proj}(R^\lambda(n))$ be the categories of finitely generated projective graded modules over $R(n)$, $R^\lambda(n)$ respectively. Let $[\mathrm{proj}(R(n))]$, $[\mathrm{proj}(R^\lambda(n))]$ be their Grothendieck groups. They are A-modules, where the action of q is given by the grade shift functor. There are natural embeddings $R(m) \otimes R(n) \subset R(m+n)$. The induction and restriction functors equip the A-module $[\mathrm{proj}(R)] = \bigoplus_{n \geqslant 0} [\mathrm{proj}(R(n))]$ with the structure of a bialgebra.

Theorem 3.4 ([32]). *The A-module $[\mathrm{proj}(R)]$ is isomorphic to U_A^- as a bialgebra.*

Composing the induction and restriction functors with the functor

$$\mathrm{proj}(R(n)) \to \mathrm{proj}(R^\lambda(n)),\ M \mapsto R^\lambda(n) \otimes_{R(n)} M,$$

Kang and Kashiwara proved the following, see also [58].

Theorem 3.5 ([24]). *There is a natural structure of U_A-module on*

$$[\mathrm{proj}(R^\lambda)] = \bigoplus_{n \geqslant 0} [\mathrm{proj}(R^\lambda(n))]$$

such that it is isomorphic to $L_A(\lambda)$.

Remark 3.6. For each $\alpha \in Q_+$ of height n we write $I^\alpha = \{\mathbf{i} = (i_1, \ldots, i_n) \in I^n;\ \sum_{k=1}^n \alpha_{i_k} = \alpha\}$, $e(\alpha) = \sum_{\mathbf{i} \in I^\alpha} e(\mathbf{i})$, $R(\alpha) = e(\alpha)\, R(n)\, e(\alpha)$ and $R(\alpha)^\lambda = e(\alpha)\, R(n)^\lambda\, e(\alpha)$. Then, the isomorphisms in Theorems 3.4, 3.5 map $[\mathrm{proj}(R(\alpha))]$ and $[\mathrm{proj}(R(\alpha)^\lambda)]$ to the weight subspaces $U_{A,\alpha}^- = U_A^- \cap U_{q,\alpha}^-$ and $L_A(\lambda)_{\lambda-\alpha} = L_A(\lambda) \cap L_q(\lambda)_{\lambda-\alpha}$.

3.2.2. Geometric realization of quiver-Hecke algebras. Let $\Gamma = (I, \Omega)$ be a locally finite quiver without loops, with a vertex set I and an oriented edge set Ω. For each arrow $h \in \Omega$ let h', h'' denote the incoming and outgoing vertex. For $i, j \in \Omega$ with $i \neq j$, let $\Omega_{ij} = \{h \in \Omega\,;\, h' = i,\, h'' = j\}$ and $h_{ij} = \sharp\Omega_{ij}$. The matrix A given by $a_{ii} = 2$ and $a_{ij} = -h_{ij} - h_{ji}$ is a symmetric generalized Cartan matrix, and any symmetric generalized Cartan matrix can be realized in this way via a quiver.

Fix a finite dimensional I-graded $\mathbb{C}$-vector space $V = \bigoplus_{i\in I} V_i$. A *representation* of Γ in V is an element of $E_V = \bigoplus_{h\in\Omega} \mathrm{Hom}(V_{h'}, V_{h''})$. The groups $G_V = \prod_{i\in I} \mathrm{GL}(V_i)$ and $T_\Omega = (\mathbb{G}_m)^\Omega$ act on the space of representations E_V by $(g,t)\cdot(x_h)_{h\in\Omega} = (t_h\, g_{h''}\, x_h\, g_{h'}^{-1})_{h\in\Omega}$. We'll abbreviate $E = E_V$.

For $\mathbf{i} = (i_1, \dots, i_m) \in I^n$, the variety of *complete flags of type* $\mathbf{i}$ is a $\mathbb{C}$-scheme whose set of $\mathbb{C}$-points is the set $\mathcal{F}_\mathbf{i}$ of tuples $\phi = (0 = \phi_0 \subset \phi_1 \subset \cdots \subset \phi_m = V)$ where ϕ_k is an I-graded subspace such that $\dim(\phi_k/\phi_{k-1}) = \alpha_{i_k}$ for $k \in [1, m]$. The group G_V acts transitively on $\mathcal{F}_\mathbf{i}$ and T_Ω acts trivially.

For $x \in E$, a flag $\phi \in \mathcal{F}_\mathbf{i}$ is *x-stable* if $x_h(\phi_k \cap V_{h'}) \subset \phi_{k-1} \cap V_{h''}$ for each $h \in \Omega$, $k \in [1, m]$. Let $\widetilde{\mathcal{F}}_\mathbf{i}$ be the set of pairs $(x, \phi) \in E \times \mathcal{F}_\mathbf{i}$ such that ϕ is x-stable. The group $G_V \times T_\Omega$ acts diagonally on $\widetilde{\mathcal{F}}_\mathbf{i}$. Let $\pi_\mathbf{i} : \widetilde{\mathcal{F}}_\mathbf{i} \to E$ be the obvious projection. We write $\mathcal{L}(\mathbf{i}) = R\pi_{\mathbf{i}!}(\mathbb{C}_{\widetilde{\mathcal{F}}_\mathbf{i}}[2\dim \widetilde{\mathcal{F}}_\mathbf{i}])$, a semisimple complex in the bounded $G_V \times T_\Omega$-equivariant derived category $\mathbf{D}^b_{G_V\times T_\Omega}(E)$ of sheaves of $\mathbb{C}$-vector spaces on E. Set $\mathcal{L}(n) = \bigoplus_{\mathbf{i}\in I^n} \mathcal{L}(\mathbf{i})$.

The $\mathbb{Z}$-graded module $\mathrm{Ext}(\mathcal{L}(n), \mathcal{L}(n)) = \bigoplus_i \mathrm{Ext}^i(\mathcal{L}(n), \mathcal{L}(n))$ is a $\mathbb{Z}$-graded k-algebra for the Yoneda multiplication. We call it the *Yoneda algebra* of $\mathcal{L}(n)$. Here the extension groups are computed in the triangulated category $\mathbf{D}^b_{G_V\times T_\Omega}(E)$.

Now, take $\mathrm{k} = H^*_{T_\Omega}(\bullet, \mathbb{C})$ as the base ring. We have $\mathrm{k} = \mathbb{C}[\chi_h\,;\, h \in \Omega]$, where χ_h is the equivariant Chern class of the 1-dimensional representation of the h-th factor $\mathbb{C}^\times$ in T_Ω. Set $Q_{ij}(u, v) = \prod_{h\in\Omega_{ij}}(v - u + \chi_h) \prod_{h\in\Omega_{ji}}(u - v + \chi_h)$ if $i \neq j$ and $Q_{ij}(u, v) = 0$ if $i = j$. Let $R(n)$ be the quiver-Hecke algebra of degree $n \geqslant 0$ associated with the generalized Cartan matrix A and the matrix $(Q_{i,j})_{i,j\in I}$.

Theorem 3.7 ([42, 55]). *There is a $\mathbb{Z}$-graded* k*-algebra isomorphism*

$$R(n) \simeq \mathrm{Ext}(\mathcal{L}(n), \mathcal{L}(n))$$

which identifies the idempotent $e(\mathbf{i})$ with the projection to the direct summand $\mathcal{L}(\mathbf{i}) \subset \mathcal{L}(n)$.

Now, set $\mathrm{k} = \mathbb{C}$, viewed as the quotient of $\mathbb{C}[\chi_h\,;\, h \in \Omega]$ by the maximal ideal generated by all elements χ_h. Fix a non-degenerate Cartan datum $(X, \Phi, X^\vee, \Phi^\vee)$ with generalized Cartan matrix equal to the matrix A above. Let $\lambda \in X_+$ be a dominant weight and U_q, $R(n)$, $R(n)^\lambda$ be the corresponding quantum group, quiver-Hecke algebra and cyclotomic quiver-Hecke algebra. Using the previous theorem and Lusztig's geometric realization of the canonical bases, see [35], we obtain the following refinement of Theorem 3.4, 3.5.

Corollary 3.8.

(a) *There is a bialgebra isomorphism $[\mathrm{proj}(R)] \simeq U_A^-$ which identifies the canonical basis in the right hand side with the set of projective indecomposable self-dual modules in the left hand side.*

(b) *There is a U_A-module isomorphism $[\mathrm{proj}(R^\lambda)] \simeq L_A(\lambda)$ which identifies the canonical basis in the right hand side with the set of projective indecomposable self-dual modules in the left hand side.*

Remark 3.9.

(a) The construction above can be generalized to allow quiver with loops, arbitrary partial flags of a quiver representation, or sheaves of vector spaces over a field of positive characteristic, see [25, 36, 49] for details. Taking a more general version of flags in representations of the quiver yields a more general version of quiver-Hecke algebras called *weighted KLR algebras* by Webster in [59].

(b) If the polynomial $Q_{ij}(u,v)$ satisfies the conditions in Section 3.2.1, but does not satisfy the conditions in Section 3.2.2, then Corollary 3.8 may not hold, see [26] for details.

(c) It is not known how to construct the canonical basis of U_q^- using quiver-Hecke algebras when the Cartan matrix A is not symmetric. However, one can construct the canonical basis of U_q^- for any non symmetric A of finite or affine type by mimicking the construction in [35]. More precisely, let Γ be a quiver with a *compatible automorphism* γ, i.e., a pair of automorphisms $\gamma : I \to I$, $\gamma : \Omega \to \Omega$ such that $\gamma(h)' = \gamma(h')$, $\gamma(h)'' = \gamma(h'')$ for each $h \in \Omega$. Assume that γ is of finite order ℓ and that h', h'' are not in the same γ-orbit for each h.

Put $[I] = I/\gamma$, and for each $i \in I$ let $[i] \in [I]$ be its γ-orbit. Let $h_{[i],[j]}$ be the number of γ-orbits in the set $\Omega_{[i],[j]} = \{h \in \Omega\,;\, h' \in [i],\, h'' \in [j]\}$. Put $d_{[i]} = \sharp[i]$. The matrix A given by $a_{[i],[i]} = 2$ and $a_{[i],[j]} = -(h_{ij} + h_{ji})/d_{[i]}$ is a symmetrizable generalized Cartan matrix, and any generalized Cartan matrix of finite or affine type can be realized in this way. Let U_q be the corresponding quantum group.

For any element $\alpha = \sum_{i\in I} a_i\,\alpha_i$ in Q_+ such that $a_{\gamma(i)} = a_i$ for all i, the quiver-Hecke algebra $R(\alpha)$ admits a natural action of γ. This yields a *periodic functor* on the category $\mathrm{proj}(R(\alpha))$, with the terminology of [35, chap. 11]. Let $\mathcal{K}(\mathrm{proj}(R(\alpha))$ be the corresponding *twisted Grothendieck group*, as defined in [35, sec. 11.1.5].

Let $\mathcal{O} \subset \mathbb{C}$ be the subring consisting of all $\mathbb{Z}$-linear combinations of ℓ-th roots of 1. There is a bialgebra isomorphism $\mathcal{K}(\mathrm{proj}(R)) \simeq U_A^- \otimes_{\mathbb{Z}} \mathcal{O}$ which identifies the canonical basis of U^- with the set of projective indecomposable self-dual modules in $\mathcal{K}(\mathrm{proj}(R)) = \bigoplus_\alpha \mathcal{K}(\mathrm{proj}(R(\alpha))$. A similar construction gives a realization of the canonical basis of all integrable simple modules of U_q.

3.2.3. Affine Hecke algebras of type A : Ariki's theorem. Consider the Cartan datum of type A_{n-1} with weight lattice $X = \bigoplus_{i=1}^n \mathbb{Z}\,\epsilon_i \simeq \mathbb{Z}^n$ and simple roots given by $\alpha_i = \epsilon_i - \epsilon_{i+1}$ with $i \in [1,n)$. The Weyl group is the symmetric group $W = S_n$.

Set $A = \mathbb{Z}[t^{-1}, t]$. The *affine Hecke algebra* of $\mathrm{GL}(n)$ is the A-algebra $\mathrm{H}_t^A(n) = \mathcal{H}_t(W, X)$ which is generated by elements $T_1, \dots, T_{n-1}$ satisfying the braid relations of $B(S_n)$, the quadratic relations $(T_i - t)(T_i + t^{-1}) = 0$, and commuting elements $X_1^{\pm 1}, \dots, X_n^{\pm 1}$ satisfying the relation $T_i X_i T_i = X_{i+1}$, $T_i X_j = X_j T_i$ if $i \in [1,n)$ and $j \neq i, i+1$.

Fix an element $\zeta \in \mathbb{C}^\times$ such that $\zeta^2 \neq 1$. Set $\mathrm{H}_\zeta^A(n) = \mathrm{H}_t^A(n) \otimes_A \mathbb{C}$, where $\chi : A \to \mathbb{C}$ is the character such that $t \mapsto \zeta$. The group $\mathbb{Z}$ acts on $\mathbb{C}^\times$ by $\mathbb{Z} \ni n : i \mapsto i\zeta^{2n}$. Let I be a $\mathbb{Z}$-invariant subset in $\mathbb{C}^\times$.

Let $\mathrm{mod}(\mathrm{H}_\zeta^A(n))$ be the category of all finitely generated $\mathrm{H}_\zeta^A(n)$-modules. Let $\mathrm{mod}_I^{\mathrm{fd}}(\mathrm{H}_\zeta^A(n))$ be the full subcategory of all finite dimensional modules *of type* I, i.e., the finite dimensional modules M such that $M = \bigoplus_{\mathbf{i}\in I^n} M_{\mathbf{i}}$ where $M_{\mathbf{i}} = \{v \in M\,;\, (X_k - i_k)^r v = 0$ for any k and for some $r \gg 0\}$. Let $[\mathrm{mod}_I^{\mathrm{fd}}(\mathrm{H}_\zeta^A(n))]$ be the Grothendieck group

of $\mathrm{mod}_I^{\mathrm{fd}}(\mathrm{H}_\zeta^A(n))$, and set $[\mathrm{mod}_I^{\mathrm{fd}}(\mathrm{H}_\zeta^A)] = \bigoplus_{n\geqslant 0}[\mathrm{mod}_I^{\mathrm{fd}}(\mathrm{H}_\zeta^A(n))]$. The group $[\mathrm{mod}_I^{\mathrm{fd}}(\mathrm{H}_\zeta^A)]$ is a bialgebra where the product and coproduct are given by the induction and restriction with respect to the obvious inclusion $\mathrm{H}_\zeta^A(m)\otimes \mathrm{H}_\zeta^A(n)\subset \mathrm{H}_\zeta^A(m+n)$.

We can view I as a quiver without loops, with vertex set I and with an arrow $i\to i\zeta^2$ for each $i\in\mathbb{C}^\times$. Let U_q be the quantum group associated with this quiver and let U be its specialization at $q=1$. We define U^-, $L(\lambda)$ in the obvious way. The following was observed by Grojnowski. It follows from the Kazhdan-Lusztig and Ginzburg works [32], [10].

Theorem 3.10 ([22]). *The group $[\mathrm{mod}_I^{\mathrm{fd}}(\mathrm{H}_\zeta^A)]$ is isomorphic to U^- as a bialgebra. Under this isomorphism, the classes of the simple modules is identified with the dual canonical basis of U^-.*

If J is another $\mathbb{Z}$-invariant subset in $\mathbb{C}^\times$ such that $I\cap J=\emptyset$, then the induction yields an equivalence of categories $\mathrm{mod}_{I\cup J}^{\mathrm{fd}}(\mathrm{H}_\zeta^A(n))\simeq \mathrm{mod}_I^{\mathrm{fd}}(\mathrm{H}_\zeta^A(n))\times \mathrm{mod}_J^{\mathrm{fd}}(\mathrm{H}_\zeta^A(n))$. Hence it is enough to assume that I is a $\mathbb{Z}$-orbit. Then, the Cartan datum associated with the quiver I above, see Section 3.2.2, is either of type A_∞ or of type $A_e^{(1)}$ for some integer $e>0$. We deduce that $U\otimes_{\mathbb{Z}}\mathbb{C}$ is either the enveloping algebra of $\mathfrak{gl}_\infty$, if ζ is not a root of 1, or the enveloping algebra of the affine Kac-Moody algebra $\widehat{\mathfrak{sl}}_e$, if ζ is a e-th primitive root of 1.

The Hecke algebra of the complex reflection group $G(d,1,n)$ with parameters $\zeta, u_1,\dots,u_d$ is isomorphic to the quotient $\mathrm{H}_\zeta^{A,u}(n)$ of the affine Hecke algebra $\mathrm{H}_\zeta^A(n)$ by the *cyclotomic relation* $(X_1-u_1)\cdots(X_1-u_d)=0$.

Let $\mathrm{mod}_I^{\mathrm{fd}}(\mathrm{H}_\zeta^{A,u}(n))$ be the category of all finite dimensional $\mathrm{H}_\zeta^{A,u}(n)$-modules of type I. Let $[\mathrm{mod}_I^{\mathrm{fd}}(\mathrm{H}_\zeta^{A,u}(n))]$ be the Grothendieck group of $\mathrm{mod}_I^{\mathrm{fd}}(\mathrm{H}_\zeta^{A,u}(n))$, and set $[\mathrm{mod}_I^{\mathrm{fd}}(\mathrm{H}_\zeta^{A,u})]=\bigoplus_{n\geqslant 0}[\mathrm{mod}_I^{\mathrm{fd}}(\mathrm{H}_\zeta^{A,u}(n))]$.

Assume that $u_p=t^{2s_p}$, with $s_p\in\mathbb{Z}$ for each $p\in[1,d]$. Let $\lambda=\sum_{p=1}^d\omega_{u_p}$, where ω_i is the i-th fundamental weight of the Cartan datum associated with I. Let $\mathrm{H}_\zeta^{A,\lambda}(n)$ denote the corresponding cyclotomic Hecke algebra. Composing the induction and restriction with the functor $\mathrm{mod}^{\mathrm{fd}}(\mathrm{H}_\zeta^{A,\lambda})\to\mathrm{mod}^{\mathrm{fd}}(\mathrm{H}_\zeta^A)$ induced by the obvious surjective algebra homomorphism $\mathrm{H}_\zeta^A\to\mathrm{H}_\zeta^{A,\lambda}$, Ariki has obtained the following, yielding a proof of a conjecture of Lascoux-Leclerc-Thibon.

Theorem 3.11 ([1]). *There is a natural structure of U-module on $[\mathrm{mod}_I^{\mathrm{fd}}(\mathrm{H}_\zeta^{A,\lambda})]$ such that $[\mathrm{mod}_I^{\mathrm{fd}}(\mathrm{H}_\zeta^{A,\lambda})]$ is isomorphic to the dual of the integrable highest weight U-module $L(\lambda)$ with highest weight λ. Under this isomorphism, the classes of the simple modules is identified with the dual canonical basis of $L(\lambda)$.*

Let $R(n)$, $R^\lambda(n)$ be the quiver-Hecke algebra and the cyclotomic quiver-Hecke algebra of degree n associated with the Cartan datum of the quiver I and the dominant integral weight λ.

Now, we specialize χ_h to 0 for all $h\in\Omega$. Let $\mathrm{mod}^0(R(n))$ be the category of finitely generated modules over $R(n)$ such that $x_k e(\mathbf{i})$ acts locally nilpotently for each $k\in[1,n]$ and $\mathbf{i}\in I^n$.

The relation between Theorems 3.10, 3.11 and Theorems 3.4, 3.5 is the following. It is a consequence of the theory of *intertwiners* of affine Hecke algebras developed in [34] to prove that affine Hecke algebras and their graded versions are Morita equivalent, see Section 2.2.4.

Theorem 3.12 ([9, 42]). *For each λ, n the following hold*

(a) *there is an equivalence of categories* $\mathrm{mod}_I(\mathrm{H}_\zeta^A(n)) \simeq \mathrm{mod}^0(R(n))$,

(b) *there is an algebra isomorphism* $\mathrm{H}_\zeta^{A,\lambda}(n) \simeq R^\lambda(n)$.

Remark 3.13.

(a) Historically, Theorems 3.10, 3.11 have been proved before Theorems 3.4, 3.5 and have been one of the major motivation for the discovery of quiver-Hecke algebras.

(b) For each λ as above, Dipper-James-Mathas have defined in [12] some *cyclotomic ζ-Schur algebras* $\mathrm{S}_\zeta^{A,\lambda}(n)$ with a Schur functor $\mathrm{S}_\zeta^{A,\lambda}(n) \to \mathrm{H}_\zeta^{A,\lambda}(n)$ which is a highest weight cover in the sense of Section 2.1.2, see also [38], [41]. Theorem 3.11 has been extended conjecturally by Yvonne in [61] in the following way.

Assume that ζ is a e-th primitive root of 1. Then, there should be a natural structure of U-module on $[\mathrm{mod}_I^{\mathrm{fd}}(\mathrm{S}_\zeta^{A,\lambda})] = \bigoplus_{n\geqslant 0}[\mathrm{mod}_I^{\mathrm{fd}}(\mathrm{S}_\zeta^{A,\lambda}(n))]$ such that $[\mathrm{mod}_I^{\mathrm{fd}}(\mathrm{S}_\zeta^{A,\lambda})]$ is isomorphic to the level d Fock space $F(s)$ of multicharge $s = (s_1, \dots, s_d)$. Under this isomorphism, the classes of the simple modules is identified with the dual canonical basis of $F(s)$. For $d = 1$ this conjecture was formulated previously by Lascoux-Leclerc-Thibon and proved in [51]. For arbitrary d it follows from the results in Section 4.2 below.

3.2.4. Affine Hecke algebras of types B, C : the conjecture of Enomoto-Kashiwara. Fix a non-degenerate Cartan datum with a symmetric generalized Cartan matrix $A = (a_{i,j})_{i,j\in I}$. Fix an involution θ of the set I such that $a_{\theta(i),\theta(j)} = a_{ij}$ for all $i, j \in I$.

Definition 3.14 ([15]). Let $\mathcal{B}_\theta$ be the associative $\mathbb{Q}(q)$-algebra with 1 generated by e_i, f_i, $i \in I$, satisfying the usual Serre relations and by commuting invertible elements l_i, $i \in I$, satisfying the relations $l_{\theta(i)} = l_i$ and

$$l_j e_i l_j^{-1} = q^{a_{ij}+a_{i\theta(j)}} e_i, \quad l_j f_i l_j^{-1} = q^{-a_{ij}-a_{i\theta(j)}} f_i, \quad e_i f_j = q^{-a_{i,j}} f_j e_i + \delta_{ij} + \delta_{\theta(i),j} l_i.$$

Lemma 3.15 ([15]). *For each dominant integral weight $\lambda = \sum_{i\in I} \lambda_i\, \omega_i$ in X_+, there is a unique irreducible $\mathcal{B}_\theta$-module $V_\theta(\lambda)$ generated by a vector v_λ such that $\{v \in V_\theta(\lambda)\,;\, e_i v = 0\} = \mathbb{Q}(q)\, v_\lambda$ and $l_i v_\lambda = q^{\lambda_i+\lambda_{\theta(i)}} v_\lambda$ for all $i \in I$.*

Now, set $A = \mathbb{Z}[t_0^{\pm 1}, t_1^{\pm 1}, t_2^{\pm 1}]$. The *affine Hecke algebra* of type C_n is the A-algebra $\mathrm{H}_t^C(n)$ which is generated by elements $T_0, \dots, T_{n-1}$ satisfying the braid relations of type B_n, i.e., the elements $T_i, \dots, T_{n-1}$ satisfy the braid relations of $B(S_n)$ and $T_0T_1T_0T_1 = T_1T_0T_1T_0$, the quadratic relations $(T_0 - t_0)(T_0 + t_1^{-1}) = 0$ and $(T_i - t_2)(T_i + t_2^{-1}) = 0$ if $i \neq 0$, and commuting elements $X_1^{\pm 1}, \dots, X_n^{\pm 1}$ satisfying the relations $T_0X_1^{-1} - X_1T_0 = (t_1^{-1} - t_0)X_1 + t_0t_1^{-1} - 1$, $T_iX_iT_i = X_{i+1}$ and $T_iX_j = X_jT_i$ if $i \neq 0$, $j \neq i, i+1$.

Fix $\zeta_0, \zeta_1, \zeta_2 \in \mathbb{C}^\times$ with $\zeta_2^2 \neq 1$. Set $\mathrm{H}_\zeta^C(n) = \mathrm{H}_t^C(n) \otimes_A \mathbb{C}$, where $\chi : A \to \mathbb{C}$ is the character such that $t_i \mapsto \zeta_i$ for $i = 0, 1, 2$. The semi-direct product $\{1, -1\} \ltimes \mathbb{Z}$ acts on $\mathbb{C}^\times$ by $(\epsilon, n) : i \mapsto i^\epsilon \zeta_2^{2n}$. Let I be a $\{1, -1\} \ltimes \mathbb{Z}$ invariant subset in $\mathbb{C}^\times$. As above, we may assume that I is a $\{1, -1\} \ltimes \mathbb{Z}$-orbit. Let $\mathrm{mod}_I(\mathrm{H}_\zeta^C(n))$ be the category of all finitely generated $\mathrm{H}_\zeta^C(n)$-modules and let $\mathrm{mod}_I^{\mathrm{fd}}(\mathrm{H}_\zeta^C(n))$ be the full subcategory of all finite dimensional modules of type I (as above). Let $[\mathrm{mod}_I^{\mathrm{fd}}(\mathrm{H}_\zeta^C(n))]$ be the Grothendieck group of $\mathrm{mod}_I^{\mathrm{fd}}(\mathrm{H}_\zeta^C(n))$, and set $[\mathrm{mod}_I^{\mathrm{fd}}(\mathrm{H}_\zeta^C)] = \bigoplus_{n\geqslant 0}[\mathrm{mod}_I^{\mathrm{fd}}(\mathrm{H}_\zeta^C(n))]$. The group $[\mathrm{mod}_I^{\mathrm{fd}}(\mathrm{H}_\zeta^C)]$

is a module over the bialgebra $[\mathrm{mod}_I^{\mathrm{fd}}(\mathrm{H}_\zeta^A)]$. The product and coproduct are given by the induction and restriction with respect to the obvious inclusion $\mathrm{H}_\zeta^C(m)\otimes\mathrm{H}_\zeta^A(n)\subset\mathrm{H}_\zeta^A(m+n)$.

Now, recall the following standard definition.

Definition 3.16 ([13]). A *quiver with involution* (or *symmetric quiver*) is a pair (Γ,θ), where Γ is a quiver and θ is an *involution* of Γ, i.e., θ consists of a pair of involutions of the sets I,Ω such that $\theta(h)''=\theta(h')$, $\theta(h)'=\theta(h'')$ for each $h\in\Omega$.

We can view the subset $I\subset\mathbb{C}^\times$ as a quiver with involution without loops, with vertex set I, with an arrow $i\to i\,\zeta^2$ for each $i\in\mathbb{C}^\times$ and with the involution $\theta: i\mapsto i^{-1}$. Assume that A is the generalized Cartan matrix associated with I, see Section 3.2.2, and set $\lambda_i=\delta_{i,\zeta_1}+\delta_{i,-\zeta_0}$. The following was conjectured in [15] and proved in [56].

Theorem 3.17 ([56]). *Assume that $1,-1\notin I$. Then, the $\mathcal{B}_\theta$-module $V_\theta(\lambda)$ has a canonical basis, the vector space $[\mathrm{mod}_I^{\mathrm{fd}}(\mathrm{H}_\zeta^{B,\lambda})]\otimes_{\mathbb{Z}}\mathbb{Q}$ is isomorphic to a specialization of $V_\theta(\lambda)$ at $q=1$, and the classes of the simple modules are identified with the dual canonical basis of $V_\theta(\lambda)$ at $q=1$.*

Remark 3.18.

(a) An analogous construction in type D has been given in [29]. The corresponding conjectures are proved in [46].

(b) If $1\in I$ or $-1\in I$ then $[\mathrm{mod}_I^{\mathrm{fd}}(\mathrm{H}_\zeta^{B,\lambda})]\otimes_{\mathbb{Z}}\mathbb{Q}$ is no longer irreducible as a $\mathcal{B}_\theta$-module.

3.2.5. Quiver-Hecke algebras of types B, C. The main ingredient in the proof of Theorem 3.17 is a $\mathbb{Z}$-graded algebra which is an analogue, for affine Hecke algebras of type B,C, of quiver-Hecke algebras.

The Weyl group of type C is the semidirect product $W=S_n\ltimes\{-1,1\}^n$. For $k\in[1,n]$ let $\varepsilon_k\in W$ be -1 placed at the k-th spot.

Fix a set I with an involution θ. The group W acts on a tuple $\mathbf{i}=(i_{1-n},\dots,i_{n-1},i_n)$ of I^{2n} in the obvious way : the reflection $s_l\in S_n$ switches the entries i_l, i_{l+1} and the entries i_{1-l}, i_{-l}, while ε_k switches the entries i_k, i_{1-k}. This action preserves the subset $I^{\theta,n}=\{\mathbf{i}\in I^{2n}\,;\,\theta(i_k)=i_{1-k}\text{ for all }k\}$. The group W also acts on algebra $P=\mathrm{k}[x_1,\dots,x_n]$ so that s_l switches x_l and x_{l+1}, while ε_k switches x_k and $-x_k$.

Now, fix $\zeta_2\in\mathbb{C}^\times\setminus\{-1,1\}$. Let $I\subset\mathbb{C}^\times$ be a $\{1,-1\}\ltimes\mathbb{Z}$ invariant subset. We view it as a quiver with involution without loops, with vertex set I, with an arrow $i\to i\,\zeta_2^2$ for each $i\in\mathbb{C}^\times$, and with the involution given by $\theta: i\mapsto i^{-1}$, compare Section 3.2.4. Assume that A is the generalized Cartan matrix associated with I, see Section 3.2.2. Fix a dominant integral weight $\lambda=\sum_{i\in I}\lambda_i\,\omega_i$ in X_+.

Definition 3.19. The *quiver-Hecke algebra* of degree n associated with Γ, θ, λ is the subalgebra $R(n)^{\theta,\lambda}\subset\mathrm{End}_{\mathrm{k}}(\bigoplus_{\mathbf{i}\in I^{\theta,n}}Pe(\mathbf{i}))$ generated by the linear operators $e(\mathbf{i})$, x_k, σ_l with $\mathbf{i}\in I^{\theta,n}$, $k\in[1,n]$, $l\in[0,n)$ such that $x_k(fe(\mathbf{i}))=x_kfe(\mathbf{i})$ and

$$\sigma_0(fe(\mathbf{i}))=\begin{cases}(2x_1)^{-1}(\varepsilon_1f-f)\,e(\mathbf{i}) & \text{if } i_1=i_0,\\ (x_1)^{\lambda_{i_0}}\varepsilon_1f\,e(\varepsilon_1\mathbf{i}) & \text{if } i_l=\zeta_2^2i_{l+1},\\ 0 & \text{otherwise,}\end{cases}$$

$$\sigma_l(fe(\mathbf{i}))=\begin{cases}(x_{l+1}-x_l)^{-1}(s_lf-f)\,e(\mathbf{i}) & \text{if } i_l=i_{l+1},\\ (x_{l+1}-x_l)f\,e(s_l\mathbf{i}) & \text{if } i_l=\zeta_2^2i_{l+1},\\ 0 & \text{otherwise,}\end{cases}$$

where $l \neq 0$. The k-algebra $R(n)^{\theta,\lambda}$ is $\mathbb{Z}$-graded, the grading being given by $\deg e(\mathbf{i}) = 0$, $\deg x_k e(\mathbf{i}) = 2d_{i_k}$, $\deg \sigma_0 e(\mathbf{i}) = \lambda_{i_0} + \lambda_{i_1} - 2\delta_{i_1,i_0}$ and $\deg \sigma_l e(\mathbf{i}) = -d_{i_k}\, a_{i_k,i_{k+1}}$.

The algebra $R(n)^{\theta,\lambda}$ has a presentation similar to the one in [56]. It also admits a geometric realization. More precisely, fix $\sigma \in \{-1, 1\}$. For any representation $x \in E_V$ of Γ, let x° be the representation on the I-graded vector space V° such that $V_i^\circ = V_{\theta(i)}^*$ and $x_h^\circ = \sigma\, x_{\theta(h)}^*$.

Definition 3.20 ([13]). A *σ-orthogonal (resp. a σ-symplectic)* representation of (Γ, θ) in V is the datum of a representation $x \in E_V$ with an isomorphism $x \to x^\circ$ such that the underlying isomorphism $V \to V^\circ$ defines a symmetric (resp. antisymmetric) non-degenerate bilinear form $V \times V \to \mathbb{C}$.

Then, the $\mathbb{Z}$-graded algebra $R(n)^{\theta,\lambda}$ is isomorphic to the Yoneda algebra of a complex of sheaves on the space $E_V^{\theta,\lambda}$ consisting of 1-symplectic representations of Γ which admit a *λ-framing* in the sense of [56, sec. 4.4].

The relation between $R(n)^{\theta,\lambda}$ and affine Hecke algebras is the following. Fix elements $\zeta_0, \zeta_1 \in \mathbb{C}^\times$. Let $\mathrm{H}_\zeta^C(n)$ be the corresponding affine Hecke algebra of type C_n. Set $\lambda_i = \delta_{i,\zeta_1} + \delta_{i,-\zeta_0}$.

We specialize χ_h to 0 for all $h \in \Omega$. Let $\mathrm{mod}^0(R(n)^{\theta,\lambda})$ be the category of finitely generated modules over $R(n)^{\theta,\lambda}$ such that $x_k e(\mathbf{i})$ acts locally nilpotently for each $k \in [1, n]$ and $\mathbf{i} \in I^{\theta,n}$. We have the following analogue of Theorem 3.12.

Theorem 3.21 ([56]). *There is an equivalence of categories*

$$\mathrm{mod}_I(\mathrm{H}_\zeta^C(n)) \simeq \mathrm{mod}^0(R(n)^{\theta,\lambda}).$$

Remark 3.22.

(a) Elements of the *2-exotic nilpotent cone* in [30] can be identified with nilpotent λ-framed 1-symplectic representations as above. Theorem 3.21 and the geometric realization of $R(n)^{\theta,\lambda}$ yield another proof of Kato's theorem which parametrizes the simple $\mathrm{H}_\zeta^C(n)$-modules via the 2-exotic nilpotent cone.

(b) In [56] the $\mathbb{Z}$-graded algebra $R(n)^{\theta,\lambda}$ is realized as the Yoneda algebra of a complex of sheaves on the space of (-1)-orthogonal representations of (Γ, θ) with a λ-framing. The space of (-1)-orthogonal representations of (Γ, θ) is also used in [14] and yields a geometric construction of some simple $\mathrm{H}_\zeta^C(n)$-modules.

4. Categorical representations and rational DAHA's

4.1. Definition. Fix a non-degenerate Cartan datum $(X, \Phi, X^\vee, \Phi^\vee)$ with a symmetrizable generalized Cartan matrix. Let U_q be the quantum group associated with $(X, \Phi, X^\vee, \Phi^\vee)$.

Let $\{c_{i,j,p,q}\}$ be a family of indeterminates with $i \neq j \in I$ and $p, q \in [0, -a_{ij})$ such that $c_{i,j,p,q} = c_{j,i,q,p}$. Set $\mathrm{k} = \mathbb{Z}[c_{i,j,p,q}][c_{i,j,-a_{i,j},0}^{-1}]$ and consider the polynomials given by $Q_{ij}(u, v) = \sum_{p,q \geqslant 0} c_{i,j,p,q}\, u^p v^q$ if $i \neq j$, and $Q_{ij}(u, v) = 0$ if $i = j$. Let $R(n)$ be the quiver-Hecke k-algebra associated with $Q_{ij}(u, v)$.

Finally, let Z be a noetherian commutative k-algebra and $\mathcal{C}$ be a Z-linear category whose Hom's are finitely generated Z-modules. We'll abbreviate $R(n)$ for the Z-algebra $Z \otimes_{\mathrm{k}} R(n)$.

Definition 4.1. An *integrable representation of U on $\mathcal{C}$* is the datum of a decomposition $\mathcal{C} = \bigoplus_{\mu\in X} \mathcal{C}_\mu$, an adjoint pair of Z-linear functors (F_i, E_i) with $E_i : \mathcal{C}_\mu \to \mathcal{C}_{\mu+\alpha_i}$, $F_i : \mathcal{C}_\mu \to \mathcal{C}_{\mu-\alpha_i}$ and elements $x_i \in \mathrm{End}(F_i)$, $\sigma_{ij} \in \mathrm{Hom}(F_iF_j, F_jF_i)$ satisfying the following conditions

(a) E_i is isomorphic to a left adjoint of F_i,

(b) E_i, F_i are locally nilpotent,

(c) the relations of the quiver-Hecke algebra $R(n)$ hold for x_i and σ_{ij},

(d) given $\mu \in X$, there are isomorphisms of functors $(E_iF_i)|_{\mathcal{C}_\mu} \simeq (F_iE_i)|_{\mathcal{C}_\mu} \oplus \mathrm{Id}_{\mathcal{C}_\mu}^{(\mu,\alpha_i^\vee)}$ if $(\mu, \alpha_i^\vee) \geqslant 0$, and $(F_iE_i)|_{\mathcal{C}_\mu} \simeq (E_iF_i)|_{\mathcal{C}_\mu} \oplus \mathrm{Id}_{\mathcal{C}_\mu}^{-(\mu,\alpha_i^\vee)}$ if $(\mu, \alpha_i^\vee) \leqslant 0$.

An integrable representation of U on $\mathcal{C}$ yields a representation of U on the Grothendieck group $[\mathcal{C}]$ of $\mathcal{C}$. We'll say that the representation of U on $\mathcal{C}$ *categorifies* the representation of U on $[\mathcal{C}]$.

Now, fix a dominant integral weight $\lambda \in X_+$. Let c_{ir} be a family of indeterminates with $i \in I$ and $r \in (0, s]$, where $s = (\lambda, \alpha_i^\vee)$. Set $Z = \mathrm{k}[c_{ir}]$ and $c_{i0} = 1$. Consider the monic polynomial in $Z[u]$ given by $a_i^\lambda(u) = \sum_{r=0}^n c_{ir}\, u^{s-r}$. Let $R(n)^\lambda$ be the cyclotomic quiver-Hecke Z-algebra associated with the quiver-Hecke Z-algebra $R(n)$, the dominant weight λ and the polynomials a_i^λ.

Let $\mathrm{proj}^0(R^\lambda(n))$ be the Z-linear category of finitely generated projective modules over $R^\lambda(n)$. We abbreviate $\mathcal{L}(\lambda) = \bigoplus_{n\geqslant 0} \mathrm{proj}^0(R^\lambda(n))$. Then it is proved in[24],[27],[58] that the induction and restriction yield functors E_i, F_i on $\mathcal{L}(\lambda)$ which satisfy the axioms above. Hence, Theorem 3.5 can be rephrased as follows.

Theorem 4.2. *The induction and restriction functors yield a categorification of the integrable U-module $L(\lambda)$ on $\mathcal{L}(\lambda)$.*

We have the following unicity result.

Theorem 4.3 ([42]). *Given an integrable categorical representation of U on a Z-linear category $\mathcal{C}$ which is idempotent-closed, and an object $M \in \mathcal{C}_\lambda$ such that $\mathrm{End}(M) = Z$ and $E_i(M) = 0$ for all i, there is a fully faithful functor $\mathcal{L}(\lambda) \otimes_{Z_\lambda} Z \to \mathcal{C}$ taking the module Z_λ over $R^\lambda(0) \simeq Z_\lambda$ to M.*

Remark 4.4.

(a) If $\mathcal{C}$ is indeed an abelian category, then the notion of a categorical representation on $\mathcal{C}$ can be formulated in a simpler way, see e.g., [43].

(b) Using the A-algebra U_A instead of the ring U, and using a *$\mathbb{Z}$-graded category* (i.e., a category enriched in $\mathbb{Z}$-graded modules) instead of the abelian category $\mathcal{C}$, we define in a similar way a notion of *categorification* of the integrable U_A-module $L_A(\lambda)$ such that the action of q is given by the grade shift functor.

(c) A proof of the bi-adjointness of the functors E_i, F_i is given in [27, 58].

In the next section we consider two remarkable applications of categorical representations for RDAHA's.

4.2. Categorical representations and CRDAHA's. We fix the integer $d \geqslant 1$ and we allow n to vary in $\mathbb{N}$. Consider the categories $\mathcal{O}_c(d,n)$'s introduced in Section 2.1.3. Since the set S/W has exactly d elements, we can view the parameter c of the algebra $H_c(d,n)$ as a d-tuple. We'll assume that this parameter c is *integral*, which means that the parameter $t = \exp(2i\pi c)$ of the cyclotomic Hecke algebra $\mathcal{H}_t(d,n)$ is a tuple $(q, q_1, \dots, q_d)$ where q is a primitive e-th root of 1 and $q_p = q^{s_p}$ for some integers $e, s_1, \dots s_d$ with $e > 0$. Here $(q_1, \dots, q_d)$ is determined modulo the diagonal action of $\mathbb{C}^\times$.

Let $F(s)$ be Fock space of multicharge $s = (s_1, \dots, s_d)$, which was introduced in Section 2.1.3. It is a level d integrable module over the affine Kac-Moody algebra of $\widehat{\mathfrak{sl}}_e$ which can be defined as follows.

Set $N = s_1 + \cdots + s_d$. Let $\ell \in [0, d)$ be the residue class of N modulo d. Let $L(\omega_\ell)$ be the ℓ-th *fundamental module* of the Lie algebra $\widehat{\mathfrak{gl}}_d$, i.e., the simple integrable module with highest weight the ℓ-th fundamental weight ω_ℓ. Recall that $\widehat{\mathfrak{gl}}_d$ is a central extension of the Lie algebra $\mathfrak{gl}_d[\varpi, \varpi^{-1}]$. The assignment $\varpi \mapsto \varpi^e$ yields a Lie algebra endomorphism of $\widehat{\mathfrak{gl}}_d$ which multiplies the central element by e. Pulling back $L(\omega_\ell)$ by this endomorphism we get a level e integrable representation of $\widehat{\mathfrak{gl}}_d$ on $L(\omega_\ell)$, which is no longer simple but only semisimple. This level e representation admits a commuting level d action of the affine Kac-Moody algebra of $\widehat{\mathfrak{sl}}_e$. The Fock space $F(s)$ is the weight space of $L(\omega_\ell)$ associated with some weight γ_s of the level e action of $\widehat{\mathfrak{sl}}_d \subset \widehat{\mathfrak{gl}}_d$ which depends on the d-tuple s. Hence, it is a level d module of $\widehat{\mathfrak{sl}}_e$.

Theorem 4.5 ([45]). *The induction and restriction functors yield a categorification of the integrable module $F(s)$ on $\bigoplus_{n\geqslant 0} \mathcal{O}_c(d,n)$.*

The next step is to identify the simple modules in $\mathcal{O}_c(d,n)$ with some *canonical basis* in $F(s)$ and to compute their dimension, for which a conjecture was formulated in [41]. This follows from theorem 4.7 below.

Another remarkable example of categorical representation, inspired by [6, 11], is the following. Assume that $s_1, \dots, s_d$ are non negative. We can consider the *parabolic category* $\mathcal{O}$ of the affine Lie algebra $\widehat{\mathfrak{gl}}_N$, denoted by $\mathcal{O}(s)^{\widehat{\mathfrak{gl}}_N}$, which consists of modules of level $-e-N$ in the usual category O of $\widehat{\mathfrak{gl}}_N$ which are integrable with respect to the parabolic subalgebra associated with the blocks decomposition $N = s_1 + \cdots + s_d$.

Theorem 4.6 ([54]). *The Kazdhan-Lusztig fusion product of $\widehat{\mathfrak{gl}}_N$-modules yields a categorical representation of $\widehat{\mathfrak{sl}}_e$ on $\mathcal{O}(s)^{\widehat{\mathfrak{gl}}_N}$.*

The categorical representations of $\widehat{\mathfrak{sl}}_e$ on $\bigoplus_{n\geqslant 0} \mathcal{O}_c(d,n)$ and $\mathcal{O}(s)^{\widehat{\mathfrak{gl}}_N}$ are different : the first one categorify an integrable module of level d and the second one an integrable module of level 0. However, using them one proves the following (see also [33] for a closely related result), which was conjectured in [55].

Theorem 4.7 ([44]). *Assume that $s_p \geqslant n$ for each $p \in [1,d]$. Then there is a fully faithful exact functor $\mathcal{O}_c(d,n) \subset \mathcal{O}(s)^{\widehat{\mathfrak{gl}}_N}$.*

Note that this theorem implies that the category $\mathcal{O}_c(d,n)$ is Koszul by [47], and it describes its Koszul dual via the *level-rank* duality of I. Frenkel.

Another remarkable application of categorical representations is the following. Let $\mathfrak{H}$ be the Heisenberg algebra. It is an infinite dimensional Lie algebra. The inclusion of the center

$\mathbb{C} \subset \mathfrak{gl}_d$ yields an inclusion $\mathfrak{H} \subset \widehat{\mathfrak{gl}}_d$. The Lie algebra $\mathfrak{H}$ acts on $F(s)$. This action lifts to an action of $\mathfrak{H}$ on the category $\bigoplus_{n\geqslant 0} \mathcal{O}_c(d,n)$. Using the latter and the level-rank duality, which yields an explicit description of the decomposition of $L(\omega_\ell)$ as a $\widehat{\mathfrak{sl}}_d \times \mathfrak{H} \times \widehat{\mathfrak{sl}}_e$-module of level (e, de, d), one proves the theorem below which was conjecture by Etingof [17].

Since the parameter c is a d-uple of complex numbers, it can be identified with a weight of $\widehat{\mathfrak{sl}}_d$. One defines a Lie subalgebra $\mathfrak{a} \subset \widehat{\mathfrak{sl}}_d$ which is generated by the weight vectors of $\widehat{\mathfrak{sl}}_d$ which are *integral with respect to the weight* c, see [17] for details. Let $L^{\mathfrak{a}}$ be the $\mathfrak{a}$-submodule of the fundamental module $L(\omega_0)$ of $\widehat{\mathfrak{gl}}_d$ which is generated by the sum of all *extremal* weight subspaces of $L(\omega_0)$. Let δ be the smallest positive imaginary root.

Theorem 4.8 ([48]). *The number of isomorphism classes of finite dimensional irreducible $H_c(d,n)$-modules is equal to the dimension of the weight subspace of $L^{\mathfrak{a}}$ associated with the weight $\omega_0 - n\delta$.*

The Etingof conjecture is more general and yields indeed a characterization of the whole filtration of the category $\mathcal{O}_c(d,n)$ by the dimension of the support of the modules, see [48] for the proof. It extends also to a larger family of algebras than the $H_c(d,n)$'s, see [3]. These algebras are not associated in any natural way to Hecke algebras any more. They are called *symplectic reflection algebras* and have been introduced in [18].

Acknowledgements. This research was partially supported by the ANR grants ANR-10-BLAN-0110, ANR-13-BS01-0001-01.

References

[1] Ariki, S., *On the decomposition numbers of the Hecke algebra of $G(m,1,n)$*, J. Math. Kyoto Univ., **36** (1996), 789–808.

[2] Bezrukavnikov, R. and Etingof, P., *Parabolic induction and restriction functors for rational Cherednik algebras*, Selecta Math. (N.S.), **14** (2009), no. 3-4, 397–425.

[3] Bezrukavnikov, R. and Losev, I., *Etingof conjecture for quantized quiver varieties*, preprint, arXiv:1309.1716.

[4] Baranovsky, V. and Evens, E., Ginzburg, V. *Representations of quantum tori and double-affine Hecke algebras*, preprint, arXiv:0005024.

[5] Baranovsky, V. and Ginzburg, V., *Conjugacy Classes in Loop Groups and G-Bundles on Elliptic Curves*, Internat. Math. Res. Notices, **15** (1996), 734–751.

[6] Bernstein, J., Frenkel, I., and Khovanov, M., *A categorification of the Temperley-Lieb algebra and Schur quotients of U(sl2) via projective and Zuckerman functors*, Selecta Math. (N.S.), **5** (1999), 199–241.

[7] Broué, M., Malle, G., and Rouquier, R., *Complex reflection groups, braid groups*, Hecke algebras, J. Reine Angew. Math., **500** (1998), 127–190.

[8] Brundan, J. and Kleshchev, A., *Schur-Weyl duality for higher levels*, Selecta Math., **14** (2008), 1–57.

[9] Brundan, Kleshchev, *Blocks of cyclotomic Hecke algebras and Khovanov-Lauda algebras*, Invent. Math., **178** (2009), 451–484.

[10] Chriss, N. and Ginzburg, V., *Representation Theory and Complex Geometry*, Birkhäuser Boston, 1997.

[11] Chuang, J. and Rouquier, R., *Derived equivalences for symmetric groups and $\mathfrak{sl}_2$-categorification*, Annals of Math., **167** (2008), 245–298.

[12] Dipper, R., James, G., and Mathas, A., *Cyclotomic q-Schur algebras*, Math. Z., **229** (1998), 385–416.

[13] Derksen, H. and Weyman, J., *Generalized quivers associated to reductive groups*, Colloq. Math., **94** (2002), 151–173.

[14] Enomoto, N., *A quiver construction of symmetric crystals*, Int. Math. Res. Not., **12** (2009), 2200–2247.

[15] Enomoto, N. and Kashiwara, M., *Symmetric Crystals and LLT-Ariki type conjectures for the affine Hecke algebras of type B*, Combinatorial representation theory and related topics RIMS Kôkyûroku Bessatsu, 1–20, Res. Inst. Math. Sci. (RIMS), Kyoto, 2008.

[16] Etingof, P., *Supports of irreducible spherical representations of rational Cherednik algebras of finite Coxeter groups*, Adv. Math., **229** (2012), 2042–2054.

[17] ———, *Symplectic reflection algebras and affine Lie algebras*, Mosc. Math. J., **12** (2012), 668–669.

[18] Etingof, P. and Ginzburg, V., *Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism*, Invent. Math., **147** (2002), 243–348.

[19] Garland, H. and Grojnowski, I., *Affine Hecke algebras associated to Kac-Moody groups*, preprint, arXiv:q-alg/9508019.

[20] Ginzburg, V., *On primitive ideals*, Selecta Math., **9** (2003), 379–407.

[21] Ginzburg, V., Guay, N., Opdam, E., and Rouquier, R., *On the category O for rational Cherednik algebras*, Invent. Math., **154** (2003), 617–651.

[22] Grojnowski, I., *Representations of affine Hecke algebras (and affine quantum GL_n) at roots of unity*, Internat. Math. Res. Notices, **5** (1994), 215–217.

[23] Kac, V.G., *Infinite dimensional Lie algebras*, third ed., Cambridge University Press, Cambridge, 1990.

[24] Kang, S.J. and Kashiwara, *Categorification of highest weight modules via Khovanov-Lauda-Rouquier algebras*, Invent. Math., **190** (2012), 699–742.

[25] Kang, S.J., Kashiwara, M., and Park, E., *Geometric realization of Khovanov-Lauda-Rouquier algebras associated with Borcherds-Cartan data*, Proc. London Math. Soc., 2013.

[26] Kashiwara, M., *Notes on parameters of quiver-Hecke algebras*, Proc. Japan Acad. Ser. A Math. Sci., **88** (2012), 97–102.

[27] ———, *Biadjointness in cyclotomic Khovanov-Lauda-Rouquier algebras*, Publ. Res. Inst. Math. Sci., **48** (2012), 501–524.

[28] ———, *On crystal bases of the q-analogue of universal enveloping algebras*, Duke J. Math., **63**, 465–516 (1991).

[29] Kashiwara, M. and Miemietz, V., *Crystals and affine Hecke algebras of type D*, Proc. Japan Acad. Ser. A Math. Sci., **83** (2007), 135–139.

[30] Kato, S., *An exotic Deligne-Langlands correspondence for symplectic groups*, Duke J. Math., **148**, 305–371 (2009).

[31] Kazhdan, D. and Lusztig, G., *Proof of the Deligne-Langlands conjecture for Hecke algebras*, Invent. Math., **87** (1987), 153–215.

[32] Khovanov, M. and Lauda, A., *A diagrammatic approach to categorification of quantum groups I*, Represent. Theory, **13** (2009), 309–347.

[33] Losev, I., *Proof of Varagnolo-Vasserot conjecture on cyclotomic categories O*, preprint, arXiv:1305.4894.

[34] Lusztig, G., *Affine Hecke algebras and their graded version*, J. Amer. Math. Soc., **2** (1989), 599–635.

[35] ———, *Introduction to quantum groups*, Progress in Mathematics 110, Birkhäuser Boston, 1993.

[36] Maksimau, R., *Canonical basis, KLR-algebras and parity sheaves*, preprint, arXiv:1301.6261.

[37] Macdonald, I.G., *Affine Hecke algebras and orthogonal polynomials*, Cambridge Tracts in Mathematics, vol. 157, Cambridge University Press, Cambridge, 2003.

[38] Mathas, A., *The representation theory of the Ariki-Koike and cyclotomic q-Schur algebras*, Representation theory of algebraic groups and quantum groups, Adv. Stud. Pure Math., **40**, 261–320, Math. Soc. Japan,Tokyo, 2004.

[39] Pressley, A. and Segal, G., *Loop groups*, Oxford Mathematical Monographs, 1986.

[40] Rouquier, R., *Representations of rational Cherednik algebras*, Infinite-dimensional aspects of representation theory and applications, Contemp. Math., 392, Amer. Math. Soc., Providence, RI, (2005), 103–131.

[41] ———, *q-Schur algebras and complex reflection groups*, Mosc. Math. J., **8** (2008), no. 1, 119–158.

[42] ———, *2-Kac Moody algebras*, preprint, arXiv:0812.5023.

[43] ———, *quiver-Hecke algebras and 2-Lie algebras*, Algebra Colloquium, **19** (2012), 359–410.

[44] Rouquier, R., Shan, P., Varagnolo, M., and Vasserot, E., *Categorifications and cyclotomic rational double affine Hecke algebras*, preprint, arXiv:1305.4456.

[45] Shan, P., *Crystals of Fock spaces and cyclotomic rational double affine Hecke algebras*, Ann. Sci. Ec. Norm. Super., **44** (2011), 147–182.

[46] Shan, P., Varagnolo, M., and Vasserot, E., *Canonical bases and affine Hecke algebras of type D*, Adv. Math., **227** (2011), 267–291.

[47] _____, *Koszul duality of affine Kac-Moody algebras and cyclotomic rational DAHA*, Adv. Math., to appear.

[48] Shan, P. and Vasserot, E., *Heisenberg algebras and rational double affine Hecke algebras*, J. Amer. Math. Soc., (2012).

[49] Stroppel, C. and Webster, B., *Quiver Schur algebras and q-Fock space*, preprint, arXiv:1110.1115.

[50] Uglov, D., *Canonical bases of higher level q-deformed Fock spaces and Kazhdan-Lusztig polynomials*, Progress in Math., **191**, Birkhauser (2000).

[51] Varagnolo, M. and Vasserot, E., *On the decomposition matrices of the quantized Schur algebra*, Duke. Math. J., **100** (1999), 267–297.

[52] _____, *Finite dimensional representations of DAHA and affine Springer fibers: the spherical case*, Duke Math. J.

[53] _____, *Double affine Hecke algebras and affine flag manifolds, I.*, (2010), 233–289.

[54] _____, *E. Cyclotomic double affine Hecke algebras and affine parabolic category O*, Adv. Math. **225** (2010),1523–1588.

[55] _____, *Canonical bases and KLR algebras*, J. Reine Angew. Math., **659** (2011), 67–100.

[56] _____, *Canonical bases and affine Hecle algebras of type B*, Invent. Math., **185** (2011), 593–693.

[57] Shephard, G.C. and Todd, J.A., *Finite unitary reflection groups*, Canadian J. Math., **6** (1954), 274–304.

[58] Webster, B., *Knot invariants and higher representation theory*, preprint, arXiv:1309.3796.

[59] _____, *Weighted Khovanov-Lauda-Rouquier algebras*, preprint, arXiv:1209.2463.

[60] Yun, Z., *Global Springer Theory*, Adv. Math., **228** (2011), 266–328.

[61] Yvonne, X., *A conjecture for q-decomposition matrices of cyclotomic v-Schur algebras*, J. Algebra, **304** (2006), 419–456.

Université de Cergy-Pontoise, Département de Mathématiques et CNRS UMR 8088, F-95000 Cergy-Pontoise, France

E-mail: michela.varagnolo@u-cergy.fr

Université Paris Diderot, UFR de Mathématiques, Bâtiment Sophie Germain, 5 rue Thomas Mann, 75205 Paris Cedex 13, France

E-mail: vasserot@math.jussieu.fr

8. Analysis and its Applications

Random matrices, log-gases and Hölder regularity

László Erdős

Abstract. The Wigner-Dyson-Gaudin-Mehta conjecture asserts that the local eigenvalue statistics of large real and complex Hermitian matrices with independent, identically distributed entries are universal in a sense that they depend only on the symmetry class of the matrix and otherwise are independent of the details of the distribution. We present the recent solution to this half-century old conjecture. We explain how stochastic tools, such as the Dyson Brownian motion, and PDE ideas, such as De Giorgi-Nash-Moser regularity theory, were combined in the solution. We also show related results for log-gases that represent a universal model for strongly correlated systems. Finally, in the spirit of Wigner's original vision, we discuss the extensions of these universality results to more realistic physical systems such as random band matrices.

Mathematics Subject Classification (2010). Primary 15B52; Secondary 82B44.

Keywords. De Giorgi-Nash-Moser parabolic regularity, Wigner-Dyson-Gaudin-Mehta universality, Dyson Brownian motion.

1. Introduction

Large complex systems with many degrees of freedom often exhibit remarkably simple universal patterns. The Gauss law describes the fluctuations of large sums of independent or weakly dependent random variables irrespective of their distribution. The Poisson point process is the universal model for many independent events in space or time. Both laws are ubiquitous in Nature thanks to their large domain of attraction but they cannot accurately model strong correlations. Can one find a universality for correlated systems?

Since correlations appear in many forms, this seems an impossible task. Nevertheless this is exactly what E. Wigner has accomplished when he discovered a universal pattern in the spectrum of heavy nuclei. Spectral measurement data for various nuclei clearly show that the density of energy levels depends on the actual nucleus. But Wigner asked a different question: he looked at the energy *gaps*, i.e. the *difference* of consecutive energy levels. He discovered that their statistics, after rescaling with the local density, showed a very similar pattern for different nuclei.

Wigner's revolutionary insight was that this coincidence does not stem from some particular property of the specific physical system but it has a profound mathematical origin. General quantum mechanics postulates that energy levels are eigenvalues of a certain hermitian matrix (or operator) $H = (h_{ij})$, the *Hamiltonian* of the system. The matrix elements h_{ij} represent quantum transition rates between two states labelled by i and j. While h_{ij}'s are specific to the system, the gap statistics largely depend only on the basic symmetry class of H, as long as h_{ij}'s are chosen somewhat generically.

Proceedings of the International Congress of Mathematicians, Seoul, 2014

To illustrate this mechanism, consider a 2×2 hermitian matrix

$$H = \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix}, \qquad a, d \in \mathbb{R},\ b \in \mathbb{C}.$$

The difference (or gap) of the two eigenvalues is $\lambda_2 - \lambda_1 = \left[(a-d)^2 + 4|b|^2\right]^{1/2}$. If the matrix elements are drawn independently from some continuous distribution, then the probability that the gap is very small;

$$\mathbb{P}(|\lambda_2 - \lambda_1| \le \varepsilon), \qquad \varepsilon \ll 1,$$

is of order ε^2 for real symmetric matrices ($b \in \mathbb{R}$) and it is of order ε^3 for complex hermitian matrices ($b \in \mathbb{C}$). The exponent of ε is thus determined by the symmetry class of H.

Very surprisingly, for large $N \times N$ matrices the *entire distribution* of the gap becomes universal as $N \to \infty$ and not only its asymptotics in the $\varepsilon \ll 1$ regime. Based upon a more precise calculation with Gaussian matrix elements, Wigner predicted that this universal law is given by a simple formula (called the *Wigner surmise*). For the real symmetric case it is

$$\mathbb{P}\Big(\widetilde{\lambda}_j - \widetilde{\lambda}_{j-1} = s + \mathrm{d}s\Big) \approx \frac{\pi s}{2} \exp\big(-\frac{\pi}{4} s^2\big)\mathrm{d}s, \tag{1.1}$$

where $\widetilde{\lambda}_j = \varrho \lambda_j$ denote the eigenvalues λ_j rescaled by the density of eigenvalues ϱ near λ_j. This law is characteristically different from the gap distribution of the Poisson point process which is the exponential distribution, $e^{-s}\mathrm{d}s$. The prefactor s in (1.1) indicates a *level repulsion* for the point process $\widetilde{\lambda}_j$, in particular the eigenvalues are strongly correlated (eigenvalues are often called (energy) levels in random matrix theory). Similar formulas hold for the joint statistics of several consecutive gaps.

Comparing measurement data from various experiments, Wigner concluded that the energy gap distribution of complicated quantum systems is essentially universal; it depends only on the basic symmetries of model (such as time-reversal invariance). This thesis has never been rigorously proved for any realistic physical system but experimental data and extensive numerics leave no doubt on its correctness (see [50] for an overview).

Once universality is expected, explicit formulas for the statistics can be computed from the most convenient model within the universality class. The simplest representatives of these universality classes are $N \times N$ random matrices with independent (up to symmetry), identically distributed Gaussian entries. These are called the *Gaussian orthogonal ensemble (GOE)* and the *Gaussian unitary ensemble (GUE)* in case of real symmetric and complex Hermitian matrices, respectively. Wigner's bold vision was to neglect all details of the actual Hamiltonian operator and replace it with a large random Gaussian matrix of the same symmetry class. As far as the gap statistics are concerned, this simple-minded model very accurately reproduced the behavior of large complex quantum systems!

Since Wigner's discovery random matrix statistics are found everywhere in physics and beyond, wherever nontrivial correlations prevail. Random matrix theory (RMT) is present in chaotic quantum systems in physics, in principal component analysis in statistics, in communication theory and even in number theory. In particular, the zeros of the Riemann zeta function on the critical line are expected to follow RMT statistics due to a spectacular result of Montgomery [54].

In retrospect, Wigner's idea should have received even more attention. For centuries, the primary territory of probability theory was to model uncorrelated or weakly correlated

systems. The surprising ubiquity of random matrix statistics is a strong evidence that it plays a similar fundamental role for correlated systems as Gaussian distribution and Poisson point process play for uncorrelated systems. RMT seems to provide essentially the only universal and generally computable pattern for complicated correlated systems.

A few years after Wigner's seminal paper [75], Gaudin [41] discovered another remarkable property of this new point process: the correlation functions have an exact determinantal structure, at least if the distributions of the matrix elements are Gaussian. The algebraic identities within the determinantal form opened up the route to obtain explicit formulas for local correlation functions. For example, in the complex Hermitian case (GUE) the n-point correlation function $p^{(n)}$ of the rescaled eigenvalues $\widetilde{\lambda}_i$ in the bulk is given by the determinant of the celebrated sine-kernel:

$$p^{(n)}(\widetilde{\lambda}_1, \widetilde{\lambda}_2, \ldots, \widetilde{\lambda}_n) = \det\big[K(\widetilde{\lambda}_i, \widetilde{\lambda}_j)\big]_{i,j=1}^n, \qquad K(x,y) := \frac{\sin \pi(x-y)}{\pi(x-y)}. \tag{1.2}$$

(The same determinantal expression with a different but closely related kernel function K holds for the real symmetric case.) As a consequence, the gap distribution is given by a Fredholm determinant involving Hermite polynomials. In fact, Hermite polynomials were first introduced in the context of random matrices by Mehta and Gaudin [52] earlier. Dyson and Mehta [19, 21, 51] have later extended this exact calculation to correlation functions and to other symmetry classes. When compared with the exact formula, the Wigner surmise (1.1), based upon a simple 2×2 matrix model, turned out to be quite accurate. While the determinantal structure is present only in Gaussian Wigner matrices, the paradigm of spectral universality predicts that the formulas for the local eigenvalue statistics obtained in the Gaussian case hold for general distributions as well.

2. Random matrix ensembles and log-gases

We consider $N \times N$ hermitian matrices H with matrix elements having mean zero and variance $1/N$, i.e.

$$\mathbb{E}\, h_{ij} = 0, \quad \mathbb{E}|h_{ij}|^2 = \frac{1}{N} \qquad i,j = 1, 2, \ldots, N. \tag{2.1}$$

The matrix elements h_{ij} are real or complex independent random variables subject to the symmetry constraint $h_{ij} = \overline{h}_{ji}$. These ensembles of random matrices are called *(standard) Wigner matrices.* The normalization (2.1) is introduced for definiteness.

An important special case of Wigner matrices is the Gaussian case (GOE or GUE), when h_{ij}'s have Gaussian distribution. In this case the matrix ensemble can also be given by the probability law

$$P(H)\mathrm{d}H = Z^{-1} e^{-\frac{\beta}{4} N \operatorname{Tr} H^2} \mathrm{d}H, \tag{2.2}$$

where $\mathrm{d}H = \prod_{i<j} \mathrm{d}h_{ij} \mathrm{d}\bar{h}_{ij} \prod_i \mathrm{d}h_{ii}$ is the standard Lebesgue measure on real symmetric or complex hermitian $N \times N$ matrices and $Z = Z_N$ is the normalization. The parameter β is chosen to be $\beta = 1$ for GOE and $\beta = 2$ for GUE to ensure the normalization (2.1).

The representation (2.2) shows that the Gaussian ensembles enjoy an invariance property; the distribution $P(H)$ is invariant under a base transformation, $H \to UHU^*$, where U is

orthogonal (in case of GOE) or unitary (in case of GUE). In fact, invariance property is not restricted to the Gaussian case; one may directly generalize (2.2) to

$$P(H)\mathrm{d}H = Z^{-1} e^{-\frac{\beta}{2} N \operatorname{Tr} V(H)} \mathrm{d}H, \tag{2.3}$$

where $V : \mathbb{R} \to \mathbb{R}$ is an arbitrary function with sufficient growth at infinity to ensure the normalizability of the measure. The ensembles of the form (2.3) are called *invariant matrix ensembles.*

Wigner ensembles and invariant ensembles represent two natural but quite different ways to equip the space of $N \times N$ matrices with a probability measure. These two families are essentially disjoint; only the Gaussian ensembles belong to their intersection.

Let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ denote the eigenvalues of H in increasing order. Since eigenvalues are complicated functions of the matrix elements, there is no explicit formula to express the probability distribution of $\boldsymbol{\lambda}$ induced by a general Wigner ensemble. However, quite remarkably, for invariant ensembles (2.3) the joint probability density of the eigenvalues is explicitly given by

$$\mu_{\beta,V}^{(N)}(\boldsymbol{\lambda}) = C \prod_{1 \le i < j \le N} (\lambda_j - \lambda_i)^\beta \prod_{j=1}^{N} e^{-\frac{\beta}{2} N V(\lambda_j)} \tag{2.4}$$

with a normalization constant C. This formula may directly be obtained from (2.3) by diagonalizing $H = U \Lambda U^*$ and integrating out the matrix of eigenvectors $U \in O(N)$ or $U \in U(N)$ with respect to the Haar measure.

From statistical physics point of view, we may consider the distribution (2.4) as a Gibbs measure for a gas of N point particles on $\mathbb{R}$. We may write

$$\mu_{\beta,V}^{(N)}(\boldsymbol{\lambda}) = C\, e^{-\beta N \mathcal{H}(\boldsymbol{\lambda})}, \qquad \mathcal{H}(\boldsymbol{\lambda}) := \sum_{k=1}^{N} \frac{1}{2} V(\lambda_k) - \frac{1}{N} \sum_{1 \le i < j \le N} \log(\lambda_j - \lambda_i), \tag{2.5}$$

where, according to the Gibbs formalism, $\mathcal{H}(\boldsymbol{\lambda})$ is the Hamiltonian (energy function) of the gas and the parameter β plays the role of the inverse temperature. The Vandermonde determinant in (2.4) translates into a logarithmic pair interaction between the particles. We may completely ignore the original random matrix ensemble behind (2.4) and consider (2.5) more generally for any parameter $\beta > 0$, not only for the specific values $\beta = 1, 2$. The Gibbs measure (2.5) is often called *β-log-gas* or *β-ensemble*.

Eigenvalue distributions of Wigner ensembles and β-log-gases are quite different mathematical entities despite their connection via (2.4) in the special Gaussian case, $V(\lambda) = \frac{1}{2}\lambda^2$ and $\beta = 1, 2$. Wigner ensembles are parametrized by the value $\beta = 1, 2$ and by the distribution of the single matrix elements, while log-gases are parametrized by β and the potential function V. The central thesis of universality asserts that the gap statistics of both families of ensembles depend only on the parameter β and are otherwise independent of any other details of the models.

For Wigner matrices this thesis is generally referred to as the *universality conjecture of random matrices* and we will call it the *Wigner-Dyson-Gaudin-Mehta conjecture*. It was first formulated in Mehta's treatise on random matrices [50] in 1967 and has remained a key question in the subject ever since. In this article we review the recent progress that has led to the proof of this conjecture and the analogous conjecture for log gases. For more details, the reader is referred to the lecture notes [22].

3. Random band matrices and Anderson model

As mentioned in the introduction, Wigner's vision extends the thesis of universality far beyond the models we just introduced. We now present an extension that was an important source of motivation in the development of the subject.

Viewed as a quantum mechanical Hamilton operator, a Wigner matrix H represents a *mean-field system*; the quantum transition rates h_{ij} between any two quantum states, labelled by i and j, are comparable in size. The quantum states of more realistic physical models have a spatial structure and typically quantum transition occurs between nearby states only.

The spatial structure is essential to understand the *metal-insulator transition* which is the fundamental phase transition of disordered quantum systems modelled by a random Hamilton operator H. According to the physical theory, in the *metallic phase* the eigenfunctions are delocalized, the quantum time evolution e^{itH} is diffusive and the local eigenvalue statistics coincide with the ones from the GUE/GOE random matrix theory (1.2). The *localization length*, which is the characteristic lengthscale of the physically relevant quantities (such as eigenfunctions or propagators), is practically infinite. In contrast, in the *insulator phase*, the eigenfunctions are localized with a localization length ℓ independent of the system size, the time evolution remains bounded for all times and the local eigenvalue statistics are Poisson. In the mathematics literature these two phases are usually called *delocalized* and *localized* regimes, respectively, and they are primarily characterized by the spectral type (absolutely continuous vs. pure point) of the corresponding infinite volume operator.

The basic model for the metal-insulator transition is the celebrated Anderson model in solid state physics [3]. The Anderson Hamiltonian is given by $-\Delta + V(x)$ on the Hilbert space $\ell^2(\mathbb{Z}^d)$, where Δ is the lattice Laplacian and $V(x)$ is a real valued random potential field such that $\{V(x) \;:\; x \in \mathbb{Z}^d\}$ are independent and identically distributed centered random variables with variance $\sigma^2 := \mathbb{E}|V(x)|^2$. The Anderson model has been extensively studied mathematically. In nutshell, the high disorder regime is relatively well understood since the seminal work of Frőhlich and Spencer [39] for localization (an alternative proof is given by Aizenman and Molchanov [1]), complemented by the work of Minami [53] proving the local Poissonian spectral statistics. In contrast, in the low disorder regime, starting from three spatial dimension and away from the spectral edges, the eigenfunctions are conjectured to be delocalized but no rigorous proof exists (*extended states conjecture*).

Random band matrices are another popular model for the metal-insulator transition [68]. For definiteness, let the state space be a finite box $\Lambda := [1, L]^d \subset \mathbb{Z}^d$ of the d-dimensional integer lattice equipped with periodic boundary condition. We consider hermitian matrices $H = (h_{ij})_{i,j\in\Lambda}$ whose rows and columns are labelled by the elements of Λ and whose matrix elements are independent. Given a parameter $W \le L/2$, called the *band width*, we assume that the matrix elements h_{ij} vanish beyond a distance $|i - j| \ge W$, i.e. we replace (2.1) with the condition

$$\mathbb{E}\, h_{ij} = 0, \;\; \forall i, j \in \Lambda; \quad \text{and} \quad h_{ij} = 0 \quad \text{for} \quad |i - j| \ge W. \tag{3.1}$$

($|\cdot|$ denotes the periodic distance on Λ). These are called *random band matrices*. We often assume a translation invariant profile for the variances, i.e. that

$$\sigma_{ij}^2 := \mathbb{E}|h_{ij}|^2 = \frac{1}{W^d} f\Big(\frac{|i - j|}{W}\Big) \tag{3.2}$$

with some compactly supported function $f \ge 0$ on $\mathbb{R}^d$ with $\int_{\mathbb{R}^d} f = 1$. Notice that the

normalization is chosen such that

$$\sum_{j\in\Lambda}\sigma_{ij}^2 = 1, \qquad \forall i \in \Lambda. \tag{3.3}$$

If the band width is maximal, $W = L/2$,, and f is constant on $[-\frac{1}{2},\frac{1}{2}]^d$, then we recover the Wigner matrices (2.1). Wigner matrices are always in the delocalized regime as it was shown that all eigenfunctions are extended with very high probability [31]. The other extreme is when W remains bounded even as the matrix size $|\Lambda| = L^d$ goes to infinity. This system behaves very similarly to the Anderson model. In particular, in $d = 1$ it exhibits Anderson localization even if W grows slowly with L as $W \ll L^{1/8}$ [60]. Therefore random band matrices with an intermediate band width, $1 \ll W \ll L$, serve as a model to study the metal-insulator transition. The fundamental conjecture in $d = 1$ is that the transition occurs at $W = L^{1/2}$. This conjecture is supported by supersymmetric (SUSY) functional integration techniques [40] which are intriguingly elegant but notoriously hard to justify with full mathematical rigour. Nevertheless, very recently sine-kernel local statistics (1.2) were proven for a Gaussian band matrix with a specifically chosen block structure [65] using SUSY approach. The details have been worked out for $W \geq L^{1-\varepsilon}$ with some small $\varepsilon > 0$. In a related problem (correlation function of the characteristic polynomial of H at two different energies) the result even holds down to the critical band width $W \geq L^{1/2+\varepsilon}$ [64], but still only for a specific block structure and Gaussian distribution.

For more general band matrices the universality of the local statistics have not yet been proven, but it was shown in $d = 1$ that the localization length is at least $W^{5/4}$, indicating band matrices with band width at least $W \gg L^{4/5}$ are in the delocalized regime [29].

4. Universality on three levels

We consider an ensemble of N (unordered) random points $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ on the real line, either given by eigenvalues of hermitian random matrices or points of a log-gas. We always choose the normalization such that all points lie in a bounded interval, independent of N, with a very high probability. The typical spacing between the points is therefore of order $1/N$.

The statistics of $\boldsymbol{\lambda}$ are characterized by the n-point functions $p_N^{(n)}$. They are defined by the following relation that holds for any function O of n variables:

$$\mathbb{E}\binom{N}{n}^{-1}\sum^{*} O(\lambda_{i_1},\lambda_{i_2},\ldots,\lambda_{i_n}) \tag{4.1}$$
$$= \int_{\mathbb{R}^n} p_N^{(n)}(x_1,x_2,\ldots,x_n)O(x_1,\ldots,x_n)\mathrm{d}x_1\ldots\mathrm{d}x_n.$$

Here the star indicates that the summation runs over all n-tuples of distinct integers, $(i_1, i_2, \ldots, i_n)$ with $1 \leq i_j \leq N$. The correlation function for $n = 1$ is called the *density*. Typically we fix n and consider the limit of the correlation functions $p_N^{(n)}$ as $N \to \infty$ to obtain the limiting statistics.

We may consider the limiting statistics of the points on three scales. For definiteness we illustrate these scales for Wigner matrices; similar results hold for the log-gases and for random band matrices, but the latter only under more restrictive conditions.

4.1. Macroscopic scale. The largest scale corresponds to observable functions O in (4.1) that are unscaled with N. For Wigner matrices (2.1) the limiting density is given by the celebrated semicircle law [75]

$$\varrho_{sc}(x) := \frac{1}{2\pi}\sqrt{(4-x^2)_+} \tag{4.2}$$

in the form of a weak limit:

$$\mathbb{E}\frac{1}{N}\sum_i O(\lambda_i) = \int_{\mathbb{R}} p_N^{(1)}(x)O(x)\mathrm{d}x \to \int_{\mathbb{R}} \varrho_{sc}(x)O(x)\mathrm{d}x, \quad \text{as } N\to\infty, \tag{4.3}$$

that holds for any continuous, compactly supported function O. In fact, the semicircle law also holds not only in expectation but also as a convergence in probability for the empirical density:

$$\mathbb{P}\Big(\Big|\frac{1}{N}\sum_i O(\lambda_i) - \int_{\mathbb{R}} \varrho_{sc}(x)O(x)\mathrm{d}x\Big| \ge \varepsilon\Big) \to 0 \tag{4.4}$$

for any $\varepsilon > 0$ as $N\to\infty$.

These results are the simplest form of spectral universality; they assert that the eigenvalue density on macroscopic scales is independent of the specific distribution of the matrix elements. In fact, this result also holds for *generalized Wigner matrices* whose matrix elements are still centered and independent, but their distributions may vary. The semicircle law (4.3) holds as long as the row sums of the variances is constant, i.e.

$$\sum_j \sigma_{ij}^2 = 1, \qquad \sigma_{ij} := \mathbb{E}|h_{ij}|^2, \tag{4.5}$$

for any i. If (4.5) does not hold but the variances have a macroscopic profile in a sense that

$$\sigma_{ij}^2 = S\big(\frac{i}{N}, \frac{j}{N}\big)$$

with some fixed function S on $[0,1]\times[0,1]$, then the limiting density still exists and can be computed from S, but it is not given by the semicircle law any more [2]. These results show that the limiting density is determined by variances of the matrix elements alone and not by their full distribution.

4.2. Mesoscopic scales. We now consider an N-dependent scaling parameter $\eta = \eta_N > 0$ and a fixed point E in the support of the limiting density, $|E| < 2$ (real numbers E in the context of location in the spectrum are often called *energy* due to the physical meaning of the spectrum). The regime $1/N \ll \eta \ll 1$ corresponds to mesoscopic scales; on these scales the fluctuation of the empirical density around the semicircle density profile is still negligible, but the effects of individual points are not yet visible.

We rescale the observable around E in a window of size η and consider

$$\mathbb{E}\frac{1}{N\eta}\sum_i O\Big(\frac{\lambda_i - E}{\eta}\Big) = \int_{\mathbb{R}} p_N^{(1)}(E+x\eta)O(x)\mathrm{d}x. \tag{4.6}$$

If $\eta \to 0$ as $N\to\infty$, then formally (4.3) would indicate that the limit of (4.6) is $\varrho_{sc}(E)\int O(x)\mathrm{d}x$. This is indeed correct, with some technical assumptions even in the

stronger sense (4.4), as long as $1/N \ll \eta \ll 1$. This is called the *local semicircle law* in the bulk of the spectrum. The first result down to the optimal scale $\eta \gg 1/N$ (modulo $\log N$ factors) was given in [31] followed by several improvements and generalizations, see [27] for a summary. In particular, local semicircle law has also been extended to the spectral edge, $|E| = 2$, where the optimal scale is $\eta \gg N^{-2/3}$ reflecting the fact that the eigenvalue spacing near the edge is of order $N^{-2/3}$.

Local semicircle laws imply, among others, that the points λ_j are very close to their *classical location* denoted by γ_j and defined as the j-th quantile of the limiting density:

$$\int_{-\infty}^{\gamma_j} \varrho_{sc}(x)\mathrm{d}x = \frac{j}{N}. \tag{4.7}$$

More precisely, we have for any j (including the extreme eigenvalues near the spectral edge) that

$$|\lambda_j - \gamma_j| \lesssim |\gamma_{j+1} - \gamma_j| \tag{4.8}$$

with a very high probability, where $\lesssim$ indicates logarithmic factors [37]. The property (4.8) is called *rigidity* and it asserts that the fluctuation of the points is essentially on the scale of the local gap $|\gamma_{j+1} - \gamma_j|$. In particular, for points in the bulk spectrum, their fluctuation is only slightly larger than $1/N$.

Local semicircle law also holds for random band matrices with (3.2), however the local density is controlled only down to scales $\eta \gg W^{-1}$, see [27] for a summary and also [66]. The regime $\eta \ll W^{-1}$ is mathematically unexplored and there is no optimal rigidity result.

While the density on mesoscopic scales behaves exactly as on the macroscopic scale, the density-density correlation exhibits a new universality. For two random variables, X, Y, let $\langle X; Y\rangle = \mathbb{E}XY - \mathbb{E}X\,\mathbb{E}Y$ denote their covariance. Consider two energies $E_2 \geq E_1$ and a scale η such that $N^{-1/7} \ll \eta \ll E_2 - E_1 \ll 1$. Then for Wigner matrices the covariance decays with a universal power-law [7, 25, 26]

$$\left\langle \frac{1}{N\eta}\sum_i O\Big(\frac{\lambda_i - E_1}{\eta}\Big)\,;\, \frac{1}{N\eta}\sum_i O\Big(\frac{\lambda_i - E_2}{\eta}\Big)\right\rangle \sim -\big[N(E_2 - E_1)\big]^{-2} \tag{4.9}$$

(for Gaussian case the result extends to $\eta \gg 1/N$ [6]). Higher order moments satisfy the Wick theorem asymptotically, i.e. the local densities at different energies converge to a Gaussian variables with a non-trivial covariance structure [25, 26].

Similar result holds for band matrices with (3.2) in d dimensions, but the power law decay in (4.9) undergoes a phase transition. For $W^{-d/7} \ll \eta \ll (W/L)^2$ the asymptotics (4.9) holds with the mean-field exponent -2, while for $(W/L)^2 \ll \eta \ll 1$ the power in the right hand side becomes $-2 + \frac{d}{2}$ for $d = 1, 2, 3$ and it is logarithmic for $d = 4$. In higher dimensions, $d \geq 5$, the universality breaks down. This feature is closely related to the quantum diffusion phenomenon for the unitary time evolution [23, 24]. In the physics literature these asymptotics are called the Altshuler-Shklovskii formulas and recently they have been rigorosly proved [25, 26].

4.3. Microscopic scale. The most intriguing regime for universality is the microscopic scale where the scaling parameter η in the observable is chosen comparable with the typical local eigenvalue spacing. In particular, individual eigenvalues are observed. This is the regime for the gap distribution in Wigner's surmise, and the original conjecture of Mehta [50] on random matrix universality also pertains microscopic scales.

Before we formulate the precise results, we make two remarks to explain why there will be different universality theorems.

First, for the local statistics we need to distinguish the bulk spectrum where $\eta \sim 1/N$ and the edge spectrum where $\eta \sim N^{-2/3}$. Not only the scaling but also the explicit formulas are different in these two regimes. The correlation functions are asymptotically determinantal (Pfaffian) in both cases, but in the bulk they are given by the Dyson sine kernel (1.2) and its real symmetric counterpart, while at the edge they are given by the Airy kernel [72, 73]. In all cases the explicit formulas have been computed in the corresponding Gaussian model which is computationally the most accessible case via orthogonal polynomials. The significance of orthogonal polynomials in random matrices has first been realized by Gaudin, Mehta and Dyson [21, 41, 52]. Their approach was later generalized and combined with the Riemann-Hilbert method to yield explicit asymptotic calculations for broader classes of invariant ensembles, see [5, 14, 17, 18, 38, 49, 57, 58] for the extensive literature in the $\beta = 2$ case and [16, 47, 62] for the more complicated $\beta = 1, 4$ case. Our universality results show that the local statistics for a general Wigner matrix or invariant ensemble (or even more generally a β-log-gas) coincide with those of the corresponding Gaussian model. Therefore all explicit asymptotic calculations apart from the simplest Gaussian case become redundant.

Second, there is a subtle difference between the universality of n-point local correlation functions around a *fixed energy* E and the universality of n consecutive points $\lambda_{j+1}, \lambda_{j+2}, \dots$ λ_{j+n} for some *fixed label* j. The former asks for identifying the limit

$$\mathbb{E}\frac{1}{(N\eta)^n}\sum\nolimits^{*} O\Big(\frac{\lambda_{i_1}-E}{\eta}, \frac{\lambda_{i_2}-E}{\eta}, \dots, \frac{\lambda_{i_n}-E}{\eta}\Big) \tag{4.10}$$
$$= \int_{\mathbb{R}^n} p_N^{(n)}(E+x_1\eta, E+x_2\eta, \dots, E+x_n\eta)O(x_1,\dots,x_n)\mathrm{d}x_1\dots\mathrm{d}x_n$$

for any smooth, compactly supported observable O, i.e. identifying the weak limit of the rescaled correlation functions $p_N^{(n)}(E+x_1\eta, E+x_2\eta, \dots)$ in the variables $x_1, \dots, x_n$, The latter asks for the joint distribution of $\lambda_{j+1}, \lambda_{j+2}, \dots \lambda_{j+n}$ with an appropriate rescaling.

The rigidity (4.8) locates the j-th eigenvalue λ_j around a fixed energy $E = \gamma_j$ but only with a precision slightly larger than $1/N$. In fact, for the Gaussian ensembles it is known [42, 56] that $\lambda_j - \gamma_j$ is Gaussian and it fluctuates on scale $\sqrt{\log N}/N$ therefore there is no direct translation between the two types of universality results. In particular, the universality of n consecutive gaps which was originally advocated by Wigner, i.e. the limit of

$$\mathbb{E}\, O\Big(\frac{\lambda_{j+1}-\lambda_j}{\eta}, \frac{\lambda_{j+2}-\lambda_{j+1}}{\eta}, \dots, \frac{\lambda_{j+n}-\lambda_{j+n-1}}{\eta}\Big) \tag{4.11}$$
$$= \int_{\mathbb{R}^n} g_N^{(j)}(x_1, x_2, \dots, x_n)O(x_1,\dots,x_n)\mathrm{d}x_1\dots\mathrm{d}x_n,$$

with the natural scaling $\eta = 1/N$, cannot be concluded from the fixed energy universality (4.10).

Given the historical importance of the Wigner surmise, it is somewhat surprising that gap universality with a fixed label did not receive much attention until very recently. The first results on the Wigner-Dyson-Gaudin-Mehta universality proved (4.10) in the sense of *average energy*, i.e. after taking average in the parameter E in a small interval of size $N^{-1+\varepsilon}$. Since $N^{-1+\varepsilon}$ is above the rigidity scale, average energy universality easily implies *average label gap universality*, i.e. the averaged version of (4.11) after averaging the label j in an interval of size N^{ε}.

Our more recent understanding shows that there is a profound difference between the weaker *"averaged"* results versus the stronger *"fixed"* ones. Obviously, *"fixed"* results are necessary for the precise statistics of individual points hence for fully characterizing the limiting process. At first sight, removing the local averaging may only seem a fine technical point; it merely requires to exclude the pathological case that a certain energy E (or a certain label j) might behave very differently than a typical one. Physicists have never worried about this situation since there is no apparent reason for such pathology (in fact Mehta's original version of the conjecture did not specify the precise formulation of universality). Mathematically, however, it turned out surprisingly involved to exclude the worst case scenarios and we needed to develop a completely new approach. Finally, we point out that, unlike their averaged counterparts, the fixed energy and the fixed label results are not equivalent, in fact each required a separate proof.

5. Universality of local statistics: the main results

5.1. Wigner ensembles. Our main results hold for a larger class of ensembles than the standard Wigner matrices, which we will call *generalized Wigner matrices*.

Definition 5.1 ([36]). The real symmetric or complex Hermitian matrix ensemble H with centred and independent matrix elements $h_{ij} = \overline{h}_{ji}$, $i \le j$, is called *generalized Wigner matrix* if the variances $\sigma_{ij}^2 = \mathbb{E}|h_{ij}|^2$ satisfy:

(A) For any j fixed

$$\sum_{i=1}^{N} \sigma_{ij}^2 = 1\,. \tag{5.1}$$

(B) There exist two positive constants, C_1 and C_2, independent of N such that

$$\frac{C_1}{N} \le \sigma_{ij}^2 \le \frac{C_2}{N}. \tag{5.2}$$

For Hermitian ensembles, we additionally require that for each i, j the 2×2 covariance matrix is bounded by C/N in matrix sense, i.e.

$$\Sigma_{ij} := \begin{pmatrix} \mathbb{E}(\mathrm{Re}h_{ij})^2 & \mathbb{E}(\mathrm{Re}h_{ij})(\mathrm{Im}h_{ij}) \\ \mathbb{E}(\mathrm{Re}h_{ij})(\mathrm{Im}h_{ij}) & \mathbb{E}(\mathrm{Im}h_{ij})^2 \end{pmatrix} \ge \frac{C}{N}.$$

The following theorem settles the average energy version of the Wigner-Dyson-Gaudin-Mehta conjecture for generalized Wigner matrices. It is formulated under the weakest moment assumptions. The same result under somewhat more restrictive assumptions were already obtained in [32, 33]; see also [69] for the complex hermitian case and for a quite restricted class of real symmetric matrices. More details on the history can be found in [22].

Theorem 5.2 (Universality with averaged energy [28, Theorem 7.2]). *Suppose that $H = (h_{ij})$ is a complex Hermitian (respectively, real symmetric) generalized Wigner matrix. Suppose that for some constants $\varepsilon > 0$, $C > 0$,*

$$\mathbb{E}\left|\sqrt{N}h_{ij}\right|^{4+\varepsilon} \le C. \tag{5.3}$$

Let $n \in \mathbb{N}$ and $O : \mathbb{R}^n \to \mathbb{R}$ be a test function (i.e. compactly supported and continuous). Fix $|E_0| < 2$ and $\xi > 0$, then with $b_N = N^{-1+\xi}$ we have

$$\lim_{N\to\infty} \int_{E_0-b_N}^{E_0+b_N} \frac{\mathrm{d}E}{2b_N} \int_{\mathbb{R}^n} \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_n \, O(\alpha_1, \ldots, \alpha_n) \times \frac{1}{\varrho_{sc}(E)^n} \left(p_N^{(n)} - p_{\mathrm{G},N}^{(n)} \right) \left(E + \frac{\alpha_1}{N\varrho_{sc}(E)}, \ldots, E + \frac{\alpha_n}{N\varrho_{sc}(E)} \right) = 0 . \quad (5.4)$$

Here ϱ_{sc} is the semicircle law defined in (4.2)*, $p_N^{(n)}$ is the n-point correlation function of the eigenvalue distribution of H* (4.1)*, and $p_{\mathrm{G},N}^{(n)}$ is the n-point correlation function of an $N \times N$ GUE (respectively, GOE) matrix.*

The additional rescaling in (5.4) with $\varrho_{sc}(E)$ is not essential, it just reflects the choice of variables under which the Gaussian correlation function is given exactly by the sine kernel (1.2) and not by some trivially rescaled version of it.

We remark that our method also provides an effective speed of convergence in (5.4). We also point out that the condition (5.2) can be relaxed, see Corollary 8.3 [27]. For example, the lower bound can be changed to $N^{-9/8+\varepsilon}$. Alternatively, under an additional symmetry condition on the law of the matrix elements, the upper bound can be relaxed to $N^{-8/9-\varepsilon}$.

For the next result, we introduce the notation $[\![A, B]\!] := \{A, A+1, \ldots, B\}$ for any integers $A < B$. A relatively straightforward consequence of Theorem 5.2 is the average gap universality:

Corollary 5.3 (Gap universality with averaged label). *Let H be as in Theorem 5.2 and O be a test function of n variables. Fix small positive constants $\xi, \alpha > 0$. Then for any integer $j_0 \in [\![\alpha N, (1-\alpha)N]\!]$ we have*

$$\lim_{N\to\infty} \frac{1}{2N^\xi} \sum_{|j-j_0|\le N^\xi} \big[\mathbb{E} - \mathbb{E}^G\big] O\big(N(\lambda_j - \lambda_{j+1}), N(\lambda_j - \lambda_{j+2}), \ldots, N(\lambda_j - \lambda_{j+n})\big) = 0. \quad (5.5)$$

Here λ_j's are the ordered eigenvalues. $\mathbb{E}$ and $\mathbb{E}^G$ denote the expectation with respect to the Wigner ensemble H and the Gaussian (GOE or GUE) ensemble, respectively.

We remark that, similarly to the explicit formulas for the correlation functions (1.2), for Gaussian (GOE or GUE) ensembles there are explicit expressions for the gap distribution even without local averaging. They are given in terms of a Fredholm determinant of the corresponding kernel K, see [14, 16, 62].

Now we present our results for fixed energy:

Theorem 5.4 (Universality at fixed energy [11]). *Theorem 5.2 holds under the same conditions without averaging, i.e. for any E with $|E| < 2$ we have*

$$\lim_{N\to\infty} \int_{\mathbb{R}^n} \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_n \, O(\alpha_1, \ldots, \alpha_n) \times \frac{1}{\varrho_{sc}(E)^n} \left(p_N^{(n)} - p_{\mathrm{G},N}^{(n)} \right) \left(E + \frac{\alpha_1}{N\varrho_{sc}(E)}, \ldots, E + \frac{\alpha_n}{N\varrho_{sc}(E)} \right) = 0 . \quad (5.6)$$

We remark that the fixed energy result (5.6) for the $\beta = 2$ (complex Hermitian) case was already known before, see [30, 70] for special cases and [34] for the general case. The $\beta = 2$ case is exceptional since the Harish-Chandra/Itzykson/Zuber identity allows one to compute correlation functions for Wigner matrices with a tiny Gaussian component. This method relies on an algebraic identity and cannot be generalized to other symmetry classes.

Finally, the gap universality with fixed label asserts that (5.5) holds without averaging.

Theorem 5.5 (Gap universality with fixed label [35, Theorem 2.2]). *Assuming subexponential decay of the matrix elements instead of* (5.3)*, Corollary 5.3 holds without averaging:*

$$\lim_{N\to\infty} \big[\mathbb{E} - \mathbb{E}^G\big] O\Big(N(\lambda_j - \lambda_{j+1}), N(\lambda_j - \lambda_{j+2}), \dots, N(\lambda_j - \lambda_{j+n})\Big) = 0, \tag{5.7}$$

for any $j \in [\![\alpha N, (1-\alpha)N]\!]$ with a fixed $\alpha > 0$.

More generally, for any $k, m \in [\![\alpha N, (1-\alpha)N]\!]$ we have

$$\begin{aligned}\lim_{N\to\infty} \Big|\mathbb{E} O\Big((N\varrho_k)(\lambda_k - \lambda_{k+1}), (N\varrho_k)(\lambda_k - \lambda_{k+2}), \dots, (N\varrho_k)(\lambda_k - \lambda_{k+n})\Big) \\ - \mathbb{E}^G O\Big((N\varrho_m)(\lambda_m - \lambda_{m+1}), \dots, (N\varrho_m)(\lambda_m - \lambda_{m+n})\Big)\Big| = 0,\end{aligned} \tag{5.8}$$

where the local density ϱ_k is defined by $\varrho_k := \varrho_{sc}(\gamma_k)$ with γ_k from (4.7)*.*

The second part (5.8) of this theorem asserts that the gap distribution is not only independent of the specific Wigner ensemble, but it is also universal throughout the bulk spectrum. This is the counterpart of the statement that the appropriately rescaled correlation functions (5.6) have a limit that is independent of E, see (1.2).

Prior to our work, universality for a single gap was only achieved in the special case of the Gaussian unitary ensemble (GUE) in [71], which statement then easily implies the same results for complex Hermitian Wigner matrices satisfying the four moment matching condition.

5.2. Log-gases. In the case of invariant ensembles, it is well-known that for V satisfying certain mild conditions the sequence of one-point correlation functions, or densities, associated with $\mu = \mu_{\beta,V}^{(N)}$ from (2.5) has a limit as $N \to \infty$ and the limiting equilibrium density $\varrho_V(s)$ can be obtained as the unique minimizer of the functional

$$I(\nu) = \int_{\mathbb{R}} V(t)\nu(t)\mathrm{d}t - \int_{\mathbb{R}}\int_{\mathbb{R}} \log|t-s|\nu(s)\nu(t)\mathrm{d}t\mathrm{d}s.$$

We assume that $\varrho = \varrho_V$ is supported on a single compact interval, $[A, B]$ and $\varrho \in C^2(A, B)$. Moreover, we assume that V is *regular* in the sense that ϱ is strictly positive on (A, B) and vanishes as a square root at the endpoints, i.e.

$$\varrho(t) = s_A\sqrt{t-A}\,(1 + O(t-A)),\ t \to A^+, \tag{5.9}$$

for some constant $s_A > 0$ and a similar condition holds at the upper edge.

It is known that these conditions are satisfied if, for example, V is strictly convex. In this case ϱ_V satisfies the equation

$$\frac{1}{2}V'(t) = \int_{\mathbb{R}} \frac{\varrho_V(s)\mathrm{d}s}{t-s} \tag{5.10}$$

for any $t \in (A, B)$. For the Gaussian case, $V(x) = x^2/2$, the equilibrium density is given by the semicircle law, $\varrho_V = \varrho_{sc}$, see (4.2).

The following result was proven in Corollary 2.2 of [8] for convex real analytic potential V, it was generalized in Theorem 1.2 of [9] for the non-convex case and further generalized for arbitrary C^4 potential in Theorem 2.5 of [10].

Theorem 5.6 (Universality with averaged energy). *Assume $V \in C^4(\mathbb{R})$, regular and let $\beta > 0$. Consider the β-ensemble $\mu_V = \mu_{\beta,V}^{(N)}$ given in* (2.5) *with correlation functions $p_{V,N}^{(n)}$ defined analogously to* (4.1). *For the Gaussian case, $V(x) = x^2/2$, the correlation functions are denoted by $p_{G,N}^{(n)}$. Let $E_0 \in (A, B)$ lie in the interior of the support of ϱ and similarly let $E_0' \in (-2, 2)$ be inside the support of ϱ_{sc}. Then for $b_N = N^{-1+\xi}$ with some $\xi > 0$ we have*

$$\lim_{N\to\infty} \int \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_n \, O(\alpha_1, \ldots, \alpha_n) \tag{5.11}$$
$$\times \left[\int_{E_0-b_N}^{E_0+b_N} \frac{\mathrm{d}E}{2b_N} \frac{1}{\varrho(E)^n} p_{V,N}^{(n)}\Big(E + \frac{\alpha_1}{N\varrho(E)}, \ldots, E + \frac{\alpha_n}{N\varrho(E)}\Big) \right.$$
$$\left. - \int_{E_0'-b_N}^{E_0'+b_N} \frac{\mathrm{d}E'}{2b_N} \frac{1}{\varrho_{sc}(E')^n} p_{\mathrm{G},N}^{(n)}\Big(E' + \frac{\alpha_1}{N\varrho_{sc}(E')}, \ldots, E' + \frac{\alpha_n}{N\varrho_{sc}(E')}\Big) \right] \;=\; 0\,,$$

i.e. the correlation functions of $\mu_{\beta,V}^{(N)}$ averaged around E_0 asymptotically coincide with those of the Gaussian case. In particular, they are independent of E_0.

Theorem 5.6 immediately implies gap universality with averaged label, exactly in the same way as Corollary 5.3 was deduced from Theorem 5.2; we refrain from stating it explicitly. The following two theorems show that these results hold without averaging.

Theorem 5.7 (Universality at fixed energy [11]). *Consider the setup of Theorem 5.6 and we additionally assume that $\beta \geq 1$. Then* (5.11) *holds without averaging, i.e. for any $E \in (A, B)$ and $E' \in (-2, 2)$ we have*

$$\lim_{N\to\infty} \int \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_n \, O(\alpha_1, \ldots, \alpha_n) \tag{5.12}$$
$$\times \left[\frac{1}{\varrho(E)^n} p_{V,N}^{(n)}\Big(E + \frac{\alpha_1}{N\varrho(E)}, \ldots, E + \frac{\alpha_n}{N\varrho(E)}\Big) \right.$$
$$\left. - \frac{1}{\varrho_{sc}(E')^n} p_{\mathrm{G},N}^{(n)}\Big(E' + \frac{\alpha_1}{N\varrho_{sc}(E')}, \ldots, E' + \frac{\alpha_n}{N\varrho_{sc}(E')}\Big) \right] \;=\; 0\,.$$

Prior to our work and with a different method, the same result was also proven in [63] for analytic potentials and for any $\beta > 0$ even if the support of ϱ has several intervals. An extension of the method to $V \in C^5$ is anticipated in [63].

To formulate the result for the gap universality with a fixed label, we define the quantiles $\gamma_{j,V}$ of the density ϱ_V by

$$\frac{j}{N} = \int_A^{\gamma_{j,V}} \varrho_V(x)\mathrm{d}x, \tag{5.13}$$

similarly to (4.7). We set

$$\varrho_j^V := \varrho_V(\gamma_{j,V}), \qquad \text{and} \qquad \varrho_j := \varrho_{sc}(\gamma_j) \tag{5.14}$$

to be the limiting densities at the j-th quantiles. Let $\mathbb{E}^{\mu_V}$ and $\mathbb{E}^G$ denote the expectation w.r.t. the measure μ_V and its Gaussian counterpart for $V(\lambda) = \frac{1}{2}\lambda^2$.

Theorem 5.8 (Gap universality with fixed label [35, Theorem 2.3]). *Consider the setup of Theorem 5.6 and we also assume $\beta \geq 1$. Set some $\alpha > 0$, then*

$$\lim_{N\to\infty} \Big| \mathbb{E}^{\mu_V} O\Big((N\varrho_k^V)(\lambda_k - \lambda_{k+1}), (N\varrho_k^V)(\lambda_k - \lambda_{k+2}), \dots, (N\varrho_k^V)(\lambda_k - \lambda_{k+n})\Big) - \mathbb{E}^{\mu_G} O\Big((N\varrho_m)(\lambda_m - \lambda_{m+1}), \dots, (N\varrho_m)(\lambda_m - \lambda_{m+n})\Big)\Big| = 0 \quad (5.15)$$

for any $k, m \in [\![\alpha N, (1-\alpha)N]\!]$. In particular, the distribution of the rescaled gaps w.r.t. μ_V does not depend on the index k in the bulk.

We point out that Theorem 5.6 holds for any $\beta > 0$, but Theorems 5.7 and 5.8 require $\beta \geq 1$. This is only a technical restriction related to a certain condition in the De Giorgi-Nash-Moser regularity theory that is the backbone of our proof. Indeed, a year after our work was completed, an alternative proof of (5.15) was given for any $\beta > 0$ but with a higher regularity assumption on V and with an additional hypothesis that can be effectively checked only for convex V, see [4].

5.3. Universalities at the edge. We stated our results for the bulk of the spectrum. Similar results hold at the edge; in this case the "averaged" results are meaningless. For completeness, we give the universality results for both ensembles.

Theorem 5.9 (Universality at the edge for Wigner matrices [10]). *Let H be a generalized Wigner ensemble with subexponentially decaying matrix elements. Fix $n \in \mathbb{N}$, $\kappa < 1/4$ and a test function O of n variables. Then for any $\Lambda \subset [\![1, N^\kappa]\!]$ with $|\Lambda| = n$, we have*

$$\Big| \big[\mathbb{E} - \mathbb{E}^G\big] O\left(\Big(N^{2/3} j^{1/3}(\lambda_j - \gamma_j)\Big)_{j\in\Lambda}\right)\Big| \leq N^{-\chi},$$

with some $\chi > 0$, where $\mathbb{E}^G$ is expectation w.r.t. the standard GOE or GUE ensemble depending on the symmetry class of H and γ_j's are semicircle quantiles.

Edge universality for Wigner matrices was first proved in [67] assuming symmetry of the distribution of the matrix elements and finiteness of all their moments. The symmetry condition was completely eliminated [37] and the optimal moment condition was obtained in [48]. All these works heavily rely on the fact that the variances of the matrix elements are identical. The main point of Theorem 5.9 is to consider generalized Wigner matrices, i.e., matrices with non-constant variances. In fact, it was shown in [37] that the edge statistics for any generalized Wigner matrix are universal in the sense that they coincide with those of a generalized Gaussian Wigner matrix with the same variances, but it was not shown that the statistics are independent of the variances themselves. Theorem 5.9 provides this missing step and thus it proves the edge universality in the broadest sense.

Theorem 5.10 (Universality at the edge for log-gases [10]). *Let $\beta \geq 1$ and V (resp. $\widetilde{V}$) be in $C^4(\mathbb{R})$, regular such that the equilibrium density ϱ_V (resp. $\varrho_{\widetilde{V}}$) is supported on a single interval $[A, B]$ (resp. $[\widetilde{A}, \widetilde{B}]$). Without loss of generality we assume that for both densities*

(5.9) *holds with* $A = 0$ *and with the same constant* s_A. *Fix* $n \in \mathbb{N}$, $\kappa < 2/5$. *Then for any* $\Lambda \subset [\![1, N^\kappa]\!]$ *with* $|\Lambda| = n$, *we have*

$$\left| (\mathbb{E}^{\mu_V} - \mathbb{E}^{\mu_{\tilde{V}}}) O\left(\left(N^{2/3} j^{1/3} (\lambda_j - \gamma_j) \right)_{j \in \Lambda} \right) \right| \leq N^{-\chi} \tag{5.16}$$

with some $\chi > 0$. *Here* γ_j *are the quantiles w.r.t. the density* ϱ_V (5.13).

The first results on edge universality for invariant ensembles concerned the classical values of $\beta = 1, 2, 4$. The case $\beta = 2$ and real analytic V was solved in [15, 17]. The $\beta = 1, 4$ cases are considerably harder than $\beta = 2$. For $\beta = 1, 4$ universality was first proved for polynomial potentials in [15], then for the real analytic case for $\beta = 1$ in [59, 61], which also give an alternative proof for $\beta = 2$. Finally, independently of our work with a completely different method, edge universality for any $\beta > 0$ and convex polynomial V was recently proved in [46].

6. Outline of the proof strategy

6.1. "Averaged" results: Dyson Brownian motion. The proof of Theorem 5.2 follows a three-step strategy that was first introduced in [30] and further developed in [32].

Step 1. *Local semicircle law and rigidity of eigenvalues.* The main tool is the resolvent of H at a spectral parameter $z = E + i\eta$ with $\eta \gg 1/N$;

$$m_N(z) := \frac{1}{N} \operatorname{Tr} \frac{1}{H - z} = \frac{1}{N} \sum_j \frac{1}{\lambda_j - z},$$

which is of the form of (4.6) with $O(x) = (x - i)^{-1}$. Using the Schur decomposition formula we may write

$$m_N(z) = \frac{1}{N} \sum_{j=1}^{N} \frac{1}{h_{jj} - z - \sum_{a,b \neq j} h_{ja} G^{(j)}_{ab}(z) h_{bj}},$$

where $G^{(j)}(z) = (H^{(j)} - z)^{-1}$ is the resolvent of the $(N-1) \times (N-1)$ minor $H^{(j)}$ of H after removing the j-th row and column. Since $G^{(j)}_{ab}(z)$ and $h_{ja} h_{bj}$ are independent, we may use concentration results to replace the double sum in the denominator by its expectation over the matrix elements in the j-th row and column. Neglecting the fluctuation, we recover $m^{(j)}_N(z)$, the normalized trace of the resolvent of $H^{(j)}$. Since $m^{(j)}_N(z)$ and $m_N(z)$ are close, we obtain the following *self-consistent equation*

$$m_N(z) = -\frac{1}{z + m_N(z)} + \text{error}. \tag{6.1}$$

If the error is neglected, then the solution of the resulting quadratic equation is exactly the Stieltjes transform

$$m_{sc}(z) := \int_{\mathbb{R}} \frac{1}{x - z} \varrho_{sc}(x) \mathrm{d}x$$

of the Wigner semicircle law $\varrho_{sc}(x)$. This allows us to conclude that $m_N(z)$ is close to m_{sc}, and a careful analysis yields

$$|m_N(z) - m_{sc}(z)| \lesssim \frac{1}{N\eta}. \tag{6.2}$$

This is the local semicircle law in resolvent form, from which the limit of (4.6) and the rigidity property (4.8) can be concluded.

Step 2. *Universality for Gaussian divisible ensembles:* The Gaussian divisible ensembles are matrices of the form

$$H_t = e^{-t/2}H + \sqrt{1-e^{-t}}U,$$

where H is a Wigner matrix and U is an independent GUE/GOE matrix. The parametrization of H_t reflects that, in the sense of distribution, it is most conveniently obtained by an Ornstein-Uhlenbeck process:

$$\mathrm{d}H_t = \frac{1}{\sqrt{N}}\mathrm{d}B_t - \frac{1}{2}H_t\mathrm{d}t, \tag{6.3}$$

where B_t is a matrix-valued Brownian motion of the appropriate symmetry class. Dyson observed [20] that the corresponding process $\boldsymbol{\lambda}_t$ of the eigenvalues of H_t remarkably satisfies a system of stochastic differential equations (SDE), called the *Dyson Brownian Motion (DBM)*:

$$\mathrm{d}\lambda_j = \frac{1}{\sqrt{N}}\mathrm{d}B_j + \Big[-\frac{1}{2}\lambda_j + \frac{1}{N}\sum_{k\neq j}\frac{1}{\lambda_k - \lambda_j}\Big]\mathrm{d}t, \tag{6.4}$$

written for $\beta = 2$, where B_j's are independent standard real Brownian motions. The key idea is to study the relaxation of the flow (6.4) to its equilibrium measure which is the distribution of the GUE eigenvalues. It turns out that, tested against observables involving only *differences of eigenvalues*, the convergence is extremely fast. Combined with the rigidity bound that guarantees a strong apriori control on the initial state, we obtain that the gap statistics are already in local equilibrium (hence universal) after a very short time $t = N^{-1+\varepsilon}$, see [32, 33].

This method substantially improves Johansson's result [44] which showed universality only with a substantial Gaussian component (essentially for $t > 0$ independent of N) and only for the $\beta = 2$ symmetry class. In fact, the first restriction can be relaxed by using our optimal rigidity bound [30, 34], but the second one cannot be removed since the proof relies on the Harish-Chandra/Itzykson/Zuber formula. The analysis of the DBM is much more robust, in particular it applies to any symmetry class. However, it yields only an averaged result (5.5) (from which (5.4) can be deduced), while [30, 34] gives the fixed energy results (5.6) but only for $\beta = 2$.

Step 3. *Approximation by Gaussian divisible ensembles:* It is a simple density argument in the space of matrix ensembles which shows that for any probability distribution of the matrix elements there exists a Gaussian divisible distribution with a small Gaussian component, as in Step 2, such that the two associated Wigner ensembles have asymptotically identical local eigenvalue statistics. The first implementation of this approximation scheme was via a reverse heat flow argument [30]; it was later replaced by the *Green function comparison theorem* [36] motivated by the four moment matching condition of [69]. This comparison

argument is very robust: it works even without averaging and for arbitrary observables not only for those of difference type.
The proof of Theorem 5.6 follows a somewhat similar path but with essential differences. Rigidity estimates still hold on the smallest scale, but their derivation cannot use resolvents since there is no matrix behind a general log-gas. Instead of (6.1) we use the loop equation from [45] or [62], but extended to smooth potentials. There is no analogue of the Gaussian divisible ensemble for log-gases, but an enhanced version of the DBM underlying the invariant measure μ_V can still be analyzed.

In summary, the DBM plays the fundamental role behind the "averaged" universality result for both models.

6.2. "Fixed" results: Hölder regularity and homogenization. For definiteness, we will present some ideas to prove Theorem 5.8, the proof of Theorems 5.4, 5.5, 5.7 and the results at the edge require additional steps.

Step 1. *Comparison of local Gibbs measures.* The basic mechanism for universality is that the microscopic structure of the measure μ_V defined in (2.4) is insensitive of the potential V, it is essentially determined by the Vandermonde determinant, i.e. the log-interaction in (2.5). In the first step we localize the problem by freezing (conditioning on) all particles at a distance $1 \ll K \ll N$ away from the fixed index j of the gap $\lambda_j - \lambda_{j+1}$ we want to study. Thus the corresponding local Gibbs measure is defined on an interval $I = [j-K, j+K]$ and it still retains the Vandermonde structure. On this mesoscopic scale the potential is locally constant, hence its effect is trivial, so the key question is to show that $\lambda_j - \lambda_{j+1}$ is largely insensitive to the boundary effects we just introduced by localization. This is a question about the long range correlation structure of the Gibbs measure.

The main difficulty is that the log-gas is a strongly correlated system in contrast to the customary setup in statistical physics where correlations often decay very fast. In fact, the covariance between two points decays only logarithmically

$$\frac{\langle \lambda_i; \lambda_j \rangle}{\sqrt{\langle \lambda_i; \lambda_i \rangle \langle \lambda_j; \lambda_j \rangle}} \sim \frac{1}{\log |i-j|}, \qquad 1 \ll |i-j| \ll N. \tag{6.5}$$

One key observation is that the correlation decay between a *gap* $\lambda_i - \lambda_{i+1}$ and a *point* λ_j is faster, it is $|i-j|^{-1}$, practically the discrete derivative of (6.5).

Step 2. *Random walk representation of the covariance.* In a more general setup, consider a Gibbs measure $\omega(\mathrm{d}\mathbf{x}) = e^{-\beta \mathcal{H}(\mathbf{x})} \mathrm{d}\mathbf{x}$ on finitely many points labelled by I and with a strictly convex Hamiltonian, $\mathcal{H}''(\mathbf{x}) \geq c > 0$. Then the covariance w.r.t. ω can be expressed as

$$\langle F(\mathbf{x}); G(\mathbf{x}) \rangle_\omega = \frac{1}{2} \int_0^\infty \mathrm{d}s \int \mathrm{d}\omega(\mathbf{x}) \mathbb{E}_{\mathbf{x}} \big[\nabla G(\mathbf{x}(s)) \cdot \mathcal{U}(s, \mathbf{x}(\cdot)) \nabla F(\mathbf{x}) \big], \tag{6.6}$$

see [43, 55]. Here $\mathbb{E}_{\mathbf{x}}$ is the expectation for the (random) paths $\mathbf{x}(\cdot)$ starting from $\mathbf{x}(0) = \mathbf{x}$ and solving the canonical SDE for the measure ω:

$$\mathrm{d}\mathbf{x}(s) = \mathrm{d}\mathbf{B}(s) - \beta \nabla \mathcal{H}(\mathbf{x}(s)) \mathrm{d}s, \tag{6.7}$$

and $\mathcal{U}(s) = \mathcal{U}(s, \mathbf{x}(\cdot))$ is the fundamental solution to the linear system of equations

$$\partial_s \mathcal{U}(s) = -\mathcal{U}(s) \mathcal{A}(s), \qquad \mathcal{A}(s) := \beta \mathcal{H}''(\mathbf{x}(s)) \tag{6.8}$$

with $\mathcal{U}(0) = \mathrm{Id}$. Notice that the coefficient matrix $\mathcal{A}(s)$, and thus the fundamental solution, depend on the random path. The SDE (6.7) is the generalization of the DBM, (6.4). Formula (6.6) turns the problem of computing the covariance $\langle F; G\rangle$ into a time-dependent question to understand the fundamental solution $\mathcal{U}$ of the parabolic equation (6.8).

In particular, if G is a function of a single gap, $G(\mathbf{x}) = O(x_j - x_{j+1})$ with some fixed j, and F represents the boundary effects, then (6.6) becomes

$$\frac{1}{2}\int_0^\infty \mathrm{d}s \int \mathrm{d}\omega(\mathbf{x}) \sum_{i\in I} \mathbb{E}_{\mathbf{x}}\Big[\, O'(x_j - x_{j+1})\big(\mathcal{U}_{i,j}(s) - \mathcal{U}_{i,j+1}(s)\big)\partial_i F(\mathbf{x})\Big]. \tag{6.9}$$

The key technical step is to show that for a typical path $\mathbf{x}(\cdot)$ the solution $\mathcal{U}(s)$ is Hölder-regular in a sense that $\mathcal{U}_{i,j}(s) - \mathcal{U}_{i,j+1}(s)$ is small if j is away from the boundary of I and s is not too small.

Step 3. *Hölder-regularity of the solution to* (6.8). For any fixed realization of the path $\mathbf{x}(\cdot)$, we will view the equation (6.8) as a finite dimensional version of a parabolic equation. The coefficient matrix, the Hessian of the local Gibbs measure, is computed explicitly. It can be written as $\mathcal{A} = \mathcal{B} + \mathcal{W}$, where $\mathcal{W} \geq 0$ is diagonal, $\mathcal{B}$ is a symmetric matrix with quadratic form

$$\langle \mathbf{u}, \mathcal{B}(s)\mathbf{u}\rangle = \frac{1}{2}\sum_{i,j\in I} B_{ij}(s)(u_i - u_j)^2, \qquad B_{ij}(s) := \frac{\beta}{(x_i(s) - x_j(s))^2}.$$

After rescaling the problem so that the gap is of order one, for a typical path and large $i - j$ we have

$$B_{ij}(s) \sim \frac{1}{(i-j)^2} \tag{6.10}$$

by rigidity. We also have a lower bound for any $i \neq j$

$$B_{ij}(s) \gtrsim \frac{1}{(i-j)^2}, \tag{6.11}$$

at least with a very high probability. If a matching upper bound were true for any $i \neq j$, then (6.8) would be the discrete analogue of the general equation

$$\partial_t u(t,x) = \int K(t,x,y)[u(t,y) - u(t,x)]\mathrm{d}y, \qquad t > 0, \quad x, y \in \mathbb{R}^d \tag{6.12}$$

considered by Caffarelli-Chan-Vasseur in [12], where the kernel K is symmetric and has a specific short distance singularity

$$C_1|x-y|^{-d-s} \leq K(t,x,y) \leq C_2|x-y|^{-d-s} \tag{6.13}$$

for some $s \in (0,2)$ and positive constants C_1, C_2. Roughly speaking, the integral operator K corresponds to the behavior of the operator $|p|^s$, where $p = -i\nabla$. The main result of [12] asserts that for any $t_0 > 0$, the solution $u(t,x)$ is ε-Hölder continuous, $u \in C^\varepsilon((t_0,\infty), \mathbb{R}^d)$, for some positive exponent ε that depends only on t_0, C_1, C_2. This is a version of the celebrated De Giorgi-Nash-Moser regularity result for a non-local operator.

Our equation (6.8) is of this type with $d = s = 1$, but it is discrete and in a finite interval I with a potential term. The key difference, however, is that the coefficient $B_{ij}(t)$ can be

singular in the sense that $B_{ij}(t)|i-j|^2$ is not uniformly bounded when i, j are close to each other. Thus the analogue of the uniform upper bound (6.13) does not even hold for a fixed t. We first need to regularize the singularity of B_{ij} on a very tiny scale. Even after that we can control the regularized B_{ij}^{reg} only in a certain average sense:

$$\sup_{0\le s\le\sigma}\sup_{0\le M\le K}\frac{1}{1+s}\int_0^s\frac{1}{M}\sum_{i\in I\,:\,|i-Z|\le M}B_{i,i+1}^{\text{reg}}(s)\mathrm{d}s\le CK^{\rho}\tag{6.14}$$

with high probability, for some small exponent ρ and for any fixed Z away from the edges of I. This estimate essentially says that the space-time maximal function of $B_{i,i+1}^{\text{reg}}(t)$ at a fixed space-time point $(Z,0)$ is bounded by K^{ρ}. Our main generalization of the result in [12] is to show that the weak upper bound (6.14) at a few space-time points together with (6.10) and (6.11) (holding up to a factor K^{ξ}) are sufficient for proving a discrete version of the Hölder continuity at the point $(Z,0)$. More precisely, there exists an $\varepsilon>0$ such that for any fixed $1\ll\sigma\ll K$ the solution to (6.8) satisfies

$$\sup_{|j-Z|+|j'-Z|\le\sigma^{1-\alpha}}|\mathcal{U}_{i,j}(\sigma)-\mathcal{U}_{i,j'}(\sigma)|\le CK^{\xi}\sigma^{-1-\varepsilon\alpha}\tag{6.15}$$

with any $\alpha\in[0,1/3]$ if we can guarantee that ρ and ξ are sufficiently small. The exponent ε plays the role of the Hölder regularity exponent. Notice that $\mathcal{U}_{i,j}(\sigma)$ decays as σ^{-1}, hence (6.15) provides an additional decay for the discrete derivative. In particular, this guarantees that the $\mathrm{d}s$ integration in (6.9) is finite. With several further technical steps, this proves Theorem 5.8.

Step 4. *Homogenization.* The proofs of Theorems 5.5, 5.7 require an additional information about the fundamental solution of (6.8). Since in the $|i-j|\gg 1$ regime we have $B_{ij}(s)\sim|i-j|^{-2}$, it is reasonable to expect that the large time and large scale behavior of $\mathcal{U}$ is given by the

$$\mathcal{U}_{ij}(t)\approx\left(e^{-t|p|}\right)_{ij}=\frac{t}{t^2+(i-j)^2},\quad |i-j|\gg 1,\ \ t\gg 1,\tag{6.16}$$

where we computed the heat kernel of $|p|=\sqrt{-\Delta}$ explicitly. This result, combined with a coupling argument, yields that

$$\lambda_i(t)-\widetilde{\lambda}_i(t)=\left(e^{-t|p|}\boldsymbol{\lambda}(0)\right)_i-\left(e^{-t|p|}\widetilde{\boldsymbol{\lambda}}(0)\right)_i+\text{error},\tag{6.17}$$

where $\boldsymbol{\lambda}$ and $\widetilde{\boldsymbol{\lambda}}$ are two solutions of the SDE (6.7) with the same Brownian motion $\mathbf{B}(s)$ but with two different initial conditions. In the applications, $\boldsymbol{\lambda}(0)$ will be GUE/GOE eigenvalues and $\widetilde{\boldsymbol{\lambda}}(0)$ will be the eigenvalues of a general Wigner matrix. Formula (6.17) allows us to express a single Wigner eigenvalue $\lambda_i(t)$ in terms of the corresponding Gaussian eigenvalue $\widetilde{\lambda}_i(t)$ and in terms of averaged quantities involving many eigenvalues. Since averaged quantities can be computed much easier and Gaussian computations can be performed by explicit formulas, we obtain nontrivial information about $\lambda_i(t)$. Finally, approximation ideas similar to Step 3. in Section 6.1 can relate general Wigner eigenvalues to Wigner eigenvalues with some Gaussian component such as $\lambda_i(t)$. In particular, these ideas can prove the logarithmic correlation decay (6.5) for any Wigner matrix.

In summary, the detailed analysis of the parabolic equation (6.8) with singular coefficients given by the Dyson Brownian motion play the crucial role behind all "fixed" universality results for both Wigner matrices and log-gases.

Acknowledgement. The author is partially supported by SFB-TR 12 Grant of the German Research Council and by ERC Advanced Grant, RANMAT 338804. Most results in this paper were obtained in collaboration with Horng-Tzer Yau, Benjamin Schlein, Jun Yin, Antti Knowles and Paul Bourgade and in some work, also with Jose Ramirez and Sandrine Péche. This article reports the joint progress with these authors.

References

[1] Aizenman, M. and Molchanov, S., *Localization at large disorder and at extreme energies, an elementary derivation*, Commun. Math. Phys. **157** (1993), 245–278.

[2] Ajanki, O., Erdős, L., Krűger, T., *Universality for Wigner random matrices with non-stochastic variance matrix*, Preprint.

[3] Anderson, P., *Absences of diffusion in certain random lattices*, Phys. Rev. **109** (1958), 1492–1505.

[4] Bekerman, F., Figalli, A., and Guionnet, A., *Transport maps for β-matrix models and universality*, Preprint, arXiv:1311.2315.

[5] Bleher, P., and Its, A., *Semiclassical asymptotics of orthogonal polynomials, Riemann-Hilbert problem, and universality in the matrix model*, Ann. of Math. **150** (1999), 185–266.

[6] Boutet de Monvel, A., and Khorunzhy, A., *Asymptotic distribution of smoothed eigenvalue density. I.*, Gaussian random matrices, Random Oper. and Stoch. Equ. **7** (1999), 1–22.

[7] ———, *Asymptotic distribution of smoothed eigenvalue density. II.*, Wigner random matrices, Random Oper. and Stoch. Equ. **7** (1999), 149–168.

[8] Bourgade, P., Erdős, L., and Yau, H.-T., *Universality of General β-Ensembles*, arXiv: 1104.2272.

[9] ———, *Bulk Universality of General β-Ensembles with Non-convex Potential*, J. Math. Phys. **53** (2012), 095221.

[10] ———, *Edge universality of beta ensembles*, Preprint, arXiv:1306.5728.

[11] Bourgade, P., Erdős, L., Yau, H.-T., and Yin, J., *Universality for Wigner matrices and log gases at fixed energy*, Preprint.

[12] Caffarelli, L., Chan, C.H., and Vasseur, A., *Regularity theory for parabolic nonlinear integral operators*, *J. Amer. Math. Soc.* **24** (2011), no. 3, 849–889.

[13] Deuschel, J.-D., Giacomin, G., and Ioffe, D., *Large deviations and concentration properties for $\nabla\varphi$ interface models*, Probab. Theor. Relat. Fields. **117** (2000), 49–111.

[14] Deift, P., *Orthogonal polynomials and random matrices: a Riemann-Hilbert approach*, Courant Lecture Notes in Mathematics **3**, American Mathematical Society, Providence, RI, 1999.

[15] Deift, P. and Gioev, D., *Universality at the edge of the spectrum for unitary, orthogonal, and symplectic ensembles of random matrices*, Comm. Pure Appl. Math. **60** (2007), no. 6, 867–910.

[16] ———, *Random Matrix Theory: Invariant Ensembles and Universality*, Courant Lecture Notes in Mathematics **18**, American Mathematical Society, Providence, RI, 2009.

[17] Deift, P., Kriecherbauer, T., McLaughlin, K.T-R, Venakides, S., and Zhou, X., *Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory*, Comm. Pure Appl. Math. **52** (1999), 1335–1425.

[18] ———, Venakides, S., and Zhou, X., *Strong asymptotics of orthogonal polynomials with respect to exponential weights*, Comm. Pure Appl. Math. **52** (1999), 1491–1552

[19] Dyson, F.J., *Statistical theory of energy levels of complex systems, I, II, and III.*, J. Math. Phys. **3** (1962) 140–156, 157–165, 166–175.

[20] ———, *A Brownian-motion model for the eigenvalues of a random matrix*, J. Math. Phys. **3** (1962), 1191–1198.

[21] ———, *Correlations between eigenvalues of a random matrix*, Commun. Math. Phys. **19** (1970), 235–250.

[22] Erdős, L., *Universality for random matrices and log-gases, In: Current Developments in Mathematics 2012*, Ed. D. Jerison, M. Kisin, T. Mrowka. R. Stanley, H.-T. Yau, S.-T. Yau, International Press, 59–132, 2013

[23] Erdős, L. and Knowles, A., *Quantum Diffusion and Eigenfunction Delocalization in a Random Band Matrix Model*, Commun. Math. Phys. **303** (2011), no. 2, 509–554.

[24] ———, *Quantum Diffusion and Delocalization for Band Matrices with General Distribution*, Annales Inst. H. Poincare **12** (7), (2011), 1227–1319.

[25] ———, *The Altshuler-Shklovskii formulas for random band matrices I.: the unimodular case*, To appear in Commun. Math. Phys., arXiv:1309.5106.

[26] ———, *The Altshuler-Shklovskii formulas for random band matrices II.: the general case*, To appear in Annals Henri Poincaré., arXiv:1309.5107.

[27] Erdős, L., A. Knowles, A., Yau, H.-T., and J. Yin., *The local semicircle law for a general class of random matrices*, Electron. J. Probab. **18** (2013), no. 59, 1–58.

[28] ———, *Spectral Statistics of Erdős-Rényi Graphs II.: Eigenvalue Spacing and the Extreme Eigenvalues*, Comm. Math. Phys. **314** (2012), no. 3. 587–640.

[29] ———, *Delocalization and Diffusion Profile for Random Band Matrices*, Preprint, arXiv:1205.5669.

[30] Erdős, L., Péché, G., Ramírez, J., Schlein, B., and Yau, H.-T., *Bulk universality for Wigner matrices*, Commun. Pure Appl. Math. **63** (2010), no. 7, 895–925.

[31] Erdős, L., Schlein, B., and Yau, H.-T., *Local semicircle law and complete delocalization for Wigner random matrices*, Commun. Math. Phys. **287** (2009), 641–655.

[32] ———, *Universality of random matrices and local relaxation flow*, Invent. Math. **185** (2011), no. 1, 75–119.

[33] Erdős, L., Schlein, B., Yau, H.-T., and Yin, J., *The local relaxation flow approach to universality of the local statistics for random matrices*, Annales Inst. H. Poincaré (B), Probability and Statistics **48** (2012), no. 1, 1–46.

[34] Erdős, L. and Yau, H.-T., *A comment on the Wigner-Dyson-Mehta bulk universality conjecture for Wigner matrices*, Electron. J. Probab. **17** (2012), no. 28, 1–5.

[35] ———, *Gap universality of generalized Wigner and β-ensembles*, Preprint, arXiv: 1211.3786.

[36] Erdős, L., Yau, H.-T., and Yin, J., *Bulk universality for generalized Wigner matrices*, Prob. Theor. Rel. Fields **154**, (2012), no. 1-2., 341–407.

[37] ———, *Rigidity of Eigenvalues of Generalized Wigner Matrices*, Adv. Math. **229** (2012), no. 3, 1435–1515.

[38] Fokas, A. S., Its, A. R., and Kitaev, A. V., *The isomonodromy approach to matrix models in 2D quantum gravity*, Comm. Math. Phys. **147** (1992), 395–430.

[39] Frőhlich, J. and Spencer, T., *Absence of diffusion in the Anderson tight binding model for large disorder or low energy*, Commun. Math. Phys. **88** (1983), 151–184.

[40] Fyodorov, Y.V. and Mirlin, A.D., *Scaling properties of localization in random band matrices: a σ-model approach*, Phys. Rev. Lett. **67** (1991), 2405–2409.

[41] Gaudin, M., *Sur la loi limit de l'espacement des valeurs propres d'une matrice aléatoire*, Nucl. Phys. **25**, 447–458.

[42] Gustavsson, J., *Gaussian fluctuations of eigenvalues in the GUE*, Ann. Inst. H. Poincare Probab. Statist **41** (2005), no. 2, 151–178.

[43] Helffer, B., Sjőstrand, J., *On the correlation for Kac-like models in the convex case*, J. Statis. Phys. **74** (1994), no. 1-2, 349–409.

[44] Johansson, K., *Universality of the local spacing distribution in certain ensembles of Hermitian Wigner matrices*, Comm. Math. Phys. **215** (2001), no. 3, 683–705.

[45] ———, *On the fluctuations of eigenvalues of random Hermitian matrices*, Duke Math. J. **91** (1998), 151–204.

[46] Krishnapur, M., Rider, B., and Virág, B., *Universality of the Stochastic Airy Operator*, Preprint, arXiv:1306.4832.

[47] Kriecherbauer, T. and Shcherbina, M., *Fluctuations of eigenvalues of matrix models and their applications*, Preprint, arXiv:1003.6121.

[48] Lee, J. O. and Yin, J., *A Necessary and Sufficient Condition for Edge Universality of Wigner matrices*, Preprint, arXiv:1206.2251.

[49] Lubinsky, D.S., *A New Approach to Universality Limits Involving Orthogonal Polynomials*, Ann. Math. **170** (2009), 915–939.

[50] Mehta, M.L., *Random Matrices.*, Third Edition, Academic Press, New York, 1991.

[51] ———, *A note on correlations between eigenvalues of a random matrix*, Commun. Math. Phys. **20**, (1971), no.3. 245–250.

[52] Mehta, M.L. and Gaudin, M., *On the density of eigenvalues of a random matrix*, Nuclear Phys. **18** (1960), 420–427.

[53] Minami, N., *Local fluctuation of the spectrum of a multidimensional Anderson tight binding model*, Commun. Math. Phys. **177** (1996), 709–725.

[54] Montgomery, H.L., *The pair correlation of zeros of the zeta function. Analytic number theory*, Proc. of Sympos. in Pure Math. **24** (1973), Amer. Math. Soc. Providence, R.I., 181–193.

[55] Naddaf, A. and Spencer, T., *On homogenization and scaling limit of some gradient perturbations of a massless free field*, Commun. Math. Phys. **183** (1997), no. 1, 55–84.

[56] O'Rourke, S., *Gaussian Fluctuations of Eigenvalues in Wigner Random Matrices*, J. Statis. Phys. **138** (2010), no. 6, 1045–1066.

[57] Pastur, L. and Shcherbina, M., *Universality of the local eigenvalue statistics for a class of unitary invariant random matrix ensembles*, J. Stat. Phys. **86** (1997), 109-147.

[58] ———, *Bulk universality and related properties of Hermitian matrix models*, J. Stat. Phys. **130** (2008), no. 2, 205–250.

[59] Pastur, L. and Shcherbina M., *On the edge universality of the local eigenvalue statistics of matrix model*, Mat. Fiz. Anal. Geom. **10** (2003), no. 3, 335–365

[60] Schenker, J., *Eigenvector localization for random band matrices with power law band width*, Commun. Math. Phys. **290** (2009), 1065–1097.

[61] Shcherbina, M., *Edge universality for orthogonal ensembles of random matrices*, J. Stat. Phys. **136** (2009), no. 1, 35–50.

[62] ———, *Orthogonal and symplectic matrix models, universality and other properties*, Comm. Math. Phys. **307** (2011), no. 3, 761–790.

[63] ———, *Change of variables as a method to study general β-models: bulk universality*, Preprint, arXiv:1310.7835.

[64] Shcherbina, T., *On the second mixed moment of the characteristic polynomials of the 1D band matrices*, Preprint, arXiv:1209.3385.

[65] ———, *Universality of the local regime for the block band matrices with a finite number of blocks*, Preprint, arXiv:1309.2120.

[66] Sodin, S., *An estimate for the average spectral measure of random band matrices*, J. Stat. Phys. **144** (2011), no. 1, 46–59.

[67] Soshnikov, A., *Universality at the edge of the spectrum in Wigner random matrices*, Comm. Math. Phys. **207** (1999), no. 3, 697–733.

[68] Spencer, T., *Random banded and sparse matrices (Chapter 23)*, to appear in "Oxford Handbook of Random Matrix Theory" edited by G. Akemann, J. Baik, and P. Di Francesco.

[69] Tao, T. and Vu, V., *Random matrices: Universality of the local eigenvalue statistics*, Acta Math. **206** (2011), no. 1, 127–204.

[70] ———, *The Wigner-Dyson-Mehta bulk universality conjecture for Wigner matrices*, Electron. J. Probab. **16** (2011), no. 77, 2104–2121.

[71] Tao, T., *The asymptotic distribution of a single eigenvalue gap of a Wigner matri*, Preprint, arXiv:1203.1605.

[72] Tracy, C. and Widom, H., *Level spacing distributions and the Airy kernel*, Commun. Math. Phys. **159** (1994), 151–174.

[73] ———, *On orthogonal and symplectic matrix ensembles*, Comm. Math. Phys. **177** (1996), no. 3, 727–754.

[74] Valkó, B. and Virág, B., *Continuum limits of random matrices and the Brownian carousel*, Invent. Math. **177** (2009), no. 3, 463–508.

[75] Wigner, E., *Characteristic vectors of bordered matrices with infinite dimensions*, Ann. of Math. **62** (1955), 548–564.

IST Austria, Am Campus 1, A-3400 Klosterneuburg, Austria
E-mail: lerdos@ist.ac.at

Quantitative stability results for the Brunn-Minkowski inequality

Alessio Figalli

Abstract. The Brunn-Minkowski inequality gives a lower bound on the Lebesgue measure of a sumset in terms of the measures of the individual sets. This inequality plays a crucial role in the theory of convex bodies and has many interactions with isoperimetry and functional analysis. Stability of optimizers of this inequality in one dimension is a consequence of classical results in additive combinatorics. In this note we describe how optimal transportation and analytic tools can be used to obtain quantitative stability results in higher dimension.

Mathematics Subject Classification (2010). Primary 49Q20; Secondary 35A23.

Keywords. Geometric and functional inequalities, quantitative stability, sumsets, Brunn-Minkowski.

1. Introduction

Geometric and functional inequalities naturally appear in several problems in the calculus of variations, partial differential equations, geometry, etc. Among the most classical inequalities, we recall the isoperimetric **(Isop)** inequality, Sobolev **(Sob)** and Gagliardo-Nirenberg **(GN)** inequalities, and the Brunn-Minkowski **(BM)** inequality.

Although different, all these inequalities are intimately related. Indeed it is well-known that the following chain of implications holds:

$$(\mathbf{BM}) \quad \Rightarrow \quad (\mathbf{Isop}) \quad \Rightarrow \quad (\mathbf{Sob}) \quad \Rightarrow \quad (\mathbf{GN}). \tag{1.1}$$

Let us introduce briefly all these inequalities and describe this connection.

The Brunn-Minkowski inequality deals with sum of sets: given A, B nonempty subsets of $\mathbb{R}^n$ one defines $A + B := \{a + b \,:\, a \in A,\, b \in B\}$. Then **(BM)** gives a sharp lower bound on the measure of $A + B$ in terms of the measures of A and B: more precisely,

$$(\mathbf{BM}) \qquad |A + B|^{1/n} \geq |A|^{1/n} + |B|^{1/n}.$$

The isoperimetric inequality, instead, deals with boundary measure and volume: if $E \subset \mathbb{R}^n$ is a smooth bounded set, its volume $|E|$ is controlled by the perimeter $P(E)$:

$$(\mathbf{Isop}) \qquad P(E) \geq C(n)|E|^{(n-1)/n},$$

where $C(n) > 0$ is an explicit dimensional constant.
Sobolev inequalities are a "generalization" of isoperimetric inequalities but they concern

Proceedings of the International Congress of Mathematicians, Seoul, 2014

functions instead of sets: they say that the gradient of a function in some L^p norm controls some L^q norm of the function itself. More precisely, for any $u \in C_c^\infty(\mathbb{R}^n)$ and $p < n$,

$$(\mathbf{Sob}) \qquad \|\nabla u\|_{L^p(\mathbb{R}^n)} \geq C(n,p)\|u\|_{L^q(\mathbb{R}^n)},$$

where $q := \frac{np}{n-p} > p$, and $C(n,p) > 0$ is an explicit constant depending only on n and p. Finally, Gagliardo-Nirenberg inequalities control a L^q norm of a function with a weaker L^r norm of the function and the L^p norm of its gradient:

$$(\mathbf{GN}) \qquad \|\nabla u\|^\theta_{L^p(\mathbb{R}^n)}\|u\|^{1-\theta}_{L^r(\mathbb{R}^n)} \geq C(n,p,r)\|u\|_{L^q(\mathbb{R}^n)},$$

where $\theta = \theta(n,p,r) \in (0,1)$, $q = q(n,p,r) \in \left(r, \frac{np}{n-p}\right)$, and $C(n,p,r) > 0$ are explicit constants depending only on n, p, and r.

We now explain the chain of implications (1.1).

• $(\mathbf{BM}) \Rightarrow (\mathbf{Isop})$. We apply **(BM)** to $A = E$ and $B = B_\epsilon(0)$ for some small $\epsilon > 0$. Then

$$|E + B_\varepsilon(0)|^{1/n} \geq |E|^{1/n} + |B_\varepsilon(0)|^{1/n} = |E|^{1/n} + \epsilon\,|B_1(0)|^{1/n}. \tag{1.2}$$

On the other hand $E + B_\varepsilon(0)$ coincides with the ϵ-neighborhood of E, hence its volume is approximately

$$|E + B_\varepsilon(0)| = |E| + \epsilon\,P(E) + o(\epsilon).$$

Inserting the above expression in (1.2), a Taylor expansion gives

$$|E|^{1/n} + \epsilon\,\frac{1}{n}\frac{P(E)}{|E|^{(n-1)/n}} + o(\epsilon) = \Big(|E| + \epsilon\,P(E) + o(\epsilon)\Big)^{1/n} \geq |E|^{1/n} + \epsilon\,|B_1(0)|^{1/n},$$

that is

$$\frac{1}{n}\frac{P(E)}{|E|^{(n-1)/n}} + \frac{o(\varepsilon)}{\varepsilon} \geq |B_1(0)|^{1/n},$$

and letting $\epsilon \to 0$ we obtain

$$\frac{1}{n}\frac{P(E)}{|E|^{(n-1)/n}} \geq |B_1(0)|^{1/n}$$

as desired.

• $(\mathbf{Isop}) \Rightarrow (\mathbf{Sob})$. The basic idea is that the perimeter of a set corresponds to the mass of the gradient of the indicator function of a set E: more precisely, if $\mathbf{1}_E$ denotes the indicator function of a set E, that is,

$$\mathbf{1}_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E, \end{cases}$$

then

$$P(E) = \int_{\mathbb{R}^n} |\nabla \mathbf{1}_E|.$$

This formula is not rigorous since $\nabla \mathbf{1}_E$ is zero a.e. (both inside and outside E) while it gives a "Dirac mass" on the boundary of E, but it can be made precise with the notions of sets of finite perimeter and of BV functions [28].

Now, to relate sets and functions, one applies the layer-cake formula: assuming for simplicity $u \geq 0$, u can be written as

$$u(x) = \int_0^\infty \mathbf{1}_{\{u>t\}}(x)\, dt, \tag{1.3}$$

from which one deduces, by differentiation,

$$\nabla u = \int_0^\infty \nabla \mathbf{1}_{\{u>t\}}\, dt.$$

Although not completely obvious it is possible to show that the above identity still holds when taking the moduli on the gradients, that is

$$|\nabla u| = \int_0^\infty |\nabla \mathbf{1}_{\{u>t\}}|\, dt,$$

so integrating this identity on $\mathbb{R}^n$ we obtain

$$\int_{\mathbb{R}^n} |\nabla u| = \int_0^\infty \left(\int_{\mathbb{R}^n} |\nabla \mathbf{1}_{\{u>t\}}| \right) dt = \int_0^\infty P(\{u>t\})\, dt$$

(see [28, Theorem 13.1] for a rigorous proof of the above identity, usually called co-area formula). Applying now **(Isop)** to the sets $\{u>t\}$ one obtains

$$\int_{\mathbb{R}^n} |\nabla u| \geq C(n) \int_0^\infty |\{u>t\}|^{(n-1/n)}\, dt = C(n) \int_0^\infty \|\mathbf{1}_{\{u>t\}}\|_{L^{(n-1/n)}(\mathbb{R}^n)}\, dt. \tag{1.4}$$

Recalling now that the norm of the integral is less than the integral of the norm, from (1.3) we get

$$\|u\|_{L^{(n-1/n)}(\mathbb{R}^n)} = \left\| \int_0^\infty \mathbf{1}_{\{u>t\}}(x)\, dt \right\|_{L^{(n-1/n)}(\mathbb{R}^n)} \leq \int_0^\infty \|\mathbf{1}_{\{u>t\}}\|_{L^{(n-1/n)}(\mathbb{R}^n)}\, dt,$$

that combined with (1.4) proves **(Sob)** when $p = 1$.

To prove the general case $p \in [1, n)$, it suffices to apply first **(Sob)** with $p = 1$ to the function $v := |u|^\gamma$ with $\gamma := \frac{p(n-1)}{n-p}$, and then use Hölder inequality: indeed, recalling that $q = \frac{np}{n-p}$, setting $s := \frac{np-p}{n-p}$ we get

$$\begin{aligned}
\|u\|_{L^q(\mathbb{R}^n)} &= \big\| |u|^s \big\|^{1/s}_{L^{n/(n-1)}(\mathbb{R}^n)} \leq C(n)^{1/s} \big\| |\nabla |u|^s| \big\|^{1/s}_{L^1(\mathbb{R}^n)} \\
&\leq (s\, C(n))^{1/s} \big\| |\nabla u|\, |u|^{s-1} \big\|^{1/s}_{L^1(\mathbb{R}^n)} \\
&\leq (s\, C(n))^{1/s} \|\nabla u\|^{1/s}_{L^p(\mathbb{R}^n)} \|u\|^{1-1/s}_{L^q(\mathbb{R}^n)},
\end{aligned}$$

as desired.

• $(\mathbf{Sob}) \Rightarrow (\mathbf{GN})$. As in the last step of the previous argument, also this implication is a consequence of Hölder inequality: more precisely, given any choice of numbers $r < q < s$ one applies Hölder inequality to get

$$\|u\|_{L^q(\mathbb{R}^n)} \leq \|u\|^\theta_{L^s(\mathbb{R}^n)} \|u\|^{1-\theta}_{L^r(\mathbb{R}^n)}, \qquad \text{for some } \theta = \theta(r,q,s) \in (0,1),$$

and then chooses $s := \frac{np}{n-p}$ to control $\|u\|_{L^s(\mathbb{R}^n)}$ with $\|\nabla u\|_{L^p(\mathbb{R}^n)}$ using **(Sob)**.

The discussion above shows how it is possible to derive some inequalities from others, and that Brunn-Minkowski is at the basis of all of them. However, it is interesting to point out that, although one inequality may imply another one, the constants we obtained from the proofs may not be sharp. More precisely, in the discussion above **(BM)** implied **(Isop)** with the sharp constant, and **(Isop)** implied **(Sob)** for $p = 1$ with sharp constant again, but all the other implications (based on Hölder inequality) are non-sharp.

The issue of the sharpness of a constant, as well as the characterization of minimizers, is a classical and important question which is by now well understood (at least for the class of inequalities we are considering). More recently, a lot of attention has been given to the stability issue:

> *Suppose that a function almost attains the equality in one of the previous inequalities. Can we prove, if possible in some quantitative way, that such a function is close (in some suitable sense) to one of the minimizers?*

In the latest years several results have been obtained in this direction, showing stability for isoperimetric inequalities [12, 13, 17, 22, 27], the Brunn-Minkowski inequality on convex sets [23], Sobolev [11, 15, 24] and Gagliardo-Nirenberg inequalities [3, 15]. We notice that, apart from their own interest, this kind of results have applications in the study of geometric problems (see for instance [9, 20, 21]) and can be used to obtain quantitative rates of convergence for diffusion equations (see for instance [3]).

The aim of this note is to describe recent results on the stability of the Brunn-Minkowski inequality [18, 19, 23]. As we shall see, the study of this problem involves an interplay between linear structure, analysis, and affine-invariant geometry of Euclidean spaces.

2. Sumsets and the Brunn-Minkowski inequality

Let $A \subset \mathbb{R}^n$ be a Borel set with $|A| > 0$. We define its semi-sum as

$$\frac{A+A}{2} := \left\{ \frac{a+a'}{2} \,:\, a, a' \in A \right\}.$$

Obviously $\frac{A+A}{2} \supset A$, hence

$$\left| \frac{A+A}{2} \right| \geq |A|.$$

In addition it is not difficult to show that equality holds if and only if "A is convex": more precisely, a necessary and sufficient condition for equality is

$$|\mathrm{co}(A) \setminus A| = 0,$$

where $\mathrm{co}(A)$ denote the convex hull of A.

This result is a particular case of a more general inequality: the *Brunn-Minkowski inequality*. Although we already introduced it in the previous section we restate it here since, for convenience, we will write it in an equivalent form with a semisum instead of a sum of sets.

Given $A, B \subset \mathbb{R}^n$ Borel sets, with $|A|, |B| > 0$, we define

$$\frac{A+B}{2} := \left\{ \frac{a+b}{2} \, : \, a \in A, \, b \in B \right\}.$$

As we already mentioned in the introduction, the Brunn-Minkowski inequality gives a control from below on the measure of $A + B$ in terms of the measures of A and B:

$$\left| \frac{A+B}{2} \right|^{1/n} \geq \frac{|A|^{1/n} + |B|^{1/n}}{2}. \tag{2.1}$$

In addition, equality holds if and only if "A and B are homothetic convex sets", that is, there exist $\alpha, \beta > 0$, $v, w \in \mathbb{R}^n$, and K convex, such that

$$A \subset \alpha K + v, \qquad |(\alpha K + v) \setminus A| = 0,$$

$$B \subset \beta K + w, \qquad |(\beta K + w) \setminus B| = 0.$$

The main question we are interested in is the following: *Are these results stable?*

For instance, assume that

$$\left| \frac{A+A}{2} \right| = |A| + \epsilon$$

with $\epsilon \ll |A|$. Is it true that A is close to its convex hull? Moreover, can we quantify the closeness in terms of ϵ?

This same kind of question can also be asked for the Brunn-Minkowski inequality: assume that (2.1) is almost an equality. Is it true that both A and B are almost convex, and that actually they are close to the same convex set?

Let us notice that the latter question has two statements in it: we are wondering if:

- The error in the Brunn-Minkowski inequality controls how far A and B are from their convex hulls **(Convexity)**.
- The error in the Brunn-Minkowski inequality controls the difference between the shapes of A and B **(Homothety)**.

The aim of this note is to address the questions raised above. We will proceed by steps as follows: in Section 3 we will focus only on the **(Homothety)** issue. More precisely, we assume that A and B are already convex and we prove that, if equality almost holds in (2.1), then A and B have almost the same shape. Then, in Section 4 we will focus on the **(Convexity)** issue in the simpler case $A = B$, and we shall prove that A is close to its convex hull. Finally, in Section 5 we will deal with the general case.

3. Stability on convex sets

Let A, B be bounded convex set with $0 < \lambda \leq |A|, |B| \leq \Lambda$, and

$$\delta(A, B) := \left| \frac{A+B}{2} \right|^{1/n} - \frac{|A|^{1/n} + |B|^{1/n}}{2}.$$

It follows from (2.1) that $\delta(A,B) \geq 0$, and we would like to show that $\delta(A,B)$ controls some kind of "distance" between the shape of A and the one of B.

In order to compare A and B, we first want them to have the same volume. Hence, we renormalize A so that it has the same measure of B: if $\gamma := \frac{|B|^{1/n}}{|A|^{1/n}}$ then

$$|\gamma A| = |B|.$$

We then define a "distance"[1] between A and B as follows:

$$d(A,B) := \min_{x\in\mathbb{R}^n} |B\Delta(x+\gamma A)|,$$

where

$$E\Delta F := (E\setminus F)\cup(F\setminus E).$$

The following result has been obtained in [22, Section 4] (see also [23]):

Theorem 3.1. *Let A, B be bounded convex set with $0 < \lambda \leq |A|, |B| \leq \Lambda$. There exists $C = C(n,\lambda,\Lambda)$ such that*

$$d(A,B) \leq C\,\delta(A,B)^{1/2}.$$

As observed in [22, Section 4], the exponent $1/2$ is optimal and the constant C is explicit.

Sketch of the proof. Notice that Theorem 3.1 contains as a corollary the validity of the Brunn-Minkowski inequality on convex sets, as it implies in particular that $\delta(A,B) \geq 0$. Hence, as a general principle, in order to hope for a stability estimate to hold, we should at least be able to prove the easier inequality $\delta \geq 0$.

Thus, we will first show how optimal transportation can be used to prove the Brunn-Minkowski inequality, and then we will explain how the same proof can be exploited to obtain the desired stability result. We notice that a proof of (2.1) using optimal transportation was first given in [29]. Here we follow the argument given in [23] since (as we shall see in Step 2 below) it is particularly suitable to be used to obtain a stability estimate.

Step 1: A proof of (2.1) **via optimal transportation.** We notice that this part of the proof does not require A and B to be convex. Hence, we consider $A, B \subset \mathbb{R}^n$ Borel with $|A|, |B| > 0$, and we define the probability measures

$$\mu := \frac{\mathbf{1}_A(x)}{|A|}\,dx, \qquad \nu := \frac{\mathbf{1}_B(y)}{|B|}\,dy.$$

Since μ is absolutely continuous with respect to the Lebesgue measure, Brenier's Theorem [1] ensures the existence of a convex function $\varphi : \mathbb{R}^n \to \mathbb{R}$ whose gradient $T := \nabla\varphi$ sends μ onto ν:

$$T_{\#}\mu = \nu, \qquad \text{i.e., } \mu(T^{-1}(E)) = \nu(E) \text{ for all } E \text{ Borel}.$$

It is easy to check that T satisfies (at least formally) the following properties:

[1] Notice that d is not properly a distance since it is not symmetric. Still, it is a natural geometric quantity which measures, up to translations, the L^1-closeness between γA and B: indeed, observe that an equivalent formulation for d is

$$d(A,B) := \min_{x\in\mathbb{R}^n} \frac{1}{2}\|\mathbf{1}_B - \mathbf{1}_{x+\gamma A}\|_{L^1(\mathbb{R}^n)}.$$

(i) $T(A) = B$;

(ii) $\det(\nabla T) = |B|/|A| = \gamma^n$.

Indeed, (i) is a consequence of the fact that μ lives on A and ν on B.

For (ii) we observe that if $\chi : \mathbb{R}^n \to \mathbb{R}$ denotes a test function, the condition $T_{\#}\mu = \nu$ gives

$$\int_{\mathbb{R}^n} \chi(T(x)) \, \frac{\mathbf{1}_A(x)}{|A|} \, dx = \int_{\mathbb{R}^n} \chi(y) \, \frac{\mathbf{1}_B(y)}{|B|} \, dy.$$

Now, assuming in addition that T is a diffeomorphism, we can set $y = T(x)$ and use the change of variable formula to obtain that the second integral is equal to

$$\int_{\mathbb{R}^n} \chi(T(x)) \, \frac{\mathbf{1}_B(T(x))}{|B|} \big|\det(\nabla T(x))\big| \, dx.$$

Hence, since $T(A) = B$, setting $\hat{\chi} := \chi \circ T$ we obtain

$$\frac{1}{|A|} \int_A \hat{\chi}(x) \, dx = \frac{1}{|B|} \int_B \hat{\chi}(x) \big|\det(\nabla T(x))\big| \, dx,$$

so (ii) follows by the arbitrariness of $\hat{\chi}$ (or equivalently of χ).

We now define

$$\frac{Id + T}{2}(A) := \left\{ \frac{a + T(a)}{2} \ : \ a \in A \right\}.$$

Then it follows by (i) that

$$\frac{Id + T}{2}(A) \subset \frac{A + B}{2},$$

from which we deduce that

$$\begin{aligned} \left| \frac{A+B}{2} \right| \geq \left| \frac{Id+T}{2}(A) \right| &= \int_A \det\left(\frac{Id + \nabla T}{2} \right) \\ &= \frac{1}{2^n} \int_A \det(Id + \nabla T). \end{aligned} \tag{3.1}$$

Up to now we never used that T is the gradient of a convex function. We now exploit this: since $T = \nabla \varphi$ with φ convex, the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $D^2\varphi = \nabla T$ are nonnegative. Thus, using the inequality between AM-GM (arithmetic mean and geometric mean), one can prove that

$$\begin{aligned} \det(Id + \nabla T) = \prod_{i=1}^n (1 + \lambda_i) &\geq \left(1 + \left(\prod_{i=1}^n \lambda_i \right)^{1/n} \right)^n \\ &\overset{(2)}{=} \left(1 + \left(|B|/|A| \right)^{1/n} \right)^n. \end{aligned}$$

Combining this inequality with (3.1) we finally obtain

$$\left| \frac{A+B}{2} \right| \geq \frac{1}{2^n} \int_A \left(1 + \left(|B|/|A| \right)^{1/n} \right)^n = \left(\frac{|A|^{1/n} + |B|^{1/n}}{2} \right)^n,$$

as desired. Notice that this argument is formal since a priori the transport map is not smooth, but there are two possible ways of fixing this: if A and B are general Borel set, then the above proof can be made rigorous by using some fine results on BV functions and sets of finite perimeter, see [22]; if instead one only wants to prove the result when A and B are convex, then one can rely on [2] to say that the map T is actually smooth.

Step 2: The quantitative estimate. In the previous step we proved that $\delta(A,B) \geq 0$ for any Borel sets A, B. We now want to control $d(A,B)$ with $\delta(A,B)$ when A and B are convex. The first observation is that, by the proof above (using the same notation), we have

$$2^n \delta \geq \int_A \Big[\prod_{i=1}^n (1+\lambda_i) - \Big(1 + \Big(\prod_{i=1}^n \lambda_i\Big)^{1/n}\Big)^n\Big].$$

Notice that before we used AM-GM to say that the integrand in the right hand side was nonnegative. Also, recall that equality holds in AM-GM if and only if all numbers are equal. Hence, by an improved version of AM-GM which quantifies the closeness of the numbers in terms of the error (see [22, Lemma 2.5]), we obtain a precise quantitative form of the following rough statement:

$$\delta \ll 1 \quad \Rightarrow \quad \lambda_i(x) \simeq \lambda_j(x) \qquad \forall\, i,j, \text{ for most } x \in A.$$

Since $\prod_{i=1}^n \lambda_i(x) = |B|/|A|$ (by (ii)), we also deduce that

$$\lambda_i(x) \simeq \gamma \qquad \forall\, i, \text{ for most } x \in A,$$

where $\gamma = (|B|/|A|)^{1/n}$. Using now that $\nabla T = D^2\varphi$ is a symmetric matrix, from the fact that all its eigenvalues are close to γ we deduce that

$$\nabla T(x) \simeq \gamma\, \mathrm{Id} \qquad \text{for most } x \in A,$$

To be more precise, by carefully performing the above estimates, one can prove that

$$\int_A |\nabla T - \gamma\, \mathrm{Id}| \leq C\, \delta(A,B)^{1/2}.$$

We now want to use the estimate above on $\nabla(T - \gamma\, x)$ to obtain a bound on $T - \gamma\, x$. For this, we wish to apply a trace inequality of the form

$$C \int_A |\nabla f| \geq \inf_{c \in \mathbb{R}} \int_{\partial A} |f - c|\, d\sigma \qquad \forall\, f \in C^\infty(\mathbb{R}^n),$$

where σ denotes the surface measure on ∂A. Since A is convex its boundary is Lipschitz, so the above trace inequality holds and we can show that, up to a translation,

$$C\, \delta(A,B)^{1/2} \geq \int_{\partial A} |T(x) - \gamma\, x|\, d\sigma.$$

In particular, since $T(x) \in \overline{B}$ for $x \in \overline{A}$, we deduce that

$$C\, \delta(A,B)^{1/2} \geq \int_{\partial A} \mathrm{dist}(x, B/\gamma)\, d\sigma. \tag{3.2}$$

A simple geometric argument then shows that

$$C \int_{\partial A} \operatorname{dist}(x, B/\gamma)\, d\sigma \geq |A \setminus (B/\gamma)| \tag{3.3}$$

(see [23, Proof of Theorem 1, Step 4]). Since $|A| = |B/\gamma|$ one observes that

$$|A \setminus (B/\gamma)| = |(B/\gamma) \setminus A| = \frac{1}{2}|A \Delta (B/\gamma)|,$$

so the desired result follows from (3.2) and (3.3). □

4. Stability when $A = B$

As we already mentioned before, the quantitative estimate in the proof of Theorem 3.1 works only if A and B are convex. There are several technical reasons why we need this assumption, but there is also a simple way to see why one cannot hope to use the above proof to solve the **(Convexity)** issue that we raised at the end of Section 2.

Indeed, assume that $A = B$. In that case the map T in the proof of Theorem 3.1 is simply the identity map, that is $T(x) = x$, and the argument given in the first part of the proof is completely "empty", in the sense that it does not introduce any new information. In particular, the proof by optimal transportation does not help in showing that $\delta(A, A)$ controls the distance between A and its convex hull. Hence a completely new strategy is needed to address this issue.

4.1. The case $n = 1$. Already in the one dimensional case the problem is far from being trivial. Up to rescale A we can always assume that $|A| = 1$. Define

$$\delta_1(A) := |A + A| - 2|A|.$$

It is easy to see that $\delta_1(A)$ cannot control in general $|\mathrm{co}(A) \setminus A|$: indeed take

$$A := [0, 1/2] \cup [L, L + 1/2]$$

with $L \gg 1$. Then

$$A + A = [0, 1] \cup [L, L + 1] \cup [2L, 2L + 1],$$

which implies that $\delta_1(A) = 1 (= |A|)$ while $|\mathrm{co}(A) \setminus A| = L - 1/2$ is arbitrarily large. Luckily, as shown by the following theorem, this is essentially the only thing that can go wrong.

Theorem 4.1. *Let $A \subset \mathbb{R}$ be a measurable set with $|A| = 1$, and denote by $\mathrm{co}(A)$ its convex hull. If $\delta_1(A) < 1$ then*

$$|\mathrm{co}(A) \setminus A| \leq \delta_1(A).$$

This theorem can be obtained as a corollary of a result of G. Freiman [25] about the structure of additive subsets of $\mathbb{Z}$. (See [26] or [31, Theorem 5.11] for a statement and a proof.) However, it turns out that to prove of Theorem 4.1 one only needs weaker results. For convenience of the reader, instead of relying on deep and intricate combinatorial results,

we will give an elementary proof of Theorem 4.1. Our proof is based on the simple observation that a subset of $\mathbb{R}$ can be discretized to a subset of $\mathbb{Z}$ starting at 0 and ending at a prime number p. This may look strange from an analytic point of view, but it considerably simplifies the combinatorial aspects.

Sketch of the proof of Theorem 4.1. The proof consists of three steps:
- first, one proves a Brunn-Minkowski type inequality in $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$;
- then, we show a simple case of the so-called Freiman's $3k-3$ Theorem;
- finally, an approximation argument proves the theorem on $\mathbb{R}$.

We give here a sketch of the proof, referring to [18, Section 2] for more details. Note that, in the discrete setting, $|A|$ will always denote the cardinality of the set A.

Step 1: Cauchy-Davenport inequality. *If $\emptyset \neq A, B \subset \mathbb{Z}_p$ with p prime, then* $|A+B| \geq \min\{|A|+|B|-1, p\}$.

The proof is by induction on the size of $|B|$, the case $|B|=1$ being trivial. To perform the induction, it is useful to define the e-transform of A and B: given $e \in A-B$, define

$$A_{(e)} := A \cup (B+e), \qquad B_{(e)} := B \cap (A-e).$$

Notice that

$$A_{(e)} + B_{(e)} \subset A+B, \qquad |A_{(e)}| + |B_{(e)}| = |A| + |B| \tag{4.1}$$

We now consider two cases:
Case 1: there exists $e \in A-B$ such that $|B_e| < |B|$. Then by the inductive step

$$|A_{(e)} + B_{(e)}| \geq \min\{|A_{(e)}| + |B_{(e)}| - 1, p\},$$

and we conclude by (4.1).
Case 2: $|B_e| = |B|$ for any $e \in A-B$. This means that $B_{(e)} = B$ for any $e \in A-B$, which implies that $B+e \subset A$ for any $e \in A-B$, that is $A+B-B \subset A$. Thus $B-B$ is contained inside to the subgroup $Sym_1(A) := \{h \in \mathbb{Z}_p : A+h = A\}$. Since $|B|>1$ and the only subgroups of $\mathbb{Z}_p$ are $\{0\}$ and $\mathbb{Z}_p$, this means that $Sym_1(A) = \mathbb{Z}_p$. This implies that $A = \mathbb{Z}_p$, so the result is trivially true since $|A+B| \geq |A| = p$.

Step 2: Freiman's $3k-3$ Theorem. *Let A be a finite nonempty subset of $\mathbb{Z}$ with* $\min(A) = 0$ *and* $\max(A) = p$*, with p prime. Assume that $|A+A| - 2|A| < |A| - 3$. Then* $|\{0,\dots,p\} \setminus A| \leq |A+A| - 2|A| + 1$.

Before proving this result, we notice that in Step 3 we will apply it to sets with very high cardinality. Hence, if we forget about the terms -3 and $+1$, the above statement says the following: if $\delta_1(A) < |A|$ then $|\{0,\dots,p\} \setminus A| \leq \delta_1(A)$. Notice that this statement is exactly what we wanted, if one thinks that $\{0,\dots,p\}$ is the "convex hull" of A in this discrete setting.

To prove this step, let $\phi_p : \mathbb{Z} \to \mathbb{Z}_p$ denote the canonical quotient map. We have

$$\begin{cases} A+A \supset A \cup (A+p) \\ \phi_p(A) = \phi_p(A+p) \end{cases} \quad \Rightarrow \quad |\phi_p(A+A)| \leq |A+A| - |A|,$$

hence

$$|\phi_p(A+A)| < 2|A| - 3 = 2|\phi_p(A)| - 1$$

(observe that $|\phi_p(A)| = |A| - 1$). Thus, by Step 1,

$$|\phi_p(A + A)| = p,$$

which implies

$$p \le |A + A| - |A|,$$

as desired.

Step 3: The discretization argument. We now prove the theorem. We first notice that, by approximation, it suffices to prove the result when A is compact.

Without loss of generality assume $\mathrm{co}(A) = [0, M]$. We then approximate A with sets A_k of the form

$$A_k := \bigcup_{j\,:\,I_{k,j}\cap A\neq\emptyset} I_{k,j}, \qquad I_{k,j} := \left[\frac{jM}{p_k+1}, \frac{(j+1)M}{p_k+1}\right],$$

where p_k is a sequence of prime numbers with $p_k \to +\infty$. Then, we define

$$B_k := \{j \in \mathbb{Z} : I_{k,j} \subset A_k\} \subset \mathbb{Z}.$$

Thanks to the assumption $\delta_1(A) < |A|$ it is easy to check that B_k satisfies the assumptions of Step 2 for k large enough. Hence, it follows by Step 2 that

$$|\{0, \dots, p_k\} \setminus B_k| \le |B_k + B_k| - 2|B_k| + 1,$$

that rewritten in terms of A_k gives

$$|[0, M] \setminus A_k| \le \delta_1(A_k) + \frac{1}{p_k + 1}.$$

Letting $k \to \infty$ concludes the proof. □

4.2. The case $n \ge 2$. Let us define the *deficit* of A as

$$\delta(A) := \frac{\left|\frac{1}{2}(A + A)\right|}{|A|} - 1 = \frac{|A + A|}{|2A|} - 1.$$

In the previous section we showed how to obtain a precise stability result in one dimension by translating it into a problem on $\mathbb{Z}$. The main result in [18] is a quantitative stability result in arbitrary dimension, showing that a power of $\delta(A)$ dominates the measure of the difference between A and its convex hull $\mathrm{co}(A)$.

Theorem 4.2. *Let $n \ge 2$. There exist computable dimensional constants $\delta_n, c_n > 0$ such that if $A \subset \mathbb{R}^n$ is a measurable set of positive measure with $\delta(A) \le \delta_n$, then*

$$\delta(A)^{\alpha_n} \ge c_n \frac{|\mathrm{co}(A) \setminus A|}{|A|}, \qquad \alpha_n := \frac{1}{8 \cdot 16^{n-2} n!(n-1)!}.$$

Sketch of the proof. Let $\mathcal{H}^k$ denote the k-dimensional Hausdorff measure on $\mathbb{R}^n$, denote by $(y, t) \in \mathbb{R}^{n-1} \times \mathbb{R}$ a point in $\mathbb{R}^n$, and let $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the canonical projection $\pi(y, t) := y$. Given $E \subset \mathbb{R}^n$ and $y \in \mathbb{R}^{n-1}$, we use the notation

$$E_y := E \cap \pi^{-1}(y).$$

We say that E is *t-convex* if E_y is a segment for every $y \in \pi(E)$.

Our proof is by induction on n, the case $n = 1$ being true by Theorem 4.1. As a preliminary observation we notice that if $L : \mathbb{R}^n \to \mathbb{R}^n$ is an affine transformation with $\det L = 1$, then $\delta(A) = \delta(L(A))$ and $|\operatorname{co}(A) \setminus A| = |\operatorname{co}(L(A)) \setminus L(A)|$. Hence it is enough to prove the theorem for $L(A)$. This simple remark will be extremely useful, as it will allow us to reduce to the case when A is bounded (see Step 3 below).

Step 1. The first argument consists in combining Theorem 4.1 with a Fubini's type argument to show that, for most $y \in \pi(A)$, the set $A_y \subset \{y\} \times \mathbb{R}$ is close to its convex hull. Since this part is elementary and also it will be useful to explain one of the main differences with the case $A \neq B$, we detail it.

By Fubini

$$\delta(A) = \left|\tfrac{1}{2}(A + A)\right| - |A| = \int_{\mathbb{R}^{n-1}} \mathcal{H}^1\Big(\big(\tfrac{1}{2}(A + A)\big)_y\Big) - \mathcal{H}^1(A_y)\, dy,$$

hence, since $\frac{1}{2}(A_y + A_y) \subset \big(\frac{1}{2}(A + A)\big)_y$, we deduce

$$\delta(A) \geq \int_{\mathbb{R}^{n-1}} \mathcal{H}^1\Big(\tfrac{1}{2}(A_y + A_y)\Big) - \mathcal{H}^1(A_y)\, dy.$$

We now distinguish between two cases, depending whether we can apply Theorem 4.1 or not:

- y is *good* if $\mathcal{H}^1\left(\frac{1}{2}(A_y + A_y)\right) - \mathcal{H}^1(A_y) \leq \mathcal{H}^1(A_y)/2$.
- y is *bad* if $\mathcal{H}^1\left(\frac{1}{2}(A_y + A_y)\right) - \mathcal{H}^1(A_y) \geq \mathcal{H}^1(A_y)/2$.

Notice that if y is "good" we can apply Theorem 4.1 to A_y, while in the "bad" case

$$\mathcal{H}^1\left(\tfrac{1}{2}(A_y + A_y)\right) - \mathcal{H}^1(A_y)$$

trivially controls $\mathcal{H}^1(A_y)$, therefore

$$\delta(A) \geq \int_{y\,\text{good}} \mathcal{H}^1\big(\operatorname{co}(A_y) \setminus A_y\big)\, dy + \int_{y\,\text{bad}} \mathcal{H}^1\big(A_y\big)/2\, dy.$$

From this estimate we deduce that A is (quantitatively) close to the t-convex set

$$A' := \bigcup_{y\,\text{good}} \operatorname{co}(A_y).$$

Now, applying the inductive hypothesis with $n-1$, an argument similar to the one above shows that

$$E_{s_0} := \{y \in \mathbb{R}^{n-1} : \mathcal{H}^1(A_y) > s_0\}$$

is close to its convex hull for some small $s_0 > 0$. Using this fact we prove that, for s_0 small enough, A is close to the t-convex set

$$A^* := \bigcup_{y \in E_{s_0}} \operatorname{co}(A_y).$$

Step 2. Define $\mathrm{co}(A_y) = \{y\} \times [a(y), b(y)]$. A careful analysis based on the assumption that $\delta(A)$ is small shows that the midpoint $c(y) := (a(y) + b(y))/2$ of A_y^* have bounded second differences as a function of y: more precisely,

$$|c(y') + c(y'') - 2c(y)| \le 6 \qquad \forall\, y, y', y'' \in E_{s_0},\, y = \frac{y' + y''}{2}.$$

Step 3. As mentioned before, it is enough to prove the result for $L(A)$ instead of A, where $L : \mathbb{R}^n \to \mathbb{R}^n$ is a measure preserving affine transformation. We show here that we can find a map L such that $L(A^*)$ is bounded.

Using the above bound for c, we prove that c is at bounded distance from an affine function ℓ. In addition, since $\pi(A^*) = E_{s_0}$ and E_{s_0} is close to its convex hull (by Step 1), a classical result in convex geometry (called John's Lemma) states that we can find a measure preserving affine transformation $T : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ such that $T(\mathrm{co}(E_{s_0}))$ is bounded. Hence, up to applying the affine measure-preserving transformation

$$L(y, t) := (Ty, t - \ell(Ty)),$$

A^* is bounded.

Step 4. We want to show that A^* is close to a convex set. For this, we need to prove a statement of the following form (the exact form of the statement proved in [18] is more involved):

Assume that

$$\frac{f(y') + f(y'')}{2} \le f(y) + \gamma \;\forall\, y, y', y'' \in E,\, y = \frac{y' + y''}{2}, |f| \le 1, \,|\,\mathrm{co}(E) \setminus E| \le \gamma$$

for some $E \subset \mathbb{R}^{n-1}$, $\gamma \ll 1$. Then there exist $C, \alpha > 0$ such that

$$\int_E |f - F| \le C\gamma^\alpha, \qquad \text{for some function } F : \mathbb{R}^{n-1} \to \mathbb{R} \text{ concave.}$$

Recalling that $\mathrm{co}(A_y) = \{y\} \times [a(y), b(y)]$, the above statement applied to $f = b$ and $f = -a$ shows that A^* (hence also A) is quantitatively close to a convex set.

Step 5. By a simple geometric argument we prove that the convex set obtained in Step 4 can be assumed to be the convex hull of A, concluding the proof. □

5. Stability when $A \neq B$

As in the case $A = B$, when $n = 1$ a sharp stability result holds as a consequence of classical theorems in additive combinatorics (an elementary proof of this result can be given using Kemperman's theorem [7, 8]):

Theorem 5.1. *Let $A, B \subset \mathbb{R}$ be measurable sets. If $|A + B| < |A| + |B| + \delta$ for some $\delta \le \min\{|A|, |B|\}$, then there exist two intervals $I, J \subset \mathbb{R}$ such that $A \subset I$, $B \subset J$, $|I \setminus A| \le \delta$, and $|J \setminus B| \le \delta$.*

Concerning the higher dimensional case, in [5, 6] M. Christ proved a *qualitative* stability result for (2.1), namely, if $|A+B|^{1/n}$ is close to $|A|^{1/n}+|B|^{1/n}$ then A and B are close to homothetic convex sets.

The main result in [19] is a quantitative version of Christ's result. After dilating A and B appropriately, we can assume $|A| = |B| = 1$ while replacing the semisum $(A+B)/2$ by a convex combination $S := tA + (1-t)B$ with $t \in (0,1)$. It follows by (2.1) that $|S| = 1+\delta$ for some $\delta \geq 0$. Since our proof is by induction on the dimension, it will be convenient to allow the measures of $|A|$ and $|B|$ not to be exactly equal, but just close in terms of δ. The main result of [19] shows that the measure of the difference between the sets A and B and their convex hull is bounded by a power δ^{ε}, confirming a conjecture of Christ [5].

Theorem 5.2. *Let A, B be measurable subsets of $\mathbb{R}^n$ with $n \geq 2$, and define $S := tA + (1-t)B$ for some $t \in [\tau, 1-\tau]$, $0 < \tau \leq 1/2$. There are computable dimensional constants N_n and computable functions $M_n(\tau), \varepsilon_n(\tau) > 0$ such that if*

$$\big||A|-1\big| + \big||B|-1\big| + \big||S|-1\big| \leq \delta \tag{5.1}$$

for some $\delta \leq e^{-M_n(\tau)}$, then there exists a convex set $\mathcal{K} \subset \mathbb{R}^n$ such that, up to a translation,

$$A, B \subset \mathcal{K} \qquad \text{and} \qquad |\mathcal{K} \setminus A| + |\mathcal{K} \setminus B| \leq \tau^{-N_n} \delta^{\varepsilon_n(\tau)}.$$

Explicitly, we may take

$$M_n(\tau) = \frac{2^{3^{n+2}} n^{3^n} |\log \tau|^{3^n}}{\tau^{3^n}}, \qquad \varepsilon_n(\tau) = \frac{\tau^{3^n}}{2^{3^{n+1}} n^{3^n} |\log \tau|^{3^n}}.$$

Sketch of the proof. For convenience we reintroduce some of the notation, although identical to the one in the proof of Theorem 4.2.

Let $\mathcal{H}^k$ denote k-dimensional Hausdorff measure on $\mathbb{R}^n$, denote by $x = (y,s) \in \mathbb{R}^{n-1} \times \mathbb{R}$ a point in $\mathbb{R}^n$, and $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ and $\bar{\pi} : \mathbb{R}^n \to \mathbb{R}$ denote the canonical projections, i.e.,

$$\pi(y,s) := y \qquad \text{and} \qquad \bar{\pi}(y,s) := s.$$

Given a compact set $E \subset \mathbb{R}^n$, $y \in \mathbb{R}^{n-1}$, and $\lambda > 0$, we use the notation

$$E_y := E \cap \pi^{-1}(y) \subset \{y\} \times \mathbb{R}, \qquad E(s) := E \cap \bar{\pi}^{-1}(s) \subset \mathbb{R}^{n-1} \times \{s\},$$

$$\mathcal{E}(\lambda) := \big\{ y \in \mathbb{R}^{n-1} : \mathcal{H}^1(E_y) > \lambda \big\}.$$

Following Christ [6], we consider different symmetrizations:
We define the *Schwarz symmetrization* E^* of E as follows. For each $t \in \mathbb{R}$,

- If $\mathcal{H}^{n-1}\big(E(s)\big) > 0$, then $E^*(s)$ is the closed disk centered at $0 \in \mathbb{R}^{n-1}$ with the same measure.
- If $\mathcal{H}^{n-1}\big(E(s)\big) = 0$, then $E^*(s)$ is empty.

We define the *Steiner symmetrization* $E^\star$ of E so that for each $y \in \mathbb{R}^{n-1}$, the set $E^\star_y$ is empty if $\mathcal{H}^1(E_y) = 0$; otherwise it is the closed interval of length $\mathcal{H}^1(E_y)$ centered at $0 \in \mathbb{R}$. Finally, we define $E^\natural := (E^\star)^*$.

The proof of Theorem 5.2 is very elaborate, combining the techniques of M. Christ with those developed by the present authors in [18] (where we proved Theorem 5.2 in the special case $A = B$ and $t = 1/2$), as well as several new ideas. Before describing the proof, we begin by showing one of the differences with respect to the case $A = B$.

Let us try to repeat Step 1 in the proof of Theorem 4.2: arguing in the very same way as we did there, one would obtain

$$\begin{aligned}\left|\tfrac{1}{2}(A+B)\right| - \frac{|A|+|B|}{2} &= \int_{\mathbb{R}^{n-1}} \mathcal{H}^1\Big(\big(\tfrac{1}{2}(A+B)\big)_y\Big) - \tfrac{\mathcal{H}^1(A_y)+\mathcal{H}^1(B_y)}{2}\,dy \\ &\geq \int_{\mathbb{R}^{n-1}} \mathcal{H}^1\Big(\tfrac{1}{2}(A_y+B_y)\Big) - \tfrac{\mathcal{H}^1(A_y)+\mathcal{H}^1(B_y)}{2}\,dy \\ &\geq 0.\end{aligned}$$

However the above inequality is *false* when $n \geq 2$, as one can immediately check by taking $A = B_1(0)$ and $B = \{0\}$, so that $\frac{1}{2}(A+B) = B_{1/2}(0)$. The mistake in the above argument is the following: in the last inequality we applied the one dimensional Brunn-Minkowski inequality

$$\mathcal{H}^1\Big(\tfrac{1}{2}(A_y+B_y)\Big) - \tfrac{\mathcal{H}^1(A_y)+\mathcal{H}^1(B_y)}{2} \geq 0,$$

but the latter is true only when both A_y and B_y are *nonempty* (since the semisum of any set with the empty set is the empty set).

As we shall see, this is just the first of several new issues that arise in the stability proof when $A \neq B$. We now give a detailed description of the proof of the theorem.

Case 1: $A = A^\natural$ and $B = B^\natural$. First we prove the theorem in the special case $A = A^\natural$ and $B = B^\natural$. In this case we have that

$$A_y = \{y\} \times [-a(y), a(y)] \quad \text{and} \quad B_y = \{y\} \times [-b(y), b(y)],$$

for some functions $a, b : \mathbb{R}^{n-1} \to \mathbb{R}^+$, and it is easy to show that a and b satisfy the "3-point concavity inequality"

$$ta(y') + (1-t)b(y'') \leq [ta + (1-t)b](y) + \delta^{1/4} \tag{5.2}$$

whenever y', y'', and $y := ty' + (1-t)y''$ belong to a large subset F of $\pi(A) \cap \pi(B)$. From this 3-point inequality and an elementary argument we show that a satisfies the "4-point concavity inequality"

$$a(y_1) + a(y_2) \leq a(y'_{12}) + a(y''_{12}) + \frac{2}{t}\delta^{1/4} \tag{5.3}$$

with $y'_{12} := t'y_1 + (1-t')y_2$, $y''_{12} := t''y_1 + (1-t'')y_2$, $t' := \frac{1}{2-t}$, $t'' := 1 - t'$, provided all four points belong to F. (The analogous inequality for b involves a different set of four points.)

Using this inequality and a variant of the argument in Step 4 of the proof of Theorem 4.2, we deduce that a is quantitatively close in L^1 to a concave function.

Once we know that a (and analogously b) is L^1-close to a concave function, we deduce that both A and B are L^1-close to convex sets K_A and K_B respectively, and we would like

to say that these convex sets are nearly the same. This is demonstrated by first showing that S is close to $tK_A + (1-t)K_B$, then applying Theorem 3.1 to deduce that K_A and K_B are almost homothetic, and then constructing a convex set $\mathcal{K}$ close to A and B and containing both of them.

This concludes the proof of Theorem 4.2 in the case $A = A^\natural$ and $B = B^\natural$.

Case 2: The general case. We now consider the general case, which we prove in several steps, culminating in induction on dimension.

Step 1. This first step is very close to the argument used by M. Christ in [6], although our analysis is more elaborate since we have to quantify every estimate.

Given A, B, and S, as in the theorem, we consider their symmetrizations $A^\natural$, $B^\natural$, and $S^\natural$, and apply Case 1 above to deduce that $A^\natural$ and $B^\natural$ are close to the same convex set. This information combined with a lemma of Christ allows us to deduce that functions $y \mapsto \mathcal{H}^1(A_y)$ and $y \mapsto \mathcal{H}^1(B_y)$ are almost equipartitioned (that is, the measure of their level sets $\mathcal{A}(\lambda)$ and $\mathcal{B}(\lambda)$ are very close). This fact combined with a Fubini argument yields that, for most levels λ, $\mathcal{A}(\lambda)$ and $\mathcal{B}(\lambda)$ are almost optimal for the $(n-1)$-dimensional Brunn-Minkowski inequality. Thus, by the inductive step, we can find a level $\bar\lambda \sim \delta^\zeta$ ($\zeta > 0$) such that we can apply the inductive hypothesis to $\mathcal{A}(\bar\lambda)$ and $\mathcal{B}(\bar\lambda)$. Consequently, after removing sets of small measure both from A and B and translating in y, we deduce that $\pi(A), \pi(B) \subset \mathbb{R}^{n-1}$ are close to the same convex set.

Step 2. This step is the analogue of Step 1 in the proof of Theorem 4.2: we apply a Fubini argument and Theorem 5.1 to most of the sets A_y and B_y for $y \in \mathcal{A}(\bar\lambda) \cap \mathcal{B}(\bar\lambda)$ to deduce that they are close to their convex hulls. Note, however, that to apply Fubini and Theorem 5.1 it is crucial that, thanks to Step 1, we found a set in $\mathbb{R}^{n-1}$ onto which both A and B project almost fully. Indeed, as we already mentioned at the beginning of the proof, to say that $\mathcal{H}^1(A_y + B_y) \geq \mathcal{H}^1(A_y) + \mathcal{H}^1(B_y)$ it is necessary to know that both A_y and B_y are nonempty, as otherwise the inequality would be false.

Step 3. To understand the properties of the barycenter of A_y and B_y (in analogy with Step 2 in the proof of Theorem 4.2), we consider the "upper" (resp. "lower") profile of A and B, that is the functions $a^+(y) := \max\{t \in \mathbb{R} : t \in A_y\}$ (resp. $a^-(y) := \min\{t \in \mathbb{R} : t \in A_y\}$) and $b^+(y) := \max\{t \in \mathbb{R} : t \in B_y\}$ (resp. $b^-(y) := \min\{t \in \mathbb{R} : t \in B_y\}$). With this notation we obtain a 3-point concavity inequality as in (5.2) for a^+ and b^+ (and the analogous one but with opposite signs for a^- and b^-). This inequality allows us to say that the barycenter of A_y satisfies the 4-point inequality (5.3) both from above and from below, and from this information we can deduce that, as a function of y, the barycenter of A_y (resp. B_y) is at bounded distance from a linear function. It follows that the barycenters of $\bar S_y$ are at bounded distance from a linear function for a set $\bar S$ which is almost of full measure inside S. Then a variation of Step 3 in the proof of Theorem 4.2 allows us to show that, after an affine measure preserving transformation, $\bar S$ is universally bounded, that is, bounded in diameter by a constant of the form $C_n \tau^{-M_n}$ where C_n and M_n are dimensional constants.

Step 4. By a relatively easy argument we find sets $A^\sim$ and $B^\sim$ of the form

$$A^\sim = \bigcup_{y \in F} \{y\} \times [a^A(y), b^A(y)] \qquad B^\sim = \bigcup_{y \in F} \{y\} \times [a^B(y), b^B(y)]$$

which are close to A and B, respectively, and are universally bounded.

Step 5. This is a crucial step: we want to show that $A^\sim$ and $B^\sim$ are close to convex sets. As in the case $A = A^\natural$ and $B = B^\natural$, we would like to deduce that b^A and b^B (resp. a^A and a^B) are L^1-close to concave (resp. convex) functions.

The main issue for proving this is to show first that the level sets of b^A and b^B are close to their convex hulls. To deduce this we wish to prove that most slices of $A^\sim$ and $B^\sim$ are nearly optimal in the Brunn-Minkowski inequality in dimension $n-1$ and invoke the inductive hypothesis. We achieve this by introducing a new inductive proof of the Brunn-Minkowski inequality, based on combining the validity of Brunn-Minkowski in dimension $n-1$ with 1-dimensional optimal transport.

An examination of this new proof of the Brunn-Minkowski inequality in the situation near equality shows that if A and B are almost optimal for the Brunn-Minkowski inequality in dimension n, then for most levels s, the slices $A(s)$ and $B(T(s))$ have comparable $(n-1)$-measure, where T is the 1-dimensional optimal transport map, and this pair of sets is almost optimal for the Brunn-Minkowski inequality in dimension $n-1$. In particular, we can apply the inductive hypothesis to deduce that most $(n-1)$-dimensional slices are close to their convex hulls.

In this way, we end up proving that $A^\sim$ and $B^\sim$ are close to convex sets, as desired.

Step 6. Since $A^\sim$ and $B^\sim$ are close to A and B respectively, as in the case $A = A^\natural$ and $B = B^\natural$ we find a convex set $\mathcal{K}$ close to A and B and containing both of them.

Step 7. Tracking down the exponents in the proof, we provide an explicit lower (resp. upper) bound on $\varepsilon_n(\tau)$ (resp. $M_n(\tau)$), which concludes the proof of the theorem. □

6. Concluding remarks

In this note we have seen three substantially different methods to obtain stability results: the proof of Theorem 3.1 relies on optimal transportation techniques, the one of Theorem 4.1 is based on additive combinatorics' arguments, while Theorems 4.2 and 5.2 involve an interplay between measure theory, analysis, and affine-invariant geometry. While Theorem 4.1 is sharp, Theorems 3.1, 4.2, and 5.2 still leave space for improvements.

First of all, in the statement of Theorem 3.1 it would be interesting to find a sharp dependence on the constant C with respect to the parameters n, λ, and Λ: if for instance λ and Λ are comparable (that is, A and B have comparable volumes) then the proof in [23] provides the estimate of the form $C(n) \approx A^n$ for some universal constant $A > 1$, while [22] improves it to a polynomial bound $C(n) \approx n^{17/2}$. As later shown in [30], a careful examination of the methods presented in [22, 23] permits to improve the constant and get a (still non-sharp) bound $C(n) \approx n^{7/2}$.

Concerning Theorems 4.2 and 5.2, there are even more fundament questions. For instance, notice that the exponents α_n and $\beta_n(\tau)$ depend on the dimension, and it looks very plausible to us that they are both non-sharp. An important question in this direction would be to improve our exponents and, if possible, understand what the sharp exponents should be.

Let us conclude by pointing out that improving the exponent in a stability estimate is

not a merely academic question. For instance, the exponent in the stability estimate for the Gagliardo-Nirenberg inequality used in [3] is related to the rate of convergence to the stationary states for solutions of the critical Keller-Segel equation, so a better exponent would give a faster rate. Even more surprisingly, the results in [9] rely in a crucial way on the sharpness of the exponent in the stability estimate for the isoperimetric inequality found in [22, 27].

It is our belief that this line of research will continue growing in the next years, producing new and powerful stability results.

References

[1] Brenier, Y., *Polar factorization and monotone rearrangement of vector-valued functions*, Comm. Pure Appl. Math., **44** (1991), no. 4, 375–417.

[2] Caffarelli, L. A., *The regularity of mappings with a convex potential*, J. Amer. Math. Soc., **5** (1992), no. 1, 99–104.

[3] Carlen, E. A. and Figalli, A., *Stability for a GN inequality and the Log-HLS inequality, with application to the critical mass Keller-Segel equation*, Duke Math. J., **162** (2013), no. 3, 579–625.

[4] Carlen, E. A. and Loss, M., *Competing symmetries, the logarithmic HLS inequality and Onofri's inequality on* $\mathbb{S}^n$, Geom. Funct. Anal., **2** (1992), 90–104.

[5] Christ, M., *Near equality in the two-dimensional Brunn-Minkowski inequality*, Preprint, 2012, Available online at http://arxiv.org/abs/1206.1965.

[6] Christ, M., *Near equality in the Brunn-Minkowski inequality*, Preprint, 2012, Available online at http://arxiv.org/abs/1207.5062.

[7] ———, *An approximate inverse Riesz-Sobolev inequality*, Preprint, 2012, Available online at http://arxiv.org/abs/1112.3715.

[8] ———, *Personal communication*.

[9] Cicalese, M. and Spadaro, E., *Droplet minimizers of an isoperimetric problem with long-range interactions*, Comm. Pure Appl. Math., **66** (2013), no. 8, 1298–1333.

[10] Cordero-Erausquin, D., Nazaret, B., and Villani, C., *A mass-transportation approach to sharp Sobolev and Gagliardo-Nirenberg inequalities*, Adv. Math., **182** (2004), no. 2, 307–332.

[11] Cianchi, A., Fusco, N., F. Maggi, and Pratelli, A., *The sharp Sobolev inequality in quantitative form*, J. Eur. Math. Soc. (JEMS), **11** (2009), no. 5, 1105–1139.

[12] ———, *A. On the isoperimetric deficit in Gauss space*, Amer. J. Math., **133** (2011), no. 1, 131–186.

[13] Cicalese, M. and Leonardi, G. P., *A Selection Principle for the Sharp Quantitative Isoperimetric Inequality*, Arch. Ration. Mech. Anal., **206** (2012), no. 2, 617–643.

[14] Del Pino, M. and Dolbeault, J., *Best constants for Gagliardo-Nirenberg inequalities and applications to nonlinear diffusions*, J. Math. Pures Appl., **81** (2002), no. 9, 847–875.

[15] Dolbeault, J. and Toscani, G., *Improved interpolation inequalities, relative entropy and fast diffusion equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **30** (2013), no. 5, 917–934.

[16] Esposito, L., Fusco, N., and Trombetti, C., *A quantitative version of the isoperimetric inequality: the anisotropic case*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), **4** (2005) no. 4, 619–651.

[17] Figalli, A. and Indrei, E., *A sharp stability result for the relative isoperimetric inequality inside convex cones*, J. Geom. Anal., **23** (2013), no. 2, 938–969.

[18] Figalli, A. and Jerison, D., *Quantitative stability for sumsets in* $\mathbb{R}^n$, Preprint, 2012.

[19] ———, *Quantitative stability for the Brunn-Minkowski inequality*, Preprint, 2013.

[20] Figalli, A. and Maggi, F., *On the shape of liquid drops and crystals in the small mass regime*, Arch. Ration. Mech. Anal., **201** (2011), no. 1, 143–207.

[21] ———, *On the isoperimetric problem for radial log-convex densities*, Calc. Var. Partial Differential Equations, **48** (2013), no. 3-4, 447–489.

[22] Figalli, A., Maggi, F., and Pratelli, A., *A mass transportation approach to quantitative isoperimetric inequalities*, Invent. Math., **182** (2010), no. 1, 167–211.

[23] ———, *A refined Brunn-Minkowski inequality for convex sets*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **26** (2009), no. 6, 2511–2519.

[24] ———, *Sharp stability theorems for the anisotropic Sobolev and log-Sobolev inequalities on functions of bounded variation*, Adv. Math., **242** (2013), 80–101.

[25] Freiman, G. A., *The addition of finite sets. I. (Russian)*, Izv. Vyss. Ucebn. Zaved. Matematika, **13** (1959), no. 6, 202–213.

[26] ———, *Foundations of a structural theory of set addition*, Translated from the Russian. Translations of Mathematical Monographs, Vol 37., American Mathematical Society, Providence, R. I., 1973.

[27] Fusco, N., Maggi, F., and Pratelli, A., *The sharp quantitative isoperimetric inequality*, Ann. of Math. (2), **168** (2008), 941–980.

[28] Maggi, F., *Sets of finite perimeter and geometric variational problems*, An introduction to geometric measure theory. Cambridge Studies in Advanced Mathematics, 135, Cambridge University Press, Cambridge, 2012.

[29] McCann, R. J., *A convexity principle for interacting gases*, Adv. Math., **128** (1997), no. 1, 153–179.

[30] Segal, A., *Remark on stability of Brunn-Minkowski and isoperimetric inequalities for convex bodies*, Geometric aspects of functional analysis, 381–391, Lecture Notes in Math., 2050, Springer, Heidelberg, 2012.

[31] Tao, T. and Vu, V., *Additive combinatorics*, Cambridge Studies in Advanced Mathematics, 105, Cambridge University Press, Cambridge, 2006.

The University of Texas at Austin, Mathematics Dept. RLM 8.100, 2515 Speedway Stop C1200, Austin, Texas 78712-1202, USA

E-mail: figalli@math.utexas.edu

Q and Q-prime curvature in CR geometry

Kengo Hirachi

Abstract. The Q-curvature has been playing a central role in conformal geometry since its discovery by T. Branson. It has natural analogy in CR geometry, however, the CR Q-curvature vanishes on the boundary of a strictly pseudoconvex domain in $\mathbb{C}^{n+1}$ with a natural choice of contact form. This fact enables us to define a "secondary" Q-curvature, which we call Q-prime curvature (it was first introduced by J. Case and P. Yang in the case $n = 1$). The integral of the Q-prime curvature, the total Q-prime curvature, is a CR invariant of the boundary. When $n = 1$, it agrees with the Burns-Epstein invariant, which is a Chern-Simons type invariant in CR geometry. For all $n \geq 1$, it has non-trivial variation under the deformation of domains. Combining the variational formula with the deformation complex of CR structures, we show that the total Q-prime curvature takes local maximum at the standard CR sphere in a formal sense.

This talk is a report in collaboration with Rod Gover, Yoshihiko Matsumoto, Taiji Marugame and Bent Ørsted.

Mathematics Subject Classification (2010). Primary 32V05; Secondary 32T15.

Keywords. Q-curvature; CR manifolds; CR invariant differential operators; strictly pseudoconvex domains, Burns-Epstein invariant.

1. Introduction

In 1979, C. Fefferman [10] proposed a program of studying the analysis and geometry of strictly pseudoconvex domains in $\mathbb{C}^{n+1}$, which is called "Parabolic invariant theory". The basic idea is to consider the asymptotic expansions of the Bergman and Szegö kernels as analogies of the heat kernel on Riemannian manifolds and develop invariant theory that leads to index theorems. Since then there has been a series of works that completes the local theory of the asymptotic expansion; see [21] for an overview. The main tool is the ambient metric, a Ricci-flat Lorentz-Kähler metric defined on the canonical bundle of the domain; all local invariants of the CR structure on the boundary of the domain can be expressed as invariant polynomials in the jets of the curvature of the ambient metric. However, the connection of the local formula to the global invariants has not been understood. In this talk, as a continuation of the program, we will give a construction of global CR invariants which we call total Q and Q-prime curvatures.

The concept of Q-curvature was first introduced by T. Branson in his study of the functional determinant of conformally invariant differential operators. In conformal geometry, it is a volume form valued local Riemannian invariant $Q(g)$ defined on even dimensional Riemannian manifolds (M, g). While Q is not a local conformal invariant, its de Rham class

Proceedings of the International Congress of Mathematicians, Seoul, 2014

is conformally invariant, i.e., the integral $\overline{Q} = \int Q$ over M is a global conformal invariant, which is called the total Q-curvature. The total Q-curvature got much attention by the seminal work of Graham-Zworski [18] on the scattering theory of conformally compact Einstein manifolds (X, g_+). The Q-curvature on the boundary at infinity $M = \partial X$ can be expressed in terms of scattering-matrix, and $\overline{Q}$ is characterized as a coefficient in the volume renormalization of (X, g_+). This approach has natural connection with AdS/CFT correspondence in string theory and gives new insights in conformal geometry. A nice overview of these progress is given in the lecture notes by Juhl [3].

The Q-curvature in CR geometry was first introduced in [20] in the description of the Szegö kernel for domains in $\mathbb{C}^2$; the general definition was given later in [15]. See also the results on the Szegö kernel along this line [4, 22]. However, it also turned out that the total Q-curvature always vanishes for the boundaries of strictly pseudoconvex domains in $\mathbb{C}^{n+1}$. A breakthrough was made recently by Case-Yang [9]; they defined a "secondary" Q-curvature Q' on 3-dimensional CR manifolds and called it Q-prime curvature. It is shown that the integral $\overline{Q}' = \int Q'$, the total Q-prime curvature, agrees with the Burns-Epstein invariant [5], which is a secondary invariant in 3-dimensional CR geometry analogous to the Chern-Simons invariant of conformal 4-manifolds. The generalization of Q' to higher dimensions was given by [23]. When $n \geq 2$, $\overline{Q}'$ is different from the secondary characteristic number defined by Burns-Epstein [6]. The crucial property of $\overline{Q}'$ is its variational formula under the deformation of domains, which enables us to study $\overline{Q}'$ as a functional on the moduli space of CR structures. In particular, we show that the critical points are exactly the boundaries of the domains with smooth solution to a Dirichlet problem for the complex Monge-Ampère equation (it can be also characterized as the vanishing of obstruction function, which is a local CR invariant of the boundary).

We can find an intimate relation between the variational formula of $\overline{Q}'$ and the deformation complex of CR structures as an application of representation theory. In the parabolic invariant theory, CR manifolds are seen as geometric structures modeled on the homogeneous space G/P, where $G = SU(n+1, 1)$ and P is a parabolic subgroup. The invariant theory of P gives much information of the geometry. The deformation complex of CR structures is the resolution of the adjoint representation $\mathfrak{g}$ by the complex of CR invariant (or equivalently G-invariant) differential operators acting on the bundles induced from irreducible representations of P; this is known as an example of (generalized) Bernstein-Gelfand-Gelfand complex [7]. The Hessian of $\overline{Q}'$ at the sphere $S^{2n+1} = G/P$ gives a CR invariant, self-adjoint, differential operator L between two bundles in the deformation complex and the CR invariance property of L forces the operator to be semidefinite. Moreover, the kernel of L can be described by using the operators in the complex. Thus we can read geometric information from the Hessian. As a result we conclude that $\overline{Q}'$ takes local maximum at the standard CR sphere in a formal sense; see Theorem 3.7. We also apply the same procedure to the total Q-curvature for partially integrable CR manifolds (which may not be embeddable). In this generalized setting, the total Q-curvature is non-trivial and the Hessian has an interesting connection to the integrability condition.

The computation of the Hessian of $\overline{Q}'$ is an analogy of that of $\overline{Q}$ in conformal geometry given in Møller-Ørsted [28]. In the conformal case, the deformation complex is simpler and it is easy to describe the kernel of the Hessian. We include an overview of the conformal Q in the next section, which should help to understand the more involved structure of CR Q and Q'.

2. Q-curvature in conformal geometry

We start with a quick review of Q-curvature in the conformal geometry with an intention to explain the relation between the total Q and the deformation complex. Many deep results in geometric analysis of Q-curvature are not mentioned; the article by Alice Chang [1] gives a clear overview in this direction.

2.1. Dimensions 2 and 4. The Q-curvature is defined as a generalization of the scalar curvature on surfaces to higher even dimensions in the context of conformal geometry. Given a Riemannian manifold (M, g) of dimension n, let Scal be the scalar curvature and $\Delta = -g^{ab}\nabla_a\nabla_b = -\nabla_a\nabla^a$ be the Laplacian (we use the Einstein summation convention). If we denote by $\widehat{\mathrm{Scal}}$ and $\widehat{\Delta}$ the ones for the scaled metric $\widehat{g} = e^{2\Upsilon}g$ for $\Upsilon \in C^\infty(M)$, then, for $n = 2$, we have

$$\widehat{\mathrm{Scal}} = e^{-2\Upsilon}(\mathrm{Scal} + 2\Delta\Upsilon), \quad \widehat{\Delta} = e^{-2\Upsilon}\Delta.$$

The factor $e^{-2\Upsilon}$ can be considered as the scaling of the volume form $d\,\mathrm{vol}_g$. So setting

$$Q_2 = \frac{1}{2}\,\mathrm{Scal}\cdot d\,\mathrm{vol}_g, \qquad P_2 f = \Delta f \cdot d\,\mathrm{vol}_g,$$

we may write the transformation rules as

$$\widehat{Q}_2 = Q_2 + P_2\Upsilon, \qquad \widehat{P}_2 = P_2.$$

If M is compact, then $\int_M P_2\Upsilon = 0$ and thus $\int_M Q_2$ is a conformal invariant. In fact, it is topological as the Gauss-Bonnet theorem gives $\int_M Q_2 = 2\pi\chi(M)$.

On a conformal manifold $(M, [g])$ of even dimension n, the Q-curvature Q_n and conformally invariant differential operator P_n are defined as the pair of objects that generalize these properties. Namely, Q_n is a volume form valued local invariant of metric g such that

$$\widehat{Q}_n = Q_n + P_n\Upsilon$$

with a conformally invariant differential operator

$$P_n\colon C^\infty(M) \to C^\infty(M, \wedge^n T^*M),$$

which is self-adjoint and $P_n 1 = 0$. The last two properties of P_n ensure that the integral

$$\overline{Q}_n = \int_M Q_n$$

is a global invariant of a conformal manifold $(M, [g])$, which is called the *total Q-curvature*. As we see below, when $n \geq 4$, $\overline{Q}_n$ gives an interesting invariant which is not just topological.

In the case $n = 4$, we can define Q_4 and P_4 by explicit formulas:

$$Q_4 = \left(\frac{1}{6}\Delta\,\mathrm{Scal} - \frac{1}{2}\,\mathrm{Ric}^{ab}\,\mathrm{Ric}_{ab} + \frac{1}{6}\,\mathrm{Scal}^2\right) d\,\mathrm{vol},$$

$$P_4 f = \left(\Delta^2 f + \nabla_a(2\,\mathrm{Ric}^{ab} - \frac{2}{3}\,\mathrm{Scal}\,g^{ab})\nabla_b f\right) d\,\mathrm{vol},$$

where Ric_{ab} is the Ricci tensor. We may also write Q_4 as

$$Q_4 = 2\,\mathrm{Pfaff}_4 - \frac{1}{4}|\,\mathrm{Weyl}\,|^2 d\,\mathrm{vol} + \frac{1}{6}\Delta\,\mathrm{Scal}\cdot d\,\mathrm{vol},$$

where Pfaff_n is the Pfaffian, which integrates to $(-2\pi)^{n/2}\chi(M)$, and $|\,\mathrm{Weyl}\,|^2$ is the squared norm of the Weyl curvature Weyl_{abcd}, the trace-free part of Riemannian curvature R_{abcd}. Hence the total Q-curvature satisfies

$$\overline{Q}_4 = 8\pi^2\chi(M) - \frac{1}{4}\int_M |\,\mathrm{Weyl}\,|^2 d\,\mathrm{vol}\,.$$

Since $|\,\mathrm{Weyl}\,|^2 d\,\mathrm{vol}$ is independent of the scale, the second term is conformally invariant itself. In particular, we see that

$$\overline{Q}_4 \le 8\pi^2\chi(M)$$

and the equality holds if and only if $[g]$ is conformally flat.

2.2. The ambient metric. To construct Q_n and P_n in higher dimensions, we use the ambient metric of Fefferman and Graham [13]. To motivate the definition of the ambient metric, we first recall the Möbius transformations of the standard sphere and associated metrics.

Let $G = SO(n+1,1)$ be the orthogonal group for the quadratic form

$$B(\zeta) = -\zeta_0^2 + \zeta_1^2 + \cdots + \zeta_{n+1}^2, \qquad (\zeta_0, \ldots, \zeta_{n+1}) \in \mathbb{R}^{n+2}.$$

Then G preserves the light cone $\mathcal{N} = \{\zeta \in \mathbb{R}^{n+2} \setminus \{0\} : B(\zeta) = 0\}$ and the hyperboloid $\mathcal{H} = \{\zeta \in \mathbb{R}^{n+2} : B(\zeta) = -1\}$. The projectivization of $\mathcal{N}$ can be identified with the unit sphere $S^n = \{x \in \mathbb{R}^{n+1} : |x|^2 = 1\}$ by

$$S^n \ni x \mapsto \mathbb{R}(1,x) \in \mathcal{N}/\mathbb{R}^* \subset \mathbb{P}^{n+1}$$

and $\mathcal{H}$ can be identified with the unit ball $B^{n+1} = \{x \in \mathbb{R}^{n+1} : |x|^2 < 1\}$ by

$$\mathcal{C} \ni (\zeta_0, \zeta') \mapsto \frac{\zeta'}{1+\zeta_0} \in B^{n+1}.$$

The action of G on $\mathbb{R}^{n+2}$ can be also seen as the isometries of the Lorentzian metric:

$$\widetilde{g} = -d\zeta_0^2 + d\zeta_1^2 + \cdots + d\zeta_{n+1}^2.$$

Since $\widetilde{g}$ induces the Poincaré metric g_+ on $\mathcal{H} \cong B_{n+1}$,

$$g_+ = 4\frac{dx_1^2 + \cdots + dx_{n+1}^2}{(1-|x|^2)^2},$$

we see that the action of G on B_{n+1} gives the isometries of g_+.

On the other hand, $\widetilde{g}$ induces a degenerate two tensor on $\mathcal{N}$. For each section of $\pi\colon \mathcal{N} \to \mathcal{N}/\mathbb{R}^* \cong S^n$, the pullback of $\widetilde{g}$ gives a Riemannian metric which is conformal to the standard metric g_0 on S^n. Thus (upper half of) $\mathcal{N}$ can be identified with the metric bundle over $(S^n, [g_0])$; hence G acts as conformal maps of S^n.

To sum up, we have three spaces of different dimensions on which G acts as automorphisms:

- Lorentzian space $(\mathbb{R}^{n+2}, \widetilde{g})$;
- Poincaré ball (B^{n+1}, g_+);
- Conformal sphere $(S^n, [g_0])$.

Fefferman-Graham [13] generalized these correspondences to a curved conformal manifold $(M, [g])$. They call $\widetilde{g}$ and g_+, respectively, the *ambient metric* and the *Poincaré-Einstein metric*. We first recall the construction of the ambient metric. Let $\widetilde{M} = \mathbb{R}_+ \times M \times (-1, 1)$ and choose a local coordinate system (t, x, ρ). For each scale $g \in [g]$, the trivial $\mathbb{R}_+$-bundle $\mathcal{M} = \mathbb{R}_+ \times M$ can be identified with the metric bundle by $(t, x) \mapsto t^2 g(x) \in S^2T^*M$ and we embed $\mathcal{M}$ into $\widetilde{M} = \mathcal{M} \times (-1, 1)$ as a hypersurface $\rho = 0$. On $\widetilde{M}$, we consider a Lorentzian metric of the form

$$\widetilde{g} = 2t dt\, d\rho + 2\rho dt\, dt + t^2 g_{ab}(x, \rho) dx^a dx^b,$$

where $g(x, \rho) = g_{ab}(x, \rho) dx^a dx^b$ is a family of Riemannian metrics on M with parameter ρ such that $g(\,\cdot\,, 0) = g$. To fix $\widetilde{g}$, we impose the Einstein equation along $\rho = 0$:

$$\operatorname{Ric}(\widetilde{g}) = \begin{cases} O(\rho^\infty) & n \text{ odd}, \\ O^+(\rho^{n/2-1}) & n \text{ even}. \end{cases} \tag{2.1}$$

Here $f = O^+(\rho^l)$ means that each component of f is $O(\rho^l)$ and $\rho^{-l} f|_{T\mathcal{M}}$ in the frame dt, dx^a is of the form

$$\begin{pmatrix} 0 & 0 \\ 0 & \phi_{ab} \end{pmatrix} \quad \text{with } g^{ab}\phi_{ab} = 0.$$

The ambient metric $\widetilde{g}$ is defined as the solution to the equation (2.1), which exists uniquely modulo $O(\rho^\infty)$ for odd n and modulo $O^+(\rho^{n/2})$ for even n. For even n, we set

$$c_n\big((-\rho)^{1-n/2} \operatorname{Ric}(\widetilde{g})\big)|_{T\mathcal{M}} = \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{O}_{ab} \end{pmatrix}, \quad c_n = 2^{n-3}(n/2-1)!^2.$$

The tensor $\mathcal{O}_{ab}$ is called the *obstruction tensor* and is shown to be a local conformal invariant in the sense that $\widehat{\mathcal{O}}_{ab} = e^{(2-n)\Upsilon}\mathcal{O}_{ab}$ under the change of scale $\widehat{g} = e^{2\Upsilon} g$ that gives the initial data on $\rho = 0$. We can show that $\operatorname{Ric}(\widetilde{g}) = O(\rho^\infty)$ admits a smooth solution if and only if $\mathcal{O}_{ab} = 0$, as its name suggests.

While this definition of $\widetilde{g}$ depends on a choice of scale g, the ambient metric is determined by the conformal class $[g]$ in the following sense: if $\widetilde{g}$ and $\widetilde{g}'$ are ambient metrics determined by $g, g' \in [g]$, then there is a diffeomorphism $\Phi(t, x, \rho) = (t', x', \rho')$ such that $\Phi(\lambda t, x, \rho) = (\lambda t', x', \rho')$ for $\lambda > 0$, $\Phi(\mathcal{M}) = \mathcal{M}$, and $\Phi^*\widetilde{g}' = \widetilde{g}$ modulo $O(\rho^\infty)$ for odd n and modulo $O^+(\rho^{n/2})$ for even n.

The Poincaré-Einstein metric g_+ on $X = M \times (0, 1) \ni (x, r)$ is then defined by the pullback of $\widetilde{g}$ by the embedding $X \hookrightarrow \widetilde{M}$, $(x, r) \mapsto (1/r, x, -r^2/2)$. In the coordinate system (x, r), we have

$$g_+ = \frac{dr^2 + h_{ab}(x, r) dx^a dx^b}{r^2}, \tag{2.2}$$

where $h_{ab}(x, r) = g_{ab}(x, -r^2/2)$. We call r the defining function normalized by the scale $h(\,\cdot\,, 0) = g \in [g]$. The equation (2.1) then implies

$$\operatorname{Ric}(g_+) + n g_+ = \begin{cases} O(r^\infty) & n \text{ odd}, \\ c_n^{-1}\mathcal{O}_{ab} r^{n-2} + O(r^{n-1}) & n \text{ even}. \end{cases}$$

Alternatively, one may define Poincaré-Einstein metric to be the solution to this equation of the form (2.2).

2.3. Total Q-curvature and volume renormalization. Now we are ready to define Q_n and P_n. For a conformal manifold $(M,[g])$ of even-dimension n, we take the ambient metric $\widetilde{g}$. Let $\widetilde{\Delta} = -\widetilde{\nabla}^A\widetilde{\nabla}_A$ be the (wave) Laplacian; here $\widetilde{\nabla}$ is the Levi-Civita connection of $\widetilde{g}$ and the index A runs through $0,1,\dots,n+1$. The Q-curvature and the invariant operator P_n for the scale $g\in[g]$ are given by

$$Q_n = -(\widetilde{\Delta}^{n/2}\log t)\big|_{t=1,\rho=0} d\operatorname{vol}_g,$$
$$P_n f = (\widetilde{\Delta}^{n/2} f)\big|_{t=1,\rho=0} d\operatorname{vol}_g .$$

Here $f\in C^\infty(M)$ is identified with a function on $\widetilde{M}$ that is free of (t,ρ). It is shown that P_n is independent of the choice of the scale $g\in[g]$ and the pair Q_n, P_n satisfies the required properties [15]. (See also §3.2 for the original definition by T. Branson.)

To study $\overline{Q}_n$, it is useful to give its characterization in terms of the Poincaré-Einstein metric g_+. The complete Riemannian manifold (X,g_+) has infinite volume; we define its finite part by considering the volume expansion of the subdomains $\{r>\epsilon\}$:

$$\int_{r>\epsilon} d\operatorname{vol}_{g_+} = a_0\epsilon^{-n} + a_2\epsilon^{-n+2} + \cdots + \begin{cases} a_{n-1}\epsilon^{-1} + V + o(1) & n \text{ odd},\\ a_{n-2}\epsilon^{-2} - L\log\epsilon + V + o(1) & n \text{ even}.\end{cases}$$

Here r is a defining function of M normalized by a scale $g\in[g]$ and a_j, V, L are constants. The constant term $V(g_+,g)$ is called the *renormalized volume* of g_+ with respect to the scale g.

Theorem 2.1 (Graham-Zworski [18]). *For odd n, V is independent of the choice of a scale $g\in[g]$. For even $n=2m$, L is independent of the choice of a scale g. Moreover, one has*

$$\overline{Q}_n = (-1)^m 2^{n-1} m!(m-1)! L.$$

The proof is based on the scattering theory of Δ_+, the Laplacian of g_+. A simpler proof using the Dirichlet problem for Δ_+ was later given by Fefferman-Graham [12].

For even n, $V(g_+,g)$ is not conformally invariant. In fact, the variation of the scale gives

$$\frac{d}{dt}\Big|_{t=0} V(g_+, e^{2t\Upsilon}g) = \int_M \Upsilon v_n(g) d\operatorname{vol}_g,$$

where $v_n(g)$ is a local invariant of g called the *holographic anomaly*. It appears in the expansion of the volume form:

$$\frac{\operatorname{vol}(h(\cdot,r))}{\operatorname{vol}(g)} = 1 + v_2 r^2 + v_4 r^4 + \cdots .$$

It is not difficult to see $L=\int_M v_n(g) d\operatorname{vol}_g$. So v_n has a similar property to Q_n, while the transformation law of v_n under the scaling is not easy to write down. In dimension 4, one has

$$Q_4 = 16 v_4 + \frac{1}{6}\Delta \operatorname{Scal}.$$

General relation between Q_n and v_j has been studied extensively by Juhl and Fefferman-Graham; see [3] and [14].

We next study the variation of $\overline{Q}_n$ under the deformation of conformal structures. Let g^t be a one parameter family of Riemannian metrics on M that preserves the volume form. Then it has an expansion

$$g^t_{ab} = g_{ab} + t\psi_{ab} + O(t^2)$$

with $g^{ab}\psi_{ab} = 0$. We shall denote the total Q-curvature for the conformal structure $[g^t]$ by $\overline{Q}_n(g^t)$.

Theorem 2.2 ([17]). *The first variation of the total Q-curvature is given by*

$$\frac{d}{dt}\Big|_{t=0} \overline{Q}_n(g^t) = (-1)^{n/2} \int_M \mathcal{O}_{ab}\psi^{ab}\, d\,\mathrm{vol}_g,$$

where $\mathcal{O}_{ab}$ is the obstruction tensor for $g = g^0$.

The critical points of the functional $\overline{Q}_n$ on the space of conformal structures are characterized by $\mathcal{O}_{ab} = 0$. At a critical point, the second variation is given by

$$\frac{d^2}{dt^2}\Big|_{t=0} \overline{Q}_n(g^t) = \int_M L_n(\psi)_{ab}\psi^{ab}\, d\,\mathrm{vol}_g,$$

where L_n is a conformally invariant differential operator of order n:

$$L_n(\psi)_{ab} = (-1)^{n/2} \frac{d}{dt}\Big|_{t=0} \mathcal{O}_{ab}(g^t).$$

By analyzing the eigenvalues of L_n, one obtain the following rigidity of the conformal sphere.

Theorem 2.3 (Møller-Ørsted [28]). *The total Q-curvature has a local maximum at the standard conformal sphere $(S^n, [g_0])$, $n \geq 4$. Namely, there exists a neighborhood U of g_0 in the space of Riemannian metrics on S^n such that*

$$\overline{Q}_n(g) \leq \overline{Q}_n(g_0) \quad \textit{for all } g \in U$$

and the equality holds if and only if $(S^n, [g])$ is conformally equivalent to $(S^n, [g_0])$.

The proof is based on an analysis of L_n using representation theory. To formulate it, let us recall the deformation complex of conformal structures. Let D_0 be the conformal Killing map

$$D_0 \colon \Gamma(T^*M) \to \Gamma(S^2_0 T^*M)$$

defined by

$$D_0(f_a) = \text{trace-free part of } \nabla_{(a} f_{b)},$$

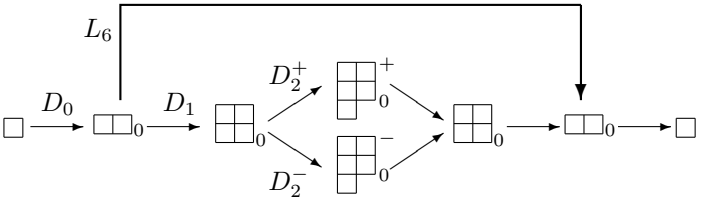
where $S^2_0 T^*M$ denote the bundle of symmetric trace-free 2-tensors. Let

$$D_1 \colon \Gamma(S^2_0 T^*M) \to \Gamma(\otimes^4 T^*M)$$

be the linearization of the Weyl curvature:

$$D_1(\varphi_{ab}) = \text{projection to } \boxplus_0 \text{ part of } \nabla_{ab}\varphi_{cd}.$$

Here $\square\!\square_0$ denotes the space of trace-free 4-tensor with symmetry given by the Young diagram $\square\!\square$. Then we have $\operatorname{Im} D_0 = \ker D_1$ and these maps can be extended to a complex, which is known as the deformation complex. On S^6, it is given by (thick long arrow omitted)



Here, we only write the symmetries of the tensor bundles in terms of the Young diagram; the superscripts $\pm$ denote the self-dual/anti-self-dual parts. If we properly put a density weight on each bundle, this becomes a complex of G-invariant differential operators. (Recall that $G = SO(n+1,1)$, which acts on S^n as conformal maps.) For general dimensions, the deformation complex has length $n+1$, like the de Rham complex. The de Rham and deformation complexes are examples of generalized Bernstein-Gelfand-Gelfand (BGG) complexes in the parabolic geometry modeled on G/P. Each BGG complex gives a resolution of a finite dimensional irreducible representation of G; the de Rham complex (resp. deformation complex) corresponds to the trivial representation $\mathbb{R}$ (resp. the adjoint representation $\mathfrak{g}$). See [7].

The Hessian L_n gives a G-invariant operator from $\square\!\square_0$ on the left to $\square\!\square_0$ on the right. Such an operator is unique up to a constant multiple and turns out to be a semi-definite, self-adjoint, operator with kernel $\operatorname{Im} D_0$. Since $\operatorname{Im} D_0 = \ker D_1$ is the tangent space to the submanifold consisting of flat conformal structures on S^n, we get an infinitesimal version of the theorem. This analysis can be applied to many other conformal functionals, e.g., the determinant of Yamabe Laplacian; this recovers an earlier result of K. Okikiolu.

3. CR geometry

Now we turn to the CR case. Many of the results outlined in Section 2 have natural analogs in CR/complex setting. However, we here put weight on the parts that are specific to the CR case and will omit many of the fundamental results, for which we refer to [19] and [16]. To simplify the exposition, we only consider the case of strictly pseudoconvex domains $\mathbb{C}^{n+1}$ and the CR structure on the boundary. More general formulation is given in [23], in which Lee's pseudo-Einstein condition plays essential role.

3.1. The ambient metric for strictly pseudoconvex domains. Let $\Omega \subset \mathbb{C}^{n+1}$ be a bounded strictly pseudoconvex domain with smooth boundary $M = \partial\Omega$. The CR structure on M is given by $T^{1,0} = T^{1,0}\mathbb{C}^{n+1} \cap \mathbb{C}TM$, a rank n complex subbundle of $\mathbb{C}TM$. It is *integrable* in the sense that

$$[\Gamma(T^{1,0}), \Gamma(T^{1,0})] \subset \Gamma(T^{1,0}).$$

Take a C^∞ defining function ρ of Ω which is positive in Ω, i.e., $\rho \in C^\infty(\mathbb{C}^{n+1})$, $\Omega = \{\rho > 0\}$ and $d\rho \neq 0$ on M. Then we can define three hermitian metrics:

- **The Levi metric**: for each $p \in M$, a hermitian form on $T^{1,0}_p$ is given by

$$L_\rho(Z,W) = -\partial\overline{\partial}\rho(Z,\overline{W}), \qquad Z,W \in T^{1,0}_p,$$

  which is positive by the definition the strictly pseudoconvexity of Ω.
- **Complete Kähler metric**: on the domain Ω, the real $(1,1)$-form

$$g_+[\rho] = -\sqrt{-1}\partial\overline{\partial}\log\rho$$

  gives a complete Kähler metric near the boundary.
- **Lorentz-Kähler metric**: let $\rho_\sharp\colon \mathbb{C}^*\times\mathbb{C}^{n+1}\to\mathbb{R}$ be the function $\rho_\sharp(z_0,z) = |z_0|^2\rho(z)$, $(z_0,z)\in\mathbb{C}^*\times\mathbb{C}^{n+1}$. Then

$$\widetilde{g}[\rho] = -\sqrt{-1}\partial\overline{\partial}\rho_\sharp$$

  gives a Lorentz-Kähler metric on $\mathbb{C}^*\times\mathbb{C}^{n+1}$ near the hypersurface $\rho_\sharp = 0$.

In the case $\rho(z) = 1-|z|^2$, Ω is the unit ball $B^{2n+2}\subset\mathbb{C}^{n+1}$,

$$(g_+[\rho])_{i\overline{j}} = \rho(z)^{-1}\delta_{i\overline{j}} - \rho(z)^{-2}\overline{z}_i z_j$$

is the complex hyperbolic metric (or the Bergman metric) and

$$\widetilde{g}[\rho] = \sqrt{-1}\big(-d\zeta_0\wedge d\overline{\zeta}_0 + d\zeta_1\wedge d\overline{\zeta}_1 + \cdots + d\zeta_{n+1}\wedge d\overline{\zeta}_{n+1}\big)$$

is a flat Lorentz-Kähler metric, where ζ is the coordinate system given by

$$\zeta_0 = z_0, \quad \zeta_j = z_0 z_j, \quad j = 1,\dots,n+1.$$

As in the conformal case, the special unitary group $G = SU(n+1,1)$ for this metric acts on $\mathbb{C}^{n+2}$ as linear transformations in ζ and induces isometries on (B^{2n+2}, g_+) and CR diffeomorphisms on $S^{2n+1} = \partial B^{2n+2}$, i.e., the diffeomorphisms that preserve the subbundle $T^{1,0}$.

For a general strictly pseudoconvex domain, we fix the defining function ρ by imposing a complex Monge-Ampère equation:

$$\mathcal{J}_z[\rho] = 1 \qquad \text{on} \quad \Omega,$$

where

$$\mathcal{J}_z[\rho] = (-1)^{n+1}\det\begin{pmatrix}\rho & \partial_j\rho\\ \partial_{\overline{k}}\rho & \partial_{j\overline{k}}\rho\end{pmatrix}_{j,k=1,\dots,n+1}.$$

The unique existence of the solution has been proved by S. Y. Cheng and S. T. Yau. For such ρ, one has

$$\begin{aligned}\mathrm{Ric}[g_+] &= -(n+1)g_+ \qquad &&\text{on} \quad \rho>0,\\ \mathrm{Ric}[\,\widetilde{g}\,] &= 0 &&\text{on} \quad \rho_\sharp>0.\end{aligned}$$

However, in general, the exact solution has weak singularity at the boundary. We thus use the best approximate solution $r\in C^\infty(\mathbb{C}^{n+1})$ constructed by Fefferman [11]. Recall that there is a smooth defining function r such that

$$\mathcal{J}_z[r] = 1 + \eta\, r^{n+2} \tag{3.1}$$

for an $\eta \in C^\infty(\mathbb{C}^{n+1})$. Such an r is unique modulo $O(r^{n+3})$ and is called *Fefferman's defining function* of Ω. It is also important to note that

$$\mathcal{O} = \eta|_M \in C^\infty(M)$$

is independent of the choice r and is called the *obstruction function*, as $\mathcal{O} = 0$ if and only if the Cheng-Yau solution is smooth up to the boundary. We define the *ambient metric* to be $\widetilde{g}[r]$, which is not Ricci-flat but $\mathrm{Ric}[\widetilde{g}] = O(r^n)$ holds.

The operator $\mathcal{J}_z$ depends on the choice of coordinates z and so does r. However, there is a simple transformation rule under the coordinate changes. If $\widehat{z} = \Phi(z)$ is another holomorphic coordinate system,

$$\widehat{r} = e^{-2\,\mathrm{Re}\,\varphi(z)} r$$

gives Fefferman's solution in $\widehat{z}$, where $\varphi(z) = (\det \Phi'(z))^{1/(n+2)}$ is the power of holomorphic Jacobian. Thus the map

$$\Phi_\sharp \colon (z_0, z) \mapsto (\widehat{z}_0, \widehat{z}) = (z_0\varphi(z), \Phi(z))$$

gives an isometry between $\widetilde{g}[r]$ and $\widetilde{g}[\widehat{r}\,]$. In other word, we can say that the ambient metric is naturally defined on an $(n+2)$-nd root of the canonical bundle.

In the following, we regard the domain $\Omega \subset \mathbb{C}^{n+1}$ as a complex manifold and consider the family of Fefferman's defining functions, each of them corresponds to a choice of coordinates. For any defining functions r and $\widehat{r}$ in the family, we have

$$\widehat{r} = e^{\Upsilon} r \quad \overline{\Omega} \quad \mathrm{mod}\ O(r^{n+3})$$

for a pluriharmonic function Υ on $\overline{\Omega}$. If we define a contact form by

$$\theta[r] = \frac{i}{2}(\partial - \overline{\partial})r\big|_{TM},$$

then we have

$$\widehat{\theta} = e^{\Upsilon}\theta \quad \text{on} \quad M.$$

Therefore the family of defining functions (or corresponding contact forms) can be seen as an analogy of conformal structure. Important fact here is that the scaling is parametrized not by $C^\infty(M)$ but by the boundary values of pluriharmonic functions, which are called *CR pluriharmonic functions*.

3.2. Q-prime curvature. We use the ambient metric $\widetilde{g} = \widetilde{g}[r]$ to construct CR invariant differential operators. Let $\widetilde{\Delta}$ be the Laplacian of $\widetilde{g}$. Then, for an integer $2m \in [-n, 0]$ and a function $f \in C^\infty(\overline{\Omega})$,

$$\left(\widetilde{\Delta}^{n+2m+1}|z_0|^{2m} f\right)\Big|_{\{1\}\times M} \in C^\infty(M)$$

is shown to depend only on the boundary value of f and gives a differential operator

$$P_{n+2m+1} \colon C^\infty(M) \to C^\infty(M).$$

While this definition depends on the choice of r, we can say that this is *CR invariant* in the sense that if $\widehat{r} = e^{\Upsilon} r$ for a pluriharmonic function Υ, then

$$\widehat{P}_{n+2m+1}(e^{m\Upsilon} f) = e^{(-n-m-1)\Upsilon} P_{n+2m+1} f.$$

We will use the density notation and write this transformation law as

$$P_{n+2m+1}\colon \mathcal{E}(m) \to \mathcal{E}(-n-m-1).$$

The case $m=0$ has special importance as we have $\mathcal{E}(-n-1) = \Gamma(\wedge^{2n+1}T^*M)$ so that

$$P_{n+1}\colon C^\infty(M) \to \Gamma(\wedge^{2n+1}T^*M).$$

Now we recall Branson's idea of defining Q-curvature from these invariant operators. Consider the 0th order term of P_{n+1} for higher dimensions $\mathbb{C}^{N+1}$ and take the "limit as $N \to n$" after factoring out $(N-n)$. This gives a formal definition of Q-curvature:

$$Q_{n+1} = \lim_{N\to n} \frac{1}{N-n}\Big(\widetilde{\Delta}^{n+1}|z_0|^{2(n-N)}\Big)\Big|_{\{1\}\times M^{2N+1}}.$$

We can justify this limit by considering Taylor expansion in $N-n$:

$$|z_0|^{2(n-N)} = \sum_{k=0}^{\infty} \frac{(N-n)^k}{k!}(-\log|z_0|^2)^k.$$

Applying $\widetilde{\Delta}^{n+1}$ on $\mathbb{C}^*\times\mathbb{C}^{n+1}$ to the both sides gives

$$\big(\widetilde{\Delta}^{n+1}|z_0|^{2(n-N)}\big)|_{\{1\}\times M} = \sum_{k=0}^{\infty} \frac{(N-n)^k}{k} Q^{(k)},$$

where

$$Q^{(k)} = \widetilde{\Delta}^{n+1}(-\log|z_0|^2)^k\big|_{\{1\}\times M}.$$

While the expansion does not have clear meaning, the coefficients $Q^{(k)}$ are standard quantities defined on M of dimension $2n+1$. Clearly, $Q^{(0)}=0$. In the conformal case, the second term $Q^{(1)}$ gives the Q-curvature, where $\log|z_0|^2$ is replaced by $\log t^2$. However, $Q^{(1)}=0$ because $\log|z_0|^2$ is pluriharmonic and $\widetilde{g}$ is Kähler. Hence the leading term of the expansion is $Q^{(2)}$, which we define to be the *Q-prime curvature* and denote by Q'.

The definition of Q' depends on the choice of r and is not a CR invariant. If $\widehat{r} = e^{\Upsilon}r$, where Υ is pluriharmonic, then

$$\widehat{Q}' = Q' + 2P'\Upsilon + P_{n+1}(\Upsilon^2).$$

Here P' is a differential operator defined on the space of CR pluriharmonic functions by

$$P'f = -\widetilde{\Delta}^{n+1}(\widetilde{f}\log|z_0|^2)\big|_{\{1\}\times M},$$

which we call the *P-prime operator*. Here $\widetilde{f}$ denotes the pluriharmonic extension of f. Again, P' is not a CR invariant operator but satisfies the transformation law:

$$\widehat{P}'f = P'f + P_{n+1}(\Upsilon f).$$

A crucial fact is that P' and P_{n+1} are formally self-adjoint and $P'1 = P_{n+1}1 = 0$. It follows that the *total Q-prime curvature*

$$\overline{Q}'(M) = \int_M Q'\theta\wedge(d\theta)^n$$

is a CR invariant of M, i.e., it is independent of the choice of r.

3.3. Explicit formulas in dimensions 3 and 5. In the case M has dimension 3, we can explicitly write down P' and Q' in terms of Tanaka-Webster connection ∇ (analogous to the Levi-Civita connection, for each choice of a contact form or the Levi metric, one can define a canonical connection of TM). With respect to the contact form $\theta = \theta[r]$ for r given as above, we have

$$P'f = \Delta_b^2 f - \operatorname{Re}\nabla^1(\operatorname{Scal}\nabla_1 f - 2\sqrt{-1}A_{11}\nabla^1 f), \tag{3.2}$$

$$Q' = \frac{1}{2}\Delta_b \operatorname{Scal} + \frac{1}{4}\operatorname{Scal}^2 - |A|^2. \tag{3.3}$$

Here Δ_b is the sub-Laplacian, and Scal, A_{11} are respectively the scalar curvature and torsion of the connection; $|A|^2 = A_{11}A^{11}$ is the squared norm of the torsion. We are still using the Einstein convention but, since $T^{1,0}$ has rank one, we only have index 1.

These formulas were first given by J. Case and P. Yang prior to the general definition in the previous subsection. Their aim was to give a CR analogue of Gursky's sphere theorem in 4-dimensional conformal geometry. While we cannot go into the details, let us recall their main theorem.

Theorem 3.1 (Case-Yang [9]). *Let (M, H, J) be a compact 3-dimensional CR manifold with a pseudo-Einstein contact form. Assume that P_3 is nonnegative and that CR Yamabe constant is nonnegative. Then*

$$\overline{Q}'(J) \le \overline{Q}'(J_0)$$

and the equality holds if and only if (M, H, J) is CR equivalent to the standard sphere (S^3, H_0, J_0).

This is a deep result of geometric analysis; the proof is based on the CR positive mass theorem of J.-H. Cheng, A. Malchiodi, and P. Yang.

From the explicit formula of Q' in 3-dimensions, we can see that $\overline{Q}'$ agrees with the Burns-Epstein invariant $\mu(M)$, [5], up to a universal constant:

$$\overline{Q}'(M) = -4\pi^2\mu(M).$$

From this fact we can also obtain the renormalized Gauss-Bennet formula for $\Omega \subset \mathbb{C}^2$:

$$\int_\Omega c_2(B) = \chi(\Omega) - \frac{1}{4\pi^2}\overline{Q}'(M), \tag{3.4}$$

where c_2 is the second Chern form for the Bochner tensor B of g_+, the trace-free part of the Kähler curvature tensor of g_+.

For higher dimensions, such equality does not hold in general. To state it precisely, let us recall a result of T. Marugame [26], which improved the renormalized Gauss-Bonnet formula of Burns-Epstein [6]. For $\Omega \subset \mathbb{C}^{n+1}$, he found a transgression formula that gives an invariant polynomial $\Pi(R, A)$ in the curvature R and torsion A of the Webster-Tanaka connection for θ such that

$$\int_\Omega c_{n+1}(B) = \chi(\Omega) - \int_M \Pi\cdot\theta\wedge(d\theta)^n,$$

where $c_{n+1}(B)$ is the $(n+1)$-st Chern form for the Bochner tensor of g_+ on Ω. When $n = 2$, we have

$$-(4\pi)^3\Pi = \frac{1}{27}\operatorname{Scal}^3 - 4R_{a\bar{c}b\bar{d}}A^{ab}A^{\bar{c}\bar{d}} + \frac{1}{3}|S|^2\operatorname{Scal}.$$

Here $|S|^2$ is the squared norm of the Chern-Moser tensor $S_{a\bar{c}b\bar{d}}$, which is the trace-free part of the Tanaka–Webster curvature $R_{a\bar{c}b\bar{d}}$. The lower indices a, b (resp. $\bar{c}, \bar{d}$) run through $1, 2, \dots, n$ (resp. $\overline{1}, \overline{2}, \dots, \overline{n}$) and correspond to $(T^{1,0})^*$ (resp. $(T^{1,0})^*$). Analogous to the Weyl curvature in the conformal case, $S_{a\bar{b}c\bar{d}} = 0$ if and only if M is *spherical*, i.e., locally CR equivalent the sphere. With this Π, we can write $\overline{Q}'$ as

$$\overline{Q}' = -\int_M \Big((4\pi)^3\Pi + \frac{1}{3}|S|^2\,\mathrm{Scal} + 4|\nabla A|^2\Big)\theta \wedge (d\theta)^2.$$

Thus we obtain the following

Theorem 3.2 ([24]). *Let $\Omega \subset \mathbb{C}^3$ be a strictly pseudoconvex domain. If the scalar curvature of the Webster-Tanaka connection for $\theta[r]$ is positive almost everywhere, then*

$$(4\pi)^3\overline{Q}' \le \chi(\Omega) - \int_\Omega c_3(B).$$

The equality holds only if $M = \partial\Omega$ is spherical.

Note that the assumption on Ω holds for domains that are sufficiently close to the ball in $\mathbb{C}^3$. So there are many examples for which $-(4\pi)^3\overline{Q}'$ and $\int_M \Pi$ are different.

3.4. Volume renormalization and variational formula. We next consider the volume renormalization of strictly pseudoconvex domains. While M has odd dimensions, CR geometry is analogous to even dimensional conformal geometry. Hence the situation is a little bit complicated. As before, let $g_+ = -\sqrt{-1}\partial\overline{\partial}\log r$ be the complete Kähler metric on Ω defined from Fefferman's defining function r.

Theorem 3.3 ([23, 24]). *Let $\Omega \subset \mathbb{C}^{n+1}$ be a strictly pseudoconvex domain and r be Fefferman's defining function. Then the integrals over the subdomains $\{r > \epsilon\}$ admit expansions as $\epsilon \to +0$:*

$$\int_{r>\epsilon} |d\log r|^2_{g_+}\, d\,\mathrm{vol}_{g_+} = a_0\epsilon^{-n-1} + \cdots + a_n\epsilon^{-1} + k_n\overline{Q}'\log\epsilon + O(1), \tag{3.5}$$

$$\int_{r>\epsilon} d\,\mathrm{vol}_{g_+} = b_0\epsilon^{-n-1} + \cdots + b_n\epsilon^{-1} + k_n'\overline{Q}' + o(1), \tag{3.6}$$

where a_j, b_j are constants given by integrals over $M = \partial\Omega$ of some local invariants of the CR structure of M and $\theta = \theta[r]$, and k_n, k_n' are non-zero universal constants depending only on the dimension.

The first formula is an analogy of the even dimensional conformal case, while the second formula says that the renormalized volume is a CR invariant, which corresponds to the odd dimensional conformal case.

Using (3.5), we can compute the variation of $\overline{Q}'$ under the perturbation of domains. Let $\{\Omega_t\}_{t\in\mathbb{R}}$ be a smooth family of strictly pseudoconvex domains in $\mathbb{C}^{n+1}$ in the sense that there is a C^∞ function $\rho_t(z)$ of $(t, z) \in \mathbb{R}\times\mathbb{C}^{n+1}$ such that $\Omega_t = \{z \in \mathbb{C}^{n+1} : \rho_t(z) > 0\}$ and $d_z\rho_t \neq 0$ on $\partial\Omega_t$. Solving the Monge-Ampère equation for each t, one may assume that ρ_t is Fefferman's defining function for each fixed t. On the boundary $M_t = \partial\Omega_t$, $\theta_t = \theta[\rho_t]$ gives a natural contact form.

Theorem 3.4 ([24]). *Let $\{\Omega_t\}_{t\in\mathbb{R}}$ be a smooth family of strictly pseudoconvex domains in $\mathbb{C}^{n+1}$. Then the total Q'-curvature $\overline{Q}'(M_t)$ of M_t satisfies*

$$\frac{d}{dt}\Big|_{t=0}\overline{Q}'(M_t) = 2\int_{M_0}\dot\rho\,\mathcal{O}\,\theta_0\wedge(d\theta_0)^n, \tag{3.7}$$

where $\dot\rho(z) = d\rho_t/dt|_{t=0}$ and $\mathcal{O}$ is the obstruction function of ρ_0.

3.5. Q-curvature for partially integrable CR structures. As we have seen, the Q-curvature vanishes for the boundary of a domain in $\mathbb{C}^{n+1}$. However, if we consider abstract CR structure which may not be embeddable, the total Q-curvature becomes non-trivial and has natural variational formula. We here recall a result of Y. Matsumoto [27].

Let θ be a contact form on a manifold M of dimension $2n+1$, that is, θ is a real one form satisfying $\theta\wedge(d\theta)^n\neq 0$. An abstract CR structure is a complex structure J on the contact distribution $H=\ker\theta\subset TM$; we denote the $\pm\sqrt{-1}$-eigenspace decomposition by $\mathbb{C}H = T^{1,0}\oplus T^{0,1}$. We assume that J is *partially integrable* in the sense that

$$[\Gamma(T^{1,0}),\Gamma(T^{1,0})]\subset\Gamma(T^{1,0}\oplus T^{0,1}).$$

This enables us to define the Levi metric L_θ on H by $d\theta(X,JY)$; we assume that it is positive definite. A choice of contact form θ gives a decomposition of cotangent bundle $\mathbb{C}T^*M = \mathbb{C}\theta\oplus(T^{1,0})^*\oplus(T^{0,1})^*$ such that the corresponding coframe $\theta,\theta^a,\theta^{\overline{a}}$ gives $d\theta=\sqrt{-1}h_{a\bar b}\theta^a\wedge\theta^{\bar b}$.

Let (M,H,J) be a partially integrable CR manifold with a contact form θ. An *asymptotically complex hyperbolic (ACH) metric* is a Riemannian metric on $M\times(0,1)$ with the following asymptotic expansion

$$g_+ = \frac{1}{4\rho^2}d\rho^2+\frac{1}{\rho^2}\theta^2+O(\rho^{-1}),\qquad g_+|_H=\frac{L_\theta}{\rho}+O(1).$$

Here θ and L_θ are identified with their pullbacks by the projection $M\times(0,1)\to M$. We also assume that g_+ is *smooth* in the sense that $\rho^2 g_+$ is C^∞ on $M\times[0,1)$.

As in the case of Poincare-Einstein metric, we consider the best approximate solution to the Einstein equation. It is shown that there is a smooth ACH metric that satisfies

$$\mathrm{Ric}(r_+)+\frac{1}{2}(n+2)g_+ = \rho^n E \tag{3.8}$$

for a symmetric 2-tensor E which is C^∞ on $M\times[0,1)$ and

$$E|_{\rho=0} = 2\,\mathrm{Re}(\mathcal{O}_{ab}\theta^a\otimes\theta^b)\mod\theta,d\rho.$$

The *obstruction tensor* in this setting is defined to be

$$\mathcal{O}_{ab}\in\Gamma(S^2(T^{1,0}M)^*).$$

It is CR invariant, i.e., $\widehat{\mathcal{O}}_{\alpha\beta}=e^{-n\Upsilon}\mathcal{O}_{\alpha\beta}$ holds under the scaling $\widehat\theta=e^\Upsilon\theta$. If $\mathcal{O}_{\alpha\beta}=0$, we can find a smooth ACH metric that satisfies Einstein equation modulo $O(\rho^\infty)$. It is important to note that $\mathcal{O}_{\alpha\beta}=0$ if $T^{1,0}$ is integrable.

Fixing a smooth ACH metric g_+ satisfying (3.8), we now define Q-curvature for θ on (M, H, J). Let Δ_+ be the Laplacian of g_+. Then there are functions $A, B \in C^\infty(M\times[0,1))$ such that

$$\Delta_+(\log\rho + A + B\rho^{n+1}\log\rho) = n+1+O(\rho^\infty).$$

The Q-curvature is now defined by

$$B|_{\rho=0} = \frac{(-1)^n}{n!(n+1)!}\,Q.$$

One can show that Q satisfies the required transformation law

$$\widehat{Q} = Q + P_{n+1}\Upsilon, \qquad \widehat{\theta} = e^{\Upsilon}\theta,$$

where $\Upsilon \in C^\infty(M)$ and P_{n+1} is a self-adjoint CR invariant differential operator of order $2n+2$ without constant term. It follows that

$$\overline{Q} = \int_M Q\theta\wedge(d\theta)^n$$

is a CR invariant. For integrable CR structures, this definition of Q agrees with the one given in [15] via the ambient metric (in this case, we can also say that CR Q is the pushforward of the conformal Q of the Fefferman space $S^1\times M$).

To state the variational formula of $\overline{Q}$, we recall the deformation of (partially integrable) CR structures. Take a frame Z_a of $T^{1,0}$ and set $Z_{\overline{a}} = \overline{Z_a} \in T^{0,1}$. Then we may define another CR structure $\widehat{J}$ by the frame of $\widehat{T}^{0,1}$,

$$\widehat{Z}_{\overline{a}} = Z_{\overline{a}} + \varphi_{\overline{a}}{}^{b} Z_b.$$

It is partially integrable if $\varphi_{\overline{a}\overline{b}}$ is symmetric, where the index b is lowered by using the Levi metric $h_{a\overline{b}}$. Thus partially integrable CR structure nearby $T^{1,0}$ is parametrized by a symmetric two tensor $\varphi_{\overline{a}\overline{b}} \in \Gamma(S^2(T^{0,1})^*)$.

Theorem 3.5 (Matsumoto [27]). *Let $\{J_t\}_{t\in\mathbb{R}}$ be a one parameter family of partially integrable CR structures parameterized by $\varphi^{(t)}_{\overline{a}\overline{b}}$. Then the total Q-curvature $\overline{Q}(J_t)$ of (M, H, J_t) satisfies*

$$\frac{d}{dt}\Big|_{t=0}\overline{Q}(J_t) = (-1)^n c_n \int \mathrm{Re}\,\mathcal{O}_{ab}\dot{\varphi}^{ab}\theta\wedge(d\theta)^n$$

with a universal constant $c_n > 0$. Here $\mathcal{O}_{ab}$ is the obstruction tensor for (M, H, J_0) in the scale θ and $\dot{\varphi}_{\overline{a}\overline{b}} = d\varphi^{(t)}_{\overline{a}\overline{b}}/dt|_{t=0}$.

In particular, if all J_t are integrable, we have $\mathcal{O}_{ab} = 0$ and $\overline{Q}(J_t)$ is constant.

3.6. Deformation complex of CR structures. We have obtained the variational formulas of $\overline{Q}$ and $\overline{Q}'$. To derive geometric consequences from them, we shall recall the deformation complex of CR structures.

We will use the Young diagram to denote the symmetries of tensor bundle. The symmetric product of $(T^{0,1})^*$ is now denoted by $\square\!\square(T^{0,1})^*$. The integrability of J is equivalent to

the vanishing of the Nijenhuis tensor $N_{\overline{a}\overline{b}c}$, which has the symmetry $\begin{smallmatrix}\square\square\\\square\end{smallmatrix}$; the linearization of $N_{\overline{a}\overline{b}c}$ gives the map

$$D_1^-: \Gamma\Big(\square\square(T^{0,1})^*\Big) \to \Gamma\Big(\begin{smallmatrix}\square\square\\\square\end{smallmatrix}(T^{0,1})^*\Big), \qquad D_1^-\varphi_{\overline{a}\overline{b}} = \nabla_{[\overline{c}}\varphi_{\overline{b}]\overline{a}}.$$

Let $\mathcal{E} = C^\infty(S^{2n+1}, \mathbb{C})$ and define

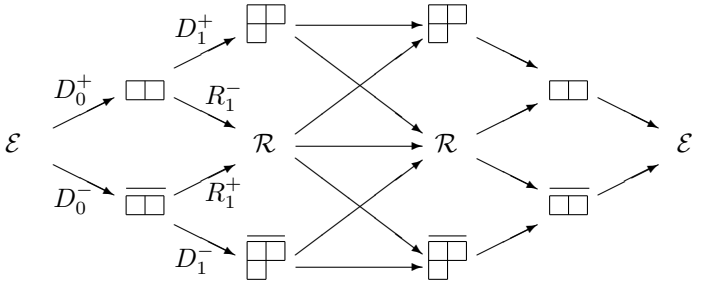
$$D_0^-: \mathcal{E} \to \Gamma\Big(\square\square(T^{0,1})^*\Big), \qquad D_0^- f = \nabla_{\overline{a}\overline{b}} f,$$
$$D_0^+: \mathcal{E} \to \Gamma\Big(\square\square(T^{1,0})^*\Big), \qquad D_0^+ f = \nabla_{ab} f.$$

These maps give a complex

$$\mathcal{E} \xrightarrow{D_0^-} \Gamma\Big(\square\square(T^{0,1})^*\Big) \xrightarrow{D_1^-} \Gamma\Big(\begin{smallmatrix}\square\square\\\square\end{smallmatrix}(T^{0,1})^*\Big).$$

This is the beginning of Kuranishi's deformation complex of CR structures in the form later improved by Akahori-Garfield-Lee [2]. The cohomology of this complex describes the moduli of the deformations of isolated singularities. In this setting, CR manifolds that bound the same singularity are identified; this equivalence is given infinitesimally by the image of D_0^-.

To study the deformation of partially integrable CR structures, we need the full deformation complex which is given as the BGG complex of the adjoint representation $\mathfrak{su}(n+1,1)$. The Kuranishi complex is contained in the BGG complex as an edge. On the 5-dimensional sphere $S^5 \subset \mathbb{C}^3$, the deformation complex is given by

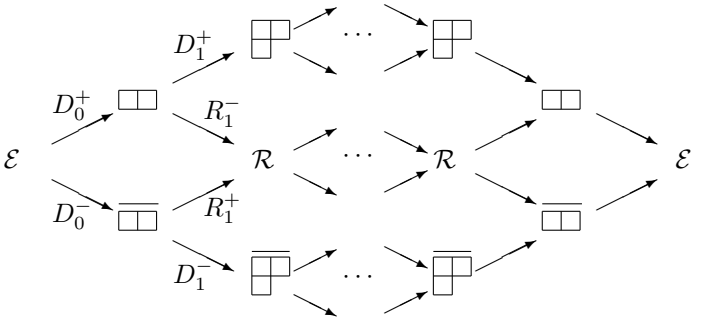


Here we have simplified the notation by omitting Γ and $(T^{1,0})^*$ or $(T^{0,1})^*$; the overline means that $(T^{0,1})^*$ is omitted. $\mathcal{R}$ denotes space of the sections of the trace-free tensors $\varphi_{ab\overline{c}\overline{d}}$ with symmetry $\square\square \otimes \overline{\square\square}$ and R_1^- is given by

$$R_1^-(\varphi_{ab}) = \text{trace-free part of } \nabla_{\overline{c}\overline{d}}\varphi_{ab}.$$

Each arrow is a CR invariant differential operator if we properly put density weight on each bundle. Moreover, it is known that there is exactly one CR invariant operator for each arrow. (Recall that $G = SU(n+1,1)$ acts on S^{2n+1} as CR automorphisms. Hence, in this setting, CR invariant operators are G-invariant operators and vice versa.)

For higher dimensions, we have a diagram of length $2n+2$. It is similar to the type decomposition of the de Rham complex, but it has more maps in the middle. The part we need here is the following:



If we write $f = u + \sqrt{-1}v$ for real valued u and v, then $D_0^- \sqrt{-1}v$ is infinitesimally given by a pullback of the CR structure by a contact diffeomorphism, while $D_0^- u$ is the first variation of Kuranishi wiggle (perturbations of S^{2n+1} within $\mathbb{C}^{n+1}$).

We set $\mathcal{Q} = \mathbb{C}T/(T^{1,0} \oplus T^{0,1})$ and denote the space of sections of $\mathcal{Q}$ by $\mathcal{E}(1)$. For each choice of a contact form we have an identification $\mathcal{E}(1) \cong \mathcal{E}$. In the deformation complex, the first $\mathcal{E}$ is $\mathcal{E}(1)$ and the last one is $\mathcal{E}(-n-2) = \Gamma(\mathcal{Q}^{-n-2})$. The symmetric two tensor on the left side $\square\!\square$ has weight $\square\!\square \otimes \mathcal{Q}$ and the one on the right side has weight $\square\!\square \otimes \mathcal{Q}^{-n}$; we set

$$\mathcal{E}_{\square\!\square}(1) = \Gamma(\square\!\square \otimes \mathcal{Q}) \quad \text{and} \quad \mathcal{E}_{\square\!\square}(-n) = \Gamma(\square\!\square \otimes \mathcal{Q}^{-n}).$$

Take a family of partially integrable CR structures J_t given by $\varphi_{ab}^{(t)}$ (the conjugate of $\varphi_{\bar{a}\bar{b}}^{(t)}$) and set

$$\dot{\varphi}_{ab} = \frac{d}{dt}\Big|_{t=0} \varphi_{ab}^{(t)} \in \mathcal{E}_{\square\!\square}(1).$$

Then the first variation of $\mathcal{O}_{ab}(J_t)$ at $t=0$ gives a CR invariant operator of order $2n+2$:

$$L_{n+1} \colon \mathcal{E}_{\square\!\square}(1) \to \mathcal{E}_{\square\!\square}(-n).$$

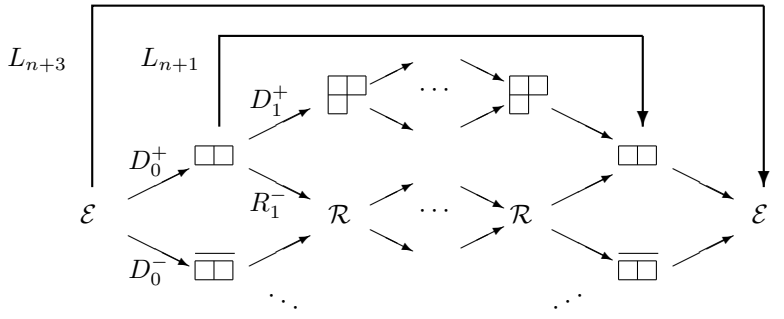
For the obstruction function $\mathcal{O}$, we take its variation under an integrable deformation $\varphi_{ab}^{(t)}$ such that

$$\frac{d}{dt}\Big|_{t=0} \varphi_{ab}^{(t)} = D_0^+ f \in \mathcal{E}_{\square\!\square}(1) \quad \text{for} \quad f \in \mathcal{E}(1).$$

Then the first variation of $\mathcal{O}(J_t)$ gives a CR invariant operator of order $2n+6$:

$$L_{n+3} \colon \mathcal{E}(1) \to \mathcal{E}(-n-2).$$

These are the operators sending the bundles on the left to the ones on the right with the same Young diagrams:



Moreover, L_{n+3} and L_{n+1} are intertwiners between G-modules $\mathcal{E}(1) \to \mathcal{E}(-n-2)$ and $\mathcal{E}_{\square\square}(1) \to \mathcal{E}_{\square\square}(-n)$; such maps are unique up to a constant multiple and the eigenvalues of these maps can be explicitly computed by using representation theory.

Theorem 3.6 ([25]).

(1) *On the standard sphere S^{2n+1} the operator $L_{n+3}\colon \mathcal{E}(1) \to \mathcal{E}(-n-2)$ is semi-negative and*

$$\ker L_{n+3} = \ker D_0^+ + \ker D_0^-.$$

(2) *On the standard sphere S^{2n+1}, $n \geq 2$, the operator $L_{n+1}\colon \mathcal{E}_{\square\square}(1) \to \mathcal{E}_{\square\square}(-n)$ is semi-negative and*

$$\ker L_{n+1} = \ker D_1^+ + \ker R_1^-.$$

As a direct consequence, we have

Theorem 3.7 ([25]).

(1) *Let $\Omega_t = \{\rho_t > 0\}$ be a smooth family of strictly pseudoconvex domains in $\mathbb{C}^{n+1}$ such that Ω_0 is the unit ball. If $\dot\rho = d\rho_t/dt|_{t=0}$ satisfies $\dot\rho|_{S^{2n+1}} \notin \ker D_0^+ + \ker D_0^-$, then*

$$\frac{d^2}{dt^2}\Big|_{t=0} \overline{Q}'(\partial\Omega_t) < 0.$$

(2) *Let $\{J_t\}_{t\in\mathbb{R}}$ be a family of partially integrable CR structures such that J_0 is the standard one on S^{2n+1}, and $\dot\varphi_{ab}$ be the first variation of J_t at $t = 0$. If $\dot\varphi_{ab} \notin \ker D_1^+ + \ker R_1^-$, then*

$$\frac{d^2}{dt^2}\Big|_{t=0} \overline{Q}(J_t) < 0.$$

We shall explain the geometric meaning of the condition on the direction of the deformations. If $n \geq 2$, we have

$$\ker D_0^+ + \ker D_0^- = \ker R_1^- D_0^+$$

and the composition $R_1^- D_0^+$ is the linearization of the Chern-Moser tensor $S_{ab\overline{c}\overline{d}}$. Thus $\dot{\rho}|_{S^{2n+1}} \in \ker D_0^+ + \ker D_0^-$ means that the family is spherical to the first order. The inequality states that if the family deformation is not trivial, then $\overline{Q}'$ takes local maximum value at the standard sphere along the family.

The similar argument can be applied to the case $n = 1$; then R_1^- is a 4-th order operator and Chern-Moser tensor has 6 indices $S_{1111\overline{11}}$. It is also not difficult to derive Theorem 3.6 (1), $n = 1$, by a direct computation as was done in Burns-Epstein [5]. J. H. Cheng and J. Lee further showed the following stronger result:

Theorem 3.8 (Cheng-Lee [8]). *For the CR structures on S^3 near the standard one, the Burns-Epstein invariant takes minimal value only for the standard sphere.*

Since $\mu = -(2\pi)^2\overline{Q}'$, this theorem is consistent with the theorem above. To prove the local minimality from the semi-positivity of the Hessian, they developed a slice theorem of the moduli space of CR structures on S^3.

The geometric meaning of Theorem 3.6 (2) is still not clear; but it gives a insight to the partially integrable CR structures. The subspace $\ker D_1^+$ is the direction of integrable CR structures and $\ker R_2^-$ is the direction with $S_{ab\overline{c}\overline{d}} = 0$ (in the partially integrable case, $S_{ab\overline{c}\overline{d}} = 0$ may not mean that the surface is spherical). Thus the vanishing of total Q does not characterize integrable CR structures — contrary to our initial hypothesis. However, this theorem suggests the existence of a natural class of partially integrable CR structures for which $\overline{Q}$ vanishes identically.

References

[1] S.-Y. Alice Chang, *Conformal invariants and partial differential equations*, Bull. Amer. Math. Soc., **42** (2005), 365–393.

[2] T. Akahori, P. Garfield, Peter and J. M. Lee, *Deformation theory of 5-dimensional CR structures and the Rumin complex*, Michigan Math. J., **50** (2002), 517–550.

[3] H. Baum and A. Juhl, *Conformal Differential Geometry: Q-Curvature and Conformal Holonomy,* Oberwolfach Seminars, Birkhäuser, 2010.

[4] L. Boutet de Monvel, *Vanishing of the logarithmic trace of generalized Szegö projectors*, Algebraic analysis of differential equations from microlocal analysis to exponential asymptotics, 67–78, Springer, 2008.

[5] D. Burns and C. Epstein, *A global invariant for three dimensional CR-manifolds*, Invent. Math., **92** (1988), 333–348.

[6] ———, *Characteristic numbers of bounded domains*, Acta Math., **164** (1990), 29–71.

[7] A. Čap, J. Slovák and V. Souček, *Bernstein–Gelfand–Gelfand sequences*, Ann. of Math., **154** (2001), 97–113; See also the preprint version: arXiv:math/0001164.

[8] J.-H. Cheng and J. M. Lee, *A local slice theorem for 3-dimensional CR structures*, Amer. J. Math., **117** (1995), 1249–1298.

[9] J. S. Case and P. Yang, *A Paneitz-type operator for CR pluriharmonic functions*, Bull. Inst. Math. Acad. Sin., **8** (2013), 285–322.

[10] C. L. Fefferman, *Parabolic invariant theory in complex analysis*, Adv. in Math., **31** (1979), 131–262.

[11] ———, *Monge-Ampère equations, the Bergman kernel, and geometry of pseudoconvex domains*, Ann. of Math., **103** (1976), 395–416.

[12] C. L. Fefferman and C. R. Graham, *Q-curvature and Poincareé metrics*, Math. Res. Lett., **10** (2003), 819–832.

[13] ———, *The ambient metric*, Princeton Univ. Press, 2011.

[14] ———, *Juhl's formulae for GJMS operators and Q-curvatures*, J. Amer. Math. Soc., **26** (2013), 1191–1207.

[15] C. L. Fefferman and K. Hirachi, *Ambient metric construction of Q-curvature in conformal and CR geometries*, Math. Res. Lett., **10** (2003), 819–832.

[16] A. R. Gover and C. R. Graham, *CR invariant powers of the sub-Laplacian*, J. Reine Angew. Math., **583** (2005), 1–27.

[17] C. R. Graham and K. Hirachi, *The ambient obstruction tensor and Q-curvature, AdS/CFT correspondence, Einstein metrics and their conformal boundaries*, 59–71, IRMA Lect. Math. Theor. Phys., **8**, Eur. Math. Soc., Zürich, 2005.

[18] C. R. Graham and M. Zworski, *Scattering matrix in conformal geometry*, Invent. Math., **152** (2003), 89–118.

[19] P. D. Hislop, P. A. Perry, and S.-H. Tang, *CR-invariants and the scattering operator for complex manifolds with boundary*, Anal. PDE, **1** (2008), 197–227.

[20] K. Hirachi, *Scalar pseudo-hermitian invariants and the Szegö kernel on three-dimensional CR manifolds*, Complex Geometry, Lect. Notes in Pure and Appl. Math., **143**, 67–76, Dekker, New York, 1993.

[21] ———, *Invariant theory of the Bergman kernel of strictly pseudoconvex domains*, Sugaku Expositions, **17** (2004), 151–169.

[22] ———, *Logarithmic singularity of the Szegö kernel and a global invariant of strictly pseudoconvex domains*, Ann. of Math., **163** (2006), 499–515.

[23] ———, *Q-prime curvature on CR manifolds*, Diff. Geom. Appl., **33** Suppl. (2014), 213–245.

[24] K. Hirachi, T. Marugame, and Y. Matsumoto (with an appendix by A. R. Gover and K. Hirachi), *Variation of total Q-prime curvature on CR manifolds*, in preparation.

[25] K. Hirachi, Y. Matsumoto, and B. Ørsted, *in preparation.*

[26] T. Marugame, *Renormalized Chern-Gauss-Bonnet formula for complete Kähler-Einstein metrics*, preprint, arXiv:1309.2766.

[27] Y. Matsumoto, *GJMS operators, Q-curvature, and obstruction tensor of partially integrable CR manifolds*, preprint, arXiv:1402.4110.

[28] N. M. Møller and B. Ørsted, *Rigidity of conformal functionals on spheres*, Int. Math. Res. Notices, published online August 2013, arXiv:0902.4067.

Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro, Tokyo 153-8914 JAPAN

E-mail: hirachi@ms.u-tokyo.ac.jp

Advances in weighted norm inequalities

Tuomas Hytönen

Abstract. The classical theory of weighted norm inequalities provides a characterization of admissible weights such that the Hilbert transform or other singular operators act boundedly from the weighted space $L^p(w)$ to itself. This lecture surveys two lines of recent development: proving sharp quantitative forms of the classical mapping properties (the A_2 theorem), and characterizing the admissible pairs of weights when the operator acts from one $L^2(u)$ space to another $L^p(v)$ (the two-weight problem).

Mathematics Subject Classification (2010). Primary 42B20; Secondary 42B25.

Keywords. Weighted norm inequality, A_2 theorem, two-weight problem, Hilbert transform, testing condition.

1. Introduction

A fundamental object in Harmonic Analysis for more than a hundred years is the Hilbert transform

$$Hf(x) = \mathrm{p.v.} \int_{-\infty}^{\infty} \frac{f(y)dy}{x-y} = \lim_{\epsilon\to 0} \Big(\int_{-\infty}^{x-\epsilon} + \int_{x+\epsilon}^{\infty} \Big) \frac{f(y)dy}{x-y}. \tag{1.1}$$

(The usual normalization involving the factor $1/\pi$ is irrelevant for the present discussion.) It appeared, in its periodic version, in the work of D. Hilbert in 1905 [13, see in particular Eqs. (7*) and (5)], and the basic L^p inequalities

$$\|Hf\|_{L^p(\mathbb{R})} \le c_p \|f\|_{L^p(\mathbb{R})}, \qquad 1 < p < \infty, \tag{1.2}$$

both in the periodic case and as stated, were first established by M. Riesz in 1928 [47]. Ever since, obtaining analogues of the norm bound (1.2) for different variants and generalizations of the Hilbert transform has been among the central themes in Harmonic Analysis. It was already in same paper of M. Riesz [47, §23] that the corresponding L^p bounds for the discrete analogue of (1.1),

$$H_d f(k) = \sum_{\substack{j\in\mathbb{Z}\\ j\neq k}} \frac{f(j)}{k-j}, \tag{1.3}$$

were also proven, namely the estimate

$$\|H_d f\|_{\ell^p(\mathbb{Z})} \le c_p' \|f\|_{\ell^p(\mathbb{Z})}, \qquad 1 < p < \infty. \tag{1.4}$$

Proceedings of the International Congress of Mathematicians, Seoul, 2014

A unified framework, which captures both the continuous and the discrete cases, is to consider the Hilbert transform with respect to a (Radon) measure μ:

$$H(f\,d\mu)(x)=\int\frac{f(y)\,d\mu(y)}{x-y},\qquad x\notin\operatorname{spt}f. \tag{1.5}$$

(The existence of principal value, as in (1.1), is a subtle issue in general, and we content ourselves by requiring that H is *some* linear operator that satisfies (1.5), although this does not specify it uniquely.) Both classical estimates (1.2) and (1.4) are then special cases of

$$\|H(f\,d\mu)\|_{L^p(\mu)}\le C\|f\|_{L^p(\mu)}, \tag{1.6}$$

where we have stopped keeping track of the parameter-dependence of the constants.

Our main emphasis in this lecture is on generalizations of (1.6) involving *different measures* rather than just one fixed μ. The reader can easily count three occurrences of μ in (2.4), and we could, in principle, try to replace each of them by an independent object, arriving at the question of characterizing the estimate

$$\|H(f\,d\mu)\|_{L^p(\omega)}\le C\|f\|_{L^p(\lambda)}. \tag{1.7}$$

The reason that this "three-weight problem" is rarely mentioned is that a little argument, essentially due to E. Sawyer [49], reveals that only two of these measures deserve to be genuinely independent, in that the essence of the matter is fully captured by the situation in which λ and μ are mutually absolutely continuous:

Lemma 1.1. *Without loss of generality, we may assume that $\lambda\ll\mu\ll\lambda$ in* (1.7).

Roughly speaking, the point is that the singular part of λ is both harmless and useless, since it does not affect the left side of (1.7), while the singular part of μ is essentially forbidden, since it cannot be controlled by the right side of (1.7). A more precise formulation of the reduction and the argument will be formulated in Section 3. Once this reduction is in force, a useful dual weight trick reduces matters to just two measures altogether:

Proposition 1.2. *For $\lambda\ll\mu\ll\lambda$ (which we may assume by Lemma 1.1), the bound* (1.7) *is equivalent to*

$$\|H(f\,d\sigma)\|_{L^p(\omega)}\le C\|f\|_{L^p(\sigma)}, \tag{1.8}$$

where

$$d\sigma=\phi\,d\mu=\phi^p\,d\lambda,\qquad \phi=\Big(\frac{d\mu}{d\lambda}\Big)^{1/(p-1)},$$

and $d\mu/d\lambda$ is the Radon–Nikodým derivative.

Proof. Let ϕ be a fixed measurable function that is finite and strictly positive everywhere. The identity $f=f'\phi$ establishes a bijective and isometric correspondence between $f\in L^p(d\lambda)$ and $f'\in L^p(\phi^p\,d\lambda)$. Making the substitution $f=f'\phi$ in (1.7), we find that this bound is equivalent to

$$\|H(f'\phi\,d\mu)\|_{L^p(\omega)}\le C\|f'\|_{L^p(\phi^p\,d\lambda)}.$$

Requiring that $\phi\,d\mu=\phi^p\,d\lambda$ leads to the choice of ϕ specified in the statement of the Proposition. Note that this is everywhere finite and strictly positive by the condition that μ and λ are mutually absolutely continuous. □

Three main lines of investigation related to the bound (1.7) are the following:

1. Classical weighted inequalities correspond to $d\mu = dx$, $d\omega = d\lambda = w\,dx$; thus $d\sigma = w^{-1/(p-1)}dx$.
2. The "geometric theory" corresponds to the same measure $d\mu = d\omega = d\lambda$ in all three positions.
3. The full two-weight theory addresses arbitrary pairs of measures in (1.8).

The aim of this lecture is to present some recent advances particularly in the first and the third research directions above. The geometric theory, which involves one fixed measure throughout, is usually not regarded as an instance of the "weighted theory", but it is also discussed at some length due to its proximity in terms of techniques.

We begin in Section 2 by providing a more detailed historical perspective for the three research domains defined above. We stop in Section 3 to give a proof of the basic reduction stated in Lemma 1.1. The recent advances on our two main topics are discussed in the next two sections: classical weighted inequalities in Section 4 and the two-weight theory in Section 5. We conclude with a list of open problems in Section 6.

Throughout the presentation, the emphasis of our discussion stays with the Hilbert transform. We will make occasional remarks about results that hold for more general Calderón–Zygmund operators, but avoid most of the details (including the definition of these operators) in this direction. When discussing the most general set-up in terms of measures — the two-weight theory — the present knowledge is essentially restricted to the Hilbert transform in any case.

2. Three historical lines of study

The main instances of (1.7) for us are the following:

2.1. Classical weighted inequalities. These correspond to computing the Hilbert transform (or another integral operator) with respect to the plain Lebesgue measure $d\mu = dx$, and evaluating the norms of both the 'input' f and the 'output' $H(f\,dx)$ with respect to *one other* independent measure $d\lambda = d\omega$. Recalling that $d\lambda$ and $d\mu = dx$ can be taken to be absolutely continuous by Lemma 1.1, we have in this case that $d\lambda = d\omega = w\,dx$, where the *weight* w is finite and strictly positive. Thus the bound (1.7) then reads as

$$\|H(f\,dx)\|_{L^p(w\,dx)} \le C\|f\|_{L^p(w\,dx)} \tag{2.1}$$

or, in the dual weight formulation (1.8), as

$$\|H(f\,w^{1-p'}dx)\|_{L^p(w\,dx)} \le C\|f\|_{L^p(w^{1-p'}\,dx)}$$

The classical (1970s) theory of R. Hunt, B. Muckenhoupt and R. Wheeden [15] provides a characterization of the admissible weights for (2.1) in terms of the celebrated A_p condition, the finiteness of the weight characteristic

$$[w]_{A_p} := \sup_{I\subset\mathbb{R}} \Big(\frac{1}{|I|}\int_I w\,dx\Big)\Big(\frac{1}{|I|}\int_I w^{1-p'}\,dx\Big)^{p-1}, \tag{2.2}$$

where here and below, $\sup_{I\subset\mathbb{R}}$ refers to a supremum over all intervals $I\subset\mathbb{R}$. The importance of this condition (2.2) is in its *universality*. Prior to [15], Muckenhoupt had shown its necessity and sufficiency for the analogue of (2.1) with the Hardy–Littlewood maximal operator, namely

$$\|M(f\,dx)\|_{L^p(w)}\le C\|f\|_{L^p(w)},\quad M(f\,d\mu):=\sup_{I\subset\mathbb{R}}\frac{1}{|I|}\int_I|f|\,d\mu, \tag{2.3}$$

and shortly after R. R. Coifman and C. Fefferman proved that the same characterization persists for the class of all Calderón–Zygmund operators.

A more recent (2000s) trend has been the quantification of the sharp dependence of the operator norm C in (2.1) and related estimates on the weight characteristic $[w]_{A_p}$. This will be discussed in Section 4.

2.2. "Geometric" theory: results with one arbitrary measure. Here we have in mind the bound (1.6) (i.e., (1.7) with $\lambda=\omega=\mu$), and we extend the defining formula (1.5) by allowing the domain of integration (which we purposely left unspecified in (1.5)) to be a subset of $\mathbb{C}$ instead of $\mathbb{R}$. Clearly, formula (1.5) makes perfect sense for complex numbers $x\ne y$ as well. Here and below, our main emphasis is on the case $p=2$.

Over the last quarter of the twentieth century, several authors contributed to the problem of classifying the measures μ on $\mathbb{C}$ for which the L^2 estimate

$$\|H(f\,d\mu)\|_{L^2(\mu)}\le C\|f\|_{L^2(\mu)}, \tag{2.4}$$

is valid. This quest was pioneered by the works of A.-P. Calderón [5] resp. R. R. Coifman, A. McIntosh and Y. Meyer [6], who proved the admissibility of $\mu=\mathcal{H}^1|_\Gamma$, the one-dimensional Hausdorff measure restricted to a Lipshitz graph $\Gamma=\{(t,A(t)):t\in\mathbb{R}\}$ with small resp. arbitrary Lipschitz constant $\|A'\|_\infty$. After the first proof of this result, several alternative arguments were found. In particular, the "geometric proof" by M. S. Melnikov and J. Verdera [33] allowed to extend the positive result to chord-arc curves Γ. This paper highlighted the role of the *curvature of a measure* in these questions, a theme that was further elaborated by the same authors and P. Mattila [32]. Assuming that $\mu=\mathcal{H}^1|_E$ is Ahlfors–David (AD) 1-regular, which means

$$cr\le\mu(D(x,r))\le Cr\qquad\forall x\in\operatorname{spt}\mu,\quad\forall r\in(0,\operatorname{diam}(\operatorname{spt}\mu)), \tag{2.5}$$

Mattila et al. [32] showed that (2.4) holds, if and only if E is contained in an AD-1-regular *curve*. Note that (2.5) makes $\operatorname{spt}\mu$, equipped with the Euclidean distance and the measure μ, into a *space of homogeneous type*, and the work of Mattila et al. [32] was essentially the ultimate stretch of the homogeneous theory of singular integrals in this set of problems.

In order to go beyond, F. Nazarov, S. Treil and A. Volberg [38] on the one hand, and X. Tolsa [52], independently, on the other, set the foundations for what is now known as *non-homogeneous harmonic analysis*. They showed that for measures μ satisfying only the upper half of (2.5), i.e.

$$\mu(D(x,r))\le Cr\qquad\forall x\in\mathbb{C},\quad\forall r>0, \tag{2.6}$$

the bound (2.4) is equivalent to the curvature condition of [32, 33], namely

$$\iiint_{D\times D\times D}\frac{d\mu(x)\,d\mu(y)\,d\mu(z)}{R(x,y,z)^2}\le C\mu(D)\qquad\forall\text{ disks }D\subset\mathbb{C},$$

where $R(x, y, z)$ is the radius of the circle through x, y and z (interpreted as ∞ if the points are collinear), or further to the *testing condition*

$$\|1_Q H(1_Q\, d\mu)\|_{L^2(\mu)} \le C\mu(Q)^{1/2} \qquad \forall \text{ squares } Q \subset \mathbb{C}. \tag{2.7}$$

The method of Nazarov et al. [38] was particularly important for the further development of the weighted theory, including most of the recent main results discussed in this lecture. It introduced the powerful method of *random dyadic decompositions*, which has become a cornerstone of non-homogeneous harmonic analysis; as a general tool, it allowed to treat not only the Hilbert transform but a whole class of next generation Calderón–Zygmund operators with respect to non-homogeneous measures [39].

As for the linear growth assumption (2.6) present in both [38, 52], this had been known as a necessary condition for (2.4) for some time (see G. David [9, Prop. III.1.4]) *provided* that the measure is non-atomic (i.e., $\mu(\{x\}) = 0$ for all x) to begin with. The removal of this side-condition, by coping with possible atoms, was another major step from Tolsa [53], who showed that the equivalence of (2.4) and (2.7) persists without any *a priori* assumptions. As Tolsa pointed out [53, p. 197], "when μ is doubling there are few differences between the case μ continuous [i.e., non-atomic] and the case μ non-continuous", whereas what he proves in [53] "cannot be derived easily from other known results for non-doubling measures without atoms." There is an interesting parallelism between this and some rather recent (2013) developments that we discuss below.

The subsequent work of Nazarov, Treil, Volberg and Tolsa on this topic led to the solution of a number of celebrated long-standing problems in Complex Analysis. This important line of development has been surveyed in the 2006 ICM lecture by Tolsa [54], and we shall not repeat it here. Instead, we mention M. Lacey, E. Sawyer and I. Uriarte-Tuero's [28] more recent related resolution of K. Astala's [2, Question 4.4] on the distortion of Hausdorff measures by quasiconformal maps: A key ingredient of [28] was a bound analogous to the classical (2.1) for the Beurling–Ahlfors transform, also known as the two-dimensional Hilbert transform,

$$Bf(z) = -\,\mathrm{p.v.} \int_{\mathbb{C}} \frac{f(\zeta)\, d\zeta_1\, d\zeta_2}{\pi(z-\zeta)^2}, \tag{2.8}$$

but with a restriction $1_E\, dx_1\, dx_2$ in place of dx and a weight $w = 1_E w \notin A_2$. These seemingly innocent changes were enough to leave this problem outside the classical framework; its solution was an important prelude to the extensive collaboration of the same trio on the two-weight problem (see Section 5). Altogether, the geometric theory here outlined had a major influence for the core topics of our present discussion in Sections 4 and 5.

2.3. Full two-weight inequalities. We finally discuss the general problem that, after the reduction of Lemma 1.1, can be formulated as (1.8). Many of the two-weight questions were already posed, and some of them solved, during the 1970s. However, there is a noteworthy difference between the classical and the recent theory: in the former one, although weighted *norms* were considered, the *operators* themselves were always evaluated with respect to the Lebesgue measure only, i.e., the measure $d\mu$ in (1.7) was always taken to be dx. Because of this, and Lemma 1.1 or its analogues in other situations, measures other than Lebesgue's with a weight seldom occurred in the theory, and the classical two-weight problem took the form

$$\|H(f\, dx)\|_{L^p(u\, dx)} \le C\|f\|_{L^p(v\, dx)}, \tag{2.9}$$

that is

$$d\mu = dx, \quad d\omega = u\,dx, \quad d\lambda = v\,dx, \quad d\sigma = v^{-1/(p-1)}dx =: \tilde{v}\,dx \tag{2.10}$$

in the language of (1.7) and (1.8). As Muckenhoupt notes in his 1979 survey [34, p. 70], replacing weighted measures by general ones "has not been particularly popular in the literature because in some cases it can be shown that only absolutely continuous measures can be used, and in other cases the results are just complicated but trivial consequences." However, as we already witnessed in §2.2, and shall see in other situations below, interesting and nontrivial results with singular measures take place as soon as we also allow a new measure *inside* the operator, i.e., in place of $d\mu$ in (1.7).

The success of the A_p condition in the one-weight situation has encouraged attempts to build a two-weight theory around suitable generalizations of this condition. It was already known to Muckenhoupt and Wheeden [35] that (2.9) implies a two-weight analogue of (2.2),

$$\begin{aligned}[\omega,\sigma]_{A_p} &:= \sup_{I\subset\mathbb{R}} \frac{\omega(I)}{|I|}\Big(\frac{\sigma(I)}{|I|}\Big)^{p-1} \\ &= \sup_{I\subset\mathbb{R}} \Big(\frac{1}{|I|}\int_I u\,dx\Big)\Big(\frac{1}{|I|}\int_I \tilde{v}\,dx\Big)^{p-1} < \infty\end{aligned} \tag{2.11}$$

and indeed a strengthening of this condition involving a "Poisson-type" (at least for $p = 2$) tail:

$${}^*[\omega,\sigma]_{A_p} := \sup_{I\subset\mathbb{R}} \Big(\int_{\mathbb{R}} \frac{d\omega(x)}{(|I| + |x - c_I|)^p}\Big)\sigma(I)^{p-1} < \infty. \tag{2.12}$$

(Restricting the integral over $\mathbb{R}$ to just I essentially reduces (2.12) to (2.11).) Simple examples show that (2.11) is not sufficient for (2.9); see [36, p. 1], which also presents a more substantial counterexample, to be discussed shortly.

One of the early positive results, a prototype for a substantial line of further research, is implicit in the work of C. Neugebauer [42]: he showed that "bumping up" the A_p condition by a power $r > 1$,

$$[u,\tilde{v}]_{A_p,r} := \sup_{I\subset\mathbb{R}} \Big(\frac{1}{|I|}\int_I u^r\,dx\Big)\Big(\frac{1}{|I|}\int_I \tilde{v}^{\,r}dx\Big)^{p-1} < \infty, \tag{2.13}$$

is sufficient (and necessary) for the existence of $w \in A_p$ with $c^{-1}u \le w \le cv$, and therefore sufficient for (2.9) by the one-weight theory. A considerable effort has been devoted to finding minimal bump conditions of this type (on the scale of Orlicz or more general Banach function spaces), which still suffice for the estimate (2.9) and its analogue for more general Calderón–Zygmund operators. Our main focus, however, is in a different line of research aimed at *characterizing* (2.9), or the more general (1.8).

Part of the interest in this problem, especially for $p = 2$, came from its connection to D. Sarason's question on the boundedness of *products of Toeplitz operators* $T_f T_{\bar{g}}$ on H^2, the analytic subspace of L^2. The Toeplitz operator with symbol f is defined as $T_f = P \circ M_f$, where M_f is the pointwise multiplication by f and P is the orthogonal projection of L^2 onto H^2. As is well known, P is essentially the same as the Hilbert transform, and precisely given by

$$P = \frac{1}{2}(I + P_0 + iH),$$

where P_0, the projection onto constants, vanishes on $L^2(\mathbb{R})$ and has rank one on $L^2(\mathbb{T})$, where these operators are usually considered. If f and g are analytic, then $T_f T_{\bar{g}} = M_f \circ P \circ M_{\bar{g}}$ is bounded on H^2 if and only if it is bounded from L^2 to H^2 (since $L^2 \ominus H^2$ is invariant for $M_{\bar{g}}$ and annihilated by P), and this operator factorizes as

$$M_f \circ P \circ M_{\bar{g}} : L^2 \xrightarrow{M_{\bar{g}}} L^2(|g|^{-2}) \xrightarrow{P} H^2(|f|^2) \xrightarrow{M_f} H^2,$$

where the multiplication operators are isometric between the indicated function spaces. Thus the boundedness of $T_f T_{\bar{g}}$ on H^2 is equivalent to the boundedness of $P : L^2(|g|^{-2}) \to H^2(|f|^2)$, or equivalently, of $P : L^2(|g|^{-2}) \to L^2(|f|^2)$, since it automatically maps into the analytic part. This factorization and its consequences were observed by D. Cruz-Uribe [7].

In the world of Toeplitz operators, D. Sarason had made a conjecture that, translated to the language of the Hilbert transform, corresponds to the characterization of (1.8) by the "double-Poisson" version of (2.12), namely

$$^*[\omega,\sigma]^*_{A_2} := \sup_{I\subset\mathbb{R}} \Big(\int_{\mathbb{R}} \frac{d\omega(x)}{(|I|+|x-c_I|)^2} \Big) \Big(\int_{\mathbb{R}} \frac{d\sigma(x)}{(|I|+|x-c_I|)^2} \Big) < \infty. \tag{2.14}$$

F. Nazarov [36] disproved the conjecture that, "as Sarason himself once confessed, appeared 'just as a wild guess'." [36, p. 2]

It was probably soon after Nazarov's counterexample that the hopes of any simple "A_p style" characterization of the two-weight problem were abandoned for good, and the activity around the *testing paradigm* began to catch fire. An obvious necessary condition for (1.8) (or indeed any other inequality) to hold for all relevant functions f, is that it should hold for any more restricted selection of $f \in \mathcal{F}$. As (1.8) is easily (at least formally, using the unweighted duality $H^* = -H$) seen to be equivalent to its dual version,

$$\|H(g\,dw)\|_{L^{p'}(\sigma)} \le C\|g\|_{L^{p'}(w)}, \tag{2.15}$$

a similar remark on restriction to $g \in \mathcal{F}$ can be made for this dual inequality. The point of the testing paradigm is to look for a minimal set of *testing functions* $\mathcal{F}$, so that the full estimate (1.8) is already implied by its (and its dual version's) specialization to all $f, g \in \mathcal{F}$. A particularly popular choice is to use the family $\mathcal{F} = \{1_I : I \subset \mathbb{R} \text{ interval}\}$ of indicators of all finite intervals (or cubes in higher dimensional analogues).

This philosophy has two independent contemporaneous historical origins in the 1980s. On the one hand, E. Sawyer obtained several characterizations of this type for two-weight inequalities involving a *positive* operator (such as the maximal operator [49], or fractional and Poisson integrals [50]) in place of the Hilbert transform in (1.8). On the other hand, for singular operators, but in the *unweighted* case, the $T(1)$ theorem of G. David and J. L. Journé takes exactly this form, especially when using its local formulation: a weakly defined operator T with a Calderón–Zygmund kernel acts boundedly on the unweighted $L^2(\mathbb{R})$ if and only if $\|T(1_I)\|_{L^2} \le C|I|^{1/2}$ and $\|T^*(1_I)\|_{L^2(I)} \le C|I|^{1/2}$ for all intervals $I \subset \mathbb{R}$. It seems that the connection of these two theories was not recognized until F. Nazarov, S. Treil and A. Volberg united them in the context of the two-weight problem. This line of development will be discussed in more detail in Section 5.

3. Why not three weights? — Proof of Lemma 1.1

We reformulate and prove Lemma 1.1 more precisely as follows:

Lemma 3.1. *Let $\mu = \mu_c + \mu_s$ and $\lambda = \lambda_c + \lambda_s$, where $\lambda_c \ll \mu_c \ll \lambda_c$ while μ_s and λ, resp. λ_s and μ are mutually singular. Then* (1.7) *holds as stated, if and only if it holds with (λ, μ) replaced by (λ_c, μ_c) and in addition:*

(1) *$\mu_s = 0$ or $\omega = 0$,*

(2) *both μ_s and ω are nonzero multiples of some δ_{x_1}, and $H(d\delta_{x_1})(x_1) = 0$.*

Note that for $\omega = 0$, the whole estimate (1.7) is vacuous, whereas the condition 2 above is not really a condition to verify but a *matter of definition* (the general formula (1.1) does not specify the Hilbert transform of a measure on the support of this measure). Thus the essence of the matter is indeed in the bound (1.7) with the absolutely continuous parts λ_c and μ_c.

Proof. It is immediate that (1.7) is equivalent to

$$\|H(f\,d\mu)\|_{L^p(\omega)} \le C\|f\|_{L^p(\lambda_c)}. \tag{3.1}$$

Namely, if we substitute $1_{\complement \operatorname{spt}\lambda_s} f$ in place of f in (1.7), the left side is unchanged (as we only altered f in a set of μ-measure zero), and the right side becomes $\|f\|_{L^p(\lambda_c)}$. Conversely, since $\|f\|_{L^p(\lambda_c)} \le \|f\|_{L^p(\lambda)}$, it is clear that (3.1) implies (1.7). So we may assume for the rest of the proof that $\lambda = \lambda_c$, and it remains to prove that we may also replace μ by μ_c. With similar substitutions, it is easy to see that (1.7) is equivalent to the two bounds

$$\|H(f\,d\mu_c)\|_{L^p(\omega)} \le C\|f\|_{L^p(\lambda)}$$

and

$$\|H(f\,d\mu_s)\|_{L^2(\omega)} = 0. \tag{3.2}$$

where the first one, with $\lambda = \lambda_c$, is precisely (1.7) with the absolutely continuous parts of the measures, and it remain to prove that (3.2) is equivalent to having condition 1 or 2 of Lemma 3.1.

Now assume (3.2). For a nonnegative f that is not μ_s-a.e. equal to zero, we have that

$$H(f\,d\mu_s)(x) = \int \frac{f(y)\,d\mu_s(y)}{x-y}$$

is strictly positive on the right of $\operatorname{spt} f$, and strictly negative on the left of $\operatorname{spt} f$. This would give a nonzero contribution to $\|H(f\,d\mu_s)\|_{L^p(\omega)}$, unless ω gives a zero measure for the infinite half-lines both on the right and the left of $\operatorname{spt} f$. By choosing different parts of $\operatorname{spt}\mu_s$ as $\operatorname{spt} f$, we find that $\omega(\mathbb{R}) = 0$ whenever $\operatorname{diam}(\operatorname{spt}\mu_s) > 0$. Thus the only nontrivial case is when $\operatorname{spt}\mu_s = \{x_1\}$ is a singleton. In this case, (3.2) requires that ω must vanish both on the left and the right of x_1, so that ω itself is a multiple of δ_{x_1}, and then (3.2) is equivalent to the requirement that

$$H(\delta_{x_1})(x_1) = 0,$$

which is condition 2 of Lemma 3.1. Altogether we checked that (3.2) implies that either condition 1 or 2 is valid. Conversely, it is immediate that either of these conditions implies (3.2). □

4. Sharp weighted inequalities

4.1. The A_2 conjecture and theorem. The size of the constants in the classical weighted inequalities seems to have been first addressed by S. Buckley [4], who proved that $C(2.3) \leq c_p[w]_{A_p}^{1/(p-1)}$, where the dependence on $[w]_{A_p}$ is best possible, and posed the similar question for $C(2.1)$ and related bounds.

Buckley's problem for singular integrals gained new momentum from the work of K. Astala, T. Iwaniec and E. Saksman [1], who were interested in the particular case of the Beurling–Ahlfors transform (2.8). This operator is fundamental in Complex Analysis, since it intertwines between the two derivatives by $B \circ \partial_{\bar{z}} = \partial_z$. Motivated by the conditional corollary below, Astala et al. posed the following conjecture:

Conjecture 4.1 ([1]). $\|Bf\|_{L^p(w)} \leq c_p[w]_{A_p}\|f\|_{L^p(w)}$ *for all* $p \in [2, \infty)$.

Corollary 4.2 ([1]). *Conditionally on the conjecture, the following implication holds for all* $q \in (1, 2)$*: If* $f \in W^{1,q}_{\mathrm{loc}}(\Omega)$ *satisfies* $|\partial_{\bar{z}} f| \leq (q-1)|\partial_z f|$ *pointwise on a domain* $\Omega \subset \mathbb{C}$*, then* $f \in W^{1,2}_{\mathrm{loc}}(\Omega)$.

Unconditionally, they were able to show a similar implication assuming that

$$f \in W^{1,q+\epsilon}_{\mathrm{loc}}(\Omega)$$

instead; on the other, $f \in W^{1,q-\epsilon}_{\mathrm{loc}}(\Omega)$ was shown to be insufficient for the conclusion. Thus Conjecture 4.1 was critical to the sharp version of this self-improvement result.

The conjecture was subsequently confirmed by S. Petermichl and A. Volberg [46]. While this was enough for the goal of proving Corollary 4.2, the question then arose whether a similar bound could be established for other operators as well. This came to be known under the name "A_2 conjecture", due to the critical role of the exponent $p = 2$ resulting from the quantitative version of a classical extrapolation theorem of J. L. Rubio de Francia [48] established by O. Dragičević, L. Grafakos, M. C. Pereyra and S. Petermichl [10]:

Theorem 4.3 ([10]). *Suppose that an operator* T*, linear or not, satisfies* $\|Tf\|_{L^2(w)} \leq c[w]_{A_2}\|f\|_{L^2(w)}$ *for all* $f \in L^2(w)$ *and all* $w \in A_2$*. Then it also satisfies*

$$\|Tf\|_{L^p(w)} \leq c_p[w]_{A_p}^{\max(1,1/(p-1))}\|f\|_{L^p(w)}$$

for all $f \in L^p(w)$*, all* $w \in A_p$ *and all* $p \in (1, \infty)$.

The A_2 conjecture for the Hilbert transform,

$$\|H(f\,dx)\|_{L^2(w)} \leq c[w]_{A_2}\|f\|_{L^2(w)}, \tag{4.1}$$

was confirmed by Petermichl [45], based on a *dyadic model* whose reincarnations were central in the subsequent developments as well. A *dyadic system of intervals* is a collection $\mathcal{D} = \bigcup_{k\in\mathbb{Z}} \mathcal{D}_k$, where

- each $\mathcal{D}_k$ is a partition of $\mathbb{R}$ into intervals of length 2^{-k}, and
- each $\mathcal{D}_{k+1}$ is a refinement of the previous $\mathcal{D}_k$.

For each such $\mathcal{D}$, the Haar functions

$$h_I := \frac{1}{\sqrt{|I|}}(1_{I_{\text{left}}} - 1_{I_{\text{right}}}), \qquad I \in \mathcal{D},$$

(where I_{left} and I_{right} are the left and right halves of I) form an orthonormal basis of $L^2(\mathbb{R})$, so that $f = \sum_{K\in\mathcal{D}} h_K\langle h_K, f\rangle$. Petermichl's dyadic model was a "shift" of this series, namely

$$S_{\mathcal{D}} f = \sum_{K\in\mathcal{D}} A_K, \qquad S_K f = (h_{K_{\text{left}}} - h_{K_{\text{right}}})\langle h_K, f\rangle. \tag{4.2}$$

Her key identity, which reduced the proof of (4.1) to its dyadic version

$$\|S_{\mathcal{D}}(f)\|_{L^2(w)} \le c[w]_{A_2}\|f\|_{L^2(w)}, \tag{4.3}$$

was the averaging formula, already in [44],

$$H = c\mathbb{E}_{\mathcal{D}} S_{\mathcal{D}}, \tag{4.4}$$

where $\mathbb{E}_{\mathcal{D}}$ is an appropriate average over all translated and dilated dyadic systems. It should be noted that proving (4.3) was still a highly nontrivial task at the time.

The averaging formula (4.4) was crucially based on the translation and dilation invariance of the Hilbert transform, and a similar strategy was successfully applied to a few other distinguished classical operators involving invariances and symmetries. It was therefore not obvious that the result could be pushed to the generality of all Calderón–Zygmund operators, which nevertheless turned out to be the case:

Theorem 4.4 (The A_2 theorem, [17]). *The estimate* (4.1) *remains valid for any Calderón–Zygmund operator in place if* H.

The proof was also based on an extension of (4.4): for a general Calderón–Zygmund operator T with kernel regularity $\alpha \in (0, 1]$, there is a representation

$$T = c\mathbb{E}_{\mathcal{D}} \sum_{m,n=0}^{\infty} 2^{-\alpha(m+n)/2} S_{\mathcal{D}}^{m,n}, \tag{4.5}$$

where the shifts $S_{\mathcal{D}}^{m,n}$ have the form

$$S_{\mathcal{D}}^{m,n} f = \sum_{K\in\mathcal{D}} S_K^{m,n}, \qquad S_K^{m,n} f = \sum_{\substack{I,J\subseteq K\\ |I|/|K|=2^{-m}\\ |J|/|K|=2^{-n}}} s_{IJ}^K h_I\langle h_J, f\rangle,$$

and the coefficients s_{IJ}^K satisfy bounds which ensure that $|S_K^{m,n} f| \le \frac{1}{|K|}\int_K |f|\,dx$; if $T1 \ne 0$ or $T^*1 \ne 0$, one needs to incorporate additional *dyadic paraproducts* in $S_{\mathcal{D}}^{0,0}$. These general shifts were already identified in the prior work of M. Lacey, S. Petermichl and M. C. Reguera [25], who also considered the bound (4.3). However, the representation of all Calderón–Zygmund operators in terms of these shifts derived its inspiration from another direction, the non-homogeneous techniques of Nazarov, Treil and Volberg [38] on the geometric one-measure theory (§2.2). For the role of non-homogeneous analysis in this context,

it is perhaps worth stressing that although the classical A_p weights are *doubling* (i.e., satisfy $\int_{2I} w\,dx \le C\int_I w\,dx$), using this property for any sharp weighted bound is essentially out of question, since this would introduce unfavourable dependence on the weight via the doubling constant $C = C(w)$. In order to complete the proof of Theorem 4.4 via the Dyadic Representation Theorem stated in (4.5), it was also necessary to have a good dependence of the constant c(4.5) on the *shift complexity* (m, n), which was another contribution of [17].

4.2. Further developments. After the first proof of Theorem 4.4, the A_2 conjecture has been reconfirmed, extended and simplified by several alternative arguments. (See [16] for an account of these developments.) These culminated in the ultimate dyadic model discovered by A. Lerner [30]:

Theorem 4.5 (Lerner's dyadic domination theorem [30]). *For an arbitrary Banach function space X and a Calderón–Zygmund operator T, we have*

$$\|Tf\|_X \le c\sup_{\mathcal{D},\mathcal{S}} \|A_{\mathcal{S}}|f|\|_X, \qquad A_{\mathcal{S}}f = \sum_{S\in\mathcal{S}} \frac{1_S}{|S|}\int_S f\,dx,$$

where the supremum is over all dyadic systems $\mathcal{D}$ and their sparse *subcollections $\mathcal{S}$: for each $S \in \mathcal{S}$, there is $E(S) \subseteq S$ such that $|E(S)| \ge \frac{1}{2}|S|$, and the sets $E(S)$, for $S \in \mathcal{S}$, are pairwise disjoint.*

In particular, this applies to $X = L^2(w)$ and reduces the proof of the A_2 theorem to the case of the very basic dyadic model operators $A_{\mathcal{S}}$. This was a substantial simplification, since a great deal of effort in the earlier contributions was always spent on keeping track of the dependence on the shift complexity parameters, which was now completely eliminated from applications of Theorem 4.5. The A_2 theorem for the operators $A_{\mathcal{S}}$ was already known to follow from an elegant few-lines argument of D. Cruz-Uribe, J. M. Martell and C. Pérez [8] that predated Theorem 4.4; at the time, it was unknown (and perhaps unimaginable) that these simple objects could be used to recover all Calderón–Zygmund operators.

The Dyadic Domination Theorem 4.5 now seems to be the most efficient route to the A_2 theorem and several related estimates. In particular, it has become a new standard tool in the part of the two-weight theory based on the bump paradigm (see §2.3); we refer to some of its applications in [20, 30, 37]. Nevertheless, also the original Dyadic Representation Theorem (4.5) may still have its independent interest as a structure theorem for singular integrals. In this spirit, T. Orponen [43] has shown that the representation (4.5) *only* exists for Calderón–Zygmund operators, whereas H. Martikainen [31] has extended and applied variants of (4.5) to bi-parameter singular integrals.

5. Two-weight theory with testing conditions

5.1. Towards a "testing" characterization. Building on the line of ideas from their work in the non-homogeneous one-measure bounds for the Hilbert transform (§2.2), F. Nazarov, S. Treil and A. Volberg set the foundations for the modern theory of the two-weight problem [40, 55]. The continuation of this endeavour was subsequently taken over by another trio of M. Lacey, E. Sawyer and I. Uriarte-Tuero [29], eventually joined by C.-Y. Shen [26].

Until very recently, all available results took the following generic form:

Theorem 5.1 ([22, 26, 40, 55]). *Let σ and ω be two Radon measures on $\mathbb{R}$. Subject to an additional* side condition*, the Hilbert transform satisfies the norm inequality*

$$\|H(f\,d\sigma)\|_{L^2(\omega)} \le C\|f\|_{L^2(\sigma)}, \tag{5.1}$$

if and only if it satisfies the testing conditions

$$\|1_I H(1_I\,d\sigma)\|_{L^2(\omega)} \le C\sigma(I)^{1/2}, \quad \|1_I H(1_I\,d\omega)\|_{L^2(\sigma)} \le C\omega(I)^{1/2}, \quad \forall\, I \subset \mathbb{R}, \tag{5.2}$$

and the weights satisfy the double-Poisson A_2 condition (2.14)*.*

The *side condition* refers to an *a priori* assumption needed for a particular proof of (5.1), but not verifiably necessary for this inequality. Of course, Theorem 5.1 has a mathematical content only after the specification of an admissible side condition.

A rather classical-style side condition was used in Volberg's book [55, Theorem 15.1]: the doubling property of both measures,

$$\sigma(2I) \le C\sigma(I), \quad \omega(2I) \le C\omega(I) \quad \forall \text{ intervals } I \subset \mathbb{R}. \tag{5.3}$$

The subsequent developments were guided by Volberg's comment after the statement of the theorem: "We hope that (5.3) is superfluous." This was achieved only gradually by the replacement of (5.3) by increasingly more general side conditions.

An influential side condition was the *pivotal property* introduced by Nazarov, Treil and Volberg in an unpublished paper [40] that, according to its arXiv website, "was written in 2005 and subsequently lost." The pivotal property is the uniform estimate

$$\sum_{i=1}^{\infty} \Big(\int_I \frac{|I_i|\,d\sigma(x)}{(|I_i| + |x - c_{I_i}|)^2} \Big)^2 \omega(I_i) \le C\sigma(I), \tag{5.4}$$

required to hold whenever the I_i are disjoint dyadic subintervals of the arbitrary interval $I \subset \mathbb{R}$, as well as the "dual" condition with the roles of σ and ω interchanged.

It was speculated in [40] that (5.4) "might turn out to be a necessary condition" for (5.1). This conjecture was disproved by Lacey, Sawyer and Uriarte-Tuero [29], who exhibited a more general *energy hypothesis*, proved its sufficiency for (5.1), and constructed an elaborate example of measures for which the energy hypothesis holds but the pivotal condition does not. Nevertheless, the pivotal condition not only played an important role in guiding the further development, but it also resurfaced as a useful necessary condition in the characterization of a two-weight inequality for the g-function, by M. Lacey and K. Li [24].

The energy hypothesis involves the following "energy": (A semiphysical motivation for the terminology is provided in [29, Remark 2.16].)

$$E(I,\omega)^2 := \frac{1}{\omega(I)} \int_I \Big(\frac{x - \langle \mathrm{id} \rangle_I^{\omega}}{|I|} \Big)^2 d\omega(x), \qquad \langle \mathrm{id} \rangle_I^{\omega} = \frac{1}{\omega(I)} \int_I x\,d\omega(x), \tag{5.5}$$

which always satisfies $E(I,\omega) \le 1$, but can be much smaller if ω is highly concentrated on I. Now, Lacey et al. [29] observed that a weakening of (5.4), namely

$$\sum_{i=1}^{\infty} \Big(\int_I \frac{|I_i|\,d\sigma(x)}{(|I_i| + |x - c_{I_i}|)^2} \Big)^2 E(I_i,\omega)^2 \omega(I_i) \le C\sigma(I), \tag{5.6}$$

is already implied by the assumptions (5.2) and (2.14) of Theorem 5.1, whereas a *hybrid condition* between (5.4) and (5.6),

$$\sum_{i=1}^{\infty} \Big(\int_I \frac{|I_i|\, d\sigma(x)}{(|I_i| + |x - c_{I_i}|)^2} \Big)^2 E(I_i, \omega)^{2(1-\epsilon)} \omega(I_i) \le C\sigma(I), \tag{5.7}$$

is an admissible side condition for any $\epsilon > 0$. (The actual *energy hypothesis* of [29] is a more technical "optimal" version of this hybrid condition.)

Finally, Lacey, Sawyer, Shen and Uriarte-Tuero [22, 26] were able to prove Theorem 5.1 with only the size condition

$$\omega(\{x\})\sigma(\{x\}) = 0 \qquad \forall x, \tag{5.8}$$

i.e., that the measures ω and σ are not allowed to have common point masses. More precisely, all four authors [26] first reduced the estimate (5.1) to a certain "local" version, and Lacey [22] completed the argument by settling the remaining local bound. Their result may be very legitimately considered a solution of the classical two-*weight* problem, given that (5.8) is manifestly satisfied be measures that are continuous with respect to Lebesgue measure. It also has important consequences, like the following:

5.2. Application to Carleson measures for "model spaces". As will be elaborated below, the results discussed in this section have certain components that are rather specific to the one-dimensional situation. This restriction has been only partially overcome in the following extension of the result of Lacey et al. [22, 26] just mentioned, where one of the two measures is required to be one-dimensional. The theorem is due to M. Lacey, E. Sawyer, C.-Y. Shen, I. Uriarte-Tuero and B. Wick [27]:

Theorem 5.2 ([27]). *Let $(\mathbb{H}, \partial\mathbb{H}) \in \{(\mathbb{R}^2_+, \mathbb{R}), (\mathbb{D}, \mathbb{T})\}$, where $\mathbb{D}$ is the unit disk and $\mathbb{T}$ the unit circle in $\mathbb{C}$. Let σ be a Radon measure on $\partial\mathbb{H}$ and ω on $\bar{\mathbb{H}}$, without common point masses. Then the Hilbert transform satisfies the norm bound* (5.1)*, if and only if analogues of the testing conditions* (5.12) *and the double-Poisson A_2 condition* (2.14) *are valid.*

We refer to [27] for a precise formulation of these "analogues"; let us only mention that the "intervals" on $\partial\mathbb{H}$ will be usual intervals or arcs, and those on $\mathbb{H}$ should be appropriate Carleson regions.

In the same paper [27], Lacey et al. provide the following application of Theorem 5.2; we follow the exposition from [27] and its forerunner by F. Nazarov and A. Volberg [41]: For an inner function $\vartheta \in H^\infty(\mathbb{D})$ (which means that $|\vartheta(z)| = 1$ for a.e. $z \in \mathbb{T}$), the *model space* is defined as

$$K_\vartheta := H^2(\mathbb{D}) \ominus \vartheta H^2(\mathbb{D}).$$

By Beurling's theorem, these spaces describe all invariant subspaces of the adjoint of the shift operator $f \mapsto zf$ in $H^2(\mathbb{D})$.

The connection to the Hilbert transform comes from the unitary equivalence $U : L^2(\sigma) \to K_\vartheta$, where σ is a suitable measure supported on $\mathbb{T} \cap \{\vartheta = 1\}$, and U is concretely given by

$$U : f \in L^2(\sigma) \mapsto Uf = (1 - \vartheta)H(f\, d\sigma) \in K_\vartheta.$$

A measure μ on $\bar{\mathbb{D}}$ is called a *Carleson measure* for K_ϑ if $K_\vartheta \hookrightarrow L^2(\mu)$, i.e. if

$$\|g\|_{L^2(\mu)} \le C\|g\|_{K_\vartheta}. \tag{5.9}$$

Writing $g = Uf = (1-\vartheta)H(f\,d\sigma)$ with $f \in L^2(\sigma)$, (5.9) is equivalent to

$$\|H(f\,d\sigma)\|_{L^2(\omega)} \le C\|f\|_{L^2(\sigma)}, \qquad d\omega = |1-\vartheta|^2\,d\mu, \tag{5.10}$$

where the measures ω on $\bar{\mathbb{D}}$ and σ on $\mathbb{T} \cap \{\vartheta = 1\}$ do not have a common point mass. Thus Theorem 5.2 applies to give a characterization of (5.10), and therefore of (5.9).

5.3. Unconditional characterization. From the point-of-view of the general two-*measure* formulation of the problem, the assumption (5.8) imposed by Lacey et al. [22, 26] is still a side condition; in particular, it fails to contain the classical example of the discrete Hilbert transform (1.3). This is not just a failure of the proof, but of the theorem; namely, the Poisson A_2 condition (2.14), even the classical version (2.11), is *too strong* in general: it already excludes the possibility of common point masses by observing that

$$\omega(\{x\})\sigma(\{x\})^{p-1} \le \lim_{\substack{I\ni x\\ |I|\to 0}} \omega(I)\sigma(I)^{p-1} \le \lim_{\substack{I\ni x\\ |I|\to 0}} [\omega,\sigma]_{A_p}\cdot |I|^p = 0.$$

Indeed, the condition (5.8) hardly appears in the work of Lacey et al. [22, 26]; it is only used to prove the *necessity* of the Poisson A_2 condition (2.14), and the difficult part of the *sufficiency* of (2.14) and (5.12) never makes a reference to (5.8), which is only used via (2.14). So the question of an unconditional characterization of (5.1) was partly a question of finding a *weaker* substitute for the classical A_2 condition. A conjecture was proposed by Lacey [21, Conjecture 1.12] (the conjecture is only found in the first arXiv version of this paper) and proven (although in a somewhat different form) by the present author [18]:

Theorem 5.3 ([18]). *Let σ and ω be two Radon measures on $\mathbb{R}$. Then the following are equivalent:*

(1) *The Hilbert transform satisfies the norm inequality* (5.1)*.*

(2) *It satisfies the testing conditions* (5.2) *and the following weakening of the Poisson A_2 conditions* (2.12)*:*

$$\sup_{I\supset\mathbb{R}} \int_{I^c} \frac{d\omega(x)}{(x-c_I)^2}\cdot\sigma(I) < \infty, \qquad \sup_{I\subset\mathbb{R}} \int_{I^c} \frac{d\sigma(x)}{(x-c_I)^2}\cdot\omega(I) < \infty. \tag{5.11}$$

(3) *It satisfies the global testing conditions*

$$\|H(1_I\,d\sigma)\|_{L^2(\omega)} \le C\sigma(I)^{1/2}, \quad \|H(1_I\,d\omega)\|_{L^2(\sigma)} \le C\omega(I)^{1/2}. \tag{5.12}$$

Although the equivalence of (2) and (3) is much less deep than the equivalence of (1) to these two, it is interesting that the condition (3) seems to have remained unrecorded before even in the absence of point masses. Incidentally, A_2 conditions similar to (5.11) had already appeared in the work of Y. Belov, T. Mengestie and K. Seip [3, Eq. (1.5)] on discrete Hilbert transforms on sparse sequences in the plane. Even in the absence of point masses, this condition may be more feasible to check in concrete cases than the double-Poisson A_2 condition (2.14).

5.4. Outline of proof. In the last part of this section, we give an outline of the proof of the main direction (2) $\Rightarrow$ (1) of Theorem 5.3. Many of its components derive from the proofs of the earlier results discussed in this lecture, and we make some indications of this history.

Step I: The probabilistic reduction. This part of the proof goes back to Nazarov, Treil and Volberg [38]. We consider the bilinear form

$$B(f,g) := \int H(f\,d\sigma)\cdot g\,d\omega = -\int H(g\,d\omega)\cdot f\,d\sigma,$$

and the goal is to prove that $|B(f,g)| \le C\|f\|_{L^2(\sigma)}\|g\|_{L^2(w)}$.

For this purpose, we expand f and g (which may be taken compactly supported with vanishing σ and ω-integrals, respectively) in terms of the *dyadic martingale differences*

$$\Delta_I^\sigma f := \sum_{u\in\{\text{left,right}\}} E_{I_u}^\sigma f - E_I^\sigma f, \qquad E_I^\sigma f := 1_I\langle f\rangle_I^\sigma := \frac{1_I}{\sigma(I)}\int_I f\,d\sigma.$$

as

$$f = \sum_{I\in\mathcal{D}} \Delta_I^\sigma f, \tag{5.13}$$

and similarly with g in terms of Δ_J^ω. As in the Section 4, $\mathcal{D}$ is going to be a random shift of the standard dyadic intervals.

An interval $I\in\mathcal{D}$ is called *bad* if it is relatively close to the boundary of a much bigger interval, more precisely, if $\operatorname{dist}(I,\partial J) < |I|^\gamma |J|^{1-\gamma}$ for some $J\in\mathcal{D}$ with $|J| > 2^r|I|$, where $\gamma\in(0,1)$ (small) and $r\in\mathbb{Z}_+$ (large) are auxiliary numbers. For a random choice of the dyadic system, we have the probability bound $\mathbb{P}_\mathcal{D}(I \text{ is bad}) \le c_\gamma 2^{-r\gamma} =: \epsilon$, which can be taken as small as we like by adjusting the parameter r. Using independence and orthogonality, it then follows that

$$\mathbb{E}\|f_{\text{bad}}\|_{L^2(\sigma)}^2 := \mathbb{E}\Big\|\sum_{\substack{I\in\mathcal{D}\\ I \text{ bad}}}\Delta_I^\sigma f\Big\|_{L^2(\sigma)}^2 = \mathbb{E}\sum_{\substack{I\in\mathcal{D}\\ I \text{ bad}}}\|\Delta_I^\sigma f\|_{L^2(\sigma)}^2 \le \epsilon\|f\|_{L^2(\sigma)}^2.$$

By a standard absorption argument ("$\|H\| \le C + \epsilon\|H\|$"), we find that it is enough to estimate $B(f_{\text{good}}, g_{\text{good}})$, where f_{good} is defined by summing only over the good intervals in (5.13). We henceforth assume without loss of generality that $f = f_{\text{good}}$ and $g = g_{\text{good}}$.

Step II: Non-homogeneous $T(1)$ argument. This is still a straightforward adaptation of the work of Nazarov et al. [38] on the case that $\omega = \sigma = \mu$. We need to redo the argument, since the new A_2 condition (5.11) replaces the conditions on measures used in earlier contributions, but it turns out that this does not present any major obstacles.

In this step, in the expansion

$$B(f,g) = \sum_{I,J\in\mathcal{D}} B(\Delta_I^\sigma f, \Delta_J^\omega g)$$

we estimate all terms with $I\cap J = \varnothing$, or $2^{-r} \le |I|/|J| \le 2^r$.

There remains the case that one of the intervals is deeply contained in the other: $I \Subset J$, by which we mean $I\subset J$ and $|I| < 2^{-r}|J|$, or $J\Subset I$. By symmetry, we deal with the first case only. Here observe that $\Delta_J^w g$ is piecewise constant on the two halves of J, and we denote the one that contains I by J_I. The part $1_{J\setminus J_I}\Delta_J^w g$ can still be estimated in the same way as the disjoint terms, so that altogether this step reduces the consideration to

$$B_{\text{below}}(f,g) := \sum_J \sum_{I\Subset J} B(\Delta_I^\sigma f, 1_{J_I})\langle\Delta_J^\omega g\rangle_{J_I}^\omega. \tag{5.14}$$

In more classical $T(1)$ contexts, this term would lead to a paraproduct, and we would already be a good way through the proof. In the two-weight situation at hand, it is only here that the difficulties begin, and the new innovations of Nazarov, Treil and Volberg [40, 55] and Lacey, Sawyer and Uriarte-Tuero [29] are needed.

Step III: Organization under stopping intervals. Without loss of generality, all dyadic intervals entering the decomposition (5.13), and its analogue for g, are contained in a maximal $I_0 \in \mathcal{D}$. We choose a collection $\mathcal{S}$ of *stopping intervals* inductively so that $I_0 \in \mathcal{S}$ and then, if S is minimal among the intervals in $\mathcal{S}$ already chosen, we also take into $\mathcal{S}$ all the maximal subintervals $S' \subset S$ that satisfy at least one of the following:

1. $\dfrac{1}{\omega(S')}\displaystyle\int_{S'} |g|\,d\omega > 4\frac{1}{\omega(S)}\int_S |g|\,d\omega$, or
2. $\dfrac{1}{\omega(S')}\displaystyle\int_{S'} |H(1_S\,d\omega)|^2\,d\sigma > 4\frac{1}{\omega(S)}\int_S |H(1_S\,d\omega)|^2\,d\sigma$

Here (1) is a typical stopping condition related to the construction of what is often called the *principal cubes*. For $\omega = \sigma$, (2) is not untypical of proofs of so-called *local $T(b)$ theorems*. Lacey et al. [26] used variants of (2) which referred to the Hilbert transform more indirectly via the energy condition (5.6).

The stopping intervals satisfy the Carleson condition:

$$\sum_{\substack{S\in\mathcal{S}\\ S\subseteq Q}} w(S) \le 2w(Q), \quad \text{thus} \quad \sum_{S\in\mathcal{S}} w(S)\cdot|\langle g\rangle_S^w|^2 \le C\|g\|^2_{L^2(w)}.$$

Indeed, ensuring that some suitable stopping intervals have this property was the key application of the side conditions in the forerunner Theorem 5.1 (see e.g. [55, Theorem 20.1]).

Once the stopping intervals are chosen, we organize all other intervals "under" them through the following definitions:

$$\pi I := \min\{S\in\mathcal{S} : S \supsetneq I\}, \quad \tilde\pi I := \min\{S\in\mathcal{S} : S \ni I\},$$

$$P_S^\sigma f := \sum_{I:\pi I=S} \Delta_I^\sigma f, \quad \tilde P_S^\sigma f := \sum_{I:\tilde\pi I=S} \Delta_I^\sigma f$$

We then reorganize the term $B_{\mathrm{below}}(f,g)$ from (5.14) as

$$B_{\mathrm{below}}(f,g) = \sum_J \sum_{I\Subset J} \cdots = \sum_{\substack{S,S'\in\mathcal{S}\\ S\subseteq S'}} \sum_{J:\pi J=S'} \sum_{\substack{I:\tilde\pi I=S\\ I\Subset J}} \cdots,$$

where we observed that the inner sum is empty unless $S \subseteq S'$. The part with $S' = S$ gives $\sum_{S\in\mathcal{S}} B_{\mathrm{below}}(\tilde P_S^\sigma f, P_S^\omega g)$ which, after estimating some "error terms", can be reduced to the *local form*

$$B_{\mathrm{local}}(f,g) := \sum_{S\in\mathcal{S}} B_{\mathrm{below}}(P_S^\sigma f, P_S^w g). \tag{5.15}$$

where both f and g are localized in the same part of the stopping tree.

On the other hand, the part with $S \subsetneq S'$ gives the *tail form*

$$B_{\mathrm{tail}}(f,g) := \sum_{S\in\mathcal{S}} B\Big(\tilde P_S^\sigma f, \sum_{\substack{S'\in\mathcal{S}\\ S'\supsetneq S}} \sum_{J:\pi J=S'} 1_{J_I}\langle\Delta_J^\omega g\rangle_{J_I}^\omega\Big) = \sum_{S\in\mathcal{S}} B(\tilde P_S^\sigma f, \Phi_S^\omega g), \tag{5.16}$$

where (using a basic telescoping identity for the martingale differences)

$$\Phi_S^\omega g = \sum_{J \supsetneq S} 1_{J_I} \langle \Delta_J^\omega g \rangle_{J_I}^\omega = \sum_{J \supsetneq S} 1_{J \setminus J_S} \langle g \rangle_J^\omega$$

satisfies $|\Phi_S^\omega g| \le 1_{S^c} M_d^\omega g$, and $M_d^\omega g := \sup_{J \in \mathcal{D}} 1_J \langle |g| \rangle_J^w$ is the dyadic maximal operator, bounded on $L^2(\omega)$.

Step IV: The tail form. This involves the bilinear form on $\tilde{P}_S^\sigma f$, whose martingale differences live "deep inside" S, and $\Phi_S^\omega g$, which is supported in S^c. It is for such situations that the fundamental "monotonicity lemma" of Lacey et al. was designed:

Lemma 5.4 ([26]). *Let ϕ be supported on $J \in \mathcal{D}$ with $\int \phi \, d\sigma = 0$, and ψ on $(3J)^c$. For a certain dyadic martingale transform $\tilde{\phi}$ of ϕ, we have the bounds*

$$|B(\phi, \psi)| \le B(\tilde{\phi}, |\psi|) \eqsim \int x \tilde{\phi}(x) \, d\sigma(x) \int_{(3J)^c} \frac{|\psi(x)| d\omega(x)}{(x - c_J)^2}. \tag{5.17}$$

This lemma is used for the estimation of the tails forms as follows: First, we split $\tilde{P}_S^\sigma f$ into parts supported on appropriate Whitney subintervals $K \subset 3K \subset S$, to ensure that the support condition of Lemma 5.4 is satisfied by $\phi = 1_K \tilde{P}_S^\sigma f$ and $\psi = \Phi_S^\omega g$ with $J = K$. After the first domination in (5.17), we can then freely estimate back and forth between the two rightmost quantities in (5.17). The last form has two important features: the functions ϕ and ψ are decoupled, and the action on $|\psi|$ is a *positive operator*; indeed, up to scaling, the Poisson integral with a "hole" in $3J$. We note that the first factor on the right of (5.17), which for $\tilde{\phi} = \tilde{P}_S^\sigma f$ becomes ("id" is the identity function $\mathrm{id}(x) = x$)

$$\int \mathrm{id} \cdot \tilde{P}_S^\sigma f \, d\sigma \le \|\tilde{P}_S^\sigma \, \mathrm{id} \, \|_{L^2(\sigma)} \|\tilde{P}_S^\sigma f\|_{L^2(\sigma)} \le \|1_S (\mathrm{id} - \langle \mathrm{id} \rangle_S^\sigma)\|_{L^2(\sigma)} \|\tilde{P}_S^\sigma f\|_{L^2(\sigma)},$$

is responsible for the emergence of the "energy" (5.5) in these considerations.

The strategy of Lacey et al. [26] to the tail form was to estimate the right side of (5.17) up by the full Poisson integral of $M_d^w g$, and to observe that the required bound $|B_{\mathrm{tail}}(f, g)| \le C \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)}$ then follows from an appropriate two-weight inequality for the Poisson integral. Now, an interval-testing characterization for this was already established by Sawyer [50]. However, one still has to check that the resulting Poisson testing conditions actually hold in the case under consideration, whereas the assumptions (5.2) "test" the Hilbert transform, not the Poisson integral. It is here that the reverse estimate in (5.17) is needed: in the testing conditions, the Poisson integral is again dominated by the Hilbert transform so that the assumptions (5.2) may be exploited. It is this two-sided domination of a singular integral by a positive operator, and vice versa, that seems to be the feature of the entire proof that is most specific to the one-dimensional Hilbert transform, and most difficult to extend to other operators.

In estimating the Poisson testing conditions, not only the Hilbert testing (5.2) but also the double-Poisson A_2 condition (2.14) was needed in [26]. Thus this argument cannot be simply repeated in the setting of Theorem 5.3, and it is necessary to proceed more carefully, without forgetting the hole that is available in (5.17). For this purpose, the key technical novelty of [18] was an interval-testing characterization of a (dyadic model of) the two-weight boundedness of the "holed" Poisson integral. Since the operator is smaller than the full Poisson integral, also the characterizing conditions are weaker, and can be verified (after an application of (5.17)) from the assumptions (5.2) and the new A_2 condition (5.11).

Step V: The local form. After easy reductions, the local form boils down to estimating

$$B_{\mathcal{Q}}(f,g)=\sum_{(I,J)\in\mathcal{Q}} B(\Delta_I^\sigma f,1_{S\setminus J})\langle\Delta_{J^{(1)}}^w g\rangle_J^w$$

for a particular collection $\mathcal{Q}$ of pairs of intervals (I,J) such that $I\Subset J$. In the case of (5.8), this was the last remaining step missing from the work of Lacey et al. [26], which was only settled in a sequel paper by Lacey alone [22].

His proof was based on an iteration scheme inspired by the proof of Carleson's theorem on Fourier series: Rather than attempting a proof for the single particular $\mathcal{Q}$ of ultimate interest, Lacey defines a class of *admissible* collections $\mathcal{Q}$ and equips them with an auxiliary quantity $\operatorname{size}(\mathcal{Q})$ with the following properties:

- Every admissible $\mathcal{Q}$ satisfies

$$\|B_{\mathcal{Q}}\|:=\sup\frac{|B_{\mathcal{Q}}(f,g)|}{\|f\|_{L^2(\sigma)}\|g\|_{L^2(w)}}\le C_\epsilon\operatorname{size}(\mathcal{Q})+c\sup\|B_{\mathcal{Q}'}\|,$$

  where the supremum is over all admissible $\mathcal{Q}'\subset\mathcal{Q}$ with $\operatorname{size}(\mathcal{Q}')\le\epsilon\cdot\operatorname{size}(\mathcal{Q})$. By absorption, this implies that

$$\|B_{\mathcal{Q}}\|\le\frac{C_\epsilon}{1-c\epsilon}\operatorname{size}(\mathcal{Q}).$$

- For all admissible collections, $\operatorname{size}(\mathcal{Q})$ is estimated by the constants in (5.2).

The final outcome is as desired, but this seems only approachable via the auxiliary quantity $\operatorname{size}(\mathcal{Q})$ on which one can iterate; figuring out a suitable notion of this "size" is a key insight of Lacey's argument.

As for Theorem 5.3, it suffices to inspect Lacey's proof [22] and observe that the assumption (5.8), either directly or via (2.14), makes no appearance in this reasoning, so that it can be simply borrowed as a black box. And this is the last step to complete the proof of Theorem 5.3.

6. Some open problems

The following list is necessarily highly incomplete.

6.1. The A_2 theorem for rough kernels. The A_2 Theorem 4.4 is now known for "all Calderón–Zygmund operators". The most common definition of these operators involves kernels with Hölder regularity of some exponent $\alpha\in(0,1]$, and this was the case already covered in [17]. Subsequent proofs have extended the class of admissible kernels , so that it is now known (see [19]) to be enough to have a modulus of continuity with a logarithmic strengthening of the classical Dini condition,

$$\int_0^1\omega(t)\log\frac{1}{t}\frac{dt}{t}<\infty. \tag{6.1}$$

However, whether this condition is optimal, is open.

More generally, for any class of operators T for which the qualitative estimate

$$\|Tf\|_{L^2(w)} \le c_w \|f\|_{L^2(w)} \qquad \forall w \in A_2 \tag{6.2}$$

is known, one may wonder about the optimal form of the quantitative bound $c_w \le c\phi([w]_{A_2})$. A case of particular interest consists of the *rough homogeneous singular integrals*

$$T_\Omega f(x) = \text{p. v.} \int_{\mathbb{R}^d} \frac{\Omega(y)}{|y|^d} f(x-y)\, dy, \tag{6.3}$$

where

$$\Omega(\lambda y) = \Omega(y) \quad \forall \lambda > 0, \qquad \Omega \in L^\infty(\mathbb{R}^d), \qquad \int_{\mathbb{S}^{d-1}} \Omega(u)\, d\sigma(u) = 0,$$

so that the kernel of T has pointwise bounds and cancellation, but no regularity. The qualitative bound (6.2) for these operators is due to X. Duoandikoexea [12]. They also map $T_\Omega : L^1 \to L^{1,\infty}$ (see S. Hofmann [14]), which seems to be an essential feature of the methods so far used in the proofs of the A_2 theorem. So it is reasonable to ask: Do these operators satisfy the A_2 Theorem 4.4; is it true that

$$\|T_\Omega f\|_{L^2(w)} \le c\|\Omega\|_\infty \cdot [w]_{A_2} \cdot \|f\|_{L^2(w)} \ ? \tag{6.4}$$

If not, what is the optimal bound of the form $c_w \le c\phi([w]_{A_2})$ in (6.2)?

6.2. Powers of the Beurling–Ahlfors operator. As we have explained above, it was around the Beurling–Ahlfors operator that the A_2 conjecture first caught fire. While this is now a theorem, and a much more general one than originally conjectured, there are still interesting questions around the very point of origin. The integer powers B^m of B are singular integrals of the form (6.3) (see [11, p. 493]) with

$$\Omega_m(e^{i\phi}) := \Omega_{B^m}(e^{i\phi}) = \frac{(-1)^m}{\pi} \cdot m \cdot e^{-i2m\phi}.$$

Of course, this kernel is smooth, but *using* the smoothness in an estimate produces an unfavourable dependence on m, due to its presence in the exponent. The modulus of continuity ω of Ω_m satisfies $\omega(t) \le cm \min(1, mt)$, for which the integral (6.1) is $cm(1+\log m)^2$. Via the existing A_2 theorems, this gives the bound

$$\|B^m f\|_{L^2(w)} \le c \cdot m \cdot (1+\log m)^2 \cdot [w]_{A_2} \cdot \|f\|_{L^2(w)}.$$

It would be interesting to eliminate the logarithmic factor, giving a bound simultaneously linear in both m and $[w]_{A_2}$. This would be a consequence of a positive answer to (6.4), or it could possibly be based on an analysis specific to the Beurling–Ahlfors operator, just like the original paper [46]. Note that it follows from extrapolation and the sharp *unweighted* L^p bounds for B^m, due to O. Dragičević, S. Petermichl and A. Volberg [11, Theorem 3], that $\|B^m f\|_{L^2(w)} \le cm[w]_{A_2}\|f\|_{L^2(w)}$ would be optimal in m at least on the scale of power functions: m cannot be replaced by $m^{1-\epsilon}$ for any $\epsilon > 0$.

6.3. Unconditional characterizations for two-weight inequalities beyond Hilbert transform in $d = 1, p = 2$. Although substantial parts of the proof of the two-weight Theorems 5.2 and 5.3 use general techniques that immediately extend to other Calderón–Zygmund operators, the central Monotonicity Lemma 5.4, at the present state of knowledge, does not have fully satisfactory counterparts in more complicated situations, and limits the existing characterizations for other operators to conditional versions under additional assumptions; see [23, 51].

Accordingly, it is an open problem to obtain unconditional testing characterizations for the two-weight inequalities of any of the following:

- The Hilbert transform for two measures with arbitrary supports in $\mathbb{C}$; in particular, neither of them required to be one-dimensional.
- Any other distinguished special operator like the Beurling–Ahlfors transform.
- General Calderón–Zygmund operators.

In another direction, the existing two-weight characterizations for singular operators are very much restricted to the L^2 setting; characterizations of the L^p inequality (1.8) for $p \neq 2$ are completely open even for the one-dimensional Hilbert transform.

In the unweighted case, simple scaling arguments show that the Hilbert transform can never map one L^p space to a different L^q, but this reasoning breaks down in the two-weight world, and there is no *a priori* reason to exclude the possibility of even more general bounds

$$\|H(f\,d\sigma)\|_{L^q(\omega)} \le C\|f\|_{L^p(\sigma)}.$$

It seems likely that if one is able to develop any two-weight theory for $p \neq 2$ at all, moving to $p \neq q$ should be an order of magnitude easier.

Acknowledgements. The author is supported by the European Union through the ERC Starting Grant "Analytic–probabilistic methods for borderline singular integrals.

References

[1] K. Astala, T. Iwaniec, and E. Saksman, *Beltrami operators in the plane*, Duke Math. J., **107**(1):27–56, 2001.

[2] K. Astala, *Area distortion of quasiconformal mappings*, Acta Math., **173**(1):37–60, 1994.

[3] Y. Belov, T. Y. Mengestie, and K. Seip, *Discrete Hilbert transforms on sparse sequences*, Proc. Lond. Math. Soc. (3), **103**(1):73–105, 2011.

[4] S. M. Buckley, *Estimates for operator norms on weighted spaces and reverse Jensen inequalities*, Trans. Amer. Math. Soc., **340**(1):253–272, 1993.

[5] A.-P. Calderón, *Cauchy integrals on Lipschitz curves and related operators*, Proc. Nat. Acad. Sci. U.S.A., **74**(4):1324–1327, 1977.

[6] R. R. Coifman, A. McIntosh, and Y. Meyer, *L'intégrale de Cauchy définit un opérateur borné sur L^2 pour les courbes lipschitziennes*, Ann. of Math. (2), **116**(2):361–387, 1982.

[7] D. Cruz-Uribe, *The invertibility of the product of unbounded Toeplitz operators*, Integral Equations Operator Theory, **20**(2):231–237, 1994.

[8] D. Cruz-Uribe, J. M. Martell, and C. Pérez, *Sharp weighted estimates for approximating dyadic operators*, Electron. Res. Announc. Math. Sci., **17**:12–19, 2010.

[9] G. David, *Wavelets and singular integrals on curves and surfaces*, volume **1465** of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1991.

[10] O. Dragičević, L. Grafakos, M. C. Pereyra, and S. Petermichl, *Extrapolation and sharp norm estimates for classical operators on weighted Lebesgue spaces*, Publ. Mat., **49**(1):73–91, 2005.

[11] O. Dragičević, S. Petermichl, and A. Volberg, *A rotation method which gives linear L^p estimates for powers of the Ahlfors-Beurling operator*, J. Math. Pures Appl. (9), **86**(6):492–509, 2006.

[12] J. Duoandikoetxea, *Weighted norm inequalities for homogeneous singular integrals*, Trans. Amer. Math. Soc., **336**(2):869–880, 1993.

[13] D. Hilbert, *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen. (Dritte Mitteilung)*, Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl., **1905**:307–338, 1905.

[14] S. Hofmann, *Weak $(1,1)$ boundedness of singular integrals with nonsmooth kernel*, Proc. Amer. Math. Soc., **103**(1):260–264, 1988.

[15] R. Hunt, B. Muckenhoupt, and R. Wheeden, *Weighted norm inequalities for the conjugate function and Hilbert transform*, Trans. Amer. Math. Soc., **176**:227–251, 1973.

[16] T. P. Hytönen, *The A_2 theorem: remarks and complements*, Preprint, arXiv:1212.3840, 2012.

[17] ———, *The sharp weighted bound for general Calderón-Zygmund operators*, Ann. of Math. (2), **175**(3):1473–1506, 2012.

[18] ———, *The two-weight inequality for the Hilbert transform with general measures*, Preprint, arXiv:1312.0843, 2013.

[19] T. P. Hytönen, M. T. Lacey, and Carlos Pérez, *Sharp weighted bounds for the q-variation of singular integrals*, Bull. Lond. Math. Soc., **45**(3):529–540, 2013.

[20] M. T. Lacey, *On the separated bumps conjecture for Calderón–Zygmund operators*, Preprint, arXiv:1310.3507, 2013.

[21] ———, *The two weight inequality for the Hilbert transform: A primer*, Preprint, arXiv:1304.5004 (version 1), 2013.

[22] ———, *Two weight inequality for the Hilbert transform: A real variable characterization, II*, Preprint, arXiv:1301.4663, 2013.

[23] M. T. Lacey and Wick B. D, *Two weight inequalities for Riesz transforms: Uniformly full dimension weights*, Preprint, arXiv:1312.6163, 2013.

[24] M. T. Lacey and K. Li, Two weight norm inequalities for the g function. Preprint, arXiv:1309.5839, 2013.

[25] M. T. Lacey, S. Petermichl, and M. C. Reguera, *Sharp A_2 inequality for Haar shift operators*, Math. Ann., **348**(1):127–141, 2010.

[26] M. T. Lacey, E. T. Sawyer, C.-Y. Shen, and I. Uriarte-Tuero, *Two weight inequality for the Hilbert transform: A real variable characterization, I*, Preprint, arXiv:1201.4319, 2012.

[27] M. T. Lacey, E. T. Sawyer, C.-Y. Shen, I. Uriarte-Tuero, and Wick B. D., *Two weight inequalities for the Cauchy transform from $\mathbb{R}$ to $\mathbb{C}_+$*, Preprint, arXiv:1310.4820, 2013.

[28] M. T. Lacey, E. T. Sawyer, and I. Uriarte-Tuero, *Astala's conjecture on distortion of Hausdorff measures under quasiconformal maps in the plane*, Acta Math., **204**(2):273–292, 2010.

[29] _____, *A two weight inequality for the Hilbert transform assuming an energy hypothesis*, J. Funct. Anal., **263**(2):305–363, 2012.

[30] A. K. Lerner, *On an estimate of Calderón-Zygmund operators by dyadic positive operators*, J. Anal. Math., **121**:141–161, 2013.

[31] H. Martikainen, *Representation of bi-parameter singular integrals by dyadic operators*, Adv. Math., **229**(3):1734–1761, 2012.

[32] P. Mattila, M. S. Melnikov, and J. Verdera, *The Cauchy integral, analytic capacity, and uniform rectifiability*, Ann. of Math. (2), **144**(1):127–136, 1996.

[33] M. S. Melnikov and J. Verdera, *A geometric proof of the L^2 boundedness of the Cauchy integral on Lipschitz graphs*, Internat. Math. Res. Notices, (**7**):325–331, 1995.

[34] B. Muckenhoupt, *Weighted norm inequalities for classical operators*, In Harmonic analysis in Euclidean spaces (Proc. Sympos. Pure Math., Williams Coll., Williamstown, Mass., 1978), Part 1, Proc. Sympos. Pure Math., XXXV, Part, pages 69–83. Amer. Math. Soc., Providence, R.I., 1979.

[35] B. Muckenhoupt and R. L. Wheeden, *Two weight function norm inequalities for the Hardy-Littlewood maximal function and the Hilbert transform*, Studia Math., **55**(3):279–294, 1976.

[36] F. Nazarov, *A counterexample to Sarason's conjecture*, Unpublished manuscript, http://www.math.msu.edu/~fedja/prepr.html, 1997.

[37] F. Nazarov, A. Reznikov, and A. Volberg, *Bellman approach to the one-sided bumping for weighted estimates of Calderón–Zygmund operators*, Preprint, arXiv:1306.2653, 2013.

[38] F. Nazarov, S. Treil, and A. Volberg, *Cauchy integral and Calderón-Zygmund operators on nonhomogeneous spaces*, Internat. Math. Res. Notices, (**15**):703–726, 1997.

[39] _____, *The Tb-theorem on non-homogeneous spaces*, Acta Math., **190**(2):151–239, 2003.

[40] _____, *Two weight estimate for the Hilbert transform and corona decomposition for non-doubling measures*, Unpublished manuscript, arXiv:1003.1596, 2005/2010.

[41] F. Nazarov and A. Volberg. *The Bellman function, the two-weight Hilbert transform, and embeddings of the model spaces K_θ*, J. Anal. Math., **87**:385–414, 2002, Dedicated to the memory of Thomas H. Wolff.

[42] C. J. Neugebauer, *Inserting A_p-weights*, Proc. Amer. Math. Soc., **87**(4):644–648, 1983.

[43] T. Orponen, *On the Haar shift representations of Calderón-Zygmund operators*, Proc. Amer. Math. Soc., **141**(8):2693–2698, 2013.

[44] S. Petermichl, *Dyadic shifts and a logarithmic estimate for Hankel operators with matrix symbol*, C. R. Acad. Sci. Paris Sér. I Math., **330**(6):455–460, 2000.

[45] _____, *The sharp bound for the Hilbert transform on weighted Lebesgue spaces in terms of the classical A_p characteristic*, Amer. J. Math., **129**(5):1355–1375, 2007.

[46] S. Petermichl and A. Volberg. *Heating of the Ahlfors-Beurling operator: weakly quasiregular maps on the plane are quasiregular*, Duke Math. J., **112**(2):281–305, 2002.

[47] M. Riesz, *Sur les fonctions conjuguées*, Math. Z., **27**(1):218–244, 1928.

[48] J. L. Rubio de Francia, *Factorization theory and A_p weights*, Amer. J. Math., **106**(3):533–547, 1984.

[49] E. T. Sawyer, *A characterization of a two-weight norm inequality for maximal operators*, Studia Math., **75**(1):1–11, 1982.

[50] _____, *A characterization of two weight norm inequalities for fractional and Poisson integrals*, Trans. Amer. Math. Soc., **308**(2):533–545, 1988.

[51] E. T. Sawyer, S.-Y. Shen, and I. Uriarte-Tuero, *A two weight theorem for fractional singular integrals with an energy side condition*, Preprint, arXiv:1302.5093, 2013.

[52] X. Tolsa. *L^2-boundedness of the Cauchy integral operator for continuous measures*, Duke Math. J., **98**(2):269–304, 1999.

[53] _____, *A $T(1)$ theorem for non-doubling measures with atoms*, Proc. London Math. Soc. (3), **82**(1):195–228, 2001.

[54] _____, *Analytic capacity, rectifiability, and the Cauchy integral*, In International Congress of Mathematicians. Vol. II, pp. 1505–1527. Eur. Math. Soc., Zürich, 2006.

[55] A. Volberg. *Calderón-Zygmund capacities and operators on nonhomogeneous spaces*, volume 100 of CBMS Regional Conference Series in Mathematics, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2003.

Department of Mathematics and Statistics, P.O. Box 68 (Gustaf Hällströmin katu 2b),
FI-00014 University of Helsinki, Finland

E-mail: tuomas.hytonen@helsinki.fi

The flecnode polynomial: a central object in incidence geometry

Nets Hawk Katz

Abstract. Recently there has been a lot of progress in point/line incidence theory in three dimension real affine spaces. Generally, this progress all happens where the lines and points lie in a ruled surface. Conversely, in many related problems we are yet unable to touch, we are just outside the threshold where the ruled condition can be enforced. This puts the celebrated flecnode polynomial of Cayley and Salmon at the center of the action.

Mathematics Subject Classification (2010). 14M99, 52C10.

Keywords. Incidence theory, flecnodes, ruled surfaces.

1. Introduction

Incidence geometry as we shall understand it in this lecture is the study of configuration of points and lines in real Euclidean space, largely with a view towards bounding the number of incidences, that is pairs of points and lines where the point lies on the line. The study of incidence geometry has a long history. One of its high points is the Szemerédi-Trotter theorem [10]:

Theorem 1.1. *A set of m distinct lines and n distinct points in the Euclidean plane has at most*

$$O(n^{\frac{2}{3}}m^{\frac{2}{3}} + n + m)$$

incidences.

One thing that is remarkable about this result, published in 1983, is that except for constants, it is entirely sharp. More delicate incidence questions, for instance those involving incidences between configurations of points and lines which were in some way forced to be higher dimensional, for a long time eluded sharp treatments. The subject has undergone a revolution recently, however, in which sharp results for many problems became possible. The revolution was started by the following result of Ze'ev Dvir in 2008. [1] (Strictly speaking, in this lecture we view Dvir's result as outside of incidence theory because it is over finite fields.)

Theorem 1.2. *Let q be a power of a prime and F_q be the finite field of q elements. Let $E \subset F_q^n$ be a set of points containing a line in every direction. Then*

$$|E| \gtrsim_n q^n.$$

Proceedings of the International Congress of Mathematicians, Seoul, 2014

Here, absolute values denote the cardinality of sets of points, and the notation $\gtrsim_n$ means that we may be concealing a constant depending on n, but certainly not on q. Dvir's proof was obtained by studying a polynomial vanishing on the set of points. It easily solved in the affirmative a conjecture of Tom Wolff's that had been considered quite hard. (It was the finite field Kakeya problem and was intended as an analog for the real Kakeya problem which arises in geometric measure theory and harmonic analysis.) Previously the behavior of polynomials had not been used much in incidence geometry and a number of breakthroughs occurred once it was realized that this is possible.

A lot was accomplished in two papers by myself and Larry Guth. [6, 7] We settled the Joints conjecture in $\mathbf{R}^3$. Three lines are said to form a joint at a point p if the lines are all incident to p and not coplanar.

Theorem 1.3 (Joints problem (Guth-K.)). *A set of N lines in $\mathbf{R}^3$ forms at most $O(N^{\frac{3}{2}})$ distinct joints.*

We settled a conjecture of Bourgain intended to serve as an analog in incidence geometry for the Kakeya problem.

Theorem 1.4 (Bourgain problem (Guth-K.)). *Let E be a set of points in $\mathbf{R}^3$ and let L be a set of N^2 lines so that no more than N lines of L lie in any common plane and so that each line of L is incident to at least N points of E. Then*

$$|E| \gtrsim N^3.$$

We obtained a near-solution to a famous problem of Erdös on distinct distances between points in the plane.

Theorem 1.5 (Erdős distance result (Guth-K.)). *Let E be a set of N points in $\mathbf{R}^2$. Let D be the set of distances between any two of the points of E then*

$$|D| \gtrsim \frac{N}{\log N}.$$

We make some remarks on what Theorems 1.3, 1.4, and 1.5 have in common. Each of theorems 1.3 and 1.4 is clearly a result about Euclidean lines in three dimensions. A completely general set of lines in three dimensions might lie in a plane, in which case, no incidence result better than Theorem 1.1 is possible. That is why each of Theorems 1.3 and 1.4 contain hypotheses preventing too many lines from lying in a plane. In the case of Theorem 1.3, this hypothesis is that each triple of lines forming a joint is noncoplanar. In the case of Theorem 1.4, things are more explicit. No more than N lines are allowed to lie in a plane. Theorem 1.5 appears to be different. It doesn't mention lines in three dimensions at all. But the proof is obtained through what's now called the Elekes-Sharir framework. (Blame me. I named it thus because I learned about it from a particular paper of Elekes and Sharir. [2]) This framework is a kind of realization of the Erlangen program. Instead of studying Theorem 1.5 in the Euclidean plane where it is stated, we study in the groups of rigid motions which is three dimensional. In fact, it contains Zariski dense open sets which can be viewed as $\mathbf{R}^3$. It turns out that proving Theorem 1.5 precisely amounts to solving an incidence problem between points and lines in $\mathbf{R}^3$ in which the lines are restricted from being too much in a two-dimensional set.

Theorems 1.3 and 1.4 were discovered before Theorem 1.5 and may each be viewed as special cases of the incidence result underlying Theorem 1.5. Their proofs provided essential

clues for discovering the proof of Theorem 1.5. It is hard to imagine the bigger result coming first. In the last few years, incidence geometry has become much more crowded and many people are working on and refining the new polynomial methods. The proof of Theorem 1.3 has been so simplified that one can obtain it without even using Bezout's lemma. But in some sense, these simplifications are merely cosmetic and serve to obscure the unity of the three theorems. The proof of Theorem 1.5 has essentially two parts. One part is topological. Roughly it serves to show that if the set of points we are investigating does not lie in the zero set of too low degree a polynomial, one gets a kind of three dimensional improvement of Theorem 1.1. While our paper was the first to really do this so that three dimensionality is expressed in terms of polynomials, and this method is now referred to as polynomial partitioning, it is very much in the spirit of the pre-existing theory of incidence geometry and of the decomposition method which provided the original proof of Theorem 1.1. The second part of the proof is about what to do when the points in question are in the zero set of a fairly low degree polynomial. Both Theorems 1.3 and 1.4 are special cases of this part. They are sufficiently easy special cases that they can be resolved quite simply. But what all results in point line incidence theory in $\mathbf{R}^3$ which we can do because the points lie in the zero set of a low degree polynomial have in common is this: if there are too many incidences, then most of the offending lines lie in an algebraic ruled surface (of fairly low degree). It is this commonality which I hope to emphasize in the current lecture.

A number of criticisms can be levelled against the current lecture. The most serious is that it doesn't contain any actual proofs. However, it does contain sketches which can be turned into actual proofs, at least by me, and which express the way I thought about the results with Guth when I was working with him on them. A number of standard uniformity arguments are entirely sloughed over and the algebraic arguments contain slightly excessive assumptions of genericity which have to be justified. Nonetheless, for that perfect reader who catches the zeitgeist, this is supposed to provide a short sweet introduction to the subject, emphasizing major ideas and removing annoying details. A few open problems are mentioned where they're related to the subject of the lecture. I can't claim any originality in posing them. A lot of ideas about the frontier were in the air at an IPAM program in Spring 2014. I thank profusely any participant I may have inadvertently stolen from.

2. The Cayley Salmon theorem

In this section, we prove the main result which allows the theory of ruled surfaces to enter incidence geometry. This is the theorem of Cayley and Salmon which says that any algebraic surface in $\mathbf{C}^3$ contain enough lines must have a ruled component. More precisely it says:

Theorem 2.1 (Cayley-Salmon theorem). *Let $p(x, y, z)$ be a polynomial of degree d on $\mathbf{C}^3$. Then there is a polynomial $Flec(p)(x, y, z)$ of degree no more than $11d-24$ which vanishes at a point $w = (x, y, z)$ of the zero set of p only if there is a line containing w so that p restricted to the line vanishes to third order at w. If $Flec(p)$ vanishes at all points of the zero set of p then the zero set is ruled. (That is, through each point of the zero set, there is a line contained in the zero set.)*

Recently, there has been a lot of confusion about the Cayley Salmon theorem. As often happens when people are confused, Terry Tao in the goodness of his heart, posted an elementary proof of the theorem on his blog, to much acclaim [11]. One thing a bit odd

about this is that Salmon also posted an elementary proof. This happened in 1862 [9], so he put his proof in an analytic geometry textbook. This book is now past the expiration of its copyright, but unfortunately amazon still charges around twenty bucks, to produce a copy by print-on-demand. This seems off-putting to people. The Tao blog admits that the theorem dates to at least 1915. (Probably this date comes from the most common reprint of the 5th edition.) One of the motives for this lecture is to defend Salmon's honor and explain his original proof.

There are a number of reasons why Salmon's proof is difficult to parse for modern readers. One is that Salmon's notation is not so good as Tao's. Another is that Salmon was not restricting his attention to ruled surfaces. He was interested in having similar statements for surfaces ruled by other classes of curves than arbitrary lines. The idea was that he assumed his surface had the desired property. Then he derived a differential equation which the surface had to satisfy. Then he observed that this differential equation had first integrals and that these imply that any surface satisfying his equation also has the desired ruling.

We begin with the first step. We follow the notation of article 437 of Salmon's book where the argument is explained. We assume that a surface is ruled. Then locally (at least away from singular points), it can be written as a one parameter family of lines. We write the equations of these lines

$$z = c_1 x + c_3, \quad y = c_2 x + c_4.$$

We view c_2, c_3, and c_4 as arbitrary functions of c_1. Of course, this doesn't work entirely in general. The projection of our family of lines into the xz plane might have constant slope. But we change coordinates so that this is true locally. Then we view the surface as being locally a graph,

$$z = f(x, y).$$

Of course, a change of variables may be required and f is not a polynomial. It is an algebraic function obtained implicitly from the equation

$$p(x, y, z) = 0.$$

At a regular point, we can use the implicit function theorem to solve this for some choice of the z-direction. Now, however, we find a third order partial differential equation satisfied by f from the parametrized description of our surface as a family of lines.

We shall be concerned for the moment with the behavior of f along a single line. Like Salmon, we adopt the traditional notation for partial derivatives. We let the first derivatives be

$$p = \frac{\partial f}{\partial x}(x, c_2 x + c_4); \quad q = \frac{\partial f}{\partial y}(x, c_2 x + c_4).$$

Here we've emphasized that these partials are being evaluated along one line, where the line is parametrized by x. We omit this same dependence on the variables in describing the rest of Salmon's notation. We let the second derivatives be

$$r = \frac{\partial^2 f}{\partial x^2}; \quad s = \frac{\partial^2 f}{\partial x \partial y}; \quad t = \frac{\partial^2 f}{\partial y^2},$$

and finally

$$\alpha = \frac{\partial^3 f}{\partial x^3}; \quad \beta = \frac{\partial^3 f}{\partial x^2 \partial y}; \quad \gamma = \frac{\partial^3 f}{\partial x \partial y^2}; \quad \delta = \frac{\partial^3 f}{\partial y^3}.$$

Now we write down what it means that an individual line lies in our surface. We have

$$c_1x + c_3 = f(x, c_2x + c_4).$$

We differentiate this equation in x, in effect differentiating along the line and we obtain Salmon's equations

$$p + mq = c_1; \quad m = c_2,$$

from the chain rule. Note that the second equation is in place to say that m is constant along the line. Thus we are free to keep differentiating along the line as long as we make sure that we follow the chain rule, producing a c_2, now called m, every time we introduce a partial of f with respect to y. A second derivative produces

$$r + 2sm + tm^2 = 0.$$

This is not yet a differential equation for f because it still involves m, one of the constants of the line. But we take a third derivative:

$$\alpha + 3\beta m + 3\gamma m^2 + \delta m^3 = 0.$$

We solve the quadratic equation, obtain two solutions

$$m = \frac{-2s \pm \sqrt{4s^2 - 4rt}}{2r}.$$

Plugging each value of m that we obtain into the cubic equation and multiplying the two equations together, we get an equation which is rational in $\alpha, \beta, \gamma, \delta, r, s$, and t. This is the PDE which we assert guarantees that a surface is ruled. (We don't have to know the constants c_1, c_2, c_3, c_4 in order to interpret the differential equation. All the partials are evaluated at the same place.) What the equation says precisely is that one of the two complex directions in which the quadratic form associated to the second derivative vanishes also annihilates the third derivative. In other words, over the complex numbers, the surface has a line tangent to it at third order. Now what remains is for us to see is that the fact that this equation is satisfied actually implies that the surface contains a line at its generic point.

So once again, we have a surface $z = f(x, y)$ which satisfies our differential equation. Rather than write it out in all its horror, we go backwards by a reversible step and observe that at a generic point, we have a once differentiable function $m(x, y)$ satisfying

$$r + 2sm + tm^2 = 0, \tag{2.1}$$

and

$$\alpha + 3\beta m + 3\gamma m^2 + \delta m^3 = 0, \tag{2.2}$$

where now the derivatives $r, s, t, \alpha.\beta, \gamma, \delta$ are viewed as being evaluated on x, y rather than on a line. In the case of planes and quadrics, the function m can be found by hand. In all other cases, it is produced for us because we have a unique line vanishing to third order at the generic point. Now as before, we can parametrize the line at a given (x, y) by c_1, c_2, c_3, c_4, where always $c_2 = m$. It is enough to show that c_1, c_3, and c_4 are also just functions of m. If this is the case, then it is the same line tangent line vanishing to third order on the points of each level set of m on the surface which implies that the level sets are in fact contained in

the lines and that the surface is ruled by these lines. Now, we can easily write down each of c_1, c_3, and c_4 in terms of x, y and m. Namely

$$c_1 = p + mq,$$

$$c_3 = z - c_1 x,$$

and

$$c_4 = y - mx.$$

Our goal now is simply to show each of these three functions is a function of m. We will do this by showing that the gradient of each one is a multiple of the gradient of m. Thus the level curves are also level curves of c_1, c_3, and c_4.

We begin with a preliminary calculation. We will differentiate equation (2.1) first with respect to x and then with respect to y. With respect to x, we get

$$\alpha + 2\beta m + \gamma m^2 + 2s\frac{\partial m}{\partial x} + 2tm\frac{\partial m}{\partial x} = 0.$$

Then differentiating with respect to y, we get

$$\beta + 2\gamma m + \delta m^2 + 2s\frac{\partial m}{\partial y} + 2tm\frac{\partial m}{\partial y} = 0.$$

Adding the first equation to m multiplied by the second equation and dividing by $2s + 2tm$ [we leave as an exercise to the reader to work the excluded case where $2s+2tm$ is identically zero], we obtain

$$\frac{\partial m}{\partial x} + m\frac{\partial m}{\partial y} = 0.$$

Now we compare the gradients of c_1, c_3, and c_4. We calculate

$$\nabla c_1 = (\alpha + m\beta + \frac{\partial m}{\partial x}q, \beta + m\gamma + \frac{\partial m}{\partial y}q).$$

Taking the dot product of this with $(1, m)$ and using equation (2.1), we say that ∇c_1 points in the same direction as ∇c_2. Further we calculate

$$\nabla c_3 = (p - c_1, q) - x\nabla c_1.$$

By dotting with $(1, m)$, we see, using the fact that ∇c_1 is already in the direction of ∇m and using the definition of c_1 as $p + mq$, we see also that ∇c_3 is in the same direction as ∇m. Finally, we calculate

$$\nabla c_4 = (-m, 1) - x\nabla m,$$

which is immediately seen to be in the direction of ∇m. This, in effect, is Salmon's argument. He refers to the equations

$$c_1 = \psi(m); \quad c_3 = \phi(m); \quad c_4 = \chi(m),$$

with ψ, ϕ, and χ as unknown functions of m as the first integrals of his differential equations for surfaces. Part of the reason this proof of Salmon's is difficult to parse is that he claims it in much greater generality for any surface ruled by curves of constant complexity. Basically,

if the curves come from a family with a fixed number of parameters and we assume that all parameters are a function of one of the parameters as we did for lines then the differential equation obtained by reducing away all parameters of the curves must imply that the surface is ruled by such curves.

Once this is done, arriving at Theorem (2.1) is merely a matter of keeping track in the case of a surface $p(x, y, z) = 0$ of the polynomial obtained from checking whether the vectors in the tangent space to a point in whose direction lines vanish to second order have the property that these lines actually vanish to third order. From the point of view of reduction theory, this is precisely analogous to the process of eliminating m from the equations (2.1) and (2.2) which we have discussed. This yields a polynomial of degree $11d - 24$ as discussed in Article 588 of Salmon.

3. On intersections between lines

In this section, we describe the applications of Theorem 2.1 to real incidence geometry. We remark that it is easy to express surprise that the theorem is applicable at all. After all, the theorem is stated over the complex numbers. Still the reals are a subfield and it is possible to exploit this. An important and basic tool is the following variant of Bezout's lemma.

Lemma 3.1. *Let $p(x, y, z)$ and $q(x, y, z)$ be two complex polynomials of three variables of degree m and n respectively. Suppose that p and q vanish simultaneously on more than mn complex lines. Then p and q have a nontrivial common factor. If p and q are both real, then their common factor must be real.*

It may be viewed as a drawback of the flecnode polynomial for investigating real geometry that it is possible to find real polynomials $p(x, y, z)$ for which $Flec(p)(x, y, z) = 0$ but nonetheless the real surface $p(x, y, z) = 0$ contains no lines. An obvious example is the unit two sphere given by

$$p(x, y, z) = x^2 + y^2 + z^2 - 1.$$

When we view the zero set of p as a complex surface, it is ruled (and in fact doubly ruled), but over the reals it contains no lines. However, this is not the way that we ever use Theorem 2.1. We don't assert that a surface contains many lines by showing its flecnode polynomial vanishes identically. Instead, we start with a surface containing many lines and conclude that it has a ruling. Indeed when the lines are real, it is often possible to show that the ruling is real. But it isn't really important. We are interested in intersections between these lines that we already know about and the presence of the ruling allows us to show that some lines don't intersect, even if the ruling is complex.

To wit, we state the following corollary of Theorem 2.1 and Lemma 3.1

Corollary 3.2. *Let $p(x, y, z)$ be an irreducible polynomial of degree d. Suppose the surface $p(x, y, z) = 0$ contains more than $11d^2 - 24d$ complex lines. Then the surface must be ruled over the complex numbers.*

The proof of Corollary 3.2 is simple. If a line l is in the zero set of p, then at each point of the line l, there is a line going through the point namely l on which p vanishes to order at least three. Thus l is in the zero set of $Flec(p)$. Applying Lemma 3.1, we conclude that p and $Flec(p)$ have a nontrivial common factor, and since p is assumed irreducible, it must

be that p is that factor. Thus $Flec(p)$ vanishes on the zero set of p and we conclude from Theorem 2.1 that the zero set of p is ruled over the complex numbers.

This raises something of an open problem. (It is probably not a very serious one.) The corollary above is written in a form that is rather easily usable by incidence geometers. If an irreducible algebraic surface of low degree contains too many lines then it is ruled. It might be useful to have such results for other curves and in higher dimensions. The result for curves inside surfaces in $\mathbf{R}^3$ is probably already contained in Salmon's Article 431. In general, if one finds a polynomial of sufficiently low degree in $\mathbf{R}^n$ whose zero set contains enough l dimensional surfaces of a certain class, does this imply that many of those l dimensional surfaces lie in a surface of dimension $l+1$ or greater ruled by the l dimensional surfaces. [We can't require something of dimension greater than $l+1$ because an $l+1$ dimensional surface ruled by the l dimensional ones already contains infinitely many.] A number of special cases are in the literature (see e.g. [5], [8]) but maybe somebody who is good at calculus should write a general theorem and greatly demystify the subject. A fun exercise might be to see whether the higher dimensional joints problem is related to ruled surfaces in the way we're about to show the regular joints problem is.

In order to utilize Corollary 3.2, we should ask how can we find a low degree polynomial that vanishes on a set of lines. One approach is simply to use surface-fitting.

Lemma 3.3 (curve-fitting). *Let Q be a set of N^3 points in $\mathbf{R}^3$. Then there is a polynomial of degree $O(N)$ vanishing on the points of Q. Let L be a set of N^2 lines then there is a polynomial of degree $O(N)$ vanishing on all the lines of L.*

The proof of the first part of the Lemma is just that the general polynomial of degree $O(N)$ has more than N^3 coefficients. The system of linear equations on the coefficients which says that the polynomial p vanishes on all the points of Q is underdetermined. To prove the second part, just pick KN points on each line, where K is constant which is large compared to the implicit constant in the $O(N)$ of the first part. Now a polynomial which has degree $O(K^{\frac{1}{3}}N)$ vanishes on all these points. Since $O(K^{\frac{1}{3}}N)$ is smaller than KN by the fundamental theorem of algebra, the polynomial must vanish on all the lines.

To apply Corollary 3.2, it should be clear that Lemma 3.3 is useless. The reason is that this fitting applies to all sets of lines, whereas we are trying to find structure in a set of lines. Luckily we have a technique for finding lower degree polynomials that vanish on sets of lines when those lines have unusually many intersections.

Lemma 3.4 (Degree reduction). *Let L_1 and L_2 be sets of at most N lines. Suppose each line l of L_2 intersects at least $QN^{\frac{1}{2}}$ lines l' of L_1 with $Q > 0$ a large real number. Then there is a polynomial of degree $O(\frac{N^{\frac{1}{2}}}{Q})$ which vanishes on all the lines of L_2.*

To prove this, we make a random selection L_3 of lines from L_1 so that each line is chosen independently with probability $\sim \frac{1}{Q^2}$. Then with high probability there are $\sim \frac{N}{Q^2}$ lines of L_3. Moreover with high probability, each line of L_2 intersects $\sim \frac{N^{\frac{1}{2}}}{Q}$ lines of L_3. But there is a polynomial p of degree $\sim \frac{N^{\frac{1}{2}}}{Q}$ which vanishes on all the lines of L_2. The reader may check that by setting the constants correctly, we can make the degree of the polynomial slightly lower than the number of lines of L_3 each line of L_2 intersects. Thus all the lines of L_2 are in the zero set of p.

A version of the Lemma above was first used in the proof of Theorem 1.3. These days, people gleefully tell me that no one ever uses degree reduction to prove the joints theorem.

There are simplifications. (See [3]). Isn't it far better just to say, "let us consider the polynomial of lowest degree vanishing on all significant lines" and not to worry at all about what that degree is. But it is a remarkable fact that the joints theorem only works (to within a constant) in the regime where we have significant degree reduction, that is where most of the lines are arranged in ruled surfaces. Similarly, we didn't even need to use degree reduction to prove Theorem 1.4. If the set E contains only $\frac{N^3}{Q}$ points, just curve-fitting guarantees that the lines are in the zero set of a polynomial of degree $\frac{N}{Q^{\frac{1}{3}}}$ and with Q sufficiently large, this already guarantees that the lines are mostly arranged in ruled surfaces. Why should we care that the degree is really $\frac{N}{Q}$. (See [4] for a partial answer.)

We used Lemma 3.4 in conjunction with Corollary 3.2 in proving the following result which played a role in the proof of Theorem 1.5.

Theorem 3.5. *Let L be a set of N^2 lines. Suppose at most $O(N)$ of the lines of L lie in a common plane and that at most $O(N)$ lines lie in a common doubly ruled surface (parabolic hyperboloid or regulus). Then letting P be the set of points contained in at least two lines then $|P| = O(N^3)$.*

We briefly sketch the proof of Theorem 3.5. Assume that $|P| = QN^3$ with Q large. The worst case is that there are $\sim N^2$ lines each meeting QN lines. (Situations where the intersections are concentrated on fewer lines end up being easier to handle since with a work this corresponds to having fewer lines account for all the intersections.) Then using degree reduction, we find a polynomial p of degree $O(\frac{N}{Q})$ which vanishes on these N^2 lines. We factor p into irreducible components (over the complex numbers) $p_1 \dots p_d$. Each line is in the zero set of one of the components. Each zero-set having its share of lines is, in fact, ruled. If a component is a plane or regulus, it has $O(N)$ lines, less than its share. We end up concluding that most lines lie in ruled components, and as before, if we show these lines aren't involved in most of the intersections, then we end up with an easier problem.

A line not lying in some ruled surface of degree k, will only intersect that ruled surface in at most k points, so it emerges that most of the intersections we have to worry about come from within an irreducible ruled surface. How often can lines in a non-planar, non-regulus ruled surface intersect? We say that a point in an irreducible ruled surface is exceptional if it intersects an infinite number of lines contained in the surface. We say a line in the ruled surface is exceptional if it meets an infinite number of lines contained in the surface. An irreducible ruled surface contains at most one exceptional point and two exceptional lines. Thus we have at most $O(N)$ exceptional lines in the whole story, which contribute at most $O(N^3)$ intersections which is harmless. Nonexceptional lines in a ruled surface of degree d (in Salmon's language: generators) meet exactly $d-2$ other generators of the surface. Again, this gives every line at most $O(N)$ intersections. We conclude that there cannot be more than $O(N^3)$ points of intersection.

The same ideas prove Theorem 1.3. To prove the joints theorem, we must take account of the fact that we have removed the restriction on the number of lines in a regulus and the number of lines in a plane. However a joint cannot come exclusively from the lines in a plane or from the lines in a regulus. Each line outside a plane or regulus can intersect the plane or regulus at most once or twice respectively. The bounds on internal intersections in other ruled components control the number of internal joints there.

At the time we proved Theorems 1.3 and 1.4, we were not aware of the Cayley-Salmon polynomial $Flec(p)$. We thought these theorems revolved around the gradient and second fundamental forms respectively. But this was short-sighted on our part. Regardless of which

polynomial we use, these results are dramatic only in the ruled regime. This creates rather big problems in operating just outside these regimes. A question frequently posed by Guth (see [5]) asks: suppose we have a set of N lines making almost $N^{\frac{3}{2}}$ joints. Must a large set of lines (say of size almost $N^{\frac{1}{2}}$ be coplanar? We have no good algebraic way of addressing this question yet because we are just outside the range where degree reduction works. We know no special algebraic properties of the lines.

4. Elekes-Sharir framework and polynomial partitioning

Theorem 3.5 plays a role in the proof of theorem 1.5, but is not the whole of the proof. We now briefly review the Elekes-Sharir framework which explains how incidences between points and lines in space give information about distances between points in the plane.

Given E a set of N points and D the set of distances, we may define $Q \subset E^4$ to be the set of distance quadruplets, namely (e_1, e_2, e_3, e_4) is a distance quadruplet if the distance between e_1 and e_2 is the same as the distance between e_3 and e_4. A simple application of the Cauchy Schwarz inequality shows that

$$|Q| \geq \frac{|E|^4}{|D|}.$$

Thus to prove Theorem 1.5, it suffices to prove

$$|Q| = O(N^3 \log N).$$

Now the ancients had a more descriptive term for distance quadruplets. They referred to them as pairs of congruent line segments. And one thing they knew is that whenever two line segments are congruent there is a rigid motion between them. The space of rigid motions is three dimensional. It consists of all rotations around a center and all translations. If we restrict to non-translations, a good coordinate system is given by the center of the rotation in Cartesian coordinates, together with the cotangent of half the angle of rotation.

Given two points of E, say e_1 and e_3, we let $l_{e_1 e_3}$ be the set of rigid motions taking e_1 to e_3. This is a one dimensional set and in the coordinate system described in the paragraph above, it is a line. We see that (e_1, e_2, e_3, e_4) form a distance quadruple precisely when $l_{e_1 e_3}$ and $l_{e_2 e_4}$ intersect. We are back to incidence theory. It is fortunate that with L the set of N^2 lines in rigid motion space of the form $l_{e_1 e_3}$, we have no more than $O(N)$ in a regulus and no more than $O(N)$ in a plane. Then Theorem 3.5 tells us, that there are at most $O(N^3)$ points of intersection of two lines from L. Unfortunately, this is not enough.

Consider a point where k lines of L meet. This contributes k^2 distance quadruplets. We need to keep the number of distance quadruplets below $N^3 \log N$. If too many of the points where at least two lines meet have many lines meeting there then we're sunk. We let P_k be the set of points where between k and $2k$ lines of L meet. As long as we can prove that

$$|P_k| = O(\frac{N^3}{k^2}),$$

by dyadically decomposing, we obtain the desired bound. However this is a tricky business. There is no purely algebraic argument. (The estimate doesn't hold in finite fields.) To take care of this, we use some topology.

Using the polynomial ham sandwich theorem, we obtain the following polynomial partitioning lemma which has proved quite useful.

Lemma 4.1. *Let F be a set of M points in $\mathbf{R}^3$. Then for any s, a power of 2, there is a real polynomial $p(x,y,z)$ of degree $O(s^{\frac{1}{3}})$ so that the complement of the zero set of p in $\mathbf{R}^3$ has at most s connected components with points of F and each connected component contains at most $\frac{M}{s}$ points.*

It is important to note that Lemma 4.1 does not guarantee us that most of the points aren't in the zero set of the polynomial. As it turns out, we will be very happy if they are.

We proceed now to sketch a proof that indeed we have the estimate

$$|P_k| = O(\frac{N^3}{k^2}).$$

We suppose not. Then there are $\frac{QN^3}{k^2}$ such points with Q large. Note that Q is certainly never larger than k^2 simply by Theorem 3.5. Better *a priori* estimates are possible. We would like to subdivide this set of points into components of size at most k. By Lemma 4.1, there is a polynomial p of degree $\frac{Q^{\frac{1}{3}}N}{k}$ which does this. We divide into two cases. In the first case, most of the points of P_k are in the complement of the zero set of p. In the second case, most of the points of P_k are on the zero set.

In the first case, we obtain upper and lower bounds on the number of incidences between components and lines. We say a line and a component are incident, if there is a point on the interior of the component which lies on the line. Let I be the number of such incidences. Since we are in the case where most points are in the interior, many components have $\sim k$ points. Each point has k lines through it. Since any two points have at most one line in common, there is not too much double counting and we get $\sim k^2$ lines incident to each cell. We conclude

$$I \gtrsim \frac{QN^3}{k}.$$

On the other hand, each line can switch components only by crossing the zero set of the polynomial. Each line does this at most as many times as the degree of p plus one. Since there are only N^2 lines, we conclude

$$I \lesssim \frac{Q^{\frac{1}{3}}N^3}{k}.$$

We have arrived at a contradiction.

Thus, we are in the second case. Most of the points are in the zero set of the polynomial. There are $\frac{QN^3}{k^2}$ of these points each incident to k lines. This gives, on average, $\frac{QN}{k}$ incidences per line. Since the polynomial has degree $\frac{Q^{\frac{1}{3}}N}{k}$, this means that average lines are in the zero set of the polynomial. As usual, the worst case is that most of the N^2 lines are average. (Because if in fact most of the incidence are created by fewer lines, we can basically redo the argument with a larger Q and smaller N.) If we are in the setting where the lines are all average, once again, we are in the domain where the lines are structured into ruled surfaces, and we can use this structure much as before. Thus we have completed our sketch of the argument for Theorem 1.5.

We make a brief remark about the partitioning part of this argument. Contrary to algebra which works best in the complex numbers, polynomial partitioning seems to work best in

the reals. This allows one to prove incidence theorems in the reals much more easily than in the complex numbers. As an open problem, we suggest considering a set of points in $\mathbf{C}^3$ no more than half of which is in the zero set of any low degree polynomial. (For instance, degree lower than $\frac{Q^{\frac{1}{3}}N}{k}$, as above.) Can one make arguments giving results analogous to the case above with most points in the complement of the zero set?

Acknowledgements. The author was partially supported by NSF grant DMS 1266104.

References

[1] Z. Dvir, *On the size of Kakeya sets in finite fields*, J. Amer. Math Soc., **22** (2009), 1093–1097.

[2] Gy. Elekes and M. Sharir, *Incidences in three dimensions and distinct distances in the plane*, Proceedings 26th ACM Symposium on Computational Geometry (2010) 413–422.

[3] Gy. Elekes, H. Kaplan, and M. Sharir, *On lines, joints, and incidences in three dimensions*, Journal of Combinatorial Theory, Series A, **118** (2011), 962–977.

[4] L. Guth, *Degree reduction and graininess for Kakeya sets in* $\mathbf{R}^3$, arXiv:1402.0518.

[5] _____, *Lecture notes on the polynomial method*, http://math.mit.edu/~lguth/PolynomialMethod.html.

[6] L. Guth and N. H. Katz, *Algebraic methods in discrete analogues of the Kakeya problem*, Adv. in Math., **225** (2010), 2828–2839.

[7] _____, *On the Erdős distinct distances problem in the plane*.

[8] J.M. Landsberg, *Is a linear space contained in a submanifold - On the number of the derivatives needed to tell*, Journal für die reine und angewandte Mathematik, **508** (1999), 53–60.

[9] G. Salmon, *A treatise on the analytic geometry of three dimensions, vol.2*, 1st. edition, 1862, Longmans, Green and co.

[10] E. Szemerédi and W. Trotter, *Extremal problems in discrete geometry*, Combinatorica, **3** (1983), 381–392.

[11] T. Tao, *The Cayley-Salmon theorem via classical differential geometry*, http://terrytao.wordpress.com/2014/03/28/the-cayley-salmon-theorem-via-classical-differential-geometry.

Departement of Mathematics, Caltech, 253-37, Pasadena, CA 91125, USA
E-mail: nets@caltech.edu

Harmonic analysis and the geometry of fractals

Izabella Łaba

Abstract. Singular and oscillatory integral estimates such as maximal theorems and restriction estimates for measures on hypersurfaces have long been a central topic in harmonic analysis. We discuss the recent work by the author and her collaborators on the analogues of such results for singular measures supported on fractal sets. The common thread is the use of ideas from additive combinatorics. In particular, the additive-combinatorial notion of "pseudorandomness" for fractals turns out to be an appropriate substitute for the curvature of manifolds.

Mathematics Subject Classification (2010). Primary 28A78, 42A38, 42B25, 26A24; Secondary 11B25, 42A45.

Keywords. Fourier analysis, Hausdorff dimension, restriction estimates, maximal operators.

1. Introduction

A recurring theme in Euclidean harmonic analysis is the connection between Fourier-analytic properties of measures and geometric characteristics of their supports. The best known classical results of this type concern estimates on singular and oscillatory integrals associated with surface measures on submanifolds of $\mathbb{R}^d$, with ranges of exponents depending on geometric features of the submanifold in question such as its dimension, smoothness and curvature.

Our focus here is on more recent lines of research that dispense with the regularity assumptions. Instead of surface measures on smooth manifolds, we will be concerned with fractal measures supported on sets of possibly non-integer dimension. This includes in particular the case of ambient dimension 1, where there are no non-trivial lower-dimensional submanifolds but many interesting fractal sets. It turns out that the dichotomy between flatness and curvature for manifolds in higher dimensions has useful analogues in dimension one. Specifically, "random" fractals (in a sense that will be made precise later) often behave like curved hypersurfaces such as spheres, whereas fractals exhibiting arithmetic structure (e.g. the middle-third Cantor set) behave like flat surfaces.

The goal of this paper is to provide an exposition of the recent work by the author and her collaborators on three specific questions of this type: restriction estimates, differentiation estimates, and Szemerédi-type results. In the context of fractal sets, the first two lines of investigation can be dated back to Mockenhaupt's restriction theorem [37] (see also Mitsis [36]) and the work of Aversa and Preiss [1, 2]. However, our work was also influenced by ideas from additive combinatorics (see [53]), where the study of "randomness" and "arithmetic structure" in sets of integers was a key part of recent major advances such as

Proceedings of the International Congress of Mathematicians, Seoul, 2014

Gowers's "quantitative" proof of Szemerédi's theorem [16] and the Green-Tao theorem on arithmetic progressions in the primes [18]. In the last section, we present Szemerédi-type results for fractal sets, motivated by number-theoretic results from additive combinatorics but also drawing on harmonic analytic techniques.

2. Fractal sets and Fourier decay

Throughout this article, we will refer to certain types of fractal sets of non-integer dimension. We now provide the pertinent definitions and examples.

For a set $E \subset \mathbb{R}^d$, we will use $\dim_H(E)$ to denote its Hausdorff dimension. The following characterization of the Hausdorff dimension, provided by Frostman's lemma, will suffice for our purposes instead of a definition; we refer the reader to [13, 35, 56] for more background. Let $\mathcal{M}(E)$ be the set of all probability measures supported on E. We will say that a measure $\mu \in \mathcal{M}(E)$ obeys the *ball condition* with exponent α if there is a constant $C(\alpha)$ such that

$$\mu(B(x,\epsilon)) \leq C(\alpha)\epsilon^\alpha \text{ for all } x \in \mathbb{R}^d,\ \epsilon > 0, \tag{2.1}$$

where $B(x,\epsilon)$ denotes the open ball of radius ϵ centered at x.

Lemma 2.1. *(Frostman) Let $E \subset \mathbb{R}^d$ be a compact set. Then*

$$\dim_H(E) = \sup\{\alpha \in [0,d] :\ \exists \mu \in \mathcal{M}(E) \text{ s.t. (2.1) holds for some } C(\alpha) > 0\} \tag{2.2}$$

If E is a smooth submanifold of $\mathbb{R}^d$, then its Hausdorff dimension coincides with its topological dimension: for instance, the sphere $S^{d-1} \subset \mathbb{R}^d$ has Hausdorff dimension $d-1$. However, there are also many sets whose Hausdorff dimension is non-integer. The following basic examples will be important in the sequel.

Example 2.2. (Self-similar Cantor sets) Construct a set $E \subset [0,1]$ via the following iteration. Fix integers N, t such that $1 < t < N$. Divide $[0,1]$ into N intervals of equal length, and choose t of them. This is our first iteration E_1 of the Cantor set.

$$E_1 = \bigcup_{a \in A} \left[\frac{a}{N}, \frac{a+1}{N}\right]$$

where A is a subset of $\{0, 1, \ldots, N-1\}$ of cardinality t. We now iterate the construction in a self-similar manner, dividing each interval of E_1 into N congruent subintervals and choosing k of them according to the same pattern, etc. We thus get a decreasing sequence of sets $E_1, E_2, \ldots$, where E_n consists of t^n intervals of length N^{-n}:

$$E_n = \bigcup_{a_1,\ldots,a_n \in A} \left[\sum_{i=1}^n \frac{a_i}{N^i}, \sum_{i=1}^n \frac{a_i}{N^i} + \frac{1}{N^n}\right]$$

Let $E = \bigcap_{n=1}^\infty E_n$, then E is a compact set of Lebesgue measure 0. If $N = 3$, $t = 2$ and $A = \{0, 2\}$, then E is the usual middle-thirds Cantor set. It is easy to see that

$$\dim_H(E) = \frac{\log t}{\log N}. \tag{2.3}$$

Furthermore, the measure $\mu \in \mathcal{M}(E)$ constructed as the weak limit of the absolutely continuous measures with densities

$$\phi_n = \frac{1}{|E_n|}\mathbf{1}_{E_n} = \frac{N^n}{t^n}\mathbf{1}_{E_n} \tag{2.4}$$

obeys (2.1) with this value of α. (We use $\mathbf{1}_X$ to denote the characteristic function of a set X.) We will refer to such μ as the "natural" measure on E.

Example 2.3. (Generalized Cantor sets) We modify the procedure from Example 2.2. As before, we start by dividing $[0,1]$ into N congruent intervals and choosing t of them to form E_1. Suppose that we have constructed E_n, consisting of t^n intervals I_j of length N^{-n} each. We subdivide each I_j into N congruent intervals and choose t of these; however, this does not need to be the same choice as for E_1 or any other preceding steps, nor do we have to use the same pattern for all intervals of E_n. This again produces a decreasing sequence of sets converging to a compact set E of Hausdorff dimension $\alpha = (\log t)/(\log N)$, and a natural probability measure $\mu = \mathrm{w} - \lim \phi_n$ on E, where ϕ_n are defined as in (2.4). However, such sets and measures are no longer self-similar, and can display a much wider range of behaviours than those from Example 2.2. Of particular importance will be "random" and "quasirandom" Cantor sets, where the choices of intervals at each step are made through some randomized procedure within specified constraints. An example of this is given in [31], Section 6.

Further modifications are possible. For instance, instead of keeping the values of N and t fixed, one could repeat the last construction with a slowly increasing sequence of N_n and t_n such that $\frac{\log t_n}{\log N_n} \to \alpha$ as $n \to \infty$; this produces Cantor sets of arbitrary dimension $0 \le \alpha \le 1$, not just of the form $\frac{\log t}{\log N}$ with t, N integer.

Analytic properties of fractal sets and measures (such as those described above) depend very strongly on their arithmetic structure, in a manner that is reminiscent of the relation between the geometry of a submanifold of $\mathbb{R}^d$ and its Fourier-analytic properties. One indicator of the arithmetic structure, or lack thereof, of a measure μ on $\mathbb{R}^d$ is the decay of its Fourier transform. Let

$$\widehat{\mu}(\xi) = \int e^{-2\pi i \xi \cdot x} d\mu(x).$$

We will be interested in pointwise estimates of the form

$$|\widehat{\mu}(\xi)| \le C(\beta)(1+|\xi|)^{-\beta/2} \text{ for all } \xi \in \mathbb{R}^d. \tag{2.5}$$

The relation between Hausdorff dimension and estimates such as (2.5) is as follows. Let $E \subset \mathbb{R}^d$ be compact. It is well known that

$$\dim_H(E) = \sup\{\beta \in [0,d] : \exists \mu \in \mathcal{M}(E) \text{ s.t. } I_\beta(\mu) < \infty\}, \tag{2.6}$$

where

$$I_\beta(\mu) = \int_{\mathbb{R}^d} |\widehat{\mu}(\xi)|^2 \, |\xi|^{-(d-\beta)} d\xi \tag{2.7}$$

Thus for any $\beta < \dim_H(E)$, there are measures supported on E that obey (2.5) "on average." On the other hand, (2.5) cannpt hold with $\beta > \dim_H(E)$.

We will say that a measure μ is a *Salem measure* if it obeys (2.5) for all $\beta < \dim_H(\operatorname{supp}\mu)$. (As indicated above, this is the best possible range of β except possibly for the endpoint.)

An easy example is provided by the Lebesgue measure on the sphere $S^{d-1} \subset \mathbb{R}^d$, or more generally on a bounded $(d-1)$-dimensional smooth manifold with non-vanishing Gaussian curvature. In this case, the estimate (2.5) with $\beta = d-1$ follows from well known stationary phase estimates. It is more difficult to produce Salem measures with supports of non-integer dimension. The first such construction was given by Salem in [44]; for other examples, see Kaufman [24], Kahane [23], Bluhm [4, 5].

The property of being a Salem measure (and indeed any pointwise estimate such as (2.5) with $\beta > 0$) is deeper than average decay as in (2.6), and indicative of the level of the arithmetic structure of the measure in question. Roughly speaking, "random" fractal measures often obey (2.5), whereas "structured" ones do not. For example, the self-similar Cantor measure μ in Example 2.2 has the Fourier transform

$$\widehat{\mu}(\xi) = \prod_{j=1}^{\infty} \left(\frac{1}{|A|} \sum_{a \in A} e^{2\pi i a \xi / N^j} \right)$$

and, since $A \subset \mathbb{Z}$, we have $\widehat{\mu}(N^j) = \widehat{\mu}(1)$ for all $j \in \mathbb{N}$, so that (2.5) does not hold for any $\beta > 0$. On the other hand, the more general construction in Example 2.3 can be randomized so that μ obeys (2.5) for all $\beta < \dim_H(E)$ (see [31]). We will see that those measures that obey (2.5) for some $\beta > 0$, and those that do not, behave very differently from the harmonic analytic point of view.

3. Restriction estimates

We define the Fourier transform of a function $f : \mathbb{R}^d \to \mathbb{C}$ by

$$\widehat{f}(\xi) = \int f(x) e^{-2\pi i x \cdot \xi} dx.$$

This maps the Schwartz space of functions $\mathcal{S}$ to itself. By the Hausdorff-Young inequality, the Fourier transform extends to a bounded operator from $L^p(\mathbb{R}^d)$ to $L^{p'}(\mathbb{R}^d)$ if $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

Let μ be a finite, compactly supported measure of $\mathbb{R}^d$. We are particularly interested in the case when μ is a singular measure, supported on a set $E \subset \mathbb{R}^d$ of d-dimensional Lebesgue measure 0. We also write $\widehat{f d\mu}(\xi) = \int f(x) e^{-2\pi i x \cdot \xi} d\mu(x)$.

Question 3.1. *(Restriction problem) For what values of p, q do we have an estimate*

$$\|\widehat{f d\mu}\|_{L^p(\mathbb{R}^d)} \leq C \|f\|_{L^q(\mathbb{R}^d, d\mu)}, \ f \in \mathcal{S}? \tag{3.1}$$

Here and below, C and other similar constants may depend on the dimension d, the measure μ, and the exponents p, q, but not on f. Whenever we use the notation $L^p(X)$ without indicating the measure, the latter is assumed to be the Lebesgue measure on X.

The restriction problem takes its name from the dual formulation, which we state now.

Question 3.2. *(Restriction problem, dual version) For what values of p', q' do we have an estimate*

$$\|\widehat{f}\|_{L^{q'}(\mathbb{R}^d, d\mu)} \leq C \|f\|_{L^{p'}(\mathbb{R}^d)}, \ f \in \mathcal{S}? \tag{3.2}$$

It is not difficult to see that (3.1) and (3.2) are equivalent if p, p' and q, q' are pairs of dual exponents: $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$. Informally, Question 3.2 can be stated in terms of *restricting* the Fourier transform of an $L^{p'}$ function f to the set E. This is trivial if $p' = 1$ and $q' = \infty$, since then $\widehat{f}$ is continuous and bounded everywhere. On the other hand, no such result is possible if $p' = 2$. This is because the Fourier transform maps L^2 *onto* L^2, so that we are not able to say anything about the behaviour of $\widehat{f}$ on a set of measure 0. For the intermediate values of $p' \in (1, 2)$ (or, equivalently, for $p > 2$ in (3.1)), the answer depends on the geometric and arithmetic pproperties of μ, as we will see in the rest of this section.

We now specialize to $q = 2$, in which case we have the following theorem.

Theorem 3.3. *Let μ be a compactly supported positive measure on $\mathbb{R}^d$ such that for some $\alpha, \beta \in (0, d)$ we have*

$$\mu(B(x, r)) \leq C_1 r^{\alpha} \text{ for all } x \in \mathbb{R}^d \text{ and } r > 0, \tag{3.3}$$

$$|\widehat{\mu}(\xi)| \leq C_2(1 + |\xi|)^{-\beta/2} \text{ for all } \xi \in \mathbb{R}^d. \tag{3.4}$$

Then for all p such that

$$p \geq p_{d,\alpha,\beta} := \frac{2(2d - 2\alpha + \beta)}{\beta} \tag{3.5}$$

there is a $C(p) > 0$ such that

$$\|\widehat{fd\mu}\|_{L^p(\mathbb{R}^d)} \leq C(p)\|f\|_{L^2(d\mu)} \tag{3.6}$$

for all $f \in L^2(d\mu)$.

The classical Stein-Thomas theorem [48, 49, 54, 55] asserts this in the prototype case when μ is the surface measure on the unit sphere S^{d-1} in $\mathbb{R}^d$, so that $\alpha = \beta = d - 1$. First proved by Stein for a smaller range of p (1967, unpublished), it was then extended to $q > \frac{2d+2}{d-1}$ by Tomas [54, 55], and finally the endpoint estimate was proved by Stein [48].

We note here that the Stein-Tomas theorem exploits the curvature of S^{d-1} via the estimate (3.4), and that the same result holds (for the same reasons) for more general $(d-1)$-dimensional hypersurfaces in $\mathbb{R}^d$ with non-vanishing Gaussian curvature. On the other hand, it is easy to see that there can be no estimates such as (3.6) (or more generally, such as (3.1)) with $p < \infty$ if E is contained in a hyperplane. For manifolds whose Gaussian curvature vanishes at some points, such as cones or polynomial surfaces of higher order, there is a range of nontrivial restriction estimates with exponents depending on the geometry of the manifold.

In the case of the sphere (and more generally, hypersurfaces with non-vanishing Gaussian curvature), the range of exponents in (3.5) is known to be optimal, in the sense that (3.6) fails for all $p < \frac{2d+2}{d-1}$. This is seen from the so-called Knapp example, where (3.6) is tested on characteristic functions of small spherical caps (see e.g. [49, 56]).

Theorem 3.3 as stated above, with exponents as above except for the endpoint, was proved by Mockenhaupt [37] (see also Mitsis [36]), and the endpoint estimate is due to Bak and Seeger [3]. Mockenhaupt's argument follows closely Tomas's proof of the non-endpoint Tomas-Stein theorem for the sphere. The point of Mockenhaupt's work is that estimates such as (3.6) can also hold for less regular measures obeying (3.3) and (3.4), including fractal measures with α, β not necessarily integer. This shifts the emphasis from properties generally

associated with differentiable manifolds, such as smoothness and curvature, to arithmetic properties that may hold for more general measures.

The question of the optimality of the estimate (3.6) for fractal sets appears to be more complicated than for hypersurfaces. The question of sharpness of the exponent in Theorem 3.3 for measures on $\mathbb{R}$ was only settled recently in [11, 19].

Theorem 3.4. *Let $0 < \beta \le \alpha < 1$. Then there is a probability measure μ on $[0,1]$ supported on a set E of dimension α and obeying (3.3) and (3.4), and a sequence of functions $\{f_\ell\}_{\ell\in\mathbb{N}}$ on $[0,1]$ (characteristic functions of finite unions of intervals), such that the restriction estimate (3.6) fails for the sequence $\{f_\ell\}$ and for every $1 \le p < p_{1,\alpha,\beta}$, in the sense that*

$$\frac{\|\widehat{f_\ell d\mu}\|_{L^p(\mathbb{R})}}{\|f_\ell\|_{L^2(d\mu)}} \to \infty \ \text{ as } \ \ell \to \infty. \tag{3.7}$$

This is due to Hambrook and the author [19] in a slightly weaker form (which already demonstrates that the dependence of p on α, β in (3.5) cannot be improved for Salem measures), and to Chen [11] as stated.

The main idea, due to [19], is that, while Salem sets behave like random sets overall, they may nonetheless contain much smaller sets that are highly structured. Specifically, we construct a set E of dimension $\alpha = \frac{\log t}{\log n}$ via a randomized Cantor iteration as in Example 2.3, following the procedure from [31] to ensure that (3.4) holds for all $\beta < \alpha$. At the same time, we also modify the construction so that each iteration E_n contains a much smaller subset F_n, where F_n is constructed as in Example 2.2 with A an arithmetic progression. This can be done without destroying the estimate (3.4) as long as $|A| \le \sqrt{t}$. If A has the maximal allowed size $\sqrt{t}$, the set $F = \bigcap F_n$ is a highly structured Cantor set of dimension $\alpha/2$. In the language of additive combinatorics, the endpoints of each finite iteration F_n lie in a generalized arithmetic progression of the lowest possible dimension. The functions f_n are then defined as the characteristic functions of F_n. The construction in [11] follows the main steps of that in [19], but with N, t varying between different stages of the iteration, allowing more flexibility with dimensions and exponents.

In a sense, this may be viewed as a one-dimensional analogue of Knapp's counterexample. The latter is based on the fact that an "almost flat" spherical cap is contained in the curved sphere, or equivalently, that the sphere is tangent to a flat hyperplane. Here, the set E may be thought of as random but nonetheless "tangent" to the arithmetically structured set F.

We also note that our lower bound on $\|\widehat{f_\ell d\mu}\|_p$ relies on arithmetic arguments, specifically on counting solutions to linear equations in the set of endpoints of the Cantor intervals in the construction. This idea appears to be new in this setting, but has been used extensively in recent work on restriction estimates in finite fields, see e.g. [21, 33, 38].

Theorem 3.4 shows that the range of p in (3.5) cannot, in this generality, be improved. It remains unknown, however, whether such improvements might be possible for *some* measures μ, and if so, how such measures might be characterized.

In this regard, we first note that a measure $\mu \in \mathcal{M}(\mathbb{R}^d)$ supported on a set of dimension α_0 cannot obey (3.6) for any $p < 2d/\alpha_0$, even if the L^2 norm on the right side is replaced by the stronger L^∞ norm. This can be seen by letting $f \equiv 1$ and considering the energy integral (2.7) (see [19] for details).

Question 3.5. *Is there a measure $\mu \in \mathcal{M}(\mathbb{R}^d)$ supported on a set of dimension α_0, obeying (3.3) and (3.4) with α and β arbitrarily close to α_0, such that (3.6) holds for (some or all)*

exponents in the intermediate range

$$\frac{2d}{\alpha_0} \le p < \frac{4d - 2\alpha_0}{\alpha_0} \quad ? \tag{3.8}$$

If so, what properties of μ determine the range of such exponents?

Chen [10] provides an example of a measure supported on a set $E \subset \mathbb{R}$ of dimension $1/2$ for which the restriction estimate (3.6) holds for the maximal possible range $p \ge 4$. Chen's example is based on Körner's construction in [30] of fractal measures whose k-fold convolutions, for an appropriate k, are absolutely continuous; in particular, the 1/2-dimensional example just mentioned depends on the existence of a measure μ supported on a set of dimension $1/2$ such that $\mu * \mu$ has an L^∞ density. However, Körner's measures do not appear to obey (3.3) and (3.4) with α, β near α_0, and it is not clear whether the construction can be modified to ensure these properties.

Another open question concerns restriction estimates beyond the Stein-Tomas range.

Question 3.6. *Let $\mu \in \mathcal{M}(\mathbb{R})$ be a Salem measure of dimension $\alpha_0 \in (0,1)$, obeying the assumptions of Theorem 3.3 with α, β arbitrarily close to α_0. Are there any restriction estimates of the form*

$$\|\widehat{fd\mu}\|_{L^p(\mathbb{R}^d)} \le C(p)\|f\|_{L^\infty(d\mu)} \tag{3.9}$$

for all $f \in L^\infty(d\mu)$, where $p < \frac{4d-2\alpha_0}{\alpha_0}$?

In the case when μ is the normalized surface measure on S^{d-1}, Stein [48] conjectured that

$$\|\widehat{fd\mu}(\xi)\|_{L^p(\mathbb{R}^d)} \le C(d,p)\|f\|_{L^\infty(S^{d-1},d\mu)}, \tag{3.10}$$

for all $p > \frac{2d}{d-1}$. This is known for $d = 2$ (due to Fefferman and Stein [14]). It remains open for all $d > 2$, but partial results are available (see e.g. [52, 56] for an overview of the subject, and [8] for the current best result for the sphere). The range of q as above, suggested by stationary phase formulas, is known to be optimal.

We do not know whether fractal measures as in Question 3.6 admit any estimates such as (3.9) with $p < \frac{4d-2\alpha_0}{\alpha_0}$. In the case of a sphere, such estimates require sophisticated geometric input related to the Kakeya problem. It is unclear how such arguments might translate to the setting of fractal sets.

4. Maximal functions and differentiation theorems

One of the most basic results in analysis is the Hardy-Littlewood maximal theorem.

Theorem 4.1. *Given $f \in L^1(\mathbb{R}^d)$, define its Hardy-Littlewood maximal function by*

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy, \tag{4.1}$$

where $B(x,r) = \{y : |x-y| \le r\}$. Then

$$\|Mf\|_p \le C_{p,d}\|f\|_p$$

for all $1 < p \le \infty$. Moreover, M is of weak type (1,1):

$$|\{x : \ Mf(x) > \lambda\}| \le C\lambda^{-1}\|f\|_1.$$

This easily implies the *Lebesgue differentiation theorem*: if $f \in L^1(\mathbb{R}^d)$, then for almost all x we have

$$\lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)dy = f(x). \tag{4.2}$$

In particular, if $f = \chi_E$ is the characteristic function of a measurable set E, (4.2) states that for almost all $x \in E$

$$\lim_{r\to 0} \frac{|E \cap B(x,r)|}{|B(x,r)|} = 1, \tag{4.3}$$

which is the Lebesgue theorem on density points.

We will be interested in analogues of Theorem 4.1 and its corollaries (4.2), (4.3) where the averages on balls $B(x,r)$ are replaced by averages with respect to singular measures supported on lower-dimensional sets. In general, such averages can be quite badly behaved, as can be seen from the consideration of Kakeya and Nikodym type examples (see e.g. [49, 56]). However, non-trivial maximal estimates can hold for certain types of singular measures. In the case of hypersurfaces and, more generally, manifolds in $\mathbb{R}^d$, the main issues are smoothness and curvature. A classic result of this type is the *spherical maximal theorem*, due to E.M. Stein [47] for $d \geq 3$ and Bourgain [6] for $d = 2$.

Theorem 4.2. *Define the spherical maximal operator in $\mathbb{R}^d$ by*

$$M^S f(x) = \sup_{t>0} \int_{S^{d-1}} |f(x-ty)| d\sigma(y), \tag{4.4}$$

where σ is the normalized Lebesgue measure on S^{d-1}. Then

$$\|M^S f(x)\|_{L^p(\mathbb{R}^d)} \leq C\|f\|_{L^p(\mathbb{R}^d)},\ p > \tfrac{d}{d-1}, \tag{4.5}$$

and this range of p is optimal.

There is a vast literature on maximal and averaging operators over families of smooth lower-dimensional submanifolds of $\mathbb{R}^d$, see e.g. [22, 39–41, 45, 46, 50]. The situation is somewhat similar to restriction estimates in that results of this type, including Stein's proof of the spherical maximal theorem for $d \geq 3$, exploit curvature via Fourier decay estimates such as (2.5) for the surface measure on the manifold. Such decay estimates are weaker for manifolds with flat directions, which is reflected in a weaker range of exponents in maximal and averaging estimates. We also note that the argument used to deduce (4.2) and (4.3) from Theorem 4.1 is very general and applies to more general averages. In particular, Theorem 4.2 implies the analogues of (4.2) and (4.3) for spherical averages, for $f \in L^p(\mathbb{R}^d)$ with $p > \frac{d}{d-1}$.

We are interested in analogues of Theorem 4.1 and its corollaries for singular measures supported on fractal sets. For $\mu \in \mathcal{M}(\mathbb{R}^d)$, define the maximal operator associated with it:

$$\mathfrak{M}f(x) := \sup_{r>0} \int |f(x+ry)|\, d\mu(y). \tag{4.6}$$

In dimensions $d \geq 2$, a theorem of Rubio de Francia [43] asserts that if μ obeys the Fourier decay condition (2.5) with $\beta > 1$, then $\mathfrak{M}$ is bounded on $L^p(\mathbb{R}^d)$ for $p > (\beta + 1)/\beta$. This in particular implies Theorem 4.2 for $d \geq 3$, and provides its analogue for Salem measures of dimension strictly greater than 1. However, it does not apply to singular measures on $\mathbb{R}$, since it is not possible for such measures to obey (2.5) with $\beta > 1$.

In [32], we prove the following.

Theorem 4.3. *(a) There is a measure $\mu \in ([1,2])$, supported on a set E of Lebesgue measure 0 (but Hausdorff dimension 1) such that $\mathfrak{M}$ is bounded on $L^p(\mathbb{R})$ for all $p > 1$.*

(b) For any $0 < \epsilon < \frac{1}{3}$, there is a measure $\mu \in ([1,2])$, supported on a set E of Hausdorff dimension $1-\epsilon$. such that $\mathfrak{M}$ is bounded on $L^p(\mathbb{R})$ for $p > \frac{1+\epsilon}{1-\epsilon}$.

As a corollary, we have a differentiation theorem for the measures constructed in [32]:

$$\lim_{r\to 0} \left| \int f(x+ry) d\mu(y) - f(x) \right| = 0 \text{ for a.e. } x \in \mathbb{R} \tag{4.7}$$

for $f \in L^p(\mathbb{R})$ with the same range of p as in Theorem 4.3. This answers a question of Aversa and Preiss [1], [2]. Note that we require μ to be supported on $[1,2]$ rather than $[0,1]$; the purpose of this is to exclude the trivial solution $\mu = \delta_0$. An argument due to Preiss, included in [32], shows that $\mathfrak{M}$ cannot be bounded on $L^1(\mathbb{R})$, and (4.7) cannot hold for all $f \in L^1(\mathbb{R})$, if μ is singular with respect to the Lebesgue measure.

Question 4.4. *What is the optimal range of ϵ and p for which there exists a measure $\mu \in \mathcal{M}([1,2])$ with $\dim_H(\mathit{supp}\,\mu) = 1-\epsilon$, such that $\mathfrak{M}$ is bounded on $L^p(\mathbb{R})$, or that (4.7) holds for all $f \in L^p(\mathbb{R})$?*

The range of ϵ and p in Theorem 4.3 is an artifact of the construction, and is likely not optimal. On the other hand, it is easy to see that if $\dim_H(\text{supp}\,\mu) = \alpha$, then (4.7) cannot hold for $f \in L^p(\mathbb{R})$ (hence $\mathfrak{M}$ cannot be bounded on $L^p(\mathbb{R})$) if $p < 1/\alpha$.

While the L^p-boundedness of $\mathfrak{M}$ implies a differentiation theorem on L^p, there is no converse implication, so that at least in principle it is possible that the range of p, ϵ for differentiation theorem might be wider than for maximal theorems. We also note that, while singular measures cannot differentiate all $L^1(\mathbb{R})$ functions as pointed out above, there might be differentiation theorems of this type on spaces such as $L \log L$.

The measures in [32] are constructed via a randomized Cantor iteration, similar to Example 2.3 but with variable numbers of intervals at different stages of the construction. Thus, again, randomness of fractal sets is a substitute for curvature. However, unlike with restriction estimates, the random behaviour of μ is not mediated via Fourier estimates such as (2.5). Instead, we work in the "physical space" and use randomization to ensure the correlation condition (4.8) below. This is somewhat similar to Bourgain's argument in [6], where the crucial geometrical input concerns intersections of pairs of thin annuli.

Specifically, let $E_n \subset [1,2]$ be the n-th iteration of the Cantor construction, $\phi_n = \frac{1}{|E_n|}\mathbf{1}_{E_n}$, and $\sigma_n = \phi_{n+1} - \phi_n$. The correlation condition we require asserts that, for an appropriate range of n depending on p and ϵ, and for "most" choices of translation and dilation parameters c_ℓ, r_ℓ, we have

$$\left| \int \prod_{\ell=1}^{k} \sigma_n\Big(\frac{z-c_\ell}{r_\ell}\Big)\, dz \right| \le C(k,n) \tag{4.8}$$

with $C(k,n)$ decayig exponentially in n. Heuristically, σ_n are highly oscillating random functions with $\int \sigma_n = 0$, so that affine copies of $\sigma_k n$ with generic translation and scaling parameters should be close to orthogonal, leading to massive cancellations in the integral in (4.8).

The condition (4.8) is reminiscent of higher-order uniformity conditions in additive combinatorics (cf. [16, 18]). A calculation from [16] shows that, at least if ϵ is small enough,

(4.8) implies that μ obeys a Fourier decay estimate (2.5) for some (not necessarily optimal) $\beta > 0$; this, however, is not used in the proof of the maximal theorem. At the same time, (4.8) is perfectly compatible with μ being a Salem measure, and it is not difficult to modify the construction in [32] along the lines of [31] to ensure that μ also has that property.

Question 4.5. *Give an explicit, deterministic example of a measure $\mu \in \mathcal{M}([1,2])$, singular with respect to the Lebesgue measure, sich that $\mathfrak{M}$ is bounded on $L^p(\mathbb{R})$ for some $p < \infty$.*

The construction in [32] is random and produces no explicit examples. By the arguments in [32], it would suffice to produce an explicit Cantor iteration for which an appropriate version of (4.8) holds. There are many "pseudorandom" arithmetic sets known in number theory that correlate poorly with their translates, and the hope would be that such sets might be used as a basis for the Cantor iteration. The main obstacle appears to be that the copies of σ_n in (4.8) are not only translated but also dilated, and this makes the correlation condition very difficult to verify for any such explicit sets.

5. Arithmetic patterns in fractal sets

We now turn to Szemerédi-type problems for fractal sets. The general question, vaguely formulated, is as follows: if $E \subset \mathbb{R}^d$ has sufficiently large Hausdorff dimension, must it contain certain specified geometric configurations? If not, what additional assumptions on E are sufficient to guarantee that? This could be viewed as continuous analogues of Szemerédi's theorem on arithmetic progressions in sets of integers of positive upper asymptotic density [51], or of its multidimensional variants [15].

It follows easily from the Lebesgue density theorem (4.3) that any set $E \subset \mathbb{R}^d$ of positive Lebesgue measure contains a similar copy of any finite set F. Erdős [12] conjectured that given any infinite sequence $\{a_n\} \subset \mathbb{R}$, there exists a set E of positive measure which does not contain any non-trivial affine copy of it. Falconer [13] proved this for sequences that decay sufficiently slowly; see also [7, 20, 28, 29] for other related results and examples. The question remains open for faster decaying sequences, such as the geometric sequence $\{2^{-n}\}$.

Our focus here is on finding finite configurations in sets $E \subset \mathbb{R}^g$ of d-dimensional Lebesgue measure 0, but Hausdorff dimension close to d. The simplest question of this type is: given a triple $F = \{x, y, z\}$ of distinct points in $\mathbb{R}$, is it true that any set $E \subset \mathbb{R}$ of dimension α sufficiently close to 1 must contain an affine copy of F? Without additional assumptions on E, the answer is negative, even if $\alpha = 1$. This is due to Keleti [25], who also constructs sets that avoid all "parallelograms" $\{x, x+y, x+z, x+y+z\}$, with $y, z \neq 0$ [25], and sets that avoid all affine copies of infinitely many 3-point configurations [26]. Similar results are known in higher dimensions: for instance, Maga [34] proved that, given a triple $F = \{x, y, z\}$ of distinct points in $\mathbb{R}^2$, there exists a compact set in $\mathbb{R}^2$ with Hausdorff dimension 2 which does not contain any similar copy of F.

Additive combinatorics suggests that sets E that are "pseudorandom" in an appropriate sense should be better behaved with regard to Szemerédi-type phenomena than generic sets of the same size. For example, Szemerédi-type results are available for sets of integers of zero asymptotic density if additional randomness or pseudorandomness conditions are assumed, see e.g. [17, 18, 27]. The nature of such conditions depends on the context and especially on the type of configurations being sought. For 3-term arithmetic progressions in sets of integers, the relevant criterion is *linear uniformity*, expressed in terms of Fourier

analytic estimates [42]; higher order uniformity norms [16] can be used to guarantee the existence of longer arithmetic progressions.

It turns out that Fourier decay estimates of the form (2.5) for fractal measures can indeed serve as analogues of the additive-combinatorial notion of linear uniformity. The following theorem is due to myself and Pramanik [31].

Theorem 5.1. *Let $E \subseteq [0,1]$ be a closed set. Assume that there is a measure $\mu \in \mathcal{M}(E)$ such that:*

$$\mu(B(x,\epsilon)) \leq C_1 \epsilon^{\alpha} \text{ for all } 0 < \epsilon \leq 1 \tag{5.1}$$

$$|\widehat{\mu}(\xi)| \leq C_2 (1+|\xi|)^{-\beta/2} \tag{5.2}$$

with $0 < \alpha < 1$ and $2/3 < \beta \leq 1$. If $\alpha > 1 - \epsilon_0(C_1, C_2, \beta)$, then E contains a 3-term arithmetic progression.

While Theorem 5.1 is stated and proved in [31] only for arithmetic progressions, the same proof works for any fixed 3-point configuration $\{x, y, z\}$. In many cases of interest including Salem measures, (5.2) is satisfied with β arbitrarily close to α. The proof in [31] shows that the dependence of ϵ_0 on β can be dropped from the statement of the theorem if β is bounded from below away from $2/3$, e.g. $\beta > 4/5$, so that in such cases the ϵ_0 in Theorem 5.1 depends only on C_1, C_2.

More recently, in a joint work with Chan and Pramanik [9], we proved a multidimensional analogue of Theorem 5.1. Roughly speaking, we consider certain types of "admissible" finite configurations defined by appropriate systems of matrices. If $E \subset \mathbb{R}^d$ supports a probability measure obeying (5.1) and (5.2) with $\alpha > d - \epsilon_0$, where $\epsilon_0 = \epsilon_0(C_1, C_2, \beta)$ is small enough depending on the configuration in question, then E must contain that configuration. We omit the precise statement, since the definition of admissible configurations is quite lengthy and technical. Instead, we mention a few corollaries of the main theorem of [9].

Corollary 5.2. *Suppose that $E \subset \mathbb{R}^2$ supports a probability measure obeying (5.1) and (5.2), with $\alpha > 2 - \epsilon_0$.*

(a) *Let $d = 2$, and let a, b, c be three distinct points in the plane. If ϵ_0 is small enough depending on the configuration a, b, c, then E must contain three distinct points x, y, z such that the triangle $\triangle xyz$ is a similar (possibly rotated) copy of the triangle $\triangle abc$.*

(b) *Let a, b, c be three distinct colinear points in $\mathbb{R}^d$. If ϵ_0 is small enough depending on a, b, c, then E must contain three distinct points x, y, z that form a similar image of the triple a, b, c.*

Maga's result [34] shows that (a) fails without the assumption (5.2), even if E has Hausdorff dimension 2.

Corollary 5.3. *Let $E \subset \mathbb{R}^d$ be as in Corollary 5.2, with ϵ_0 small enough. Then E contains a parallelogram $\{x, x+y, x+z, x+y+z\}$, where the four points are all distinct.*

Again, this should be compared to a result of Maga [34], which shows that the result is false without the Fourier decay assumption. More complicated examples are also possible, see [9] for details.

Question 5.4. *Is there an analogue of Theorem 5.1 for k-term arithmetic progressions with $k \geq 4$? If so, what are the appropriate higher order uniformity conditions on μ?*

Question 5.5. *The main theorem of [9] provides a class of finite configurations in $\mathbb{R}^d$ that are "controlled" (in the sense of e.g. [16]) by the Fourier transform. Can this class be extended? (The constraints on the various parameters in [9] are unlikely to be optimal.) Is there a characterization of those configurations that are not controlled by the Fourier transform?*

Acknowledgements. The author was supported in part by an NSERC Discovery Grant.

References

[1] Aversa, V. and Preiss, D., *Hearts density theorems*, Real Analysis Exchange, **13**(1) (1987), 28–32.

[2] _____, *Sistemi di derivazione invarianti per affinita*, preprint, Complesso Universitario di Monte S. Angelo, Napoli, 1995.

[3] Bak, J.-G. and Seeger, A., *Extensions of the Stein-Tomas theorem*, Math. Res. Lett., **18** (2011), no. 4, 767–781.

[4] Bluhm, C., *Random recursive construction of Salem sets*, Ark. Mat., **34** (1996), 51–63.

[5] _____, *On a theorem of Kaufman: Cantor-type construction of linear fractal Salem sets*, *Ark. Mat.*, **36** (1998), 307–316.

[6] Bourgain, J., *Averages in the plane over convex curves and maximal operators*, J. Analyse Math., **47** (1986), 69–85.

[7] _____, *Construction of sets of positive measure not containing an affine image of a given infinite structure*, Israel J. Math., **60** (1987), 333–344.

[8] Bourgain, J. and Guth, L., *Bounds on oscillatory integral operators based on multilinear esstimates*, Geom. Funct. Anal., **21** (2011), 1239–1295.

[9] Chan, V., Łaba, I., and Pramanik, M., *Finite configurations in sparse sets*, J. Anal. Math., to appear.

[10] Chen, X., *A Fourier restriction theorem based on convolution powers*, Proc. Amer. Math. Soc., to appear.

[11] _____, *Sets of Salem type and sharpness of the L^2-Fourier restriction theorem*, Trans. Amer. Math. Soc., to appear.

[12] Erdős, P., *Remarks on some problems in number theory*, in: Papers presented at the Fifth Balkan Mathematical Congress (Belgrade, 1974), Math. Balkanica, **4** (1974), 197–202.

[13] Falconer, K.J., *On a problem of Erdős on sequences and measurable sets*, Proc. Amer. Math. Soc., **90** (1984), 77–78.

[14] Fefferman, C., *Inequalities for strongly singular operators*, Acta Math., **124** (1970), 9–36.

[15] Furstenberg, H. and Katznelson, Y., *An ergodic Szemerédi theorem for commuting transformations*, J. Analyse Math., **34** (1979), 275–291.

[16] Gowers, W.T., *A new proof of Szemerédi's theorem*, Geom. Funct. Anal., **11** (2001), 465–588.

[17] Green, B., *Roth's theorem in the primes*, Ann. Math., **161** (2005), 1609–1636.

[18] Green, B. and Tao, T., *The primes contain arbitrarily long arithmetic progressions*, Ann. Math., **167** (2008), 481–547.

[19] Hambrook, K. and Łaba, I., *On the sharpness of Mockenhaupt's restriction theorem*, Geom. Funct. Anal., **23** (2013), 1262–1277.

[20] Humke, P.D. and M. Laczkovich, M., *A visit to the Erdős problem*, Proc. Amer. Math. Soc., **126** (1998), 819–822.

[21] Iosevich, A. and Koh, D., *Extension theorems for paraboloids in the finite field setting*, Math. Z., **266** (2010), 471–487.

[22] Iosevich, A. and Sawyer, E., *Maximal averages over surfaces*, Adv. in Math., **132** (1997), 46–119.

[23] Kahane, J.P., *Some Random Series of Functions*, Cambridge Univ. Press, 1985.

[24] Kaufman, L., *On the theorem of Jarnik and Besicovitch*, Acta Arith., **39** (1981), 265–267.

[25] Keleti, T., *A 1-dimensional subset of the reals that intersects each of its translates in at most a single point*, Real Anal. Exchange, **24** (1998/99), 843–844.

[26] _____, *Construction of one-dimensional subsets of the reals not containing similar copies of given patterns*, Analysis & PDE, **1** (2008), 29–33.

[27] Kohayakawa, Y., Łuczak, T., Rödl, V., *Arithmetic progressions of length three in subsets of a random set*, Acta Arith., **75** (1996), 133–163.

[28] Kolountzakis, M., *Infinite patterns that can be avoided by measure*, Bull. London Math. Soc., **29** (1997), 415–424.

[29] Kómjáth, P., *Large sets not containing images of a given sequence*, Canad. Math. Bull., **26** (1983), 41–43.

[30] Körner, T.W., *On a theorem of Saeki concerning convolution squares of singular measures*, Bull. Soc. Math. France, **136** (2008), 439–464.

[31] Łaba, I. and Pramanik, M., *Arithmetic progressions in sets of fractional dimension*, Geom. Funct. Anal., **19** (2009), 429–456.

[32] _____, *Maximal theorems and differentiation theorems for sparse sets*, Duke Math. J., **158** (2011), 347–411.

[33] Lewko, A. and Lewko, M., *Endpoint restriction estimates for the paraboloid over finite fields*, Proc. Amer. Math. Soc., **140** (2012), 2013–2028.

[34] Maga, P., *Full dimensional sets without given patterns*, Real Anal. Exchange, **36** (2010), 79–90.

[35] Mattila, P., *Geometry of sets and measures in Euclidean spaces*, Cambridge Studies in Advanced Mathematics, vol. 44, Cambridge University Press, 1995.

[36] Mitsis, T., *A Stein-Tomas restriction theorem for general measures*, Publ. Math. Debrecen, **60** (2002), 89–99.

[37] Mockenhaupt, G., *Salem sets and restriction properties of Fourier transforms*, Geom. Funct. Anal., **10** (2000), 1579–1587.

[38] Mockenhaupt, G. and Tao, T., *Restriction and Kakeya phenomena for finite fields*, Duke Math. J., **121** (2004), 35–74.

[39] Nagel, A., Seeger, A., and Wainger, S., *Averages over convex hypersurfaces*, Amer. J. Math., **115** (1993), 903–927.

[40] Phong, D.H. and Stein, E.M., *Hilbert integrals, singular integrals, and Radon transforms I,* Acta Math., **157** (1986), 99–157.

[41] ———, *Hilbert integrals, singular integrals, and Radon transforms II,* Invent. Math., **86** (1986), 75–113.

[42] Roth, K., *On certain sets of integers*, J. London Math. Soc., **28** (1953), 245–252.

[43] Rubio de Francia, J.L., *Maximal functions and Fourier transforms*, Duke Math. J., **53** (1986), 395–404.

[44] Salem, R., *On singular monotonic functions whose spectrum has a given Hausdorff dimension*, Ark. Mat., **1** (1950), 353–365.

[45] Sogge, C. and Stein, E.M., *Averages of functions over hypersurfaces in* $\mathbb{R}^n$, Invent. Math., **82** (1985), 543–556.

[46] ———, *Averages over hypersurfaces II*, Invent. Math., **86** (1986), 233–242.

[47] Stein, E.M., *Maximal functions: Spherical means*, Proc. Nat. Acad. Sci. U.S.A., **73** (1976), 2174–2175.

[48] ———, *Oscillatory integrals in Fourier analysis*, in Beijing Lectures in Harmonic Analysis (E.M. Stein, ed.), Ann. Math., Study 112, Princeton Univ. Press, 1986, pp. 307–355.

[49] ———, *Harmonic Analysis*, Princeton Univ. Press, Princeton, 1993.

[50] Stein, E.M. and Wainger, S., *Problems in harmonic analysis related to curvature*, Bull. Amer. Math. Soc., **8**4 (1978), 1239–1295.

[51] Szemerédi, E., *On sets of integers containing no* k *elements in arithmetic progression*, Acta Arith., **27** (1975), 299–345.

[52] Tao, T., *Recent Progress on the Restriction Conjecture. Fourier analysis and convexity*, Appl. Numer. Harmon. Anal., Birkhauser, Boston, MA, 2004, pp. 217–243.

[53] Tao, T. and Vu, V., *Additive Combinatorics*, Cambridge University Press, 2006.

[54] Tomas, P.A., *A restriction theorem for the Fourier transform*, Bull. Amer. Math. Soc., **81** (1975), 477–478.

[55] ———, *Restriction theorems for the Fourier transform*, in Harmonic Analysis in Euclidean Spaces, G. Weiss and S. Wainger, eds., Proc. Symp. Pure Math., # 35, Amer. Math. Soc., 1979, vol. I, pp. 111–114.

[56] Wolff, T., *Lectures on Harmonic Analysis*, I. Łaba and C. Shubin, eds., Amer. Math. Soc., Providence, R.I., 2003.

Department of Mathematics, University of British Columbia, Vancouver, B.C. V6T 1Z2, Canada
E-mail: ilaba@math.ubc.ca

Mean field equations, hyperelliptic curves and modular forms

Chang-Shou Lin

Abstract. In this paper, we survey some recent joint works with C. L. Chai and C. L. Wang [3, 17], in which we have developed a theory to connect the mean field equation, Green function and Lame equation. In this theory, we have constructed a family of hyperelliptic curves and a premodular form of degree $\frac{1}{2}n(n+1)$ and proved that the nonlinear elliptic PDE on a flat torus E_τ has a solution iff τ is a zero of this premodular form. As a consequence, we show that the Green function of E_τ has either three critical points or five critical points. Furthermore, $\Omega_5 = \{[\tau] \mid G \text{ has five critical points}\}$ is simply-connected.

Mathematics Subject Classification (2010). Primary 35J60; Secondary 58E11.

Keywords. Mean field equation, Liouville equation, Green function, hyperelliptic curve, modular form.

1. Mean field equations and the Green function

Let $E = E_\tau = \mathbb{C}/\Lambda_\tau$ be a two-dimensional flat torus, where $\Lambda_\tau = \mathbb{Z} + \mathbb{Z}\tau$, $\tau \in \mathbb{C}$ and $\mathrm{Im}\tau > 0$. In this lecture, we follow the conventional notations: $\omega_1 = 1$, $\omega_2 = \tau$ and $\omega_3 = 1 + \tau$. Consider the following equation

$$\Delta u + e^u = \rho\delta_0 \quad \text{in } E, \tag{1.1}$$

where $\rho > 0$, $\Delta = \sum_{i=1}^2 \frac{\partial^2}{\partial x_i^2}$ and δ_0 stands for the Dirac measure on E at the lattice point 0. We want to study how the geometry of E is related to equation (1.1).

From the analytic point of view, we could study the geometry of E via the Green function G:

$$\begin{cases} \triangle G = -\delta_0 + \dfrac{1}{|E|} & \text{on } E, \\ \int_E G = 0. \end{cases} \tag{1.2}$$

It is easy to see that G is an even function, i.e. $G(-z) = G(z)$. Let $\omega_i/2$, $i = 1, 2, 3$, be the three half periods. By the evenness, we have

$$G\left(-z + \frac{\omega_i}{2}\right) = G\left(z - \frac{\omega_i}{2}\right) = G\left(z + \frac{\omega_i}{2}\right),$$

where the last identity is due to the double period of G. Thus, G is also an even function

Proceedings of the International Congress of Mathematicians, Seoul, 2014

with respect to any half period, which implies

$$\nabla G\left(\frac{\omega_i}{2}\right)=0,\quad 1\le i\le 3,$$

that is, the Green function G always has three critical points at their three half periods. The connection of (1.1) with the Green function is the following theorem.

Theorem 1.1. *Equation (1.1) with $\rho=8\pi$ has one solution if and only if the Green function has non-half period critical points.*

We refer the proof of Theorem 1.1 to [8, 15]. An immediate question is that does G have critical points rather than the three half periods? For example, if E is a rectangle, i.e., $\tau\in i\mathbb{R}$, then we can prove that G has exactly three critical points. However, if $\tau=\frac{1}{2}+\frac{\sqrt{3}}{2}i=e^{\frac{\pi}{3}i}$, then G has at least an extra pair of critical points at $\pm\frac{1+\tau}{3}$. A deep result is about the exact number of critical points G might have.

Theorem 1.2. *For any torus E, the Green function G has either three critical points or five critical points.*

In order to find critical points of G, we have to know the analytic express of G. Let $\wp(z)$, $\zeta(z)$ and $\vartheta_1(z)$ be the Weierstrass $\wp$-function, $zeta$ function and the Jacobi theta fucntion:

$$\begin{aligned}
&\wp(z)=\frac{1}{z^2}+\sum_{\omega\in\Lambda_\tau\setminus\{0\}}\left(\frac{1}{(z-\omega)^2}-\frac{1}{\omega^2}\right),\\
&\zeta(z)=-\int^z\wp,\quad\text{and}\\
&\vartheta_1(z)=-i\sum_{n=-\infty}^{+\infty}(-1)^n q^{(n+\frac{1}{2})^2}e^{(2n+1)\pi iz},
\end{aligned}$$

where $q=e^{\pi i\tau}$. The $zeta$ function $\zeta(z)$ is an odd meromorphic function in $\mathbb{C}$, satisfying the following quasi-periodic condition:

$$\zeta(z+\omega_i;\tau)=\zeta(z;\tau)+\eta_i(\tau).$$

These quasi-period $\eta_i(\tau)$ is holomorphic in $\tau\in\mathbb{H}=\{\tau\in\mathbb{C}\mid \mathrm{Im}\tau>0\}$, and satisfies the classic Legendre relation:

$$\eta_1\omega_2-\eta_2\omega_1=2\pi i.$$

See [4, 24]. The Green function $G(z;\tau)$ can be expressed in terms of ϑ_1:

$$G(z;\tau)=-\frac{1}{2\pi}\log|\vartheta_1(z)|+\frac{1}{2b}x_2^2+C(\tau),\tag{1.3}$$

where $\tau=a+bi$, $z=x_1+x_2i$ and $C(\tau)$ is a constant which matches the condition $\int_E G(z;\tau)dz=0$. By using the formula $(\log\vartheta_1(z))=\zeta(z)-\eta_1 z$ and the Legendre relation, we have

$$-4\pi G_z(z)=\zeta(z)-t\eta_1-s\eta_2,\tag{1.4}$$

if $z=t\omega_1+s\omega_2$. Here $G_z=\frac{1}{2}(\frac{\partial}{\partial x_1}-i\frac{\partial}{\partial x_2})G$. From (1.4), a critical point $z=t\omega_1+s\omega_2$ of G is a solution to

$$\zeta(t\omega_1+s\omega_2)=t\eta_1+s\eta_2.\tag{1.5}$$

Clearly, if z is a half period or equivalently, a 2-torsion point, then (1.5) holds always. At the first sight, equation (1.5) looks naive. However, the proof of Theorem 1.2 is not so simple. In [15], we actually apply the nonlinear PDE (1.1) to solve the two-dimensional equation (1.5). To explain the idea of the proof, we note that (1.1) is known to be an integrable system in the literature through the Liouville theorem. The Liouville theorem for (1.1) is the claim: For any smooth solution $u(z)$ of (1.1) with $\rho = 8n\pi$, there is a meromorphic function $f(z)$ in $\mathbb{C}$ such that

$$u(z) = \log \frac{8|f'(z)|^2}{(1+|f(z)|^2)^2}, \quad z \in E. \tag{1.6}$$

Furthermore, under some normalization, $f(z)$ satisfies

$$f(z+\omega_j) = e^{2i\theta_j} f(z), \quad z \in E \tag{1.7}$$

for two real constants θ_j. The function $f(z)$ is called a developing map for the solution. Please see [3, 15–17] for a complete account. In the following, we will survey the theory developed in [3, 15] to explore the connection among (1.3), the Green function and the classic Lame equation. We start with the question how to recover $f(z)$ by u. By differentiating (1.6), we obtain

$$u_{zz} - \frac{1}{2}u_z^2 = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f}\right)^2, \tag{1.8}$$

where the right hand side is the Schwartz derivative of f, denoted by $S(f)$. Since u is smooth except at lattice points, the function $u_{zz} - \frac{1}{2}u_z^2$ is meromorphic with the only pole of order 2 at 0. Therefore, (1.8) implies

$$S(f) = -2(\eta(\eta+1)\wp(z) + B), \tag{1.9}$$

where B is a complex constant and $\eta = \rho/(8\pi) = n$. We note that the complex constant B depends on u, thus, it is not known a priori. Classically, η and B are known as the index and the accessary parameter of the Lame equation:

$$y''(z) = (n(n+1)\wp(z) + B)y. \tag{1.10}$$

It has been known since 19th century that for any two linearly independent solutions y_1 and y_2 of (1.10), the ratio $g(z) = \frac{y_2(z)}{y_1(z)}$ satisfies

$$S(g) = -2(\eta(\eta+1)\wp(z) + B).$$

Therefore (1.9) indicates that any developing map $f(z)$ of a solution to (1.1) is a ratio of two linearly independent solutions of the Lame equation (1.10), although B is unknown. This connection plays a fundamentally important role in [3, 4, 17].

The striking consequence of this integrability is the simple observation: If $f(z)$ is a developing map of $u(z)$, satisfying (1.7), then $e^{\lambda}f(z)$ also satisfies (1.7) for any $\lambda \in \mathbb{R}$. Thus (1.6) gives

$$u^{\lambda}(z) = \log \frac{8e^{2\lambda}|f'(z)|^2}{(1+e^{2\lambda}|f(z)|^2)^2}$$

is also a solution of (1.1) with $\rho = 8\pi n$, i. e., once (1.1) with $\rho = 8\pi n$ has one solution, then it has one parameter family of solutions. When $n = 1$, as $\lambda \to \pm\infty$, u^{λ} blow up at the

zero q (or the pole $-q$) of f. By the method from PDE, we can prove that q is a non-half period critical point of G. We could further prove among this family solutions u^λ, there is a unique even solution to (1.1). From the integrability, we conclude that those is a one-to-one correspondence between the non-half period critical points and even solutions to (1.1). Therefore, Theorem 1.2 is equivalent to the following uniqueness theorem for equation (1.1) with $\rho \in (0, 8\pi]$.

Theorem 1.3. *For any $\rho \in (0, 8\pi]$, equation (1.1) has at most one even function. If $\rho \in (0, 8\pi)$, then any solution of (1.1) must be even.*

We remark that for $\rho \in (0, 8\pi)$, equation (1.1) has at least one solution. Hence Theorem 1.3 says that for $\rho \in (0, 8\pi)$, equation (1.1) has a unique solution. However, for $\rho = 8\pi$, (1.3) might have no solutions, but if it has a solution, then there is only one family of solutions. We use the classical Bol-Alexander inequality and the symmetrization to show the uniqueness part if $\rho \geq 4\pi$. For the complete proof, see [15, 17].

It is not really surprised that (1.1) is closely related to the geometry of E. Equation (1.1) is originated from the prescribed curvature problem in conformal geometry. In general, for any compact Riemann surface (M, g) we may consider the following equation:

$$\triangle u + e^u - 2K = 4\pi \sum_{j=1}^{n} \alpha_j \, \delta_{Q_j} \quad \text{on } M, \tag{1.11}$$

where $K(x)$ is the Gaussian curvature of the given metric g at $x \in M$, $Q_j \in M$ are distinct points, and $\alpha_j > -1$ are constants. For any solution $u(x)$ to (1.11), equation (1.11) is equivalent to saying that the Gaussian curvature of the new metric $\tilde{g} := e^v g$ (where $2v = u - \log 2$) has constant Gaussian curvature $\tilde{K} = 1$ outside those Q_j's. Since (1.11) has singular source at Q_j, the conformal metric $e^u g$ degenerates at Q_j and is called a metric on M with conic singularity at those Q_j's.

Equation (1.11) belongs to a general class of equations, the so-called *mean field equations*:

$$\Delta u + \rho \left(\frac{h e^u}{\int h e^u} - \frac{1}{|M|} \right) = 4\pi \sum_{j=1}^{n} \alpha_j \left(\delta_{Q_j} - \frac{1}{|M|} \right) \quad \text{on } M, \tag{1.12}$$

where $h(x)$ is a positive C^1 function on M. Equation (1.12) arises not only from geometry, but also from many applications in physics. For example, it also comes from the statistical physics as the mean field limits of the Euler flow, hence the name. Recently it was shown to be related to the self-dual condensation of the Chern-Simons-Higgs model. We refer the readers to [1, 5–9, 18–20, 22, 23] and the references therein for the recent development on this subject.

2. Mean field equations and modular forms

Fix $(s, t) \in [0, 1] \times [0, 1]$. We denote the function in (1.5) by $Z_{(s,t)}(\tau)$:

$$Z(s, t; \tau) := Z_{(s,t)}(\tau) = \wp(s\omega_1 + t\omega_2; \tau) - s\eta_1(\tau) - t\eta_2(\tau). \tag{2.1}$$

Since $\wp(\cdot; \tau)$, $\eta_1(\tau)$ and $\eta_2(\tau)$ are holomorphic in τ, $Z_{(s,t)}(\tau)$ is holomorphic in the upper half plane $\mathbb{H}$. Indeed, $Z_{(s,t)}(\tau)$ is a "pre-modular" form, that is, $Z_{(s,t)}(\tau)$ is a modular form of degree 1 with respect to $\Gamma(N)$ provided that (s, t) is a N-torsion point, i.e.

$(Ns, Nt) \in \mathbb{N}$, here

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(Z) \;\Big|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}.$$

Historically, $Z_{(s,t)}(\tau)$ was introduced first by Hecke in 1926 [13], where the modularity of $Z_{(s,t)}(\tau)$ was proved. The connection of the Hecke form and the Lame equation was founded in [2, 10, 11]. Later on, it was also due to Hecke that $Z_{(s,t)}$ is identical to the Eisenstein series of degree 1. By the quasi-period of $\zeta(z)$, $Z_{(s,t)}(\tau) \equiv 0$ if (s,t) is a 2-torsion point. Suppose that (s,t) is not a 2-torsion point, then $Z_{(s,t)}$ has a zero at τ if and only if $G(z;\tau)$ has five critical points.

For $\tau \in \mathbb{H}$, we denote by $[\tau]$ either one point in $M = SL_2(Z) \setminus \mathbb{H}$ or the orbit of τ under the action of $SL_2(Z)$. Let

$$\Omega_5 = \{[\tau] \mid G(z;\tau) \text{ has five critical points}\} \quad \text{and} \quad \Omega_3 = M \setminus \Omega_5.$$

Obviously, Ω_3 is the set of $[\tau]$ such that $G(z;\tau)$ has only three critical points.

Theorem 2.1. *The followings hold.*

(i) *the set Ω_5 is open and simply connected.*

(ii) *$\partial\Omega_5 = \{[\tau] \mid$ one of half periods is a degenerate critical point of $G(z;\tau)\}$. Furthermore, $\partial\Omega_5 \approx S^1$, where S^1 is the one-dimensional circle.*

(iii) *$M \setminus \overline{\Omega}_5 = \{[\tau] \mid G(z,\tau)$ has only three critical points and all of them are non-degenerate critical points of $G(z;\tau)\}$. Furthermore, $M \setminus \overline{\Omega}_5$ is also simply connected.*

To study the Hecke form $Z(s,t;\tau)$, we consider $\tau \in F$:

$$F = \left\{ \tau \in \mathbb{H} \;\Big|\; 0 \le \mathrm{Im}\tau \le 1, \;\; |\tau - \frac{1}{2}| \ge \frac{1}{2} \right\}. \tag{2.2}$$

Geometrically, F is a fundamental domain for $\Gamma_0(2)$, where $\Gamma_0(2) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ $\in SL_2(Z) \mid c \equiv 0 \mod 2\}$. It was proved that equation (1.1) with $\rho = 8\pi$ has no solution if $\tau \in \partial F$. Obviously this statement is equivalent to the following: For any (s,t) is not a 2-torsion point,

$$Z_{(s,t)}(\tau) \neq 0 \quad \forall\, \tau \in \partial F. \tag{2.3}$$

Hence if $Z_{(s,t)}(\tau)$ does not tend to 0 as $\tau \to \infty, 0, 1$ for a set of (s,t), then by the argument principle, $Z_{(s,t)}(\tau)$ has the same number of zeros in F for this set of (s,t), counting with the multiplicities. This is a simple, but important application of nonlinear PDE (1.1) to the Hecke form $Z(s,t;\tau)$. As discussed above, we have to study the asymptotic behavior of $Z(s,t;\tau)$ at the three cusps of F, $0, 1, \infty$. For this purpose, we use the following q-expansion of Z:

$$\begin{aligned} Z_{(t,s)}(\tau) =& 2\pi i s - \pi i \frac{1+e^{2\pi i z}}{1-e^{2\pi i z}} \\ & - 2\pi i \sum_{n=1}^{\infty} \left(\frac{e^{2\pi i z} q^n}{1-e^{2\pi i z} q^n} - \frac{e^{-2\pi i z} q^n}{1-e^{-2\pi i z} q^n} \right), \end{aligned} \tag{2.4}$$

where $z = t\omega_1 + s\omega_2$ and $q = e^{\pi i\tau}$.

For any fixed $s \in [0,1)$, (2.4) implies that

$$\lim_{\tau\to\infty} Z_{(t,s)}(\tau) = \begin{cases} 2\pi i(s-\frac{1}{2}) & \text{if } s \neq 0, \\ \pi\cot\pi t & \text{if } s = 0. \end{cases} \tag{2.5}$$

Note that if (s,t) is not a 2-torsion point, then $Z_{(t,s)}(\tau)$ uniformly tend to its limit in any compact set of $[0,1]\times[0,1]\setminus\{(0,0),(\frac{1}{2},0),(0,\frac{1}{2}),(\frac{1}{2},\frac{1}{2})\}$ as $\tau\to+\infty$.

To compute the limit as $\tau\to 0$, we observe that under the involution $\tau\mapsto S\tau := -1/\tau$,

$$\begin{aligned} \zeta(z;-1/\tau) &= \tau\zeta(z\tau;\tau), \\ \eta_1(-1/\tau) &= \tau\eta_2(\tau), \\ \eta_2(-1/\tau) &= -\tau\eta_1(\tau). \end{aligned}$$

Therefore

$$Z_{t,s}(-1/\tau) = \tau Z_{-s,t}(\tau), \tag{2.6}$$

and for $t\in(0,1)$,

$$Z_{t,s}(\tau) = \frac{-1}{\tau}Z_{-s,t}(-1/\tau) = \frac{2\pi i}{\tau}\big(\tfrac{1}{2} - t + o(1)\big) \tag{2.7}$$

as $\tau\to 0$. For $t=0$, a contribution $-\pi\cot\pi s/\tau$ appears as the dominant term instead. For other t, the value is determined by periodicity.

It is also easy to see that under the translation $\tau\mapsto T\tau := \tau+1$,

$$\begin{aligned} \zeta(z;\tau+1) &= \zeta(z;\tau), \\ \eta_1(\tau+1) &= \eta_1(\tau), \\ \eta_2(\tau+1) &= \eta_1(\tau)+\eta_2(\tau). \end{aligned}$$

Thus

$$Z_{t,s}(\tau+1) = Z_{t+s,t}(\tau), \tag{2.8}$$

and for $t+s\in(0,1)$,

$$Z_{t,s}(\tau) = Z_{t+s,s}(\tau-1) = \frac{2\pi i}{\tau-1}\big(\tfrac{1}{2} - (t+s) + o(1)\big) \tag{2.9}$$

as $\tau\to 1$. For $t+s=0$, the dominant term is replaced by $-\pi\cot\pi(t+s)/(\tau-1)$. For general $t+s$, the value is again determined by periodicity.

We summarize the above computations as the following lemma.

Lemma 2.2. *$Z_{(s,t)}(\tau)$ uniformly tends to a non-zero complex number as $\tau\to 0,1$ or ∞ in any compact set (s,t) of $[0,1]\times[0,1]\backslash$the lines $\{s=0,1\}\cup\{t=0,1\}\cup\{s+t=1/2\}$.*

For the proof of Theorem 2.1, we need another lemma.

Lemma 2.3. *For any $\tau\in\mathbb{H}$,*

(i) $\zeta(\frac{3}{4}\omega_1+\frac{1}{4}\omega_2)) \neq \frac{3}{4}\eta_1+\frac{1}{4}\eta_2$.

(ii) $\zeta(\frac{1}{6}\omega_1+\frac{1}{6}\omega_2)) \neq \frac{1}{6}\eta_1+\frac{1}{6}\eta_2$.

(iii) $\zeta(\frac{2}{6}\omega_1+\frac{3}{6}\omega_2))\neq\frac{2}{6}\eta_1+\frac{3}{6}\eta_2$.

Proof. We will use the addition formula

$$\frac{\wp'(z)}{\wp(z)-\wp(u)}=\zeta(z+u)+\zeta(z-u)-2\zeta(z).$$

For (i), we choose $z=\frac{1}{4}(3\omega_1+\omega_2)=\frac{1}{2}\omega_1+\frac{1}{4}\omega_3$ and $u=\frac{1}{4}\omega_3$. Then $\zeta(z-u)=\zeta(\frac{1}{2}\omega_1)=\frac{1}{2}\eta_1$ and $\zeta(z+u)=\zeta(\omega_1+\frac{1}{2}\omega_2)=\eta_1+\frac{1}{2}\eta_2$. Hence

$$\begin{aligned}\zeta(\tfrac{3}{4}\omega_1+\tfrac{1}{4}\omega_2))-(\tfrac{3}{4}\eta_1+\tfrac{1}{4}\eta_2)&=\zeta(z)-\tfrac{1}{2}(\zeta(z+u)+\zeta(z-u))\\&=-\frac{1}{2}\frac{\wp'(\frac{3}{4}\omega_1+\frac{1}{4}\omega_2)}{\wp(\frac{3}{4}\omega_1+\frac{1}{4}\omega_2)-\wp(\frac{1}{4}\omega_3)}\neq 0.\end{aligned}$$

This proves (i).

For (ii), we choose $z=\frac{1}{6}(\omega_1+\omega_2)=\frac{1}{6}\omega_3$ and $u=\frac{1}{3}\omega_3$. Then

$$\begin{aligned}0\neq\frac{\wp'(z)}{\wp(z)-\wp(u)}&=\zeta(\tfrac{1}{2}\omega_3)+\zeta(-\tfrac{1}{6}\omega_3)-2\zeta(\tfrac{1}{6}\omega_3)\\&=-3(\zeta(\tfrac{1}{6}\omega_1+\tfrac{1}{6}\omega_2)-\tfrac{1}{6}\eta_1-\tfrac{1}{6}\eta_2).\end{aligned}$$

This proves (ii).

For (iii), we choose $z=\frac{1}{3}\omega_1+\frac{1}{2}\omega_2$ and $u=\frac{1}{3}\omega_1$. Then $\wp'(z)\neq 0$ and

$$\begin{aligned}0&\neq\zeta(\tfrac{2}{3}\omega_1+\tfrac{1}{2}\omega_2)+\zeta(\tfrac{1}{2}\omega_2)-2\zeta(\tfrac{1}{3}\omega_1+\tfrac{1}{2}\omega_2)\\&=\zeta(-\tfrac{1}{3}\omega_1-\tfrac{1}{2}\omega_2)+(\eta_1+\eta_2)+\tfrac{1}{2}\eta_2-2\zeta(\tfrac{1}{3}\omega_1+\tfrac{1}{2}\omega_2)\\&=-3(\zeta(\tfrac{1}{3}\omega_1+\tfrac{1}{2}\omega_2)-\tfrac{1}{3}\eta_1-\tfrac{1}{2}\eta_2).\end{aligned}$$

This proves (iii). □

Now we are in the position to prove Theorem 2.1. Since $Z_{(t,s)}(\tau)=0$ iff $Z_{(1-s,1-t)}(\tau)=0$, we might assume $(t,s)\in[0,1]\times[0,\frac{1}{2}]$ in the following proof.

Theorem 2.4. *Suppose that* $(t,s)\in[0,1]\times[0,\frac{1}{2}]\backslash\{(0,0),(\frac{1}{2},0),(0,\frac{1}{2}),(\frac{1}{2},\frac{1}{2})\}$. *Then* $Z_{t,s}(\tau)=0$ *has a solution* $\tau\in\mathbb{H}$ *if and only if that*

$$(t,s)\in\triangle:=\{(t,s)\mid 0<t,s<\tfrac{1}{2},\ t+s>\tfrac{1}{2}\}.$$

Moreover, the solution $\tau\in F$ *is unique for any* $(t,s)\in\triangle$.

Proof. We separate the proof into three steps.

Step 1. We will show that $Z_{t,s}(\tau)$ has no solutions if $(t,s)\notin\overline{\triangle}$.

Indeed, if $s,t,t+s\neq\frac{1}{2}$, then (2.5), (2.7) and (2.9) imply that

$$Z_{t,s}(\tau)\not\to 0\quad\text{as }\tau\to\infty,0,1$$

respectively. Furthermore, the pole order at $\tau=0,1$ is unchanged among such (t,s)'s.

Thus an extended version of the argument principle shows that the number of zero of $Z_{t,s}(\tau)$ is constant in the region

$$\triangle_1:=\{(t,s)\mid t>0,s>0,t+s<\tfrac{1}{2}\}.$$

By Lemma (2.3) (ii), $Z_{1/6,1/6}(\tau)$ has no solutions. Since $(\frac{1}{6},\frac{1}{6}) \in \triangle_1$, this implies that $Z_{t,s}(\tau)$ has no solutions for any $(t,s) \in \triangle_1$.

Similarly $Z_{t,s}(\tau)$ has no solutions for $(t,s) \in \square$, where

$$\square := \{(t,s) \mid \tfrac{1}{2} < t < 1 \text{ and } 0 < s < \tfrac{1}{2}\}.$$

This follows from Lemma (2.3) (i) and the fact that $(\frac{3}{4},\frac{1}{4}) \in \square$.

Step 2. $Z_{t,s}(\tau)$ has no solutions if $(t,s) \notin \triangle$.

Indeed, it follows easily form the argument principle that the points (t,s) such that $Z_{t,s}(\tau)$ has only finite solutions form an open set. In particular, by Step 1, for $(t,s) \in \overline{\triangle_1} \cup \square$, the function $Z_{t,s}(\tau)$ either has no solutions or has infinite solutions (which corresponds to the trivial case $t,s \in \frac{1}{2}\mathbb{Z}$ and $Z_{t,s} \equiv 0$).

Step 3. In order to conclude the proof of the theorem, by the same reasoning as in Step 1 we only need to establish the existence and uniqueness of solution $Z_{t,s}(\tau) = 0$ in $\tau \in F$ for one special point $(t,s) \in \triangle$. For this purpose we take $(t,s) = (\frac{1}{3},\frac{1}{3}) \in \triangle$.

By an easy symmetry argument (c.f. [15]), $Z_{\frac{1}{3},\frac{1}{3}}(\tau) = 0$ for $\tau = \rho := e^{\pi i/3}$. Conversely we will prove that $\rho \in F$ is the unique zero of $Z_{\frac{1}{3},\frac{1}{3}}$ and it is a simple zero. The following argument motivated by [2, 13] is the only place where the theory of modular forms is used.

Define

$$Z_{(3)}(\tau) = \prod{}' Z_{\frac{k_1}{3},\frac{k_2}{3}}(\tau),$$

where the product is over all pairs (k_1,k_2) with $0 \le k_1,k_2 \le 2$ and with $\gcd(k_1,k_2,3) = 1$. In this case it simply means $(k_1,k_2) \neq (0,0)$. There are 8 factors in the product and in fact $Z_{(3)}$ is a modular function of weight 8 with respect to the full modular group $\mathrm{SL}(2,\mathbb{Z})$. The counting formula for the zeros of $Z_{(3)}$ then reads as

$$\nu_\infty(Z_{(3)}) + \frac{1}{2}\nu_i(Z_{(3)}) + \frac{1}{3}\nu_\rho(Z_{(3)}) + \sum_{p\neq\infty,i,\rho} \nu_p(Z_{(3)}) = \frac{8}{12}.$$

Since $Z_{\frac{1}{3},\frac{1}{3}}(\rho) = Z_{\frac{2}{3},\frac{2}{3}}(\rho) = 0$, we have $\nu_\rho(Z_{(3)}) \ge 2$. The counting formula then implies that all the other terms vanish and $\tau = \rho$ is a simple (and unique) zero for $Z_{\frac{1}{3},\frac{1}{3}}(\tau)$ (as well as for $Z_{\frac{2}{3},\frac{2}{3}}(\tau)$).

The proof of the theorem is complete. □

Corollary 2.5. *The set $\widetilde{\Omega}_5 = \{\tau \in F \mid G(z;\tau)$ has five critical points$\}$ is an unbounded simply connected domain.*

Proof. Let $\tilde{\Omega}_5$ be the lifting of Ω_5 in F. The theorem establishes a continuous map $\phi : (s,t) \mapsto \tau$ from $\triangle$ onto $\tilde{\Omega}_5$. The map ϕ is one to one due to the uniqueness theorem of extra pair (non half-period points) of critical points of the Green function G proved in [15]. Being the continuous image of a simply connected domain $\triangle$ under a one to one continuous function ϕ on $\mathbb{R}^2$, $\tilde{\Omega}_3$ must also be a simply connected domain. (This is the classic result on "Invariance of Domain" proved in algebraic topology. In the current case it follows easily from the inverse function theorem since ϕ is differentiable.)

It is also proven in [15] that the domain $\tilde{\Omega}_5$ contains the vertical line $\frac{1}{2} + ib$ for $b \ge b_1$ where $b_1 \in (1/2, \sqrt{3}/2)$, hence it is unbounded.

The corresponding statement for Ω_5 follows from the obvious $\mathbb{Z}_3$ identification. □

Clearly, Corollary 2.5 implies (i) of Theorem 2.1. For other parts of Theorem 2.1, we refer [17]. Indeed, in [17] we have also discussed the minimality or the degeneracy of critical points of G. In particular, we can prove the following deep result:

While deforming τ, G could bifurcate its critical points only at the half periods. (2.10)

3. Hyperellptic curves and their modular forms

As we discussed in Section 1, equation (1.1) is originated from the curvature problem in two-dimensional space. There is another relation with the d-dimensional complex Monge-Ampere equaiton:

$$\det\left(\frac{\partial^2\omega}{\partial z_i\partial z_i}\right)=e^{-w}\quad\text{on }\ (E\setminus\{0\})^d, \tag{3.1}$$

the d-th Cartesian product of $E\setminus\{0\}$. For any solution u to (1.1), the function

$$\omega(z_1,\cdots,z_n)=-\sum_{i=1}^{d}u_i(z_i)+d\log 4 \tag{3.2}$$

satisfies (3.1) with a logarithmic singularity along the normal crossing divisor $D=E^d\setminus(E\setminus\{0\})^d$. In particular, bubbling solutions to (3.1) give examples of bubbling solutions to the complex Monge-Ampere equation (3.1). Those bubbling behaviors of (3.1) have never been studied in the literature.

The above discussions indicate the fundamental importance to study the concentration phenomena of bubbling solutions to (1.1) in details. Indeed, this is the heart inside the connection between (1.1) and the Green function $G(z;\tau)$ which we want to explore more. Suppose taht $\{u_k\}$ is a sequence of blow-up solutions to (1.1) with $\rho=\rho_k$, $\rho_k\to 8\pi n$, $n\in\mathbb{N}$. Then the blow-up set $\{a_1,\cdots,a_n\}$ of u_k must satisfies

$$n\nabla G(a_i)=\sum_{j=1,\neq i}^{n}\nabla G(a_i-a_j),\quad i=1,\cdots,n. \tag{3.3}$$

See [5] for a proof. For $n=1$, (3.3) is the equation for critical points of G. Like the case $n=1$, (3.3) is not a system of algebraic equations.

Theorem 3.1. *Let $a=(a_1,\cdots,a_n)\in(E\setminus\{0\})^n$ be a solution to (3.3), then either $\{a_1,\cdots,a_n\}\cap\{-a_1,\cdots,-a_n\}=\emptyset$ or $\{a_1,\cdots,a_n\}=\{-a_1,\cdots,-a_n\}$. Moreover, (3.3) is equivalent to*

$$\sum_{j=1}^{n}\nabla G(a_j)=0, \tag{3.4}$$

and the following holomorphic system

$$\sum_{j=1,\neq i}^{n}(\zeta(a_i-a_j)+\zeta(a_j)-\zeta(a_i))=0,\qquad i=1,\cdots,n. \tag{3.5}$$

For equation (3.5), we could introduce a hyper-elliptic curve $\overline{Y}_n(\tau)$ for each τ as follows. Let

$$Y_n = \{\,(a_1,\cdots,a_n) \mid a_i \in E\setminus\{0\} \text{ for all } i,\ a_i - a_j \neq 0 \text{ for all } i\neq j, \\ \text{and } (a_1,\cdots,a_n) \text{ satisfies (3.5)}\,\},$$

and $\bar{Y}_n$ be the closure of Y_n in $\mathrm{Sym}^n E = E^n/S_n$, the n-th symmetric product of E. Then

$$\bar{Y}_n = Y_n \cup \{(0,\cdots,0)\}$$

The hyper-ellipticity comes from the map $B : Y_n \to \mathbb{C}$

$$a \longrightarrow B(a) = (2n-1)\sum_{i=1}^{n} \wp(a_i), \tag{3.6}$$

which is two to one from $X_n \subset Y_n \to \mathbb{C}$, where

$$X_n = \{(a_1,\cdots,a_n) \in Y_n \mid \{a_1,\cdot,a_n\} \cap \{-a_1,\cdots,-a_n\} = \emptyset\}.$$

Furthermore, $Y_n \setminus X_n$ consists of $2n+1$ points (counting with multiplicities), and each of them is called a branch point of Y_n's. Thus, $\overline{X}_n = \overline{Y}_n$. Furthermore, it can be parameterized by

$$Y_n \simeq \{(B,C) \mid C^2 = l_n(B)\},$$

where $l_n(B)$ is a polynomial of degree $2n+1$ and B is given by (3.6). For the details of the above statements, we refer the readers to [3]. Those branch points correspond to the zeros of $l_n(B)$ under this parametrization. Therefore, Theorem 3.1 says that $(a_1,\cdots,a_n)$ is a solution of (3.3) iff $a \in Y_n$ and satisfies (3.4). We remark that any branch point a of Y_n automatically satisfies (3.4). Thus, (3.3) has $2n+1$ solutions at least. A solution a of (3.3) is called non-trivial if a is not a branch point of $Y_n(\tau)$. Then we have the following result. See [3] for the proof.

Theorem 3.2. *Equation (1.1) with $\rho = 8\pi n$ has a solution iff (3.3) has a nontrivial solution.*

Remark 3.3. Let us take the example $n=2$. We recall the additional formula

$$\frac{1}{2}\frac{\wp'(z)-\wp'(u)}{\wp(z)-\wp(u)} = \zeta(z+u)-\zeta(z)-\zeta(u) \tag{3.7}$$

provided that $\wp(z) \neq \wp(u)$, i.e. $z \neq \pm u$.

If $a=(a_1,a_2) \in Y_2$ is not a branch point, then (3.5) is equivalent to

$$\wp'(a_1)+\wp'(a_2)=0, \quad a_1 \neq a_2. \tag{3.8}$$

Thus a nontrivial solution a of (3.3) is equivalent to both a_i are not half periods, and satisfy

$$\wp'(a_1)+\wp'(a_2)=0, \quad \nabla G(a_1)+\nabla G(a_2)=0 \quad \text{and} \quad a_1 \neq a_2. \tag{3.9}$$

Hence Theorem 3.2 says that equation (1.1) with $\rho = 16\pi$ has a solution iff (3.9) has a solution.

If (a_1, a_2) is a branch point, then either both a_i are half periods or $(a_1, a_2) = (q, -q)$ such that

$$q \text{ is not a half period and } \zeta(2q) - 2\zeta(q) = 0. \tag{3.10}$$

By (3.7), we can deduce that (3.10) is equivalent to $\wp''(q) = 0$, i.e. $\wp(q) = \pm\sqrt{g_2/3}$. We denote $q_\pm$ to be $\wp(q_\pm) = \pm\sqrt{g_2/3}$. Hence $(\frac{w_i}{2}, \frac{w_j}{2})$, $i \neq j$, and $(q_\pm, -q_\pm)$ are five saddle points of $Y_2(\tau)$. From the information, we could reconstruct the polynomial $l_2(B)$, which is the defining function of Y_2. Because the five zeros of $l_2(B)$ are those corresponding branch points. Hence the five zeros of $l_2(B)$ are $-3e_k$, $k = 1, 2, 3$, and $\pm\sqrt{12g_2}$. Therefore,

$$l_2(B) = (B^2 - 12g_2)(B^3 - g_2B + 27g_3).$$

For (3.3) with $n \geq 2$, we want to study the solution structure during the process of doforming τ. We conjecture:

$$\begin{aligned} &\textit{In the deforming process, equation (3.3) could bifurcate its} \\ &\textit{solutions only at the trivial solutions}. \end{aligned} \tag{3.11}$$

For $n = 1$, it is identical to (2.10) in Section 2. For our purpose, we will construct a "pre-modular" form $Z_n(\sigma;\tau)$ of degree $\frac{n(n+1)}{2}$, which is naturally associated with $\overline{Y}_n(\tau)$ and $\sigma = \sum_{i=1}^{n} a_i$, $(a_1, \cdots, a_n) \in Y_n$. When $n = 1$, $Z_1(\sigma;\tau)$ is the old Heche form. The function $Z_n(\sigma;\tau)$ is called *pre-modular* because $Z_n(\sigma;\tau)$ is a modular form with respect to $\Gamma(N)$ if σ is N-torsion point. The most important property of $Z_n(\sigma;\tau)$ is the following:

$$\begin{aligned} &a = (a_1, \cdots, a_n) \textit{ is a non-trivial solution of (3.3) with } E = E_\tau \\ &\textit{iff } \tau \textit{ is a zero of } Z_n(\sigma;\tau) \textit{ where } \sigma = \sum_{i=1}^{n} a_i. \end{aligned} \tag{3.12}$$

Our conjecture about Z_n is

$$\begin{aligned} &\textit{Suppose } \sigma \textit{ is not a 2-torsion point. Then any zero } \tau \textit{ of } Z_n(\sigma;\tau) \\ &\textit{is simple}. \end{aligned} \tag{3.13}$$

Obviously, (3.13) implies (3.11). For $N = 2, 3$, the premodular form $Z_n(\sigma;\tau)$ was explicitly constructed by S. Dahmen [10, 11]

$$Z_2(\sigma;\tau) = Z^3 - 3\wp Z - \wp',$$

and

$$Z_3(\sigma;\tau) = Z^6 - 15\wp Z^4 - 20\wp' Z^3 + (\frac{27}{4}g_2 - 45\wp^2)Z^2 - 12\wp'\wp Z - \frac{5}{4}(\wp')^2,$$

where $Z =$ the Hecke form$Z(\sigma;\tau)$, $\wp = \wp(\sigma;\tau)$ and $\wp' = \wp'(\sigma;\tau)$. The construction of Z_2 and Z_3 are based on the addition formula and a classical cubic formula. See [10]. However, for $n \geq 4$, his method can not work and it requires more techniques from algebraic geometry such as cubic theorem and other methods. For details, we refer the readers to [17].

References

[1] D. Bartolucci and G. Tarantello, *Liouville type equations with singular data and their applications to periodic multi-vortices for the electro-weak theory*, Comm. Math. Phys. **229** (2002), 3–47.

[2] F. Beukers and A.V.D. Waall, *Lamé equations with algebraic solutions*, J. Diff. Equation **197** (2004), 1–25.

[3] C.-L. Chai, C.-S. Lin, and C.-L. Wang, *Mean field equations, hyperelliptic curves, and modular forms: I*, preprint, 2013.

[4] ——, *Algebraic aspects of mean field equations with multiple singular sources*, preprint, 2014.

[5] C.-C. Chen and C.-S. Lin, *Sharp estimates for solutions of multi-bubbles in compact Riemann surfaces*, Comm. Pure Appl. Math. **55** (2002), 728–771.

[6] ——, *Topological degree for a mean field equation on a Riemann surface*, Comm. Pure Appl. Math. **56** (2003), 1667–1727.

[7] ——, *Topological degree for a mean field equation with singular sources*, to appear in CPAM.

[8] C.-C. Chen, C.-S. Lin, and G. Wang, *Concentration phenomenon of two-vortex solutions in a Chern–Simons model*, Ann. Scuola Norm. Sup. Pisa CI. Sci. (5) Vol. III (2004), 367–379.

[9] K. Choe, *Asymptotic behavior of condensate solutions in the Chern–Simons–Higgs theory*, J. Math. Phys. **48**, (2007), 103501.

[10] S. Dahmen, *Counting integral Lamé equations with finite monodromy by means of modular forms*, Master Thesis, Utrecht University 2003.

[11] ——, *Counting integral Lamé equations by means of dessins d'enfants*, Trans. Amer. Math. Soc. **359**, no. 2 (2007), 909–922.

[12] G.-H. Halphen, *Traité des Fonctions Elliptique II*, 1888.

[13] E. Hecke, *Zur Theorie der elliptischen Modulfunctionen*, Math. Ann. **97** (1926), 210–242.

[14] S. Lang, *Elliptic Functions*, Graduate Text in Mathematics **112**, Springer–Verlag 1987.

[15] C.-S. Lin and C.-L. Wang, *Elliptic functions, Green functions and the mean field equations on tori*, Annals of Math. **172** (2010), no.2, 911–954.

[16] ——, *A function theoretic view of the mean field equations on tori*, in "Recent advances in geometric analysis", 173–193, Adv. Lect. Math. 11, Int. Press, Somerville MA, 2010.

[17] ——, *Mean field equations, hyperellptic curves and modular forms, Part II*, preprint, 2014.

[18] C.-S. Lin and S. Yan, *Bubbling solutions for relativistic abelian Chern–Simons model on a torus*, Comm. Math. Phys. **297** (2010), no. 3, 733–758.

[19] ——, *Existence of bubbling solutions for Chern–Simons model on a torus*, Arch. Ration. Mech. Anal. bf 207 (2013), no. 2, 352–392.

[20] ——, *On the self-dual condensate of the Chern–Simons–Higgs model, Part I*, preprint 2014.

[21] D. Mumford, *Abelian Varieties, 2nd ed*, Oxford University Press, 1974.

[22] M. Nolasco and G. Tarantello, *Double vortex condensation in the Chern–Simons–Higgs theory*, Calc. Var. Partial Diff. Equ. **9** (1999), 31–94.

[23] M. Nolasco and G. Tarantello, *Vortex condensates for the $SU(3)$ Chern–Simons theory*, Comm. Math. Phys. **213** (2000), 599–639.

[24] E.T. Whittaker and G.N. Watson, *A Course of Modern Analysis*, 4th edition, Cambridge University Press, 1927.

Taida Institute of Mathematical Sciences (TIMS) and Center for Advanced Studies in Theoretic Sciences (CASTS), National Taiwan University, Taipei

E-mail: cslin@math.ntu.edu.tw

Liouville equations from a variational point of view

Andrea Malchiodi

Abstract. After discussing the role of Liouville equations in both Conformal Geometry and Mathematical Physics, we will explore some of their variational features. In particular we will show the role of the Moser-Trudinger inequality, as well as of some of its improved versions, in characterizing the Euler-Lagrange energy levels of the problems under interest. This description reduces the study of PDEs of Liouville type to topological properties of explicit finite-dimensional objects.

Mathematics Subject Classification (2010). Primary 35J20, 35J75; Secondary 35Q40, 53A30.

Keywords. Liouville equations, variational methods, conformal geometry, singular PDEs.

1. Introduction

Liouville equations are of elliptic type and involve exponential nonlinear terms. They play a prominent role in the study of problems in Conformal Geometry and in Mathematical Physics.

In Conformal Geometry they rule the transformation law for Gaussian curvature: considering a compact Riemannian surface (Σ, g) with Gaussian curvature K_g and a conformal metric on Σ written as $\tilde{g} = e^{2w} g$, K_g transforms according to the law

$$-\Delta_g w + K_g = K_{\tilde{g}} e^{2w}. \tag{1.1}$$

Here $K_{\tilde{g}}$ stands for the Gaussian curvature of the new metric $\tilde{g}$.

Natural questions associated to (1.1) are the classical *Uniformization Problem*, namely to get constant Gaussian curvature from an arbitrary surface, or the *Kazdan-Warner* or *Nirenberg problem*, which consists in prescribing $K_{\tilde{g}}$ as a given function on Σ. There is indeed also natural higher-order counterpart of (1.1) in four dimensions (in fact, in all even dimensions), which concerns the prescription of *Q-curvature*: we will discuss this specific problem in Section 2.

Liouville equations also appear in models from Mathematical Physics, for example in the description of mean field vorticity in steady flows ([12, 20]), Chern-Simons vortices in superconductivity or Electroweak theory ([55, 59]). On compact surfaces (the torus would for example model a periodic system in the plane) one is led to study equations of the form

$$-\Delta_g u = \rho \left(\frac{h(x) e^{2u}}{\int_\Sigma h(x) e^{2u} dV_g} - a(x) \right), \tag{1.2}$$

which include (1.1) as well. Here ρ is a positive parameter, $a, h : \Sigma \to \mathbb{R}$ two smooth functions, with $h(x)$ strictly positive on Σ.

Proceedings of the International Congress of Mathematicians, Seoul, 2014

Solutions to (1.2) can be found as critical points of the Euler-Lagrange energy $I_\rho : W^{1,2}(\Sigma) \to \mathbb{R}$ defined by

$$I_\rho(u) = \int_\Sigma |\nabla_g u|^2 dV_g + 2\rho \int_\Sigma a(x) u \, dV_g - \rho \log \int_\Sigma h(x) e^{2u} dV_g. \tag{1.3}$$

We recall that in two dimensions the Sobolev space $W^{1,2}(\Sigma)$ embeds into every L^p space: however the embedding can be pushed up to exponential class. Indeed one has the well-known Moser-Trudinger inequality ([48, 57])

$$\log \int_\Sigma e^{2(u-\overline{u})} dV_g \leq \frac{1}{4\pi} \int_\Sigma |\nabla_g u|^2 dV_g + C_{\Sigma,g}, \tag{1.4}$$

where $\overline{u} = \fint_\Sigma u \, dV_g$ stands for the average of u on Σ. The latter inequality makes indeed the energy I_ρ well defined, and actually also Frechet-differentiable with continuous derivative.

Using (1.4) and Poincaré's inequality, together with the Direct Methods of the Calculus of Variations, one finds immediately existence of minimizers of I_ρ for ρ sufficiently small.

Proposition 1.1. *If $\rho < 4\pi$ I_ρ is coercive (i.e. $I_\rho(u) \to +\infty$ as $\|u\| \to +\infty$) and admits a minimizer, which is a solution of* (1.9).

If $\rho = 4\pi$ then I_ρ is still lower bounded but coercivity is lost, and hence minimizers might not exist. This case is analysed in [28, 51], where the asymptotic behaviour of miniminzers of I_{ρ_n}, $\rho_n \nearrow 4\pi$, is studied.

We are mainly interested here in cases for which a lower bound on the Euler-Lagrange energy is not available ($\rho > 4\pi$ in (1.2)), and to describe some general methods to attack the problem under these circumstances. We will see in particular the role of some improved versions of (1.4) in terms of the distribution over the surface of the *conformal volume* e^{2u}. There are two well-known results in this spirit from the 70's. The first, due to J.Moser ([49]), asserts that on the standard sphere one gains a better constant on antipodally symmetric functions, namely

$$\log \int_{S^2} e^{2(u-\overline{u})} dV_{g_{S^2}} \leq \frac{1}{8\pi} \int_{S^2} |\nabla_{g_{S^2}} u|^2 dV_{g_{S^2}} + C_{S^2} \qquad \text{for every } u \text{ even}.$$

The second is due to T.Aubin, [2], who proved that for every positive ε there exists $C_\varepsilon > 0$ such that

$$\log \int_{S^2} e^{2(u-\overline{u})} dV_{g_{S^2}} \leq \frac{1+\varepsilon}{8\pi} \int_{S^2} |\nabla_{g_{S^2}} u|^2 dV_{g_{S^2}} + C_\varepsilon \qquad \text{for every } u \text{ balanced}.$$

A function defined on S^2 is said to be balanced provided $\int_{S^2} e^{2u} x_i dV_{g_{S^2}} = 0$ for all $i = 1, 2, 3$, where x_i stands for the restriction of the i-th coordinate of $\mathbb{R}^3$ to S^2. Such a result was extended to arbitrary surfaces by W.Chen and C.Li in [22] provided the conformal volume e^{2u} is sufficiently *spread* over Σ, in a proper quantitative way. The argument relies on localizing the Moser-Trudinger inequality near each set supporting a finite part of the volume. We will see in Section 2 (specifically, for the Q-curvature equation, which possesses analogous variational features) how to exploit this kind of improved inequalities in order to characterize the volume accumulation for functions with low Euler-Lagrange energy. More precisely, if the parameter ρ is between the k-th and the $(k+1)$-th multiple

of the threshold energy, 4π, and if $I_\rho(u)$ is largely negative, then one can show that e^{2u}, normalized in $L^1(\Sigma)$, concentrates near at most k points of Σ. From a distributional point of view, this means that the normalization of e^{2u} resembles one of the following probability measures

$$\Sigma_k = \left\{ \sum_{j=1}^{k} t_j \delta_{x_j} \,:\, \sum_{j=1}^{k} t_j = 1,\ x_j \in \Sigma \right\}, \tag{1.5}$$

where δ_{x_j} stands for the Dirac delta at x_j. It turns out that one could also construct test functions modelled on this set for which the Euler-Lagrange energy is arbitrarily low (if ρ is still in the above range). Somehow, it is possible to construct maps going back and forth from Σ_k to the low energy levels of I_ρ. This fact together with a topological information on Σ_k, namely its non-contractibility, makes it possible to use min-max methods to prove existence of solutions to (1.2) for generic values of ρ on any surface.

Remark 1.2. Another approach to the study of (1.9) relies on computing the Leray-Schauder degree of the equation (see [23, 24, 39]). In some cases the degrees of different solutions might cancel and a Morse-theoretical approach would give more precise information on their structure. For more details and a comparison of the two approaches we refer to [44]. See also [26] for some multiplicity results.

We turn next to a singular version of (1.2), namely

$$-\Delta_g u = \rho \left(h(x) e^{2u} - a(x) \right) - 2\pi \sum_{j=1}^{m} \alpha_j \delta_{p_j}. \tag{1.6}$$

Here ρ is again a positive parameter and $a, h : \Sigma \to \mathbb{R}$ two smooth functions with $h(x) > 0$ for every $x \in \Sigma$. The p_j's are given points of Σ and α_j are real numbers.

In Chern-Simons theory singular sources as in (1.6) represent *vortices* in the physical system, namely points where the material described in the model would have null conductivity. In this case the numbers α_j would be integer, and would stand for the vanishing order of the scalar field. Singularities in the form of Dirac masses also appear in fluid dynamics, see for example [58].

In Conformal Geometry instead, (1.6) represents a singular version of (1.1), in which one prescribes a conical geometry at the points p_j. Precisely, if one could solve (1.6), the conical angle at p_j would be given by $2\pi(1 + \alpha_j)$, so in this geometric context it is interesting to consider any $\alpha_j > -1$. Negative values of α correspond to *standard cones*, while positive α's endow Σ with an orbifold structure. In general, the formal sum $\underline{\alpha} := \sum_j \alpha_j p_j$ is referred to as a *divisor*, and encodes the singular structure of the desired conformal metric.

Problem (1.6) presents in general more difficulties compared to (1.2), as there are well-known (*Kazdan-Warner type*) obstructions to the existence of solutions. On the sphere these obstructions are based on integrations by parts arguments, (see also [42]), but there are more recent examples on surfaces of positive genus as well, see [13].

To attack the existence problem it is convenient to *desingularize* the equation, considering the Green's function of the Laplacian with pole at $p \in \Sigma$, namely the solution to

$$-\Delta_g G_p(x) = \delta_p - \frac{1}{|\Sigma|} \quad \text{on } \Sigma, \qquad \text{with} \quad \int_\Sigma G_p(x)\, dV_g = 0. \tag{1.7}$$

Making the substitution

$$u \mapsto u + 2\pi \sum_{j=1}^{m} \alpha_j G_{p_j}(x), \qquad h(x) \mapsto \tilde{h}(x) = h(x) e^{-2\pi \sum_{j=1}^{m} \alpha_j G_{p_j}(x)} \tag{1.8}$$

(1.2) becomes equivalent to

$$-\Delta_g u = \rho \left(\tilde{h}(x) e^{2u} - \tilde{a}(x) \right) \qquad \text{on } \Sigma. \tag{1.9}$$

Here $\tilde{a}(x)$ is a smooth function on Σ, while $\tilde{h}$ is such that

$$\tilde{h} > 0 \text{ on } \Sigma \setminus \cup_j \{p_j\}; \qquad \tilde{h}(x) \simeq \gamma_j d(x, p_j)^{2\alpha_j} \quad \text{near } p_j \tag{1.10}$$

for some constant $\gamma_j > 0$, where $d(\cdot, \cdot)$ stands for the distance induced by g. The singular behaviour of $\tilde{h}$ is determined by the asymptotic profile of the Green's function near the poles $G_{p_j}(x) \simeq \frac{1}{2\pi} \log \frac{1}{d(x,p_j)}$. Although completely equivalent to the original problem, the advantage of (1.9) consists in inheriting a variational structure: in fact its solutions are critical points of the energy

$$\tilde{I}_\rho(u) = \int_\Sigma |\nabla_g u|^2 dV_g + 2\rho \int_\Sigma \tilde{a}(x) u \, dV_g - \rho \log \int_\Sigma \tilde{h}(x) e^{2u} dV_g. \tag{1.11}$$

Even for a singular coefficient $\tilde{h}$, there is a variant of (1.4) from [21] and [56] (see also [18]).

Proposition 1.3 ([21, 56]). *Let $\alpha_j > -1$ for all j, and let $\tilde{h} : \Sigma \to \mathbb{R}$ be as in* (1.10)*. Then one has*

$$\log \int_\Sigma \tilde{h}(x) e^{2(u - \overline{u})} dV_g \leq \frac{1}{4\pi \min\{1, 1 + \min_j \alpha_j\}} \int_\Sigma |\nabla u|^2 + C_{\tilde{h},g} \tag{1.12}$$

for all $u \in W^{1,2}(\Sigma)$.

As one can see, the constant in (1.12) is determined by the most singular behaviour of $\tilde{h}$ near the points p_j. As for (1.4), it is still possible to localize inequality (1.12) near each region containing some portion of the total volume. The local constant will then be $\frac{1}{4\pi}$ if this region only contains points with negative weights α_j: it will instead be equal to $\frac{1}{4\pi(1+\alpha_j)}$ if the region is localized near a point p_j with negative α_j, see [14] for details.

When all weights are negative, following this reasoning one is induced to assign to any given point $q \in \Sigma$ a weighted cardinality as follows:

$$\tau(q) = \begin{cases} 1 + \alpha_j & \text{if } q = p_j \text{ for some } j = 1, \dots, m; \\ 1 & \text{otherwise.} \end{cases}$$

The cardinality of any finite set of (pairwise distinct) points on Σ can be defined extending τ by additivity. As we will see in Section 3, this weighted cardinality enables to describe functions with low energy $\tilde{I}_\rho$, in the sense that their (normalized) conformal volume will be distributionally close to an element belonging to the set of probability measures

$$\Sigma_{\rho, \underline{\alpha}} = \left\{ \sum_{q_j \in J} t_j \delta_{q_j} : \sum_{q_j \in J} t_j = 1, \ t_j \geq 0, \ q_j \in \Sigma \quad 4\pi \, \tau(J) < \rho \right\}. \tag{1.13}$$

We will see that the non-contractibility of $\Sigma_{\rho,\underline{\alpha}}$, as a topological space endowed with the weak topology of distributions, will give a sufficient condition for the existence of solutions to (1.9).

When some coefficient α_j is positive instead, if e^{2u} *concentrates* near p_j one would expect an improvement of the local Moser-Trudinger constant by the fact that $\tilde{h}$ vanishes at p_j with a certain rate. On the other hand, there is no sign of this improvement from (1.12) in such a case. We will see that indeed one can take a better constant provided some information on the *microscopic* structure of the conformal volume is available. In particular this information relies on evaluating a certain number (depending on α_j) of *angular moments* of the volume measure near p_j, see [4, 45]. There is indication that the non-contractibility of $\Sigma_{\rho,\underline{\alpha}}$ for arbitrarily signed coefficients could still lead to existence for (1.9).

2. Conformal metrics with constant Q-curvature

Q-curvature is a scalar quantity introduced by Thomas Branson in [9] which recently attracted a lot of attention as it appears in several contexts. These include functional determinants, compactification of locally conformally flat manifolds, scattering theory for Poincaré-Einstein metrics, ambient metrics, volume renormalization: we refer to the survey paper [15], where other geometric applications are discussed (see also [16, 17, 35]).

On a four dimensional manifold (M, g), with Ricci tensor Ric_g and scalar curvature R_g the Q-curvature is defined by

$$Q_g = -\frac{1}{12}\left(\Delta_g R_g - R_g^2 + 3|Ric_g|^2\right). \tag{2.1}$$

Under a conformal change of metric $\tilde{g} = e^{2w} g$ Q transforms according to the formula

$$P_g w + 2Q_g = 2Q_{\tilde{g}} e^{4w}, \tag{2.2}$$

where P_g is the *Paneitz operator*

$$P_g(\varphi) = \Delta_g^2 \varphi + div_g\left(\frac{2}{3}R_g g - 2Ric_g\right) d\varphi; \qquad \varphi \in C^\infty(M),$$

introduced in [52] to derive conformally covariant versions of Maxwell's equations.

In addition to the similarity with (1.1), Q-curvature shares other analogies with the Gaussian curvature, like appearing in a Gauss-Bonnet type formula. Indeed, if W_g denotes the Weyl tensor of g one has

$$\int_M \left(Q_g + \frac{|W_g|^2}{8}\right) dV_g = 4\pi^2 \chi(M). \tag{2.3}$$

As W_g is pointwise conformally covariant, it follows that the total Q-curvature of M

$$q_P := \int_M Q_g dV_g, \tag{2.4}$$

is a global conformal invariant.

Similarly to the uniformization problem, in four dimensions one might ask whether it would be possible to find conformal metrics with constant Q-curvature. Writing $\tilde{g} = e^{2w}g$, by (2.2) the question amounts to finding a solution of the equation

$$P_g w + 2Q_g = 2\overline{Q}e^{4w}, \tag{2.5}$$

where $\overline{Q}$ is a real constant. Problem (2.5) is variational, and solutions can be found as critical points of the following functional

$$II(u) = \langle P_g u, u\rangle + 4\int_M Q_g u dV_g - q_P \log \int_M e^{4u} dV_g; \qquad u \in W^{2,2}(M). \tag{2.6}$$

Here $W^{2,2}(M)$ is the space of real-valued functions on M which are of class L^2 together with their first and second derivatives, and the symbol $\langle P_g u, v\rangle$ stands for

$$\langle P_g u, v\rangle = \int_M \left(\Delta_g u \Delta_g v + \frac{2}{3} R_g \nabla_g u \cdot \nabla_g v - 2(Ric_g \nabla_g u, \nabla_g v)\right) dV_g. \tag{2.7}$$

The last term in the functional II is controlled by means of the *Adams inequality* (see [1, 19])

$$\log \int_M e^{4(u-\overline{u})} dV_g \leq \frac{1}{8\pi^2}\langle P_g u, u\rangle + C, \qquad P_g \geq 0, \quad u \in W^{2,2}(M). \tag{2.8}$$

When P_g is not positive-definite one can still treat the logarithmic term by a minor modification of the proof in [19].

Problem (2.5) was solved in [19] when $P_g \geq 0$ and $q_P < 8\pi^2$. Under these assumptions, by (2.8) the functional II is bounded from below and coercive, so one has an analogue of Proposition 1.1. The result in [19] has also been extended in [10] to higher-dimensional cases (regarding the so-called GJMS operators, [34]) in [9] using a geometric flow.

We describe next the following theorem, which relaxes the assumptions in [19] and allows to consider generic manifolds (see also [50] for a higher-dimensional version).

Theorem 2.1 ([30]). *Suppose ker $P_g = \{constants\}$, and assume that $q_P \neq 8k\pi^2$ for $k = 1, 2, \ldots$. Then (M, g) admits a conformal metric with constant Q-curvature.*

The assumptions in Theorem 2.1 are conformally invariant and by their general character they apply to a large class of four manifolds. One of the main differences between (1.1) and (2.5) is that in four dimensions the integral of Q_g, which we denoted by q_P, could be arbitrarily large (as for example on products of negatively-curved surfaces). As it was discussed for I_ρ, here if either $q_P > 8\pi^2$ or if P_g has some negative eigenvalue, then II becomes unbounded from below.

For brevity we only treat here the case of non-negative P_g's, as the main difficulty and peculiarity of the problem is caused by the condition $q_P > 8\pi^2$ (referring to [30] for details when negative eigenvalues are present). To see the lack of lower bounds on II one can fix a point $x \in M$ and consider a test function of the following form

$$\varphi_{\lambda,x}(y) \simeq \frac{1}{4}\log\left(\frac{2\lambda}{1+\lambda^2 d(y,x)^2}\right); \qquad y \in M.$$

It turns out that for $\lambda \to +\infty$ one has that $II(\varphi_{\lambda,x}) \to -\infty$.

In the next two subsections we will sketch the main ideas for the proof of Theorem 2.1. We first describe how the *distribution* over M of e^{4u} affects the multiplicative constant in (2.8), and then how to derive existence via a min-max scheme based on the four-dimensional counterpart of (1.5).

2.1. Effects of volume distribution. As we anticipated in the introduction, spreading of conformal volume leads to improvements of the Moser-Trudinger inequality. Here is a rigorous statement for (2.8).

Lemma 2.2. *For a fixed integer ℓ, let $\Omega_1, \dots, \Omega_{\ell+1}$ be subsets of M satisfying $d(\Omega_i, \Omega_j) \geq \delta_0$ for $i \neq j$, where δ_0 is a positive real number, and let $\gamma_0 \in \left(0, \frac{1}{\ell+1}\right)$. Then, for any $\tilde{\varepsilon} > 0$ there exists a constant $C = C(\ell, \tilde{\varepsilon}, \delta_0, \gamma_0)$ such that*

$$\log \int_M e^{4(u-\overline{u})} dV_g \leq C + \frac{1}{8(\ell+1)\pi^2 - \tilde{\varepsilon}} \langle P_g u, u \rangle$$

for all the functions $u \in W^{2,2}(M)$ satisfying

$$\frac{\int_{\Omega_i} e^{4u} dV_g}{\int_M e^{4u} dV_g} \geq \gamma_0, \quad \forall\, i \in \{1, \dots, \ell+1\}. \tag{2.9}$$

The original version of the above lemma was given for two dimensions in [22] for $\ell = 1$. The main idea of the proof consists in constructing cut-off functions g_i which are identically equal to 1 on Ω_i and which have mutually disjoint supports. Then one applies (2.8) to $g_i u$ and sums over i. The main term in $\langle P_g g_i u, g_i u \rangle$ is given by $\int_M g_i^2 (\Delta u)^2 dV_g$, while the remaining ones can be treated via interpolation inequalities.

A consequence of the above lemma is that if $q_P < 8(k+1)\pi^2$ for some natural k and if u satisfies (2.9), then $II(u)$ stays bounded from below. Since we can choose δ_0 and the Ω_i's arbitrarily, this suggests that if $II(u)$ is sufficiently low, then e^{4u} should *concentrate* near at most k points of M. Using a covering argument this idea can indeed be made rigorous, yielding the following characterization of low-energy functions.

Lemma 2.3. *Assuming $P_g \geq 0$ and $q_P \in (8k\pi^2, 8(k+1)\pi^2)$ with $k \geq 1$, the following property holds. For any $\varepsilon > 0$ and any $r > 0$ there exists a large positive $L = L(\varepsilon, r)$ such that for every $u \in W^{2,2}(M)$ with $II(u) \leq -L$ there exist k points $p_{1,u}, \dots, p_{k,u} \in M$ such that*

$$\int_{M \setminus \cup_{i=1}^k B_r(p_{i,u})} e^{4u} dV_g < \varepsilon. \tag{2.10}$$

As we saw in the Introduction, if we impose the normalization $\int_M e^{4u} dV_g = 1$, the fact that almost all conformal volume is contained in k small balls in M suggests to consider the following set of probability measures on M.

$$M_k = \left\{ \sum_{i=1}^k t_i \delta_{x_i} \ : \ x_i \in M, t_i \in [0,1], \sum_{i=1}^k t_i = 1 \right\}.$$

More precisely, (exponentials of) functions with low energy will approach elements in M_k in the sense of measures.

Endowing naturally M_k with the weak topology of distributions, we obtain a finite-dimensional object: for $k = 1$ M_1 is homeomorphic to M, but for larger k's this will be a *stratified set*, namely union of open manifolds of different dimensions, whose maximal one is $5k - 1$. While it is difficult in general to obtain a full description of the topology of M_k, it will be enough for us to use the following characterization.

Lemma 2.4. *If M is a compact closed manifold, for any $k \geq 1$ the set M_k is non-contractible.*

Using Lemma 2.3 one can construct a natural continuous map from large negative sublevels of II into M_k, as it is described in the next result.

Proposition 2.5. *For $k \geq 1$ there exists a large $L > 0$ and a continuous map Ψ from the sublevel $\{II \leq -L\}$ into M_k which acts non-trivially on the homology groups of $\{II \leq -L\}$.*

2.2. The min-max scheme. In order to proceed we need some preliminary notation. For $\delta > 0$ small, consider a smooth non-decreasing cut-off function $\chi_\delta : \mathbb{R}_+ \to \mathbb{R}$ satisfying the following properties

$$\begin{aligned} \chi_\delta(t) &= t \qquad \text{for } t \in [0, \delta]; \\ \chi_\delta(t) &= 2\delta \qquad \text{for } t \geq 2\delta; \\ \chi_\delta(t) &\in [\delta, 2\delta] \qquad \text{for } t \in [\delta, 2\delta]. \end{aligned} \tag{2.11}$$

Then, given $\sigma \in M_k$ $\left(\sigma = \sum_{i=1}^k t_i \delta_{x_i}\right)$ and $\lambda > 0$, we define the function $\varphi_{\lambda,\sigma} : M \to \mathbb{R}$ as

$$\varphi_{\lambda,\sigma}(y) = \frac{1}{4} \log \sum_{i=1}^k t_i \left(\frac{2\lambda}{1 + \lambda^2 \chi_\delta^2 (d_i(y))} \right)^4; \qquad y \in M, \tag{2.12}$$

where $d_i(y) = d(y, x_i)$, $y \in M$; the cut-off χ_δ has been introduced for regularity reasons. We have then the following result.

Proposition 2.6. *Let $\varphi_{\lambda,\sigma}$ be defined as in* (2.12)*. Then, as $\lambda \to +\infty$ one has the following properties*

(i) $e^{4\varphi_{\lambda,\sigma}} \rightharpoonup \sigma$ *weakly in the sense of distributions;*

(ii) $II(\varphi_{\lambda,\sigma}) \to -\infty$ *uniformly for $\sigma \in M_k$.*

Moreover, if Ψ is given in Proposition 2.5, the map $\sigma \mapsto \varphi_{\lambda,\sigma} \mapsto \Psi(\varphi_{\lambda,\sigma})$ converges to the identity on M_k as λ tends to infinity.

We next describe the variational scheme yielding existence of solutions to (2.5). Let $\widehat{M_k}$ denote the (contractible) topological cone over M_k, which can be realized as $\widehat{M_k} = M_k \times [0, 1]$, with $M_k \times \{0\}$ collapsed to a single point. Choose first L be so large that Proposition 2.5 applies, and then let $\overline{\lambda}$ be so large that $II(\varphi_{\lambda,\sigma}) \leq -4L$ for all $\sigma \in M_k$, which is possible by Proposition 2.6. Fixing this number $\overline{\lambda}$, we define the following class of maps

$$\Pi_{\overline{\lambda}} = \left\{ \theta : \widehat{M_k} \to W^{2,2}(M) \ : \ \theta \text{ is continuous, } \theta(\cdot \times \{1\}) = \varphi_{\overline{\lambda},\cdot} \text{ on } M_k \right\}. \tag{2.13}$$

We have the following properties.

Lemma 2.7. *The set $\Pi_{\overline{\lambda}}$ is non-empty and moreover, letting*

$$\overline{\Pi}_{\overline{\lambda}} = \inf_{\theta \in \Pi_{\overline{\lambda}}} \sup_{m \in \widehat{M_k}} II(\theta(m)), \qquad \text{one has} \qquad \overline{\Pi}_{\overline{\lambda}} > -2L. \tag{2.14}$$

As a consequence, the functional II possesses a Palais-Smale sequence at level $\overline{\Pi}_{\overline{\lambda}}$.

To check that $\Pi_{\overline{\lambda}}$ is non empty, it is sufficient to consider the map $\theta(s,\sigma) = s\,\varphi_{\overline{\lambda},\sigma}$, which is well defined on $\widehat{M_k}$, $s \in [0,1]$. The inequality in (2.14) follows by a contradiction argument. If there would be an admissible map in $\Pi_{\overline{\lambda}}$ for which the energy is always below $-2L$, then its image would belong to the domain of Ψ. We would obtain then a retraction of the cone $\widehat{M_k}$ onto its base, which (letting the *height* vary from the vertex to the base) would realize a homotopy between the identity map on Σ_k and a constant map. This is indeed impossible since Σ_k is non contractible.

A *Palais-Smale sequence at level* c is a sequence $(u_n)_n$ such that $II(u_n) \to c$ and such that $II'(u_n) \to 0$ as $n \to +\infty$. To produce it one considers any map in $\Pi_{\overline{\lambda}}$ with $\sup_{m\in\widehat{M_k}} II(\theta(m))$ sufficiently close to $\overline{\Pi}_{\overline{\lambda}}$ and let it evolve via the gradient flow of II, keeping its boundary fixed. By (2.14) the maximum value of II on the evolved set, which still belongs to $\Pi_{\overline{\lambda}}$, cannot decrease below $\overline{\Pi}_{\overline{\lambda}}$, so the gradient of II along the evolution has to become small somewhere along the flow. In this reasoning, it is *crucial* to have a *gap* in energy between the maximal energy at the interior of $\widehat{M_k}$ and at its boundary, which is guaranteed by the (2.14), see for example [54] for further details.

If Palais-Smale sequences are bounded, then it is easy to see that they must converge to a solution of (2.5). Unfortunately boundedness of Palais-Smale sequences is still an open problem, but this issue can be bypassed using an argument due to M.Struwe in [53]. For t in a neighbourhood of 1 we define the functional $II_t : W^{2,2}(M) \to \mathbb{R}$ by

$$II_t(u) = \langle P_g u, u\rangle + 4t\int_M Q_g dV_g - 4tq_P \log \int_M e^{4u} dV_g, \qquad u \in W^{2,2}(M),$$

whose critical points give rise to solutions of

$$P_g u + 2tQ_g = 2tq_P e^{4u} \qquad \text{in } M. \tag{2.15}$$

Running the above variational scheme for t close to 1 one finds a min-max value as in Lemma 2.7, which we call $\overline{\Pi}_t$, and a corresponding Palais-Smale sequence. One can easily show that $\frac{\overline{\Pi}_t}{t}$ is non-increasing, and hence a.e. differentiable, in $[1-t, 1+t]$ provided t is sufficiently small. One has then the following result, which can be proved as in [27].

Lemma 2.8. *Let $\Lambda \subset [1-t, 1+t]$ be the (dense) set of t's for which the function $\frac{\overline{\Pi}_t}{t}$ is differentiable. Then, for $t \in \Lambda$, II_t possesses a bounded Palais-Smale sequence $(u_l)_l$ at level $\overline{\Pi}_t$.*

By the last lemma and the comments we made before it one can then deduce existence of solutions u_{t_l} to (2.15) for a sequence $t_l \to 1$. To show that then u_{t_l} converges to a solution of (2.5) one can use the following result from [32, 43] (together with some standard regularity theory), where it is proved that compactness holds provided q_P stay bounded away from $8\pi^2\mathbb{N}$. The proof relies on blow-up analysis of solutions, and has previous two-dimensional counterparts in [11, 39, 40].

Theorem 2.9 ([43]). *Suppose $ker\ P_g = \{constants\}$ and that $(u_l)_l$ is a sequence of solutions of*

$$P_g u_l + 2Q_l = 2q_l e^{4u_l} \qquad \text{in } M, \tag{2.16}$$

satisfying $\int_M e^{4u_l} dV_g = 1$, where $q_l = \int_M Q_l dV_g$, and where $Q_l \to Q_0$ in $C^0(M)$. Assume also that $k_0 := \int_M Q_0 dV_g \neq 8k\pi^2$ for $k = 1, 2, \dots$. Then $(u_l)_l$ is bounded in $C^\alpha(M)$ for any $\alpha \in (0,1)$.

It is an interesting open problem to study the case when q_P is an integer multiple of $8\pi^2$. Some results in this direction can be found in [38].

3. Singular Liouville equations

We showed in the Introduction how to desingularize equation (1.6) and to transform it into (1.9), which has variational structure, with Euler-Lagrange energy $\tilde{I}_\rho$ (see (1.11)). Similarly to Lemma 2.2, one has the following result when portions of the conformal volume accumulate near both regular and singular points, see [14].

Lemma 3.1. *Suppose $\alpha_j > 0$ for all j. Let $n \in \mathbb{N}$ and let $I \subseteq \{1, \dots, m\}$ with $n + card\,(I) > 0$. Assume there exists $r > 0$, $\delta_0, \gamma_0 > 0$ and pairwise distinct points $\{q_1, \dots, q_n\} \subseteq \Sigma \setminus \{p_1, \dots, p_m\}$ such that:*

- *for any $\{a, b\} \subseteq \{q_1, \dots, q_n \cup (\cup_{i \in I} p_i)\}$ with $a \neq b$ $d(B_r(a), B_r(b)) \geq 4\delta_0$;*
- *for any $a \in \{q_1, \dots, q_m\}$ one has $d(p_i, B_r(a)) \geq 4\delta_0$ for any $i \in \{1, \dots, m\} \setminus I$.*

Then, for any $\widetilde{\epsilon} > 0$ there exists a constant $C := C(\Sigma, g, n, I, r, \delta_0, \gamma_0, \widetilde{\epsilon})$ such that

$$\log \int_\Sigma \widetilde{h} e^{2(u - \overline{u})}\, dV_g \leq \frac{1}{4\pi \left(n + \sum_{i \in I}(1 + \alpha_i) - \widetilde{\epsilon}\right)} \int_\Sigma |\nabla_g u|^2\, dV_g + C \tag{3.1}$$

for all $u \in W^{1,2}(\Sigma)$ satisfying $\frac{\int_{B_r(a)} \widetilde{h} e^{2u}\, dV_g}{\int_\Sigma \widetilde{h} e^{2u}\, dV_g} \geq \gamma_0$; $a \in \{q_1, \dots, q_n \cup (\cup_{i \in I} p_i)\}$.

With this version of the improved inequality, it is possible to argue as for Lemma 2.3, but due to the different local Moser-Trudinger constant in presence of a weight, singular points play a separate role. Precisely, if all the weights are negative, one can show that conformal volumes of functions with low energy are close in the distributional sense to the set $\Sigma_{\rho, \underline{\alpha}}$ introduced in (1.13).

We say that $\overline{\rho} > 0$ is a *singular value* for problem (1.9) if

$$\overline{\rho} = 4\pi n + 4\pi \sum_{i \in I} (1 + \alpha_i) \tag{3.2}$$

for some $n \in \mathbb{N}$ and $I \subseteq \{1, \dots, m\}$ (possibly empty) satisfying $n + card\,(I) > 0$. The set of singular values will be denoted by $\mathfrak{S} = \mathfrak{S}(\underline{\alpha})$. It appears when studying the compactness properties of solutions to (1.9). As it was proven in [5], [6], [7], solutions stay compact when $\rho \notin \mathfrak{S}$, and this allows to apply min-max theory to $\tilde{I}_\rho$ analogously to as it was done in the previous section.

Theorem 3.2 ([14]). *Suppose that $\alpha_j > 0$ for all j, and that $\rho \in \mathbb{R} \setminus \mathfrak{S}$. Then, if the set $\Sigma_{\rho, \underline{\alpha}}$ is not contractible, problem* (1.9) *admits a solution.*

We expect the same theorem to hold also for general values of the α's, namely when some (or all) of them are positive. The case of positive α's is somehow more delicate since, as it was already noticed, there is no local improvement in the Moser-Trudinger constant from (1.12). We will see in the next subsection how some new kind of improved inequality might play a role in this problem.

3.1. A new improved inequality. We illustrate an example in which one can derive different improved inequalities in presence of positive α's. Instead of looking at compact surfaces, we consider the following model problem in the unit ball B of $\mathbb{R}^2$ (which however contains the peculiarity of the problem)

$$\begin{cases} -\Delta u = \rho \frac{\tilde{h} e^{2u}}{\int_B \tilde{h} e^{2u} dx} & \text{in} \quad B \\ u = 0 & \text{on} \quad \partial B. \end{cases} \tag{3.3}$$

We assume that there is only one singular point at the origin with weight $\alpha > 0$, and that $\tilde{h}$ has still the same asymptotics as in (1.10). Also this problem is variational, with Euler-Lagrange energy given by

$$\tilde{I}_{\rho,B}(u) = \int_B |\nabla_g u|^2 dx - \rho \log \int_B \tilde{h}(x) e^{2u} dx. \tag{3.4}$$

Notice that a denominator appears in the right-hand side of (3.3) since, differently from the case of compact surfaces, here we are fixing Dirichlet boundary data and we cannot translate solutions by suitable constants in order to normalize it.

Under radial symmetry, it was proved in [31] that the best constant in (1.12) is $\frac{1}{4\pi(1+\alpha)}$ even for positive α's (see also [47] for a $\mathbb{Z}_k$-symmetric version). One can indeed substitute the radiality assumption, which is quite stringent, with a constraint of finite codimension in $W_0^{1,2}(B)$. This consists in looking at suitable angular moments of the right-hand side of (3.3). To introduce this condition we define the probability measure on B

$$\tilde{f}_u := \frac{\tilde{h} e^{2u}}{\int_B \tilde{h} e^{2u} dx},$$

and from this the probability measure on the unit circle

$$\mu_u(A) = \int_{\tilde{A}} \tilde{f}_u dx; \qquad A \subseteq S^1, \quad \tilde{A} = \cup_{t \in (0,1]} tA. \tag{3.5}$$

Using complex notation, we then let

$$F_k(\tilde{f}_u) = \left(\int_{S^1} z \, d\mu_u, \int_{S^1} z^2 d\mu_u, \dots, \int_{S^1} z^k d\mu_u \right), \tag{3.6}$$

map the probability measures on S^1 into $\mathbb{C}^k$. One has then the following result.

Proposition 3.3 ([4, 45]). *For $\alpha > 0$, let k_α denote the smaller integer greater or equal to α, let $k \in \{1, \dots, k_\alpha\}$ and let F_k denote the map in* (3.6)*. Then for any $\varepsilon > 0$ there exists a constant $C = C(\varepsilon, k, \alpha)$ such that*

$$\log \int_B \tilde{h}(x) e^{2u} dx \le \frac{1+\varepsilon}{4\pi(1+k)} \int_B |\nabla u|^2 dx + C,$$

whenever $F_k(\tilde{f}_u) = 0$.

We notice that the main feature of the last proposition is that the assumptions are scaling invariant, and hence they might apply also to functions u which are arbitrarily concentrated near the singularity. As a corollary we obtain a lower bound on $\tilde{I}_{\rho,B}$ provided a sufficient number of angular moments vanishes.

Corollary 3.4. *Suppose that $\rho \in (4k\pi, 4(k+1)\pi) \cap (4\pi, 4\pi(1+\alpha))$ for some $k \in \mathbb{N}$. Then there exists a constant $L = L(\rho, k, \alpha)$ such that*

$$\tilde{I}_{\rho,B} \geq -L \qquad \text{for all } u \text{ such that } F_k(\tilde{f}_u) = 0.$$

3.2. An application. As an application of the previous result we would like to illustrate the following theorem.

Theorem 3.5 ([4, 45]). *Suppose $\alpha > 0$, and that $\rho \in (4\pi, 4\pi(1+\alpha)) \setminus 4\pi\mathbb{N}$. Then problem* (3.3) *is solvable.*

Remark 3.6. When $\tilde{h}(x) = |x|^{2\alpha}$, one can show using Pohozaev-type identities that (3.3) has no solutions if $\rho \geq 4\pi(1+\alpha)$, see [4]. Therefore the upper bound on ρ in the previous theorem is optimal.

To prove Theorem 3.5 one can still use a min-max scheme as in the previous section. Consider first the circle

$$\gamma := \partial B_{1/2}$$

centred at the origin of $\mathbb{R}^2$. For $k \in \mathbb{N}$, similarly to (1.5) one can consider the family of unit measures supported on at most k points of γ

$$\gamma_k = \left\{ \sum_{j=1}^{k} t_j \delta_{x_j} \,:\, \sum_{j=1}^{k} t_j = 1,\ x_j \in \gamma \right\}. \tag{3.7}$$

While for two-dimensional surfaces (or four-dimensional manifolds) it is in general difficult to describe the topology of these formal combinations of Dirac masses, for a circle there is a complete characterization.

Proposition 3.7 ([4]). *For $k \geq 1$, there exists a homeomorphism $\mathcal{S}_k$ from the sphere S^{2k-1} onto γ_k.*

Also, as for Proposition 2.6, one has the following result.

Proposition 3.8. *There exists a family of functions $\tilde{\varphi}_{\lambda,\sigma}$, $\sigma \in \gamma_k$, which have the following properties as $\lambda \to +\infty$*

(i) $\tilde{h}(x) e^{2\tilde{\varphi}_{\lambda,\sigma}} \rightharpoonup \sigma$ *weakly in the sense of distributions;*

(ii) $\tilde{I}_{\rho,B}(\tilde{\varphi}_{\lambda,\sigma}) \to -\infty$ *uniformly for $\sigma \in \gamma_k$.*

We next introduce a min-max scheme suitable for this problem. Let $\rho \in (4k\pi, 4(k+1)\pi) \cap (4\pi, 4\pi(1+\alpha))$, and let L be as in Corollary 3.4. Choose then $\tilde{\lambda} > 0$ so large that $\tilde{I}_{\rho,B}(\tilde{\varphi}_{\tilde{\lambda},\sigma}) < -4L$ for every $\sigma \in \gamma_k$, which is possible by Proposition 3.8. If B^{2k} stands for the unit ball of $\mathbb{R}^{2k}$ and if $\mathcal{S}_k$ is as in Proposition 3.7, consider the class of maps

$$\Pi_{\tilde{\lambda}} = \left\{ \theta : \overline{B}^{2k} \to W_0^{1,2}(B) \ :\ \theta \text{ continuous}, \theta(y) = \varphi_{\tilde{\lambda},\mathcal{S}_k(y)} \text{ for } y \in \partial B^{2k} \right\}. \tag{3.8}$$

Similarly to Lemma 2.7 we have the following result.

Lemma 3.9. *The set $\Pi_{\tilde\lambda}$ is non-empty and moreover, letting*

$$\overline{\Pi}_{\tilde\lambda} = \inf_{\theta\in\Pi_{\tilde\lambda}} \sup_{z\in B^{2k}} \tilde I_{\rho,B}(\theta(m)), \qquad \textit{one has} \qquad \overline{\Pi}_{\tilde\lambda} > -2L. \tag{3.9}$$

Moreover, $\tilde I_{\rho,B}$ possesses a Palais-Smale sequence at level $\overline{\Pi}_{\tilde\lambda}$.

To show inequality (3.9) one can argue by contradiction and assume the existence of an admissible map $\tilde\theta$ in $\Pi_{\tilde\lambda}$ for which the supremum of the energy stays below $-2L$.

On the other hand, by Proposition 3.7 and the properties of $\tilde\varphi_{\tilde\lambda,\sigma}$ one can prove that the topological degree $\deg(F_k, B^{2k}, 0)$ is equal to 1. This implies that F_k vanishes somewhere on $\tilde\theta(B^{2k})$, and therefore Corollary 3.4 would contradict the above upper bound on the energy $(-2L)$.

For a degree-theoretical approach to (1.6) we refer to [25]. It would be also interesting to explore the case in which ρ belongs to the *critical set* $\mathfrak{S}$, see [41] for a specific problem on the flat torus. When one prescribes constant Gaussian curvature in presence of conical singularities, deriving general non-existence results is also an interesting open question, see [33, 42].

We also mention a natural system of Liouville equations, the *Toda system*, related to (1.6) and motivated from the study of non-abelian Chern-Simons vortices. Its variational structure is only partially explored, see [8, 46] and the references therein, and many questions remain open for systems of this type.

Acknowledgements. The author is supported by the PRIN project *Variational and perturbative aspects of nonlinear differential problems.*

References

[1] Adams D., *A sharp Inequality of J. Moser for Higher Order Derivatives*, Ann. of Math. **128**-2 (1988), 385–398.

[2] Aubin T., *Meilleures constantes dans le theoreme d'inclusion de Sobolev et un theoreme de Fredholm non lineaire pour la transformation conforme de la courbure scalaire*, J. Funct. Anal. **32** (1979), 148–174.

[3] Bahri A. and Coron J. M., *On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain*, Comm. Pure Appl. Math., **41**-3 (1988), 253–294.

[4] Bartolucci, D. and Malchiodi, A., *An improved geometric inequality via vanishing moments, with applications to singular Liouville equations*, Comm. Math. Phys. **322** (2013), no. 2, 415–452.

[5] Bartolucci D. and Montefusco E., *On the shape of blow-up solutions to a mean field equation*, Nonlinearity **19** (2006), 611–631.

[6] ———, *Blow up analysis, existence and qualitative properties of solutions for the two dimensional Emden-Fowler equation with singular potential*, M^2.A.S. **30**(18) (2007), 2309–2327.

[7] Bartolucci D. and Tarantello G., *Liouville type equations with singular data and their application to periodic multivortices for the electroweak theory*, Comm. Math. Phys. **229** (2002), 3–47.

[8] Battaglia, L., Jevnikar, A., Malchiodi, A., and Ruiz, D., *A general existence result for the Toda system on compact surfaces*, preprint, 2013, http://arxiv.org/abs/1307.3921

[9] Branson T.P., *Differential operators canonically associated to a conformal structure*, Math. Scand. **57**-2 (1985), 293–345.

[10] Brendle S., *Global existence and convergence for a higher order flow in conformal geometry*, Ann. of Math. **158** (2003), 323–343.

[11] Brezis H. and Merle F., *Uniform estimates and blow-up behavior for solutions of* $-\Delta u = V(x)e^u$ *in two dimensions*, Comm. Part. Diff. Eq. **16** (8-9) (1991), 1223–1253.

[12] Caglioti E., Lions P.L., Marchioro C., and Pulvirenti M., *A special class of stationary flows for two dimensional Euler equations: a statistical mechanics description. II.*, Comm. Math. Phys. **174** (1995), 229–260.

[13] Carlotto A., *On the solvability of singular Liouville equations on compact surfaces on arbitrary genus*, Trans. A.M.S. **366** (2014), no. 3, 1237–1256.

[14] Carlotto A. and Malchiodi A., *Weighted barycentric sets and singular Liouville equations on compact surfaces*, J. Funct. Anal., **262**(2) (2012), 409–450.

[15] Chang, S.Y.A., Eastwood, M., Ørsted, B., and Yang, P.C., *What is Q-Curvature?*, Acta Appl. Math., July 2008, Volume 102, Issue 2-3, pp. 119–125.

[16] Chang S.Y.A., Gursky M.J., and Yang P.C., *A conformally invariant sphere theorem in four dimensions*, Publ. Math. Inst. Hautes Études Sci. **98** (2003), 105–143.

[17] Chang S.Y.A., Qing J., and Yang, P.C., *Compactification of a class of conformally flat 4-manifold*, Invent. Math. **142**-1 (2000), 65–93.

[18] Chang S.Y.A. and Yang, P. C., *Conformal deformation of metrics on* S^2, J. Diff. Geom. **27** (1988), 259–296.

[19] ———, *Extremal metrics of zeta functional determinants on* 4*-manifolds*, Ann. of Math. **142** (1995), 171–212.

[20] Chanillo S. and Kiessling M., *Rotational symmetry of solutions of some nonlinear problems in statistical mechanics and in geometry*, Comm. Math. Phys. **160** (1994), 217–238.

[21] Chen W. X., *A Trudinger inequality on surfaces with conical singularities*, Proc. Amer. Math. Soc. **108** (1990), 821–832.

[22] Chen W. and Li C., *Prescribing Gaussian curvatures on surfaces with conical singularities*, J. Geom. Anal. **1**(4) (1991), 359–372.

[23] Chen C.C. and Lin C.S., *Sharp estimates for solutions of multi-bubbles in compact Riemann surfaces*, Comm. Pure Appl. Math. **55**(6) (2002), 728–771.

[24] ———, *Topological degree for a mean field equation on Riemann surfaces*, Comm. Pure Appl. Math. **56**(12) (2003), 1667–1727.

[25] ———, *A degree counting formulas for singular Liouville-type equation and its application to multi vortices in electroweak theory*, Comm. Pure Appl. Math., to appear.

[26] De Marchis F., *Generic multiplicity for a scalar field equation on compact surfaces*, J. Funct. Anal. **259** (2010), 2165–2192.

[27] Ding W., Jost J., Li J., and Wang G., *Existence results for mean field equations*, Ann. Inst. Henri Poincaré, Anal. Non Linéaire **16**(5) (1999), 653–666.

[28] ———, *An analysis of the two-vortex case in the Chern- Simons-Higgs model*, Calc. Var. P.D.E.s **7** (1998), 87–97.

[29] Djadli Z., *Existence result for the mean field problem on Riemann surfaces of all genuses*, Comm. Contemp. Math. **10**(2) (2008), 205–220.

[30] Djadli Z., Malchiodi A., *Existence of conformal metrics with constant Q-curvature*, Ann. of Math. **168**(3) (2008), 813–858.

[31] Dolbeault J., Esteban M.J., and Tarantello G., *The role of Onofri type inequalities in the symmetry properties of extremals for Caffarelli-Kohn-Nirenberg inequalities, in two space dimensions*, Ann. Sc. Norm. Super. Pisa Cl. Sci., Vol. VII (2008), 313–341.

[32] Druet O. and Robert F., *Bubbling phenomena for fourth-order four-dimensional pdes with exponential growth*, Proc. Amer. Math. Soc., 134 (2006), no. 3, 897–908.

[33] Eremenko A., *Metrics of positive curvature with conical singularities on the sphere*, Proc. A.M.S. **132**(11), (2004), 3349–3355.

[34] Graham C.R., Jenne R., Mason L.J., and Sparling G., *Conformally invariant powers of the Laplacian. I. Existence*, J. London Math. Soc. **46**-3 (1992), 557–565.

[35] Gursky M., *The principal eigenvalue of a conformally invariant differential operator, with an application to semilinear elliptic PDE*, Comm. Math. Phys. **207**-1 (1999), 131–143.

[36] Hong J., Kim Y., and Pac P.Y., *Multivortex Solutions of the Abelian Chern-Simons Theory*, Phys. Rev. Lett. **64** (1990), 2230–2233.

[37] Jackiw R. and Weinberg E.J., *Selfdual Chern-Simons vortices*, Phys. Rev. Lett. **64** (1990), 2234–2237.

[38] Li J., Li Y. and Liu, P., *The Q-curvature on a 4-dimensional Riemannian manifold (M, g) with $\int_M Q dV_g = 8\pi^2$*, Adv. Math. **231** (2012), 2194–2223.

[39] Li Y.Y., *Harnack type inequality: The method of moving planes*, Comm. Math. Phys. **200**(2) (1999), 421–444.

[40] Li Y.Y. and Shafrir I., *Blow-up analysis for solutions of $-\Delta u = Ve^u$ in dimension two*, Indiana Univ. Math. J. **43**(4) (1994), 1255–1270.

[41] Lin C.S. and Wang C.L., *Elliptic functions, Green functions and the mean field equations on tori*, Ann of Math. **172**(2) (2010), 911–954.

[42] Luo F. and Tian G., *Liouville Equation And Spherical Convex Polytopes*, Proc. Amer. Math. Soc. **116** (1992), no. 4, 1119–1129.

[43] Malchiodi A., *Compactness of solutions to some geometric fourth-order equations*, J. Reine Angew. Math. **594** (2006), 137–174.

[44] ———, *Morse theory and a scalar field equation on compact surfaces*, Adv. Diff. Eq. **13**(11-12) (2008), 1109–1129.

[45] Malchiodi A., Ruiz D., *New improved Moser-Trudinger inequalities and singular Liouville equations on compact surfaces*, G.A.F.A. **21**-5 (2011), 1196–1217.

[46] ———, *A variational analysis of the toda system on compact surfaces*, Comm. Pure Appl. Math. **66** (2013), no. 3, 332–371.

[47] McOwen R., *Conformal metrics in $\mathbb{R}^2$ with prescribed Gaussian curvature and positive total curvature*, Indiana Univ. Math. J. **34** (1985), 97–104.

[48] Moser J., *A sharp form of an inequality by N.Trudinger*, Indiana Univ. Math. J. **20** (1971), 1077–1091.

[49] ———, *On a nonlinear problem in differential geometry*, Dynamical Systems, (M. Peixoto ed.), Academic Press, New York, 1973, 273–280.

[50] Ndiaye C.B., *Constant Q-curvature metrics in arbitrary dimensions*, J. Funct. Anal. **251** (2007), no. 1, 1–58.

[51] Nolasco M. and Tarantello G., *Vortex condensates for the SU(3) Chern-Simons theory*, Comm. Math. Phys. **213** (2000), 599–639.

[52] Paneitz S., *A quartic conformally covariant differential operator for arbitrary pseudo-Riemannian manifolds*, preprint, 1983.

[53] Struwe M., *The existence of surfaces of constant mean curvature with free boundaries*, Acta Math. **160**(1-2) (1988), 19–64.

[54] ———, *Variational methods. Applications to nonlinear partial differential equations and Hamiltonian systems*, Third edition, Springer-Verlag, Berlin, 2000.

[55] Tarantello G., *Self-Dual Gauge Field Vortices: An Analytical Approach*, PNLDE 72, Birkhäuser Boston, Inc., Boston, MA, 2007.

[56] Troyanov M., *Prescribing Curvature on Compact Surfaces with Conical Singularities*, Trans. Am. Math. Soc. **324**(2) (1991), 793–821.

[57] Trudinger N., *On imbeddings into Orlicz spaces and some applications*, J. Math. Mech. **17** 1967 473–483.

[58] Tur A., Yanovsky V., *Point vortices with a rational necklace: New exact stationary solutions of the two-dimensional Euler equation*, Phys. Fluids **16**(8) (2004), 2877–2885.

[59] Yang Y., *Solitons in Field Theory and Nonlinear Analysis*, Springer Monographs in Mathematics, Springer, New York, 2001.

SISSA, Via Bonomea 265, 34136 Trieste, ITALY; Mathematics Institute, The University of Warwick, Coventry CV4 7AL, UK

E-mail: malchiod@sissa.it, A.Malchiodi@warwick.ac.uk

Ramanujan graphs and the solution of the Kadison–Singer problem

Adam W. Marcus, Daniel A. Spielman, and Nikhil Srivastava

Abstract. We survey the techniques used in our recent resolution of the Kadison–Singer problem and proof of existence of Ramanujan Graphs of every degree: mixed characteristic polynomials and the method of interlacing families of polynomials. To demonstrate the method of interlacing families of polynomials, we give a simple proof of Bourgain and Tzafriri's restricted invertibility principle in the isotropic case.

Mathematics Subject Classification (2010). Primary, 05C50, 46L05; Secondary, 26C10.

Keywords. Interlacing polynomials, Kadison–Singer, mixed characteristic polynomials, Ramanujan graphs, mixed discriminants, restricted invertibility.

1. Introduction

In a recent pair of papers [30, 31], we prove the existence of infinite families of bipartite Ramanujan graphs of every degree and we affirmatively resolve the Kadison–Singer Problem. The techniques that we use in the papers are closely related. In both we must show that certain families of matrices contain particular matrices of small norm. In both cases, we prove this through a new technique that we call the *method of interlacing families of polynomials*. In the present survey, we review this technique and the polynomials that we analyze with it, the *mixed characteristic polynomials*.

We begin by defining Ramanujan Graphs, explaining the Kadison–Singer Problem, and explaining how these problems are related. In particular, we connect the two by demonstrating how they are both related to the problem of sparsifying graphs.

1.1. Ramanujan graphs. Let G be an undirected graph with vertex set V and edge set E. The *adjacency matrix* of G is the symmetric matrix A whose rows and columns are indexed by vertices in V with entries

$$A(a,b) = \begin{cases} 1 & \text{if } (a,b) \in E \\ 0 & \text{otherwise.} \end{cases}$$

Since A is symmetric it has $|V|$ real eigenvalues, which we will also refer to as the *eigenvalues of* G.

Consider a function $f : V \to \mathbb{R}$. Multiplication by A corresponds to the operator that replaces the value of f at a given vertex with the sum of the values at its neighbors in G. In

Proceedings of the International Congress of Mathematicians, Seoul, 2014

this way, A is related to random walks and diffusion on G. It is well known that the speed of the convergence of these processes is determined by the eigenvalues of A and related matrices.

We will restrict our attention to graphs that are connected and d-regular. When $|V|$ is finite, it is easy to check that every such graph has an eigenvalue of d corresponding to the eigenvector of all 1's. Furthermore, in the case that G is bipartite, one can check that the eigenvalues of A are symmetric about the origin. Thus every finite bipartite d-regular graph must also have an eigenvalue of $-d$. Because these eigenvalues are unavoidable (they are an artifact of being finite), they are often referred to as the *trivial eigenvalues*.

The graphs on which random walks mix the fastest are those whose non-trivial eigenvalues are as small as possible. An infinite family of connected d-regular graphs all of whose non-trivial eigenvalues are at most α for some constant $\alpha < d$ is called a family of *expander graphs*. Constructing d-regular expanders with a small number of vertices (relative to d) is easy: for example, the complete graph on $d+1$ vertices has all non-trivial eigenvalues -1 and the complete bipartite graph with $2d$ vertices has all non-trivial eigenvalues 0. The interesting problem is to construct d-regular expanders with an arbitrarily large number of vertices. Margulis [32] was the first to find an explicit construction of such an infinite family.

Expander graphs have proved to be incredibly useful in a variety of contexts. We refer the reader who is interested in learning more about expander graphs, with a focus on their applications in computer science, to the survey of Hoory, Linial, and Wigderson [25]. Many applications of expanders depend upon the magnitudes of their non-trivial eigenvalues. A theorem of Alon and Boppana provides a bound on how small the non-trivial eigenvalues can be.

Theorem 1.1 ([3, 35]). *For every integer $d \geq 3$ and every $\epsilon > 0$, there exists an n_0 so that every d-regular graph G with more than n_0 vertices has a non-trivial eigenvalue that is greater than $2\sqrt{d-1} - \epsilon$.*

The number $2\sqrt{d-1}$ in Theorem 1.1 has a meaning: it is the spectral radius of the infinite $d-$regular tree, whose spectrum is the closed interval $[-2\sqrt{d-1}, 2\sqrt{d-1}]$ (it has no trivial eigenvalues because it is not finite) [25]. Since Theorem 1.1 says that no infinite family of d-regular graphs can have eigenvalues that are asymptotically smaller than $2\sqrt{d-1}$, we may view this infinite tree as being the "ideal" expander. A natural question is whether there exist infinite families of finite d-regular graphs whose eigenvalues are actually as small as those of the tree.

Lubotzky, Phillips and Sarnak [29] and Margulis [33] were the first to construct infinite families of such graphs. Their constructions were Cayley graphs, and they exploited the algebraic properties of the underlying groups to prove that all of the nontrivial eigenvalues of their graphs have absolute value at most $2\sqrt{d-1}$. Their proofs required the proof of the Ramanujan Conjecture, and so they named the graphs they obtained *Ramanujan graphs*. As of 2013, all known infinite families of Ramanujan graphs were obtained via constructions similar to [29, 33]. As a result, all known families of Ramanujan graphs had degree $p^k + 1$ for p a prime and k a positive integer.

The main theorem of [30] is that there exist infinite families of d-regular bipartite Ramanujan graphs for every integer $d \geq 3$. This is achieved by proving a variant of a conjecture of Bilu and Linial [9], which implies that every $d-$regular Ramanujan graph has a $2-$cover which is also Ramanujan, immediately establishing the existence of an infinite sequence. In contrast to previous results, the proof is completely elementary, and we will sketch most of

it in this survey.

Bilu and Linial's conjecture is a purely linear algebraic statement about *signings* of adjacency matrices. To define a signing, recall that we can write the adjacency matrix of any graph $G = (V, E)$ as

$$A = \sum_{(a,b)\in E} A_{(a,b)},$$

where $A_{(a,b)}$ is the adjacency matrix of a single edge (a, b). Then, a signing is any matrix of the form

$$\sum_{(a,b)\in E} s_{(a,b)} A_{(a,b)},$$

where $s_{(a,b)} \in \{-1, +1\}$ are signs. A graph with m edges has exactly 2^m signings.

Bilu and Linial conjectured that every $d-$regular adjacency matrix A has a signing A_s with $\|A_s\| \leq 2\sqrt{d-1}$. We prove the following weaker statement, which is equivalent to their conjecture in the bipartite case, as in this case the eigenvalues are symmetric about zero.

Theorem 1.2. *Every $d-$regular adjacency matrix A has a signing A_s with*

$$\lambda_{max}(A_s) \leq 2\sqrt{d-1}.$$

This is a statement about the existence of a certain sum of rank two matrices of type $s_{(a,b)}A_{(a,b)}$, but it it useful to rewrite it as a statement about a sum of rank one matrices by making the substitution

$$s_{(a,b)}A_{(a,b)} = (e_a + s_{(a,b)}e_b)(e_a + s_{(a,b)}e_b)^T - e_a e_a^T - e_b e_b^T,$$

where e_a is the standard basis vector with a 1 in position a. For a $d-$regular graph, we now have

$$A_s = \sum_{(a,b)\in E} s_{(a,b)}A_{(a,b)} = \sum_{(a,b)\in E} (e_a + s_{(a,b)}e_b)(e_a + s_{(a,b)}e_b)^T - dI. \tag{1.1}$$

So, Theorem 1.2 is equivalent to the statement that there is a choice of $s_{(a,b)}$ for which

$$\lambda_{max}\left(\sum_{(a,b)\in E} (e_a + s_{(a,b)}e_b)(e_a + s_{(a,b)}e_b)^T\right) \leq d + 2\sqrt{d-1}.$$

The existence of such a choice can be written in probabilistic terms by defining for each $(a, b) \in E$ a random vector

$$r_{(a,b)} := \begin{cases} (e_a + e_b) & \text{with probability } 1/2 \text{ and} \\ (e_a - e_b) & \text{with probability } 1/2. \end{cases} \tag{1.2}$$

Notice that

$$\mathbb{E} \sum_{(a,b)\in E} r_{(a,b)} r_{(a,b)}^T = dI. \tag{1.3}$$

Thus, Theorem 1.2 is equivalent to the statement that for every $d-$regular $G = (V, E)$,

$$\lambda_{max}\left(\sum_{(a,b)\in E} r_{(a,b)} r_{(a,b)}^T\right) \leq \lambda_{max}\left(\mathbb{E} \sum_{(a,b)\in E} r_{(a,b)} r_{(a,b)}^T\right) + 2\sqrt{d-1} \tag{1.4}$$

with positive probability.

Such a sum may be analyzed using tools of random matrix theory, but this approach does not give the sharp bound we require, and it is known that it cannot in general as there are graphs for which the desired signing is exponentially rare (consider a union of disjoint cliques on d vertices).

The main subject of this survey is an approach that succeeds in proving (1.4) exactly. The methodology also succeeds in resolving several other important questions about sums of independent random rank one matrices, including Weaver's conjecture and thereby the Kadison–Singer problem. We review these first and describe their connection to Ramanujan graphs before proceeding to describe the actual technique. The proof of (1.4) and Theorem 1.2 will be sketched in Section 5.1.

1.2. Sparse approximations of graphs. Spielman and Teng [39] observed that one can view an expander graph as an approximation of a complete graph, and asked if one could find analogous approximations of arbitrary graphs. In this context, it is more natural to consider the class of general weighted graphs rather than just unweighted $d-$regular graphs, and to study the *Laplacian matrix* instead of the adjacency matrix. Recall that the Laplacian of a weighted graph $G = (V, E, w)$ may be defined as the following sum of rank one matrices over the edges:

$$L_G = \sum_{(a,b)\in E} w_{(a,b)}(e_a - e_b)(e_a - e_b)^T.$$

In the unweighted $d-$regular case, it is easy to see that $L = dI - A$, so the eigenvalues of the Laplacian are just d minus the eigenvalues of the adjacency matrix. The Laplacian matrix of a graph always has an eigenvalue of 0; this is a trivial eigenvalue, and the corresponding eigenvectors are the constant vectors.

Following Spielman and Teng, we say that two graphs G and H on the same vertex set V are *spectral approximations* of each other if their Laplacian quadratic forms multiplicatively approximate each other:

$$\kappa_1 \cdot x^T L_H x \le x^T L_G x \le \kappa_2 \cdot x^T L_H x \qquad \forall x \in \mathbb{R}^V,$$

for some approximation factors $\kappa_1, \kappa_2 > 0$. We will write this as

$$\kappa_1 \cdot L_H \preceq L_G \preceq \kappa_2 \cdot L_H,$$

where $A \preceq B$ means that $B - A$ is positive semidefinite, i.e., $x^T(B - A)x \ge 0$ for every x.

The complete graph on n vertices, K_n, is the graph with an edge of weight 1 between every pair of vertices. All of the eigenvalues of L_{K_n} other than 0 are equal to n. If G is a d-regular non-bipartite Ramanujan graph, then 0 is the trivial eigenvalue of its Laplacian matrix, L_G, and all of the other eigenvalues of L_G are between $d-2\sqrt{d-1}$ and $d+2\sqrt{d-1}$. After a simple rescaling, this allows us to conclude that

$$(1 - 2\sqrt{d-1}/d)L_{K_n} \preceq (n/d)L_G \preceq (1 + 2\sqrt{d-1}/d)L_{K_n}.$$

So, $(n/d)L_G$ is a good approximation of L_{K_n}.

Batson, Spielman and Srivastava proved that every weighted graph has an approximation that is almost this good.

Theorem 1.3 ([7]). *For every $d > 1$ and every weighted graph $G = (V, E, w)$ on n vertices, there exists a weighted graph $H = (V, F, \tilde{w})$ with $\lceil d(n-1) \rceil$ edges that satisfies:*

$$\left(1 - \frac{1}{\sqrt{d}}\right)^2 L_G \preceq L_H \preceq \left(1 + \frac{1}{\sqrt{d}}\right)^2 L_G. \tag{1.5}$$

However, their proof had very little to do with graphs. In fact, they derived their result from the following theorem about sparse weighted approximations of sums of rank one matrices.

Theorem 1.4 ([7]). *Let $v_1, v_2, \ldots, v_m$ be vectors in $\mathbb{R}^n$ with*

$$\sum_i v_i v_i^T = V.$$

For every $\epsilon \in (0, 1)$, there exist non-negative real numbers s_i with

$$|\{i : s_i \neq 0\}| \leq \lceil n/\epsilon^2 \rceil$$

so that

$$(1-\epsilon)^2 V \preceq \sum_i s_i v_i v_i^T \preceq (1+\epsilon)^2 V. \tag{1.6}$$

Taking V to be a Laplacian matrix written as a sum of outer products and setting $\epsilon = 1/\sqrt{d}$ immediately yields Theorem 1.3.

Theorem 1.4 is very general and turned out to be useful in a variety of areas including graph theory, numerical linear algebra, and metric geometry (see, for instance, the survey of Naor [34]). One of its limitations is that it provides no guarantees on the weights s_i that it produces, which can vary wildly. So it is natural to ask: is there a version of Theorem 1.4 in which all the weights are the same?

This may seem like a minor technical point, but it is actually a fundamental difference. In particular, Gil Kalai observed that the statement of Theorem 1.4 with $V = I$ is similar to Weaver's Conjecture, which was known to imply a positive solution to the Kadison–Singer Problem. It turns out that the natural unweighted variant of it is essentially *the same* as Weaver's conjecture. We discuss the Kadison–Singer problem and this connection in the next section.

1.3. The Kadison-Singer problem and Weaver's conjecture. In 1959, Kadison and Singer [26] asked the following fundamental question: does every pure state on the abelian von Neumann algebra $D(\ell_2)$ of diagonal operators on ℓ_2 have a unique extension to a pure state on $B(\ell_2)$, the von Neumann algebra of all bounded operators on ℓ_2? In their original paper, they suggested an approach to resolving this question: they showed that the answer is yes if every operator in $B(\ell_2)$ can be 'paved' by a constant number of operators which are strictly smaller in the operator norm. Beginning with the work of Anderson [4–6], this was shown to be equivalent to several combinatorial questions about decomposing *finite* matrices into a small number of strictly smaller pieces.

Among these questions is Akemann and Anderson's "projection paving conjecture" [2], which Nik Weaver [44] later showed was equivalent to the following discrepancy-theoretic conjecture that he called KS_2.

Conjecture 1.5. *There exist positive constants α and ϵ so that for every n and d and every set of vectors $v_1, \ldots, v_n \in \mathbb{C}^d$ such that $\|v_i\| \leq \alpha$ for all i and*

$$\sum_i v_i v_i^* = I,$$

there exists a partition of $\{1, \ldots, n\}$ into two sets S_1 and S_2 so that for each $j \in \{1, 2\}$

$$\left\| \sum_{i \in S_j} v_i v_i^* \right\| < 1 - \epsilon. \tag{1.7}$$

To see the similarity between this conjecture and Theorem 1.4, observe that for any partition $S_1 \cup S_2$:

$$\sum_{i \in S_1} v_i v_i^* + \sum_{i \in S_2} v_i v_i^* = I,$$

so that condition (1.7) is equivalent to

$$\epsilon I \preceq \sum_{i \in S_1} v_i v_i^* \preceq (1 - \epsilon) I.$$

Thus, choosing a subset of the weights s_i to be non-zero in Theorem 1.4 is similar to choosing the set S_1. The difference is that Conjecture 1.5 assumes a bound on the lengths of the vectors v_i and in return requires the stronger conclusion that all of the s_i are either 0 or 1. It is easy to see that long vectors are an obstacle to the existence of a good partition; an extreme example is provided by considering an orthonormal basis $e_1, \ldots, e_n$. Weaver's conjecture asserts that this is the only obstacle.

Overcoming this seemingly small difference turns out to require substantial new machinery beyond the techniques used in the proof of Theorem 1.4. However, much of this machinery is built on two key ideas which are contained in [7]. The first is the use of "barrier functions" to bound the roots of polynomials, which is discussed in Section 3.2. The second, which was presented purely for motivational purposes in [7], is the examination of expected characteristic polynomials.

As in the case of Ramanujan graphs, Weaver's conjecture can be written in terms of sums of independent random rank one matrices. Given vectors $v_1, \ldots, v_m \in \mathbb{C}^d$, define for each i the random vector $r_i \in \mathbb{C}^{2d}$

$$r_i = \begin{pmatrix} v_i \\ 0_d \end{pmatrix} \quad \text{with probability } 1/2 \quad \text{and} \quad \begin{pmatrix} 0_d \\ v_i \end{pmatrix} \quad \text{with probability } 1/2, \tag{1.8}$$

where $0_d \in \mathbb{C}^d$ is the zero vector. Then it is easy to see that every realization of $r_1, \ldots, r_m$ corresponds to a partition $S_1 \cup S_2 = [m]$ in the natural way, and that

$$\sum_i r_i r_i^* = \begin{pmatrix} \sum_{i \in S_1} v_i v_i^* & 0 \\ 0 & \sum_{i \in S_2} v_i v_i^* \end{pmatrix}.$$

Moreover, the norm of this matrix is the maximum of the norms of the matrices in the upper-left and lower-right blocks. Thus, Weaver's conjecture is equivalent to the statement that when the $\|v_i\| \leq \alpha$, the following holds with positive probability:

$$\lambda_{max} \left(\sum_{i=1}^m r_i r_i^* \right) \leq 1 - \epsilon \tag{1.9}$$

Once again, it is possible to apply tools of random matrix theory to analyze this sum. This gives a proof of the conjecture with $\alpha = 1/\log n$, essentially recovering a result of Bourgain and Tzafriri [14], which was essentially the best partial solution to Kadison–Singer until recently.

The main result of [31] is the following strong form of Weaver's conjecture.

Theorem 1.6. *Let $v_1, \ldots, v_m \in \mathbb{C}^d$ satisfy $\sum_i v_i v_i^* = I$ and $\|v_i\|^2 \leq \alpha$ for all i. Then, there exists a partition of $\{1, \ldots, m\}$ into sets S_1 and S_2 so that for $j \in \{1, 2\}$,*

$$\left\| \sum_{i \in S_j} v_i v_i^* \right\| \leq \frac{(1 + \sqrt{2\alpha})^2}{2}. \tag{1.10}$$

We will sketch the proof of Theorem 1.6, which is closely related to the proof of Theorem 1.2, in Sections 4 and 5.2.

1.4. Sums of independent rank one random matrices. As witnessed by equations (1.1) and (1.9), the common thread in the problems described above is that they can all be resolved by showing that a certain sum of independent random rank one matrices has small eigenvalues with nonzero probability. Prior to this line of work, there were already well-developed tools in random matrix theory for reasoning about such sums, generally called Matrix Chernoff Bounds [1, 37, 42]. As mentioned earlier, these provide bounds that are worse than those we require by a factor that is logarithmic in the dimension. However, they hold with high probability rather than the merely positive probability that we obtain.

Our approach to analyzing the eigenvalues of sums of independent rank one random matrices rests on the following connection between possible values of any particular eigenvalue, and the corresponding root of its expected characteristic polynomial. We will use $\lambda_1 \geq \lambda_2, \ldots, \geq \lambda_n \in \mathbb{R}$ to denote the eigenvalues of a Hermitian matrix as well as the roots of a real-rooted polynomial.

Theorem 1.7 (Comparison with Expected Polynomial). *Suppose $r_1, \ldots, r_m \in \mathbb{C}^n$ are independent random vectors. Then, for every k,*

$$\lambda_k \left(\sum_{i=1}^m r_i r_i^* \right) \leq \lambda_k \left(\mathbb{E}\chi \left[\sum_{i=1}^m r_i r_i^* \right] (x) \right),$$

with positive probability, and the same is true with $\geq$ instead of $\leq$.

In the special case when the r_i are identically distributed with $\mathbb{E} r_i r_i^* = I$, there is short proof of Theorem 1.7 that only requires univariate interlacing. We present this proof as Lemma 3.2 and and use it to establish a variant of Bourgain and Tzafriri's restricted invertibility theorem. In Section 4 we prove the theorem in full generality using tools from the theory of real stable polynomials. This yields mixed characteristic polynomials, which are then analyzed in Sections 5.1 and 5.2 to prove the existence of infinite families of Bipartite Ramanujan Graphs as well as Weaver's Conjecture

2. Interlacing polynomials

A defining characteristic of the proofs in [30] and [31] is that they analyze matrices solely through their characteristic polynomials. This is perhaps a counterintuitive way to proceed;

on the surface, we are losing information by considering characteristic polynomials, which only know about eigenvalues and not eigenvectors. However, the structure we gain far outweighs the losses in two ways: the characteristic polynomials satisfy a number of algebraic identities which make calculating their averages tractable, and they are amenable to a set of analytic tools that do not naturally apply to matrices.

As hinted at earlier, we study the roots of *averages* of polynomials. In general, averaging polynomials coefficient-wise can do unpredictable things to the roots. For instance, the average of $(x-1)(x-2)$ and $(x-3)(x-4)$, which are both real-rooted quadratics, is x^2-5x+7, which has complex roots $2.5 \pm \sqrt{3}i$. Even when the roots of the average are real, there is in general no simple relationship between the roots of two polynomials and the roots of their average.

The main insight is that there are nonetheless many situations where averaging the coefficients of polynomials also has the effect of averaging each of the roots individually, and that it is possible to identify and exploit these situations. The key to doing this systematically is the classical notion of *interlacing*.

Definition 2.1 (Interlacing). Let f be a degree n polynomial with real roots $\{\alpha_i\}$, and let g be degree n or $n-1$ with real roots $\{\beta_i\}$ (ignoring β_n in the degree $n-1$ case). We say that g interlaces f if their roots alternate, i.e.,

$$\beta_n \leq \alpha_n \leq \beta_{n-1} \leq \ldots \beta_1 \leq \alpha_1,$$

and the largest root belongs to f.

If there is a single g which interlaces a family of polynomials $f_1, \ldots, f_m$, we say that they have a *common interlacing*.

It is an easy exercise to show that $f_1, \ldots, f_m$ of degree n have a common interlacing iff there are closed intervals $I_n \leq I_{n-1} \leq \ldots I_1$ (where $\leq$ means to the left of) such that the ith roots of all the f_j are contained in I_i. It is also easy to see that a set of polynomials has a common interlacing iff every pair of them has a common interlacing (this may be viewed as Helly's theorem on the real line).

We now state our main theorem about averages of polynomials with common interlacings.

Theorem 2.2 (Lemma 4.1 in [30]). *Suppose $f_1, \ldots, f_m$ are real-rooted of degree n with positive leading coefficients. Let $\lambda_k(f_j)$ denote the k^{th} largest root of f_j and let μ be any distribution on $[m]$. If $f_1, \ldots, f_m$ have a common interlacing, then for all $k = 1, \ldots, n$*

$$\min_j \lambda_k(f_j) \leq \lambda_k(\mathbb{E}_{j \sim \mu} f_j) \leq \max_j \lambda_k(f_j).$$

The proof of this theorem is a three line exercise, which essentially amounts to applying the intermediate value theorem inside each interval I_i.

An important feature of common interlacings is that their existence is *equivalent* to certain real-rootedness statements. Often, this characterization gives us a systematic way to argue that common interlacings exist. The following seems to have been discovered a number of times. It appears as Theorem 2.1 of Dedieu [16], (essentially) as Theorem $2'$ of Fell [17], and as (a special case of) Theorem 3.6 of Chudnovsky and Seymour [15]. The proof of it included below assumes that the roots of a polynomial are continuous functions of its coefficients (which may be shown using elementary complex analysis).

Theorem 2.3. *If $f_1, \ldots, f_m$ are degree n polynomials and all of their convex combinations $\sum_{i=1}^m \mu_i f_i$ have real roots, then they have a common interlacing.*

Proof. Since common interlacing is a pairwise condition, it suffices to handle the case of two polynomials f_0 and f_1. Let

$$f_t := (1-t)f_0 + tf_1$$

with $t \in [0,1]$. Assume without loss of generality that f_0 and f_1 have no common roots (if they do, divide them out and put them back in at the end). As t varies from 0 to 1, the roots of f_t define n continuous curves in the complex plane $C_1, \ldots, C_n$, each beginning at a root of f_0 and ending at a root of f_1. By our assumption the curves must all lie in the real line. Observe that no curve can cross a root of either f_0 or f_1 in the middle: if $f_t(r) = 0$ for some $t \in (0,1)$ and $f_0(r) = 0$, then immediately we also have $f_t(r) = tf_1(r) = 0$, contradicting the no common roots assumption. Thus, each curve defines a closed interval containing exactly one root of f_0 and one root of f_1, and these intervals do not overlap except possibly at their endpoints, establishing the existence of a common interlacing. □

It is worth mentioning that the converse of Theorem 2.3 is true as well, but we will not use this fact.

While interlacing and real-rootedness are entirely univariate notions as discussed above, the most powerful ways to apply them arise by viewing them as restrictions of multivariate phenomena. There are two important generalizations of real-rootedness to more than one variable: real stability and hyperbolicity.

We were inspired by the development of the theory of real stability in the works of Borcea and Brändén, including [10–12]. Their results center primarily around characterizations of stable polynomials, including closure properties (that is, operations that preserve real stability of polynomials) and showing that properties of various mathematical structures an be related to the stability of some "generating polynomial" of that structure.

There is an isomorphism between real stable polynomials and *hyperbolic polynomials*, a concept that originated in a series of papers by Gårding [18] in his investigation of partial differential equations. The theory of hyperbolic polynomials was developed further in the optimization community (see the survey of Renegar [36]). However, it was not until Gurvits's use of hyperbolic polynomials in his proof of the van der Waerden conjecture [21], that their combinatorial power was revealed.

While it is well known that the concepts of real stability and hyperbolicity are essentially equivalent (one can translate easily between the two), various features of the way each property is defined have led to a natural separation of results: algebraic closure properties and characterization in real stability and analytic properties such as convexity in hyperbolicity. The "method of interlacing polynomials" discussed in this survey, is in many ways a recipe for mixing the ideas from these two communities into a single proof technique.

The method of interlacing polynomials consists of two somewhat distinct parts. The first is to show that a given collection of polynomials forms what we call an *interlacing family*, which is broadly speaking any class of polynomials for which the roots of its average can be related to those of the individual polynomials. This falls naturally into the realm of results regarding real stable polynomials as it often reduces to that showing various linear combinations of polynomials are real-rooted. The second part is to bound one of the roots of the expected polynomial under some distribution. This is more of an analytic task, for which the convexity properties studied in the context of hyperbolicity are relevant. For instance,

in [31], the analysis of the largest root is based on understanding the evolution of the root surfaces defined by a multivariate polynomial as certain differential operators are applied to it, and draws on the same convexity properties that are at the core of hyperbolic polynomials.

3. Restricted invertibility

The purpose of this section is to give the simplest possible demonstration of the method of interlacing families of polynomials. It will be completely elementary and self-contained, relying only on classical facts about univariate polynomials, and should be accessible to an undergraduate. Nonetheless, it is structurally almost identical to the proof of Weaver's conjecture and contains most of the same conceptual components in a primitive form.

Bourgain and Tzafriri's restricted invertibility theorem [13] states that any square matrix B with unit length columns and small operator norm contains a large column submatrix B_S which is well-invertible on its span. That is, the least singular value of the submatrix, $\sigma_{|S|}(B_S)$, is large. This may be seen as a robust, quantitative version of the fact that any matrix contains an invertible submatrix of size equal to its rank. The theorem was generalized to arbitrary rectangular B by Vershynin [43], and further sharpened in [38, 45]. We will give a proof of the following theorem from [38], which corresponds to the important case $BB^T = I$, when the columns of B are isotropic.

Theorem 3.1. *Suppose $v_1, \ldots, v_m \in \mathbb{C}^n$ are vectors with $\sum_{i=1}^m v_i v_i^T = I_n$. Then for every $k < n$ there is a subset $S \subset [m]$ of size k with*

$$\lambda_k \left(\sum_{i \in S} v_i v_i^T \right) \geq \left(1 - \sqrt{\frac{k}{n}} \right)^2 \frac{n}{m}.$$

The proof of this theorem has two parts. The first part is the special case of Theorem 1.7 in which $r_1, \ldots, r_n$ are independent and identically distributed (i.i.d.) and $\mathbb{E} r_i r_i^* = cI$. It reduces the problem of showing the existence of a good subset to that of analyzing the roots of the expected characteristic polynomial.

Lemma 3.2. *Suppose $r_1, \ldots, r_k$ are i.i.d. copies of a finitely supported random vector r with $\mathbb{E} rr^* = cI$. Then, with positive probability,*

$$\lambda_k \left(\sum_{i=1}^k r_i r_i^* \right) \geq \lambda_k \left(\mathbb{E}\chi \left[\sum_{i=1}^k r_i r_i^* \right] \right).$$

The second part is the calculation of the expected polynomial and the derivation of a bound on its roots.

Lemma 3.3. *Suppose $r_1, \ldots, r_k$ are i.i.d. copies of a random vector r with $\mathbb{E} rr^* = I$. Then,*

$$\mathbb{E}\chi \left[\sum_{i=1}^k r_i r_i^* \right](x) = (1 - D)^k x^n = x^{n-k}(1 - D)^n x^k.$$

Moreover,

$$\lambda_k \left((1 - D)^n x^k \right) \geq \left(1 - \sqrt{\frac{k}{n}} \right)^2 n.$$

3.1. Interlacing and $(1 - D)$ operators. Let us begin with the first part. To relate the expected characteristic polynomial to its summands, we will inductively apply Theorem 2.2, which requires the existence of certain common interlacings. These will be established by a combination of two ingredients. The first is the following classical fact, which says that rank-one updates naturally cause interlacing.

Lemma 3.4 (Cauchy's Interlacing Theorem). *If A is a symmetric matrix and v is a vector then $\chi[A](x)$ interlaces $\chi[A + vv^*](x)$.*

One can easily derive this from the *matrix determinant lemma*:

Lemma 3.5. *If A is an invertible matrix and u, v are vectors, then*

$$\det(A + uv^*) = \det(A)(1 + v^*A^{-1}u)$$

The second ingredient is the following correspondence between isotropic random rank one updates and differential operators.

Lemma 3.6. *Suppose r is a random vector with $\mathbb{E}rr^* = cI$ for some constant $c \geq 0$. Then for every matrix A, we have*

$$\mathbb{E}\chi[A + rr^*](x) = (I - cD)\chi[A](x),$$

where D denotes differentiation with respect to x.

Proof. Using Lemma 3.5, we obtain

$$\begin{aligned}\mathbb{E}\det(xI - A - rr^*) &= \mathbb{E}\det(xI - A)(1 - r^*(xI - A)^{-1}r)\\ &= \det(xI - A)(1 - \mathrm{Tr}\left[(\mathbb{E}rr^*)(xI - A)^{-1}\right])\\ &= \det(xI - A)\left(1 - c\mathrm{Tr}(xI - A)^{-1}\right)\end{aligned}$$

Letting $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of A, this quantity becomes

$$\prod_{i=1}^{n}(x - \lambda_i)\left(1 - c\sum_{i=1}^{n}\frac{1}{x - \lambda_i}\right) = \chi(A)(x) - c\sum_{i=1}^{n}\prod_{j\neq i}(x - \lambda_j) = (1 - cD)\chi(A)(x),$$

as desired. □

The purpose of Lemma 3.6 is twofold. First, it allows us to easily calculate expected characteristic polynomials, which a priori could be intractably complicated sums. Second, the operators $(1 - cD)$ have other nice properties which witness that the expected polynomials we generate have real roots and common interlacings.

Lemma 3.7 (Properties of Differential Operators).

(1) *If f has real roots then so does $(I - cD)f$.*

(2) *If $f_1, \ldots, f_m$ have a common interlacing, then so do $(I - cD)f_1, \ldots, (1 - cD)f_m$.*

Proof. For part (1), assume that f and f' have no common roots (otherwise, these are also common roots of f and $f - cf'$ which are clearly real). Consider the rational function

$$\frac{f(x) - cf'(x)}{f(x)} = 1 - c\frac{f'(x)}{f(x)} = 1 - c\sum_{i=1}^{n}\frac{1}{x - \lambda_i}$$

where λ_i are the roots of f. Inspecting the poles of this function and applying the intermediate value theorem shows that $f - cf'$ has the same number of zeros as f, all distinct from those of f.

For part (2), Theorem 2.2 tells us that all convex combinations $\sum_{i=1}^m \mu_i f_i$ have real roots. By part (1) it follows that all

$$(1 - cD)\sum_{i=1}^m \mu_i f_i = \sum_{i=1}^m \mu_i (1 - cD) f_i$$

also have real roots. By Theorem 2.3, this means that the $(1 - cD)f_i$ must have a common interlacing. □

With these facts in hand, we can easily complete the proof of Lemma 3.2.

Proof. Assume r is uniformly distributed on some set $v_1, \ldots, v_m \in \mathbb{C}^n$. We need to show that there is a choice of indices $j_1, \ldots, j_k \in [m]$ for which

$$\lambda_k\left(\sum_{i=1}^k v_{j_i} v_{j_i}^*\right) \geq \lambda_k\left(\mathbb{E}\chi\left[\sum_{i=1}^k r_i r_i^*\right]\right).$$

For any partial assignment $j_1, \ldots, j_\ell$ of the indices, consider the "conditional expectation" polynomial:

$$q_{j_1,\ldots,j_\ell}(x) := \mathbb{E}_{r_{\ell+1},\ldots,r_k}\chi\left[\sum_{i=1}^\ell v_{j_i} v_{j_i}^* + \sum_{i=\ell+1}^k r_i r_i^*\right].$$

Since the r_i are independent, and $\mathbb{E} r_i = (1/m)I$, applying Lemma 3.6 $k - \ell$ times reveals that:

$$q_{j_1,\ldots,j_\ell}(x) = (1 - (1/m)D)^{k-\ell}\chi\left[\sum_{i=1}^\ell v_{j_i} v_{j_i}^*\right](x).$$

We will show that there exists a $j_{\ell+1} \in [m]$ such that

$$\lambda_k(q_{j_1,\ldots,j_{\ell+1}}) \geq \lambda_k(q_{j_1,\ldots,j_\ell}), \tag{3.1}$$

which by induction will complete the proof. Consider the matrix

$$A = \sum_{i=1}^\ell v_{j_i} v_{j_i}^*,$$

By Lemma 3.4, $\chi[A]$ interlaces $\chi[A + v_{j_{\ell+1}} v_{j_{\ell+1}}^*]$ for every $j_{\ell+1} \in [m]$. Lemma 3.7 tells us $(1 - (1/m)D)$ operators preserve common interlacing, so the polynomials

$$(1 - (1/m)D)^{k-(\ell+1)}\chi(A + v_{j_{\ell+1}} v_{j_{\ell+1}}^*) = q_{j_1,\ldots,j_\ell,j_{\ell+1}}(x)$$

must also have a common interlacing. Thus, some $j_{\ell+1} \in [m]$ must satisfy (3.1), as desired. □

3.2. Laguerre polynomials and the univariate barrier argument. We now move on to the second part, Lemma 3.3, in which we prove a bound on the kth root of the expected polynomial, which after rescaling by a factor of m is just:

$$\mathbb{E}\chi\left[m\cdot\sum_{i=1}^{k}r_ir_i^*\right](x)=(1-D)^kx^n.$$

We begin by observing that $(1-D)^kx^n = x^{n-k}(1-D)^nx^k$. This may be verified by term-by-term calculation, or by appealing to the correspondence between $(1-D)$ operators and random isotropic rank one updates established in Lemma 3.6 as follows. Let G be an n-by-k matrix of random, independently distributed, $N(0,1)$ entries. The covariance matrix of each column is the n-dimensional identity matrix, and the covariance of each row is the k-dimensional identity. So,

$$\begin{aligned}(1-D)^kx^n&=\mathbb{E}_G\chi(GG^*)(x)\\&=\mathbb{E}_Gx^{n-k}\chi(G^*G)(x)\\&=x^{n-k}(1-D)^nx^k.\end{aligned}$$

Thus, we would like to lower bound the least root of $(1-D)^nx^k$. The easiest way to do this is to observe that it is a constant multiple of a known polynomial, namely an *associated Laguerre polynomial* $\mathcal{L}_k^{(n-k)}(x)$. These are classical orthogonal polynomials and a lot is known about the locations of their roots; in particular, they are known to be contained in the interval $[n(1-\sqrt{k/n})^2, n(1+\sqrt{k/n})^2]$ (see, for instance, [27]).

In order to keep the presentation self-contained, and also because it is a key tool in the proof of Kadison–Singer and more generally in the analysis of expected characteristic polynomials, we now give a direct proof of Lemma 3.3 based on the "barrier method" introduced in [7]. The basic idea is to study the effect of each $(1-D)$ operator on the roots of a polynomial f via the associated rational function

$$\Phi_f(b):=-\frac{f'(b)}{f(b)}=-\frac{\partial\log f(b)}{\partial b}=\sum_{i=1}^{n}\frac{1}{\lambda_i-b},\tag{3.2}$$

which we will refer to as the *lower barrier function*. The poles of this function are the roots $\lambda_1,\dots,\lambda_n$ of f, and we remark that it is the same up to a multiplicative factor of $(-1/n)$ as the Stieltjes transform of the discrete measure supported on these roots. It is immediate from the above expression that $\Phi_f(b)$ is positive, monotone increasing, and convex for b is strictly less than the roots of f, and that it tends to infinity as b approaches the smallest root of f from below.

We now use the *inverse* of Φ_f to define a robust lower bound for the roots of a polynomial f:

$$\operatorname{smin}_\varphi(f):=\min\{x\in\mathbb{R}:\Phi_f(x)=\varphi\},$$

where $\varphi>0$ is a sensitivity parameter. Since $\Phi_f(b)\to 0$ as $b\to-\infty$, it is immediate that we always have $\operatorname{smin}_\varphi(f)\le\lambda_{min}(f)$. The number φ controls the tradeoff between how accurate a lower bound $\operatorname{smin}_\varphi$ is an how smoothly it varies — in particular the extreme cases are $\operatorname{smin}_\infty(f)=\lambda_{min}(f)$, which is not always well-behaved, and $\operatorname{smin}_0(f)=-\infty$, which doesn't even depend on f. This quantity was implicitly introduced and used in [7]

and explicitly defined in [41], where it was called the 'soft spectral edge'; for an intuitive discussion of its behavior in terms of an electrical repulsion model, we refer the reader to the latter paper.

We also remark that the inverse Stieltjes transform was used by Voiculescu in his development of Free Probability theory to study the limiting spectral distributions of certain random matrix ensembles as the dimension tends to infinity. We view the use of smin as a non-asymptotic analogue of that idea, except that we use it to reason about the edge of the spectrum rather than the bulk.

The following lemma tells us that $\mathrm{smin}_\varphi(f)$ grows in a smooth and predictable way when we apply a $(1-D)$ operator to f. It is similar to Lemma 3.4 of [7], which was written in the language of random rank one updates of matrices.

Lemma 3.8. *If f has real roots and $\varphi > 0$, then*

$$\mathrm{smin}_\varphi((1-D)f) \geq \mathrm{smin}_\varphi(f) + \frac{1}{1+\varphi}.$$

Proof. Let $b = \mathrm{smin}_\varphi(f)$. To prove the claim it suffices to find a $\delta \geq (1+\varphi)^{-1}$ such that $b+\delta$ is below the roots of f and $\Phi_{(1-D)f}(b+\delta) \leq \varphi$. We begin by writing the barrier function of $(1-D)$ in terms of the barrier function of f:

$$\Phi_{(1-D)f} = -\frac{(f-f')'}{f-f'} = -\frac{(f(1+\Phi_f))'}{f(1+\Phi_f)} = -\frac{f'}{f} - \frac{\Phi_f'}{1+\Phi_f} = \Phi_f - \frac{\Phi_f'}{1+\Phi_f}. \quad (3.3)$$

This identity tells us that for any $\delta \geq 0$:

$$\Phi_{(1-D)f}(b+\delta) = \Phi_f(b+\delta) - \frac{\Phi_f'(b+\delta)}{1+\Phi_f(b+\delta)},$$

which is at most $\varphi = \Phi_f(b)$ whenever

$$\frac{\Phi_f'(b+\delta)}{1+\Phi_f(b+\delta)} \geq \Phi_f(b+\delta) - \Phi_f(b).$$

This is in turn equivalent to

$$\frac{\Phi_f'(b+\delta)}{\Phi_f(b+\delta) - \Phi_f(b)} - \Phi_f(b+\delta) \geq 1.$$

Expanding each Φ_f as a sum of terms as in (3.2) and applying Cauchy-Schwartz appropriately reveals[1] that the left-hand side of this inequality it at least

$$1/\delta - \Phi_f(b)$$

This is at least 1 for all $\delta \leq (1+\varphi)^{-1}$.

We conclude that $\Phi_{(1-D)f}(b+\delta)$ is bounded by φ for all $\delta \in [0, (1+\varphi)^{-1}]$, which implies in particular that $b+\delta$ is below the roots of $(1-D)f$. □

[1]The simple but slightly cumbersome calculation appears as Claim 3.6 of [7]; we have chosen to omit it here for the sake of brevity.

Applying the lemma n times immediately yields the following bound on our polynomial of interest:

$$\begin{aligned}\lambda_k\left((1-D)^n x^k\right) &\geq \mathrm{smin}_\varphi\left((1-D)^n x^k\right)\\ &\geq \mathrm{smin}_\varphi(x^k) + \frac{n}{1+\varphi}\\ &= -\frac{k}{\varphi} + \frac{n}{1+\varphi} \qquad \text{since } \Phi_{x^k}(b) = -k/b.\end{aligned}$$

Setting $\varphi = \frac{\sqrt{k}}{\sqrt{n}-\sqrt{k}}$ yields Lemma 3.3, completing the proof of Theorem 3.1. We remark that we have, as a byproduct, derived a sharp bound on the least root of an associated Laguerre polynomial.

In Lemma 5.2 we use a multivariate version of the analogous bound for the largest root of the associated Laguerre polynomial. A crucial aspect of the proof of the upper bound on the largest root is that it essentially depends only on the convexity and monotonicity of the barrier function. For a real-rooted polynomial f, we define the *upper barrier function* as $\Phi^f(b) = f'(b)/f(b)$ and

$$\mathrm{smax}_\varphi(f) := \max\{x \in \mathbb{R} : \Phi^f(x) = \varphi\}.$$

Lemma 3.9. *If f has real roots and $\varphi > 0$, then*

$$\mathrm{smax}_\varphi((1-D)f) \leq \mathrm{smax}_\varphi(f) + \frac{1}{1-\varphi}.$$

Proof. Let $b = \mathrm{smax}_\varphi(f)$. As before, we may derive

$$\Phi^{(1-D)f} = \Phi^f - (D\Phi^f)/(1-\Phi^f).$$

So, to show that

$$\mathrm{smax}_\varphi((1-D)f) \leq b + \delta,$$

it suffices to prove that

$$\Phi^f(b) - \Phi^f(b+\delta) \geq \frac{-D\Phi^f(b+\delta)}{1-\Phi^f(b+d)}.$$

As $\Phi^f(b)$ is monotone decreasing for b above the roots of f, $D\Phi^f(b+\delta)$ is negative. As $\Phi^f(b)$ is convex for the same b,

$$\Phi^f(b) - \Phi^f(b+\delta) \geq \delta(-D\Phi^f(b+\delta)).$$

Thus, we only require

$$\delta \geq \frac{1}{1-\Phi^f(b+d)}.$$

As $\Phi^f(b)$ is monotone decreasing, this is satisfied for $\delta = 1/(1-\varphi)$. □

Setting $\varphi = \frac{\sqrt{k}}{\sqrt{n}+\sqrt{k}}$, we obtain our upper bound the largest root of an associated Laguerre polynomial.

Lemma 3.10. *The largest root of $(1-D)^n x^k$ is at most $n(1+\sqrt{k/n})^2$.*

4. Mixed characteristic polynomials

The argument given in the previous section is a special case of a more general principle: that the expected characteristic polynomials of certain random matrices can be expressed in terms of differential operators, which can then be used to establish the existence of common interlacings as well as to analyze the roots of the expected polynomials themselves. In the isotropic case of Bourgain–Tzafriri, this entire chain of reasoning can be carried out by considering univariate polynomials only. Morally, this is because the covariance matrices of all of the random vectors involved are multiples of the identity (which trivially commute with each other), and all of the characteristic polynomials involved are simple univariate linear transformations of each other (of type $(I - cD)$).

On the other hand, the proofs of Kadison-Singer and existence of Ramanujan graphs involve analyzing sums of independent rank one matrices which come from *non-identically distributed* distributions whose covariance matrices do not commute. This leads to a much more general family of expected polynomials which we call *mixed characteristic polynomials*. The special structure of these polynomials is revealed crisply when we view them as restrictions of certain multivariate polynomials. Their qualitative and quantitative properties are, correspondingly, established using multivariate differential operators and barrier functions, which are analyzed using tools from the theory of real stable polynomials.

In the remainder of this section we will sketch a proof of Theorem 1.7. The proof hinges on the following central identity, which describes the general correspondence between sums of independent random rank one matrices and (multivariate) differential operators.

Theorem 4.1. *Let $r_1, \ldots, r_m$ be independent random column vectors in $\mathbb{C}^d$. For each i, let $A_i = \mathbb{E} r_i r_i^*$. Then,*

$$\mathbb{E}\chi\left[\sum_{i=1}^{m} r_i r_i^*\right](x) = \left(\prod_{i=1}^{m} 1 - \partial_{z_i}\right) \det\left(xI + \sum_{i=1}^{m} z_i A_i\right)\Big|_{z_1=\cdots=z_m=0}. \tag{4.1}$$

In particular, the expected characteristic polynomial of a sum of independent rank one Hermitian random matrices is a function of the covariance matrices A_i. We call this polynomial the *mixed characteristic polynomial* of $A_1,\ldots,A_m$, and denote it by $\mu\left[A_1,\ldots,A_m\right](x)$. The name *mixed characteristic polynomial* is inspired by the fact that the expected determinant of this matrix is called the mixed discriminant. Notice that when $A_1 = A_2 = \ldots = A_m = I$, it is just a multiple of an associated Laguerre polynomial as in Section 3.

Theorem 4.1 may be proved fairly easily by inductively applying an identity similar to Lemma 3.6 or by appealing to the Cauchy-Binet formula; we refer the reader to [31] for a short proof. We remark that it and all of the other results in this section depend crucially on the fact that the $r_i r_i^*$ are rank one, and fail rather spectacularly for rank 2 or higher matrices.

The most important consequence of Theorem 4.1 is that mixed characteristic polynomials always have real roots. To prove this, we will need to consider a multivariate generalization of real-rootedness called real stability.

Definition 4.2. A multivariate polynomial $f \in \mathbb{R}[z_1, \ldots, z_m]$ is *real stable* if it has no roots with all coordinates strictly in the upper half plane, i.e., if

$$\mathsf{Im}(z_i) > 0 \quad \forall i \quad \Rightarrow f(z_1, \ldots, z_m) \neq 0.$$

Notice that stability is the same thing as real rootedness in the univariate case, since complex roots occur in conjugate pairs.

A natural and relevant example of real stable polynomials is the following:

Lemma 4.3 ([10]). *If $A_1, \ldots, A_m$ are positive semidefinite matrices, then*

$$f(z_1, \ldots, z_m) = \det\left(\sum_{i=1}^{m} z_i A_i\right)$$

is real stable.

One reason real stability is such a useful notion for us is that it has remarkable closure properties which are extremely well-understood In particular, Borcea and Brändén have completely characterized the linear operators preserving real stability [12]. What this means heuristically is that proofs of stability can often be reduced to a formal exercise: to prove that a particular polynomial is stable, one must simply write it as a composition of known stability-preserving operations.

To prove that mixed characteristic polynomials are real stable, we will only require the following elementary closure properties.

Lemma 4.4 (Closure Properties). *If $f(z_1, \ldots, z_m)$ is real stable, then so are*

$$(1 - \partial_{z_i}) f \qquad \textit{for every } i$$

and

$$f(\alpha, z_2, \ldots, z_m) \qquad \textit{for every } \alpha \in \mathbb{R}.$$

The first part was essentially established by Lieb and Sokal in [28]. It follows easily by considering a univariate restriction to z_i and studying the associated rational function, as in the the (entirely univariate) proof of Lemma 3.7. The second part is trivial for α strictly in the upper half plane, and may be extended to the real line by appealing to Hurwitz's theorem.

Combining these properties with Theorem 4.1 instantly establishes the following important fact.

Theorem 4.5. *If $A_1, \ldots, A_m$ are positive semidefinite, then $\mu[A_1, \ldots, A_m](x)$ is real-rooted.*

We are now in a position to prove Theorem 1.7. As in Lemma 3.2, we will do this inductively by showing that the relevant "conditional expectation" polynomials have common interlacings. However, instead of explicitly finding these common interlacings using Cauchy's theorem, we will guarantee their existence implicitly using Theorem 4.5.

Proof of Theorem 1.7. For any partial assignment $v_1, \ldots, v_\ell$ of $r_1, \ldots, r_\ell$, consider the conditional expected polynomial

$$q_{v_1,\ldots,v_\ell}(x) := \mathbb{E}\chi\left[\sum_{i=1}^{\ell} v_i v_i^* + \sum_{i=\ell+1}^{m} r_i r_i^*\right](x).$$

Suppose $r_{\ell+1}$ is supported on $w_1, \ldots, w_N$. Then, for all convex coefficients $\sum_{i=1}^{N} \mu_i = 1, \mu_i \geq 0$, the convex combination

$$\sum_{i=1}^{N} \mu_i q_{v_1,\ldots,v_\ell,w_i}(x)$$

is itself a mixed characteristic polynomial, namely

$$\mu\left[v_1v_1^*,\ldots,v_\ell v_\ell^*,\sum_{i=1}^N \mu_i w_i w_i^*,\mathbb{E}r_{\ell+2}r_{\ell+2}^*,\ldots,\mathbb{E}r_m r_m^*\right](x),$$

which has real roots by Theorem 4.5. This establishes that the $q_{v_1,\ldots,v_\ell,w_i}(x)$ have a common interlacing, which by Theorem 2.2 implies that for every k there exists an $i\in[N]$ for which

$$\lambda_k\left(q_{v_1,\ldots,v_m,w_i}(x)\right)\le\lambda_k\left(q_{v_1,\ldots,v_m}(x)\right),$$

completing the induction. □

The above proof highlights the added flexibility of allowing the r_i to have different distributions: by taking some of these distributions to be deterministic, we can encode any conditioning and more generally any addition of a positive semidefinite matrix while remaining in the class of mixed characteristic polynomials.

5. Analysis of expected polynomials

In this section, we describe two situations in which we are able to bound the largest roots of mixed characteristic polynomials. The first is very specific: we observe that the expected characteristic polynomial of a random signing of an adjacency matrix of a graph is equal, up to a shift, to the matching polynomial of the graph. The zeros of this polynomial have been studied for decades and elementary combinatorial arguments due to Heilmann and Lieb [23] can be used to give a sharp bound on its largest root. The main consequence of this bound is the existence of infinite families of bipartite Ramanujan graphs of every degree.

The second situation is almost completely general. We show that given any collection of matrices satisfying $\sum_{i=1}^m A_i = I$, the mixed characteristic polynomial $\mu\left[A_1,\ldots,A_m\right](x)$ has roots bounded by $(1+\sqrt{\max_i \mathrm{Tr}(A_i)})^2$. This is achieved by a direct multivariate generalization of the barrier function argument that we used in Section 3 to upper bound the roots of associated Laguerre polynomials. The main consequence of this bound is a proof of Weaver's conjecture and thereby a positive solution to the Kadison–Singer problem.

5.1. Matching polynomials. We are now ready to prove the bound (1.4) and thereby Theorem 1.2. For any $d-$regular graph $G=(V,E)$, let the random vectors $\{r_{(a,b)}\}_{(a,b)\in E}$ be defined as in (1.2). Applying Theorem 1.7 with $k=1$ and subtracting d from both sides, we find that:

$$\begin{aligned}\lambda_{max}\left(\sum_{(a,b)\in E} r_{(a,b)}r_{(a,b)}^* - dI\right) &= \lambda_{max}\left(\sum_{(a,b)\in E} r_{(a,b)}r_{(a,b)}^*\right) - d\\ &\le \lambda_{max}\left(\mathbb{E}\chi\left[\sum_{(a,b)\in E} r_{(a,b)}r_{(a,b)}^*\right](x)\right) - d\\ &= \lambda_{max}\left(\mathbb{E}\chi\left[\sum_{(a,b)\in E} r_{(a,b)}r_{(a,b)}^* - dI\right](x)\right),\end{aligned}$$

with positive probability. Switching back to signed adjacency matrices by applying (1.1), we conclude that

$$\lambda_{max}(A_s) \leq \lambda_{max}\left(\mathbb{E}\chi\left[A_s\right](x)\right) \tag{5.1}$$

with positive probability for a uniformly random signing A_s.

We now observe that this expected characteristic polynomial is equal to the *matching polynomial* of the graph. A matching is a graph in which every vertex has degree at most one. The matching polynomial is a generating function which counts the number of matchings that are subgraphs of a graph; for a graph on n vertices, it is defined as

$$\mu_G(x) := \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i x^{n-2i} m_i,$$

where m_i is the number of subgraphs of G with i edges that are matchings.

Godsil and Gutman [20] showed that the matching polynomial of a graph is equal to the expected characteristic polynomial of a random signing of its adjacency matrix:

$$\mathbb{E}\chi\left[A_s\right](x) = \mu_G(x). \tag{5.2}$$

This identity may be proved easily by expanding $\chi\left[A_s\right](x) = \det(xI - A_s)$ as a sum of permutations and observing that the only terms that do not vanish are the permutations with all orbits of size two, which correspond to the matchings.

About a decade before this, Heilmann and Lieb [23] studied the matching polynomial in the context of monomer-dimer systems in statistical physics. In that paper, they showed that $\mu_G(x)$ always has all real roots (a fact which we have also just proved by writing it as a shift of a mixed characteristic polynomial), and that

$$\lambda_{max}(\mu_G(x)) \leq 2\sqrt{d-1} \tag{5.3}$$

for a graph with maximum degree d. They proved this bound by finding certain simple combinatorial recurrences satisfied by $\mu_G(x)$, induced by edge and vertex deletions. The appearance of the number $2\sqrt{d-1}$ is not a coincidence; Godsil [19] later showed using similar recurrences that $\mu_G(x)$ divides the characteristic polynomial of a certain tree associated with G, which is an induced subgraph of the infinite $d-$regular tree.

Combining (5.1), (5.2), and (5.3) yields Theorem 1.2. There is also a generalization of this theorem which proves the existence of "irregular" Ramanujan graphs, which were not previously known to exist; we refer the interested reader to [30] for details.

5.2. The multivariate barrier argument. The tight bound of $2\sqrt{d-1}$ obtained above relies heavily on the fact that the random vectors $r_{(a,b)}$ of interest come from a graph and have combinatorial structure. Remarkably, it turns out that we can prove a bound that is almost as sharp by completely ignoring this structure and relying only on the much weaker property that the rr^* are rank one matrices of bounded trace. This type of generic bound is precisely what one needs to control the roots of the quite general mixed characteristic polynomials which arise in the proof of Weaver's conjecture, and thereby prove Kadison–Singer.

Theorem 5.1. *Suppose $A_1, \ldots, A_m$ are positive semidefinite matrices with $\sum_{i=1}^m A_i = I$ and $\mathrm{Tr}(A_i) \leq \epsilon$. Then,*

$$\lambda_{max}\left(\mu\left[A_1, \ldots, A_m\right](x)\right) \leq (1+\sqrt{\epsilon})^2. \tag{5.4}$$

At a high level, the proof of this theorem is very similar to that of Lemma 3.10: we express $\mu[A_1, \ldots, A_m](x)$ as a product of differential operators applied to some nice initial polynomial, and show that each differential operator perturbs the roots in a predictable way. The difference is that the differential operators and roots are now multivariate rather than univariate.

To deal with this issue, we begin by defining a notion of multivariate upper bound: we say that $b \in \mathbb{R}^m$ is *above* the roots of a real stable polynomial $f(z_1, \ldots, z_m)$ if $f(z) > 0$ for all $z \geq b$ coordinate-wise. It is best to think of an "upper bound" for the roots of f as a set rather than as a single point — the set of all points above the roots of f.

As we did in the univariate case, we soften this notion by studying certain rational functions associated with f which interact naturally with the $(1-\partial_{z_j})$ operators we are interested in. For each coordinate j, define the *multivariate barrier function*

$$\Phi_j^f(z_1, \ldots, z_m) = \frac{\partial z_j f(z_1, \ldots, z_m)}{f(z_1, \ldots, z_m)},$$

and notice that

$$\Phi_j^f(z_1, \ldots, z_m) = \sum_{i=1}^{d} \frac{1}{z_j - \lambda_i},$$

where $\lambda_1, \ldots, \lambda_d$ are the roots of the univariate restriction obtained by fixing all the coordinates other than z_j.

For a sensitivity parameter $\varphi < 1$, we define a *φ-robust upper bound* on $f(z_1, \ldots, z_m)$ to be any point b above the roots of f with $\Phi_j^f(b) \leq \varphi$ for all j. We denote the set of all such robust upper bounds by $\overrightarrow{\operatorname{smax}}_\varphi(f)$. The following multivariate analogue of Lemma 3.9 holds for $\overrightarrow{\operatorname{smax}}$. It says that applying an $(1 - \partial_{z_j})$ operator simply moves the set of robust upper bounds in direction j by a small amount.

Lemma 5.2. *If $f(z_1, \ldots, z_m)$ is real stable and $\varphi < 1$, then*

$$\overrightarrow{\operatorname{smax}}_\varphi\left((1 - \partial_{z_j})f\right) \supseteq \overrightarrow{\operatorname{smax}}_\varphi(f) + \frac{1}{1-\varphi} \cdot e_j,$$

where e_j is the elementary basis vector in direction j.

The proof of this lemma is syntactically almost identical to that of Lemma 3.9, except that it is less obvious that the barrier functions Φ_j^f are monotone and convex in the coordinate directions. In [31] we prove this by appealing to a powerful representation theorem of Helton and Vinnikov [24], which says that bivariate restrictions of real stable polynomials can always be written as determinants of positive semidefinite matrices, which are easy to analyze. Later, elementary proofs of this fact were given by James Renegar (using tools from the theory of hyperbolic polynomials [8]) and Terence Tao (using a combination of elementary calculus and complex analysis, along with Bezout's theorem).

With Lemma 5.2 in hand, one can prove Theorem 5.1 by an induction similar to the one we used in Lemma 3.3. We refer the reader to [31] for details.

Applying Theorems 1.7 and 5.1 to the random vectors defined in (1.8) immediately yields Theorem 1.6.

6. Ramanujan graphs and Weaver's conjecture

We conclude by showing how the generic bound derived above may be used to analyze the random signings that occur in the proof of Theorem 1.2. This turns out to be very instructive and is quite natural, since when $G = (V, E)$ is $d-$regular, (1.3) tells us that

$$\mathbb{E} \sum_{(a,b)\in E} \frac{r_{(a,b)} r_{(a,b)}^*}{d} = I.$$

Thus, each vector has the same norm $\|r_{(a,b)}\|^2 = 2/d$, and applying Theorems 1.7 and 5.1 shows that

$$\sum_{(a,b)\in E} r_{(a,b)} r_{(a,b)}^* \leq d \left(1 + \sqrt{\frac{2}{d}}\right)^2 = d + 2 + 2\sqrt{2d}$$

with positive probability. This bound has asymptotically the same dependence on d as the correct bound established using matching polynomials. Moreover, it immediately proves that the dependence on ϵ in Theorem 5.1 cannot be improved: if it could, the above argument would imply the existence of signings with largest eigenvalue $o(\sqrt{d})$, contradicting the Alon–Boppana bound. Thus, the matrices arising in the study of Ramanujan graphs witness the sharpness of our bounds on mixed characteristic polynomials. [2]

Acknowledgements. Adam W. Marcus: Research partially supported by an NSF Mathematical Sciences Postdoctoral Research Fellowship, Grant No. DMS-0902962. Daniel A. Spielman: Research partially supported by NSF grants CCF-0915487 and CCF-1111257, a Simons Investigator Award, and a MacArthur Fellowship.

References

[1] R. Ahlswede and A. Winter, *Strong converse for identification via quantum channels*, Information Theory, IEEE Transactions on, **48**(3) (2002), 569–579.

[2] C. A. Akemann and J. Anderson, *Lyapunov theorems for operator algebras*, Number 458. American Mathematical Soc., 1991.

[3] N. Alon, *Eigenvalues and expanders*, Combinatorica, **6**(2) (1986), 83–96.

[4] J. Anderson, *Extensions, restrictions, and representations of states on C^*-algebras*, Transactions of the American Mathematical Society, **249**(2) (1979), 303–329.

[5] _____, *Extreme points in sets of positive linear maps on $\mathcal{B}(\mathcal{H})$*, Journal of Functional Analysis, **31**(2) (1979), 195–217.

[6] _____, *A conjecture concerning the pure states of $\mathcal{B}(\mathcal{H})$ and related theorem*, Topics in modern operator theory (Timisoara/Herculane, 1980), Birkhäuser, Basel-Boston, Mass, 1981, pp. 27–43.

[2] We remark that the Alon–Boppana bound can also be used to show that the dependence on α in Theorem 1.6 itself is tight by recursively applying it to the Laplacian of the complete graph. We point the reader to [40] or [22] for a complete argument.

[7] J. Batson, D. A. Spielman, and N. Srivastava, *Twice-Ramanujan sparsifiers*, SIAM Journal on Computing, **41**(6) (2012), 1704–1721.

[8] H. H. Bauschke, O. Güler, A. S. Lewis, and H. S. Sendov, *Hyperbolic polynomials and convex analysis*, Canadian Journal of Mathematics, **53**(3) (2001), 470–488.

[9] Y. Bilu and N. Linial, *Lifts, discrepancy and nearly optimal spectral gap*, Combinatorica, **26**(5) (2006), 495–519.

[10] J. Borcea and P. Brändén, *Applications of stable polynomials to mixed determinants: Johnson's conjectures, unimodality, and symmetrized fischer products*, Duke Mathematical Journal, **143** (2008), 205–223.

[11] ———, *The Lee–Yang and Pólya–Schur programs, I. Linear operators preserving stability*, Invent. Math., **177**(3) (2009), 541–569.

[12] ———, *Multivariate Polya-Schur classification problems in the Weyl algebra*, Proc. London Mathematical Society, (3) **101**(1) (2010), 73–104.

[13] J. Bourgain and L. Tzafriri, *Invertibility of large submatrices with applications to the geometry of Banach, spaces and harmonic analysis*, Israel journal of mathematics, **57**(2) (2010), 137–224.

[14] ———, *On a problem of Kadison and Singer*, J. reine angew. Math, **420** (1991), 1–43.

[15] M. Chudnovsky and P. Seymour, *The roots of the independence polynomial of a clawfree graph*, Journal of Combinatorial Theory, Series B, **97**(3) (2007), 350–357.

[16] J. P. Dedieu, *Obreschkoff's theorem revisited: what convex sets are contained in the set of hyperbolic polynomials?*, J. Pure and Applied Algebra, **81**(3) (1992), 269–278.

[17] H. J. Fell, *Zeros of convex combinations of polynomials*, Pacific J. Math., **89**(1) (1980), 43–50.

[18] L. Gårding, *Linear hyperbolic partial differential equations with constant coefficients*, Acta Mathematica, **85**(1) (1951), 1–62.

[19] C. D. Godsil, *Matchings and walks in graphs*, J. Graph Theory, **5**(3) (1981), 285–297.

[20] C. D. Godsil and I. Gutman, *On the matching polynomial of a graph*, In L. Lovász and V. T. Sós, editors, Algebraic Methods in graph theory, volume I of Colloquia Mathematica Societatis János Bolyai, 25, 1981, pp. 241–249.

[21] L. Gurvits, *Van der Waerden/Schrijver-Valiant like conjectures and stable (aka hyperbolic) homogeneous polynomials: one theorem for all*, the electronic journal of combinatorics, **15**(R66) (2008), 1.

[22] N. J. A. Harvey and N. Olver, *Pipage rounding, pessimistic estimators and matrix concentration*, In ACM-SIAM Symposium on Discrete Algorithms, 2014.

[23] O. J. Heilmann and E. H. Lieb, *Theory of monomer-dimer systems*, Communications in Mathematical Physics, **25**(3) (1972), 190–232.

[24] J. W. Helton and V. Vinnikov, *Linear matrix inequality representation of sets*, Communications on pure and applied mathematics, **60**(5) (2007), 654–674.

[25] S. Hoory, N. Linial, and A. Wigderson, *Expander graphs and their applications*, Bulletin of the American Mathematical Society, **43**(4) (2006), 439–561.

[26] R. V. Kadison and I. M. Singer, *Extensions of pure states*, American Jour. Math., **81** (1959), 383–400.

[27] I. Krasikov, *On extreme zeros of classical orthogonal polynomials*, Journal of computational and applied mathematics, **193**(1) (2006), 168–182.

[28] E. H. Lieb and A. D. Sokal, *A general Lee-Yang theorem for one-component and multicomponent ferromagnets*, Communications in Mathematical Physics, **80**(2) (2006), 153–179.

[29] A. Lubotzky, R. Phillips, and P. Sarnak, *Ramanujan graphs*, Combinatorica, **8**(3) (1988), 261–277.

[30] A. W. Marcus, D. A. Spielman, and N. Srivastava, *Interlacing families I: Bipartite Ramanujan graphs of all degrees*, In Proceedings of the 54th IEEE Symposium on Foundations of Computer Science, 2013, pp. 529–537.

[31] ———, *Interlacing families II: Mixed characteristic polynomials and the Kadison–Singer problem*, preprint arXiv:1306.3969, 2013.

[32] G. A. Margulis, *Explicit constructions of concentrators*, Problemy Peredachi Informatsii, **9**(4) (October-December 1973), 71–80.

[33] ———, *Explicit group theoretical constructions of combinatorial schemes and their application to the design of expanders and concentrators*, Problems of Information Transmission, **24**(1) (July 1988), 39–46.

[34] A. Naor, *Sparse quadratic forms and their geometric applications (after batson, spielman and srivastava)*, preprint arXiv:1101.4324, 2011.

[35] A. Nilli, *On the second eigenvalue of a graph*, Discrete Math, **91** (1991), 207–210.

[36] J. Renegar, *Hyperbolic programs, and their derivative relaxations*, Foundations of Computational Mathematics, **6**(1) (2006), 59–79.

[37] M. Rudelson and R. Vershynin, *Sampling from large matrices: An approach through geometric functional analysis*, Journal of the ACM, **54**(4) (2007), 21.

[38] D. Spielman and N. Srivastava, *Graph sparsification by effective resistances*, SIAM Journal on Computing, **40**(6) (2011), 1913–1926.

[39] D. A. Spielman and S.-H. Teng, *Spectral sparsification of graphs*, SIAM Journal on Computing, **40**(4) (2011), 981–1025.

[40] N. Srivastava, *Windows on theory: Discrepancy, graphs, and the kadison-singer problem, 2013*, from http://windowsontheory.org/2013/07/11/discrepancy-graphs-and-the-kadison-singer-conjecture-2/.

[41] N. Srivastava and R. Vershynin, *Covariance estimation for distributions with* $2 + \varepsilon$ *moments*, The Annals of Probability, **41**(5) (2013), 3081–3111.

[42] J. A. Tropp, *User-friendly tail bounds for sums of random matrices*, Foundations of Computational Mathematics, **12**(4) (2012), 389–434.

[43] R. Vershynin, *John's decompositions: Selecting a large part*, Israel Journal of Mathematics, **122**(1) (2001), 253–277.

[44] N. Weaver, *The Kadison–Singer problem in discrepancy theory*, Discrete Mathematics, **278**(1–3) (2004), 227–239.

[45] P. Youssef, *Restricted invertibility and the Banach–Mazur distance to the cube*, Mathematika, 2012, pp. 1–18.

Department of Mathematics, Yale University, New Haven, Connecticut 06520 USA
E-mail: adam.marcus@yale.edu

Department of Computer Science, Yale University, PO Box 208285, New Haven, Connecticut, 06520-8285 USA
E-mail: spielman@cs.yale.edu

Microsoft Research India, Lavelle Rd, Bangalore 560 001 India
E-mail: niksri@microsoft.com

Carleson measures and elliptic boundary value problems

Jill Pipher

Abstract. In this article, we highlight the role of Carleson measures in elliptic boundary value problems, and discuss some recent results in this theory. The focus here is on the Dirichlet problem, with measurable data, for second order elliptic operators in divergence form. We illustrate, through selected examples, the various ways Carleson measures arise in characterizing those classes of operators for which Dirichlet problems are solvable with classical non-tangential maximal function estimates.

Mathematics Subject Classification (2010). Primary 42B99, 42B25, 35J25, 42B20.

Keywords. Carleson measures, elliptic divergence form equations, A_∞, boundary value problems.

1. Introduction

Measures of Carleson type were introduced by L. Carleson in [9] and [10] to solve a problem in analytic interpolation, via a formulation that exploited the duality between Carleson measures and non-tangential maximal functions (defined below). Carleson measures have since become one of the most important tools in harmonic analysis, playing a fundamental role in the study of singular integral operators in particular, through their connection with BMO, the John-Nirenberg space of functions of bounded mean oscillation. We aim to describe, through some specific examples, the ubiquitous role of measures of this type in the theory of boundary value problems, especially with regard to sharp regularity of "elliptic" measure, the probability measure arising in the Dirichlet problem for second order divergence form elliptic operators. Perhaps the first connection between Carleson measures and boundary value problems was observed by C. Feffeman in [18], namely that every BMO function on $\mathbb{R}^n$ has a harmonic extension to the upper half space $\mathbb{R}^{n+1}_+$ which satisfies a certain Carleson measure condition. This established an important link between solutions to boundary value problems for the Laplacian and the function space BMO. It may be surprising to see the extent to which this link exists for operators other than the Laplacian, and in the context of more general domains.

In order to define Carleson measures, we introduce the geometric notion of a Carleson region above a cube. If $Q \subset \mathbb{R}^n$ is a cube with side length $l(Q)$ set $T_Q = \{(x,t) \in \mathbb{R}^{n+1}_+ : x \in I, 0 < t < l(Q)\}$, a cube sitting above its boundary face Q. (The notation T_Q comes from an equivalent formulation involving "tents" over cubes.)

Definition 1.1. The measure $d\mu$ is a Carleson measure in the upper half space $\mathbb{R}^{n+1}_+$ if there exists a constant C such for all cubes $Q \subset \mathbb{R}^n$, $\mu(T(Q)) < C|Q|$, where $|Q|$ denotes the

Proceedings of the International Congress of Mathematicians, Seoul, 2014

Lebesgue measure of the cube Q.

The classical theory of harmonic functions in the upper half space, or the ball, considers solutions to the Dirichlet problem with measurable, specifically L^p, data. Given a function $f \in L^p(\mathbb{R}^n)$, the convolution of f and the Poisson kernel is an absolutely convergent integral when $1 < p < \infty$, giving meaning to the harmonic extension $u(x,t)$ of an L^p function. And the sense in which this extension u converges to its boundary values is "non-tangential". That is, for every $x_0 \in \mathbb{R}^n$, one can define a non-tangential approach region to x_0, $\Gamma_a(x_0) = \{(x,t) : |x - x_0| < at\}$. Then if $u(x,t)$ is the Poisson extension of $f \in L^p(\mathbb{R}^n)$, for almost every x_0, $u(x,t) \to f(x_0)$ as $(x,t) \in \Gamma_a(x_0)$ approaches x_0. Moreover, one has a non-tangential maximal function estimate, specified below, which yields solvability and uniqueness of this L^p Dirichlet problem.

The result of C. Fefferman about harmonic functions, which proved to be a powerful tool in harmonic function theory, is this: if $u(x,t)$ is the Poisson extension of $f \in BMO$, then $d\mu = t|\nabla u|^2 dxdt$ is a Carleson measure in the upper half space $\mathbb{R}^{n+1}_+$. The converse also holds for functions that are not too large at ∞.

In the last several decades, there have been many significant developments in the theory of boundary value problems with data in L^p spaces, for harmonic (or poly-harmonic) functions defined in very general domains, and for solutions to second order divergence form (and higher order) elliptic operators with non-smooth coefficients. We will highlight a selection of these developments in which the role of Carleson measures has been decisive.

For simplicity of notation, we will formulate the results in the upper half plane, $\mathbb{R}^{n+1}_+$, but in fact these results are more naturally formulated on Lipschitz domains - see the cited references for this generality. In some cases, the perturbation results hold in more general (chord-arc) domains: [41–43].

2. Definitions and background

A divergence form elliptic operator

$$L := -\operatorname{div} A(x)\nabla,$$

defined in $\mathbb{R}^{n+1}$, where A is a (possibly non-symmetric) $(n+1)\times(n+1)$ matrix of bounded real coefficients, satisfies the uniform ellipticity condition

$$\lambda|\xi|^2 \leq \langle A(x)\xi,\xi\rangle := \sum_{i,j=1}^{n+1} A_{ij}(x)\xi_j\xi_i, \quad \|A\|_{L^\infty(\mathbb{R}^n)} \leq \lambda^{-1}, \tag{2.1}$$

for some $\lambda > 0$, and for all $\xi \in \mathbb{R}^{n+1}$, $x \in \mathbb{R}^n$. As usual, the divergence form equation is interpreted in the weak sense, i.e., we say that $Lu = 0$ in a domain Ω if $u \in W^{1,2}_{loc}(\Omega)$ and

$$\int A\nabla u \cdot \nabla\Psi = 0\,,$$

for all $\Psi \in C_0^\infty(\Omega)$.

For notational simplicity, Ω will henceforth be the half-space $\mathbb{R}^{n+1}_+ := \{(x,t) \in \mathbb{R}^n \times (0,\infty)\}$ even though the results are more naturally formulated on Lipschitz domains. See the cited references for this generality.

The solvability of the Dirichlet problem for L with data in $L^p(dx)$ is a function of a precise relationship between the elliptic measure ω associated to L and Lebesgue measure.

The elliptic measure associated to L is analogous to the harmonic measure: it is the representing measure for solutions to L with continuous data on the boundary.

Definition 2.2. A non-negative Borel measure ω defined on $\mathbb{R}^n$ is said to belong to the class A_∞ if there are positive constants C and θ such that for every cube Q, and every Borel set $F \subset Q$, we have

$$\omega(F) \leq C \left(\frac{|F|}{|Q|} \right)^\theta \omega(Q). \tag{2.3}$$

A real variable argument shows that a measure, ω, belongs to $A_\infty(dx)$ if and only if it is absolutely continuous with respect to Lebesgue measure and there is an exponent $q > 1$ such that the Radon-Nikodym derivative $k := d\omega/dx$ satisfies

$$\left(\fint_Q k(x)^q dx \right)^{1/q} \leq C \fint_Q k(x)\, dx\,, \tag{2.4}$$

uniformly for every cube Q. This property is called a reverse-Hölder estimate of order q.

If ω is the elliptic measure associated to an operator L, then the existence of such a $q > 1$ is, in turn, equivalent to the solvability of the Dirichlet problem for L with boundary data $f \in L^p$ (for p dual to q), in the sense of non-tangential convergence and non-tangential estimates on the boundary. These non-tangential estimates are expressed in terms of L^p bounds on two classical operators associated to solutions: the square function

$$S^\alpha(u)(x) := \left(\iint_{|x-y|<\alpha t} |\nabla u(y,t)|^2 \frac{dydt}{t^{n-1}} \right)^{1/2}, \tag{2.5}$$

and the non-tangential maximal function

$$N_*^\alpha(u)(x) := \sup_{(y,t):|x-y|<\alpha t} |u(y,t)| \tag{2.6}$$

Precisely, the elliptic measure satisfies a reverse Hölder estimate of order q if and only if the following L^p Dirichlet problem is solvable, for p dual to the exponent q:

$$\begin{cases} Lu = 0 \text{ in } \mathbb{R}^{n+1}_+ \\ \lim_{t\to 0} u(\cdot,t) = f \text{ in } L^p(\mathbb{R}^n) \text{ and n.t.} \\ \|N_*(u)\|_{L^p(\mathbb{R}^n)} < C\|f\|_p. \end{cases} \tag{D_p}$$

Here, the notation "$u \to f$ n.t." means that $\lim_{(y,t)\to(x,0)} u(y,t) = f(x)$, for $a.e.\ x \in \mathbb{R}^n$, where the limit runs over $(y,t) \in \Gamma(x) := \{(y,t) \in \mathbb{R}^{n+1}_+ : |y-x| < t\}$. The constant C depends only on ellipticity and dimension.

We will usually suppress the dependence on the aperture α, since the choice of aperture does not affect the range of available L^p estimates.

Solutions to L are said to satisfy De Giorgi-Nash-Moser bounds when the following local Hölder continuity estimates hold. Assume that $Lu = 0$ in $\mathbb{R}^{n+1}_+$ in the weak sense and

$B_{2R}(X) \subset \mathbb{R}^{n+1}_+$, $X \in \mathbb{R}^{n+1}_+$, $R > 0$. Then

$$|u(Y) - u(Z)| \leq C \left(\frac{|Y-Z|}{R}\right)^{\mu} \left(\int\limits_{B_{2R}(X)} |u|^2 \frac{\mathrm{dx}}{|B_2 R(X)} \right)^{\frac{1}{2}}, \quad \text{for all} \quad Y, Z \in B_R(X), \tag{2.7}$$

for some constants $\mu > 0$ and $C > 0$. In particular, one can show that for any $p > 0$

$$|u(Y)| \leq C \left(\int\limits_{B_{2R}(X)} |u|^p \frac{\mathrm{dx}}{|B_2 R(X)} \right)^{\frac{1}{p}}, \quad \text{for all} \quad Y, Z \in B_R(X). \tag{2.8}$$

The De Giorgi-Nash-Moser bounds always hold when the coefficients of the underlying equation are real [14, 40, 44], and the constants depend quantitatively only upon ellipticity and dimension. We will assume that for the complex equations considered later on (t-independent coefficients), that solutions satisfy the De Giorgi-Nash-Moser bounds, which may not in general obtain ([25, 39]).

3. Perturbations of elliptic operators

In this section, we briefly discuss some background which will motivate certain topics treated later, and for which Carleson measure estimates have played a decisive role.

In the upper half space, the Dirichlet problem is uniquely solvable for the Laplacian when the boundary data belongs to $L^p(dx)$, $1 < p < \infty$, in the sense that the Poisson extension $u(x,t)$ of f satisfies the estimate $\|N(u)\|_p \leq C\|f\|_p$. The same holds for solutions to $L := -\operatorname{div} A(x)\nabla$, when coefficients of A are smooth, or even just C^1 ([20]). However, without some regularity assumptions, the elliptic measure associated to L may be singular with respect to Lebesgue measure ([7])), and no estimate of this type will hold.

Many interesting examples of elliptic operators in divergence form arise as pullbacks of the Laplacian from a change of variable. From the viewpoint of complex function theory, it is natural to consider boundary behavior of harmonic functions in domains other than the ball or the upper half space. One approach to solving boundary value problems for harmonic functions in, say, a domain above a graph, is to invoke a change variables, mapping the harmonic function v to a solution u of a new divergence form elliptic operator, L. Thus, if the domain were bounded by a smooth curve, an appropriate change of variables results in a real symmetric divergence form operator with smooth coefficients. But if the boundary of the domain is not regular, the resulting operator has non-smooth coefficients, and the problem has not become easier. For a variety of reasons, including scale invariance and naturally arising geometric constructions, attention focused on the class of Lipschitz domains. In [12], Dahlberg showed that harmonic measure on any Lipschitz domain belonged to A_∞ with respect to the surface measure on the boundary. In fact, he showed that the L^2 Dirichlet problem, D_2, was solvable, but that D_p was not uniformly solvable on all Lipschitz domains when $p < 2$. More recently, the theory has developed to include a body of results for non-graph domains described by geometric conditions (non-tangentially accessible, chord-arc, Reifenberg flat.).

Consider the following example of a particularly straightforward change of variables. The domain is the region above a graph $t = \phi(x)$, where $\phi(x)$ is Lipschitz. The change of variables, $(x,t) \to (x, t-\phi(x))$, "flattens" it to the upper half space. Under this change of variable, from the Lipschitz domain to $\mathbb{R}^{n+1}_+$, harmonic functions are mapped to solutions of a symmetric elliptic divergence form operator L whose coefficients involve the Jacobian of this transformation and are therefore merely bounded and measurable. However, the coefficients have one redeeming feature: they are independent of the transverse variable t. Jerison and Kenig (JK) discovered how to put Dahlberg's result in a larger context when they showed that D_2 was solvable in $\mathbb{R}^{n+1}_+$ for all elliptic symmetric t-independent operators. Their well known result was based on an L^2 identity (a "Rellich" identity) which decisively used these three properties of the (real) operator L: symmetry, ellipticity, t-independence of the coefficients. Specifically, if $Lu = 0$, and $\vec{e}$ is the unit normal at the boundary of $\mathbb{R}^{n+1}_+$, then

$$\operatorname{div}(A\nabla u.\nabla u\,\vec{e}) = 2\operatorname{div}(D_{n+1}(u)A\nabla u). \tag{3.1}$$

Integrating this identity and applying the divergence theorem results in a boundary identity that can be used to show that the normal and tangential derivatives of a solution are comparable in L^2 norm. This boundary identity scales to show that the elliptic measure is not only absolutely continuous but satisfies a reverse Hölder condition of order two. Therefore, the Dirichlet problem with data in L^2 is solvable.

Many subsequent advances in the theory of boundary value problems for real symmetric elliptic equations and systems were based on variants of this Rellich identity.

The theory of perturbations of elliptic operators arose from several separate points of view. One source was T. Kato's interest in the analyticity of square roots of complex second order divergence form elliptic operators, which led to a question about analyticity of small L^∞ perturbations of self-adjoint elliptic operators. There is extensive literature on this subject which we are not going to delve into in this article. (See [4] for the solution to Kato's conjecture.) Another, and related, source of interest, stemmed from the the discovery that independence in the t variable in $\mathbb{R}^{n+1}_+$ (or similarly, of the radial variable in the unit ball) endows the elliptic measure ω with good properties. One may then try to relax this condition and understand more precisely the relationship between the smoothness that is required in the t direction and good estimates for elliptic measure. This was the approach taken in [13, 19, 30, 34], and see also [2, 3, 26] for later developments in perturbation theory. Dahlberg, [13], imposed a "vanishing" condition on the Carleson discrepancy between the coefficients and proved strong results about preservation of reverse Hölder estimates for the elliptic measure. An entirely new approach to the vanishing Carleson condition was taken in [2] that provided major extensions of the perturbation theory to complex coefficient operators.

Consider an operator $L_1 := -\operatorname{div} A(x,t)\nabla$, in $\mathbb{R}^{n+1}_+$, regarded as a perturbation of $L_0 := -\operatorname{div} A(x,0)\nabla$, and suppose one asks for some quantitative conditions on $|A(x,t) - A(x,0)|$ that yield good estimates for the elliptic measure ω_{L_1}. More generally, one can formulate the question as follows: what are the optimal conditions on the difference of the coefficients such that the perturbation L_1 of a "good" operator L_0, not necessarily t-independent, also satisfies good estimates for solvability of a boundary value problem. In [19], optimal conditions were found.

Theorem 3.2. *Let* $L_0 = \operatorname{div} A_0\nabla$ *and* $L_1 = \operatorname{div} A_1\nabla$ *and define the disagreement function*

$a(x,t)$ *by*

$$a(x,t) = sup\{|A_0(y,s) - A_1(y,s)| : |y-x| < t, t/2 < s < 2t\}. \tag{3.3}$$

If $a^2(x,t)t^{-1}dxdt$ *is a Carleson measure, then* $\omega_{L_0} \in A_\infty$ *implies* $\omega_{L_1} \in A_\infty$.

4. Linking A_∞ to Carleson measure estimates

Prior to the approach taken in [32], the regularity of elliptic measure for an operator L was essentially derived either from a Rellich identity, or as a consequence of the perturbation theory. There were two obvious classes of operators of interest where these L^2-identities were not valid: operators with complex coefficients and operators with non-symmetric coefficients. In the case of operators with complex coefficients, one of the most compelling outstanding questions was the Kato conjecture. This decades-old problem was finally resolved in the series of papers [4, 5, 24]. The solution of the Kato conjecture is a long story, summarized well in C. Kenig's review [31]. We will only mention that the solution also relied on a critical use of Carleson measures. The situation regarding (non-symmetric) t-independent operators is discussed in the next section.

In [32], it was shown that the elliptic measure associated to adivergence form operator $L := -\operatorname{div} A(x)\nabla$, belongs to the class A_∞ if and only if every bounded solution could (locally) be approximated arbitrarily well by a continuous function whose gradient satisfied a Carleson measure condition. This criteria was dubbed "ϵ-approximability", and was immediately applied to t-independent operators in dimension two.

Definition 4.1. Let $u \in L^\infty(\mathbb{R}^{n+1}_+)$, with $\|u\|_\infty \leq 1$. Given $\epsilon > 0$, we say that u is ϵ-**approximable** if for every cube $Q_0 \subset \mathbb{R}^n$, there is a $\varphi = \varphi_{Q_0} \in W^{1,1}(T_{Q_0})$ such that

$$\|u - \varphi\|_{L^\infty(T_{Q_0})} < \epsilon\,, \tag{4.2}$$

and

$$\sup_{Q \subset Q_0} \frac{1}{|Q|} \iint_{T_Q} |\nabla \varphi(x,t)|\, dxdt \leq C_\epsilon\,, \tag{4.3}$$

where C_ϵ depends also upon dimension and ellipticity, but not on Q_0.

To motivate this definition, we recall that harmonic functions in the upper half space possess the property of ϵ-approximability ([21, 45]). Although bounded harmonic functions in $\mathbb{R}^{n+1}_+$ satisfy an L^2-Carleson measure condition, the (technically more desirable) L^1-Carleson condition fails to hold. It turns out that the approximation property is a good substitute for certain applications. In [11], Dahlberg showed that ϵ-approximability holds for bounded harmonic functions on Lipschitz domains as well. His proof used the previously established equivalence in L^p-norm between the square function and the non-tangential maximal function on LIpschitz domains.

Theorem 4.4 ([32]). *Let* $L := -\operatorname{div} A(x)\nabla$, *be an elliptic divergence form operator, not necessarily symmetric, with bounded measurable coefficients, defined in* $\mathbb{R}^{n+1}_+$. *Then there exists an* ϵ, *depending on the ellipticity constant of* L *such that if every solution to* $Lu = 0$ *in* $\mathbb{R}^{n+1}_+$ *with* $|u| \leq 1$ *is* ϵ*-approximable then* ω *belongs to* A_∞.

We will now sketch the main steps in the proof in [32] of this result, and then describe a recent modification of these ideas that yields a much stronger statement. The references give details, including certain technicalities, that we shall not describe in detail here.

The A_∞ class has many equivalent characterizations, and it will be convenient to work with this one:

Given any $\eta > 0$, there exists a $\delta > 0$ such that for any cube $Q \subset \mathbb{R}^n$ and any $E \subset Q$, we have that $|E|/|Q| < \eta$ whenever $\omega(E)/\omega(Q) < \delta$.

The main idea in the proof of Theorem 4.4 is as follows. Fix a cube Q of side length r, and suppose that E is a set whose elliptic measure, $\omega(E)$, is small. Let ϕ denote the ϵ-approximation of u. If E has sufficiently small measure, it will be shown that a truncated L^1-version of the square function of ϕ is large. That is, the r-truncated $A_r(\phi)(x) := \left(\iint_{|x-y|<t<r} |\nabla\phi(y,t)| \frac{dydt}{t^n}\right)^{1/2}$ will be be larger than some prescribed value $k = k(\epsilon)$. The desired conclusion will follow from the Carleson measure estimate by integrating:

$$|E|k^2 < \int_E A_r^2(\phi)(x)dx < \int_Q A_r^2(\phi)(x)dx < \iint_{T_Q} |\nabla\varphi(x,t)|\, dxdt$$

By the Carleson measure property, this latter expression is bounded by a constant C_ϵ times $|Q|$. and thus $|E|/|Q| < \eta$ where $\eta \approx 1/k^2$.

In order to show that $A(\phi)$ is large on sets of small elliptic measure, a solution u to $Lu = 0$ was constructed with the property that u that oscillates by at least some fixed value a large number of times in cones over points $x \in E$. Because u can be approximated arbitrarily well by ϕ, this entailed that ϕ also oscillates a large number of times. This lower bound on oscillation translated, via interior estimates, into an estimate from below for $\nabla\phi$ in disjoint layers of a truncated cone over x.

There are several constructions that drive this proof, the first of which is Christ's construction of dyadic grids on spaces of homogeneous type. Thus $Q \subset \mathbb{R}^n$ possesses a *dyadic grid* adapted to ω, which is a collection of subsets $\{I_{j,l}\}$ of $Q \subset \mathbb{R}^n$ such that for each fixed $j \geq 0$,

(1) $\mathbb{R}^n = \bigcup_l I_{j,l}$, and $I_{j,l_1} \cap I_{j,l_2} = \emptyset$ if $l_1 \neq l_2$.

(2) Each $I_{j,l}$ contains $B(2^{-j}, x_l)$, and is contained in an M-fold dilate $B(M2^{-j}, x_l)$, where $B(2^{-j}, x_l)$ denotes the ball of radius 2^{-j} about the point $x_l \in \mathbb{R}^n$.

(3) If $I_{j,l} \cap I_{j',l} \neq \emptyset$ then either $I_{j,l} \subset I_{j',l}$ or $I_{j',l} \subset I_{j,l}$. Moreover, there exists a $C_M < 1$ such that $\omega(I_{j,l}) < C_M\omega(I_{j',l})$whenever $I_{j,l} \subset I_{j',l}$.

(4) Any open set $\mathcal{O}$ can be decomposed as $\mathcal{O} = \bigcup I_{j,l}$ where the $I_{j,l}$ are non-overlapping. For each $I_{j,l}$ in this decomposition, there exists a point $p_{j,l}$ such that the distance from $p_{j,l}$ to $I_{j,l}$ is comparable to diam($I_{j,l}$).

Definition 4.5. Let ϵ be small and given. If $E \subset Q$, a *good ϵ-cover of E of length k* is a collection of nested open sets $\{\mathcal{O}_i\}_{i=1}^k$ with $E \subset \mathcal{O}_k \subset \mathcal{O}_{k-1} ... \subset \mathcal{O}_0 \subset Q$ where each $\mathcal{O}_i = \bigcup S_l^i$ such that

(1) each S_l^i belongs to the dyadic grid, and

(2) for all $0 < i < k$, $\omega(\mathcal{O}_i \cap S_l^{i-1}) < \epsilon\omega(S_l^{i-1})$.

Note that a good ϵ-cover has the property that each S_j^i is properly contained in some S_l^{i-1}, as well as the further nesting property that for $k > i > m > 0$, $\omega(S_j^m \cap \mathcal{O}_i) < \epsilon^{i-m}\omega(S_j^m)$.

Lemma 4.6 ([32]). *Given $\epsilon > 0$, there exists a $\delta > 0$ such that if $\omega(E) < \delta$, then E has a good ϵ-cover of length k where $k \to \infty$ as $\omega(E) \to 0$.*

The good ϵ-cover of length k is used to construct the boundary data f which will give rise to a bounded, oscillating solution u to L. Set:

$$f = \sum_{i=0}^{k} (-1)^i \mathcal{X}_{\mathcal{O}_i}. \tag{4.7}$$

and let u be the solution to $Lu = 0$, with $u(x,0) = f$.

Note that $f \leq 1$, and so $0 \leq u \leq 1$. For each point $x \in E$, we find a sequence of points, $X_m = (x_m, t_m)$ in the cone $\Gamma(x)$ with the property that, for $0 < m < k$ even, $u(X_m) > c_1$, and for $0 < m < k$ odd, $u(X_m) < c_2$ and $c_1 - c_2 > c(\epsilon)$. To define these X_m, collect the dyadic grid cubes $S_l^m \subset \mathcal{O}_m$ that contain the given point x. Let $l(S)$ denote the side length of S. The point X_m, when m is even, is essentially any point in the top half of the Carleson region over S_l^m. When m is odd, the point $X_m = (x_m, t_m)$ will also be in this Carleson region, but t_m will be closer to the boundary, that is, $t \approx \eta l(S_l^m)$. (In order to make sure that these points X_m descend in the cone, i.e., have the property that $t_m < \rho t_{m-1}$ for some $\rho < 1$, we may have to skip a finite number of levels m. Details are in [32].)

We give a rough sketch of these estimates. Recall the integral representation of solutions: $u(x,t) = \int K(x,t;y,0) f(y) d\omega(y)$.

Fix an even m. We can then write $u(X_m) = u_1(X_m) + u_2(X_m)$ where $u_1(x,0) = f_1(x) := \sum_{i=0}^{m} \mathcal{X}_{\mathcal{O}_i}$.. Moreover, since $u > 0$, we have that, for some c_1 depending only ellipticity,

$$u(X_m) > \int K(x_m, t_m; y, 0) f(y) d\omega(y) \geq c \frac{1}{\omega(S_l^m)} \int_{S_l^m} f(y) d\omega(y).$$

Because m is even, the function $f_1 = 1$ on S_l^m, and so $u_1 > c_1'$. By the nesting property of the cover,

$$\frac{1}{\omega(S_l^m)} \int_{S_l^m} f_2(y) d\omega(y) < \frac{1}{\omega(S_l^m)} \sum_{i=m+1}^{k} \omega(\mathcal{O}_i \cap S_l^m) < 2\epsilon,$$

and thus $u(X_m) > c_1' - 2\epsilon > c_1$. When m is odd, the boundary function f is split similarly, and a more technical analysis is needed to show that the main term is indeed given by f_1, which vanishes on the dyadic cube S_l^m. Since (x_m, t_m) was chosen so that $t_m \approx l(S_l^m)$, the Hölder decay of the solution near the boundary where it vanishes will be used to show that $u(x_m, t_m) < c_2 < c_1 - \epsilon$, if ϵ and η are chosen appropriately.

In conclusion, one can extract from this construction a sequence of points $\{x_m, t_m\}_{m=0}^{k} \in \Gamma(x)$ such that $|u(x_m, t_m) - u(x_{m-1}, t_{m-1})| > \epsilon$, and such that $t_m < \rho t_{m-1}$. One can then derive a lower bound for the L^1-square function $A(u)$, and likewise for $A(\phi)$ where ϕ is the approximate to u.

This approximation theorem, and its proof, yielded several applications to specific classes of operators ([15, 22, 35, 37]): [22] is explained in more detail in the next section. Since one

cannot expect the actual solution to L to satisfy an L^1-Carleson condition (as the approximate does), this program left open the question of the role of classical Carleson measure estimates for solutions.

In [16], it was shown that the A_∞ propoerty of elliptic measure is equivalent to the existence of Carleson measure estimates for solutions with boundary data in BMO. The result was proven in Lipschitz domains (and will likely hold for chord-arc domains as well).

Theorem 4.8 ([16]). *Let $L := -\operatorname{div} A(x)\nabla$, be an elliptic divergence form operator, not necessarily symmetric, with bounded measurable coefficients, defined in $\mathbb{R}^{n+1}_+$. Then $\omega \in A_\infty$ if and only if, for every solution u to $Lu = 0$ with boundary data $f \in BMO$, we have the Carleson measure estimate:*

$$\sup_Q \frac{1}{|Q|} \iint_{T_Q} t|\nabla u(x,t)|^2 \, dxdt \leq C||f||^2_{BMO}, \tag{4.9}$$

The proof of Theorem 4.8 used a dual formulation of the A_∞ condition:

> *Given any $\eta > 0$, there exists a $\delta > 0$ such that for any cube $Q \subset \mathbb{R}^n$ and any $E \subset Q$, we have that $\omega(E)/\omega(Q) < \eta$ whenever $|E|/|Q| < \delta$.*

To verify this condition, a construction of [28] was invoked to produce, for any such E, a BMO function $f \geq \mathcal{X}_E$ with small BMO norm. An upper estimate on $\omega(E)/\omega(Q)$ in terms of the (small) Carleson measure bound on f required a lemma in [34]. See [16] for details.

In turn, this left open the question of whether the A_∞ property of elliptic measure could be characterized by the existence of Carleson measure conditions for solutions to bounded data, as opposed to data in the larger class, BMO.

The solution u, with boundary data f as in (4.7), has only $A(u)$ large on the set $E \subset Q$ when $\omega(E)$ is small, but not necessarily $S(u)$ large as well. To see why, suppose Q has side length 1, and cut the cone into dyadic layers: $\Gamma_j(x) = \{(y,t) \in \Gamma(x) : 2^{-j} < t < 2^{-j+1}\}$.

We write

$$S(u)(x) = \sum_j \int_{\Gamma_j(x)} t^{1-n}|\nabla u|^2 dydt$$

Each piece $\int_{\Gamma_j(x)} t^{1-n}|\nabla u|^2 dydt$ is a scaled average of the gradient of u which, by a Poincaré estimate, can be bounded from below by the oscillation of u over this dyadic layer of the cone. However, this construction doesn't yield any information about the oscillation of u on such dyadic regions because there is no control on the distance between the the points $\{x_m, t_m\}_{m=0}^k \in \Gamma(x)$ that belong to different levels $\mathcal{O}_m$.

The linking of A_∞ to Carleson measure estimates for L^∞ functions, is the subject of [33]. Essentially, one can use the same cover, and define a new function f as follows. Each $\mathcal{O}_m$ is a union of dyadic intervals S_l^m, and each S_l^m has a (bounded) number of immediate dyadic subintervals. For each S_l^m choose one of its dyadic children and call it $\tilde{S}_l^m$. If m is even, define f_m to take the value 1 on $\bigcup_l (S_l^m \setminus \tilde{S}_l^m)$ and 0 elsewhere. If m is odd, we define f_m to "zero out" the values of f_{m-1}: $f_m = -1$ where $f_m = 1$ and is) elsewhere. Now set $f = \sum_{m=0}^k f_m$ and let u be the solution to $Lu = 0$ with boundary data f. On each even level m, f takes on both the values 0 and 1 on dyadic children. Thus, arguments modeled on those of [32] will yield the following: for some $C, c > 0$, and every $x \in E$, there are sequences $\{x_m, t_m\}_{m=0}^k$ with $ct_{m-1} < t_m < Ct_{m-1}$ for which $|u(x_m, t_m) - u(x_{m-1}, t_{m-1})| > \epsilon$.

From this construction, it can be concluded that if solutions to L with bounded data satisfy classical Carleson measure estimates, then the elliptic measure associated to L is A_∞, and thus the Dirichlet problem with data in L^p is uniquely solvable for some $p > 1$. As a corolloary, we see that solutions with BMO data posses C. Fefferman-type Carleson estimates if and only if solutions with L^∞ data posses these Carleson estimates.

Theorem 4.10 ([33]). *Let $L := -\operatorname{div} A(x)\nabla$, be an elliptic divergence form operator, not necessarily symmetric, with bounded measurable coefficients, defined in $\mathbb{R}^{n+1}_+$. Then $\omega \in A_\infty$ if and only if, for every solution u to $Lu = 0$ with boundary data $f \leq 1$, we have the Carleson measure estimate:*

$$\sup_Q \frac{1}{|Q|} \iint_{T_Q} t|\nabla u(x,t)|^2 \, dxdt \leq C. \tag{4.11}$$

5. Application to time-independent operators

The ϵ-approximability theorem of [32] was established by showing the equivalence in L^p norm between the the non-tangential maximal function and the square function, and invoking a stopping time construction due to Dahlberg ([11]). Examples were given to demonstrate that, for $p \to \infty$, there exists elliptic operators in this class for which D_p is not solvable.. In other words, no stronger conclusion than A_∞ of the elliptic measure can be concluded from ϵ-approximability. A more precise study of these counterexamples was undertaken in [1], where it was shown that the boundary equation method and the Lax-Milgram method may construct different solutions, thus underscoring the differences between the symmetric and the non-symmetric situation.

As an application of the consequences of norm equivalence between non- tangential maximal function and the square function, [32] contained a proof that two-dimensional t-independent divergence form non-symmetric elliptic operators had elliptic measure belonging to A_∞. This was a first step in establishing regularity of elliptic measure without recourse to L^2 identities of Rellich type. Although the proof only worked in $\mathbb{R}^2$, it worked under a surprisingly flexible condition on the matrix.

Theorem 5.1 ([32]). *Let $L := -\operatorname{div} A(x)\nabla$ be an elliptic operator in $\mathbb{R}^2$ with bounded measurable coefficients. Suppose that there exists a fixed unit vector $\vec{e}$ such that $A(x,t) = A((x;,t) \cdot \vec{e})$. Then the elliptic measure ω_L belongs to A_∞ in a domain in any Lipschitz domain in $\mathbb{R}^2$.*

At this point, we note that the development of the theory of non-symmetric operators has had several motivations. First of all, the boundary value problem for general non-symmetric elliptic operators cannot be solved in L^2, and L^p solvability requires a different approach than that of Rellich identities. Second, the well-posedness results for equations with real non-symmetric coefficients and associated estimates on solutions are the first step towards understanding operators with complex coefficients in the non-Hermitian case, a case of interest for Kato's analyticity program. Finally, many problems arising in homogenization theory have non-symmetric coefficients [6]. Solving the Dirichlet problem with data in L^p is the first step in the study of the uniform bounds, independent of the scaling parameter in homogenization theory, in the absence of symmetry ([6]).

It is therefore desirable to develop approaches to solving L^p boundary value problems that are neither perturbative nor rely on symmetry of the matrices. However, the proof of Theorem 5.1 did not generalize to higher dimensions, as it relied on a special change of variable to put the matrix of coefficients in upper triangular form. It took almost fifteen years, and the development of the tools used to solve Kato's conjecture (the square root estimates), to be able to prove this result in all dimensions.

Theorem 5.2 ([22]). *Let L be a divergence form elliptic operator as above, with t-independent coefficients. Then there is a $p < \infty$ such that the Dirichlet problem D_p is well-posed; equivalently, for each cube $Q \subset \mathbb{R}^n$, the L-harmonic measure $\omega_L \in A_\infty(Q)$, with constants that are uniform in Q.*

The proof in [22] proceeded, as in two dimensions, by establishing A_∞ of the elliptic measure as consequence of ϵ-approximability of bounded solutions. The boundedness in norm of the non-tangential maximal function by the square function had previously been established (globally) in [2] so the main contribution of [22] was the converse, which had the immediate corollary:

Corollary 5.3 ([22]). *Under the same hypotheses as in Theorem 5.2, for a bounded solution u, we have the Carleson measure estimate*

$$\sup_Q \frac{1}{|Q|} \iint_{T_Q} |\nabla u(x,t)|^2 t dt dx \leq C \|u\|_{L^\infty(\Omega)}\,, \tag{5.4}$$

where C depends only upon dimension and ellipticity.

Theorem 4.10 implies that this Carleson measure estimate alone is now sufficient to conclude A_∞, somehwat simplifying the proof of A_∞ for this class of elliptic measures.

In [32], it was shown that the equivalence between non-tangential maximal functions and square functions implied A_∞, for that equivalence was necessary to prove ϵ-approximation of bounded solutions. We see now that only half of this information is required, namely the bounds on the square function in terms of the non-tangential maximal function.

Remark 5.5. Most of the discussion in this article has centered on the Dirichlet problem. Over the years, there has been a parallel development for boundary value problems such as the Neumann and the regularity problems for second order operators, and for higher order operators and elliptic systems. There is a vast literature on the solvabilty of these (even more) challenging problems, which is beyond the scope of the present article.

Acknowledgements. The main topics treated in this article are the result of many fruitful collaborations over the years. In particular, I would like to acknowledge the impact of recent collaborations with M. Dindos, C. Kenig, S. Hofmann, S. Mayboroda, and T. Toro.

References

[1] A. Axelsson, *Non-unique solutions to boundary value problems for nonsymmetric divergence form equations,* Trans. Amer. Math. Soc., 362 (2010), no. 2, 661–672.

[2] P. Auscher and A. Axelsson, *Weighted maximal regularity estimates and solvability of non-smooth elliptic systems*, to appear, Invent. Math.

[3] M. Alfonseca, P. Auscher, A. Axelsson, S. Hofmann, and S. Kim, *Analyticity of layer potentials and L^2 Solvability of boundary value problems for divergence form elliptic equations with complex L^∞ coefficients*, Advances in Math, **226** (2011), 4533–4606.

[4] P. Auscher, S. Hofmann, M. Lacey, A. McIntosh, and P. Tchamitchian, *The solution of the Kato Square Root Problem for Second Order Elliptic operators on $\mathbb{R}^n$*, Ann. of Math., **156** (2002), 633–654.

[5] P. Auscher, S. Hofmann, J. Lewis, and P. Tchamitchian, *Extrapolation of Carleson measures and the analyticity of Kato's square-root operators*, Acta Math., **187** (2001), no. 2, 161–190.

[6] A. Bensoussan, J.-L. Lions, and G. Papanicolaou, *Asymptotic analysis for periodic structures*, Studies in Mathematics and its Applications, 5., North-Holland Publishing Co., Amsterdam-New York, 1978.

[7] Luis A. Caffarelli, Eugene B. Fabes, and Carlos E. Kenig, *Completely singular elliptic-harmonic measures*, Indiana Univ. Math. J., **30** (1981), no. 6, 917-924.

[8] L. Caffarelli, E. Fabes, Mortola, and S. Salsa, *Boundary behavior of nonnegative solutions of elliptic operators in divergence form*, Indiana Univ. Math. J., **30** (1981), no. 4, 621–640.

[9] L. Carleson, *An interpolation problem for bounded analytic functions*, Amer. J. Math., **80** (1958), 921–930.

[10] _____, *Interpolation by bounded analytic functions and the corona problem*, Ann. of Math., **76** (1962), 547–559.

[11] B.E.J. Dahlberg, *Approximation of harmonic functions*, Ann. Inst. Fourier (Grenoble), **30** (1980) 97–107.

[12] _____, *Estimates of harmonic measure*, Arch. Rational Mech. Anal., **65** (1977), 275–288.

[13] _____, *On the absolute continuity of elliptic measure*, American Journal of Mathematics, **108** (1986), 1119–1138.

[14] E. De Giorgi, *Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari*, Mem. Accad. Sci. Torino., Cl. Sci. Fis. Mat. Nat., **3** (1957), 25–43.

[15] M. Dindoš, S. Petermichl, and J. Pipher, *The L^p Dirichlet problem for second order elliptic operators and a p-adapted square function*, J. of Funct. Anal., **249** (2007), 372–392

[16] _____, C. Kenig, and J. Pipher, *BMO solvability and the A_∞ condition for elliptic operators*, J. Geom. Anal., **21** (2011), no. 1, 78–95.

[17] E. Fabes, D. Jerison, and C. Kenig, *Necessary and sufficient conditions for absolute continuity of elliptic-harmonic measure*, Ann. of Math., (2) **119** (1984), no. 1, 121–141.

[18] C. Fefferman, *Characterizations of bounded mean oscillation*,Bulletin of the American Mathematical Society, **77** (1971), no. 4, 587–588.

[19] R. Fefferman, C. Kenig., and J. Pipher, *The theory of weights and the Dirichlet problem for elliptic equations*, Ann. Math., **134** (1991), 65–124.

[20] E. Fabes, M. Jodeit, and N. Riviere, *Potential techniques for boundary value problems on C^1 domains*, Acta Math., **141** (1978), 165–186.

[21] J. B. Garnett, *Bounded Analytic Functions*, Academic Press, New York, 1981.

[22] S. Hofmann, C. Kenig, S. Mayboroda, and J. Pipher, *Square function/Non-tangential maximal function estimates and the Dirichlet problem for non-symmetric elliptic operators*, to appear in JAMS.

[23] S. Hofmann, C. Kenig, S. Mayboroda, and J. Pipher, *The Regularity problem for second order elliptic operators with complex-valued bounded measurable coefficients*, preprint.

[24] S. Hofmann, M. Lacey, and A. McIntosh, *The solution of the Kato problem for divergence form elliptic operators with Gaussian heat kernel bounds*, Annals of Math., **156** (2002), pp. 623–631.

[25] S. Hofmann, S. Mayboroda, and A. McIntosh, *Second order elliptic operators with complex bounded measurable coefficients in Lp*, Sobolev and Hardy spaces, Ann. Sci. Éc. Norm. Supér., (4) **44** (2011), no. 5, 723–800.

[26] S. Hofmann, S. Mayboroda, and M. Mourgoglou, *L^p and endpoint solvability results for divergence form elliptic equations with complex L^∞ coefficients*, preprint.

[27] S. Hofmann and A. McIntosh, The solution of the Kato problem in two dimensions, Proceedings of the conference on harmonic analysis and PDE held at El Escorial, June 2000, Publ. Mat. Vol., (2002), 143–160.

[28] P. Jones and J.-L. Journé, *Weak Convergence in $H^1(\mathbb{R}^n)$*, Proc. AMS, V. 120, No. 1, (2009), 137–138.

[29] (with Jerison, D. S.) *An identity with applications to harmonic measure*, Bull. AMS, Vol. 2, No. 3 (1980), 447–451.

[30] D. Jerison and C. Kenig, *The Dirichlet problem in nonsmooth domains*, Ann. of Math. (2), **113** (1981), no. 2, 367–382.

[31] C. Kenig, *MathSciNet Review*.

[32] C. Kenig, H. Koch, H. J. Pipher, and T. Toro, *A new approach to absolute continuity of elliptic measure, with applications to non-symmetric equations*, Adv. Math., **153** (2000), no. 2, 231–298.

[33] C. Kenig, B. Kirchheim, J. Pipher, and T. Toro, preprint.

[34] C. Kenig and J. Pipher, *The Neumann problem for elliptic equations with nonsmooth coefficients*, Invent. Math., **113** (1993), no. 3, 447–509.

[35] ———, *The Dirichlet problem for elliptic equations with drift terms*, Publ. Mat., **45** (2001), no. 1, 199–217.

[36] ———, *The Neumann problem for elliptic equations with nonsmooth coefficients. II*, A celebration of John F. Nash, Jr. Duke Math. J., **81** (1995), no. 1, 227–250.

[37] C. Kenig and D. Rule, *The regularity and Neumann problem for non-symmetric elliptic operators*, Trans. Amer. Math. Soc., **361** (2009), 125–160.

[38] C. Kenig and T. Toro, *Harmonic measure on locally at domains*, Duke Math. J, **87** (1997), no. 3, 509–551.

[39] V. G. Maz'ya, S. A. Nazarov, and B. A. Plamenevskiĭ, *Absence of a De Giorgi-type theorem for strongly elliptic equations with complex coefficients*, Boundary value problems of mathematical physics and related questions in the theory of functions, 14., Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), **115** (1982), 156–168, 309.

[40] J. Moser, *On Harnack's theorem for elliptic differential operators*, Comm. Pure and Appl. Math., **14** (1961), 577–591.

[41] E. Milakis and T. Toro, *Divergence form operators in Reifenberg flat domains*, Mathematische Zeitschrift., **264** (1) (2010), 15–41.

[42] E. Milakis, J. Pipher, and T. Toro, *Harmonic analysis on chord-arc domains*, Journal of Geometric Analysis, **23** (4), 2091–2157, 2, 2013.

[43] ———, *Perturbation of elliptic operators in chord arc domains*, Contemporary Mathematics (Amer. Math. Soc.), **612** (2014).

[44] J. Nash, *Continuity of the solutions of parabolic and elliptic equations*, Amer. J. Math., **80** (1957), 931–954.

[45] N. Varopoulos, *A remark on BMO and bounded harmonic functions*, Princeton Univ. Press, Princeton, NJ, 1970.

Department of Mathematics, Brown University, Box 1917, Providence, RI 02912, USA
E-mail: jpipher@math.brown.edu

Roth's theorem: an application of approximate groups

Tom Sanders

Abstract. We discuss Roth's theorem on arithmetic progressions through the lens of approximate groups.

Mathematics Subject Classification (2010). 11B25.

Keywords. Approximate groups, Roth's theorem, additive combinatorics, Bourgain systems.

1. Introduction

Our starting point is the following question.

> *Do the primes contain infinitely many arithmetic progressions? That is to say, are there infinitely many distinct primes p_1, p_2, p_3 such that $p_1 + p_3 = 2p_2$?*

There are two common ways of establishing existence: algebraic constructions, and counting. It turns out that with the primes it is the second approach which is the most fruitful, but to make sense of counting primes we need to settle a few things. In particular, we shall need a range over which to count and a natural choice is the interval $[N] := \{1, \dots, N\}$. The number of primes in this interval is denoted $\pi(N)$ and the Prime Number Theorem asserts that

$$\pi(N) \sim \frac{N}{\log N} \text{ as } N \to \infty.$$

To be clear, throughout the paper we shall be interested in the case $N \to \infty$, and all the counting we do will be rough rather than exact.

In general, given a set $A \subset [N]$ we define

$$T_N(A) := \sum_{\substack{x_1,x_2,x_3 \leqslant N \\ x_1+x_3=2x_2}} 1_A(x_1)1_A(x_2)1_A(x_3);$$

this is the number of arithmetic progressions in A where we *do* care about order so $(x + 2d, x + d, x)$ is, in general, a different progression to $(x, x + d, x + 2d)$. We should note two things immediately: first, this count includes the so called 'trivial progressions' of the form (x, x, x); and secondly, by arithmetic progression we shall (almost) always mean three-term arithmetic progression.

Proceedings of the International Congress of Mathematicians, Seoul, 2014

Writing P_N for the set of primes in the interval $[N]$ van der Corput [68] answered the question above by proving the following asymptotic.

$$T_N(P_N) \sim 2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)\cdot\frac{1}{\log^3 N}\cdot T_N([N]). \tag{1.1}$$

We should take a moment both to explain the statement and also why it answers the question. The asymptotic on the right has three parts:

- **(Total count)** $T_N([N])$ is the total number of arithmetic progressions in the interval $[N]$, so

$$T_N([N]) = \sum_{x\leqslant N}\sum_{\frac{1-x}{2}\leqslant d\leqslant\frac{N-x}{2}} 1 = \sum_{x\leqslant N}\left(\frac{N}{2}+O(1)\right) = \frac{N^2}{2}+O(N).$$

- **(Random sets)** $\log^{-3} N$ is the proportion of the total number of progressions in $[N]$ we would expect to have if P_N were a random subset of $[N]$ of size $\pi(N)$. Letting A be a random subset of $[N]$ of size $\pi(N)$ is *roughly* the same as letting A be a random set chosen by letting each $x\in[N]$ be in A independently and with probability $\log^{-1} N$. In this case we have

$$\begin{aligned}\mathbb{E}T_N(A) &= \sum_{\substack{x_1,x_2,x_3\leqslant N\\ x_1+x_3=2x_2}}\mathbb{E}1_A(x_1)1_A(x_2)1_A(x_3)\\ &= \frac{1}{\log^3 N}\cdot(T_N([N])-N)+\frac{1}{\log N}\cdot N \sim \frac{1}{\log^3 N}\cdot T_N([N])\end{aligned}$$

  by linearity of expectation.

- **(Local factors)** $2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)$ measures the deviation of P_N from being random for so called local reasons – that is reasons $\pmod p$ for small primes p. The probabilistic model for the primes due to Cramér (see [25] for a discussion) tells us to expect the count of progressions to be rescaled by a local factor β_p for each prime p, and that factor is the ratio of the number of (modular) arithmetic progressions in the residues $R_p := \{x : (x,p)=1\}$ divided by the expected number of arithmetic progressions in a random set of size $|R_p|$. These factors can be easily computed and we have $\beta_2 = 2$ and $\beta_p = 1-(p-1)^{-2}$ for all other p leading to the claimed product.

It follows from these remarks that (1.1) becomes

$$T_N(P_N) \sim 0.66\ldots\frac{N^2}{\log^3 N}.$$

Now to see how this answers the question at the start of the introduction we note that if the primes contained finitely many non-trivial arithmetic progressions then we would have $T_N(P_N) = |P_N| + O(1) \lesssim N$ contradicting the above asymptotic for sufficiently large N.

2. Roth's theorem

Although van der Corput answered the question of whether the primes contain infinitely many arithmetic progressions, there is an underlying question which remains.

> *Do the primes contain infinitely many arithmetic progressions simply because there are 'lots' of primes?*

As it stands this question is a little vague; to make sense of it we make a definition.

$$r_3(N) := \max\{A \subset [N] : A \text{ contains no non-trivial progressions}\}.$$

Now, suppose that we could show that

$$r_3(N) \leqslant \pi(N) - \omega(N) \tag{2.1}$$

for some function ω with $\omega(N) \to \infty$ as $N \to \infty$. If the primes contained only finitely many non-trivial arithmetic progressions then we could remove them and take the intersection with a sufficiently large interval to get a set of size $\pi(N) - O(1)$ containing no non-trivial arithmetic progressions. This would tell us that $r_3(N) \geqslant \pi(N) - O(1)$ contradicting (2.1). It follows that if we could prove this conjecture then we would know that the primes contain arithmetic progressions simply because there are many primes.

Nothing like (2.1) is known and, indeed, it is not even obvious that $r_3(N) = o(N)$. One way of establishing an upper bound on $r_3(N)$ is to show that sufficiently large sets are guaranteed to contain non-trivial arithmetic progressions. It turns out that this will be a fruitful, although equivalent, way of looking at the problem.

To get a flavour of the sort of result one might start with we have the following for which is is useful to introduce a definition: the **density** of a set $A \subset [N]$ is defined to be $|A|/N$.

Proposition 2.1. *Suppose that $A \subset [N]$ has density $\alpha > 0$. Then*

$$T_N(A) = \Omega((2\alpha - 1)T_N([N])).$$

This result only has content when $\alpha > 1/2$. In this case, if N is sufficiently large – say $N \geqslant C(2\alpha - 1)^{-1}$ for some large absolute constant $C > 0$ – then $|A| < T_N(A)$ and we conclude that A contains a non-trivial arithmetic progression.

We shall return to Proposition 2.1 later, but of more interest to us will be a bound of the form $\Omega_\alpha(T_N([N]))$. This is the content of Roth's theorem [42, 43] (technically coupled with an averaging argument due to Varnavides [69]).

Theorem 2.2 (Roth's theorem). *Suppose that $A \subset [N]$ has density $\alpha > 0$. Then*

$$T_N(A) = \exp(-\exp(O(\alpha^{-1})))T_N([N]).$$

Suppose that $A \subset [N]$ has size $r_3(N)$ and contains no non-trivial arithmetic progressions. Then $T_N(A) = |A| = \alpha N$ and inserting the bound for $T_N(A)$ from the above theorem we can rearrange to get

$$r_3(N) = |A| = O\left(\frac{N}{\log\log N}\right).$$

This bound was improved independently by Heath-Brown [34] and Szemerédi [65] who showed that

$$r_3(N) = O\left(\frac{N}{\log^c N}\right)$$

for some absolute $c > 0$. It seems that $c = 1/20$ was the first explicitly recorded value and then this was pushed up to around $1/4$ by Szemerédi. By a rather different approach Bourgain in two successive papers [7] and [8] improved c first to $1/2 - o(1)$ and then $2/3 - o(1)$. Finally, the best known bound is of the form

$$r_3(N) = O\left(\frac{N}{\log^{1-o(1)} N}\right);$$

it comes from the following theorem from [50].

Theorem 2.3 (Roth's theorem, better bounds). *Suppose that $A \subset [N]$ has density $\alpha > 0$. Then*

$$T_N(A) \geqslant \exp(-O(\alpha^{-1+o(1)}))T_N([N]).$$

3. Why arithmetic progressions?

An arithmetic progression can be seen as a solution to the linear equation $x + z = 2y$ and there are certainly other equations we could have chosen; with the primes firmly in mind then, we shall look at three other equations.

> *Do the primes contain infinitely many non-trivial solutions to $x + y + z = w$? That is to say are there infinitely many quadruples p_1, p_2, p_3, p_4 of distinct primes such that $p_1 + p_2 + p_3 = p_4$?*

It turns out that the answer is yes. Indeed, by essentially the same argument as was used by van der Corput one can show that the number of such quadruples is asymptotically

$$c\frac{N^3}{\log^4 N} \tag{3.1}$$

for some absolute constant $c > 0$ (which can be computed).

That being said, in this case the primes do *not* contain infinitely many such quadruples for reasons of size alone. Indeed, suppose that $A = \{\frac{1}{3}N + 1, \dots, N\}$. Then $|A| \sim \frac{2}{3}N$, but for any $x, y, z \in A$ we have $x + y + z > N$ and so certainly $x + y + z \notin A$. It follows that there are large subsets of $[N]$ not containing any solutions to $x + y + z = w$.

The reason that this example exists is because the equation $x+y+z = w$ is *not* translation invariant, in contrast to $x + z = 2y$.

> *Do the primes contain infinitely many non-trivial solutions to $x + y = z + w$?*

Here, again, van der Corput's method can be used to get an asymptotic of the same form as (3.1). However, a simple argument using the Cauchy-Schwarz inequality shows that if

$A \subset [N]$ has density $\alpha > 0$ then the number of quadruples $(x, y, z, w) \in A^4$ such that $x + y = z + w$ is

$$\sum_{x+y=z+w} 1_A(x)1_A(y)1_A(z)1_A(w) = \sum_t \left(\sum_{x+y=t} 1_A(x)1_A(y)\right)^2 \geqslant \frac{1}{2N}\left(\sum_t \sum_{x+y=t} 1_A(x)1_A(y)\right)^2 = \alpha^4 N^3/2.$$

It follows that if $\alpha > CN^{-1/4}$ for a large absolute constant $C > 0$ then A contains a non-trivial quadruple of the required form – the primes certainly have greater density than this and so certainly contain non-trivial solutions to this equation simply because there are lots of primes.

The final sort of equation we want to consider is somewhere between the two above.

Do the primes contain infinitely many non-trivial solutions to $x + y + z = 3w$?

Here, again, van der Corput's method can be used to get an asymptotic of the same form as (3.1). The equation is also translation invariant, but nothing as simple as the argument we just used works to show that any sufficiently large set contains a solution. Bloom, however, in [5] showed that if $A \subset [N]$ has density $\alpha > 0$ then

$$\sum_{x,y,z,w \leqslant N : x+y+z=3w} 1_A(z)1_A(y)1_A(z)1_A(w) = \exp(-\alpha^{-1/2} + o(1))N^3$$

and it follows that the primes do contain infinitely many solutions to $x + y + z = 3w$ simply because there are lots of them. It is worth remarking that versions of this equation with more summands have been considered by Schoen and Shkredov in [56], and they are able to get yet stronger bounds.

The problem of the behaviour of general linear equations (in place of those considered above) is a hard one although the examples we considered give some indication of the sort of things which can happen.

Ruzsa began the investigation with a single linear equation in [46, 47] and there has been considerable work since then although there are still many open questions. (See, for example, [9, 35, 51, 53].) Finally, the question of multiple linear equations can be even harder and includes results such as Szermerédi's theorem [63, 64], but we shall not discuss this here.

4. Lower bounds and graphs

So far we have largely focused our discussion on upper bounds for $r_3(N)$, but it is instructive to also ask about lower bounds. That is to say, we are interested in constructing large sets $A \subset [N]$ free of arithmetic progressions. A natural idea is to take a probabilistic approach. It turns out that this yields rather small sets: in particular if we pick $x \in [N]$ to lie in A independently with probability α then by linearity of expectation we have

$$\mathbb{E}(2|A| - T_N(A)) = \alpha N - \alpha^3(T_N([N]) - N).$$

It follows that we can pick $\alpha \sim N^{-1/2}$ such that the right hand side is at least $\alpha N/2$ and hence there is a choice of A such that $|A| - (T_N(A) - |A|)$ has size at least $\alpha N/2$. The number $T_N(A) - |A|$ is the number of non-trivial progressions in A and so we can delete one element from each non-trivial progression in A from A, and end up with a set free of arithmetic progressions and of size $\alpha N/2 = \Omega(\sqrt{N})$.

In light of the random construction one might guess that $r_3(N) = O(\sqrt{N})$. It turns out that this is not true. Indeed, it turns out that $r_3(N) = N^{1-o(1)}$ as a result of a construction of Behrend [3] improving on work of Salem and Spencer[49]. The basic idea is that a strictly convex body contains no three points in a line, and therefore contains no arithmetic progressions. A high-dimensional strictly convex body is then fattened up and embedded into $[N]$ in a way which preserves this lack of progressions; the body chosen is a sphere. In particular, if we write

$$S_r := \{(x_1, \dots, x_d) \in [M]^d : x_1^2 + \cdots + x_d^2 = r\}$$

then by averaging there is some $r \leqslant dM^2$ such that $|S_r| \geqslant M^d/dM^2$. We now consider the map

$$S_r \to [N]; (x_1, \dots, x_d) \mapsto x_1 + (2M)x_2 + \cdots + (2M)^{d-1}x_d. \tag{4.1}$$

This is well-defined provided $(2M)^d \gtrsim N$, and it is injective and preserves the lack of arithmetic progressions in S_r. It follows that there is a set of size $N^{1-2/d}/d2^d$ in $[N]$ containing no non-trivial arithmetic progressions. Optimising for d we then have that

$$r_3(N) = \Omega\left(\frac{N}{\sqrt{\log N}\exp(\sqrt{2\log 2 \log N})}\right).$$

Remarkably this bound has remained almost static for nearly 70 years. There was a recent improvement by Elkin [14] for which there is an appealing exposition by Green and Wolf [33], but it seems possible that this lower bound for $r_3(N)$ represents something like the truth.

The embedding in (4.1) can seem a little unnatural at first, but it turns out that high-dimensional spheres also appear naturally without this embedding as an example for a graph theoretic result which implies Roth's theorem.

Theorem 4.1 (Triangle removal lemma). *There is an increasing function $\epsilon : (0,1] \to (0,1]$ with $\epsilon(\delta) \to 0$ as $\delta \to 0$, such that if $\mathcal{G}$ is a graph on N vertices containing δN^3 triangles then there is a set of $\epsilon(\delta)N^2$ edges which can be removed to make the graph triangle-free.*

This result is due to Ruzsa and Szemerédi [48], and the question of the dependence of the function ϵ on δ is unknown. For a long time the dependence was of the form

$$\epsilon(\delta) = \log_*^{-\Omega(1)} \delta^{-1} \tag{4.2}$$

(where $\log_* M$ is the number of times one has to take logs to get M below 2) and it was a significant breakthrough when Fox [15] proved that

$$\epsilon(\delta) = \exp(-O(\log_* \delta^{-1})).$$

The reason this is so important is that all previous proofs of the triangle removal lemma had used something called the regularity lemma which provably [21] requires bounds of the

shape given in (4.2). There is an enormous amount more to be said about this topic, and we do not have time to do so here. The interested reader is directed to the survey [11].

Bounds for the triangle removal lemma translate easily to bounds for Roth's theorem and the Behrend construction above also provides something close to the best known lower bound on the function $\epsilon(\delta)$. (The extra strength in the best known lower bound comes from not needed to use the embedding (4.1).)

If one believes that Behrend's construction is close to the best for Roth's theorem then one might be tempted to believe that it is also best possible for the triangle removal lemma, suggesting that Roth's theorem has nothing to do with arithmetic at all! It may simply be that the best bounds for Roth's theorem follow from those for the triangle removal lemma.

To close this section it is worth mentioning a wonderful problem which sits somewhere between the triangle removal lemma and Roth's theorem.

> *Suppose that $A \subset [N]^2$ contains no three points in an axis-aligned right-angled triangle, that is no three distinct points $(x, y), (x + d, y), (x, y + d)$. How large can $|A|$ be?*

Ajtai and Szemerédi in [1] showed that $|A| = o(N^2)$, and then Vu [70] and Solymosi [60] gave some explicit dependencies. After this it was a huge breakthrough when Shkredov in [58, 59] showed that

$$|A| = O\left(\frac{N^2}{(\log\log N)^{\Omega(1)}}\right).$$

Shkredov's result implies Roth's theorem, although with slightly weaker bounds than Roth himself showed. Nevertheless, improving the bounds in Shkredov's result seems like it would be of considerable interest.

5. Other groups and the model setting

At the end of the last section we discussed generalisations of Roth's theorem which apply not just to the integers but more generally to graphs. There is an intermediate generalisation to finite Abelian groups which it will be helpful for us to consider here; suppose that G is such.

An **arithmetic progression** in G is exactly what one expects: it is a triple $x, y, z \in G$ such that $x + z = 2y$ and we count them in a similar way to before. although it will be helpful to normalise our counting gadget slightly differently:

$$T_G(A) := \mathbb{E}_{x,d\in G} 1_A(x)1_A(x + d)1_A(x + 2d).$$

It turns out that there is an analogue of Roth's theorem in this setting due to Lev [39].

Theorem 5.1. *Suppose that G is a finite Abelian group and $A \subset G$ has density $\alpha > 0$. Then*

$$T_G(A) = \exp(-\alpha^{-O(1)}).$$

With the additional assumption that G has odd order, this result was proved by Meshulam [40] somewhat earlier. If $G = (\mathbb{Z}/2\mathbb{Z})^n$ then we have $T_G(A) = \alpha^2$ for all $A \subset G$ of density α since (x, y, x) is an arithmetic progression for all pairs (x, y). In light of this it

seems that 2-torsion should make things easier, but it turns out that there are some additional technicalities in the proof. An illuminating model case is $G := (\mathbb{Z}/4\mathbb{Z})^n$.

In the paper [28], Green discusses the finite field philosophy for this area of mathematics. This is the powerful idea that problems for the integers can be well modelled by considering them in suitably chosen groups where the order of every (non-identity) element is the same (and small). For Roth's theorem the natural group is $G := (\mathbb{Z}/3\mathbb{Z})^n$. An arithmetic progression in G is then just a line, and Meshulam's argument from [40] applied to this group is particularly clean.

Theorem 5.2 (Roth-Mehulam Theorem). *Suppose that $A \subset G := (\mathbb{Z}/3\mathbb{Z})^n$ has density $\alpha > 0$. Then*

$$T_G(A) = \exp(-O(\alpha^{-1})).$$

This result should be compared with Theorem 2.3, and we shall discuss the proof later in §6.

It was a highly significant breakthrough when Bateman and Katz [2] improved Theorem 5.2 establishing the following.

Theorem 5.3. *Suppose that $A \subset G := (\mathbb{Z}/3\mathbb{Z})^n$ has density $\alpha > 0$. Then*

$$T_G(A) = \exp(-O(\alpha^{-1+\Omega(1)})).$$

We do not have the space to discuss their arguments here, but they involve a rather careful analysis of the structure of the Fourier spectrum. These methods have since been extended by Schoen and Shkredov a series of papers beginning with [57].

As a final remark for this section it is possible to ask about Roth's theorem in non-Abelian groups too. Here one has to decide what one might mean by an arithmetic progression: is it a triple (x, y, z) such that $xz = y^2$, or such that $z = yx^{-1}y$? It is possible to work with both; the first is *not* translation invariant in contrast to the second. The interested reader is directed to the papers [38] and [61].

6. The Roth-Meshulam theorem

Suppose that G is a finite Abelian group of odd order. The main tool for estimating counts of arithmetic progressions is the Fourier transform. The reason for this can be seen by noting that

$$T_G(A) = \langle 1_{2\cdot A}, 1_A * 1_A\rangle_{L_2(G)} \tag{6.1}$$

where $*$ denotes convolution and in this instance is defined by

$$1_A * 1_A(x) = \mathbb{E}_{y\in G} 1_A(y)1_A(x-y);$$

and $2 \cdot A := \{2a : a \in A\}$. The fact that G has odd order is not essential, but it makes it slightly easier to write down this expression for $T_G(A)$.

The Fourier transform is the (essentially unique) representation of $L_2(G)$ which simultaneously diagonalises all convolution operators. Concretely, we write $\widehat{G}$ for the dual group of G, that is the group of homomorphisms (**characters**) $G \to S^1 := \{z \in \mathbb{Z} : |z| = 1\}$, and for $f \in L_1(G)$ define the **Fourier transform** of f to be

$$\widehat{f}(\gamma) := \mathbb{E}_{x\in G} f(x)\gamma(x) \text{ for all } \gamma \in \widehat{G}.$$

Then

$$T_G(A) = \sum_{\gamma \in \widehat{G}} \widehat{1_A}(\gamma)^2 \widehat{1_A}(2\gamma);$$

the operator $f \mapsto 1_A * f$ in (6.1) has been diagonalised.

As an example of the utility of the Fourier transform we pause to prove Proposition 2.1 for finite Abelian groups.

Proposition 6.1. *Suppose that G is a finite Abelian group of odd order and $A \subset G$ has density $\alpha > 0$. Then*

$$T_G(A) \geqslant (2\alpha - 1)\alpha^2.$$

Proof. It is a short calculation to check that $\widehat{1_A}(1) = \alpha$ where 1 denotes the constant function on G taking the value 1. Then

$$T_G(A) = \alpha^3 + \sum_{\gamma \neq 1} \widehat{1_A}(\gamma)^2 \widehat{1_A}(2\gamma).$$

By the triangle inequality (or Hausdorff-Young if you prefer) $|\widehat{1_A}(2\gamma)| \leqslant \mathbb{E}1_A = \alpha$. It follows that

$$T_G(A) \geqslant \alpha^3 - \alpha \sum_{\gamma \neq 1} |\widehat{1_A}(\gamma)|^2.$$

On the other hand by Parseval's theorem we have

$$\sum_{\gamma} |\widehat{1_A}(\gamma)|^2 = \mathbb{E}_{x \in G} 1_A(x)^2 = \alpha.$$

We conclude that

$$T_G(A) \geqslant \alpha^3 - \alpha(\alpha - \alpha^2) = (2\alpha - 1)\alpha^2,$$

as required. □

Rather similar arguments are at the heart of the proof of the Roth-Meshulam theorem which we now sketch. Once we have discussed approximate groups we shall return to this sketch and flesh it out in that more general setting. To start with we recall the Roth-Meshulam theorem.

Theorem 6.2 (Roth-Meshulam, Theorem 5.2). *Suppose that $A \subset G := (\mathbb{Z}/3\mathbb{Z})^n$ has density $\alpha > 0$. Then*

$$T_G(A) = \exp(-O(\alpha^{-1})).$$

The argument is based around the following key lemma.

Lemma 6.3 (Iteration lemma). *Suppose that $A \subset G$ has density $\alpha > 0$. Then at least one of the following holds.*

(1) $T_G(A) > \alpha^3/2$;

(2) *there is a character γ on G such that*

$$\|1_A * \mu_{\ker \gamma}\|_{L_\infty(G)} \geqslant \alpha(1 + \Omega(\alpha)).$$

To understand this lemma it is helpful to note that $1_A * \mu_{\ker\gamma}(x)$ is the relative density of $A \cap (x + \ker\gamma)$ in the coset $x + \ker\gamma$. In words, then, the lemma states that either A has many arithmetic progressions – we are in the first case – or else A has increased relative density on a coset of $\ker\gamma$.

Given this lemma it is helpful to sketch quickly how we arrive at a proof of the Roth-Meshulam theorem. The idea is to repeatedly apply the lemma, terminating when we are in the first case; when in the second the density increases and this cannot happen without limit.

Of course this strategy can only work if the lemma also holds for G replaced by a coset of $\ker\gamma$. This *can* be done because arithmetic progressions are translation invariant – the number of arithmetic progressions in $A\cap(x+\ker\gamma)$ is the same as in $(A-x)\cap\ker\gamma$. This is the only place we use the translation invariance of arithmetic progressions, but as we have remarked before it is essential.

Every time we iterate in the above argument the density goes from α to $\alpha(1+\Omega(\alpha))$; this can happen at most $O(\alpha^{-1})$ times. It remains to estimate the size of the subgroup we have ended up with. Since $(\mathbb{Z}/3\mathbb{Z})^n$ has exponent 3, the characters take only three values and the kernel of a character therefore has index 3. It follows that the iteration terminates with a subgroup of index $3^{O(\alpha^{-1})}$ and the result follows from this once all the parts are pieced together.

Pretty much all arguments regarding these sorts of problems are iterative in this way, and it is not hard to port such arguments from the model setting so we shall not dwell on this further. What is harder to extend from the model setting to the integers is the proof of the iteration lemma. This, in turn, splits into three parts, and while some of them may seem laboured, they are are set out this way because they generalise in different ways.

Given a function $f : G \to \mathbb{C}$ we define the **Gowers uniformity norm of order** 2 of f by

$$\|f\|_{U^2}^4 := \mathbb{E}_{x_0,x_1,x_2\in G} f(x_1)\overline{f(x_1+x_0)f(x_2)}f(x_2+x_0).$$

This is a bonafide norm, and was defined by Gowers as part of a wider family in the celebrated work [22, 23]. We shall return to this hierarchy later, but for now we use this definition to record the first ingredient in Lemma 6.3.

Lemma 6.4. *Suppose that $A \subset G$ has density $\alpha > 0$. Then at least one of the following holds.*

(1) $T_G(A) > \alpha^3/2$;
(2) $\|1_A - \alpha\|_{U^2}^4 = \Omega(\alpha^5)$.

This is a simple application of the Cauchy-Schwarz inequality in a way which is very common for the field. We record it because it lets us separate out the so-called U^2-inverse theorem which we shall want to discuss later.

Lemma 6.5 (U^2-inverse theorem). *Suppose that $A \subset G$ has density $\alpha > 0$ and $\|1_A - \alpha\|_{U^2}^4 \geqslant \epsilon^2\alpha^3$. Then there is a character γ such that $|(1_A-\alpha)^\wedge(\gamma)| \geqslant \epsilon\alpha$.*

The proof of this is a simple application of Fourier inversion.

Finally,

Lemma 6.6. *Suppose that $A \subset G$ has density $\alpha > 0$ and γ is a character such that $|(1_A - \alpha)^\wedge(\gamma)| \geqslant \epsilon\alpha$. Then*

$$\|1_A * \mu_{\ker\gamma}\|_{L_\infty(G)} \geqslant \alpha(1+\Omega(\epsilon)).$$

All three of these lemmas apply to any finite Abelian group, and they easily dovetail to give Lemma 6.3. That being said, the lemma cannot always be effectively iterated because the kernel of the character may be small. Indeed, if $G = \mathbb{Z}/p\mathbb{Z}$ for p a prime then $\ker \gamma$ is trivial. It turns out, however, that the hypothesis of Lemma 6.6 also give rise to a density increment on something which is roughly – approximately – like a subgroup, and much larger than a genuine subgroup might be. These approximate (sub-)groups are the topic of the next section.

7. Approximate groups

Approximate groups are an area of study which has received considerable attention over the last five years, and the interested reader might like to consult the survey [29] of Green.

In a nutshell, approximate groups are groups in which the closure axiom has been relaxed to a sort of 'approximate' closure. To start with suppose that G is an Abelian group and B is a non-empty symmetric subset of G. Then B is a subgroup of G if and only if it is closed under addition. We can write this using additive notation as $B + B = B$ where $B + B := \{b + b' : b, b' \in B\}$. One might then reasonably ask what happens if we relax this to

$$|B + B| \leqslant (1 + \epsilon)|B|, \tag{7.1}$$

for some small ϵ. As it happens, if $\epsilon < 1/2$ then a result of Freĭman [17] (also appearing as [67, Exercise 2.6.5]) tells us that $B + B$ is a subgroup.

At first one might be rather satisfied with this state of affairs: sets satisfying a sort of approximate closure axiom can be completed into a subgroup without enlarging them too much. One reason for being a little less enthusiastic is indicated by Lemma 6.6. If these were the only approximate subgroups, then we should know that they were all close to bonafide subgroups, but our problem with Lemma 6.6 was precisely that there might not be any large (non-trivial) subgroups. If all approximate subgroups are close to bonafide subgroups then there would be groups with no large approximate subgroups.

Bourgain in [7] has an ingenious solution to the limitations of (7.1): he introduced an asymmetry in the group operation and considers pairs of symmetric sets (B, B') such that

$$|B + B'| \leqslant (1 + \epsilon)|B|.$$

Of course for this to add any generality we need B' to be somewhat smaller than B, but this is possible and leads to a genuinely new class of structure.

Our prototype for approximate groups then will be nested intervals. We write $I \subset \mathbb{Z}$ for a symmetric interval about 0 of length N; I' for the same of length N'; and so on for I'' *etc*. We think of $N'' \ll N' \ll N$. Then

$$|I + I'| \leqslant |I|(1 + O(N'/N)) \approx |I| \textit{ etc}.$$

We shall make further use of intervals when we describe Roth's proof of his theorem in the framework of §6, but for now we shall turn to more general approximate groups.

Following [31] which attempts to abstract some of the important ideas of Bourgain from [7], we define a **Bourgain system** to be a vector $\mathcal{B} = (B_\rho)_{\rho \in (0,2]}$ of subsets of G satisfying the following axioms.

(1) *(Symmetric neighbourhoods of the identity)* For all $\rho \in (0,2]$ the set B_ρ is a symmetric set containing the identity, that is to say $0_G \in B_\rho$ and $-B_\rho = B_\rho$;

(2) *(Nesting/Sub-additivity)* For all $\rho, \rho' > 0$ with $\rho + \rho' \leqslant 2$ we have $B_\rho + B_{\rho'} \subset B_{\rho+\rho'}$;

(3) *(Growth)* For every $\rho \in (0,1]$ we have that $B_{2\rho}$ can be covered by at most 2^d translates of B_ρ.

We say that the system is d-**dimensional** and has **size** $|B_1|$.

A Bourgain system may be thought of as providing a base for a topology around the identity. In this light the symmetry axiom may be seen as asking that inversion is continuous, and the sub-additivity axiom may be seen as asking that multiplication is continuous. Finally, the growth condition asks for a sort of local compactness.

Example 7.1 (Intervals). *Given an integer N we can write $B_\rho := \{x \in \mathbb{Z} : |x| \leqslant \rho N\}$. Then $\mathcal{B}$ is an $O(1)$-dimensional Bourgain system of size $2N$.*

The example of intervals generalises in two directions, both of which are rather useful.

Example 7.2 (Balls in $\mathbb{Z}^d$). *Let $\mathcal{B} := (B_\rho)_{\rho\in(0,2]}$ where B_ρ is the ball of radius ρ in $\mathbb{Z}^d$ (considered as a subset of $\mathbb{R}^d$ in any norm). It is easy to check the various axioms of a Bourgain system hold.*

Example 7.3 (Bohr sets). *For $z \in S^1$ put $\|z\| := (2\pi)^{-1}|\arg z|$, where* arg *is taken as mapping into $(-\pi,\pi]$. If $\Gamma \subset \widehat{G}$ and $\delta \in (0,1]$ then we put*

$$B(\Gamma,\delta) := \{x \in G : \|\gamma(x)\| \leqslant \delta \text{ for all } \gamma \in \Gamma\},$$

and call such a set a ***Bohr set****; the set Γ is called the* ***frequency set*** *and the parameter δ is called the* ***width****.*

It turns out that the system $(B(\Gamma,\rho\delta))_\rho$ is a Bourgain system of density at least $\delta^{|\Gamma|}$ and dimension $2|\Gamma|$. This can be shown by an averaging argument which can be found in [67, Lemma 4.20].

With Example 7.2 we can check directly that there is an absolute constant $c > 0$ such that

$$\left|\frac{|B_{\rho+\rho'}|}{|B_\rho|} - 1\right| = O(d\eta) \text{ for all } |\rho'| \leqslant c\rho/d. \tag{7.2}$$

On the other hand, this need not be the case with Bohr sets as can be seen by considering Bohr sets in groups of bounded exponent. Fortunately Bourgain recovered the situation in [7] by showing that (7.2) holds almost all the time for all Bourgain systems – he called the radii ρ for which (7.2) holds **regular**.

That ends our introduction to approximate groups and Bourgain systems which has been rather quick and dirty. The main idea is that these are structures which support sufficient harmonic analysis that certain arguments can be pushed through as we shall see in §9. That being said the reader may wish to consult the papers [30, 32] or [24] for some different perspectives and more thorough discussions.

8. Freĭman's theorem and another question

The reader only interested in Roth's theorem can skip this section, but for those with a little more time it is worth making a couple of additional remarks. Had we relaxed (7.1) further in the last section asking only that $|B+B| \leqslant K|B|$ for some possibly larger value of K then it would turn out that B is necessarily close to a Bourgain system. Indeed, in that case we have the following remarkable result of Freĭman.

Theorem 8.1 (Freĭman's theorem). *Suppose that G is an Abelian group and $A \subset G$ is finite with $|A+A| \leqslant K|A|$. Then there is an $O_K(1)$-dimensional Bourgain system $\mathcal{B}$ of size $\Omega_K(|A|)$ such that $|A \cap B_1| = \Omega_K(|B_1|)$.*

As it happens one can be rather more precise: it turns out that the Bourgain system is of a particular type called a generalised arithmetic progression; and there are reasonable dependencies available for the implied constants, particularly following the work of Schoen [52] and Croot and Sisask [13].

It is also worth saying that Freĭman's theorem is a tremendously important and powerful result, and our few short remarks cannot do it justice. The interested reader is directed towards [4, 18, 19, 26] for more information and applications.

Having established his result Freĭman himself proved the following theorem in [16] which is closer to the sort of question we are considering in this paper.

Theorem 8.2. *Suppose that $A \subset \mathbb{Z}$ is finite and contains no non-trivial arithmetic progressions. Then $\frac{|A+A|}{|A|} \to \infty$ as $|A| \to \infty$.*

This result immediately implies that $r_3(N) = o(N)$ since if $A \subset [N]$ has density $\alpha > 0$ and no non-trivial arithmetic progressions then

$$2\alpha^{-1} \geqslant \frac{|A+A|}{|A|} \to \infty \text{ as } |A| \to \infty.$$

In fact Ruzsa, in [45], showed that there is a sort of converse so that bounds for $r_3(N)$ can be used to get a rate of growth for the ratio:

$$\frac{|A+A|}{|A|} = \Omega\left(\frac{r_3(|A|)}{|A|}\right)^{-1/4}.$$

Although one can get fairly good bounds for this ratio by inserting our best bounds for Roth's theorem, Henriot [36] has shown the following further improvement by direct argument and Freĭman's theorem.

$$\frac{|A+A|}{|A|} = \Omega(\log^{1-o(1)}|A|). \tag{8.1}$$

Finally Henriot's result also gives the best known bound for a lovely problem considered by Stanchescu [62].

> *Suppose that $A \subset \mathbb{Z}^2$ contains no three co-linear points. How small can the ratio $|A+A|/|A|$ be?*

Three points in arithmetic progression certainly form a line and so such a set A is certainly free of non-trivial arithmetic progressions. Moreover, Henriot's arguments are actually set

in the full generality of Abelian groups and so, in particular, tell us that (8.1) holds in this setting too.

Remarkably, nothing better than the bound following immediately from Henriot's work is known here, despite the fact that if A has no co-linear triples then not only does it avoid the equation $x+y=2z$, but it also avoids

$$tx+sy=(t+s)z$$

for *any* $s,t\in\mathbb{N}$! It is worth noting that, again, it is the sphere of Behrend's construction which provides the best know construction. (See [62] for details of this.)

9. A proof of Roth's theorem

We now return to the framework of the argument in §6 and consider it through the lens of §7 by transferring it to intervals. Suppose that I, I', I'', and I''' are symmetric intervals of integers around 0 and having sizes $N\gg N'\gg N''\gg N'''$ respectively. We shall make use of the Fourier transform on $\mathbb{Z}$ defined on functions $f\in\ell_1(\mathbb{Z})$ by

$$\widehat{f}(\theta):=\sum_{z\in\mathbb{Z}}f(z)\gamma(z)\text{ for all }\gamma\in\widehat{\mathbb{Z}}.$$

Perhaps we should remark that $\widehat{\mathbb{Z}}\cong\mathbb{R}/\mathbb{Z}$; every character $\gamma:\mathbb{Z}\to S^1$ has the form $z\mapsto\exp(2\pi iz\theta)$ for some $\theta\in\mathbb{T}$.

Although we have defined the Fourier transform for $\mathbb{Z}$, we shall think of it as applied to the intervals I, I', *etc*. To this end, it is useful record the fact that μ_I denotes the uniform measure on I and the (a) natural version of the Fourier transform on I is given by taking $f\in L_1(\mu_I)$ to

$$\widehat{fd\mu_I}(\theta)=\mathbb{E}_{x\in I}f(x)\gamma(x).$$

Now, turning to the arguments of §6, it was Lemma 6.6 which showed us how to go from a large Fourier coefficient to a density increment on a subgroup, which had the main weakness as regards extending the Roth-Meshulam theorem from groups of bounded exponent to cyclic groups. We have the following.

Lemma 9.1. *Suppose that $A\subset I$ has density $\alpha>0$ and $\gamma:\mathbb{Z}\to S^1$ is a character such that $|((1_A-\alpha)d\mu_I)\widehat{\ }(\gamma)|\geqslant\epsilon\alpha$. Then there is an arithmetic progression P of length $\Omega(\sqrt{N'})$ such that*

$$\|1_A*\mu_P\|_{L_\infty(I)}\geqslant\alpha(1+\Omega(\epsilon))-O(N'/N).$$

The proof of this lemma is relatively simple and follows Lemma 6.6 but with the addition of the pigeonhole principle. Ultimately we shall return again and discuss something even stronger here, but for now we examine the other parts of §6.

Our next task is to formulate a version of the U^2-norm for approximate groups and at the same time it will be useful to define a version of the related U^1-(semi-)norm which we also now define. The U^2-norm is one of a hierarchy of (semi-)norms introduced by Gowers in cite [23], the first of which is defined on functions $f:G\to\mathbb{C}$ by

$$\|f\|_{U^1}^2:=\mathbb{E}_{x,y\in G}f(x)\overline{f(x+y)}.$$

It is easy to check that for a set $A' \subset G$ of density α' we always have $\|1_A - \alpha'\|_{U^1} = 0$; in the setting of approximate groups this need not be the case.

Given $f : I \to \mathbb{C}$ we define

$$\|f\|^2_{U^1(I,I')} := \mathbb{E}_{x_0 \in I'} \mathbb{E}_{x_1 \in I} f(x_1)\overline{f(x_1 + x_0)}$$

and

$$\|f\|^4_{U^2(I,I')} := \mathbb{E}_{x_0 \in I'} \mathbb{E}_{x_1,x_2 \in I} f(x_1)\overline{f(x_1 + x_0)f(x_2)}f(x_2 + x_0).$$

In addition we shall require an analogue of T_G which captures the approximate group structure – it is *not* enough to use T_N. To this end put

$$\begin{aligned} T_{I,I',I''}(A) :=& \mathbb{E}_{y \in I} \langle 1_{2\cdot(A\cap(y+I''))}, 1_{A\cap(y+I')} * (1_{A\cap(y+I')} d\mu_{y+I'}) \rangle_{L_2(2y+2I'')} \\ =& \mathbb{E}_{y \in I} \mathbb{E}_{z \in 2y+2I''} 1_{2\cdot(A\cap(y+I''))}(z) \\ & \times \mathbb{E}_{x \in y+I'} 1_{A\cap(y+I')}(x) 1_{A\cap y+I')}(z - x). \end{aligned} \tag{9.1}$$

This is a complicated expression, but it is natural and it is worth taking a moment to examine the parts. If G were a finite group and $A' \subset G$ then we would be considering

$$T_G(A') = \langle 1_{2\cdot A'}, 1_{A'} * 1_{A'} \rangle_{L_2(G)}.$$

For us the set A is contained in I which we think of as 'compact' and so it is natural to normalise convolution as an expectation. That means that

$$1_{A'} * 1_{A'} = \mathbb{E}_{z \in G} 1_{A'}(z) 1_{A'}(\cdot - z) \mapsto 1_A * (1_A d\mu_I) = \mathbb{E}_{z \in I} 1_A(z) 1_A(\cdot - z).$$

(In fact we do not quite do this, and we shall explain why in a moment.) The sensible domain of definition of $1_A * (1_A d\mu_I)$ is then any smaller interval I'. Indeed, we have

$$1_I * (1_I d\mu_I)(x) = 1 + O(N'/N) \text{ for all } x \in I'$$

which should be compared with

$$1_G * 1_G(x) = 1 \text{ for all } x \in G.$$

Of course $2 \cdot A$ is actually naturally defined on $2 \cdot I$ and we should like it to be defined on $2 \cdot I'$. Now, $2 \cdot (A \cap I')$ is defined on $2 \cdot I'$, but the former might be empty. This is why we introduce the additional averaging over $y \in I$ in (9.1). It gives us elements y such that $A \cap (y + I')$ and $A \cap (2y + 2I')$ are both large. (Actually, it does not quite do this, but if the U^1-norm is small then it does.)

Some of that may seem rather obscure and technical so perhaps it is time to record some lemmas.

Lemma 9.2 (Analogue of Lemma 6.4). *Suppose that $A \subset I$ has density $\alpha > 0$. Then at least one of the following holds.*

(1) $T_{I,I',I''}(A) > \alpha^3/2$;

(2) $\mathbb{E}_{y \in I} \|1_{A\cap(y+I)} - \alpha\|^4_{U^2(y+I',2y+2I'')} = \Omega(\alpha^5) - O(N''/N') - O(N'/N)$;

(3) $\|1_A - \alpha\|_{U^1(I,I')} = \Omega(\alpha^3) - O(N''/N') - O(N'/N)$.

The proof of this is really just the Cauchy-Schwarz inequality again, but with a few extra ingredients to take care of the fact we are in an approximate group.

Lemma 9.3 (U^2-inverse theorem for intervals/approximate groups). *Suppose that $A \subset I$ has density $\alpha > 0$ and*

$$\mathbb{E}_{y\in I}\|1_{A\cap(y+I')} - \alpha\|^4_{U^2(y+I',2y+2I'')} \geqslant \epsilon^2\alpha^3.$$

Then there is a character $\gamma : \mathbb{Z} \to S^1$ and a translate $y \in I$ such that

$$|((1_{A\cap(y+I''')} - \alpha)d\mu_{y+I'''})^\wedge(\gamma)| \geqslant \epsilon\alpha - O(N'/N) - O(N''/N') - O(N'''/N'').$$

With a little tweaking to deal with the fact that α is *not* the relative density of A in $y + I'''$ (as seems to be required in Lemma 9.1), and what to do in the event that the U^1-norm is large (this is even easier than when the U^2-norm is large), these results combine to give the following iteration lemma.

Lemma 9.4 (Iteration lemma). *Suppose that $A \subset I$ has density $\alpha > 0$. Then at least one of the following holds.*

(1) $T_{I,I',I''}(A) > \alpha^3/2$;

(2) *there is an arithmetic progressions P of size $\Omega(\sqrt{N'''})$ such that*

$$\|1_A * \mu_P\|_{L_\infty(G)} \geqslant \alpha(1 + \Omega(\alpha)) - O(N'/N) - O(N''/N') - O(N'''/N'').$$

As is typical when dealing with approximate groups, one choses $N' = o(\alpha^2 N)$, $N'' = o(\alpha^2 N')$, *etc.* and conclude that we have an iteration as before which terminates after $O(\alpha^{-1})$ steps. At each stage we have $N \mapsto N^{\Omega(1)}$, and so we end up concluding that A contains a non-trivial arithmetic progression provided $N^{-\exp(O(\alpha^{-1}))} \gg 1$. This leads to Roth's theorem.

Before closing this section we note that while we are interested in bounds, there are several other proofs of Roth's theorem with different strengths. We direct the reader's attention to the papers [12, 66] and [20] for some examples. The last of these, in particular, covers the ergodic theoretic proof of Roth's theorem which leads to an enormously powerful theory which we do not have time to discuss here.

10. Improvements to Roth's theorem

In the previous section we gave an account of Roth's proof of Roth's theorem which was certainly not expressed in the language used by Roth and which appears to introduce a number of technicalities which are not visible in Roth's original work. We did this for three reasons: first, it is natural within the framework of approximate groups; secondly, it provides a good model framework in which to practice the arguments typical of approximate groups; finally it makes discussion of improvements much simpler.

We shall begin with Bourgain's improvement the the bounds in Roth's theorem from [7]. He noted that in Lemma 9.1, instead of passing to a long arithmetic progression, one can actually pass to the level set of the character which is much larger. Suppose that $\mathcal{B} = (B_\rho)_{\rho\in(0,2]}$ is a d-dimensional Bourgain system, then we (almost) have the following.

Lemma 10.1. *Suppose that $A \subset B_1$ has density $\alpha > 0$ and $\gamma : \mathbb{Z} \to S^1$ is a character such that $|((1_A - \alpha)d\mu_{B_1})^\wedge(\gamma)| \geqslant \epsilon\alpha$. Then there is a Bourgain system $\mathcal{B}' = (B'_\rho)_{\rho\in(0,2]}$ of size $\delta^{-O(d)}|B|$ and dimension $d+1$ such that*

$$\|1_A * \mu_{B'_1}\|_{L_\infty(I)} \geqslant \alpha(1 + \Omega(\epsilon)) - O(\delta).$$

The advantage of our more complicated discussion in the previous section is that almost exactly the same arguments as worked there apply to more general Bourgain systems, and gives an iteration lemma analogous to Lemma 9.4.

Lemma 10.2 (Iteration lemma). *Suppose that $A \subset B_1$ has density $\alpha > 0$. Then at least one of the following holds.*

(1) $T_{B_1,B_\delta,B_{\delta'}}(A) > \alpha^3/2$;

(2) *there is a Bourgain system $\mathcal{B}'$ of dimension $d+1$ and size $(\delta\delta'\delta'')^{O}(d)|B_1|$ such that*

$$\|1_A * \mu_{B'_1}\|_{L_\infty(G)} \geqslant \alpha(1 + \Omega(\alpha)) - O(\delta + \delta'\delta^{-1} + \delta''\delta'^{-1}).$$

We pick $\delta = o(\alpha^2)$, $\delta' = o(\alpha^2\delta)$, and $\delta'' = o(\alpha^2\delta')$ and we have an iteration which, as before, can only proceed $O(\alpha^{-1})$ times. To conclude we end up with a Bourgain system of size

$$\alpha^{O(1+2+3+\cdots+O(\alpha^{-1}))}N = \alpha^{-O(\alpha^{-2})}N,$$

and hence $T_N(A) = \alpha^{-O(\alpha^{-2})}N^2$ which leads to Bourgain's bound.

Bourgain's argument improves on Roth's by making the cost of each iteration less. Another natural idea is to try to reduce the number of iterations. The obvious place to do this is with the use of the U^2-inverse theorem. Instead of finding just one character at which the Fourier transform is large, we can see if we can make use of all of them.

This is exactly what Heath-Brown and Szemerédi did in their papers [34] and [65], and to describe it it is useful to return to the setting of G a finite Abelian group and $A \subset G$ of density α. Suppose that $T_G(A) \leqslant \alpha^3/2$. Then

$$\sum_{\gamma:|\widehat{1_A}(2\gamma)|=\Omega(\alpha^2)} |\widehat{1_A}(\gamma)|^3 = \Omega(\alpha^3).$$

If we let B be the Bohr set (see Example 7.3 for details) with frequency set

$$\Gamma := \{\gamma : |\widehat{1_A}(2\gamma)| = \Omega(\alpha^2)\},$$

(and a suitably small width parameter) then it turns out that we can arrange things so that

$$\|1_A * \mu_B\|_{L_\infty(G)} \geqslant \alpha(1 + \Omega(1)).$$

This is a very large density increment for A; indeed it is so large that it can only be done $O(\log \alpha^{-1})$ times (assuming, as is the case, that the above argument can be run relative to a Bourgain system). The Bohr set B is a Bourgain system of dimension $O(|\Gamma|)$ and to estimate $|\Gamma|$ we can use Parseval's theorem:

$$|\Gamma|(\Omega(\alpha^2))^2 \leqslant \sum_{\gamma\in\Gamma} |\widehat{1_A}(2\gamma)|^2 \leqslant \sum_{\gamma} |\widehat{1_A}(\gamma)|^2 = \alpha$$

(provided G has odd order so that multiplication by 2 is an injection on $\widehat{G}$). It follows that $|\Gamma| = O(\alpha^{-3})$. Unfortunately this means that the resulting Bourgain system has density $\alpha^{O(\alpha^{-3})}$ which means that already after one iteration we have a worse estimate that Bourgain's argument.

One can do a little better than the bound from Parseval's theorem by applying a result called Chang's theorem [10].

Theorem 10.3 (Chang's theorem). *Suppose that $A \subset G$ has density $\alpha > 0$ and $\epsilon \in (0,1]$ is a parameter. Then there is a set Γ of size $O(\epsilon^{-2}\log\alpha^{-1})$ such that*

$$\{\gamma : |\widehat{1_A}(\gamma)| \geqslant \epsilon\alpha\} \subset \{\sum_{\gamma\in\Gamma}\sigma_\gamma\gamma : \sigma \in \{-1,0,1\}^\Gamma\}.$$

This is a tremendously useful result, foreshadowed in the work of Bourgain, particularly [6], and popularised by Green. It is essentially equivalent to something called Rudin's inequality (see [44]).

Using this result and Bourgain's framework one can recover a bound *almost* as good Bourgain's original bound in [7] but using $O(\log\alpha^{-1})$ iterations. Unfortunately Chang's theorem is best possible (see [27, 41]), so it seems as if we are stuck.

It turns out, however, if we manipulate the set A a little before using the Fourier transform then we can do better and arrive at Theorem 2.3. In particular, a rather nice combinatorial trick of Katz and Koester [37] (discovered independently and a little later by Schoen [52] and used to great effect by him and others in such work as [54, 55, 57]) can be used to make these manipulations.

Specifically our aim is to transform the set A into two sets L and S where L is thick, S is not too thin, and $L + S \subset A - 2.A$. The main idea is to construct such sets L and S iteratively using the 'Katz-Koester transformation'. Suppose that L and S have density λ and σ respectively and $L + S \subset A + A$. Unless A is 'quite structured' one expects there to be very few x for which

$$1_L * 1_{-A}(x) \geqslant \alpha/2;$$

on the other hand, by averaging, there are many $x \in G$ such that

$$1_{-S} * 1_A(x) \geqslant \sigma\alpha/2.$$

It follows that unless A is 'quite structured' one may find an $x \in G$ such that

$$1_L * 1_{-A}(x) \leqslant \alpha/2 \text{ and } 1_{-S} * 1_A(x) \geqslant \sigma\alpha/2.$$

Now, if we put

$$L' := L \cup (x + A) \text{ and } S' := S \cap (A - x),$$

then we have

$$\mu_G(L') \geqslant \mu_G(L) + \mu_G(x + A) - 1_L * 1_{-A}(x) \geqslant \lambda + \alpha/2 \text{ and } \mu_G(S') \geqslant \alpha\sigma/2,$$

and also

$$L' + S' \subset (L + S') \cup ((x + A) + S') \subset (L + S) \cup (x + A + A - x) \subset A + A.$$

We see that unless A is quite structured we have a new pair (L', S') whose sumset is contained in $A + A$, but for which L' is somewhat larger (than L) while S' is not too much

smaller (than S). In particular the density of L has increased by $\alpha/2$ while the denisty of S has reduced by at most a factor of $\alpha/2$. The first sort of increment can only happen $O(\alpha^{-1})$ times after which the resulting set L has density $\Omega(1)$ and the set S has density $\alpha^{O(\alpha^{-1})}$ as we claimed.

With the sets S and L which have $S + L \subset A + A$ and hence *essentially* $(S + L) \cap (2 \cdot A) = \emptyset$ we can proceed with a Fourier argument (in Bourgain systems). Chang's theorem is particularly strong with thin sets and so can be applied to S, and this coupled with the Croot-Sisask lemma [13] are the main two remaining ingredients.

References

[1] M. Ajtai and E. Szemerédi, *Sets of lattice points that form no squares*, Stud. Sci. Math. Hungar., **9**:9–11 (1975), 1974.

[2] M. Bateman and N. H. Katz, *New bounds on cap sets*, J. Amer. Math. Soc., **25**(2):585–613, 2012.

[3] F. A. Behrend, *On sets of integers which contain no three terms in arithmetical progression*, Proc. Nat. Acad. Sci. U. S. A., **32**:331–332, 1946.

[4] Y. Bilu, *Structure of sets with small sumset*, Astérisque, (258):xi, 77–108, 1999, Structure theory of set addition.

[5] T. F. Bloom, *Translation invariant equations and the method of Sanders*, Bulletin of the London Mathematical Society, **44**(5):1050–1067, 2012.

[6] J. Bourgain, *On arithmetic progressions in sums of sets of integers*, In A tribute to Paul Erdős, pp. 105–109, Cambridge Univ. Press, Cambridge, 1990.

[7] ———, *On triples in arithmetic progression*, Geom. Funct. Anal., **9**(5):968–984, 1999.

[8] ———, *Roth's theorem on progressions revisited*, J. Anal. Math., **104**:155–192, 2008.

[9] B. Bukh, *Non-trivial solutions to a linear equation in integers*, Acta Arith., **131**(1):51–55, 2008.

[10] M.-C. Chang, *A polynomial bound in Freĭman's theorem*, Duke Math. J., **113**(3):399–419, 2002.

[11] D Conlon and J. Fox, *Graph removal lemmas*, 2012.

[12] E. Croot and O. Sisask, *A new proof of Roth's theorem on arithmetic progressions*, Proc. Amer. Math. Soc., **137**(3):805–809, 2009.

[13] ———, *A probabilistic technique for finding almost-periods of convolutions*, Geom. Funct. Anal., **20**(6):1367–1396, 2010.

[14] M. Elkin, *An improved construction of progression-free sets*, In Symposium on Discrete Algorithms, pp. 886–905, 2010.

[15] J. Fox, *A new proof of the graph removal lemma*, Ann. of Math. (2), **174**(1):561–579, 2011.

[16] G. A. Freĭman, *Foundations of a structural theory of set addition*, American Mathematical Society, Providence, R. I., 1973, Translated from the Russian, Translations of Mathematical Monographs, Vol 37.

[17] ———, *Groups and the inverse problems of additive number theory*, In Number-theoretic studies in the Markov spectrum and in the structural theory of set addition (Russian), pp. 175–183, Kalinin. Gos. Univ., Moscow, 1973.

[18] ———, *Structure theory of set addition*, In Proc. Conf. Structure theory of set addition, pp. 299–318, June 1993.

[19] ———, *Structure theory of set addition*, Astérisque, (258):xi, 1–33, 1999, Structure theory of set addition.

[20] H. Furstenberg, *Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions*, J. Analyse Math., **31**:204–256, 1977,

[21] W. T. Gowers, *Lower bounds of tower type for Szemerédi's uniformity lemma*, Geom. Funct. Anal., **7**(2):322–337, 1997.

[22] ———, *A new proof of Szemerédi's theorem for arithmetic progressions of length four*, Geom. Funct. Anal., **8**(3):529–551, 1998.

[23] ———, *A new proof of Szemerédi's theorem*, Geom. Funct. Anal., **11**(3):465–588, 2001.

[24] W. T. Gowers and J. Wolf, *Linear forms and quadratic uniformity for functions on* $\mathbb{Z}_N$, J. Anal. Math., **115**:121–186, 2011.

[25] A. Granville, *Harald Cramér and the distribution of prime numbers*, Scand. Actuar. J., (1):12–28, 1995, Harald Cramér Symposium (Stockholm, 1993).

[26] B. J. Green. *Structure theory of set addition*, Available at http://people.maths.ox.ac.uk/greenbj/papers/icmsnotes.pdf, 2002.

[27] ———, *Spectral structure of sets of integers*, In Fourier analysis and convexity, Appl. Numer. Harmon. Anal., pp. 83–96, Birkhäuser Boston, Boston, MA, 2004.

[28] ———, *Finite field models in additive combinatorics*, In Surveys in combinatorics 2005, volume 327 of London Math. Soc. Lecture Note Ser., pp. 1–27, Cambridge Univ. Press, Cambridge, 2005.

[29] ———, *Approximate groups and their applications: work of Bourgain, Gamburd, Helfgott and Sarnak*, 2009.

[30] B. J. Green and S. V. Konyagin, *On the Littlewood problem modulo a prime*, Canad. J. Math., **61**(1):141–164, 2009.

[31] B. J. Green and T. Sanders, *A quantitative version of the idempotent theorem in harmonic analysis*, Ann. of Math. (2), **168**(3):1025–1054, 2008.

[32] B. J. Green and T. C. Tao, *An inverse theorem for the Gowers $U^3(G)$ norm.*, Proc. Edinb. Math. Soc. (2), **51**(1):73–153, 2008.

[33] B. J. Green and J. Wolf, *A note on Elkin's improvement of Behrend's construction*, In Additive number theory: Festschrift in honor of the sixtieth birthday of Melvyn B. Nathanson, pp. 141–144, Springer-Verlag, 1st. edition, 2010.

[34] D. R. Heath-Brown, *Integer sets containing no arithmetic progressions*, J. London Math. Soc. (2), **35**(3):385–394, 1987.

[35] P. Hegarty, *Extremal subsets of $\{1, \ldots, n\}$ avoiding solutions to linear equations in three variables*, Electron. J. Combin., **14**(1):Research Paper 74, 22, 2007.

[36] K. Henriot, *Arithmetic progressions in sets of small doubling*, 2013.

[37] N. H. Katz and P. Koester, *On additive doubling and energy*, SIAM J. Discrete Math., **24**(4):1684–1693, 2010.

[38] D. Král, O. Serra, and L. Vena, *A combinatorial proof of the removal lemma for groups*, J. Combin. Theory Ser. A, **116**(4):971–978, 2009.

[39] V. F. Lev, *Progression-free sets in finite abelian groups*, J. Number Theory, **104**(1):162–169, 2004.

[40] R. Meshulam, *On subsets of finite abelian groups with no 3-term arithmetic progressions*, J. Combin. Theory Ser. A, **71**(1):168–172, 1995.

[41] F. L. Nazarov, *The Bang solution of the coefficient problem*, Algebra i Analiz, **9**(2):272–287, 1997.

[42] K. F. Roth, *Sur quelques ensembles d'entiers*, C. R. Acad. Sci. Paris, **234**:388–390, 1952.

[43] ———, *On certain sets of integers*, J. London Math. Soc., **28**:104–109, 1953.

[44] W. Rudin, *Fourier analysis on groups*, *Wiley Classics Library*, John Wiley & Sons Inc., New York, 1990. Reprint of the 1962 original, A Wiley-Interscience Publication.

[45] I. Z. Ruzsa, *Arithmetical progressions and the number of sums*, Period. Math. Hungar., **25**(1):105–111, 1992.

[46] ———, *Solving a linear equation in a set of integers. I*, Acta Arith., **65**(3):259–282, 1993.

[47] ———, *Solving a linear equation in a set of integers. II*, Acta Arith., **72**(4):385–397, 1995.

[48] I. Z. Ruzsa and E. Szemerédi, *Triple systems with no six points carrying three triangles*, In Combinatorics (Proc. Fifth Hungarian Colloq., Keszthely, 1976), Vol. II, volume 18 of Colloq. Math. Soc. János Bolyai, pp. 939–945, North-Holland, Amsterdam, 1978.

[49] R. Salem and D. C. Spencer, *On sets of integers which contain no three terms in arithmetical progression*, Proc. Nat. Acad. Sci. U. S. A., **28**:561–563, 1942.

[50] T. Sanders, *On Roth's theorem on progressions*, Ann. of Math. (2), **174**(1):619–636, 2011.

[51] T. Schoen, *On sets of natural numbers without solution to a noninvariant linear equation*, Acta Arith., **93**(2):149–155, 2000.

[52] ———, *Near optimal bounds in Freĭman's theorem*, Duke Math. J., **158**:1–12, 2011.

[53] ———, *Linear equations and sets of integers*, Acta Math. Hungar., **135**(3):229–235, 2012.

[54] T. Schoen and I. D. Shkredov, *Additive properties of multiplicative subgroups in* $\mathbb{F}_p$, Quart. J. Math., 2010. to appear.

[55] ———, *On sumsets of convex sets*, Combin. Probab. Comput., **20**(5):793–798, 2011.

[56] ———, *Roth's theorem in many variables*, 2011.

[57] ———, *Higher moments of convolutions*, J. Number Theory, **133**(5):1693–1737, 2013.

[58] I. D. Shkredov, *On a generalization of Szemerédi's theorem*, Proc. London Math. Soc. (3), **93**(3):723–760, 2006.

[59] ———, *On a problem of Gowers*, Izv. Ross. Akad. Nauk Ser. Mat., **70**(2):179–221, 2006.

[60] J. Solymosi, *Note on a generalization of Roth's theorem*, In Discrete and computational geometry, volume 25 of Algorithms Combin., pp. 825–827. Springer, Berlin, 2003.

[61] ———, *Roth-type theorems in finite groups*, European J. Combin., **34**(8):1454–1458, 2013.

[62] Y. V. Stanchescu, *Planar sets containing no three collinear points and non-averaging sets of integers*, Discrete Math., **256**(1-2):387–395, 2002.

[63] E. Szemerédi, *On sets of integers containing no four elements in arithmetic progression*, Acta Math. Acad. Sci. Hungar., **20**:89–104, 1969.

[64] ———, *On sets of integers containing no* k *elements in arithmetic progression*, Acta Arith., **27**:199–245, 1975, Collection of articles in memory of Juriĭ Vladimirovič Linnik.

[65] ———, *Integer sets containing no arithmetic progressions*, Acta Math. Hungar., **56**(1-2):155–158, 1990.

[66] ———, *An old new proof of Roth's theorem*, In Additive combinatorics, volume 43 of CRM Proc. Lecture Notes, pp. 51–54, Amer. Math. Soc., Providence, RI, 2007.

[67] T. C. Tao and H. V. Vu, *Additive combinatorics*, volume 105 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2006.

[68] J. G. van der Corput, *Über Summen von Primzahlen und Primzahlquadraten*, Math. Ann., **116**(1):1–50, 1939.

[69] P. Varnavides, *On certain sets of positive density*, J. London Math. Soc., **34**:358–360, 1959.

[70] V. H. Vu, *On a question of Gowers*, Ann. Comb., **6**(2):229–233, 2002.

Mathematical Institute, 24-29 St Giles' Oxford OX1 3LB, UK
E-mail: tom.sanders@maths.ox.ac.uk

Semilinear wave equations

Wilhelm Schlag

Abstract. This survey reviews some of the recent work on semilinear wave equations, in particular the wave map equation. We discuss global wellposedness, as well as the construction of special solutions and their stability.

Mathematics Subject Classification (2010). 35L05, 35L52, 37K40, 37K45, 53Z05.

Keywords. semilinear wave equations, wave maps, wellposedness, asymptotic stability.

1. Introduction

In this article we shall survey some recent developments concerning the long-term dynamics of semi-linear wave equations. These results concern the well-known wave maps system, which is a geometric equation, as well as the semi-linear wave equation with a power nonlinearity. We begin with the basic variational formulation of these models.

1.1. Lagrangians. Consider the Lagrangian

$$\mathcal{L}(u, \partial_t u) := \int_{\mathbb{R}^{1+d}_{t,x}} \frac{1}{2}\big(- u_t^2 + |\nabla u|^2\big)(t,x)\, dtdx \tag{1.1}$$

Substitute $u = u_0 + \varepsilon v$. Then

$$\mathcal{L}(u, \partial_t u) = \mathcal{L}(u_0, \partial_t u_0) + \varepsilon \int_{\mathbb{R}^{1+d}_{t,x}} (\Box u_0)(t,x) v(t,x)\, dtdx + O(\varepsilon^2)$$

where $\Box = \partial_{tt} - \Delta$. Thus u_0 is a critical point of $\mathcal{L}$ if and only if $\Box u_0 = 0$, the latter being the free wave equation on the flat Minkowski space $\mathbb{R}^{1+d}_{t,x}$. The wave equation is also a *Hamiltonian equation* with conserved energy

$$E(u, \partial_t u) = \frac{1}{2} \int_{\mathbb{R}^d} \big(|u_t|^2 + |\nabla u|^2\big)\, dx$$

Amongst other things, the Lagrangian formulation has the following significance:

- By Nöther's theorem underlying symmetries of the Lagrangian, more precisely 1-parameter groups of symmetries, yield continuity equations or *conservation laws*. The conservation of energy, momentum, angular momentum are a result of time-translation, space-translation, and rotation invariance of the Lagrangian, respectively.

Proceedings of the International Congress of Mathematicians, Seoul, 2014

- The Lagrangian formulation has a universal character, and is both flexible and versatile.

To illustrate the latter point, let (M, g) be a Riemannian manifold, and $u : \mathbb{R}^{1+d}_{t,x} \to M$ a smooth map. What does it mean for u to satisfy a wave equation?

While it is very non-obvious how to define such an object on the level of the equation, it is easy by modifying (1.1):

$$\mathcal{L}(u, \partial_t u) = \int_{\mathbb{R}^{1+d}_{t,x}} \frac{1}{2}(-|\partial_t u|_g^2 + \sum_{j=1}^{d} |\partial_j u|_g^2)\, dtdx$$

The critical points $\mathcal{L}'(u, \partial_t u) = 0$ satisfy a manifold-valued wave equation. If $M \subset \mathbb{R}^N$ is imbedded, this equation is

$$\Box\, u \perp T_u M \quad \text{or} \quad \Box\, u = A(u)(\partial u, \partial u),$$

A being the second fundamental form. This is the *extrinsic formulation*. For example, if $M = \mathbb{S}^{n-1}$, then

$$\Box\, u = u(|\partial_t u|^2 - |\nabla u|^2) \tag{1.2}$$

This gives rise to a nonlinear wave equation in a canonical way, the nonlinearity exhibits a so-called null-form structure. Harmonic maps are time-independent solutions. The nonlinearity appears naturally, and is given by the geometry of the target.

There is also an *intrinsic formulation* of the wave map system, namely

$$D^\alpha \partial_\alpha u = \eta^{\alpha\beta} D_\beta \partial_\alpha u = 0,$$

where D_α is the covariant derivative on the pull-back bundle. This refers to the pull-back of the connection defined on M to the Minkowski space $\mathbb{R}^{1+d}_{t,x}$ using the wave map u itself. In coordinates we obtain

$$-u^i_{tt} + \Delta u^i + \Gamma^i_{jk}(u)\partial_\alpha u^j \partial^\alpha u^k = 0 \tag{1.3}$$

with $\eta = (-1, 1, 1, \ldots, 1)$ being the flat Minkowski metric. Note the following points:

- There is a formal similarity between (1.3) and the geodesic equation. This similarity yields the following conclusion concerning solutions: $u = \gamma \circ \varphi$ is a wave map provided $\Box\varphi = 0$ and γ is a geodesic in M.
- Energy conservation for wave maps:

$$E(u, \partial_t u) = \int_{\mathbb{R}^d} (|\partial_t u|_g^2 + \sum_{j=1}^{d} |\partial_j u|_g^2)\, dx$$

  is constant in time.

Of central importance is the *Cauchy problem*, which we may now state in the following way for the extrinsic formulation:

$$\Box\, u = A(u)(\partial^\alpha u, \partial_\alpha u), \quad (u(0), \partial_t u(0)) = (u_0, u_1)$$

with smooth data, where u_0 is a fixed map into the manifold M, and u_1 a vector-field in the pull-back tangent bundle. One typically also imposes a compact support assumption. For u_0 this means that outside of some compact set u_0 equals a fixed point $p \in M$, whereas for u_1 this requirement is just the usual vanishing condition.

The most basic question concerning the Cauchy problem is the following one: *Does there exist a smooth local or global-in-time solution?* In addition, we would like the solution to be robust in a suitable sense. This precise meaning of this is captured by well-posedness theory.

A very condensed answer to this question reads as follows:

- One has local existence for all data as above, and global well-posedness for small data. These results are not sensitive to the geometry of the target (such as positive vs. negative curvature).
- For large data, the question about global-in-time wellposedness is much more involved, and does depend crucially on the geometry of the target manifold and the dimension of the underlying Minkowski space.

Another Lagrangian relevant to this survey is the following one, which does not involve any curvature but rather a directly inserted nonlinearity:

$$\mathcal{L}(u, \partial_t u) := \int_{\mathbb{R}^{1+d}_{t,x}} \Big\{ \frac{1}{2}\big(- u_t^2 + |\nabla u|^2\big)(t,x) + \frac{k}{p+1}|u(t,x)|^{p+1} \Big\}\, dtdx \tag{1.4}$$

where $k \in \mathbb{R}$ is some constant. The critical points of this Lagrangian are given by the semi-linear wave equation

$$u_{tt} - \Delta u + k|u|^{p-1}u = 0$$

on $\mathbb{R}^{1+d}_{t,x}$. The sign of k is essential at least for large data. This is reflected in the conserved energy

$$E = \int_{\mathbb{R}^d} \Big\{ \frac{1}{2}\big(u_t^2 + |\nabla u|^2\big)(t,x) + \frac{k}{p+1}|u(t,x)|^{p+1} \Big\}\, dx$$

If $k > 0$ then the energy is positive definite, whereas for $k < 0$ it is indefinite. In the latter case, which is referred to us the *focusing equation*, the dynamics is incomparably more complicated than for the defocusing equation ($k > 0$).

1.2. Symmetries and solvability. The wave equation is invariant under the Poincaré group. This group is the symmetry group of special relativity and is generated by the Lorentz transforms and rigid motions of $\mathbb{R}^3$.

However, conformal invariance is also essential for the understanding of these equations. Of particular importance to the well-posedness problem is the dilation symmetry. If $u(t,x)$ is a wave map, then so is $u(\lambda t, \lambda x)\ \ \forall \lambda > 0$. Suppose the data belong to the Sobolev space $\dot{H}^s \times \dot{H}^{s-1}(\mathbb{R}^d)$. The unique s for which this space remains invariant under the natural scaling is $s = \frac{d}{2}$. On the other hand, the energy remains invariant under the following scaling: $u(t,x) \mapsto \lambda^{\frac{d-2}{2}} u(\lambda t, \lambda x)$ same as $\dot{H}^1 \times L^2(\mathbb{R}^d)$. The interplay between the natural scaling of the wave-map equation, on the one hand, and the scaling of the energy, on the other hand, is essential for the solution theory.

- *Subcritical case* $d = 1$. The natural scaling is associated with less regularity than that of the conserved energy. We therefore expect global existence. The logic being that the local time of existence only depends on the energy of the data, which is preserved.

- *Critical case* $d = 2$. The conserved energy exactly keeps the balance with the natural scaling of the equation. The geometry of the target plays the decisive role. For example, for the sphere $\mathbb{S}^2$ large data may exhibit finite-time blowup (singularity formation), whereas for the hyperbolic plane $\mathbb{H}^2$ as a target we have global existence. These results are the culmination of many years' worth of developments, carried out by numerous researchers, see Klainerman, Selberg [41, 42], Tataru [74], Tao [70, 71], Krieger, Schlag, Tataru [47], Krieger, Schlag [46], Rodnianski, Sterbenz [61], Raphael, Rodnianski [59], Sterbenz, Tataru [67, 68], Tao [72].
- *Supercritical case* $d \geq 3$. For this, as well as energy supercritical equations in general, the dynamics is poorly understood. Self-similar blowup of the form $Q(r/t)$ for the sphere as target was observed in the 1980s by Shatah [62]. Negatively curved manifolds in high dimensions admit the same type of phenomenon, see [13]. Donninger [21] established the stability of the Shatah-type blowup relative to suitable norms.

From a mathematical perspective, the study of nonlinear Hamiltonian evolution equations focuses on the following problems, broadly stated:

- *Wellposedness:* Existence and uniqueness of solutions, the continuous dependence of these solutions on the data, and the persistence of regularity. At first, one needs to understand these properties locally in time.
- *Global behavior:* Does finite time break down occur? In the usual classical interpretation, this question means the following: does there exist a finite time T_* so that a smooth solution exists for all $0 < t < T_*$, but it is impossible to extend the solution smoothly beyond T_*. Typically, for semi-linear equations this property can be shown to reduce to the question whether or not some norm, such as L^∞ or a suitable space-time norm, becomes unbounded in finite time.

  If finite-time breakdown does not occur, then we have global existence: smooth solutions exist for all times for smooth data. In some instances, such as energy subcritical equations, this property can be deduced from two ingredients: (i) the local time of existence only depends on the size of the data as expressed by an "energy norm" (ii) this norm is dominated by a conserved quantity, typically the energy.

  However, such a time-stepping scenario does not admit any conclusion about the nature of the long-term dynamics. Other methods are required in order to determine that.
- *Blow up dynamics:* If the solution breaks down in finite time, can one describe the mechanism by which it does so? For example, via energy concentration at the tip of a light cone? Often, symmetries (in a wider sense) play a crucial role in the process of singularity formation.
- *Scattering to a free wave:* If the solution exists for all times $t \geq 0$, does it approach a free wave? In more formal notation, suppose we are given a solution $u(t)$ to a nonlinear equation $\Box u = N(u)$, and we assume that u lies in a suitable space X. Does there then exist $v \in X$ with $\Box v = 0$ and such that $(\vec{u} - \vec{v})(t) \to 0$ as $t \to \infty$ in X? Here $\vec{u} = (u, \partial_t u)$. If scattering occurs, then we have local energy decay.

Of great importance are equations that admit special "soliton" solutions. This refers to standing waves in a wide sense, stationary solutions not depending on time being included. For wave maps, these would be given by harmonic maps.

- *Special solutions:* If a global solution does not approach a free wave, does it scatter to something else? A stationary nonzero solution, for example? Focusing equations often exhibit nonlinear bound states.
- *Stability theory:* If special solutions exist such as stationary or time-periodic ones, are they orbitally stable? Are they asymptotically stable?
- *Multi-bump solutions:* Is it possible to construct solutions which asymptotically split into moving "solitons" plus radiation? Lorentz invariance dictates the dynamics of the single solitons.
- *Resolution into multi-bumps:* Do all solutions decompose in this fashion (as in linear *asymptotic completeness*)? To rephrase the question: suppose solutions exist for all times $t \geq 0$. Is it then true that they either scatter to a free wave, or decompose into (moving) standing waves (solitons)? For the latter, the symmetries are essential: for the wave equation the movement would be determined by Lorentz symmetries, whereas for the Schrödinger equation Galilean symmetries determine the movement. Each soliton would consume a fixed quantum of energy, thus limiting the number of these (moving) standing waves.

For the remainder of this survey, we shall describe some of the answers to these questions that are known today for the wave map system and scalar semi-linear equations, respectively. But first, we present an indispensable tool in the study of wave equations, namely the point-wise decay of free waves.

1.3. Dispersion. In $\mathbb{R}^3$, the Cauchy problem $\Box u = 0, u(0) = 0, \partial_t u(0) = g \in C^\infty(\mathbb{R}^3)$ has the unique smooth solution

$$u(t,x) = t \fint_{tS^2} g(x+y)\, \sigma(dy)$$

If g is supported on the unit ball $B(0,1)$, then the solution $u(t,x)$ is supported on $\big||t| - |x|\big| \leq 1$. This is a manifestation of Huygens' principle.

Since the energy is conserved and is spread out evenly over a volume of size t^2, we expect point-wise decay at the rate t^{-1}. The technical estimate in $\mathbb{R}^3$ reads as follows:

$$\|u(t,\cdot)\|_\infty \leq Ct^{-1}\|Dg\|_1 \tag{1.5}$$

In dimension $= d$ the decay is $t^{-\frac{d-1}{2}}$. Generally speaking, (1.5) is not suitable for nonlinear problems, since $L^1(\mathbb{R}^d)$ is not invariant under the nonlinear flow. Rather, one uses the following energy based variant

$$\|u\|_{L_t^p L_x^q(\mathbb{R}^3)} \lesssim \|(u(0), \dot{u}(0))\|_{\dot{H}^1 \times L^2(\mathbb{R}^3)} + \|\Box u\|_{L_t^1 L_x^2(\mathbb{R}^3)}$$

where $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}$, $\frac{1}{p} + \frac{3}{q} = \frac{1}{2}$. These are *Strichartz estimates* which play a fundamental role in the study of nonlinear problems. Examples of these estimates are given by $L_t^\infty L_x^6(\mathbb{R}^{1+3})$, $L_{t,x}^8(\mathbb{R}^{1+3})$. In principle, $L_t^2 L_x^\infty(\mathbb{R}^{1+3})$ also belongs to this class although this particular endpoint fails, see Keel, Tao [32].

2. Global well-posedness for wave maps

In this section, we give an overview of the known results on the Cauchy problem for the wave-map system. We begin with solutions obeying special rotational symmetry.

2.1. Equivariant solutions. Let $M \subset \mathbb{R}^{d+1}$ for $d \geq 2$ be a surface of revolution. This means that there exists a line in $\mathbb{R}^{d+1}$ so that rotations about this axis leave M invariant. Denote such rotations by R, and we identify the group of these symmetries with $\mathrm{SO}(d)$. We call a smooth map $u : \mathbb{R}^{1+d} \to M$ a k-equivariant map, where $k \geq 1$ is some integer provided

$$u \circ R = R^k \circ u$$

for any such $R \in \mathrm{SO}(d)$. The understanding here is that R on the left-hand side acts on the domain $\mathbb{R}^d$ by rotation about the origin.

Under this symmetry, $u = u(t, r)$ where $r = |x|$ which simplifies matters considerably. For $d = 2$ with ϕ being arc-length along a generator, the wave map equation now reads as follows:

$$\phi_{tt} - \phi_{rr} - \frac{1}{r}\phi_r + \frac{1}{r^2} f(\phi) = 0 \tag{2.1}$$

with the conserved energy

$$\mathcal{E}(\vec{\phi}) = \int_0^\infty \left(\phi_t^2(t,r) + \phi_r^2(t,r) + \frac{g(\phi(t,r))^2}{r^2} \right) r\, dr \tag{2.2}$$

The function g is defined by the metric on M which is of the form

$$ds^2 = d\phi^2 + g^2(\phi) ds^2_{S^2}.$$

The function f in (2.1) is $f = gg'$. For example, for $M = S^2$ we obtain $f(\phi) = \frac{1}{2}\sin(2\phi)$. Notice that unlike the full wave map system (2.1) does not contain any derivatives in the nonlinearity, further highlighting the semi-linear nature of these equations.

A special case are geodesically convex targets, i.e., those for which $f(\phi) > 0$ for $\phi > 0$. The one-sheeted hyperboloid is an example of such a surface.

Theorem 2.1. *For geodesically convex targets M, the equation* (2.1) *has smooth global solutions for all smooth data. In other words, the equivariant wave map system from $\mathbb{R}^{1+2}_{t,x} \to M$ is globally well-posed.*

This result goes back to the pioneering work of Christodoulou, Tahvildar-Zadeh, and Shatah from the 1990s, see [14, 15, 65, 66]. See the book by Shatah, Struwe [63] for an exposition of this work. As we shall see below, Theorem 2.1 does not hold for general targets such as the sphere for which singularities may form in finite time. So a geometric condition on the target such as that of being geodesically convex, is of intrinsic importance for a global regularity result. We remark that Theorem 2.1 deals with the energy critical case, which is $d = 2$. As noted above, this case stands out as being of special analytical as well as geometric interest; the latter being evidenced by the geometrical properties of the target M entering into the analysis. This is not to say that the wave map equations in dimensions $d \geq 3$ are not of interest. These supercritical equations are much less understood. From the equivariant formulation Shatah observed in the 1980s that there exists self-similar blowup solutions [62]. We shall now describe how the dynamics of equivariant wave maps exterior to a ball in $\mathbb{R}^3$ can be completely characterized in the energy class.

2.2. Exterior wave maps. We now consider wave maps exterior to the unit ball $B(0,1)$. By this we mean a smooth equivariant map $u : \mathbb{R}^3 \setminus B(0,1) \to S^3$ satisfying the wave maps equation on $r > 1$ and a Dirichlet condition at $r = 1$; in other words, for all times we have $u(t,1) = p \in S^3$, a fixed point on the sphere.

In the equivariant formulation, we thus have an equation

$$\psi_{tt} - \psi_{rr} - \frac{2}{r}\psi_r + \frac{\sin(2\psi)}{r^2} = 0 \tag{2.3}$$

with $\psi(t,1) = 0$ for all times $t \geq 0$. The conserved energy is

$$\mathcal{E}(\psi,\psi_t) = \int_1^\infty \frac{1}{2}\big(\psi_t^2 + \psi_r^2 + 2\frac{\sin^2(\psi)}{r^2}\big) r^2\, dr \tag{2.4}$$

Any $\psi(t,r)$ of finite energy and continuous dependence on $t \in I := (t_0,t_1)$ must satisfy $\psi(t,\infty) = n\pi$ for all $t \in I$ where $n \geq 0$ is fixed. This integer n plays the role of the degree in this context.

The natural space to place the solution into for $n = 0$ is the *energy space*

$$\mathcal{H} := (\dot{H}_0^1 \times L^2)((1,\infty))$$

with norm

$$\|(\psi,\dot{\psi})\|_{\mathcal{H}}^2 := \int_1^\infty (r^2\psi_r^2(r) + \dot{\psi}^2(r))\, dr \tag{2.5}$$

Here $\dot{H}_0^1((1,\infty))$ is the completion of the smooth functions on $(1,\infty)$ with compact support under the first norm on the right-hand side of (2.5).

The exterior equation (2.3) was proposed by Bizon, Chmaj, and Maliborski [10] as a model in which to study the problem of relaxation to the ground states given by the various equivariant harmonic maps. In the physics literature, this model was introduced in [5] as an easier alternative to the Skyrmion equation. Numerical simulations described in [10] indicate that in each equivariance class and topological class given by the boundary value $n\pi$ at $r = \infty$ *every solution* scatters to the unique harmonic map that lies in this class. The existence of these harmonic maps follows from a phase-plane analysis. The conjecture from [10] was verified by Lawrie and the author in [50] for the zero degree class, and then by Kenig, Lawrie and the author [33] in full generality.

Theorem 2.2. *For any smooth energy data in the class of degree n there exists a unique global and smooth evolution to* (2.3) *which scatters to the unique harmonic map in that degree class.*

The existence of a global smooth solutions is easy, since the removal of a ball around the origin renders the equation subcritical. The difficult part is to describe the asymptotic state of the solution. For degree zero solutions the scattering amounts to the property that viewed on any fixed compact set, the energy of the solution on the set tends to zero. In other words, the solutions tends to zero. For higher degrees, it means that the solution asymptotically tends to the unique harmonic map of that degree class.

The methods employed are those that fall under the name "concentration compactness", as developed in the works of Kenig, Merle [34, 35] and Duyckaerts, Kenig, Merle [22–26].

For the zero degree class, a variant of the Kenig, Merle argument in which the compact element is excluded via a virial identity based rigidity argument, suffices. But for the higher degrees this does not suffice, since the virial identity is not available. To circumvent this road block, one uses the exterior energy estimate method of [25].

2.3. Small data theory. We shall now briefly describe the by now classical *global-in-time* results on *non-equivariant* wave maps for small data. These were preceded by the local-in-time wellposedness obtained by Klainerman, Machedon [36–39] and Klainerman, Selberg [41, 42] in the 1990s. These local results improved dramatically on the easy energy methods which required much regularity on the data since the nonlinearity was controlled by Sobolev embedding. More specifically, these authors were able to reduce the regularity requirement to $H^s \times H^{s-1}(\mathbb{R}^d)$ with $s > \frac{d}{2}$ for local wellposedness.

The latter condition goes all the way down to the critical scaling $\frac{d}{2}$, but does not achieve this endpoint. The argument relies on the contraction principle in a suitably chosen space. This turns out not to be energy and Strichartz spaces, but rather the $X^{s,b}$ spaces which take the geometry of the characteristic variety of the wave equation into account, which is the light-cone. This is crucial in order to capture the cancellations exhibited by the highly structured nonlinearity of the wavemap system. In fact, the right-hand side of (1.2) contains the term $|\partial_t u|^2 - |\nabla u|^2$ which is the Minkowski metric applied to the space-time gradient $\nabla_{t,x} u$. The relevance of this lies with the fact that the Minkowski metric vanishes on null-vectors, which are characterized by lying in a light-cone.

On the level of the wave equation, this means that the nonlinearity cancels self-interactions of plane waves. Without this property, it would not be possible to lower the regularity all the way down to $s > \frac{d}{2}$, which is called *subcritical regularity*.

After these important developments the focus shifted to the difficult question of well-posedness at the critical level $s = \frac{d}{2}$. The interest of this question hinges on the scaling invariance of the equation. Indeed, in contrast to any wellposedness theory at the subcritical level, wellposedness at the critical level is automatically global in time simply by rescaling the solution.

The key breakthroughs here were achieved by Tataru [74] and Tao [70, 71] about 15 years ago. In Tataru's work the regularity is expressed in terms of the Besov regularity $\dot{B}^1_{2,\frac{d}{2}} \times \dot{B}^1_{2,\frac{d}{2}-1}$ which is precisely at the scaling critical level. But it is stronger than the Sobolev regularity $\dot{H}^{\frac{d}{2}} \times \dot{H}^{\frac{d}{2}-1}$. In technical terms, Tataru solved the *division problem*, but not the *summation problem* (which refers to the summation over the dyadic frequency scales). Tao resolved the summation problem for the sphere as target by means of the important device of a *gauge transform*. Without going into too many details, this amounts to removing "dangerous" interactions in the nonlinearity by exploiting a freedom in the choice of coordinates (or the choice of a frame). In dimensions 5 and higher, Tao observed that Strichartz theory suffices to close the argument due to the stronger dispersion in those cases. In particular, the null form structure of the nonlinearity is not crucial. However, in low dimensions, especially in dimension $d = 2$ for which dispersion is very weak, much more technical heavy lifting is needed and the nullform becomes essential.

An important hallmark of Tao's work is the fact that the global wellposedness is *not achieved* by means of a contraction argument at the critical regularity. Rather, it is based on the device of *frequency envelope*. While one assume smallness of the data in the critical norm, the data are also assumed to have slightly more regularity than the critical level. The subcritical wellposedness theory then gives the local existence of a solution. The key is now

to show that this slightly higher degree of regularity is preserved by the flow; in this way the subcritical wellposedness theory allows one to solve globally in time. The Tataru, Tao theory was extended to more general target manifolds than the sphere, see Krieger [43, 44], Tataru [77, 78], Klainerman, Rodnianski [40]. Nahmod, Stepanov, Uhlenbeck [55] obtain a small data theorem in spatial dimensions $d \geq 4$ for targets given by compact groups or symmetric spaces.

Our current understanding is that small data wellposedness holds for all Riemannian manifolds as targets satisfying reasonable assumptions.

Shatah, Struwe [64] made the important observation that the *Coulomb gauge* can be used in dimensions $d \geq 4$ to obtain global regularity for small data. That particular gauge is natural form the theory of harmonic maps and exploits the formalism of a moving orthonormal frame. The Coulomb gauge refers to the choice of such frame which "twists" the least; in more technical terms, the Dirichlet energy is minimized by such a frame.

These developments set the stage for the next step, namely determining the different possible types of dynamics for large data wave maps. In the following two sections we will describe the two main phenomena that may appear in the energy critical setting, namely finite-time blowup on the one hand, vs. global regularity on the other hand. The energy supercritical wave map system is still very poorly understood.

2.4. Blowup for wave maps. In [47] Krieger, Tataru, and the author exhibited regularity breakdown for equivariant wave maps $u : \mathbb{R}^{2+1} \to S^2$ of co-rotation index 1 with certain H^{1+} regular initial data. More precisely, the data (u, u_t) are of class $H^{1+\delta} \times H^{\delta}$ for some $\delta > 0$. By a theorem of Struwe [69] such data result in unique local solutions of the same regularity until possible breakdown occurs via an *energy-concentration scenario*. More precisely, Struwe's result shows that if the solution is indeed C^{∞}–smooth before breakdown, such a scenario can only happen by the bubbling off of a harmonic map [69]: specifically, let $Q(r) : \mathbb{R}^2 \to S^2$ be an equivariant harmonic map, which can be constructed for every co-rotation index $k \in \mathbb{Z}$ (for example, for $k = 1$ one may use stereographic projection). We shall identify $Q(r)$ with the longitudinal angle, as above. Then according to [69], if an equivariant wave map u of co-rotation index $k = 1$, again identified with the longitudinal angle, with smooth initial data at some time $t_0 > 0$ breaks down at time $T = 0$, then energy focuses at the origin, and there is a decomposition

$$u(t, r) = Q(\lambda(t)r) + \epsilon(t, r),\ Q(r) \text{ a co-rotation } k = 1 \text{ index equivariant harmonic map}$$

where there is a sequence of times $t_i \to 0$, $t_i < 0$, $i = 1, 2, \ldots$, with $\lambda(t_i)|t_i| \to \infty$, such that the rescaled functions $u(t_i, \frac{r}{\lambda(t_i)})$ converge to $Q(r)$ locally in the strong energy topology.

We now describe the theorem of Krieger, Tataru and the author which *constructs* this type of non self-similar blowup for energy critical wave maps. We let $Q(r)$ represent the standard harmonic map of co-rotation $k = 1$, i.e., $Q(r) = 2 \arctan r$. Recall that in the equivariant formulation the energy is

$$\mathcal{E}(u) = \int_{\mathbb{R}^2} \Big[\frac{1}{2}(u_t^2 + u_r^2) + \frac{\sin^2(u)}{2r^2}\Big]\, r\, dr$$

The *local* energy relative to the origin is defined as

$$\mathcal{E}_{\mathrm{loc}}(u) = \int_{r<t} \Big[\frac{1}{2}(u_t^2 + u_r^2) + \frac{\sin^2(u)}{2r^2}\Big] r\, dr$$

It is well-known that for equivariant wave-maps singularities can only develop at the origin and this happens at time zero if and only if

$$\liminf_{t\to 0} \mathcal{E}_{\rm loc}(u)(t) > 0$$

One of the main features of the following theorem is that we need to "renormalize" the profile $Q(r\lambda(t))$ by means of a large perturbation (denoted u^e below). While this usage of the term "renormalize" may be at odds with the physics literature, it is quite common in applied mathematics and perturbation theory. What we mean here is that we can apply perturbative arguments only after a non-perturbative step that changes Q to $Q + u^e$, see Theorem 2.3. We find it convenient to solve backwards in time, with blow-up as $t \to 0+$. The equivariant formulation of the wave map equation from $\mathbb{R}^{1+2}_{t,x} \to S^2$ is

$$-u_{tt} + u_{rr} + \frac{u_r}{r} = \frac{\sin(2u)}{2r^2} \tag{2.6}$$

Theorem 2.3. *Let $\nu > \frac{1}{2}$ be arbitrary and $t_0 > 0$ be sufficiently small. Define $\lambda(t) = t^{-1-\nu}$ and fix a large integer N. Then there exists a function u^e satisfying*

$$u^e \in C^{\nu+1/2-}(\{t_0 > t > 0,\ |x| \le t\}), \qquad \mathcal{E}_{\rm loc}(u^e)(t) \lesssim (t\lambda(t))^{-2}\,|\log t|^2 \ \text{ as } \ t \to 0$$

and a blow-up solution u to (2.6) *in $[0, t_0]$ which has the form*

$$u(t,r) = Q(\lambda(t)r) + u^e(t,r) + \epsilon(t,r), \qquad 0 \le r \le t$$

where ϵ decays at $t = 0$; more precisely,

$$\epsilon \in t^N H^{1+\nu-}_{\rm loc}(\mathbb{R}^2), \qquad \epsilon_t \in t^{N-1} H^{\nu-}_{\rm loc}(\mathbb{R}^2), \qquad \mathcal{E}_{\rm loc}(\varepsilon)(t) \lesssim t^N \ \text{ as } \ t \to 0$$

with spatial norms that are uniformly controlled as $t \to 0$. Also, $u(t,0) = 0$ for all $0 < t < t_0$. The solution $u(t,r)$ extends as an $H^{1+\nu-}$ solution to all of $\mathbb{R}^2$ and the energy of u concentrates in the cuspidal region $0 \le r \lesssim \frac{1}{\lambda(t)}$ leading to blow-up at $r = t = 0$.

A somewhat surprising feature of our theorem is that the blow-up rate is prescribed as $\lambda(t) = t^{-1-\nu}$. This is in stark contrast to the usual modulation theoretic approach where the rate function is used to achieve orthogonality to all unstable modes of the linearized problem. Heuristically speaking, there are two types of instabilities which typically arise in linearized problems: those due to symmetries of the nonlinear equation (typically leading to algebraic growth of the linear evolution) and those that produce exponential growth in the linear flow (due to some kind of discrete spectrum). For example, the latter arises in the recent work on "center-stable manifolds" for orbitally unstable equations (see the discussion of scalar semi linear Hamiltonian equations below), whereas for the former see [45]. Both types can lead to blow up.

In the case of Theorem 2.3 ones does not have any discrete spectrum in the linearized equation, but rather a *zero-energy resonance* which is due to the scaling symmetry. Intuitively speaking, it is unclear at this point which role the resonance plays in the formation of the blow-up, since the approach of [47] is based on a crucial non-perturbative component, namely the elliptic profile modifier produces a *large* perturbation of the basic profile Q. The perturbative component of our proof then deals with the removal of errors produced by the elliptic profile modifier (it is essential that these errors decay rapidly in time).

The restriction $\nu > \frac{1}{2}$ is a technical one and can be relaxed to $\nu > 0$ which is optimal by Struwe's aforementioned bubbling off theorem. For this see [12]. Due to the continuum of allowed blowup rates in Theorem 2.3 the solutions constructed are expected to be highly unstable; in fact, their stability should be associated to a finite codimension condition.

In contrast to these unstable solutions, Rodnianski, Sterbenz [60] and Raphael, Rodnianski [59] studied the problem of finding stable blowup regimes. The following theorem is the main result from [59]. The affine Sobolev space relative to the harmonic map Q is defined as.

$$\mathcal{H}^2_a = \mathcal{H}^2 + Q. \tag{2.7}$$

For a pair of functions $(\epsilon(y), \sigma(y))$, we let

$$\|(\epsilon,\sigma)\|^2_{\mathcal{H}} = \int \left[\sigma^2 + (\partial_y \epsilon)^2 + \frac{\epsilon^2}{y^2}\right] \tag{2.8}$$

define the energy space. The k-equivariant formulation of the wave map problem $\mathbb{R}^{1+2}_{t,r} \to S^2$ is

$$\begin{cases} \partial_t^2 u - \partial_r^2 u - \frac{\partial_r u}{r} + k^2 \frac{f(u)}{r^2} = 0, \\ u_{|t=0} = u_0, \ \ (\partial_t u)_{|t=0} = v_0 \end{cases} \quad \text{with } \ f = gg' \tag{2.9}$$

The main theorem in [59] also makes a reference to the equivariant (in a suitable sense) Yang-Mills equation, which we however we skip here. In effect, it amounts to the previous equation with $k = 2$.

Theorem 2.4 (Stable blow up dynamics of co-rotational Wave Maps). *Let $k \geq 1$. Let $\mathcal{H}^2_a$ denote the affine Sobolev space from above.There exists a set $\mathcal{O}$ of initial data which is open in $\mathcal{H}^2_a$ and a universal constant $c_k > 0$ such that the following holds true. For all $(u_0, v_0) \in \mathcal{O}$, the corresponding solution to* (2.9) *blows up in finite time $0 < T = T(u_0, v_0) < +\infty$ according to the following universal scenario:*

(i) Sharp description of the blow up speed : *There exists $\lambda(t) \in \mathcal{C}^1([0,T), \mathbb{R}^*_+)$ such that:*

$$u(t, \lambda(t)y) \to Q \ \text{ in } \ H^1_{r,loc} \ \text{ as } \ t \to T \tag{2.10}$$

with the following asymptotics:

$$\lambda(t) = c_k(1 + o(1)) \frac{T-t}{|\log(T-t)|^{\frac{1}{2k-2}}} \ \text{ as } \ t \to T \ \text{ for } \ k \geq 2, \tag{2.11}$$

$$\lambda(t) = (T-t)e^{-\sqrt{|\log(T-t)|}+O(1)} \ \text{ as } \ t \to T \ \text{ for } \ k = 1. \tag{2.12}$$

Moreover,

$$b(t) := -\lambda_t(t) = \frac{\lambda(t)}{T-t}(1 + o(1)) \to 0 \ \text{ as } \ t \to T$$

(ii) Quantization of the focused energy*: Let $\mathcal{H}$ be the energy space* (2.8)*, then there exist $(u^*, v^*) \in \mathcal{H}$ such that the following holds true. Pick a smooth cut off function χ with $\chi(y) = 1$ for $y \leq 1$ and let $\chi_{\frac{1}{b(t)}}(y) = \chi(b(t)y)$, then:*

$$\lim_{t \to T} \left\| u(t,r) - \left(\chi_{\frac{1}{b(t)}} Q\right)\left(\frac{r}{\lambda(t)}\right) - u^*, \partial_t \left[u(t,r) - \left(\chi_{\frac{1}{b(t)}} Q\right)\left(\frac{r}{\lambda(t)}\right) - v^*\right] \right\|_{\mathcal{H}} = 0. \tag{2.13}$$

Moreover, one has the following quantization of the focused energy:

$$E_0 = E(u, \partial_t u) = E(Q, 0) + E(u^*, v^*). \tag{2.14}$$

This theorem thus gives a complete description of a stable blow up regime for all homotopy numbers $k \geq 1$. Stable blow up solutions in $\mathcal{O}$ decompose into a singular part with a universal structure and a regular part which has a strong limit in the scale invariant space. Moreover, the amount of energy which is focused by the singular part is a universal amount independent of the Cauchy data.

2.5. Characterization of blowup for equivariant wave maps. We now describe large data asymptotic behavior of equivariant wave maps taking values in $\mathbb{S}^2$. The setting is 1-equivariant (co-rotational), so u takes the special form $u(t, r, \phi) = (\psi(t, r), \phi)$ in polar coordinates, where ψ measures the angle from the north pole. This angle then satisfies the equivariant wave map equation

$$\psi_{tt} - \psi_{rr} - \frac{1}{r}\psi_r + \frac{\sin(2\psi)}{2r^2} = 0, \quad (\psi, \dot{\psi})(0) = (\psi_0, \psi_1),$$

Struwe's bubbling theorem [69] states: If a solution, $\psi(t, r)$, with smooth initial data $\vec{\psi}(0) = (\psi(0), \dot{\psi}(0))$, breaks down at $t = 1$, then the energy concentrates at the origin and there is a sequence of times $t_j \nearrow 1$ and scales $\lambda_j > 0$ with $\lambda_j \ll 1 - t_j$ so that the rescaled sequence of wave maps

$$\vec{\psi}_j(t, r) := \Big(\psi(t_j + \lambda_j t, \lambda_j r), \lambda_j \dot{\psi}\,(t_j + \lambda_j t, \lambda_j r)\Big)$$

converges *locally* to $\pm Q(r/\lambda_0)$ in the space-time norm $H^1_{\text{loc}}((-1, 1) \times \mathbb{R}^2; \mathbb{S}^2)$ for some $\lambda_0 > 0$. An important consequence is that any wave map that blows up must concentrate at least the energy of Q at the blow-up point.

Struwe's result gives a *local* characterization of blow-up behavior. To obtain a global picture, one needs to take into account the topological structure carried by an $\mathbb{S}^2$-valued wave map – in particular, each co-rotational wave map of finite energy has a fixed topological integer degree. Indeed, for the energy $\mathcal{E}(\psi, \dot{\psi})$ to be finite for a solution $\vec{\psi}$ means that we must have $\psi(t, 0) = m\pi$ and $\psi(t, \infty) = n\pi$. These integers are fixed by continuity and thus determine a homotopy class, or topological degree. Letting $m = 0$, n is the degree and $\mathcal{H}_n$ are all finite energy data of degree n, i.e.,

$$\mathcal{H}_n := \{(\psi_0, \psi_1) \mid \mathcal{E}(\psi_0, \psi_1) < \infty,\ \psi_0(0) = 0,\ \text{ and } \psi_0(\infty) = n\pi\}.$$

2.5.1. Degree 0 initial data. An immediate consequence of Struwe's theorem is that a degree 0 solution $\vec{\psi}(t) \in \mathcal{H}_0$ is global-in-time if $\mathcal{E}(\vec{\psi}(0)) < 2\mathcal{E}(Q)$. Indeed, a wave map in $\mathcal{H}_0$ with energy below $2\mathcal{E}(Q)$ stays away from the south pole and hence cannot converge to a degree 1 rescaled harmonic map. $\mathcal{E}(Q)$ is the minimal energy in $\mathcal{H}_1$, so any map which sends $r = 0$ to the north pole uses at least the energy of Q to reach the south pole. One the other hand, since blow-up is a local phenomenon, one can modify the solutions constructed in the previous section outside the light cone to obtain a blow-up solution in $\mathcal{H}_0$ which has energy $\mathcal{E}(\vec{\psi}) = 2\mathcal{E}(Q) + \delta$ for any $\delta > 0$. Thus the energy $2\mathcal{E}(Q)$ forms a threshold for data in $\mathcal{H}_0$ under which every solution is global and above which blow-up may occur. In [17], Côte, Kenig, Lawrie and the author addressed the *global dymanics* of subthresold solutions

in $\mathcal{H}_0$ by showing that in fact every solution with energy below $2\mathcal{E}(Q)$ must scatter to a free wave.

Theorem 2.5 (Global Existence and Scattering in $\mathcal{H}_0$ below $2\mathcal{E}(Q)$). *For any smooth data $\vec{\psi}(0) \in \mathcal{H}_0$ with $\mathcal{E}(\vec{\psi}(0)) < 2\mathcal{E}(Q)$, there exists a unique global evolution $\vec{\psi} \in C^0(\mathbb{R}; \mathcal{H}_0)$. Moreover, $\vec{\psi}(t)$ scatters to zero in the sense that the energy of $\vec{\psi}(t)$ on any arbitrary, but fixed compact region vanishes as $t \to \infty$. In other words, one has*

$$\vec{\psi}(t) = \vec{\varphi}_L(t) + o_{\mathcal{H}}(1) \quad as \quad t \to \infty \tag{2.15}$$

where $\vec{\varphi}_L \in \mathcal{H}$ solves the linearized equation, i.e.,

$$\varphi_{tt} - \varphi_{rr} - \frac{1}{r}\varphi_r + \frac{1}{r^2}\varphi = 0 \tag{2.16}$$

The proof follows the concentration compactness/rigidity method of Kenig and Merle, [34, 35]. The key ingredient in the proof is a rigidity statement: any equivariant wave maps with a pre-compact trajectory (modulo symmetries) must be a harmonic map. Before [17], Cote, Kenig and Merle [19] proved scattering in $\mathcal{H}_0$ for energies below $\mathcal{E}(Q) + \delta$ for small δ. We also note that Theorem 2.5 can be deduced from the more general work of Sterbenz and Tataru [67, 68] by restricting to the equivariant setting in $\mathcal{H}_0$.

2.5.2. Degree 1 initial data. Next consider initial data in $\mathcal{H}_1$. The harmonic map Q uniquely minimizes the energy in this degree class and thus there cannot be an energy threshold in $\mathcal{H}_1$ under which blow-up is excluded – indeed the blow up solutions from [47] described in the previous section can have energy $E(Q) + \delta$ for any $\delta > 0$. Note also that solutions in $\mathcal{H}_1$ cannot scatter to free waves – nontrivial degree is a topological obstruction to scattering.

The question of characterizing the possible dynamics in $\mathcal{H}_1$ is then one of determining the role that Q plays in asymptotic situations. In this regard there is an energy threshold under which this question of characterizing dynamics is most natural in $\mathcal{H}_1$, namely $3\mathcal{E}(Q)$. Indeed, consider a degree one wave map $\vec{\psi}(t)$ that blows up at $t = 1$. The result of Struwe [69] extracts the blow up profile $\pm Q_{\lambda_n} := \pm Q(\cdot/\lambda_n)$ along a sequence of times $t_n \to 1$. If $\vec{\psi}$ has $\mathcal{E} < 3\mathcal{E}(Q)$ the profile must be $+Q(\cdot/\lambda_n)$, and since $Q \in \mathcal{H}_1$ as well, we infer that $\psi(t_n) - Q_{\lambda_n} \in \mathcal{H}_0$. Since this object converges locally to zero, the energy of the difference is roughly the difference of the energies, at least for large n. Hence, if $\psi(t)$ has energy below $3\mathcal{E}(Q)$ the difference $\psi(t_n) - Q_{\lambda_n}$ is degree zero and has energy below $2\mathcal{E}(Q)$. More complicated dynamics are thus excluded by the degree zero scattering result, i.e., Theorem 2.5. The situation is similar in the case that the solution $\psi(t) \in \mathcal{H}_1$ is global in time.

Theorem 2.6 ([17, 18]). *Let $\vec{\psi}(0) := (\psi_0, \psi_1) \in \mathcal{H}_1$ be smooth, finite energy degree 1 data with energy $\mathcal{E}(\vec{\psi}) < 3\mathcal{E}(Q)$.*

(1) ***Finite time blowup:*** *If the solution $\psi(t)$ blows up at, say, $t = 1$, then there exists a continuous function, $\lambda : [0, 1) \to (0, \infty)$ with $\lambda(t) = o(1 - t)$, a map $\vec{\varphi} = (\varphi_0, \varphi_1) \in \mathcal{H}_0$ with $\mathcal{E}(\vec{\varphi}) = \mathcal{E}(\vec{\psi}) - \mathcal{E}(Q, 0)$, and a decomposition*

$$\vec{\psi}(t) = \vec{\varphi} + (Q(\cdot/\lambda(t)), 0) + o_{\mathcal{H}_0}(1) \quad as \ t \to 1$$

(2) ***Global solutions:*** *If the solution $\vec{\psi}(t) \in \mathcal{H}_1$ exists globally in time then there exists a continuous function, $\lambda : [0,\infty) \to (0,\infty)$ with $\lambda(t) = o(t)$ as $t \to \infty$, a solution $\vec{\varphi}_L(t) \in \mathcal{H}_0$ to the linear wave equation* (2.16)*, and*

$$\vec{\psi}(t) = \vec{\varphi}_L(t) + (Q(\cdot/\lambda(t)), 0) + o_{\mathcal{H}_0}(1) \ \text{ as } \ t \to \infty$$

We remark that Duyckaerts, Kenig, and Merle in [22, 23, 25] established analogous classification results for $\Box u = u^5$ in $\dot{H}^1 \times L^2(\mathbb{R}^3)$ with $W(x) = (1+|x|^2/3)^{-\frac{1}{2}}$ instead of Q. The techniques developed there motivated the proof of Theorem 2.6 as certain elements of their ideology, in particular concentration compactness techniques, are essential. The proof also relies explicitly on several classical results in the field of equivariant wave maps. In particular, crucial roles are played by the vanishing of the kinetic energy proved by Shatah, Tahvildar-Zadeh [65], and Struwe's bubbling theorem, [69], in the finite time blow-up result.

A fundamental role in the degree 1 argument is played by a property of the linear wave equation. To be specific, consider $\Box u = 0$, $u(0) = f \in \dot{H}^1(\mathbb{R}^d)$, $u_t(0) = g \in L^2(\mathbb{R}^d)$ for radial functions f, g. Then Duyckaerts, Kenig, and Merle showed the following: If the dimension d is odd, there exists $c > 0$ such that for all $t \geq 0$ or all $t \leq 0$ one has

$$E_{ext}(\vec{u}(t)) \geq cE(f,g) \tag{2.17}$$

In *even dimensions* this property fails, see [20]. To be precise, in dimensions $d = 2, 6, 10, \ldots$ (2.17) holds for data $(0, g)$, but fails in general for data $(f, 0)$. On the other hand, for dimensions $d = 4, 8, 12, \ldots$ (2.17) holds for data $(f, 0)$ but fails in general for data $(0, g)$.

The proof of both the positive and negative results is based on the Fourier representation, which in our radial context becomes a Bessel transform. The dimension d is then reflected in the phase of the Bessel asymptotics. Due to the monotonicity of the energy over the regions $\{|x| \geq t\}$ the key calculation is that of the asymptotic exterior energy as $t \to \pm\infty$.

For the Theorem 2.6 we need the $d = 4$ result rather than $d = 2$ due to the repulsive $\frac{\psi}{r^2}$-potential coming from $\frac{\sin(2\psi)}{2r^2}$. Crucially, the result from [20] for $(f, 0)$ suffices for the argument because of the classical theory of Christodoulou, Tahvildar-Zadeh, and Shatah [14, 15, 65, 66] about equivariant wave maps; see also the book by Shatah, Struwe [63]. Amongst other things, these authors showed that at the blowup $t = 1$ one has vanishing kinetic energy:

$$\lim_{t \to 1} \frac{1}{1-t} \int_t^1 \int_0^t |\dot{\psi}(t,r)|^2 \, r dr \, dt = 0$$

One has a similar averaged vanishing in the case of a global solution. This vanishing (modulo many other arguments) then allows us to work with the more restrictive form of (2.17) for data $(f, 0)$.

2.5.3. Characterization of dynamics at higher energies. Recently, a more general version of Theorem 2.6 has been established by Côte [16] which holds for data of arbitrary degree and energy.

Theorem 2.7 ([16]). *Let $\vec{\psi}(t)$ be a finite energy wave map with maximal forward time of existence $T^+(\vec{\psi})$. Then there exist a sequence of times $t_n \uparrow T^+(\vec{\psi})$, an integer $J \geq 0$, J sequences of scales $\lambda_{J,n} \ll \cdots \ll \lambda_{2,n} \ll \lambda_{1,n}$ and J harmonic maps $Q_1, \ldots, Q_J$ such that*

$$Q_J(0) = \psi(0), \quad Q_{j+1}(\infty) = Q_j(0) \quad \text{for } j = 1, \ldots, J-1,$$

and that one of the following holds:

(1) *If* $T^+(\vec{\psi}) = +\infty$, *denote* $\ell = \lim_{r\to\infty} \psi(t,r) = Q_1(\infty)$. *Then* $\lambda_{1,n} \ll t_n$ *and there exists a solution* $\vec{\phi}_L(t)$ *to the linearized wave equation* (2.16) *such that*

$$\vec{\psi}(t_n) = \sum_{j=1}^{J} \left(Q_j\left(\cdot/\lambda_{j,n}\right) - Q_j(\infty), 0\right) + (\ell, 0) + \vec{\phi}_L(t_n) + o_n(1),$$

(2) *If* $T^+(\vec{\psi}) < +\infty$, *then* $\lambda_{1,n} \ll T^+(\vec{\psi}) - t_n$ *and there exists a function* $\vec{\phi} \in \mathcal{H}_0$ *such that* $\phi(0) = \lim_{t\uparrow T^+(\vec{\psi})} \psi(t, T^+(\vec{\psi}) - t)$ *and* $Q_1(\infty) = \phi(0)$ *and*

$$\vec{\psi}(t_n) = \sum_{j=1}^{J} \left(Q_j\left(\cdot/\lambda_{j,n}\right) - Q_j(\infty), 0\right) + \vec{\phi} + o_n(1) \text{ in } \mathcal{H}_0,$$

Note that the energy of the of the wave map $\vec{\psi}$ together with the orthogonality of the scales $\lambda_{j,n}$ gives an upper bound on the number of large profiles J. An analogous result for the semi-linear equation $\Box u = u^5$ was proved in [23] and the proof of this result along with the proof of Theorem 2.6 provide a rough blueprint for the proof of Theorem 2.7. There are several key differences however, with perhaps the most significant being a generalization of Struwe's bubbling result proved in [16], which allows one to extract a bubble at each scale that carries a nontrivial amount of energy. The proof also relies on a generalization of the degree zero scattering theory, which states that any solution in $\mathcal{H}_0$ which stays bounded away from the south pole scatters to a free wave.

2.6. Large data global regularity. In contrast to the sphere as a target, for negatively curved manifolds as targets one has global existence of smooth solutions for smooth data of any size. This result is the culmination of many years' worth of effort, and was achieved in varying forms by three different groups. Sterbenz, Tataru [67, 68] proved the following very satisfactory result.

Theorem 2.8. *Consider the wave map equation for functions* $u : \mathbb{R}^{1+2}_{t,x} \to M$ *where* M *is a Riemannian manifold. Let* E_0 *be the infimum of all possible energies of nonconstant harmonic maps* $\mathbb{R}^2 \to M$. *Given smooth data* $u_0 : \mathbb{R}^2 \to M$ *and* $u_1 : \mathbb{R}^2 \to TM$ *so that the energy satisfies* $E(u_0, u_1) < E_0$, *there exists a unique global smooth evolution of the wave map* u.

In particular, if M is of negative curvature then it does not admit nontrivial harmonic maps by a result of Eels and Simpson, so we do indeed have the global regularity for all data. By different methods Tao [72] proved the global regularity theorem for hyperbolic spaces as targets, and Krieger and the author [46] obtained this result for the hyperbolic plane. The method of [46] is different from the aforementioned works since it relies on the concentration-compactness ideas of Bahouri, Gerard [4] and Kenig, Merle [34, 35]. To be more specific, one obtains both implicit space-time bounds which yield scattering for the derivative components in a suitable gauge, purely in terms of the energy, as well as a type of profile decomposition for sequences of wave maps with bounded energy. The key new aspect of the work [46] is that the Bahouri-Gérard approach to the profile decomposition,

which depends crucially on the property that factors of widely separated frequency interact only weakly in the nonlinearity, needs to be replaced by a suitable form of a "covariant" Bahouri-Gérard approach. More precisely, the wave maps nonlinearity features certain low-high frequency interactions in the nonlinearity which cannot be shown to be negligible. In order to still be able to obtain a profile decomposition (precisely, one does so for the derivative components in the Coulomb gauge), one needs to replace the free wave propagator by suitable magnetic potential wave operators of the form

$$\Box_A = \Box + 2iA_\alpha \partial^\alpha$$

with $\Box_A$ operating on (essentially) unit frequency functions, while A_α is of extremely low frequency. The precise theorem obtained in [46] then reads as follows.

Theorem 2.9. *There exists a function $K : (0,\infty) \to (0,\infty)$ with the following property: Let M be a hyperbolic Riemann surface. Suppose $(\mathbf{u}_0, \mathbf{u}_1) : \mathbb{R}^2 \to M \times TM$ are smooth and $\mathbf{u}_0 = const$, $\mathbf{u}_1 = 0$ outside of some compact set. Then the wave map evolution $\mathbf{u}$ of these data as a map $\mathbb{R}^{1+2} \to M$ exists globally as a smooth function and, moreover, for any $\frac{1}{p} + \frac{1}{2q} \leq \frac{1}{4}$ with $2 \leq q < \infty$, $\gamma = 1 - \frac{1}{p} - \frac{2}{q}$,*

$$\sum_{\alpha=0}^{2} \|(-\Delta)^{-\frac{\gamma}{2}} \partial_\alpha \mathbf{u}\|_{L_t^p L_x^q} \leq C_q \, K(E) \tag{2.18}$$

Moreover, in the case when $M \hookrightarrow \mathbb{R}^N$ is a compact Riemann surface, one has scattering:

$$\max_{\alpha=0,1,2} \|\partial_\alpha \mathbf{u}(t) - \partial_\alpha S(t)(f,g)\|_{L_x^2} \to 0 \quad \text{as } t \to \pm\infty$$

where $S(t)(f,g) = \cos(t|\nabla|)f + \frac{\sin(t|\nabla|)}{|\nabla|} g$ and suitable $(f,g) \in (\dot{H}^1 \times L^2)(\mathbb{R}^2;\mathbb{R}^N)$. Alternatively, if M is non-compact, then lifting $\mathbf{u}$ to a map $\mathbb{R}^{1+2} \to \mathbb{H}^2$ with derivative components ϕ_α^j with respect to a suitable global frame, one has

$$\max_{\alpha=0,1,2} \|\phi_\alpha^j(t) - \partial_\alpha S(t)(f^j, g^j)\|_{L_x^2} \to 0 \quad \text{as } t \to \pm\infty$$

where $(f^j, g^j) \in (\dot{H}^1 \times L^2)(\mathbb{R}^2;\mathbb{R})$. Finally, denoting the derivative components of the Wave Map with respect to the Coulomb Gauge by ψ_α, $\alpha = 0,1,2$, then given a sequence of Wave Maps of bounded energy $\mathbf{u}_n : \mathbb{R}^{2+1} \to \mathbb{H}^2$, with corresponding components $\psi_{n,\alpha}$, there is an inductive procedure to construct concentration profiles, so that the $\psi_{n,\alpha}$ can be represented as sum of the suitably modulated concentration profiles, up to an error which can be made small in a suitable sense.

For a review of this work see Tao's Bulletin article [73].

2.7. Wave maps from curved Minkowski space. One can also consider the wave maps equation on a curved domain. Let $(\mathcal{N}, h)$ be a Riemannian manifold of dimension d. Denote by $(\tilde{\mathcal{N}}, \eta)$ the Lorentzian manifold $\tilde{\mathcal{N}} = \mathbb{R} \times \mathcal{N}$, with the metric η represented in local coordinates by $\eta = (\eta_{\alpha\beta}) = \mathrm{diag}(-1, h_{ij})$. Let $(\mathcal{M}, g)$ be a complete Riemannian manifold without boundary of dimension n and consider maps $u : \tilde{\mathcal{N}} \to \mathcal{M}$. Here wave maps are formal critical points of the Lagrangian

$$\mathcal{L}(u, du) = \frac{1}{2} \int_{\tilde{\mathcal{N}}} \langle du, du \rangle_{T^*\tilde{\mathcal{N}} \otimes u^* T\mathcal{M}} \, \mathrm{dvol}_\eta.$$

The differential, du, of the map u is a section of the vector bundle $(T^*\tilde{\mathcal{N}} \otimes u^*T\mathcal{M}, \eta \otimes u^*g)$, where $u^*T\mathcal{M}$ is the pullback of $T\mathcal{M}$ by u and u^*g is the pullback metric. In local coordinates (t, x) on $\tilde{\mathcal{N}}$ and $u = (u^1, \dots, u^n)$ on $\mathcal{M}$ this becomes

$$\mathcal{L}(u, du) = \frac{1}{2} \int_{\tilde{\mathcal{N}}} \eta^{\alpha\beta}(t, x) g_{ij}(u(t, x)) \partial_\alpha u^i(t, x) \partial_\beta u^j(t, x) \sqrt{|h|}\, dt\, dx.$$

In this intrinsic formulation critical points satisfy

$$\Box_\eta u^k := -\partial_{tt} u^k + \Delta_h u^k = -\eta^{\alpha\beta} \Gamma^k_{ij}(u) \partial_\alpha u^i \partial_\beta u^j, \tag{2.19}$$

where Δ_h is the Laplace-Beltami operator on $\mathcal{N}$. One can also consider the extrinsic formulation in which case the equation for u is given by

$$\Box_\eta u \perp T_u\mathcal{M} \text{ or } \Box_\eta u = -\eta^{\alpha\beta} S(u)(\partial_\alpha u, \partial_\beta u). \tag{2.20}$$

Here S is the second fundamental form of the embedding $\mathcal{M} \hookrightarrow \mathbb{R}^m$.

It is apparent from the left-hand sides of (2.19) and (2.20) that one must first understand the dynamics of the free wave equation

$$\Box_\eta u = 0 \tag{2.21}$$

before considering the nonlinear wave maps equation. In general, (2.21) can present extremely challenging analytical aspects. For example, the presence of trapped geodesics can lead to the loss of a dispersive estimate such as (1.5). There are also difficulties from a purely technical point of view as important tools from harmonic analysis used to study dispersive equations on flat backgrounds, such as the Fourier transform, do not extend easily to the global geometric setting.

In light of these difficulties, we highlight two natural starting places where global-in-time dispersive estimates for the free equation (2.21) have been established: (1) $\mathcal{N}$ is a small, asymptotically flat perturbation of Euclidean space, $\mathbb{R}^d$; and (2) $\mathcal{N}$ is d-dimensional real hyperbolic space, $\mathbb{H}^d$.

2.7.1. Small asymptotically flat perturbations of Euclidean space. First consider the free wave equation (2.21) on a small, asymptotically flat perturbation of Euclidean space. The smallness assumption is used to avoid issues such as the trapping of bi-characteristic rays and also to handle the dynamics of low frequencies – in general one can expect significant departures from the Euclidean theory at the level of the low frequencies since these see the global geometry of $\mathcal{N}$. For $\mathcal{N}$ as in (1), *global* Strichartz estimates without loss are established by Metcalfe and Tataru in [52] for linear variable coefficient equations such as (2.21). Using a time dependent FBI transform and a delicate analysis of the evolution in phase space, the authors construct an outgoing parametrix that satisfies global-in-time dispersive estimates. These estimates are paired with a localized energy estimate to control the errors generated in the parametrix construction. See also [79] for a similar analysis of the Schrödinger evolution on curved space as well as [76] and the references therein for more background on phase space transforms and the microlocal framework in which these objects are considered.

Now consider the wave map equation (2.19) or (2.20) where $\mathcal{N}$ is a small, asymptotically flat perturbation of Euclidean space, $\mathbb{R}^d$ with $d = 4$ and $\mathcal{M}$ a smooth manifold with bounded

geometry. Using the Shatah-Struwe approach from [64], the small data global theory in the critical space $H^2 \times H^1$ is established in [48]. The idea is to use the method of moving frames to derive a wave equation for the u^*TN-valued 1-form, du, with the Coulomb gauge as the choice of frame on u^*TN. In the global geometric setting, the resulting equation for du is an equation of 1-forms,

$$\Box(du) = d(\eta^{\alpha\beta} A_\alpha du_\beta) + \delta(-A \wedge du) \tag{2.22}$$

where d is the exterior derivative on $\tilde{\mathcal{N}}$, δ is its adjoint, $\Box = d\delta + \delta d$, and A is the connection form associated to the Coulomb gauge. The components of du satisfy a system of variable coefficient nonlinear wave equations for which the Metcalfe-Tataru Strichartz estimates can be used to obtain a-priori estimates in the case of small initial data. Here the Coloumb frame is crucial to estimate the right-hand side of (2.22) since the components of A satisfy a system of variable coefficient *elliptic* equations which are then used to estimate A in terms of du; see for example the classic work [80].

2.7.2. Hyperbolic space. If $\mathcal{N}$ is d-dimensional real hyperbolic space $\mathbb{H}^d$, the free wave equation on $\mathcal{N} = \mathbb{H}^d$ is an appealing object due to the geometric significance of $\mathbb{H}^d$, but also from a technical standpoint because of the existence of the Helgason-Fourier transform; see [28, 29]. In addition to this technical advantage, the negative curvature of $\mathbb{H}^d$ suggests that there should be better dispersion for solutions to (2.21) on $\mathbb{H}^d$ than for their Euclidean counterparts. Intuitively, the exponential volume growth of concentric spheres gives more room for a wave to spread out into. This is indeed the case, and Strichartz estimates for an improved range of admissible exponents have recently been established for the linear Schrödinger equation on $\mathbb{H}^d$ in [1, 31] and for the linear wave and Klein-Gordon equations in [2, 53, 54]; see also [3, 6–9, 58, 75]. Moreover, in [30] a Bahouri-Gerard type profile decomposition was developed for the Schrödinger equation on $\mathbb{H}^d$.

Next, consider the wave maps equation on the domain $\mathcal{N} = \mathbb{R} \times \mathbb{H}^d$. Since $\mathbb{H}^d$ is rotationally symmetric one can consider *equivariant* wave maps as in the Euclidean case when the target manifold $\mathcal{M}$ is also rotationally symmetric. Restricting to the energy critical dimension, $d = 2$, the usual 1-equivariant formulation is

$$\psi_{tt} - \psi_{rr} - \coth r\, \psi_r + \frac{g(\psi)g'(\psi)}{\sinh^2 r} = 0, \tag{2.23}$$

where (ψ, θ) are geodesic polar coordinates on the target surface $\mathcal{M}$, and g determines the metric, $ds^2 = d\psi^2 + g^2(\psi)d\theta^2$.

Despite the relative simplicity of the equivariant model, this problem exhibits markedly different phenomena than its Euclidean counterpart. The cases of two model targets $\mathcal{M} = \mathbb{S}^2$ and $\mathcal{M} = \mathbb{H}^2$ are considered in [49].

When the target is $\mathbb{S}^2$, there exists a family of equivariant harmonic maps $Q_\lambda : \mathbb{H}^2 \to \mathbb{S}^2$, indexed by a parameter $\lambda \geq 0$ that measures how far the image of each harmonic map wraps around the sphere, i.e., $Q_\lambda(r) \to 2\arctan(\lambda)$ as $r \to \infty$ and have energies

$$\mathcal{E}(Q_\lambda) \to 0 \ \text{ as } \ \lambda \to 0, \ \text{ and } \ \mathcal{E}(Q_\lambda) \to \mathcal{E}_{\text{euc}}(Q_{\text{euc}}) \ \text{ as } \ \lambda \to 1$$

where Q_{euc} is the unique nontrivial co-rotational Euclidean harmonic map from $\mathbb{R}^2$ to $\mathbb{S}^2$, given by stereographic projection and $\mathcal{E}_{\text{euc}}(Q_{\text{euc}})$ is its energy. The Q_λ are asymptotically stable for values of λ smaller than a threshold that is large enough to allow for maps that

wrap more than halfway around the sphere. However, as $\lambda \to \infty$, asymptotic stability via a perturbative argument based on Strichartz estimates is precluded by the existence of *gap eigenvalues* in the spectrum of the operator obtained by linearization about Q_λ. On the other hand, a Struwe-type bubbling argument as in [69] suggests that any solution $\vec{\psi}(t)$ to (2.23) that blows up in finite time must bubble off a Euclidean harmonic map Q_{euc}, and therefore must have enough energy to wrap completely around the sphere. Indeed finite time blow-up via energy concentration is a local phenomena and the global geometry of the domain plays little role. This gives evidence towards a conjecture that in fact every Q_λ is stable – as small perturbations of Q_λ will not have enough energy to bubble off a Q_{euc} – but for large λ, the stability manifests nonlinearly.

When the target is $\mathbb{H}^2$, there exists a continuous family of asymptotically stable equivariant harmonic maps $P_\lambda : \mathbb{H}^2 \to \mathbb{H}^2$ indexed by a parameter $\lambda \in (0,1)$ with $P_\lambda(r) \to 2\text{arctanh}(\lambda)$ as $r \to \infty$ and

$$\mathcal{E}(P_\lambda) \to 0 \text{ as } \lambda \to 0, \text{ and } \mathcal{E}(P_\lambda) \to \infty \text{ as } \lambda \to 1.$$

This stands in sharp contrast to the corresponding problem on Euclidean space, where all finite energy solutions scatter to zero as time tends to infinity. The presence of the nontrivial stable, stationary solutions together with the lack of scaling symmetry and the defocusing nature of the nonlinearity make this an interesting setting to study the large data dynamics. In particular one may expect that any solution $\psi(t,r)$ with $\psi(t,r) \to 2\text{arctanh}(\lambda)$ as $r \to \infty$ scatters to P_λ as $t \to \infty$ – in other words, solution resolution.

3. Scalar semi-linear equations

In the section we present a very small selection of recent results on a much-studied family of problems, namely semilinear equations of the form

$$\Box u + f(u) = 0$$

where $f(u)$ is a suitable power nonlinearity. We will in fact consider only a very special model equation, which is however representative of the kind of phenomena we wish describe.

3.1. The defocusing cubic Klein-Gordon equation. In $\mathbb{R}^{1+3}_{t,x}$ consider the cubic defocusing Klein-Gordon equation

$$\Box u + u + u^3 = 0, \quad (u(0), \dot{u}(0)) = (f,g) \in \mathcal{H} := H^1 \times L^2(\mathbb{R}^3) \tag{3.1}$$

with conserved energy

$$E(u,\dot{u}) = \int_{\mathbb{R}^3} \Big(\frac{1}{2}|\dot{u}|^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|u|^2 + \frac{1}{4}|u|^4 \Big)\, dx$$

With $S(t)$ denoting the linear propagator of $\Box + 1$ we have

$$\vec{u}(t) = (u,\dot{u})(t) = S(t)(f,g) - \int_0^t S(t-s)(0,u^3(s))\, ds \tag{3.2}$$

By contraction mapping for small times T one obtains local wellposedness for $\mathcal{H}$ data. The means that there is a unique solution $(u, \dot{u}) \in C([0,T]; H^1) \times C([0,T]; L^2)$ which satisfies (3.1) in the Duhamel sense. Note that T depends only on the $\mathcal{H}$-size of data. From energy conservation we obtain global existence by time-stepping. By Strichartz estimates, one can easily show that the solution in fact scatters to a free wave (solution of the Klein-Gordon equation without the nonlinearity) in the norm of $\mathcal{H}$ for all small data. For large data, the classical approach to proving scattering proceeds by means of Morawetz estimates, see Ginibre,Velo [27]. Alternatively, a general and elegant method is based on induction in energy, which was first introduced by Bourgain [11]. The modern blueprint of this method is due to Kenig and Merle [34, 35], a key component of which is the powerful concentration compactness decomposition of Bahouri, Gérard [4], see also Merle,Vega [51]. See [56, Capter 2] for the implementation of this method in the context of (3.1).

3.2. The focusing cubic Klein-Gordon equation. The dynamics of the focusing equation

$$\Box u + u - u^3 = 0, \quad (u(0), \dot{u}(0)) = (f, g) \in \mathcal{H} := H^1 \times L^2(\mathbb{R}^3) \tag{3.3}$$

has been know for a long time to be very different from that of the defocusing equation. Note the conserved energy is indefinite,

$$E(u, \dot{u}) = \int_{\mathbb{R}^3} \Big(\frac{1}{2}|\dot{u}|^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|u|^2 - \frac{1}{4}|u|^4 \Big)\, dx$$

While small data again lead to global existence and scattering, large data may lead to blowup. Indeed, setting $u = u(t)$ (no spatial dependence) leads to an ODE which blows up in finite time. Truncation of the corresponding data by means of a smooth bump function which equals 1 on a large enough ball yields finite energy data that blow up in finite time. Equation (3.3) also admit time-independent solutions, which solve the elliptic PDE

$$-\Delta\varphi + \varphi - \varphi^3 = 0$$

which is the equation of a critical point of the stationary energy

$$J(\varphi) \int_{\mathbb{R}^3} \Big(\frac{1}{2}|\nabla\varphi|^2 + \frac{1}{2}|\varphi|^2 - \frac{1}{4}|\varphi|^4 \Big)\, dx$$

Amongst all nonzero solutions of this equation there exists a class which minimizes $J(\varphi)$. This class is of the form $\{\pm Q(\cdot + y) \mid y \in \mathbb{R}^3\}$ where $Q > 0$ is radial, exponentially decaying. It is also unique with this property. Many years ago, Payne and Sattinger [57] gave a characterization of all possible dynamics below the energy $E(Q, 0)$. In the regime of energies above $E(Q, 0)$ one has the following description of the dynamics for radial data, due to Nakanishi and the author [56].

Theorem 3.1. *Let $E(u_0, u_1) < E(Q, 0) + \varepsilon^2$, $(u_0, u_1) \in \mathcal{H}_{\mathrm{rad}}$. Then for $t \geq 0$ the solutions to* (3.3) *exhibit the following trichotomy:*

(1) *finite time blowup*

(2) *global existence and scattering to* 0

(3) *global existence and scattering to Q: $u(t) = Q + v(t) + o_{H^1}(1)$ as $t \to \infty$, and $\dot{u}(t) = \dot{v}(t) + o_{L^2}(1)$ as $t \to \infty$, $\Box v + v = 0$, $(v, \dot{v}) \in \mathcal{H}$.*

All nine combinations of this trichotomy occur as $t \to \pm\infty$.

Similar results can be established in the nonradial case (where Lorentz symmetries come into play), as well as several other nonlinear unstable Hamiltonian wave equations, see [56]. This theorem is not perturbative. In fact, a key step is to exclude almost homoclinic orbits emanating near $\pm Q$ and returning to these equilibria. This is based on an indirect argument using the hyperbolic dynamics near these points (which are a result of the unique negative eigenvalue of the linearized equation), together with the virial functional. The scattering statement is again obtained via concentration compactness arguments. The question of what happens for even larger energies is wide open. For the dissipative version of (3.3) one may obtain similar results, in fact a more complete picture of the dynamics emerges in that case, see the forthcoming work of Burq, Raugel, and the author.

Acknowledgements. Support of the National Science Foundation DMS-1160817 is gratefully acknowledged. The author thanks Andrew Lawrie for his substantial help in preparing this survey, and Joachim Krieger for helpful remarks on a preliminary version.

References

[1] J.-P. Anker and V. Pierfelice, *Nonlinear Schrödinger equation on real hyperbolic spaces*, Ann. Inst. H. Poincaré Anal. Non Linéaire **26** (2009), no. 5, 1853–1869. MR 2566713 (2010m:35416)

[2] J. P. Anker and V. Pierfelice, *Wave and Klein-Gordon equations on hyperbolic spaces*, Preprint (2011).

[3] J.-P. Anker, V. Pierfelice, and M. Vallarino, *The wave equation on hyperbolic spaces*, J. Differential Equations **252** (2012), no. 10, 5613–5661. MR 2902129

[4] H. Bahouri and P. Gérard, *High frequency approximation of solutions to critical nonlinear wave equations*, Amer. J. Math. **121** (1999), 131–175.

[5] B. Balakrishna, V. Schechter Sanyuk, J., and A. Subbaraman, *Cutoff quantization and the skyrmion*, Physical Review D **45** (1992), no. 1, 344–351.

[6] V. Banica, *The nonlinear Schrödinger equation on hyperbolic space*, Comm. Partial Differential Equations **32** (2007), no. 10-12, 1643–1677. MR 2372482 (2008m:35323)

[7] V. Banica, R. Carles, and T. Duyckaerts, *On scattering for NLS: from Euclidean to hyperbolic space*, Discrete Contin. Dyn. Syst. **24** (2009), no. 4, 1113–1127. MR 2505694 (2010m:35478)

[8] V. Banica, R. Carles, and G. Staffilani, *Scattering theory for radial nonlinear Schrödinger equations on hyperbolic space*, Geom. Funct. Anal. **18** (2008), no. 2, 367–399. MR 2421543 (2010h:35364)

[9] V. Banica and T. Duyckaerts, *Weighted Strichartz estimates for radial Schrödinger equation on noncompact manifolds*, Dyn. Partial Differ. Equ. **4** (2007), no. 4, 335–359. MR 2376801 (2008j:58036)

[10] P. Bizoń, T. Chmaj, and M. Maliborski, *Equivariant wave maps exterior to a ball*, Nonlinearity **25** (2012), no. 5, 1299–1309.

[11] J. Bourgain, *Global wellposedness of defocusing critical nonlinear Schrödinger equation in the radial case*, J. Amer. Math. Soc. **12** (1999), no. 1, 145–171. MR 1626257 (99e:35208)

[12] G. Can and J. Krieger, *Optimal polynomial blow up range for critical wave maps*, in preparation (2014).

[13] T. Cazenave, J. Shatah, and A. S. Tahvildar-Zadeh, *Harmonic maps of the hyperbolic space and development of singularities in wave maps and Yang-Mills fields*, Ann. Inst. H. Poincaré Phys. Théor. **68** (1998), no. 3, 315–349. MR 1622539 (2000g:58042)

[14] D. Christodoulou and A. S. Tahvildar-Zadeh, *On the asymptotic behavior of spherically symmetric wave maps*, Duke Math. J. **71** (1993), no. 1, 31–69.

[15] ———, *On the regularity of spherically symmetric wave maps*, Comm. Pure Appl. Math. **46** (1993), no. 7, 1041–1091.

[16] R. Côte, *Soliton resolution for equivariant wave maps to the sphere*, Preprint (2013).

[17] R. Côte, C. Kenig, A. Lawrie, and W. Schlag, *Characterization of large energy solutions of the equivariant wave map problem: I*, To appear in Amer. J. Math (Preprint 2012).

[18] ———, *Characterization of large energy solutions of the equivariant wave map problem: II*, To appear in Amer. J. Math (Preprint 2012).

[19] R. Côte, C. Kenig, and F. Merle, *Scattering below critical energy for the* $4d$ *Yang-Mills equation and for the* $2d$ *corotational wave map system*, Comm. Math. Phys. **284** (2008), no. 1, 203–225.

[20] R. Côte, C. Kenig, and W. Schlag, *Energy partition for the linear radial wave equation*, Preprint (2012).

[21] R. Donninger, *On stable self-similar blowup for equivariant wave maps*, Comm. Pure Appl. Math. **64** (2011), no. 8, 1095–1147.

[22] T. Duyckaerts, C. Kenig, and F. Merle, *Universality of the blow-up profile for small radial type II blow-up solutions of the energy critical wave equation*, J. Eur math. Soc. (JEMS) **13** (2011), no. 3, 533–599.

[23] ———, *Profiles of bounded radial solutions of the focusing, energy-critical wave equation*, Geom. Funct. Anal. **22** (2012), no. 3, 639–698. MR 2972605

[24] ———, *Universality of the blow-up profile for small type II blow-up solutions of the energy-critical wave equation: the nonradial case*, J. Eur. Math. Soc. (JEMS) **14** (2012), no. 5, 1389–1454. MR 2966655

[25] ———, *Classification of radial solutions of the focusing, energy critical wave equation*, Cambridge Journal of Mathematics **1** (2013), no. 1, 75–144.

[26] _____, *Scattering for radial, bounded solutions of focusing supercritical wave equations*, To appear in I.M.R.N (Preprint, 2012).

[27] J. Ginibre and G. Velo, *Scattering theory in the energy space for a class of nonlinear wave equations*, Comm. Math. Phys. **123** (1989), no. 4, 535–573. MR 1006294 (90i:35172)

[28] S. Helgason, *Topics in harmonic analysis on homogeneous spaces*, Progress in Mathematics, vol. 13, Birkhäuser, Boston, Mass., 1981. MR 632696 (83g:43009)

[29] _____, *Geometric analysis on symmetric spaces*, second ed., Mathematical Surveys and Monographs, vol. 39, American Mathematical Society, Providence, RI, 2008. MR 2463854 (2010h:22021)

[30] A. D. Ionescu, B. Pausader, and G. Staffilani, *On the global well-posedness of energy-critical Schrödinger equations in curved spaces*, Preprint (2010).

[31] A. D. Ionescu and G. Staffilani, *Semilinear Schrödinger flows on hyperbolic spaces: scattering H^1*, Math. Ann. **345** (2009), no. 1, 133–158. MR 2520054 (2010c:35031)

[32] M. Keel and T. Tao, *Endpoint Strichartz estimates*, Amer. J. Math. **120** (1998), no. 5, 955–980. MR 1646048 (2000d:35018)

[33] C. Kenig, A. Lawrie, and W. Schlag, *Relaxation of wave maps exterior to a ball to harmonic maps for all data*, Preprint (2013).

[34] C. Kenig and F. Merle, *Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case*, Invent. Math. **166** (2006), no. 3, 645–675.

[35] _____, *Global well-posedness, scattering and blow-up for the energy-critical focusing non-linear wave equation*, Acta Math. **201** (2008), no. 2, 147–212.

[36] S. Klainerman and M. Machedon, *Space-time estimates for null forms and the local existence theorem*, Comm. Pure Appl. Math. **46** (1993), no. 9, 1221–1268.

[37] _____, *Smoothing estimates for null forms and applications*, Internat. Math. Res. Notices (1994), no. 9.

[38] _____, *Smoothing estimates for null forms and applications*, Duke Math. J. **81** (1995), no. 1, 99–133.

[39] _____, *On the regularity properties of a model problem related to wave maps*, Duke Math. J. **87** (1997), no. 3, 553–589.

[40] S. Klainerman and I. Rodnianski, *On the global regularity of wave maps in the critical Sobolev norm*, Internat. Math. Res. Notices (2001), no. 13, 655–677.

[41] S. Klainerman and S. Selberg, *Remark on the optimal regularity for equations of wave maps type*, Comm. Partial Differential Equations **22** (1997), no. 5-6, 901–918.

[42] _____, *Bilinear estimates and applications to nonlinear wave equations*, Commun. Contemp. Math. **4** (2002), no. 2, 223–295.

[43] J. Krieger, *Global regularity of wave maps from* $\mathbf{R}^{2+1}$ *to* H^2. *Small energy*, Comm. Math. Phys. **250** (2004), no. 3, 507–580.

[44] ———, *Global regularity and singularity development for wave maps*, Surveys in differential geometry. Vol. XII. Geometric flows, Surv. Differ. Geom., vol. 12, Int. Press, Somerville, MA, 2008, pp. 167–201.

[45] J. Krieger and W. Schlag, *Non-generic blow-up solutions for the critical focusing nls in 1-d*, J. Eur. Math. Soc. (JEMS) **11** (2009), no. 1, 1–125.

[46] ———, *Concentration compactness for critical wave maps*, EMS Monographs, European Mathematical Society, Zürich, 2012.

[47] J. Krieger, W. Schlag, and D. Tataru, *Renormalization and blow up for charge one equivariant wave critical wave maps*, Invent. Math. **171** (2008), no. 3, 543–615.

[48] A. Lawrie, *The Cauchy problem for wave maps on a curved background*, Calc. Var. Partial Differential Equations **45** (2012), no. 3–4, 505–548.

[49] A. Lawrie, S.-J. Oh, and S. Shahshahani, *Stability of stationary equivariant wave maps from the hyperbolic plane*, Preprint (2014).

[50] A. Lawrie and W. Schlag, *Scattering for wave maps exterior to a ball*, Advances in Mathematics **232** (2013), no. 1, 57–97.

[51] F. Merle and L. Vega, *Compactness at blow-up time for* L^2 *solutions of the critical nonlinear Schrödinger equation in 2D*, Internat. Math. Res. Notices (1998), no. 8, 399–425. MR 1628235 (99d:35156)

[52] J. Metcalfe and D. Tataru, *Global parametrices and dispersive estimates for variable coefficient wave equations*, Math. Ann. **353** (2012), no. 4, 1183–1237.

[53] J. Metcalfe and M. Taylor, *Nonlinear waves on 3D hyperbolic space*, Trans. Amer. Math. Soc. **363** (2011), no. 7, 3489–3529.

[54] ———, *Dispersive wave estimates on 3D hyperbolic space*, Proc. Amer. Math. Soc. **140** (2012), no. 11, 3861–3866.

[55] A. Nahmod, A. Stefanov, and K. Uhlenbeck, *On the well-posedness of the wave map problem in high dimensions*, Comm. Anal. Geom. **11** (2003), no. 1, 49–83.

[56] K. Nakanishi and W. Schlag, *Invariant manifolds and dispersive Hamiltonian evolution equations*, Zurich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich, 2011. MR 2847755 (2012m:37120)

[57] L. E. Payne and D. H. Sattinger, *Saddle points and instability of nonlinear hyperbolic equations*, Israel J. Math. **22** (1975), no. 3-4, 273–303. MR 0402291 (53 #6112)

[58] V. Pierfelice, *Weighted Strichartz estimates for the Schrödinger and wave equations on Damek-Ricci spaces*, Math. Z. **260** (2008), no. 2, 377–392. MR 2429618 (2009d:35272)

[59] P. Raphael and I. Rodnianski, *Stable blow up dynamics for the critical corotational wave map and equivariant Yang-Mills problems*, Publi. I.H.E.S., in press (2009).

[60] I. Rodnianski and W. Schlag, *Time decay for solutions of Schrödinger equations with rough and time-dependent potentials*, Invent. Math. **155** (2004), no. 3, 451–513. MR 2038194 (2005h:35295)

[61] I. Rodnianski and J. Sterbenz, *On the formation of singularities in the critical $O(3)$ σ-model*, Ann. of Math. **172** (2010), 187–242.

[62] J. Shatah, *Weak solutions and development of singularities of the* $\mathrm{SU}(2)$ *σ-model*, Comm. Pure Appl. Math. **41** (1988), no. 4, 459–469.

[63] J. Shatah and M. Struwe, *Geometric wave equations*, Courant Lecture notes in Mathematics, New York University, Courant Institute of Mathematical Sciences, New York, American Mathematical Society, Providence RI, 1998.

[64] ———, *On the Cauchy problem for wave maps*, Int. Math. Res. Not. (2002), no. 11, 555–571.

[65] J. Shatah and A. Tahvildar-Zadeh, *Regularity of harmonic maps from the Minkowski space into rotationally symmetric manifolds*, Comm. Pure Appl. Math. **45** (1992), no. 8, 947–971.

[66] J. Shatah and A. S. Tahvildar-Zadeh, *On the Cauchy problem for equivariant wave maps*, Comm. Pure Appl. Math. **47** (1994), no. 5, 719–754.

[67] J. Sterbenz and D. Tataru, *Energy dispersed large data wave maps in* $2+1$ *dimensions*, Comm. Math. Phys. (2010), no. 1, 139–230.

[68] ———, *Regularity of wave maps in* $2+1$ *dimensions*, Comm. Math. Phys. (2010), no. 1, 231–264.

[69] M. Struwe, *Equivariant wave maps in two space dimensions*, Comm. Pure Appl. Math. **56** (2003), no. 7, 815–823.

[70] T. Tao, *Global regularity of wave maps. I. Small critical Sobolev norm in high dimension*, Internat. Math. Res. Notices (2001), no. 6, 299–328.

[71] ———, *Global regularity of wave maps II: Small energy in two dimensions*, Comm. Math. Phys. **224** (2001), no. 2, 443–544.

[72] ———, *Global regularity of wave maps III–VII*, Preprints (2008–2009).

[73] T Tao, *Concentration compactness for critical wave maps [book review]*, Bull. Amer. Math. Soc. (N.S.) **50** (2013), no. 4, 655–662.

[74] D. Tataru, *On global existence and scattering for the wave maps equation*, Amer. J. Math. **123** (2001), no. 1, 37–77.

[75] ———, *Strichartz estimates in the hyperbolic space and global existence for the semilinear wave equation*, Trans. Amer. Math. Soc. **353** (2001), no. 2, 795–807 (electronic). MR 1804518 (2001k:35218)

[76] ———, *Phase space transforms and microlocal analysis*, Phase space analysis of partial differential equations. Vol. II, Pubbl. Cent. Ric. Mat. Ennio Giorgi, Scuola Norm. Sup., Pisa, 2004, pp. 505–524. MR 2208883 (2007e:35309)

[77] ———, *The wave maps equation*, Bull. Amer. Math. Soc. (N.S.) **41** (2004), no. 2, 185–204 (electronic).

[78] ———, *Rough solutions for the wave maps equation*, Amer. J. Math. **127** (2005), no. 2, 293–377.

[79] ———, *Parametrices and dispersive estimates for Schrödinger operators with variable coefficients*, Amer. J. Math. **130** (2008), no. 3, 571–634. MR 2418923 (2009m:35077)

[80] Karen K. Uhlenbeck, *Connections with L^p bounds on curvature*, Comm. Math. Phys. **83** (1982), no. 1, 31–42. MR 648356 (83e:53035)

The University of Chicago, Department of Mathematics, 5734 South University Avenue, Chicago, IL 60637, U.S.A.

E-mail: schlag@math.uchicago.edu

Several applications of the moment method in random matrix theory

To Yonatan Naim

Sasha Sodin

Abstract. Several applications of the moment method in random matrix theory, especially, to local eigenvalue statistics at the spectral edges, are surveyed, with emphasis on a modification of the method involving orthogonal polynomials.

Mathematics Subject Classification (2010). 60B20, 44A60.

Keywords. Moment method; random matrices, orthogonal polynomials.

1. Introduction

The goal of this article is to survey a few of the applications of the moment method (and its variants) to the study of the spectral properties of random matrices, particularly, local eigenvalue statistics at the spectral edges.

Section 2 is a brief introduction to the moment method, which we understand as the variety of ways to extract the properties of a measure μ from integrals of the form

$$\int \xi^m \, d\mu(\xi) \, . \tag{1.1}$$

Examples, selected from the narrow part of random matrix theory in which the author feels competent, are intended to illustrate two theses. First, the moment method can be applied beyond the framework of weak convergence of a sequence of probability measures. Second, it is often convenient to replace the monomials ξ^m in (1.1) with a better-conditioned sequence, such as the sequence of orthogonal polynomials with respect to a measure μ_∞ which is an approximation to μ.

In Section 3 we review some applications to the local eigenvalue statistics at the spectral edges, starting from the work of Soshnikov [64]. Tracy and Widom [70, 71] and Forrester [25] introduced the Airy point processes (see Section 3.1) and showed that they describe the limiting distribution of the largest eigenvalues for special families of large Hermitian random matrices with independent entries (the Gaussian invariant ensembles). Soshnikov [64] extended these results to Wigner matrices (Hermitian random matrices with independent entries and no invariance assumptions). In the terminology of Ibragimov and Linnik [37, Chapter VI], the result of [64] is a limit theorem of collective character; it is one of the

Proceedings of International Congress of Mathematicians, 2014, Seoul

instances of the ubiquity (universality) of the Airy point processes within and outside random matrix theory (as surveyed, for example, by Johansson [38], Tracy and Widom [72], and Borodin and Gorin [9]).

In Section 3.2, we consider Wigner processes, a class of matrix-valued random processes. Informally, a random matrix $H(\mathbf{x})$ is attached to every point $\mathbf{x}$ of an underlying space $\mathbf{X}$. The statistical properties of the eigenvalues of every $H(\mathbf{x})$ are described by the theory of Wigner matrices; the joint distribution of the eigenvalues of a tuple $(H(\mathbf{x}_r))_{r=1}^k$ leads to limiting objects which depend on the geometry of $\mathbf{X}$ (which arises from the correlations of the matrix elements of H) in a non-trivial way. The moment method allows to derive limit theorems of collective character (such as Theorem 3.4) pertaining to the spectral edges of $H(\mathbf{x})$ (the result of [64] corresponds to a singleton, $\#\mathbf{X} = 1$).

In Section 3.3, we turn to the spectral edges of random band matrices. A random band matrix is (3.14) a random $N \times N$ Hermitian matrix with non-zero entries in a band of width W about the main diagonal. When W is small, a band matrix inherits the structure of the integer lattice $\mathbb{Z}$; when W is large, it is similar to a Wigner matrix. The threshold at which the local eigenvalue statistics in the bulk of the spectrum exhibit a crossover is described by precise conjectures (see Fyodorov and Mirlin [33, 34], Spencer [67, 68]). The moment method allows to prove the counterpart of these conjectures for the spectral edges (the result of [64] corresponds to the special case $W = N$).

The content of Section 2 is mostly known. The modified moment method of Section 2.4 is a version of self-energy renormalisation in perturbation theory (see Spencer [67]), related to the arguments of Bai and Yin [4]. Orthogonal polynomials were explicitly used in this context in the work of Li and Solé [44], and further in [58] (where more references may be found). Some observations are incorporated from [21]. The content of Section 3.2 is an extension of [62], whereas Section 3.3 is based on [60]. The proofs of the results stated in both of these sections build on the combinatorial arguments of [21].

2. Preliminaries and generalities

The moment method is the collection of techniques inferring the properties of a measure μ on the k-dimensional space $\mathbb{R}^k$ from the moments

$$s(m_1, \cdots, m_k; \mu) = \int_{\mathbb{R}^k} \xi_1^{m_1} \cdots \xi_k^{m_k} \, d\mu(\xi) \quad (m_1, \cdots, m_k = 0, 1, 2, \cdots) \,. \tag{2.1}$$

Introduced by Chebyshev as a means to establish Gaussian approximation for the distribution of a sum of independent random variables, the moment method achieved its first major success with the proof, given by Markov [46], of Lyapunov's Central Limit Theorem and its extension to sums of weakly dependent random variables. Some of the more recent applications are surveyed by Diaconis [13].

2.1. Convergence of probability measures. In the traditional setting of the moment method, one considers a sequence of probability measures $(\mu_N)_{N \geq 1}$ on $\mathbb{R}^k$. Suppose that the limit

$$s(m_1, \cdots, m_k) = \lim_{N \to \infty} s(m_1, \cdots, m_k; \mu_N) \tag{2.2}$$

exists for every $m_1, \cdots, m_k \geq 0$. Then the sequence $(\mu_N)_{N \geq 1}$ is tight, i.e. precompact in weak topology (defined by bounded continuous functions), and every one of its limit points

μ satisfies

$$s(m_1, \cdots, m_k; \mu) = s(m_1, \cdots, m_k) \quad (m_1, \cdots, m_k \geq 0) \,. \tag{2.3}$$

If, for example,

$$s(m_1, \cdots, m_k) \leq \prod_{r=1}^{k} (Cm_r)^{m_r} \quad (m_1, \cdots, m_k \geq 0) \,, \tag{2.4}$$

the moment problem (2.3) is determinate, i.e. there is a unique measure μ_∞ on $\mathbb{R}^k$ satisfying (2.3); in this case the convergence of moments (2.2) implies that $\mu_N \to \mu_\infty$ in weak topology; cf. Feller [23, §VIII.6].

Hardy's sufficient condition (2.4) may be somewhat relaxed; we refer to the addenda to the second chapter of the book [1] of Akhiezer for various sufficient criteria for determinacy in the case $k = 1$, and to the survey of Berg [5] for some extensions to $k \geq 1$.

2.1.1. Random measures. Suppose $(\mu_N)_{N\geq 1}$ is a sequence of random measures on $\mathbb{R}^\ell$ (i.e. random variables taking values in the space of Borel probability measures). We denote $\boldsymbol{\xi} = (\xi_1, \cdots, \xi_\ell)$, $\mathbf{m} = (m_1, \cdots, m_\ell)$, and $\boldsymbol{\xi}^{\mathbf{m}} = \xi_1^{m_1} \cdots \xi_\ell^{m_\ell}$; thus $\boldsymbol{\xi}_r^{\mathbf{m}_r} = \xi_{r,1}^{m_{r,1}} \cdots \xi_{r,\ell}^{m_{r,\ell}}$. If

$$\mathbb{E} \int_{\mathbb{R}^{\ell k}} \prod_{r=1}^{k} [d\mu_N(\boldsymbol{\xi}_r) \boldsymbol{\xi}_r^{\mathbf{m}_r}] \to \mathbb{E} \int_{\mathbb{R}^{\ell k}} \prod_{r=1}^{k} [d\mu_\infty(\boldsymbol{\xi}_r) \boldsymbol{\xi}_r^{\mathbf{m}_r}] \quad (k \geq 1) \tag{2.5}$$

for some random measure μ_∞ on $\mathbb{R}^\ell$, and the moment problem for every moment measure $\mathbb{E}\mu_\infty^{\otimes k}$ is determinate, then

$$\mathbb{E}\mu_N^{\otimes k} \xrightarrow{\text{weak}} \mathbb{E}\mu_\infty^{\otimes k} \quad (k \geq 1) \,. \tag{2.6}$$

If, for every Borel set $K \in \mathcal{B}(\mathbb{R}^\ell)$, the moment problem for the distribution of $\mu_\infty(K)$ is determinate, then (2.6) implies that $\mu_N \to \mu_\infty$ (weakly in distribution). Finally, if μ_∞ is deterministic (i.e. its distribution is supported on one deterministic measure), it is sufficient to verify (2.6) for $k = 1, 2$. See further Zessin [78].

2.2. Example: Wigner's law. The application (going back to Chebyshev) of the moment method to sums of independent random variables is based on the identity

$$\left[\sum_{j=1}^{N} X_j \right]^m = \sum_{m_1 + \cdots + m_N = m} \frac{m!}{m_1! \cdots m_N!} \prod_{j=1}^{N} X_j^{m_j} \tag{2.7}$$

expressing powers of a sum of numbers as a sum over partitions. Similarly, the application (going back to Wigner) of the moment method to random matrix theory is based on the relation

$$\begin{aligned} \operatorname{tr} H^m &= \sum_{p=(u_0, u_1, \cdots, u_{m-1}, u_m = u_0)} \prod_{j=0}^{m-1} H(u_j, u_{j+1}) \\ &= \sum_{p} \prod_{1 \leq u \leq v \leq N} H(u,v)^{\#\{(u_j, u_{j+1}) = (u,v)\}} \overline{H(u,v)}^{\#\{(u_j, u_{j+1}) = (v,u)\}} \end{aligned}$$

expressing traces of powers of an Hermitian matrix $H = (H(u,v))_{u,v=1}^N$ as a sum over paths.

Let $\big(G_N = (V_N, E_N)\big)_{N\geq 1}$ be a sequence of graphs, so that G_N is κ_N-regular (meaning that every vertex is adjacent to exactly κ_N edges), and the connectivity κ_N tends to infinity:

$$\lim_{N\to\infty} \#V_N = \infty, \quad \lim_{N\to\infty} \kappa_N = \infty\,. \tag{2.8}$$

For every N, consider a random matrix $H^{(N)} = (H(u,v))_{u,v\in V_N}$ with rows and columns indexed by the elements of V_N, so that $(H(u,v))_{u,v\in V_N}$ are independent up to the constraint $H(v,u) = \overline{H(u,v)}$; the diagonal entries $\{H(u,u)\}_u$ are sampled from a distribution $\mathcal{L}_{\text{diag}}$ on $\mathbb{R}$ satisfying

$$\mathbb{E}H(u,u) = 0\,, \quad \mathbb{E}H(u,u)^2 < \infty\,; \tag{2.9}$$

the off-diagonal entries $\{H(u,v)\}_{(u,v)\in E_N}$ are sampled from a distribution $\mathcal{L}_{\text{off-diag}}$ on $\mathbb{C}$ satisfying

$$\mathbb{E}H(u,v) = 0\,, \quad \mathbb{E}|H(u,v)|^2 = 1\,; \tag{2.10}$$

and all the other entries $H(u,v)$ are set to zero.

Let $\xi_1^{(N)} \geq \xi_2^{(N)} \geq \cdots \xi_{\#V_N}^{(N)}$ be the eigenvalues of $H^{(N)}$, and let

$$\mu_N = \frac{1}{\#V_N} \sum_{j=1}^{\#V_N} \delta\left(\xi - \frac{\xi_j^{(N)}}{2\sqrt{\kappa_N - 1}}\right)\,.$$

(The scaling is natural since, for instance, the ℓ_2 norm of every column of the $N\times N$ matrix is of order $\sqrt{\kappa_N}$.)

Theorem (Wigner's law). *In the setting of this paragraph (i.e. assuming (2.8), (2.9), (2.10)), the sequence of random measures $(\mu_N)_N$ converges (weakly, in distribution) as $N\to\infty$ to the (deterministic) semicircle measure σ_{Wig} with density*

$$\frac{d\sigma_{\text{Wig}}}{d\xi} = \frac{2}{\pi}\sqrt{(1-\xi^2)_+}\,. \tag{2.11}$$

Wigner considered [75–77] the case when G_N is the complete graph on N vertices (Wigner matrices), and the entries satisfy some additional assumptions, the important of them being that all moments are finite. Wigner's argument is based on the moment method.

Bogachev, Molchanov, and Pastur [6] observed (in the context of random band matrices) that a similar argument can be applied as long as (2.8) is satisfied. The first argument for Wigner matrices without additional restrictions on the distribution of the entries was given by Pastur [51], using the Stieltjes transform method introduced by Marchenko and Pastur [47, 48] (see Pastur [52] and the book of Pastur and Shcherbina [53] for some of the further applications of the method). Khorunzhiy, Molchanov, and Pastur [39] applied the Stieltjes transform method to prove Wigner's law for random band matrices; their argument is applicable in the setting described here.

Let us outline a proof of Wigner's law in the form stated above, following [6] (and incorporating Markov's truncation argument [46]). We refer for details to the book of Anderson, Guionnet, and Zeitouni [3, Chapter 2.1], where similar arguments are also applied to questions such as the Central Limit Theorem for linear statistics $\phi(\xi_1^{(N)}) + \cdots + \phi(\xi_{\#V_N}^{(N)})$.

Proof of Wigner's law. Due to (2.8), (2.9) and (2.10) one can find a sequence $\delta_N \to +0$ so that

$$\mathbb{E}|H(u,v)|^2 \mathbb{1}_{|H(u,v)|\geq \delta_N\sqrt{\kappa_N}} \leq \delta_N\,.$$

Consider the matrix $H_{\gtrdot}^{(N)}$ with truncated matrix elements

$$H_{\gtrdot}(u,v) = \begin{cases} H(u,v)\,, & |H(u,v)| \le \sqrt{\kappa_N} \\ 0\,, & |H(u,v)| > \sqrt{\kappa_N} \end{cases}\,.$$

Then

$$\mathbb{P}\left\{H_{\gtrdot}(u,v) \ne H(u,v)\right\} \le \delta_N \kappa_N^{-1}\,,$$

whence, bounding rank by the number of non-zero matrix elements and applying the Chebyshev inequality,

$$\mathbb{P}\left\{\operatorname{rank}(H_{\gtrdot}^{(N)} - H^{(N)}) \ge \delta_N^{1/2} \# V_N\right\} \le \delta_N^{1/2}\,.$$

For any $\xi \in \mathbb{R}$, the interlacing property of rank-one perturbation yields

$$\left|\mu_N(-\infty,\xi] - \mu_{\gtrdot,N}(-\infty,\xi]\right| \le \#V_N^{-1} \operatorname{rank}(H_{\gtrdot}^{(N)} - H^{(N)})\,, \tag{2.12}$$

therefore it is sufficient to establish the result for $H_{\gtrdot}^{(N)}$ in place of $H^{(N)}$. For large N, the elements of $H_{\gtrdot}^{(N)}$ enjoy the following estimates:

$$|EH_{\gtrdot}(u,v)| \le \delta_N \kappa_N^{-\frac{1}{2}}\,; \tag{2.13}$$

$$\left|\mathbb{E}|H_{\gtrdot}(u,v)|^2 - 1\right| \le \delta_N \quad ((u,v) \in E_N)\,; \quad \mathbb{E}|H_{\gtrdot}(u,u)|^2 \le \text{const}\,; \tag{2.14}$$

$$\mathbb{E}\left|H_{\gtrdot}(u,v)\right|^k \le 2\delta_N \kappa_N^{\frac{k-2}{2}} \quad (k \ge 3)\,. \tag{2.15}$$

Next, consider the expansion

$$\begin{aligned} s(m_1,\cdots,m_k; \mathbb{E}\mu_{\gtrdot,N}^{\otimes k}) &= \mathbb{E}\prod_{r=1}^{k}\int \xi^{m_r} d\mu_{\gtrdot,N}(\xi) \\ &= \mathbb{E}\frac{1}{(\#V_N)^k}\prod_{r=1}^{k}\operatorname{tr}\left(\frac{H_{\gtrdot}^{(N)}}{2\sqrt{\kappa_N - 1}}\right)^{m_r} \\ &= \sum \frac{1}{(\#V_N)^k}\mathbb{E}\prod_{r=1}^{k}\prod_{j=0}^{m_r-1}\frac{H_{\gtrdot}(u_{r,j},u_{r,j+1})}{2\sqrt{\kappa_N-1}}\,, \end{aligned} \tag{2.16}$$

where the sum is over k-tuples of closed paths

$$\begin{cases} u_{1,0}, u_{1,1}, \cdots, u_{1,m_1-1}, u_{1,m_1} \\ u_{2,0}, u_{2,1}, \cdots, u_{2,m_2-1}, u_{2,m_2} \\ \cdots \\ u_{k,0}, u_{k,1}, \cdots, u_{k,m_k-1}, u_{k,m_k} \end{cases} \qquad [u_{1,m_1} = u_{1,0}, \cdots, u_{k,m_k} = u_{k,0}]$$

in the augmented (multi-)graph $G_N^+ = (V_N, E_N^+)$, $E_N^+ = E_N \bigcup \{(u,u) \mid u \in V_N\}$. Two such k-tuples are called isomorphic if one is obtained from one another by a permutation of the vertices V_N. For example, the pair $(1\,3\,1\,,\ 2\,1\,4\,2)$ is isomorphic to $(7\,4\,7\,,\ 1\,7\,2\,1)$.

According to (2.13), (2.14) and (2.15), the contribution of an isomorphism class consisting of k-tuples spanning a graph $\mathfrak{g}$ with $\mathfrak{v}$ vertices and $\mathfrak{e}$ edges, of which $\mathfrak{e}_2$ are traversed exactly twice, is bounded by $\kappa_N^{\mathfrak{v}-k}(\delta_N)^{\mathfrak{e}-\mathfrak{e}_2}(\kappa_N/\text{const})^{-\mathfrak{e}}$.

The graph $\mathfrak{g}$ has at most k connected components, whence

$$\mathfrak{v} - \mathfrak{e} \le k \,, \tag{2.17}$$

with equality for graphs which are vertex-disjoint unions of k trees. For fixed $m_1, \cdots, m_k$, the number of isomorphism classes remains bounded as $N \to \infty$, therefore the limit of (2.16) is given by the contribution of vertex-disjoint k-tuples of paths corresponding to graphs with

$$\mathfrak{v} - \mathfrak{e} = k \,, \quad \mathfrak{e}_2 = \mathfrak{e} \,. \tag{2.18}$$

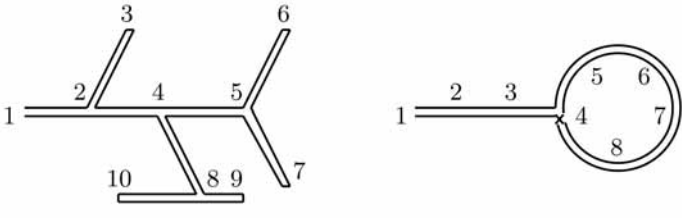


Figure 2.1. The tree-like path 1 2 3 2 4 5 6 5 7 5 4 8 9 8 10 8 4 2 1 with $\mathfrak{v} = 10$ and $\mathfrak{e} = \mathfrak{e}_2 = 9$ (left) and the non-backtracking path 1 2 3 4 5 6 7 8 4 5 6 7 8 4 3 2 1 with $\mathfrak{v} = 8$ and $\mathfrak{e} = \mathfrak{e}_2 = 8$ (right). Among the two, only the first one contributes to the semi-circle limit.

Every path in such a k-tuple is tree-like (see Figure 2.1, left); each isomorphism class contributes $2^{-\sum_p m_p}$ (due to (2.14)), and the number of classes is given by a product of Catalan numbers:

$$\prod_{r=1}^{k} \begin{cases} \frac{2}{m_r+2}\binom{m_r}{m_r/2} \,, & m_r \text{ is even} \\ 0 \,, & m_p \text{ is odd} \end{cases} = \prod_{r=1}^{k} \left[2^{m_r} s(m_r; \sigma_{\mathrm{Wig}})\right] \,.$$

Thus

$$\lim_{N\to\infty} s(m_1, \cdots, m_k; \mathbb{E}\mu_{\mathfrak{S}_N,N}^{\otimes k}) = s(m_1, \cdots, m_k; \sigma_{\mathrm{Wig}}^{\otimes k}) \,. \tag{2.19}$$

Applying the relation (2.19) with $k = 1, 2$, we conclude (cf. Section 2.1.1) that $\mu_{\mathfrak{S}_N,N}$ converge to σ_{Wig} weakly in distribution, and thus (by (2.12)) so do μ_N. □

2.3. Some quantitative aspects. Whenever the moment convergence (2.2) is a consequence of the stronger property

$$s(m_1, \cdots, m_k; \mu_N) = s(m_1, \cdots, m_k; \mu_\infty) \quad (N \ge N_0(m_1, \cdots, m_k)) \,, \tag{2.20}$$

the arguments quoted in Section 2.1 can be recast in quantitative form. This is illustrated by the following inequality due to Sonin [63]. Let γ be the Gaussian measure,

$$\frac{d\gamma}{d\xi} = \frac{1}{\sqrt{2\pi}} \exp(-\xi^2/2) \quad (\xi \in \mathbb{R}) \,,$$

and assume that

$$s(m; \mu_N) = s(m; \gamma) \left[= \begin{cases} 0, & m \text{ is odd} \\ \frac{m!}{(\frac{m}{2})! 2^{\frac{m}{2}}}, & m \text{ is even} \end{cases} \right] \quad (N \ge N_0(m)) \,. \tag{2.21}$$

Then

$$\sup_{\xi\in\mathbb{R}} |\mu_N(\xi) - \gamma(\xi)| \le \sqrt{\frac{\pi}{m-1}} \quad (N \ge \max_{m'\le m} N_0(m')) \,. \tag{2.22}$$

Measures μ_N of random matrix origin for which (2.21) holds may be found in the survey of Diaconis [13]. Inequalities of the form (2.22) may be also derived for other measures μ_∞ (see Akhiezer [1, Section II.5.4] for the general framework of Chebyshev–Markov–Stieltjes inequalities, and Krawtchouk [40] for additional examples).

Similar inequalities can be derived for $k > 1$. On the other hand, already in the setting of the Central Limit Theorem for sums of independent random variables, (2.21) is not valid (unless the addends are Gaussian themselves); the correct relation $s(m;\mu_N) \approx s(m;\gamma)$, even with the optimal dependence of the error term on m and N, yields a poor bound on the rate of convergence of μ_N to γ (the sharp Berry –Esseen bound, see Feller [23, §XVI.5], was proved by the Fourier-analytic approach). The reason is that monomials form an ill-conditioned basis; see Gautschi [35] for a discussion of computational aspects (and of remedies similar to the one discussed in the next section).

2.4. A modification of the moment method. The following modification makes the moment method better conditioned. Let $(\mu_N)_{N\ge1}$ be a sequence of probability measures on $\mathbb{R}$, and suppose μ_∞ is a candidate for the weak limit of the sequence $(\mu_N)_{N\ge1}$. Let $P_n(\xi)$ $(n = 0, 1, 2, \cdots)$ be the orthogonal polynomials with respect to μ_∞:

$$\deg P_n = n \,, \quad \int P_n(\xi)P_{n'}(\xi)d\mu_\infty(\xi) = \delta_{nn'} \,.$$

Also set

$$\widetilde{s}(n;\mu;\mu_\infty) = \int_{-\infty}^{\infty} P_n(\xi)d\mu(\xi) \,. \tag{2.23}$$

Then the convergence of moments

$$\lim_{N\to\infty} s(m;\mu_N) = s(m;\mu_\infty) \quad (m \ge 0) \tag{2.24}$$

is equivalent to

$$\lim_{N\to\infty} \widetilde{s}(n;\mu_N;\mu_\infty) = \delta_{n0} \quad (n \ge 0) \,. \tag{2.25}$$

Thus (2.25) implies that $\mu_N \to \mu_\infty$, provided that the moment problem for μ_∞ is determinate.

While the modification of the moment method advertised here seems to have no general counterpart in dimension $k > 1$, in the special case when μ_∞ is the k-th power of a one-dimensional measure with orthogonal polynomials P_n we define:

$$\widetilde{s}(n_1,\cdots,n_k;\mu;\mu_\infty) = \int_{\mathbb{R}^k} \prod_{r=1}^{k} P_{n_r}(\xi_r)\, d\mu(\xi) \,.$$

2.4.1. A random matrix example. If $X_1, \cdots, X_n$ are independent random variables with zero mean, unit variance, and finite moments, one may give a combinatorial interpretation to

$$\mathbb{E}\frac{1}{\sqrt{n!}}\,\mathrm{He}_n\left[\frac{X_1+\cdots+X_N}{\sqrt{N}}\right] \,, \tag{2.26}$$

where

$$\mathrm{He}_n(\xi) = (-1)^n e^{\xi^2/2} \frac{d^n}{d\xi^n} e^{-\xi^2/2}$$

are the Hermite polynomials; the three-term recurrent relation

$$\mathrm{He}_{n+1}(\xi) = \xi \,\mathrm{He}_n(\xi) - n \,\mathrm{He}_{n-1}(\xi)$$

eliminates the asymptotically leading terms of the moments (2.7) of $X_1 + \cdots + X_N$. Here we focus on a different example, pertaining to random matrices of the form considered in Section 2.2.

Denote

$$P_n^{(\kappa)}(\xi) = U_n(\xi) - \frac{1}{\kappa - 1} U_{n-2}(\xi) \,,$$

where

$$U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}$$

are the Chebyshev polynomials of the second kind (orthogonal with respect to σ_{Wig}), and $U_{-1} \equiv U_{-2} \equiv 0$. Let $G = (V, E)$ be a regular graph of connectivity κ, and let H be an $\#V \times \#V$ Hermitian matrix, such that

$$|H(u,v)| = \mathbb{1}_{(u,v)\in E} \,, \quad (u, v \in V) \,. \tag{2.27}$$

The three-term recurrent relation

$$P_{n+1}^{(\kappa)}(\xi) = 2\xi P_n^{(\kappa)}(\xi) - (1 + (\kappa-1)^{-1} \mathbb{1}_{n=1}) P_{n-1}^{(\kappa)}(\xi)$$

for $P_n^{(\kappa)}$ leads to

Proposition 2.1 (cf. [58, Lemma 2.7], [21, Claim II.1.2]). *For any Hermitian matrix H satisfying (2.27),*

$$P_n^{(\kappa)}\left[\frac{H}{2\sqrt{\kappa-1}}\right](u,v) = \sum \prod_{j=1}^{n} \frac{H(u_j, u_{j+1})}{\sqrt{\kappa-1}} \,, \tag{2.28}$$

where the sum is over paths $u_0, u_1, \cdots, u_{n-1}, u_n$ in G from $u_0 = u$ to $u_n = v$ which satisfy the non-backtracking condition $u_j \neq u_{j+2}$ ($0 \le j \le n-2$).

Consider a sequence of random matrices $H^{(N)}$ associated to a sequence of graphs G_N with $\kappa_N \to \infty$ as in Section 2.2; let us assume that the entries of H satisfy the unimodality assumptions (2.27). A non-backtracking path can not be tree-like (see Figure 2.1), therefore the modified moments tend to zero; this provides an alternative proof to Wigner's law in the form of Section 2.2 under the additional assumptions (2.27).

The generalisation of Propostion 2.1 to matrices which do not satisfy (2.27) is somewhat technical, and we do not present it here. In the context of Wigner (and sample covariance) matrices, it is described in [21, Part III]; for the (more involved) case of band matrices we refer to the work of Erdős and Knowles [17].

2.4.2. Advantages of modified moments. Although the convergence of modified moments (2.25) is equivalent to the convergence of moments (2.24), quantitative forms of the former yield better estimates on the rate of convergence $\mu_N \to \mu_\infty$. As an illustration, we recall a variant of the Erdős–Turán inequality [20] proved in [22]. Consider again the semi-circle measure σ_{Wig} with density (2.11).

Proposition 2.2 ([22, Proposition 5]). *Let μ be a probability measure on $\mathbb{R}$. Then, for any $\xi \in \mathbb{R}$ and any $n_0 \geq 1$,*

$$|\mu(-\infty,\xi] - \sigma_{\mathrm{Wig}}(-\infty,\xi]| \leq C \left\{ \frac{\rho(\xi;n_0)}{n_0} + \sqrt{\rho(\xi;n_0)} \sum_{n=1}^{n_0} \frac{|\widetilde{s}(n;\mu;\sigma_{\mathrm{Wig}})|}{n} \right\} ,$$

where $C > 0$ is a numerical constant, and $\rho(\xi;n_0) = \max(1 - |\xi|, n_0^{-2})$.

The original Erdős–Turán inequality provides a bound of similar structure for the measure with density $d\mu_\infty/d\xi = \pi^{-1}\left((1-\xi^2)_+\right)^{-1/2}$ (in this case, $\rho(\xi;n_0)$ should be set to 1.)[1]

2.5. Convergence of rescaled probability measures. The rescaling $R_\eta^{\xi_0}[\mu]$ of a measure μ on $\mathbb{R}^k$ about $\xi_0 \in \mathbb{R}^k$ by $\eta > 0$ is defined by

$$R_\eta^{\xi_0}[\mu](K) = \mu(\eta(K - \xi_0)) \quad \left(K \in \mathcal{B}(\mathbb{R}^k)\right) . \tag{2.29}$$

In a class of questions outside the narrow framework of Section 2.1, one is interested in vague limits (weak limits with respect to the topology defined by compactly supported continuous functions) of

$$\left(\epsilon_N^{-1} R_{\eta_N}^{\xi_0}[\mu_N]\right)_{N\geq 1} , \tag{2.30}$$

where $(\mu_N)_{N\geq 1}$ is a sequence of probability measures on $\mathbb{R}^k$, $\xi_0 \in \mathbb{R}^k$, and two sequences $\epsilon_N, \eta_N \to +0$ determine the scaling of μ_N on the value ($\updownarrow$) and variable ($\leftrightarrow$) axes, respectively.

2.5.1. Edges (corners) of the support. Moments allow to study the rescaling of μ_N about a point ξ_0 which is close to the corners of the cube supporting μ_N. Variants of this observation were used, for example, by Sinai and Soshnikov [56, 57].

Assume that we are given a sequence $(\mu_N)_{N\geq 1}$ of probability measures on $\mathbb{R}^k$, two sequences $\epsilon_N, \eta_N \to +0$ which determine the scaling (2.30), and 2^k continuous functions $\phi_\varepsilon : (\alpha_0,\infty)^k \to \mathbb{R}_+$ ($\varepsilon \in \{-1,1\}^k$) which will describe the limiting Laplace transform at the 2^k corners of the cube.

Proposition 2.3. *Suppose*

$$\epsilon_N^{-k} s(m_{1,N},\cdots,m_{k,N};\mu_N) - \sum_{\varepsilon\in\{-1,1\}^k} \prod_{r=1}^{k} \varepsilon_r^{m_r}\, \phi_\varepsilon(\alpha_1,\cdots,\alpha_k) \longrightarrow 0 \quad (N\to\infty)$$

for any sequence $(m_{1,N},\cdots,m_{k,N})_{N\geq 1}$ for which

$$\lim_{N\to\infty} \eta_N m_{r,N} = \alpha_r > \alpha_0 \quad (1 \leq r \leq k).$$

[1] A similar inequality for the Gaussian measure, combined with a careful estimate of the modified moments (2.26), could perhaps yield a proof of the Berry–Esseen theorem along the lines suggested by Chebyshev.

Then, for any $\varepsilon \in \{-1,1\}^k$, the sequence $(\epsilon_N^{-1} R^{\varepsilon}_{\eta_N}[\mu_N])_{N\geq 1}$ converges vaguely to a measure ν^{ε} which is uniquely determined by the equations

$$\int \exp(\alpha_1\lambda_1 + \cdots + \alpha_k\lambda_k) d\nu^{\varepsilon}(\varepsilon_1\lambda_1, \cdots, \varepsilon_k\lambda_k) = \phi_{\varepsilon}(\alpha_1, \cdots, \alpha_k) \quad (\alpha \in (\alpha_0, \infty)^k) \, .$$

Remark 2.4. Convergence actually holds in the stronger topology defined by continuous functions supported (for some $R > 0$) in

$$\prod_{r=1}^{k} \begin{cases} (-R, \infty) \, , & \varepsilon_r = 1 \\ (-\infty, R) \, , & \varepsilon_r = -1 \end{cases} .$$

The counterparts of Proposition 2.3 for modified moments depend on the structure of the limiting measure μ_∞. For the case $\mu_\infty = \sigma_{\mathrm{Wig}}^{\otimes k}$ such a statement was proved in [60, Section 6]. It is somewhat technical, and we do not reproduce it here; instead of the Laplace transform, the limiting measures ν^{ε} are characterised in terms of the transform

$$\int_{\mathbb{R}^k} \prod_{r=1}^{k} \frac{\sin \alpha_r \sqrt{-\lambda_r}}{\sqrt{-\lambda_r}} \, d\nu^{\varepsilon}(\varepsilon_1\lambda_1, \cdots, \varepsilon_k\lambda_k) \tag{2.31}$$

(which becomes convergent after a certain regularisation). The system of functions $\lambda \mapsto \frac{\sin \alpha\sqrt{-\lambda}}{\sqrt{-\lambda}}$ forms a continuous analogue of orthogonal polynomials (as introduced by Krein, see Denisov [12]) with respect to the measure $\frac{2\sqrt{2}}{\pi}\sqrt{-\lambda_-}$ (obtained by rescaling σ_{Wig} about $\xi_0 = 1$).

Uniqueness theorems for the transform (2.31) were proved (in dimension $k = 1$) in the 1950-s by Levitan [42], Levitan and Meiman [43], and Vul [73] (listed in order of increasing generality); the argument in [60] builds on [42].

One advantage of the approach based on modified moments is that, for a measure supported on several intervals, it allows to consider the rescaling about edges (corners) which are not maximally distant from the origin, and even internal edges. In the context of random matrices, this was exploited in [21].

2.5.2. Interior points of the support. If ξ_0 is an interior point of the support of μ_∞, it seems impossible to extract any information regarding the measures $\epsilon_N^{-1} R^{\xi_0}_{\eta_N}[\mu_N]$ from the asymptotics of the moments of μ_N. The modified moments $\widetilde{s}$ carry such information. For example, Proposition 2.2 shows that if one can find a sequence $(n_0(N))_{N\geq 1}$ so that

$$\lim_{N\to\infty} \epsilon_N n_0(N) = +\infty \, , \quad \lim_{N\to\infty} n_0(N) \sum_{n=1}^{n_0(N)} \frac{|\widetilde{s}(n; \mu_N; \sigma_{\mathrm{Wig}})|}{n} = 0 \, , \tag{2.32}$$

then

$$\epsilon_N^{-1} R^{\xi_0}_{\epsilon_N}[\mu_N] \xrightarrow[N\to\infty]{\text{vague}} \frac{1}{\pi}\sqrt{1-\xi_0^2} \ \mathrm{mes} \qquad (-1 < \xi_0 < 1) \tag{2.33}$$

(where mes is the Lebesgue measure on the real line).

Let us briefly comment on the shorter scales ϵ_N, for which (2.32) fails. The challenge is to give meaning to the expansion

$$\mu_N[\xi', \xi''] \sim \sum_{n\geq 0} \widetilde{s}(n; \mu_N; \mu_\infty) \int_{\xi'}^{\xi''} P_n(\xi) d\mu_\infty(\xi) \tag{2.34}$$

when $|\xi' - \xi''|$ is small. For $\mu_\infty = \sigma_{\rm Wig}$, a regularisation procedure suggested in [61] allows to establish (2.33) (and even to determine the subleading asymptotic terms) in the cases when (2.32) is violated due to divergent contribution to

$$\widetilde{s}(n; \mu_N; \sigma_{\rm Wig}) = \int_{-\infty}^{\infty} U_n(\xi) d\mu_N(\xi)$$

coming from the neighbourhood of $\xi = \pm 1$. It would be interesting to find a way to consider even shorter scales ϵ_N, for which the limit of $\epsilon_N^{-1} R_{\epsilon_N}^{\xi_0}[\mu_N]$ is distinct from that of $\epsilon_N^{-1} R_{\epsilon_N}^{\xi_0}[\mu_\infty]$. In the random matrix applications, such a method would allow to study the local eigenvalue statistics in the bulk of the spectrum via modified moments (in particular, in problems where alternative methods are not currently available).

3. Spectral edges of random matrices

3.1. Wigner matrices. The application of the moment method to local eigenvalues statistics originates in the work of Soshnikov [64] on universality for Wigner matrices. Let us recall the result of [64], after some preliminaries.

As before, we consider a sequence $(H^{(N)})_{N \geq 1}$ of Wigner matrices, i.e. random Hermitian matrices such that the diagonal entries of every $H^{(N)}$ are sampled from a probability distribution $\mathcal{L}_{\rm diag}$ satisfying (2.9), and the off-diagonal entries are sampled from a probability distribution $\mathcal{L}_{\text{off-diag}}$ satisfying (2.10); the eigenvalues of $H^{(N)}$ are denoted

$$\xi_1^{(N)} \geq \xi_2^{(N)} \geq \cdots \geq \xi_N^{(N)} .$$

Consider the random point process (i.e. a random collection or points, or, equivalently, a random integer-valued measure)

$$\Lambda^{(N)} = \sum_{j=1}^{N} \delta\left(\lambda - N^{1/6}\left[\xi_j^{(N)} - 2\sqrt{N}\right]\right) \tag{3.1}$$

(the scaling is natural in view of the square-root singularity of $\sigma_{\rm Wig}$ at 1).

Two special cases, the Gaussian Orthogonal Ensemble (GOE), and the Gaussian Unitary Ensemble (GUE) [as well as the Gaussian Symplectic Ensemble (GSE, not discussed here)], enjoy an invariance property which allows to apply the method of orthogonal polynomials (see Mehta [49]). The limits of $\Lambda^{(N)}$ for GOE and GUE, called the Airy$_1$ ($\mathfrak{Ai}_1$) and the Airy$_2$ ($\mathfrak{Ai}_2$) point processes, respectively, were found by Tracy and Widom [70, 71] and Forrester [25]. The correlation functions, which are (by definition) the densities

$$\rho_{\beta,k}(\lambda_1, \cdots, \lambda_k) = \frac{d}{d\ \mathrm{mes}_k} \mathbb{E}\mathfrak{Ai}_\beta^{\otimes k}|_{\lambda_1 < \cdots < \lambda_k}$$

of the off-diagonal parts of the moment measures $\mathbb{E}\mathfrak{Ai}_\beta^{\otimes k}$, are expressed via determinants involving the Airy function Ai:

$$\rho_{2,k}(\lambda_1, \cdots, \lambda_k) = \det_{k \times k}(A(\lambda_p, \lambda_r))_{p,r=1}^k , \tag{3.2}$$

$$\rho_{1,k}(\lambda_1, \cdots, \lambda_k) = \sqrt{\det_{2k\times 2k} (A_1(\lambda_p, \lambda_r))_{p,r=1}^k}\,, \tag{3.3}$$

where

$$A(\lambda,\lambda') = \int_0^\infty \mathrm{Ai}(\lambda+u)\,\mathrm{Ai}(\lambda'+u)du\,,\ A_1(\lambda,\lambda') = \begin{pmatrix} A(\lambda,\lambda') & DA(\lambda,\lambda') \\ JA(\lambda,\lambda') & A(\lambda,\lambda') \end{pmatrix},$$
$$DA(\lambda,\lambda') = \frac{\partial}{\partial\lambda'} A(\lambda,\lambda')\,,\ JA(\lambda,\lambda') = -\int_\lambda^\infty A(\lambda'',\lambda')d\lambda'' - \frac{1}{2}\,\mathrm{sign}(\lambda-\lambda')\,.$$

Theorem 3.1 (Soshnikov [64]). *Let $(H^{(N)})_{N\geq 1}$ be a sequence of Wigner matrices satisfying the additional assumptions*

$$H(u,v) \overset{distr}{=} -H(u,v)\,; \qquad \textit{(symmetry)} \tag{3.4}$$
$$\mathbb{E}|H(u,v)|^{2k} \leq (Ck)^k \qquad \textit{(subgaussian tails)} \tag{3.5}$$

on $\mathcal{L}_{diag}$ and $\mathcal{L}_{off\text{-}diag}$. If $\mathcal{L}_{off\text{-}diag}$ is supported on the real line, the point processes $\Lambda^{(N)}$ converge (in the topology of Remark 2.4) to $\mathfrak{Ai}_1$; otherwise, $\Lambda^{(N)} \to \mathfrak{Ai}_2$.

Remark 3.2. Lee and Yin [41] have shown that the theorem remains valid if (3.4) and (3.5) are replaced with the assumption

$$\lim_{R\to\infty} R^4\,\mathbb{P}\{|H(1,2)| \geq R\} = 0\,, \tag{3.6}$$

which they have shown to be necessary and sufficient. Their argument makes use of the methods developed in the works of Erdős, Bourgade, Knowles, Schlein, Yau, and Yin on universality in the bulk for Wigner matrices, cf. Erdős [15].

Remark 3.3. The work of Soshnikov was followed by numerous other applications of the moment method to local eigenvalue statistics in random matrix theory (see Soshnikov [65], Péché [54]) as well as outside it (see Okounkov [50]).

The strategy of [64] is to compute the asymptotics of moments and to show, using a version of Proposition 2.3, that the limit of $\Lambda^{(N)}$ exists and does not depend on the distribution of the entries. Thus the theorem is reduced to its special case appertaining to the Gaussian invariant ensembles.

3.1.1. An argument based on modified moments. In [21], modified moments were used to re-prove Soshnikov's theorem quoted above (the method was also applied to sample covariance matrices, to re-prove the results of Soshnikov [65] and Péché [54] on the largest eigenvalues, and to prove a new result on the smallest ones). Let us outline the argument of [21] (incorporating modifications from [60]), which serves as the basis for the extensions described later in this section.

Let us assume that the diagonal entries $H(u,u)$ are identically zero, and that the off-diagonal entries $H(u,v)$ are randomly chosen signs ± 1. Then Proposition 2.1 identifies

$$\mathbb{E}\prod_{r=1}^k \mathrm{tr}\, P_{n_r}^{(N-1)}\left(\frac{H^{(N)}}{2\sqrt{N-2}}\right) \tag{3.7}$$

as $(N-2)^{-\sum n_r/2}$ times the number of k-tuples of closed non-backtracking paths in the complete graph on N vertices, in which every edge is traversed an even number of times (in total). Such k-tuples are divided in topological equivalence classes (k-diagrams of Section 3.2.1 below). For $n_r \asymp N^{1/3}$, the contribution of every equivalence class can be asymptotically evaluated.

In the regime $n_r \asymp N^{1/3}$, (3.7) captures the asymptotics of the transform (2.31) of the moment measures of $\Lambda^{(N)}$. The combinatorial classification yields a convergent series for this transform. This allows to describe the vague limit of $\Lambda^{(N)}$. A more general argument making use of an extension of Proposition 2.1 allows to show that the same limit appears for any sequence of matrices satisfying the assumptions of Soshnikov's theorem (with $\mathcal{L}_{\text{off-diag}}$ supported on the real line) in particular, for the GOE for which the answer is already identified as the Airy_1 point process $\mathfrak{Ai}_1$.

In the remainder of this section we describe (without proofs) two results which may be seen as generalisations of [64].

3.2. Wigner processes. Instead of a single Wigner matrix $H^{(N)}$, let us consider a family $H^{(N)}(\mathbf{x}) = (H(\mathbf{x};u,v))_{1\le u\le v\le N}$ of Wigner matrices depending on a parameter $\mathbf{x} \in \mathbf{X}$; then we are interested in the eigenvalues

$$\xi_1^{(N)}(\mathbf{x}) \ge \xi_2^{(N)}(\mathbf{x}) \ge \cdots \ge \xi_N^{(N)}(\mathbf{x})$$

as a random process on $\mathbf{X}$.

Let us assume that $(\mathbf{x} \mapsto H(\mathbf{x};u,u))_{1\le u\le N}$ are independent copies of a random process $\operatorname{diag} : \mathbf{X} \to \mathbb{R}$,

$$\mathbb{E}\operatorname{diag}(\mathbf{x}) = 0\,, \quad \mathbb{E}\operatorname{diag}(\mathbf{x})^2 < \infty\,,$$

and that $(\mathbf{x} \mapsto H(\mathbf{x};u,v))_{1\le u<v\le N}$ are independent copies of $\text{off–diag} : \mathbf{X} \to \mathbb{C}$,

$$\mathbb{E}\,\text{off–diag}(\mathbf{x}) = 0\,, \quad \mathbb{E}\,|\text{off–diag}(\mathbf{x})|^2 = 1\,.$$

The process $\text{off–diag}(\mathbf{x})$ equips $\mathbf{X}$ with the L_2 metric

$$\rho(\mathbf{x},\mathbf{x}') = \sqrt{\frac{1}{2}\mathbb{E}|\text{off–diag}(\mathbf{x}) - \text{off–diag}(\mathbf{x}')|^2}\,.$$

The local properties of the eigenvalues rescaled about $\mathbf{x}_0 \in \mathbf{X}$ depend on the behaviour of ρ near $\mathbf{x}_0$, which may be captured by the tangent cone $T_{\mathbf{x}_0}\mathbf{X}$ to $\mathbf{X}$ at $\mathbf{x}_0$ (the tangent cone to a metric space was introduced by Gromov [36, Section 7]).

Moment-based methods allow to obtain rigorous results at the spectral edges. Here we focus on the special case in which $\mathbf{X} = \mathbb{R}^d$ and

$$\rho(\mathbf{x},\mathbf{x}')^2 = \|\mathbf{x}-\mathbf{x}'\|_p + o(\|\mathbf{x}\|_p + \|\mathbf{x}'\|_p) \quad (\mathbf{x},\mathbf{x}' \to 0) \tag{3.8}$$

for some $1 \le p \le 2$, which includes, for example, the Ornstein–Uhlenbeck sheet. In this case the tangent cone at the origin is the space $\mathbf{X}_p^d = (\mathbb{R}^d, \sqrt{\|\cdot\|_p})$.

Theorem 3.4. *Let $d \in \mathbb{N}$ and $1 \le p \le 2$. Let $\mathbf{X} = \mathbb{R}^d$, and suppose the processes* $\operatorname{diag}(\mathbf{x})$ *and* $\text{off–diag}(\mathbf{x})$ *have symmetric distribution (3.4) and subgaussian tails (3.5) at every point*

$\mathbf{x} \in \mathbb{R}^d$, *and that the covariance of* $\mathrm{off\text{–}diag}(\mathbf{x})$ *has the asymptotics (3.8) near the origin. Then the processes*

$$\Lambda^{(N)}(\mathbf{x}) = \sum_{j=1}^{N} \delta\left(\lambda - N^{1/6}\left[\xi_j^{(N)}(\mathbf{x}N^{1/3}) - 2\sqrt{N}\right]\right)$$

converge (as $N \to \infty$*, in the sense of finite-dimensional distributions) to a limiting process*

$$\mathrm{AD}_\beta[\mathbf{X}_p^d](\mathbf{x}) = \sum_{j=1}^{\infty} \delta(\lambda - \lambda_j(\mathbf{x})) \quad (\mathbf{x} \in \mathbb{R}^d)$$

taking values in sequences $\lambda_1(\mathbf{x}) \geq \lambda_2(\mathbf{x}) \geq \cdots$*, where* $\beta = 1$ *if* $\mathrm{off\text{–}diag}(0)$ *is real-valued, and* $\beta = 2$ *— otherwise.*

The level of generality chosen here is motivated in particular by the following result proved in [62]: the process $\mathrm{AD}_\beta[\mathbf{X}_1^2]$ also describes the edge scaling limit of corners of time-dependent random matrices (for a discussion of these, see Borodin [7, 8]).

Proposition 3.5. *For* $\beta \in \{1, 2\}$*, the process* $\mathrm{AD}_\beta[\mathbf{X}_p^d]$ *boasts the following properties:*

(1) *There exists a modification of* $\mathrm{AD}_\beta[\mathbf{X}_p^d](\mathbf{x})$ *in which every* $\lambda_j(\mathbf{x})$ *is a continuous function of* $\mathbf{x} \in \mathbb{R}^d$*.*

(2) *At a fixed* $\mathbf{x} \in \mathbb{R}^d$*,* $\mathrm{AD}_\beta[\mathbf{X}_p^d](\mathbf{x})$ *is equal in distribution to the Airy*$_\beta$ *point process* $\mathfrak{Ai}_\beta$*.*

(3) *The distribution of* $\mathrm{AD}_\beta[\mathbf{X}_p^d](\mathbf{x})$ *at a* k*-tuple of points* $(\mathbf{x}_q)_{q=1}^k$ *in* $\mathbb{R}^d$ *depends only on* β *and on the distances* $\|\mathbf{x}_q - \mathbf{x}_r\|_p$ *(*$1 \leq q < r \leq k$*).*

The last item implies that the distribution of the restriction of $\mathrm{AD}_\beta[\mathbf{X}_p^d]$ to a geodesic in ℓ_d^p does not depend on the choice of geodesic (and neither on p and d), and thus coincides in distribution with $\mathrm{AD}_\beta[\mathbf{X}^1]$.

We note that $\mathrm{AD}_2[\mathbf{X}^1]$ admits a concise determinantal description. Indeed, the $\beta = 2$ Dyson Brownian motion satisfies the assumptions of Theorem 3.4; thus its edge scaling limit is described by the process $\mathrm{AD}_2[\mathbf{X}^1]$. On the other hand, Macêdo [45] and Forrester, Nagao, and Honner [29] (see further Forrester [26, 7.1.5]) found this limit directly. This process, the moment measures of which are given by determinants, appeared again in the work of Prähofer and Spohn [55] on models of random growth (see the lecture notes of Johansson [38] for further limit theorems in which it appears); Corwin and Hammond [11] studied its properties, and coined the term 'Airy line ensemble'. Thus the distribution of the restriction of $\mathrm{AD}_2[\mathbf{X}_p^d]$ to any geodesic in ℓ_p^d is given by the Airy line ensemble.

3.2.1. Construction of the processes AD_β. With the exception of the case $\beta = 2$, $d = 1$, the process $\mathfrak{P} = \mathrm{AD}_\beta[\mathbf{X}_p^d]$ does not seem to be described by determinantal formulæ. The construction presented here is motivated by the combinatorial arguments of Soshnikov [64] and further by the work of Okounkov [50], and makes use of the results of [21].

Let $\mathfrak{P}(\mathbf{x})$ ($\mathbf{x} \in \mathbb{R}^d$) be a random process which takes values in point configurations on the line (i.e. locally finite sums of δ-functions). That is, for every $\mathbf{x} \in \mathbb{R}^d$ the random variable $\mathfrak{P}(\mathbf{x})$ is a point process on $\mathbb{R}$. Denote

$$\widetilde{\rho}_{\mathfrak{P},k}(\mathbf{x}_1, \cdots, \mathbf{x}_k) = \mathbb{E} \prod_{r=1}^{k} \mathfrak{P}(\mathbf{x}_r) \quad (\mathbf{x}_1, \cdots, \mathbf{x}_k \in \mathbb{R}^d) \tag{3.9}$$

be the moment measures of $\mathfrak{P}$, and

$$\widetilde{R}_{\mathfrak{P},k}(\mathbf{x}_1,\cdots,\mathbf{x}_k;\alpha_1,\cdots,\alpha_k)=\int\prod_{r=1}^{k}\frac{\sin\alpha_r\sqrt{-\lambda_r}}{\sqrt{-\lambda_r}}d\widetilde{\rho}_{\mathfrak{P},k}(\mathbf{x}_1,\cdots,\mathbf{x}_k;\lambda_1,\cdots,\lambda_k)$$

in our case, (the divergent integral can be regularised, cf. [60, Section 6]; the transform appears from the asymptotics of orthogonal polynomials, cf. (2.31)). Then, let

$$\widetilde{R}^{\#}_{\mathfrak{P},k}(\bar{\mathbf{x}};\alpha)=\sum_{I\subset\{1,\cdots,k\}}\widetilde{R}_{\mathfrak{P},\#I}(\bar{\mathbf{x}}|_I,\alpha|_I)\widetilde{R}_{\mathfrak{P},k-\#I}(\bar{\mathbf{x}}|_{I^c},\alpha|_{I^c})\,, \tag{3.10}$$

where $\bar{\mathbf{x}}=(\mathbf{x}_1,\cdots,\mathbf{x}_k)\in(\mathbb{R}^d)^k$, and $\bar{\mathbf{x}}|_I=(\mathbf{x}_r)_{r\in I}$. (The sum over partitions has to do with the contribution of the two spectral edges to the asymptotics.) We define AD_β via a formula (3.12) for $\widetilde{R}^{\#}_{\mathrm{AD}_\beta[\mathbf{X}_p^d],k}$ (uniqueness follows from the considerations of [60, Section 6]).

Let us consider the collection of k-tuples of non-backtracking walks for which every edge of the spanned graph $\mathfrak{g}$ (cf. Section 2.2) is traversed exactly twice, and every vertex in $\mathfrak{g}$ has degree at most three. Such k-tuples can be divided into topological equivalence classes (k-diagrams). For example, for $k=1$, the paths $1\,2\,3\,4\,5\,6\,7\,8\,4\,5\,6\,7\,8\,4\,3\,2\,1$ on Figure 2.1 (right) and the path $8\,7\,2\,3\,9\,2\,3\,9\,2\,7\,8$ belong to the same equivalence class, schematically depicted on Figure 3.1 (left). The formal definition is given in [21, 60].

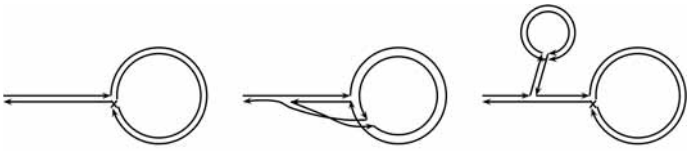


Figure 3.1. Three 1-diagrams. The diagram on the left corresponds to the projective plane with $s=1$ (left). The two diagrams in the centre and on the right correspond to surfaces with $s=2$; the one is the centre is the torus.

Different k-diagrams correspond to homotopically distinct ways to glue k disks with a marked point on the boundary. The result of such a gluing is a two-dimensional manifold. Thus to every k-diagram one can associate a number s, which is related to the Euler characteristic χ of the manifold by the formula $s=2k-\chi$; for $k=1$ the number s is the non-oriented genus. The multi-graph associated to a diagram with a certain value of s (see Figures 3.1 and 3.2) has $2s$ vertices and $3s-k$ edges. The number $D_k(s)$ of k-diagrams with a given value of s satisfies the estimates ([21, Proposition II.3.3])

$$\frac{(s/C)^{s+k-1}}{(k-1)!}\le D_k(s)\le\frac{(Cs)^{s+k-1}}{(k-1)!}\,; \tag{3.11}$$

the upper bound guarantees that the series (3.12) which we derive below converges.

Next, we associate to a k-diagram $\mathcal{D}$ and to $\alpha\in(0,\infty)^k$ a $(3s-2k)$-dimensional polytope $\Delta_{\mathcal{D}}(\alpha)$ in $\mathbb{R}^{3s-k}$, as follows. The variables $w(e)$ are labeled by the edges e of $\mathcal{D}$; the polytope is defined by the inequalities

$$\begin{cases} w(e)\ge 0 & (e\in\mathrm{Edges}(\mathcal{D}))\\ \displaystyle\sum_e c_r(e)w(e)=\alpha_r & (1\le r\le k)\end{cases}$$

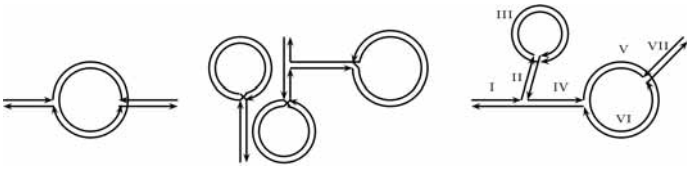


Figure 3.2. Three 2-diagrams: $s = 2$ (left), $s = 3$ (centre, right). The leftmost diagram, corresponding to a sphere glued from two disks, is often responsible for fluctuations of linear eigenvalue statistics on global and mesoscopic scales.

where $c_r(e) \in \{0, 1, 2\}$ is the number of times the edge e is traversed by the p-th path in the diagram. For example, the polytope associated with the rightmost 2-diagram of Figure 3.2 is given by

$$\begin{cases} w(\mathrm{I}), w(\mathrm{II}), \cdots, w(\mathrm{VII}) \geq 0 \\ 2w(\mathrm{I}) + 2w(\mathrm{II}) + 2w(\mathrm{III}) + 2w(\mathrm{IV}) + w(\mathrm{V}) + w(\mathrm{VI}) = \alpha_1 \\ w(\mathrm{V}) + w(\mathrm{VI}) + 2w(\mathrm{VII}) = \alpha_2 \, . \end{cases}$$

Let $\mathfrak{D}_1(k)$ be the collection of all k-diagrams, and let $\mathfrak{D}_2(k) \subset \mathfrak{D}_1(k)$ be the sub-collection of diagrams in which every edge is traversed once in one direction and once in another one (such as Figure 3.1, centre, and Figure 3.2, left; these diagrams correspond to gluings preserving orientation). Now we can finally write the series for $\widetilde{R}^{\#}$:

$$\begin{aligned} &\widetilde{R}^{\#}_{\mathrm{AD}_\beta, k}(\bar{\mathbf{x}}; \alpha) \\ &= \sum_{\mathcal{D} \in \mathfrak{D}_\beta} \int_{\Delta_{\mathcal{D}}(\alpha)} \exp \left\{ - \sum_{e \in \mathrm{Edges}(\mathcal{D})} \| \mathbf{x}_{r_+(e)} - \mathbf{x}_{r_-(e)} \|_p w(e) \right\} d \, \mathrm{mes}_{3s-2k}(w) \, , \end{aligned} \quad (3.12)$$

where $k \geq r_+(e) \geq r_-(e) \geq 1$ are the indices of the two paths traversing e in $\mathcal{D}$.

For example, when $\mathbf{X}$ is a singleton, all the terms in the exponent vanish, and (3.12) yields an expression for the Airy point process in terms of volumes of the polytopes $\Delta_{\mathcal{D}}(\alpha)$, which may be compared to the one given by Okounkov [50, §2.5.4].

3.3. Band matrices. In this section, we discuss an extension of Soshnikov's theorem to a class of matrices of the form considered in Section 2.2. First, we recall a conjecture, based on the Thouless criterion [69]. Then we discuss a particular case, the spectral edges of random band matrices, in which the conjecture can be proved. Finally, we comment on mesoscopic scales.

3.3.1. Thouless criterion. The Thouless criterion [69], originally introduced in the context of Anderson localisation, can be applied to predict the behaviour of local eigenvalue statistics; cf. Fyodorov and Mirlin [33, 34]. Consider a sequence of matrices $H^{(N)}$ associated with a sequence of graphs $G_N = (V_N, E_N)$ as in Section 2.2. Then the measures

$$\mu_N = \frac{1}{\# V_N} \sum_{j=1}^{N} \delta \left(\xi - \frac{\xi_j^{(N)}}{2\sqrt{2W_N}} \right)$$

converge to the semi-circle measure σ_{Wig}.

Let $\xi_0 \in \mathbb{R}$, and let $\eta_N > 0$ be chosen so that the sequence of (random) measures

$$\#V_N R^{\xi_0}_{\eta_N}[\mathbb{E}\mu_N] = \mathbb{E}\sum_{j=1}^{\#V_N} \delta\left(\lambda - \frac{1}{\eta_N}\left[\frac{\xi_j^{(N)}}{2\sqrt{2W_N - 1}} - \xi_0\right]\right)$$

will have a non-trivial vague limit (cf. Section 2.5). Thus chosen, η_N measures the mean spacing between eigenvalues, whereas

$$(\text{spacing/DOS})(\xi_0) = \eta_N^2 \#V_N$$

measures the mean spacing in units of the density of states. Let us compare the inverse of this quantity with the mixing time T^{mix} of the random walk on G_N.[2] In many cases the following seems to be correct: the eigenvalue statistics of $H^{(N)}$ near ξ_0 are described by random matrix theory if and only if

$$T^{\text{mix}}(G_N) \ll \frac{1}{(\text{spacing/DOS})(1)} \,. \tag{3.13}$$

This interpretation of the Thouless criterion is based on the assumption that the semi-classical approximation is valid up to the scales governing the local eigenvalue statistics; we refer to the reviews of Spencer [66, 67] for a discussion of various aspects of Thouless scaling and its mathematical justification, and to the work of Spencer and Wang [74] for some rigorous results. Here we focus our attention on the particular case of

3.3.2. Random band matrices. Denote $\|u - v\|_N = \min_{\ell \in \mathbb{Z}} |u - v - \ell N|$. A (one-dimensional) random band matrix of bandwidth W is for us a random Hermitian $N \times N$ matrix $H^{(N)} = (H(u,v))_{1 \le u,v \le N}$ such that

$$\begin{cases} H(u,v) = 0 \,, & \|u - v\|_N > W \,, \\ H(u,v) \sim \mathcal{L}_{\text{off-diag}} \,, & 1 \le \|u - v\|_N \le W \,, \\ H(u,u) \sim \mathcal{L}_{\text{diag}} \,, \end{cases} \tag{3.14}$$

where $\mathcal{L}_{\text{diag}}$ and $\mathcal{L}_{\text{off-diag}}$ satisfy the normalisation conditions (2.9) and (2.10), respectively. In the setting of Section 2.2, it corresponds to the graph $G_N = (V_N, E_N)$,

$$V_N = \{1, \cdots, N\} \,, \quad (u,v) \in E_N \iff 1 \le \|u - v\|_N \le W_N \,, \tag{3.15}$$

More general band matrices are discussed, for example, in [17, 39, 67].

For $-1 < \xi_0 < 1$ (the bulk of the spectrum),

$$(\text{spacing/DOS})(\xi_0) \asymp \frac{1}{N} \,, \quad T^{\text{mix}} \asymp \frac{N^2}{W^2} \,,$$

therefore the criterion (3.13) suggests the following: the eigenvalue statistics of $H^{(N)}$ near ξ_0 are described by random matrix theory if and only if $W \gg \sqrt{N}$. This prediction is supported by the detailed super-symmetric analysis performed by Fyodorov and Mirlin [33, 34]. Mathematical justification remains a major challenge, cf. Spencer [67, 68] and references therein.

[2] equivalently, the ratio of the mixing time and the density of states, which is interpreted as the energy-dependent mixing time, is compared to the usual inverse eigenvalue spacing η_N.

3.3.3. Spectral edges. The (modified) moment method allows to confirm the criterion (3.13) at the spectral edges of random band matrices.

Theorem 3.6 (cf. [60, Theorem 1.1]). *Let $(H^{(N)})_{N\geq 1}$ be a sequence of random band matrices satisfying the unimodality assumptions (2.27). If the bandwidth W_N of $H^{(N)}$ satisfies*

$$\lim_{N\to\infty} \frac{W_N}{N^{5/6}} = \infty \,, \tag{3.16}$$

then

$$\sum_{j=1}^{N} \delta\left(\lambda - \frac{N^{2/3}}{\sqrt{2W_N}}\left[\xi_j^{(N)} - 2\sqrt{2W_N}\right]\right) \to \mathfrak{Ai}_\beta \,,$$

where $\beta = 1$ if $\operatorname{supp} \mathcal{L}_{\text{off-diag}} \subset \mathbb{R}$, and $\beta = 2$ otherwise.

The threshold $N^{5/6}$ in (3.16) is sharp, see [60, Theorem 1.2]. The same [60, Theorem 1.2] implies that

$$\eta_N \asymp \min(W_N^{2/5} N^{-1}, N^{-2/3}) \,,$$

therefore

$$(\text{spacing}/\text{DOS})(1) \asymp \min(W_N^{4/5} N^{-1}, N^{-1/3}) \,,$$

and (3.16) is consistent with (3.13).

The unimodality conditions (2.27) simplify the analysis (cf. Proposition 2.1); we expect that they can be relaxed using the methods of [21, Part III] and [17].

3.3.4. Mesoscopic scales. On mesoscopic scales $1 \gg \epsilon_N \gg 1/\#V_N$, the following counterpart of the Thouless criterion goes back to the (physical) work of Altshuler and Shklovskii [2]. Let η_N be such that the sequence $(\epsilon_N^{-1} R_{\eta_N}[\mathbb{E}\mu_N])_N$ has a non-trivial vague limit. If

$$\epsilon_N \eta_N^{-2} \gg T^{\text{mix}} \,, \tag{3.17}$$

the fluctuations of linear eigenvalue statistics should be described by a log-correlated Gaussian field, whereas when (3.17) is violated, one expects a more regular field depending on the geometry of the underlying lattice. We refer to the works of Fyodorov, Le Doussal, and Rosso [32] and of Fyodorov and Keating [30] for a discussion of the significance of log-correlated fields within and outside random matrix theory, and to the work of Fyodorov, Khoruzhenko, and Simm [31] for results pertaining to the Gaussian Unitary Ensemble.

Erdős and Knowles proved a series of results pertaining to mesoscopic statistics for a wide class of d-dimensional band matrices. In the works [16, 17], they developed a moment-based approach which allowed them to control the quantum dynamics associated for time scales $t \leq W_N^{d/3-\delta}$. In [18, 19], they gave mathematical justification to the criterion (3.17) in the range $\epsilon_N \geq W_N^{-d/3+\delta}$. It would be interesting to extend the results of [16, 17] and [18, 19] to the full mesoscopic range.

4. Some further questions

Other limiting measures. The spectral measures in this article converge to the semicircle distribution $\mu_\infty = \sigma_{\text{Wig}}$. The modified moment method described here has been also applied to the Kesten–McKay measure (the orthogonality measure for $P_n^{(\kappa)}$), the Godsil–Mohar

measure (its bipartite analogue), and the Marchenko–Pastur measure (the infinite connectivity limit of the Godsil–Mohar measure); see e.g. [58,59]. It would be interesting to adapt the method to situations in which the recurrent relation has less explicit form.

β-ensembles. The (convincing, although so far unrigorous) ghost and shadows formalism introduced by Edelman [14] strongly suggests that the construction (3.12) should have an extension to general $\beta > 0$. See Forrester [26] and [27] for background on β-ensembles, and Borodin and Gorin [10] for a recent result pertaining to the spectral statistics of submatrices of β-Jacobi random matrices.

Time-dependent invariant ensembles. It seems plausible that, for general (non-Gaussian) invariant ensembles undergoing Dyson-type evolution, the spectral statistics near a soft edge should be described by the processes AD_β of Section 3.2.1. Currently, there seem to be no proved results of this form (even for the case $\beta = 2$ in which determinantal formalæ for finite matrix size are given by the Eynard–Mehta theorem [49, Chapter 23]).

Beyond random matrices. Motivated by the proof of the Baik–Deift–Johansson conjecture given by Okounkov [50], one may look for the appearance of (3.12) outside random matrix theory, particularly, in the context of random growth models, for a discussion of the subtle connection between which and random matrix theory we refer to the lecture notes of Ferrari [24].

Bulk of the spectrum. We are not aware of any derivation (rigorous or not) of the local eigenvalue statistics in the bulk of the spectrum using any version of the moment method. Even for the test case of the Gaussian Unitary Ensemble (tractable by other means), perturbative methods such as Chebyshev expansions have not been of use beyond the scales $\epsilon_N \gg N^{-1+\delta}$. For random band matrices the expansion (2.34) has been only regularised for $\epsilon_N \gg W^{-1+\delta}$ (see [61]).

Acknowledgment. It is a great pleasure to thank Leonid Pastur, who introduced me to the theory of random matrices and encouraged me to work on local eigenvalue statistics, and Tom Spencer, who spent an immeasurable amount of time to share with me his knowledge, intuition, and taste in various parts of mathematical physics.

A significant part of the results stated here are based on the joint work [21,22] with Ohad Feldheim. Yan Fyodorov explained me the relation between (3.16) and Thouless scaling. The results in Section 3.2 are motivated by the talks given by Alexei Borodin at the IAS, and by the subsequent discussions with him and with Vadim Gorin. Yan Fyodorov, Vadim Gorin, Antti Knowles, Alon Nishry, Mira Shamis, Misha Sodin, and Ofer Zeitouni kindly commented on a preliminary version of this text. I thank them very much.

The author is supported in part by NSF grant PHY-1305472.

References

[1] N. I. Akhiezer, *The classical moment problem and some related questions in analysis*, Translated by N. Kemmer, Hafner Publishing Co., New York, 1965.

[2] B. L. Altshuler and B. I. Shklovskii, *Repulsion of energy levels and conductivity of small metal samples*, Sov. Phys. JETP, **64** (1986), no. 1, 127–135.

[3] G. W. Anderson, A. Guionnet, and O. Zeitouni, *An introduction to random matrices*, Cambridge Studies in Advanced Mathematics, vol. 118, Cambridge University Press, Cambridge, 2010.

[4] Z. D. Bai and Y. Q. Yin, *Limit of the smallest eigenvalue of a large-dimensional sample covariance matrix*, Ann. Probab, **21** (1993), no. 3, 1275–1294.

[5] Ch. Berg, *The multidimensional moment problem and semigroups*, Moments in mathematics (San Antonio, Tex., 1987), Proc. Sympos. Appl. Math., vol. 37, Amer. Math. Soc., Providence, RI, 1987, pp. 110–124.

[6] L. V. Bogachev, S. A. Molchanov, and L. A. Pastur, *On the density of states of random band matrices*, Mat. Zametki, **50** (1991), no. 6, 31–42, 157 (Russian); English transl., Math. Notes, **50** (1991), no. 5-6, 1232–1242 (1992).

[7] A. Borodin, *CLT for spectra of submatrices of Wigner random matrices*, available at arXiv:1010.0898.

[8] ———, *CLT for spectra of submatrices of Wigner random matrices II. Stochastic evolution*, available at arXiv:1011.3544.

[9] A. Borodin and V. Gorin, *Lectures on integrable probability* (2012), available at arXiv: 1212.3351.

[10] ———, *General beta Jacobi corners process and the Gaussian Free Field*, available at arXiv:1305.3627.

[11] I. Corwin and A. Hammond, *Brownian Gibbs property for Airy line ensembles*, Inventiones mathematicae (2013), 1–68 (English).

[12] S. A. Denisov, *Continuous analogs of polynomials orthogonal on the unit circle and Kreĭn systems*, IMRS Int. Math. Res. Surv. (2006), Art. ID 54517, 148.

[13] P. Diaconis, *Application of the method of moments in probability and statistics*, Moments in mathematics (San Antonio, Tex., 1987), Proc. Sympos. Appl. Math., vol. 37, Amer. Math. Soc., Providence, RI, 1987, pp. 125–142.

[14] A. Edelman, *The random matrix technique of ghosts and shadows*, Markov Process. Related Fields, **16** (2010), no. 4, 783–790.

[15] L. Erdësh [Erdős], *Universality of Wigner random matrices: a survey of recent results*, Uspekhi Mat. Nauk **66** (2011), no. 3(399), 67–198 (Russian, with Russian summary); English transl., Russian Math. Surveys, **66** (2011), no. 3, 507–626.

[16] L. Erdős and A. Knowles, *Quantum diffusion and eigenfunction delocalization in a random band matrix model*, Comm. Math. Phys., **303** (2011), no. 2, 509–554.

[17] ———, *Quantum diffusion and delocalization for band matrices with general distribution*, Ann. Henri Poincaré, **12** (2011), no. 7, 1227–1319.

[18] ———, *The Altshuler-Shklovskii formulas for random band matrices I: the unimodular case* (2013), available at arXiv:1309.5106.

[19] ———, *The Altshuler-Shklovskii formulas for random band matrices II: the general case* (2013), available at arXiv:1309.5107.

[20] P. Erdős and P. Turán, *On a problem in the theory of uniform distribution. I–II*, Nederl. Akad. Wetensch., Proc. **51** (1948), 1146–1154 = Indagationes Math. 10, 370–378 (1948), 1262–1269 = Indagationes Math. 10, 406–413 (1948).

[21] O. N. Feldheim and S. Sodin, *A universality result for the smallest eigenvalues of certain sample covariance matrices*, Geom. Funct. Anal., **20** (2010), no. 1, 88–123.

[22] ———, *One more proof of the Erdős-Turán inequality, and an error estimate in Wigner's law*, Concentration, functional inequalities and isoperimetry, Contemp. Math., vol. 545, Amer. Math. Soc., Providence, RI, 2011, pp. 69–75.

[23] W. Feller, *An introduction to probability theory and its applications. Vol. II.*, Second edition, John Wiley & Sons Inc., New York, 1971.

[24] P. L. Ferrari, *Why random matrices share universal processes with interacting particle systems?*, available at arXiv:1312.1126.

[25] P. J. Forrester, *The spectrum edge of random matrix ensembles*, Nuclear Phys. B, **402** (1993), no. 3, 709–728.

[26] ———, *Log-gases and random matrices*, London Mathematical Society Monographs Series, vol. 34, Princeton University Press, Princeton, NJ, 2010.

[27] ———, *Beta ensembles*, The Oxford handbook of random matrix theory, Oxford Univ. Press, Oxford, 2011, pp. 415–432.

[28] P. J. Forrester and T. Nagao, *Multilevel dynamical correlation functions for Dyson's Brownian motion model of random matrices*, Physics Letters A, **247** (1998), no. 1, 42–46.

[29] P. J. Forrester, T. Nagao, and G. Honner, *Correlations for the orthogonal-unitary and symplectic-unitary transitions at the hard and soft edges*, Nuclear Phys. B, **553** (1999), no. 3, 601–643.

[30] Y.V. Fyodorov and J. P. Keating, *Freezing transitions and extreme values: random matrix theory, and disordered landscapes*, Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences, **372** (2014), no. 2007, 20120503.

[31] Y. V Fyodorov, B. A Khoruzhenko, and N. J. Simm, *Fractional Brownian motion with Hurst index $H = 0$ and the Gaussian Unitary Ensemble*, available at arXiv:1312.0212.

[32] Y. V. Fyodorov, P. Le Doussal, and A. Rosso, *Counting function fluctuations and extreme value threshold in multifractal patterns: the case study of an ideal* $1/f$ *noise*, J. Stat. Phys., **149** (2012), no. 5, 898–920.

[33] Y. V. Fyodorov and A. D. Mirlin, *Scaling properties of localization in random band matrices: a* σ*-model approach*, Phys. Rev. Lett., **67** (1991), no. 18, 2405–2409.

[34] _____, *Statistical properties of eigenfunctions of random quasi 1d one-particle Hamiltonians*, International Journal of Modern Physics B, **8** (1994), no. 27, 3795–3842.

[35] W. Gautschi, *On the construction of Gaussian quadrature rules from modified moments.*, Math. Comp., **24** (1970), 245–260.

[36] M. Gromov, *Groups of polynomial growth and expanding maps*, Inst. Hautes Études Sci. Publ. Math., **53** (1981), 53–73.

[37] I. A. Ibragimov and Yu. V. Linnik, *Independent and stationary sequences of random variables*, Wolters-Noordhoff Publishing, Groningen, 1971. With a supplementary chapter by I. A. Ibragimov and V. V. Petrov; Translation from the Russian edited by J. F. C. Kingman.

[38] K. Johansson, *Random matrices and determinantal processes*, Mathematical statistical physics, Elsevier B. V., Amsterdam, 2006, pp. 1–55.

[39] A. M. Khorunzhiĭ [Khorunzhiy], S. A. Molchanov, and L. A. Pastur, *Distribution of the eigenvalues of random band matrices in the limit of their infinite order*, Teoret. Mat. Fiz., **90** (1992), no. 2, 163–178 (Russian, with English and Russian summaries); English transl., Theoret. and Math. Phys. **90** (1992), no. 2, 108–118.

[40] M Krawtchouk, *Sur le problème de moments*, ICM Proceedings, Zürich, 1932, available at http://www.mathunion.org/ICM/ICM1932.2/Main/icm1932.2.0127.0128.ocr.pdf.

[41] J. O. Lee and J. Yin, *A Necessary and Sufficient Condition for Edge Universality of Wigner matrices*, Duke Math. J., **163** (2014), no. 1, 117–173.

[42] B. M. Levitan, *On a uniqueness theorem*, Doklady Akad. Nauk SSSR (N.S.), **76** (1951), 485–488 (Russian).

[43] B. M. Levitan and N. N. Meĭman [Meiman], *On a uniqueness theorem*, Doklady Akad. Nauk SSSR (N.S.), **81** (1951), 729–731 (Russian).

[44] W.-Ch. W. Li and P. Solé, *Spectra of regular graphs and hypergraphs and orthogonal polynomials*, European J. Combin., **17** (1996), no. 5, 461–477.

[45] A. M. S. Macêdo, *Universal parametric correlations at the soft edge of the spectrum of random matrix ensembles*, EPL (Europhysics Letters), **26** (1994), no. 9, 641.

[46] A. A. Markov, *Calculus of probabilities*, 1913 (Russian).

[47] V. A. Marčenko [Marchenko] and L. A. Pastur, *The spectrum of random matrices*, Teor. Funkciĭ Funkcional. Anal. i Priložen. Vyp., **4** (1967), 122–145 (Russian).

[48] _____, *Distribution of eigenvalues in certain sets of random matrices*, Mat. Sb. (N.S.), **72 (114)** (1967), 507–536 (Russian).

[49] M. L. Mehta, *Random matrices*, 3rd ed., Pure and Applied Mathematics (Amsterdam), vol. 142, Elsevier/Academic Press, Amsterdam, 2004.

[50] A. Okounkov, *Random matrices and random permutations*, Internat. Math. Res. Notices, **20** (2000), 1043–1095.

[51] L. A. Pastur, *The spectrum of random matrices*, Teoret. Mat. Fiz., **10** (1972), no. 1, 102–112 (Russian, with English summary).

[52] _____, *Spectra of random selfadjoint operators*, Uspehi Mat. Nauk, **28** (1973), no. 1(169), 3–64 (Russian).

[53] L. Pastur and M. Shcherbina, *Eigenvalue distribution of large random matrices*, Mathematical Surveys and Monographs, vol. 171, American Mathematical Society, Providence, RI, 2011.

[54] S. Péché, *Universality results for the largest eigenvalues of some sample covariance matrix ensembles*, Probab. Theory Related Fields, **143** (2009), no. 3-4, 481–516.

[55] M. Prähofer and H. Spohn, *Scale invariance of the PNG droplet and the Airy process*, J. Statist. Phys., **108** (2002), no. 5-6, 1071–1106. Dedicated to David Ruelle and Yasha Sinai on the occasion of their 65th birthdays.

[56] Ya. Sinai and A. Soshnikov, *Central limit theorem for traces of large random symmetric matrices with independent matrix elements*, Bol. Soc. Brasil. Mat. (N.S.), **29** (1998), no. 1, 1–24.

[57] Ya. G. Sinaĭ [Sinai] and A. B. Soshnikov, *A refinement of Wigner's semicircle law in a neighborhood of the spectrum edge for random symmetric matrices*, Funktsional. Anal. i Prilozhen., **32** (1998), no. 2, 56–79, 96 (Russian, with Russian summary); English transl., Funct. Anal. Appl. **32** (1998), no. 2, 114–131.

[58] S. Sodin, *Random matrices, nonbacktracking walks, and orthogonal polynomials*, J. Math. Phys., **48** (2007), no. 12, 123503, 21.

[59] _____, *The Tracy-Widom law for some sparse random matrices*, J. Stat. Phys., **136** (2009), no. 5, 834–841.

[60] _____, *The spectral edge of some random band matrices*, Ann. of Math. (2), **172** (2010), no. 3, 2223–2251.

[61] _____, *An estimate for the average spectral measure of random band matrices*, J. Stat. Phys., **144** (2011), no. 1, 46–59.

[62] ———, *A limit theorem at the spectral edge for corners of time-dependent Wigner matrices*, available at arXiv:1312.1007.

[63] N. Ya. Sonin, *On the precision of determination of the limiting values of integrals*, Mémoires Russian Academy of Sciences, **69** (1892), 1–30 (Russian).

[64] A. Soshnikov, *Universality at the edge of the spectrum in Wigner random matrices*, Comm. Math. Phys., **207** (1999), no. 3, 697–733.

[65] ———, *A note on universality of the distribution of the largest eigenvalues in certain sample covariance matrices*, J. Statist. Phys., **108** (2002), no. 5-6, 1033–1056. Dedicated to David Ruelle and Yasha Sinai on the occasion of their 65th birthdays.

[66] T. Spencer, *Mathematical aspects of Anderson localization*, Internat. J. Modern Phys. B, **24** (2010), no. 12-13, 1621–1639.

[67] ———, *Random banded and sparse matrices*, The Oxford handbook of random matrix theory, Oxford Univ. Press, Oxford, 2011, pp. 471–488.

[68] ———, *SUSY statistical mechanics and random band matrices*, Quantum many body systems, Lecture Notes in Math., vol. 2051, Springer, Heidelberg, 2012, pp. 125–177.

[69] D. J. Thouless, *Maximum metallic resistance in thin wires*, Physical Review Letters **39.18** (1977), 1167–1169.

[70] C. A. Tracy and H. Widom, *Level-spacing distributions and the Airy kernel*, Comm. Math. Phys., **159** (1994), no. 1, 151–174.

[71] ———, *On orthogonal and symplectic matrix ensembles*, Comm. Math. Phys., **177** (1996), no. 3, 727–754.

[72] ———, *Distribution functions for largest eigenvalues and their applications*, Proceedings of the International Congress of Mathematicians, Vol. I. (Beijing, 2002), Higher Ed. Press, Beijing, 2002, pp. 587–596.

[73] E. B. Vul, *Uniqueness theorems for a certain class of functions represented by integrals*, Dokl. Akad. Nauk SSSR, **129** (1959), 722–725 (Russian).

[74] W.-M. Wang, *On localization and density of states for the random Schrödinger operator*, Ph.D. thesis, Princeton University, 1992.

[75] E. P. Wigner, *Characteristic vectors of bordered matrices with infinite dimensions*, Ann. of Math. (2), **62** (1955), 548–564.

[76] ———, *Characteristic vectors of bordered matrices with infinite dimensions. II*, Ann. of Math. (2), **65** (1957), 203–207.

[77] ———, *On the distribution of the roots of certain symmetric matrices*, Ann. of Math. (2), **67** (1958), 325–327.

[78] H. Zessin, *The method of moments for random measures*, Z. Wahrsch. Verw. Gebiete, **62** (1983), no. 3, 395–409.

Department of Mathematics, Princeton University, Fine Hall, Washington Road, Princeton, NJ 08544-1000, USA; School of Mathematical Sciences, Tel Aviv University, Tel Aviv 6997801, Israel
E-mail: asodin@princeton.edu

Free probability and random matrices

Roland Speicher

Abstract. The concept of freeness was introduced by Voiculescu in the context of operator algebras. Later it was observed that it is also relevant for large random matrices. We will show how the combination of various free probability results with a linearization trick allows to address successfully the problem of determining the asymptotic eigenvalue distribution of general selfadjoint polynomials in independent random matrices.

Mathematics Subject Classification (2010). Primary 46L54; Secondary 60B20.

Keywords. free probability, random matrices, linearization trick, free cumulants, operator-valued free probability.

1. Introduction

Free probability theory was introduced by Voiculescu around 1983 in order to attack the isomorphism problem of von Neumann algebras of free groups. Voiculescu isolated a structure showing up in this context which he named "freeness". His fundamental insight was to separate this concept from its operator algebraic origin and investigate it for its own sake. Furthermore, he promoted the point of view that freeness should be seen as an (though noncommutative) analogue of the classical probabilistic concept of "independence" for random variables. Hence freeness is also called "free independence" and the whole subject became to be known as "free probability theory".

The theory was lifted to a new level when Voiculescu discovered in 1991 that the freeness property is also present for many classes of random matrices, in the asymptotic regime when the size of the matrices tends to infinity. This insight, bringing together the apriori entirely different theories of operator algebras and of random matrices, had quite some impact in both directions. Modelling operator algebras by random matrices resulted in some of the deepest results about operator algebras of the last decades; whereas tools developed in operator algebras and free probability theory could now be applied to random matrix problems, yielding in particular new ways to calculate the asymptotic eigenvalue distribution of many random matrices. Since random matrices are also widely used in applied fields, like wireless communications or statistics, free probability is now also quite common in those subjects.

Whereas Voiculescu's original approach to free probability is quite analytic and operator algebraic in nature, I provided another, more combinatorial, approach. This rests on the notion of "free cumulants" and is intimately connected with the lattice of "non-crossing partitions".

Proceedings of the International Congress of Mathematicians, Seoul, 2014

In this lecture we will give an introduction to free probability theory, with focus on its random matrix and combinatorial side. Freeness will be motivated not by its initial occurrence in operator algebras, but by its random matrix connection. The main result we are aiming at is also a very general random matrix problem, namely how to calculate the distribution of selfadjoint polynomials in independent random matrices. Whereas there exists a vast amount of literature on solving this problem for various special cases, often in an ad hoc way, we will see that free probability gives a conceptual way to attack this problem in full generality.

For more information on other aspects of the subject one might consult the earlier ICM contributions of Voiculescu [47], Haagerup [21], and Shlyakhtenko [34] (for the operator algebraic aspects of free probability) or of Biane [14] (for applications to the asymptotics of representations of symmetric groups). Extensions of the theory to rectangular matrices can be found in [11], and to "second order freeness" (describing fluctuations of random matrices) in [16]. The monographs [23, 27, 30, 51] give general introductions to free probability; [30] has its main emphasis on the combinatorial side of the subject. The applied side of free probability is addressed, for example, in [17, 31, 40].

2. Motivation of freeness via random matrices

In this chapter we want to motivate the definition of freeness on the basis of random matrices.

2.1. Asymptotic eigenvalue distribution of random matrices. We are interested in computing the eigenvalue distribution of $N \times N$ random matrices as $N \to \infty$. Here and in the following we will only consider selfadjoint random matrices. This guarantees that the eigenvalues are real and strong analytical tools are available to deal with such situations. For non-selfadjoint matrices the eigenvalues are in general complex and the situation, especially in the case of non-normal matrices, is more complicated. We will make some remarks on this situation at the very end of our lecture, in Sect. 9.

The typical feature for many basic random matrix ensembles is the almost sure convergence to some limiting eigenvalue distribution. Furthermore, quite often this limiting distribution can be effectively calculated.

Example 2.1. We consider an $N \times N$ *Gaussian random matrix*. This is a selfadjoint matrix $X_N = \frac{1}{\sqrt{N}}(x_{ij})_{i,j=1}^N$ such that the entries $\{x_{ij}\}_{i \geq j}$ are independent and identically distributed complex (real for $i = j$) Gaussian random variables with mean $E[x_{ij}] = 0$ and variance $E[x_{ij}\bar{x}_{ij}] = 1$.

The following figure shows typical histograms of the eigenvalues of Gaussian random matrices, for different values of N.

One sees that, whereas for small N there is no clear structure, for large N the eigenvalue histogram is approaching a smooth curve. Actually, this curve is deterministic, it is (almost surely) always the same, independent of the actual realized matrix from the ensemble. What we see here, is one of the first and most famous results of random matrix theory: for such matrices we have almost sure convergence to *Wigner's semicircle law*, given by the densitiy $\rho(t) = \frac{1}{2\pi}\sqrt{4 - t^2}$. In Figure 2.2 we compare one realization of a 4000×4000 Gaussian random matrix with the semicircle.

Example 2.2. An other important class of random matrices are*Wishart matrices*; those are

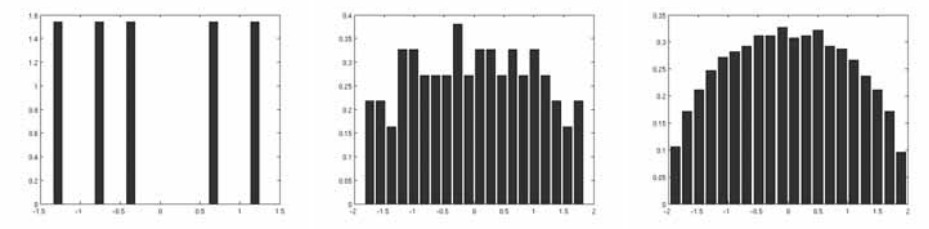


Figure 2.1. Histogram of the N eigenvalues for a realization of an $N \times N$ Gaussian random matrix; for N= 5, 100, 1000

of the form $X_N = A_N A_N^*$, where A_N is an $N \times M$ matrix with independent Gaussian entries. If we keep the ratio M/N fixed, its eigenvalue distribution converges almost surely to the *Marchenko-Pastur distribution*; see Figure 2.2.

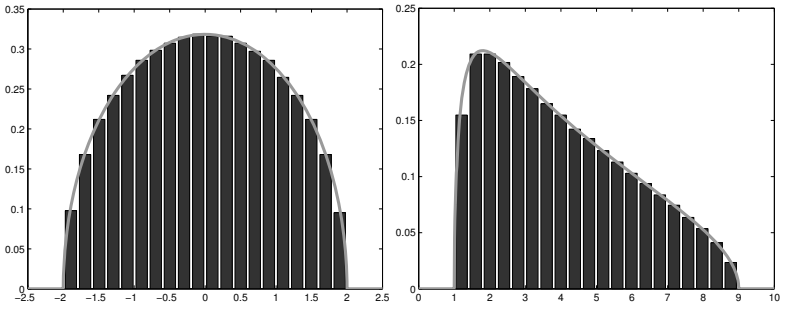


Figure 2.2. Comparision between the histogram of one realization of a random matrix and the analytic form of the density in the $N \to \infty$ limit; left: 4000×4000 Gaussian random matrix versus semicircle distribution; right: Wishart random matrix with $N = 3000$, $M = 12000$ versus the corresponding Marchenko-Pastur distribution

2.2. Polynomials in several random matrices. Instead of looking at one-matrix ensembles we are now interested in the case of several matrices. Let us consider two sequences X_N and Y_N of selfadjoint $N \times N$ matrices such that both sequences have an asymptotic eigenvalue distribution for $N \to \infty$. We are interested in the asymptotic eigenvalue distribution of sequences $p(X_N, Y_N)$ for some non-trivial functions p of two non-commuting variables. We will restrict to the simplest class of functions, namely p will be a (non-commutative) polynomial. As mentioned before, we are only dealing with selfadjoint matrices, thus p should be a selfadjoint polynomial in order to ensure that also $p(X_N, Y_N)$ is selfadjoint.

In general, the distribution of $p(X_N, Y_N)$ will depend on the relation between the eigenspaces of X_N and of Y_N. However, by the concentration of measure phenomenon, we expect that for large N this relation between the eigenspaces concentrates on *typical* or *generic positions*, and that then the asymptotic eigenvalue distribution of $p(X_N, Y_N)$ depends in a deterministic way only on the asymptotic eigenvalue distribution of X_N and on the asymptotic eigenvalue distribution of Y_N. Free probability theory replaces this vague notion of *generic position* by the mathematical precise concept of *freeness* and provides general tools for calculating the asymptotic distribution of $p(X_N, Y_N)$ out of the asymptotic distribution

of X_N and the asymptotic distribution of Y_N.

One can convince oneself easily of the almost sure convergence to a deterministic limit by simulations. Two examples are shown in Figure 2.3. Actually, usually it is also not too hard to prove this almost sure convergence by appropriate variance estimates. However, what is not clear at all is the calculation of the form of this limit shape in general. In some cases, like the left example of Figure 2.3, this was known, but in others, like the right example of Figure 2.3, no solution was known.

Our goal is to get a *conceptual way of understanding* the asymptotic eigenvalue distributions in general and also to find an *algorithm for calculating* the corresponding asymptotic eigenvalue distributions.

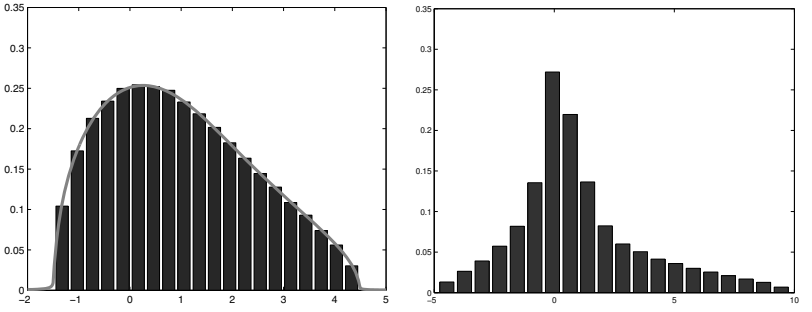


Figure 2.3. Histogram for a generic realization of a 3000×3000 random matrix $p(X, Y)$, where X and Y are independent Gaussian and, respectively, Wishart random matrices: $p(X, Y) = X + Y$ (left); $p(X, Y) = XY + YX + X^2$ (right). In the left case, the asymptotic eigenvalue distribution is relatively easy to calculate; in the right case, no such solution was known, this case will be reconsidered in Figure 8.1.

2.3. The moment method. There are different methods to analyze limit distributions of random matrices. One technique, analytical in nature, is the so called resolvent method. The main idea of this method is to derive an equation for the resolvent of the limit distribution. The advantage of this method is that there is a powerful complex analysis machinery to deal with such equations. This method also allows to look at eigenvalue distributions without finite moments. Its drawback, however, is that one cannot deal uniformly with all polynomials in X and Y; one has to treat each $p(X, Y)$ separately. On the other side, there is a more combinatorical technique, the so-called *moment method*, for calculating the limit distribution. This has the advantage that it allows, in principle, to deal in a uniform way with all polynomials in X and Y. In the following we will first concentrate on the moment method in order to motivate the concept of freeness. Later we will then come back to more analytic questions.

By $\mathrm{tr}(A)$ we denote the normalized trace of an $N \times N$ matrix A. If we want to understand the eigenvalue distribution of a selfadjoint matrix A, it suffices to know the trace $\mathrm{tr}(A^k)$ of all powers of A: because of the invariance of the trace under conjugation with unitaries, we have for $k \in \mathbb{N}$ that $\frac{1}{N}(\lambda_1^k + \cdots + \lambda_N^k) = \mathrm{tr}(A^k)$, where λ_i are the eigenvalues of A. Therefore, instead of studying the eigenvalue distribution of a matrix A directly, the moment method looks at traces of powers, $\mathrm{tr}(A^k)$.

Consider now our sequences of random matrices X_N and Y_N, each of which is assumed

to have almost surely an asymptotic eigenvalue distribution. We want to understand, in the limit $N \to \infty$, the eigenvalue distribution of $p(X_N, Y_N)$, not just for one p, but for all non-commutative polynomials. By the moment method, this asks for the investigation of the limit $N \to \infty$ of $\mathrm{tr}\big(p(X_N, Y_N)^k\big)$ for all $k \in \mathbb{N}$ and all polynomials p. Then it is clear that the basic objects which we have to understand in this approach are the asymptotic *mixed moments*

$$\lim_{N\to\infty} \mathrm{tr}(X_N^{n_1} Y_N^{m_1} \cdots X_N^{n_k} Y_N^{m_k}) \qquad (k \in \mathbb{N};\, n_1, \ldots, n_k, m_1, \ldots, m_k \in \mathbb{N}). \tag{2.1}$$

Thus our fundamental problem is the following. If X_N and Y_N each have an asymptotic eigenvalue distribution, and if X_N and Y_N are in generic position, do the asymptotic mixed moments (2.1) exist? If so, can we express them in a deterministic way in terms of the *individual moments*

$$\Big(\lim_{N\to\infty} \mathrm{tr}(X_N^k)\Big)_{k\in\mathbb{N}} \qquad \text{and} \qquad \Big(\lim_{N\to\infty} \mathrm{tr}(Y_N^k)\Big)_{k\in\mathbb{N}} ?$$

In order to get an idea how this might look like in a generic situation, we will consider the simplest case of two such random matrices.

2.4. The example of two independent Gaussian random matrices. Consider, for example, $N \times N$ random matrices X_N and Y_N such that X_N and Y_N have asymptotic eigenvalue distributions for $N \to \infty$, X_N and Y_N are independent (i.e., the entries of X_N are independent from the entries of Y_N) and Y_N is an unitarily invariant ensemble (i.e., the joint distribution of its entries does not change under unitary conjugation and thus, for any unitary $N \times N$-matrix U_N, $U_N Y_N U_N^*$ is equivalent to the original ensemble Y_N in all relevant aspects). But then we can use this U_N to rotate the eigenspaces of Y_N against those of X_N into a generic position, thus for typical realizations of X_N and Y_N the eigenspaces should be in a generic position.

The simplest example of two such random matrix ensembles are two independent Gaussian random matrices X_N and Y_N. In this case one can calculate everything concretely: in the limit $N \to \infty$, $\mathrm{tr}(X_N^{n_1} Y_N^{m_1} \cdots X_N^{n_k} Y_N^{m_k})$ is almost surely given by the number of non-crossing (aka planar) pairings of the word

$$\underbrace{X \cdot X \cdots X}_{n_1\text{-times}} \cdot \underbrace{Y \cdot Y \cdots Y}_{m_1\text{-times}} \cdots \underbrace{X \cdot X \cdots X}_{n_k\text{-times}} \cdot \underbrace{Y \cdot Y \cdots Y}_{m_k\text{-times}},$$

in two letters X and Y, such that no X is paired with a Y. A pairing is a decomposition of the word into pairs of letters; if we connect the two letters from each pair by a line, drawn in the half-plane below the word, then "non-crossing" means that we can do this without getting crossings between lines for different pairs.

For example, we have $\lim_{N\to\infty} \mathrm{tr}(X_N X_N Y_N Y_N X_N Y_N Y_N X_N) = 2$ because there are two such non-crossing pairings:

After some contemplation, one realizes that the above combinatorial description of the limit of $\mathrm{tr}(X_N^{n_1} Y_N^{m_1} \cdots X_N^{n_k} Y_N^{m_k})$ implies that the trace of a corresponding product of centered powers,

$$\lim_{N\to\infty} \mathrm{tr}\Big(\big(X_N^{n_1} - \lim_{M\to\infty} \mathrm{tr}(X_M^{n_1})\cdot 1\big) \cdot \big(Y_N^{m_1} - \lim_{M\to\infty} \mathrm{tr}(Y_M^{m_1})\cdot 1\big) \cdots \\ \cdots \big(X_M^{n_k} - \lim_{M\to\infty} \mathrm{tr}(X_M^{n_k})\cdot 1\big) \cdot \big(Y_N^{m_k} - \lim_{M\to\infty} \mathrm{tr}(Y_M^{m_k})\cdot 1\big)\Big) \quad (2.2)$$

is given by the number of non-crossing pairings as above, but with the additional property that each group of X^{n_i} must be connected by at least one pair to some other group X^{n_j} (with $i \neq j$) and, in the same way, each group of Y's must be connected to some other group of Y's. However, since the groups of X's and the groups of Y's are alternating, it is obvious that if we want to connect the groups in this way we will necessarily get crossings between some pairs. Thus there are actually no pairings of the required form and we have that the term (2.2) is equal to zero.

One might wonder what advantage is gained by trading the explicit formula for mixed moments (2.1) of independent Gaussian random matrices for the implicit relations (2.2)? The drawback of the explicit formula is that the asymptotic formula for $\mathrm{tr}(X_N^{n_1} Y_N^{m_1} \cdots X_N^{n_k} Y_N^{m_k})$ will be different for different random matrix ensembles (and in many cases an explicit formula fails to exist). However, the vanishing of (2.2) remains valid for many matrix ensembles. The vanishing of (2.2) gives a precise meaning to our idea that the random matrices should be in generic position; it constitutes Voiculescu's definition of asymptotic freeness.

Definition 2.3. Two sequences of matrices $(X_N)_{N\in\mathbb{N}}$ and $(Y_N)_{N\in\mathbb{N}}$ are *asymptotically free* if we have the vanishing of (2.2) for all $k \geq 1$ and all $n_1, m_1, \ldots, n_k, m_k \geq 1$.

Provided with this definition, the intuition that unitarily invariant random matrices should give rise to generic situations becomes now a rigorous theorem. This basic observation was proved by Voiculescu in 1991.

Theorem 2.4 (Voiculescu [44]). *Consider $N \times N$ random matrices X_N and Y_N such that: both X_N and Y_N have almost surely an asymptotic eigenvalue distribution for $N \to \infty$; X_N and Y_N are independent; Y_N is a unitarily invariant ensemble. Then, X_N and Y_N are almost surely asymptotically free.*

Extensions of this statement to other classes of random matrices can, for example, be found in [2, 19, 27].

We are now ready to give, in the next chapter, a more abstract definition of freeness. We will also see how it allows to deduce mixed moments from the individual moments.

3. Free probability and non-crossing partitions

3.1. Freeness. The starting point of free probability was the definition of freeness, given by Voiculescu in 1983. However, this happened in the context of operator algebras, related to the isomorphism problem of free group factors. A few years later, in 1991, Voiculescu discovered the relation between random matrices and free probability, as outlined in the last chapter. These connections between operator algebras and random matrices led, among others, to deep results on free group factors. In 1994, I developped a combinatorial theory of freeness, based on free cumulants; many consequences of this approach were worked out together with Nica, see [28, 30]. In the following we concentrate on this combinatorial way of understanding freeness.

Definition 3.1. A pair $(\mathcal{A}, \varphi)$ consisting of a unital algebra $\mathcal{A}$ and a linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ with $\varphi(1) = 1$ is called a *non-commutative probability space*. Often the adjective "non-commutative" is just dropped. Elements from $\mathcal{A}$ are addressed as *(non-commutative) random variables*, the numbers $\varphi(a_{i(1)} \cdots a_{i(n)})$ for such random variables $a_1, \ldots, a_k \in \mathcal{A}$ are called *moments*, the collection of all moments is called the *joint distribution of* $a_1, \ldots, a_k$.

Definition 3.2. Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space and let I be an index set.

1) Let, for each $i \in I$, $\mathcal{A}_i \subset \mathcal{A}$, be a unital subalgebra. The subalgebras $(\mathcal{A}_i)_{i \in I}$ are called *free* or *freely independent*, if $\varphi(a_1 \cdots a_k) = 0$ whenever we have: k is a positive integer; $a_j \in \mathcal{A}_{i(j)}$ (with $i(j) \in I$) for all $j = 1, \ldots, k$; $\varphi(a_j) = 0$ for all $j = 1, \ldots, k$; and neighboring elements are from different subalgebras, i.e., $i(1) \neq i(2), i(2) \neq i(3), \ldots, i(k-1) \neq i(k)$.
2) Let, for each $i \in I$, $x_i \in \mathcal{A}$. The random variables $(x_i)_{i \in I}$ are called *free* or *freely independent*, if their generated unital subalgebras are free, i.e., if $(\mathcal{A}_i)_{i \in I}$ are free, where, for each $i \in I$, $\mathcal{A}_i$ is the unital subalgebra of $\mathcal{A}$ which is generated by x_i. In the same way, subsets $(\mathcal{X}_i)_{i \in I}$ of $\mathcal{A}$ are free, if their generated unital subalgebras are so.

Freeness between x and y is, by definition, an infinite set of equations relating various moments in x and y. However, one should notice that freeness between x and y is actually a *rule for calculating mixed moments* in x and y from the moments of x and the moments of y. In this sense, freeness is analogous to the concept of independence for classical random variables. Hence freeness is also called *free independence*. Free probability theory investigates these freeness relations abstractly, inspired by the philosophy that freeness should be considered and treated as a non-commutative analogue of the classical notion of independence.

The following examples show some calculations of mixed moments. That this works also in general should be clear.

Example 3.3. Let us calculate, for $m, n \geq 1$, the mixed moment $\varphi(x^n y^m)$ of some free random variables x and y. By the definition of freeness it follows that $\varphi[(x^n - \varphi(x^n)1)(y^m - \varphi(y^m)1)] = 0$. This gives

$$\varphi(x^n y^m) - \varphi(x^n \cdot 1)\varphi(y^m) - \varphi(x^n)\varphi(1 \cdot y^m) + \varphi(x^n)\varphi(y^m)\varphi(1 \cdot 1) = 0,$$

and hence $\varphi(x^n y^m) = \varphi(x^n) \cdot \varphi(y^m)$.

The above is the same result as for independent classical random variables. However, this is misleading. Free independence is a different rule from classical independence; free independence occurs typically for *non-commuting random variables*, like operators on Hilbert spaces or (random) matrices.

Example 3.4. Let x and y be some free random variables. By definition of freeness we get

$$\varphi[(x - \varphi(x)1) \cdot (y - \varphi(y)1) \cdot (x - \varphi(x)1) \cdot (y - \varphi(y)1)] = 0,$$

which results after some elementary, but lengthy calculations and many cancellations in

$$\varphi(xyxy) = \varphi(xx) \cdot \varphi(y) \cdot \varphi(y) + \varphi(x) \cdot \varphi(x) \cdot \varphi(yy) - \varphi(x) \cdot \varphi(y) \cdot \varphi(x) \cdot \varphi(y). \quad (3.1)$$

We see that this result is different from the one for independent classical (and thus commuting) random variables. It is important to note that freeness plays a similar role in the non-commutative world as independence plays in the classical world, but that freeness is not a generalization of independence: independent random variables can be free only in very trivial situations. Freeness is a theory for genuinely non-commuting random variables.

3.2. Understanding the freeness rule: the idea of cumulants. The main idea in this section is to write moments in terms of other quantities, which we call *free cumulants*. We will see that freeness is much easier to describe on the level of free cumulants, namely by the vanishing of mixed cumulants. There is also a nice relation between moments and cumulants, given by summing over *non-crossing or planar partitions*.

Definition 3.5.

1) A *partition* of $\{1,\ldots,n\}$ is a decomposition $\pi = \{V_1,\ldots,V_r\}$ of $\{1,\ldots,n\}$ into subsets V_i with

$$V_i \neq \emptyset, \qquad V_i \cap V_j = \emptyset \quad (i \neq j), \qquad \bigcup_i V_i = \{1,\ldots,n\}.$$

The V_i are called the *blocks* of π. The set of all such partitions is denoted by $\mathcal{P}(n)$.

2) A partition π is *non-crossing* if we do not have $p_1 < q_1 < p_2 < q_2$ such that p_1, p_2 are in a same block, q_1, q_2 are in a same block, but those two blocks are different. By $NC(n)$ we will denote the set of all non-crossing partions of $\{1,\ldots,n\}$.

Let us remark that $NC(n)$ is actually a lattice with respect to refinement order.

Definition 3.6. For a unital linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ we define the *free cumulants* $\kappa_n : \mathcal{A}^n \to \mathbb{C}$ (for all $n \geq 1$) as multi-linear functionals by the *moment-cumulant relation*

$$\varphi(a_1 \cdots a_n) = \sum_{\pi \in NC(n)} \kappa_\pi(a_1,\ldots,a_n).$$

Here, κ_π is a product of cumulants: one term for each block of π, and the arguments are given by the elements corresponding to the respective blocks. This, as well as the fact that these equations define the free cumulants uniquely, will be illustrated by the following examples.

This definition is motivated by a similar formula for classical cumulants. The only difference is that in the classical case non-crossing partitions $NC(n)$ are replaced by all partitions $\mathcal{P}(n)$.

Example 3.7. Let us calculate some examples for cumulants for small n.

For $n = 1$ there exists only one partition, |, so that the first moment and the first cumulant are the same: $\varphi(a_1) = \kappa_1(a_1)$.

For $n = 2$ there are two partitions, ⊔ and || and both are non-crossing. By the moment-cumulant formula we get $\varphi(a_1 a_2) = \kappa_2(a_1, a_2) + \kappa_1(a_1)\kappa_1(a_2)$, and thus κ_2 is nothing but the covariance $\kappa_2(a_1, a_2) = \varphi(a_1 a_2) - \varphi(a_1)\varphi(a_2)$.

In the same recursive way, we are able to compute the third cumulant. There are five partitions of the set of three elements:

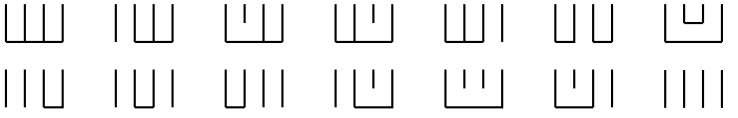
Still, they are all non-crossing and the moment-cumulant formula gives

$$\varphi(a_1a_2a_3) = \kappa_3(a_1,a_2,a_3) + \kappa_1(a_1)\kappa_2(a_2,a_3) + \kappa_2(a_1,a_2)\kappa_1(a_3) + \kappa_2(a_1,a_3)\kappa_1(a_2) + \kappa_1(a_1)\kappa_1(a_2)\kappa_1(a_3)$$

and hence

$$\kappa_3(a_1,a_2,a_3) = \varphi(a_1a_2a_3) - \varphi(a_1)\varphi(a_2a_3) - \varphi(a_1a_2)\varphi(a_3) - \varphi(a_1a_3)\varphi(a_2) + 2\varphi(a_1)\varphi(a_2)\varphi(a_3).$$

The first difference to the classical theory occurs now for $n = 4$; there are 15 partitions of the set of four elements, but one is crossing and there are only 14 non-crossing partitions:



Hence the moment-cumulant formula yields

$$\begin{aligned}\varphi(a_1a_2a_3a_4) = {} & \kappa_4(a_1,a_2,a_3,a_4) + \kappa_1(a_1)\kappa_3(a_2,a_3,a_4) + \kappa_1(a_2)\kappa_3(a_1,a_3,a_4) \\ & + \kappa_1(a_3)\kappa_3(a_1,a_2,a_4) + \kappa_3(a_1,a_2,a_3)\kappa_1(a_4) + \kappa_2(a_1,a_2)\kappa_2(a_3,a_4) \\ & + \kappa_2(a_1,a_4)\kappa_2(a_2,a_3) + \kappa_1(a_1)\kappa_1(a_2)\kappa_2(a_3,a_4) + \kappa_1(a_1)\kappa_2(a_2,a_3)\kappa_1(a_4) \\ & + \kappa_2(a_1,a_2)\kappa_1(a_3)\kappa_1(a_4) + \kappa_1(a_1)\kappa_2(a_2,a_4)\kappa_1(a_3) + \kappa_2(a_1,a_4)\kappa_1(a_2)\kappa_1(a_3) \\ & + \kappa_2(a_1,a_3)\kappa_1(a_2)\kappa_1(a_4) + \kappa_1(a_1)\kappa_1(a_2)\kappa_1(a_3)\kappa_1(a_4).\end{aligned}$$

As before, this can be resolved for κ_4 in terms of moments.

The reader should by now be convinced that one can actually rewrite the moment-cumulant equations also the other way round as cumulant-moment equations. More precisely, this can be achieved by a Möbius inversion on the poset of non-crossing partitions resulting in

$$\kappa_n(a_1,\dots,a_n) = \sum_{\pi\in NC(n)} \varphi_\pi(a_1,\dots,a_n)\mu(\pi,1_n),$$

where φ_π is a product of moments according to the block structure of π and μ is the Möbius function of $NC(n)$.

Whereas κ_1, κ_2, and κ_3 are the same as the corresponding classical cumulants, the free cumulant κ_4 and all the higher ones are different from their classical counterparts.

3.3. Freeness corresponds to vanishing of mixed cumulants. The following theorem shows that freeness is much easier to describe on the level of cumulants than on the level of moments. This characterization is at the basis of most calculations with free cumulants.

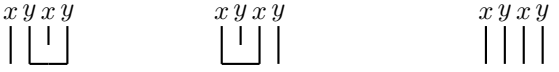
Theorem 3.8 (Speicher [36]). *The fact that x and y are free is equivalent to the fact that $\kappa_n(a_1,\dots,a_n) = 0$ whenever: $n \geq 2$, $a_i \in \{x,y\}$ for all i, and there are at least two indices i,j such that $a_i = x$ and $a_j = y$.*

A corresponding statement is also true for more than two random variables: freeness is equivalent to the vanishing of mixed cumulants.

Example 3.9. If x and y are free, then we have

$$\varphi(xyxy) = \kappa_1(x)\kappa_1(x)\kappa_2(y,y) + \kappa_2(x,x)\kappa_1(y)\kappa_1(y) + \kappa_1(x)\kappa_1(y)\kappa_1(x)\kappa_1(y),$$

corresponding to the three non-crossing partitions of $xyxy$ which connect x only with x and y only with y:



Rewriting the cumulants in terms of moments recovers of course the formula (3.1).

This description of freeness in terms of free cumulants is related to the planar approximations in random matrix theory. In a sense some aspects of this theory of freeness were anticipated (but mostly neglected) in the physics community in the paper [18].

4. Sum of free variables: free convolution

Let x, y be two free random variables. Then, by freeness, the moments of $x+y$ are uniquely determined by the moments of x and the moments of y. But is there an effective way to calculate the distribution of $x+y$ if we know the distribution of x and the distribution of y?

4.1. Free convolution. We usually consider this question in a context where we have some more analytic structure. Formally, a good frame for this is a *C^*-probability space* $(\mathcal{A}, \varphi)$, where $\mathcal{A}$ is a C^*-algebra (i.e., a norm-closed $*$-subalgebra of the algebra of bounded operators on a Hilbert space) and φ is a state, i.e. it is positive in the sense $\varphi(aa^*) \geq 0$ for all $a \in \mathcal{A}$. Concretely this means that our random variables can be realized as bounded operators on a Hilbert space and φ can be written as a vector state $\varphi(a) = \langle a\xi, \xi\rangle$ for some unit vector ξ in the Hilbert space.

In such a situation the distribution of a selfadjoint random variable x can be identified with a compactly supported probability measure μ_x on $\mathbb{R}$, via

$$\varphi(x^n) = \int_{\mathbb{R}} t^n d\mu_x(t) \qquad \text{for all } n \in \mathbb{N}. \tag{4.1}$$

Then we say that the distribution of $x+y$, if x and y are free, is the *free convolution* of the distribution μ_x of x and the distribution μ_y of y and denote this by $\mu_{x+y} = \mu_x \boxplus \mu_y$. By considering unbounded selfadjoint operators (and replacing moments of x by bounded functions of x in (4.1)) one can extend this free convolution also to a binary operation on arbitrary probability measures on $\mathbb{R}$, see [12].

In principle, freeness determines $\mu_x \boxplus \mu_y$ in terms of μ_x and μ_y, but the concrete nature of this connection on the level of moments is not apriori clear. However, by Theorem 3.8, there is an easy rule on the level of free cumulants: if x and y are free then we have for all $n \geq 1$ that $\kappa_n(x+y, x+y, \ldots, x+y) = \kappa_n(x,x,\ldots,x) + \kappa_n(y,y,\ldots,y)$, because all mixed cumulants in x and y vanish.

Thus, the description of the free convolution has now been shifted to understanding the relation between moments and cumulants. A main step for this understanding is the fact that the combinatorial relation between moments and cumulants can also be rewritten easily as a relation between corresponding formal power series.

4.2. Relation between moments and free cumulants. We denote the n-th moment of x by $m_n := \varphi(x^n)$ and the n-th free cumulant of x by $\kappa_n := \kappa_n(x, x, \ldots, x)$. Then, the combinatorical relation between them is given by the moment-cumulant formula

$$m_n = \sum_{\pi \in NC(n)} \kappa_\pi, \tag{4.2}$$

where $\kappa_\pi = \kappa_{|V_1|} \cdots \kappa_{|V_s|}$ for $\pi = \{V_1, \ldots, V_s\}$. The next theorem shows that this combinatorial relation can be rewritten into a functional relation between the corresponding formal power series.

Theorem 4.1 (Speicher [36]). *Consider formal power series $M(z) = 1 + \sum_{n=1}^{\infty} m_n z^n$ and $C(z) = 1 + \sum_{n=1}^{\infty} \kappa_n z^n$. Then the relation* (4.2) *between the coefficients is equivalent to the relation $M(z) = C[zM(z)]$.*

The main step in the proof of this is to observe that a non-crossing partition can be described by its first block (i.e., the block containing the point 1) and by the non-crossing partitions of the points between the legs of the first block. This leads to the following recursive relation between free cumulants and moments:

$$m_n = \sum_{s=1}^{n} \sum_{\substack{i_1, \ldots, i_s \geq 0 \\ i_1 + \cdots + i_s + s = n}} \kappa_s m_{i_1} \cdots m_{i_s}.$$

An early instance of the functional relation in Theorem 4.1 appeared also in the work of Beissinger [6], for the special problem of counting non-crossing partitions by decomposing them into irreducible components.

Remark 4.2. Classical cumulants c_k are combinatorially defined by the analogous formula $m_n = \sum_{\pi \in \mathcal{P}(n)} c_\pi$. In terms of exponential generating power series $\tilde{M}(z) = 1 + \sum_{n=1}^{\infty} \frac{m_n}{n!} z^n$ and $\tilde{C}(z) = \sum_{n=1}^{\infty} \frac{c_n}{n!} z^n$ this is equivalent to $\tilde{C}(z) = \log \tilde{M}(z)$.

4.3. The Cauchy transform. For a selfadjoint random variable x, with corresponding probability measure μ_x according to Eq. (4.1), we define the Cauchy transform G by

$$G(z) := \varphi\left(\frac{1}{z - x}\right) = \int_{\mathbb{R}} \frac{1}{z - t} d\mu_x(t).$$

If μ_x is compactly supported we can expand this into a formal power series:

$$G(z) = \sum_{n=0}^{\infty} \frac{\varphi(x^n)}{z^{n+1}} = \frac{M\left(\frac{1}{z}\right)}{z}.$$

Therefore, on a formal level $M(z)$ and $G(z)$ contain the same information. However, $G(z)$ has many advantages over $M(z)$. Namely, the Cauchy transform is an analytic function $G : \mathbb{C}^+ \to \mathbb{C}^-$ and we can recover μ_x from G by using the *Stieltjes inversion formula*:

$$d\mu_x(t) = -\frac{1}{\pi} \lim_{\varepsilon \to 0} \Im G(t + i\varepsilon) dt.$$

Here, $\Im$ denotes the imaginary part and the convergence in this equation is weak convergence of probability measures; the right hand side is, for any $\varepsilon > 0$, the density of a probability measure.

4.4. The R-transform. Voiculescu showed in [42] the existence of the free cumulants of a random variable by general arguments, but without having a combinatorial interpretation for them. There he defined the following variant of the cumulant generating series $C(z)$.

Definition 4.3. For a random variable $x \in \mathcal{A}$ we define its *R-transform* by

$$R(z) = \sum_{n=1}^{\infty} \kappa_n(x, \ldots, x) z^{n-1}.$$

Then by a simple application of our last theorem we get the following result. The original proof of Voiculescu was much more analytical.

Theorem 4.4 (Speicher [36], Voiculescu [42]).

1) *For a random variable we have the relation $\frac{1}{G(z)} + R[G(z)] = z$ between its Cauchy and R-transform.*
2) *If x and y are free, then we have $R_{x+y}(z) = R_x(z) + R_y(z)$.*

4.5. The R-transform as an analytic object. In the last sections we considered the R-transform only as a formal power series. But for more advanced investigations as well as for explicit calculations it is necessary to study the analytic properties of this object. It is easy to see that for bounded selfadjoint random variables the R-transform can be established as an analytic function via power series expansions around the point infinity in the complex plane. But there are some problems with the analytic properties of the R-transform. One problem is that the R-transform can, in contrast to the Cauchy transform, in general not be defined on all of the upper complex half-plane, but only in some truncated cones (which depend on the considered variable). Another problem is that the equation $\frac{1}{G(z)} + R[G(z)] = z$ does in general not allow explicit solutions and there is no good numerical algorithm for dealing with this. Therefore one is in need of other tools, which allow to compute free convolutions in a more efficient way.

4.6. An alternative to the R-transform: subordination. Let x and y be free. Put $w := R_{x+y}(z) + 1/z$, then

$$G_{x+y}(w) = z = G_x[R_x(z) + 1/z] = G_x[w - R_y(z)] = G_x[w - R_y[G_{x+y}(w)]].$$

Thus, with $\omega(z) := z - R_y[G_{x+y}(z)]]$, we have the subordination $G_{x+y}(z) = G_x\big(\omega(z)\big)$. Though the above manipulations were just on a formal level, it turns out that this subordination function ω is, for selfadjoint x and y in a C^*-probability space, always a nice analytic object and amenable to robust calculation algorithms. The subordination property has first been proved in [45] by Voiculescu under a genericity assumption, and in full generality by Biane [13].

It was noted, and for the first time explicitly formulated in [15], that the subordination property is equivalent to the R-transform approach, but has better analytic properties. A particularly nice feature is that the subordination function can be recovered by fixed point arguments, as shown in [7].

Theorem 4.5 (Belinschi, Bercovici [7]). *Let $(\mathcal{A}, \varphi)$ be a C^*-probability space and let $x = x^*$ and $y = y^*$ in $\mathcal{A}$ be free. Put $F(z) := \frac{1}{G(z)}$. Then there exists an analytic map $\omega : \mathbb{C}^+ \to$*

$\mathbb{C}^+$ *(depending both on* x *and* y*) such that*

$$F_{x+y}(z) = F_x\big(\omega(z)\big) \qquad \text{and} \qquad G_{x+y}(z) = G_x\big(\omega(z)\big).$$

The subordination function $\omega(z)$ *is given as the unique fixed point in the upper half-plane of the map* $f_z : \mathbb{C}^+ \to \mathbb{C}^+$*, given by*

$$f_z(w) = F_y(F_x(w) - w + z) - (F_x(w) - w).$$

5. Polynomials in several random matrices

Our original problem was to calculate the asymptotic eigenvalue distribution of selfadjoint polynomials in several independent random matrices in generic position. We have now a conceptual grasp on this problem by relating it to free probability theory via the basic result of Voiculescu which tells us that our random matrices become almost surely asymptoticially free. This allows us to reduce our random matrix problem to the problem of polynomials in free variables: If the random matrices $X_1, \ldots, X_k$ are asymptotically freely independent, then the eigenvalue distribution of a polynomial $p(X_1, \ldots, X_k)$ is asymptotically given by the distribution of $p(x_1, \ldots, x_k)$, where $x_1, \ldots, x_k$ are freely independent variables, and the distribution of x_i is the asymptotic distribution of X_i.

So now the question is: Can we calculate the distribution of polynomials in free variables? We have seen that free convolution gives effective analytic tools for dealing with the simplest polynomial, the sum of two matrices. By using this, we calculated for example the form of the limiting eigenvalue distribution for the sum of an independent Gaussian and Wishart matrix in the left figure of Figure 2.3. But what can we say for more general polynomials, like the one considered in the right figure of Figure 2.3.

For this problem, both from the random matrix and the free probability side, there is a long list of contributions which provide solutions for special choices of the polynomial p. In the context of free probability, Voiculescu solved it in [42] and [43] for the cases of $p(x, y) = x+y$ and $p(x, y) = xy^2x$ (corresponding to the additive and multiplicative free convolution) with the introduction of the R- and S-transform, respectively. Nica and Speicher could give in [29] a solution for the problem of the free commutator, $p(x, y) = i(xy - yx)$.

In the random matrix context, this problem was addressed for various polynomials – and usually, also for specific choices of the distributions of the $X_i^{(N)}$ – by many authors, starting with the work of Marchenko-Pastur [26]. For a more extensive list of contributions in this context we refer to the books [3, 17, 20, 40]. Some of those situations were also treated by operator-valued free probability tools, see in particular [10, 33, 38].

All those investigations were specific for the considered polynomial and up to now there has not existed a master algorithm which would work for all polynomials.

Actually, there is no hope to calculate effectively general polynomials in freely independent variables within the frame of free probability theory as presented up to now. However, there is a possible way to deal with such a general situation, by the use of a linearization trick. This trick will be the main topic of the next chapter.

6. The linearization trick

The idea of this trick is: in order to understand general polynomials in non-commuting variables, it suffices to understand matrices of *linear* polynomials in those variables. Such linearization ideas seem to be around in many different communities. In the context of operator algebras, Voiculescu used such a linearization philosophy as one motivation for his work on operator-valued free probability [46]. A seminal concrete form is due to Haagerup and Thorbjørnsen [22], who used such techniques to study the largest eigenvalue of polynomials in independent Gaussian random matrices. In 2011, based on the Schur complement, Anderson [1] developped a selfadjoint version of the linearization trick, which turns out to be the right tool in our context. We present this version of Anderson in the following.

Definition 6.1. Consider a polynomial p in several non-commuting variables. A *linearization* of p is an $N \times N$ matrix (with $N \in \mathbb{N}$) of the form

$$\hat{p} = \begin{pmatrix} 0 & u \\ v & Q \end{pmatrix},$$

where:

- u, v, Q are matrices of appropriate sizes: u is $1 \times (N-1)$; v is $(N-1) \times 1$; and Q is $(N-1) \times (N-1)$.
- Q is invertible and we have $p = -uQ^{-1}v$.
- The entries of $\hat{p}$ are polynomials in the variables, each of degree ≤ 1.

A linearization is of course not uniquely determined by the above requirements. The crucial fact is that such linearizations always exist. Furthermore, they can be chosen in such a way that they preserve selfadjointness.

Theorem 6.2 (Anderson [1]). *For each p there exists a linearization $\hat{p}$ (with an explicit algorithm for finding those). Moreover if p is selfadjoint, then this $\hat{p}$ is also selfadjoint.*

Example 6.3. We consider the selfadjoint non-commutative polynomial $p = xy + yx + x^2$. Then a selfadjoint linearization of p is the matrix

$$\hat{p} = \begin{pmatrix} 0 & x & \frac{x}{2} + y \\ x & 0 & -1 \\ \frac{x}{2} + y & -1 & 0 \end{pmatrix}, \tag{6.1}$$

because we have

$$\begin{pmatrix} x & \frac{x}{2} + y \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} x \\ \frac{x}{2} + y \end{pmatrix} = -(xy + yx + x^2).$$

At this point it might not be clear what this linearization trick has to do with our problem. What we are interested in is the distribution of p, which can be recovered from the Cauchy transform of p, which is given by taking expectations of resolvents of p. Thus we need control of inverses of $z - p$. How can the linearization $\hat{p}$ give information on those?

For $z \in \mathbb{C}$ we put $b = \left(\begin{smallmatrix} z & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and then it follows

$$b - \hat{p} = \begin{pmatrix} z & -u \\ -v & -Q \end{pmatrix} = \begin{pmatrix} 1 & uQ^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z - p & 0 \\ 0 & -Q \end{pmatrix} \begin{pmatrix} 1 & 0 \\ Q^{-1}v & 1 \end{pmatrix}.$$

One should now note that matrices of the form $\left(\begin{smallmatrix} 1 & 0 \\ a & 1 \end{smallmatrix}\right)$ are always invertible with $\left(\begin{smallmatrix} 1 & 0 \\ a & 1 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ -a & 1 \end{smallmatrix}\right)$. Thus the above calculation shows that $z - p$ is invertible if and only if $b - \hat{p}$ is invertible. Moreover, the inverses are related as follows:

$$(b-\hat{p})^{-1} = \begin{pmatrix} 1 & 0 \\ -Q^{-1}v & 1 \end{pmatrix} \begin{pmatrix} (z-p)^{-1} & 0 \\ 0 & -Q^{-1} \end{pmatrix} \begin{pmatrix} 1 & -uQ^{-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} (z-p)^{-1} & \cdots \\ \cdots & \cdots \end{pmatrix},$$

and so we can get $G_p(z) = \varphi((z-p)^{-1})$ as the (1,1)-entry of the matrix-valued Cauchy-transform

$$G_{\hat{p}}(b) = \mathrm{id} \otimes \varphi((b-\hat{p})^{-1}) = \begin{pmatrix} \varphi((z-p)^{-1}) & \cdots \\ \cdots & \cdots \end{pmatrix}.$$

We consider again the polynomial $p = xy + yx + x^2$ of our last example. Its selfadjoint linearization can be written in the form

$$\hat{p} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \otimes 1 + \begin{pmatrix} 0 & 1 & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix} \otimes x + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \otimes y.$$

It is a linear polynomial in free variables, but with matrix-valued coefficients, and we need to calculate its matrix-valued Cauchy transform $G_{\hat{p}}(b) = \mathrm{id} \otimes \varphi((b-\hat{p})^{-1})$. This leads to the question if there exists a suitable matrix-valued version of free probability theory, with respect to the matrix-valued conditional expectation $E = \mathrm{id} \otimes \varphi$.

7. Operator-valued extension of free probability

7.1. Basic definitions. An operator-valued generalization of free probability theory was provided by Voiculescu from the very beginning in [41, 46]. The idea is that we replace our expectations, which yield numbers in $\mathbb{C}$, by conditional expectations, which take values in a fixed subalgebra $\mathcal{B}$. This is the analogue of taking conditional expectations with respect to sub-σ-algebras in classical probability. Let us also remark that the concept of (operator-valued) freeness is distinguished on a conceptual level by symmetry considerations. In the same way as the classical de Finetti theorem equates conditionally independent and identically distributed random variables with random variables whose joint distribution is invariant under permutations, a recent non-commutative version by Köstler and myself [25] shows that in the non-commutative world one gets a corresponding statement by replacing "conditionally independent" by "free with amalgamation" and "permutations" by "quantum permutations". This has triggered quite some investigations on more general quantum symmetries and its relations to de Finetti theorems, see [4, 5, 32, 52].

Definition 7.1.

1) Let $\mathcal{B} \subset \mathcal{A}$ be a unital subalgebra. A linear map $E : \mathcal{A} \to \mathcal{B}$ is a *conditional expectation* if $E[b] = b$ for all $b \in \mathcal{B}$ and $E[b_1 a b_2] = b_1 E[a] b_2$ for all $a \in \mathcal{A}$ and $b_1, b_2 \in \mathcal{B}$.
2) An *operator-valued probability space* consists of $\mathcal{B} \subset \mathcal{A}$ and a conditional expectation $E : \mathcal{A} \to \mathcal{B}$. If in addition $\mathcal{A}$ is a C^*-algebra, $\mathcal{B}$ is a C^*-subalgebra of $\mathcal{A}$, and E is completely positive, then we have an *operator-valued C^*-probability space*.

Example 7.2. Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. Put

$$M_2(\mathcal{A}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathcal{A} \right\}$$

and consider $\psi := \mathrm{tr} \otimes \varphi$ and $E := \mathrm{id} \otimes \varphi$, i.e.:

$$\psi \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] = \frac{\varphi(a) + \varphi(d)}{2}, \qquad E \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] = \begin{pmatrix} \varphi(a) & \varphi(b) \\ \varphi(c) & \varphi(d) \end{pmatrix}.$$

Then $(M_2(\mathcal{A}), \psi)$ is a non-commutative probability space, and $(M_2(\mathcal{A}), E)$ is an $M_2(\mathbb{C})$-valued probability space.

Of course, we should also have a notion of distribution and freeness in the operator-valued sense.

Definition 7.3. Consider an operator-valued probability space $(\mathcal{A}, E : \mathcal{A} \to \mathcal{B})$.

(1) The *operator-valued distribution* of $a \in \mathcal{A}$ is given by all operator-valued moments $E[ab_1ab_2 \cdots b_{n-1}a] \in \mathcal{B}$ $(n \in \mathbb{N}, b_1, \ldots, b_{n-1} \in \mathcal{B})$.

(2) Random variables $x_i \in \mathcal{A}$ $(i \in I)$ are *free with respect to* E or *free (with amalgamation) over* $\mathcal{B}$ if $E[a_1 \cdots a_n] = 0$ whenever $a_i \in \mathcal{B}\langle x_{j(i)} \rangle$ are polynomials in some $x_{j(i)}$ with coefficients from $\mathcal{B}$, $E[a_i] = 0$ for all i, and $j(1) \neq j(2) \neq \cdots \neq j(n)$.

Remark 7.4. Polynomials in x with coefficients from $\mathcal{B}$ are linear combinations of expressions of the form $b_0xb_1xb_2 \cdots b_{n-1}xb_n$. It is important to note that the "scalars" $b \in \mathcal{B}$ do not commute in general with the random variables $x \in \mathcal{A}$.

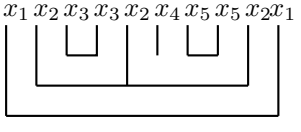
One can see that operator-valued freeness works mostly like ordinary freeness, one only has to take care of the order of the variables. This means in all expressions they have to appear in their original order.

Example 7.5.

1) Note that in scalar-valued free probability one has the rule

$$\varphi(x_1yx_2) = \varphi(x_1x_2)\varphi(y) \qquad \text{if } \{x_1, x_2\} \text{ and } y \text{ are free.}$$

By iteration, this leads to a simple factorization of all "non-crossing" moments in free variables. For example, if $x_1, \ldots, x_5$ are free, then we have for the moment corresponding to



the factorization

$$\varphi(x_1x_2x_3x_3x_2x_4x_5x_5x_2x_1) = \varphi(x_1x_1) \cdot \varphi(x_2x_2x_2) \cdot \varphi(x_3x_3) \cdot \varphi(x_4) \cdot \varphi(x_5x_5).$$

This is actually the same as for independent classical random variables. The difference between classical and free shows up only for "crossing moments".

In the operator-valued setting one has the same factorizations of all non-crossing moments in free variables; but now one has to respect the order of the variables, the final expression is of a nested form, corresponding to the nesting of the non-crossing partition. Here is the operator-valued version of the above example.

$$\begin{aligned} &E[x_1x_2x_3x_3x_2x_4x_5x_5x_2x_1] \\ &= E\Big[x_1 \cdot E\big[x_2 \cdot E[x_3x_3] \cdot x_2 \cdot E[x_4] \cdot E[x_5x_5] \cdot x_2\big] \cdot x_1\Big] \end{aligned}$$

2) For "crossing" moments one also has analogous formulas as in the scalar-valued case. But again one has to take care to respect the order of the variables. For example, the formula

$$\begin{aligned} \varphi(x_1x_2x_1x_2) = \varphi(x_1x_1)\varphi(x_2)\varphi(x_2) + \varphi(x_1)\varphi(x_1)\varphi(x_2x_2) \\ - \varphi(x_1)\varphi(x_2)\varphi(x_1)\varphi(x_2) \end{aligned}$$

for free x_1 and x_2 has now to be written as

$$\begin{aligned} E[x_1x_2x_1x_2] = E\big[x_1E[x_2]x_1\big] \cdot E[x_2] + E[x_1] \cdot E\big[x_2E[x_1]x_2\big] \\ - E[x_1] \cdot E[x_2] \cdot E[x_1] \cdot E[x_2]. \end{aligned}$$

We see that, unlike in the scalar-valued theory, the freeness property in the operator-valued case uses the full nested structure of non-crossing partitions.

7.2. Freeness and matrices. It is an easy but crucial fact that freeness is compatible with going over to matrices. For example if $\{a_1, b_1, c_1, d_1\}$ and $\{a_2, b_2, c_2, d_2\}$ are free in $(\mathcal{A}, \varphi)$, then $\left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right)$ are in general not free in the scalar-valued probability space $(M_2(\mathcal{A}), \mathrm{tr} \otimes \varphi)$, but they are free with amalgamation over $M_2(\mathbb{C})$ in the operator-valued probability space $(M_2(\mathcal{A}), \mathrm{id} \otimes \varphi)$.

Example 7.6. Let $\{a_1, b_1, c_1, d_1\}$ and $\{a_2, b_2, c_2, d_2\}$ be free in $(\mathcal{A}, \varphi)$, consider

$$X_1 := \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \qquad \text{and} \qquad X_2 := \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}.$$

Then

$$X_1X_2 = \begin{pmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{pmatrix}$$

and for $\psi = \mathrm{tr} \otimes \varphi$

$$\begin{aligned} \psi(X_1X_2) &= \big(\varphi(a_1)\varphi(a_2) + \varphi(b_1)\varphi(c_2) + \varphi(c_1)\varphi(b_2) + \varphi(d_1)\varphi(d_2)\big)/2 \\ &\neq (\varphi(a_1) + \varphi(d_1))(\varphi(a_2) + \varphi(d_2))/4 \\ &= \psi(X_1) \cdot \psi(X_2), \end{aligned}$$

but for $E = \mathrm{id} \otimes \varphi$

$$\begin{aligned} E(X_1X_2) &= \begin{pmatrix} \varphi(a_1a_2 + b_1c_2) & \varphi(a_1b_2 + b_1d_2) \\ \varphi(c_1a_2 + d_1c_2) & \varphi(c_1b_2 + d_1d_2) \end{pmatrix} = \begin{pmatrix} \varphi(a_1) & \varphi(b_1) \\ \varphi(c_1) & \varphi(d_1) \end{pmatrix} \begin{pmatrix} \varphi(a_2) & \varphi(b_2) \\ \varphi(c_2) & \varphi(d_2) \end{pmatrix} \\ &= E(X_1) \cdot E(X_2). \end{aligned}$$

Note that there is no comparable classical statement. Matrices of independent random variables do not show any reasonable structure, not even in an "operator-valued" or "conditional" sense.

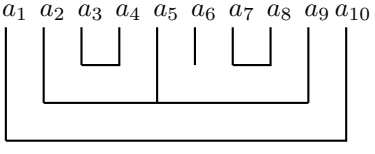
7.3. Operator-valued free cumulants. In [37] it was shown that the combinatorial description of free probability theory in terms of free cumulants can also be extended to the operator-valued setting.

Definition 7.7. Consider $E : \mathcal{A} \to \mathcal{B}$. We define the *free cumulants* $\kappa_n^{\mathcal{B}} : \mathcal{A}^n \to \mathcal{B}$ by

$$E[a_1 \cdots a_n] = \sum_{\pi \in NC(n)} \kappa_\pi^{\mathcal{B}}[a_1, \ldots, a_n].$$

The arguments of $\kappa_\pi^{\mathcal{B}}$ are distributed according to the blocks of π. But now the cumulants are also nested inside each other according to the nesting of the blocks of π.

Example 7.8. We consider $\pi = \big\{\{1, 10\}, \{2, 5, 9\}, \{3, 4\}, \{6\}, \{7, 8\}\big\} \in NC(10)$:



For this we have

$$\kappa_\pi^{\mathcal{B}}[a_1, \ldots, a_{10}] = \kappa_2^{\mathcal{B}}\Big(a_1 \cdot \kappa_3^{\mathcal{B}}\big(a_2 \cdot \kappa_2^{\mathcal{B}}(a_3, a_4), a_5 \cdot \kappa_1^{\mathcal{B}}(a_6) \cdot \kappa_2^{\mathcal{B}}(a_7, a_8), a_9\big), a_{10}\Big).$$

7.4. Operator-valued Cauchy and R-transform. Now we consider operator-valued analogues of the Cauchy and R-transform. Again, those were introduced by Voiculescu, but without having the combinatorial meaning for the coefficients of the R-transform.

Definition 7.9. For $a \in \mathcal{A}$, we define its *operator-valued Cauchy transform*

$$G_a(b) := E[\frac{1}{b-a}] = \sum_{n \geq 0} E[b^{-1}(ab^{-1})^n]$$

and *operator-valued R-transform*

$$R_a(b) := \sum_{n \geq 0} \kappa_{n+1}^{\mathcal{B}}(ab, ab, \ldots, ab, a) = \kappa_1^{\mathcal{B}}(a) + \kappa_2^{\mathcal{B}}(ab, a) + \kappa_3^{\mathcal{B}}(ab, ab, a) + \cdots$$

As in the scalar-valued case we get as a relation between those two:

$$bG(b) = 1 + R(G(b)) \cdot G(b) \qquad \text{or equivalently} \qquad G(b) = \frac{1}{b - R(G(b))}.$$

If one reconsiders the combinatorial proof of these statements from the scalar-valued case, one notices that it respects the nesting of the blocks, so it works also in the operator-valued case.

If one treats these concepts on the level of formal power series one gets all the main results as in the scalar-valued case, see [13, 37, 46, 48].

Theorem 7.10. *If x and y are free over $\mathcal{B}$, then: mixed $\mathcal{B}$-valued cumulants in x and y vanish; it holds that $R_{x+y}(b) = R_x(b) + R_y(b)$; we have the subordination $G_{x+y}(z) = G_x(\omega(z))$.*

7.5. Free analysis. In the last section we introduced the operator-valued R-transform and Cauchy transform on the level of formal power series. In order to use them in an efficient way, we want to look at these objects in a more analytical way. This leads to the theory of "free analysis". This subject aims at developping a non-commutative generalization of holomorphic functions in the setting of operator-valued variables (or in the setting of several variables with the highest degree of non-commutativity). Free analysis was started by Voiculescu in the context of free probability around 2000 [48–50]; it builds on the seminal work of J.L. Taylor [39]. Similar ideas are also used in work of Helton, Vinnikov and collaborators around non-commutative convexity, linear matrix inequalities, or descriptor systems in electrical engineering, see, e.g., [24].

7.6. Subordination in the operator-valued case. Even more as in the scalar-valued theory it is hard to deal with the operator-valued R-transform in an analytical way. Also, the operator-valued equation $G(b) = \frac{1}{b-R(G(b))}$ has hardly ever explicit solutions and, from the numerical point of view, it becomes quite intractable: instead of one algebraic equation we have now a system of algebraic equations. However, there is also a subordination version for the operator-valued case which was treated by Biane [13] and, more conceptually, by Voiculescu [48].

The following theorem shows that the analytic properties of the subordination function in the operator-valued situation are as nice as in the scalar-valued case.

Theorem 7.11 (Belinschi, Mai, Speicher [8]). *Let $(\mathcal{A}, E : \mathcal{A} \to \mathcal{B})$ be an operator-valued C^*-probability space and let x and y be selfadjoint operator-valued random variables in $\mathcal{A}$ which are free over $\mathcal{B}$. Then there exists a Fréchet analytic map $\omega\colon \mathbb{H}^+(\mathcal{B}) \to \mathbb{H}^+(\mathcal{B})$ so that*

$$G_{x+y}(b) = G_x(\omega(b)) \text{ for all } b \in \mathbb{H}^+(\mathcal{B}).$$

Moreover, if $b \in \mathbb{H}^+(\mathcal{B})$, then $\omega(b)$ is the unique fixed point of the map

$$f_b\colon \mathbb{H}^+(\mathcal{B}) \to \mathbb{H}^+(\mathcal{B}), \quad f_b(w) = h_y(h_x(w) + b) + b,$$

and

$$\omega(b) = \lim_{n\to\infty} f_b^{\circ n}(w) \qquad \text{for any } w \in \mathbb{H}^+(\mathcal{B}).$$

Here, $\mathbb{H}^+(\mathcal{B}) := \{b \in \mathcal{B} \mid (b - b^*)/(2i) > 0\}$ denotes the operator-valued upper halfplane of $\mathcal{B}$, $h(b) := \frac{1}{G(b)} - b$, and $f_b^{\circ n}$ is the n-th composition power of f_b

A similar description for the product of free variables in the operator-valued setting was shown by Belinschi, Speicher, Treilhard, and Vargas in [10].

8. Polynomials of independent random matrices and polynomials in free variables

Now we are able to solve the problem of calculating the distribution of a polynomial p in free variables (and thus also the limiting eigenvalue distribution of the polynomial in asymptotically free random matrices). The idea is to linearize the polynomial and to use operator-valued convolution for the linearization $\hat{p}$. We only present this for our running example. The general case works in the same way.

Example 8.1. A linearization $\hat{p}$ of $p = xy + yx + x^2$ was given in Eq. (6.1) As we pointed out there, this means that the Cauchy transform $G_p(z)$ is given as the (1,1)-entry of the $M_3(\mathbb{C})$-valued Cauchy transform of $\hat{p}$:

$$G_{\hat{p}}(b) = \mathrm{id} \otimes \varphi \left[(b - \hat{p})^{-1} \right] = \begin{pmatrix} G_p(z) & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix} \quad \text{for} \quad b = \begin{pmatrix} z & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

But now we can write $\hat{p}$ as $\hat{p} = \hat{x} + \hat{y}$ with selfadjoint

$$\hat{x} = \begin{pmatrix} 0 & x & \frac{x}{2} \\ x & 0 & 0 \\ \frac{x}{2} & 0 & 0 \end{pmatrix} \quad \text{and} \quad \hat{y} = \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & -1 \\ y & -1 & 0 \end{pmatrix}.$$

According to Sect. 7.2, $\hat{x}$ and $\hat{y}$ are free over $M_3(\mathbb{C})$. Furthermore, the distribution of x determines the operator-valued distribution of $\hat{x}$ and the distribution of y determines the operator-valued distribution of $\hat{y}$. This gives us the operator-valued Cauchy transforms of $\hat{x}$ and of $\hat{y}$ as inputs and we can use our results on operator-valued free convolution, in the form of Theorem 7.11, to calculate the operator-valued Cauchy transform of $\hat{x} + \hat{y}$ in the subordination form $G_{\hat{p}}(b) = G_{\hat{x}}(\omega(b))$, where $\omega(b)$ is the unique fixed point in the upper half plane $\mathbb{H}^+(M_3(\mathbb{C}))$ of the iteration

$$w \mapsto G_{\hat{y}}(b + G_{\hat{x}}(w)^{-1} - w)^{-1} - (G_{\hat{x}}(w)^{-1} - w).$$

There are no explicit solutions of those fixed point equations in $M_3(\mathbb{C})$, but a numerical implementation relying on iterations is straightforward. One point to note is that b as defined above is not in the open set $\mathbb{H}^+(M_3(\mathbb{C}))$, but lies on its boundary. Thus, in order to be in the frame as needed in Theorem 7.11, one has to move inside the upper halfplane, by replacing

$$b = \begin{pmatrix} z & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{by} \qquad \begin{pmatrix} z & 0 & 0 \\ 0 & i\varepsilon & 0 \\ 0 & 0 & i\varepsilon \end{pmatrix}$$

and send $\epsilon > 0$ to zero at the end.

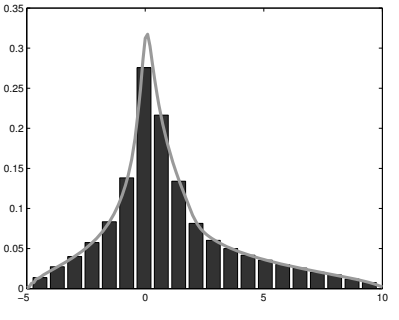


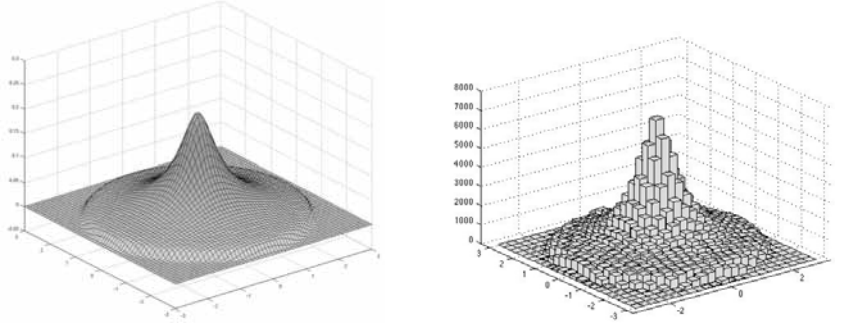
Figure 8.1. Comparision between result of our algorithm for $p(x, y) = xy + yx + x^2$ (x semicircular, y Marchenko-Pastur) and histogram of eigenvalue distribution of 4000×4000 random matrix $p(X, Y)$ (where X and Y are independent Gaussian and, respectively, Wishart random matrices)

9. Further questions and outlook

There are some canonical questions arising from this approach.

Firstly, our approach gives in principle a system of equations for the Cauchy transform of the wanted distribution. Whereas we can provide an efficient numerical fixed point algorithm for solving those equations, one would also like to derive qualitative properties of the solutions from this description. This will be pursued in the future. Prominent questions in this context are about the existence of atoms and regularity properties of the density of the distribution. One should note that, by other approaches, Shlyakhtenko and Skoufranis made in [35] some progress on such questions.

Secondly, we concentrated here only on selfadjoint polynomials of selfadjoint variables, to ensure that we are dealing with selfadjoint operators. Then the spectrum is a subset of the real line and thus the Cauchy transform contains all relevant information. In joint work with Belinschi and Sniady [9] we are presently extending our ideas to non-selfadjoint polynomials, yielding non-normal operators. Then the spectral distribution of the operators has to be replaced by the so-called Brown measure. By combining hermitian reduction ideas with the linearization trick and our subordination results one can then also extend our approach to this situation. An example for such a calculation is shown in Figure 9.1.



(a) (b)

Figure 9.1. (a) : histogram of eigenvalues of $p(X_N, Y_n, Z_n)$, where X_N, Y_N, Z_N are independent non-selfadjoint Gaussian 1000×1000 random matrices; averaged over 100 realizations; (b) : Brown measure, calculated by our algorithm, of the operator $p(x, y, z) = xyz - 2yzx + zxy$, where x, y, z are free semicircular elements

Our methods should also work for more general classes of functions in non-commuting variables. In joint work with Mai, we are presently investigating the class of non-commutative rational functions.

Acknowledgement. The author was partially supported by ERC Advanced Grant NCDFP 339760 during the preparation of this article. I am grateful to Alexandru Nica and the University of Waterloo for their hospitality during the writing of this paper; and to Tobias Mai for preparing some of the figures. I thank Tobias Mai, Claus Köstler, and Moritz Weber for many useful comments on the first draft of this manuscript.

References

[1] G. Anderson, *Convergence of the largest singular value of a polynomial in independent Wigner matrices*, Preprint, 2011, arXiv:1103.4825v2.

[2] G. Anderson, A. Guionnet, and O. Zeitouni, *An Introduction to Random Matrices*, Cambridge University Press, 2010.

[3] Z. Bai and J. Silverstein, *Spectral Analysis of Large Dimensional Random Matrices*, Science Press, 2006.

[4] T. Banica, S. Curran, and R. Speicher, *De Finetti theorems for easy quantum groups*, Ann. Probab. **40**, 401–435.

[5] T. Banica and R. Speicher, *Liberation of orthogonal Lie groups*, Advances Math., **222** (2009), 1461–1501.

[6] J. S. Beissinger, *The enumeration of irreducible combinatorial objects*, J. Comb. Th. A, **38** (1985), 143–169.

[7] S. Belinschi and H. Bercovici, *A new approach to subordination results in free probability*, Journal d'Analyse Mathematique, **101** (2007), 357–365.

[8] S. Belinschi, T. Mai, and R. Speicher, *Analytic subordination theory of operator-valued free additive convolution and the solution of a general random matrix problem*, arXiv:1303.3196.

[9] S. Belinschi, P. Sniady, and R. Speicher, *Brown measure of polynomials in free variables: hermitian reduction and linearization method*, in preparation.

[10] S. Belinschi, R. Speicher, J. Treilhard, and C. Vargas, *Operator-valued free multiplicative convolution: analytic subordination theory and applications to random matrix theory*, arXiv:1209.3508.

[11] F. Benaych-Georges, *Rectangular random matrices, related convolution*, Prob. Th. Rel. Fields, **144** (2009), 471–515.

[12] H. Bercovici and D. Voiculescu, *Free convolution of measures with unbounded support*, Indiana Univ. Math. J., **42** (1993), 733–774.

[13] P. Biane, *Processes with free increments*, Math. Z., **227** (1998), 143–174.

[14] _____, *Free probability and combinatorics*, Proceedings of the International Congress of Mathematicians, Vol II. (Beijing, 2002), Higher Ed. Press, Beijing, 2002, 765–774.

[15] G. Chistyakov and F. Götze, *The arithmetic of distributions in free probability theory*, Cent. Eur. J. Math., **9** (2011), 997–1050.

[16] B. Collins, J. Mingo, P. Sniady, and R. Speicher, *Second order freeness and fluctuations of random matrices III. Higher order freeness and free cumulants*, Documenta Math., **12** (2007), 1–70.

[17] R. Couillet and M. Debbah, *Random matrix methods for wireless communications*, Cambridge University Press, 2011.

[18] P. Cvitanovic, P.G. Lauwers, and P.N. Scharbach, *The planar sector of field theories*, Nucl. Phys. B, **203** (1982), 385–412.

[19] K. Dykema, *On certain free product factors via an extended matrix model*, J. Funct. Anal., **112** (1993), 31–60.

[20] V.L. Girko, *Theory of Stochastic Canonical Equations*, Kluwer Academic Publishers, 2001.

[21] U. Haagerup, *Random matrices, free probability and the invariant subspace problem relative to a von Neumann algebra*, Proceedings of the International Congress of Mathematicians, Vol. I (Beijing, 2002), Higher Ed. Press, Beijing, 2002, 273–290.

[22] U. Haagerup and S. Thorbjørnsen, *A new application of Random Matrices:* $\mathrm{Ext}(C^*_{\mathrm{red}}(F_2))$ *is not a group*, Ann. of Math. (2), **162** (2005), 711–775.

[23] F. Hiai and D. Petz, *The semicircle law, free random variables and entropy*, American Mathematical Society, 2000.

[24] D. Kaliuzhnyi-Verbovetskyi and V. Vinnikov, *Foundations of non-commutative function theory*, arXiv:1212.6345.

[25] C. Köstler and R. Speicher, *A noncommutative de Finetti theorem: invariance under quantum permutations is equivalent to freeness with amalgamation*, Commun. Math. Phys., **291** (2009), 473–490.

[26] V. Marchenko and L. Pastur, *Distributions of eigenvalues for some sets of random matrices*, Math. USSR-Sbornik, **1** (1967), 457–483.

[27] J. Mingo and R. Speicher, *Free Probability and Random Matrices*, Fields Monograph Series, to appear.

[28] A. Nica, *R-transforms of free joint distributions and non-crossing partitions*, J. Funct. Anal., **135** (1996), 271–296.

[29] A. Nica and R. Speicher, *Commutators of free random variables*, Duke Math. J., **92** (1998), no. 3, 553–592.

[30] A. Nica and R. Speicher, *Lectures on the combinatorics of free probability*, Cambridge University Press, 2006.

[31] N. Raj Rao and A. Edelman, *The polynomial method for random matrices*, Foundations of Computational Mathematics, **8** (2008), 649–702.

[32] S. Raum and M. Weber, *Easy quantum groups and quantum subroups of a semi-direct product quantum group*, arXiv:1311:7630.

[33] D. Shlyakhtenko, *Random Gaussian band matrices and freeness with amalgamation*, Internat. Math. Res. Notices, **1996**, no. 20, 1013–1025.

[34] ———, *Free probability, planar algebras, subfactors and random matrices*, Proceedings of the International Congress of Mathematicians, Volume III., Hindustan Book Agency, New Delhi, 2010, 1603–1623.

[35] D. Shlyakhtenko and P. Skoufranis, *Freely independent random variables with non-atomic distributions*, arXiv:1305.1920.

[36] R. Speicher, *Multiplicative functions on the lattice of non-crossing partitions and free convolution*, Math. Ann., **298** (1994), 611–628.

[37] ———, *Combinatorial theory of the free product with amalgamation and operator-valued free probability theory*, Mem. Amer. Math. Soc., **132** (1998), no. 627, x+88.

[38] R. Speicher and C. Vargas, *Free deterministic equivalents, rectangular random matrix models, and operator-valued free probability theory*, Random Matrices: Theory and Applications, **1** (2012), 1150008 (26 pages).

[39] J.S. Taylor, *Functions of several noncommuting variables*, Bull. Amer. Math. Soc., **79** (1973), 1–34.

[40] A. Tulino and S. Verdù, *Random Matrix Theory and Wireless Communications*, Foundations and Trends in Communication and Information Theory, **1** (2004), 1-182.

[41] D. Voiculescu, *Symmetries of some reduced free product C^*-algebras*, Operator algebras and their connections with topology and ergodic theory (Buşteni, 1983), Lecture Notes in Math., vol. 1132, Springer, Berlin, 1985, pp. 556–588.

[42] ———, *Addition of certain non-commutative random variables, J. Funct. Anal.*, **66** (1986), 323–346.

[43] D. Voiculescu, *Multiplication of certain noncommuting random variables*, J. Operator Theory, **18** (1987), 223–235.

[44] ———, *Limit laws for random matrices and free products*, Invent. Math., **104** (1991), 201–220.

[45] ———, *The analogues of entropy and of Fisher's information measure in free probability theory. I.*, Comm. Math. Phys., **155** (1993), 411–440.

[46] ———, *Operations on certain non-commutative operator-valued random variables*, Astérisque, **232** (1995), 243–275.

[47] ———, *Free probability theory: random matrices and von Neumann algebras*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), Birkhäuser, Basel, 1995, 227–241.

[48] ———, *The coalgebra of the free difference quotient and free probability*, Internat. Math. Res. Notices, (2000), no. 2, 79–106.

[49] ———, *Free analysis questions I: duality transform for the coalgebra of $\partial_{X:B}$*, Int. Math. Res. Not., **2004**, no. 16, 793–822.

[50] ———, *Free analysis questions II: the Grassmannian completion and the series expansions at the origin*, J. Reine Angew. Math., **645** (2010), 155–236.

[51] D. Voiculescu, K. Dykema and A. Nica, *Free random variables*, CRM Monograph Series, Vol. 1, AMS, 1992.

[52] M. Weber, *On the classification of easy quantum groups*, Advances Math., **245** (2013), 500–533.

Saarland University, Fachrichtung Mathematik, Postfach151150, 66041 Saarbrücken, Germany
E-mail: speicher@math.uni-sb.de

The h-principle and turbulence

László Székelyhidi Jr.

Abstract. It is well known since the pioneering work of Scheffer and Shnirelman that weak solutions of the incompressible Euler equations exhibit a wild behaviour, which is very different from that of classical solutions. Nevertheless, weak solutions in three space dimensions have been studied in connection with a long-standing conjecture of Lars Onsager from 1949 concerning anomalous dissipation and, more generally, because of their possible relevance to the K41 theory of turbulence.

In recent joint work with Camillo De Lellis we established a connection between the theory of weak solutions of the Euler equations and the Nash-Kuiper theorem on rough isometric immersions. Through this connection we interpret the wild behaviour of weak solutions of Euler as an instance of Gromov's h-principle. In this lecture we explain this connection and outline recent progress towards Onsager's conjecture.

Mathematics Subject Classification (2010). Primary 35D30; Secondary 76F05, 34A60, 53B20.

Keywords. Euler equations, anomalous dissipation, h-principle, Onsager's conjecture.

1. Introduction

1.1. Gromov's h-principle and the local-to-global principle. The homotopy principle was introduced by M. Gromov [37] as a general principle encompassing a wide range of existence problems in differential geometry. Roughly speaking, the h-principle applies to situations where the problem of existence of a certain object in differential geometry can be reduced to a purely topological question and thus treated with homotopic-theoretic methods. To quote Gromov [38]:

> *The infinitesimal structure of a medium, abiding by this principle does not effect the global geometry but only the topological behaviour of the medium.*

In a sense the h-principle replaces the classical local-to-global principle, where global behaviour is directly affected by infinitesimal laws. A paradigm example where the interaction of both principles can be seen is that of isometric embeddings. To fix ideas, let us consider embeddings of the standard 2-sphere S^2 into $\mathbb{R}^3$, i.e. maps

$$u : S^2 \hookrightarrow \mathbb{R}^3.$$

A continuous map u is said to be isometric if it preserves the length of curves:

$$\ell(u \circ \gamma) = \ell(\gamma) \qquad \text{for all rectifiable curves } \gamma \subset S^2. \tag{1.1}$$

Proceedings of the International Congress of Mathematicians, Seoul, 2014

If u is continuously differentiable, i.e. $u \in C^1(S^2;\mathbb{R}^3)$, this condition is equivalent to preserving the metric, which in local coordinates amounts to the system of partial differential equations

$$\partial_i u \cdot \partial_j u = g_{ij} \qquad i,j = 1,2, \tag{1.2}$$

with g_{ij} being the metric on S^2. The equivalence of (1.1) and (1.2) is a first simple instance of the local-to-global principle: the length of a curve (a global quantity) can be obtained from the metric (an infinitesimal quantity) by integrating. If u is merely Lipschitz, the system (1.2) still makes sense *almost everywhere*, since by Rademacher's theorem u is differentiable almost everywhere. However, in this case (1.2) a.e. is not equivalent to (1.1) – see below.

It is easy to construct Lipschitz isometric embeddings of S^2 which are not equivalent to the standard embedding: consider reflecting a spherical cap cut out by a plane slicing the standard sphere. More generally, one can imagine a sphere made out of paper, and crumpling it. This process will necessarily create creases, meaning that the associated embedding is only Lipschitz but not C^1. Nevertheless, such maps will still easily satisfy both (1.1) and also (1.2) almost everywhere. More generally, the class of isometric embeddings very much depends on the regularity assumption on u:

Theorem 1.1.

(i) **Rigidity.** *[39, §32] If $u \in C^2$ is isometric, then u is equal to the standard embedding of $S^2 \subset \mathbb{R}^3$, modulo rigid motion.*

(ii) **h-principle.** *[43, 49] Any short embedding can be uniformly approximated by isometric embeddings of class C^1.*

(iii) **Lipschitz maps.** *[37, p218] There exist $u \in Lip$ such that* (1.2) *is satisfied almost everywhere, but* (1.1) *fails: certain curves on S^2 get mapped to a single point.*

A short embedding is simply one that shrinks the length of curves, i.e. $\ell(u \circ \gamma) \leq \ell(\gamma)$ for all rectifiable curves $\gamma \subset S^2$.

The rigidity statement (i) is a prominent example of the local-to-global principle in geometry: a local, differential condition leads to a strong restriction of the global behaviour. The theorem of Nash-Kuiper in (ii) signifies the *failure* of this local-to-global principle if u is not sufficiently differentiable, whereas (iii) shows that for Lipschitz maps satisfying (1.2) almost everywhere even the simple local-to-global principle on the length of curves fails.

1.2. Unexpected solutions. The differential geometric local-to-global principle may be viewed as an analogue of Hadamard's principle of well-posedness for partial differential equations, in the following sense. Partial differential equations and systems arising from classical physics usually comprise a formally determined system and, given appropriate boundary and initial conditions, one expects a well-defined unique solution. This is at variance with many of the problems in differential geometry, where either due to a large invariance group or by looking at the number of variables versus number of equations one does not expect uniqueness [32, Introduction]. Quoting once more from [38]:

> *The class of infinitesimal laws subjugated by the homotopy principle is wide, but it does not include most partial differential equations (expressing infinitesimal laws) of physics with a few exceptions in favour of this principle leading to unexpected solutions. In fact, the presence of the h-principle would invalidate the*

very idea of a physical law as it yields very limited global information effected by the infinitesimal data.

Nevertheless, there are certainly problems arising from classical physics where non-uniqueness is expected. A prominent example is nonlinear elasticity, where non-uniqueness may arise from a bifurcation such as in Euler buckling or from the appearance of microstructures [3, 46].

Let us recall the basic setting. A three-dimensional elastic material is described by a reference configuration $\Omega \subset \mathbb{R}^3$ and a deformation

$$u : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3.$$

The basic assumption in nonlinear elasticity is that the material minimizes the bulk elastic energy, which in its simplest form (e.g. ignoring thermal effects) is

$$E(u) = \int_\Omega W(Du(x))\, dx.$$

Here $W : \mathbb{R}^{3\times 3} \to [0, \infty]$ is the stored energy density function of the material in question. Looking at deformations with zero bulk elastic energy one is led to consider the differential inclusion

$$Du(x) \in K \quad \text{a.e. } x \in \Omega \tag{1.3}$$

subject to appropriate boundary conditions, where $K := \{W = 0\}$. In other words K denotes the possible zero-energy deformations of the material at the microscopic level. For ordinary materials one can take $K = SO(3)$ and one simply observes a regular crystalline structure on the microscopic scale. On the other hand for materials forming microstructures, such as shape memory alloys, K will typically consist of several copies of $SO(3)$, each copy representing one phase in the underlying solid-solid phase transformation. The "local-to-global" question in this context can be formulated as follows:

To what extent does the microstructure of the material affect the macroscopic behaviour?

Here are two typical results.

Theorem 1.2.

(i) **Rigidity.** *[35, 51] If $K = SO(3)$, any solution of* (1.3) *is affine.*

(ii) **Liquid-like behaviour.** *[30, 48] If*

$$K = SO(3)\begin{pmatrix} \alpha^2 & 0 & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & 0 & \alpha^{-1} \end{pmatrix} \cup SO(3)\begin{pmatrix} \alpha^{-1} & 0 & 0 \\ 0 & \alpha^2 & 0 \\ 0 & 0 & \alpha^{-1} \end{pmatrix} \cup SO(3)\begin{pmatrix} \alpha^{-1} & 0 & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & 0 & \alpha^2 \end{pmatrix}$$

then there exists $\delta > 0$ such that for any $F \in \mathbb{R}^{3\times 3}$ with $\det F = 1$ and $|F - Id| < \delta$ there exist Lipschitz maps $u : \Omega \to \mathbb{R}^3$ with $Du \in K$ a.e. and $u(x) = Fx$ on $\partial\Omega$.

Observe that, whereas in case (i) one has a kind of local-to-global principle at work (the microscopic rigid crystalline structure leading to macroscopic rigidity), in case (ii) the macroscopic behaviour does not reflect any of the microscopic constraints.

1.3. Convex integration. The construction leading to Theorem 1.2 (ii) follows techniques developed in [47], which in turn were motivated by the construction of Nash-Kuiper for Theorem 1.1 (ii) and Gromov's convex integration [37]. The same techniques also apply to the differential inclusion (1.3) with $K = O(3)$ (the set of linear isometries in $\mathbb{R}^3$) - this amounts to constructing Lipschitz maps $\Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ satisfying the weak isometry condition (1.2) almost everywhere. In terms of the analysis a key point in such constructions is to note that the stark difference between (i) and (ii) in Theorem 1.2 cannot be seen on the linearization of the corresponding system of equations, which will be formally elliptic in both cases. To use the flexibility provided by sets K as in (ii) or for $K = O(3)$ one needs large jumps in the gradient ∇u.

The basic idea of the constructions can be explained on the following simple example: consider one-dimensional inclusion problem

$$u(x) \in \{-1, 1\} \quad \text{for a.e. } x \in [0, 1]. \tag{1.4}$$

Let

$$X := \big\{u \in L^\infty(0,1) : |u| \leq 1 \text{ a.e. } \big\}.$$

Then X is a bounded subset of $L^\infty(0,1)$ and hence the weak* topology of L^∞ is metrizable on X. In this way X becomes a compact metric space, and using Baire category arguments one can easily prove that a typical function in X satisfies (1.4). Indeed, such a statement is closely related to the Krein-Milman theorem, since X is compact and convex in L^∞ endowed with the weak* topology.

Baire category arguments for differential inclusions have a long history, see [11, 22] for ordinary differential inclusions and [13, 21, 42] for partial differential inclusions. We also refer to the survey [12]. Note that the Lipschitz solutions produced by such methods are in general highly non-smooth, c.f. [42, Proposition 3.35].

The Nash-Kuiper theorem (Theorem 1.1 (ii) above) seems not to be accessible by Baire category arguments. Although the mappings obtained are still highly irregular, a constructive scheme with estimates on the C^0 and C^1 norms is necessary. Let us recall the basic idea. For simplicity, let (M, g) be a smooth compact Riemannian manifold and we consider embeddings $u : M \hookrightarrow \mathbb{R}^{n+2}$ - this was the case dealt with by Nash in [49], the case of embeddings $u : M \hookrightarrow \mathbb{R}^{n+1}$ requires a modification [43]. Given a strictly short map u_q, $q \in \mathbb{N}$, a better approximation will be obtained with the perturbation

$$u_{q+1}(x) = u_q(x) + \frac{a_q(x)}{\lambda_{q+1}}\Big(\sin(\lambda_{q+1} x \cdot \nu_q)\eta_q(x) + \cos(\lambda_{q+1} x \cdot \nu_q)\zeta_q(x)\Big), \tag{1.5}$$

where a_q is an amplitude, λ_{q+1} a (large) frequency, ν_q is a coordinate direction (with respect to a suitable local parametrization of the manifold) and η_q, ζ_q are normal vectorfields to the embedded image $u_q(M)$. A short calculation gives

$$\partial_i u_{q+1} \cdot \partial_j u_{q+1} = \partial_i u_q \cdot \partial_j u_q + a_q^2 \nu_q^i \nu_q^j + O(\lambda_{q+1}^{-1}),$$

so that, choosing the frequency λ_{q+1} sufficiently large, one can achieve a correction to the metric by $a_q^2(x)\nu_q^i\nu_q^j$ plus a small error. On the other hand a decomposition of the metric error as

$$(g_{ij} - \partial_i u_q \partial_j u_q)(x) = \sum_{k=1}^{n_*} a_q^k(x)\nu_k^i\nu_k^j \tag{1.6}$$

allows one to choose ν_q and a_q at each step $q \in \mathbb{N}$ suitably to achieve an iterative correction of the error. The final map will have the form

$$u(x) = \sum_{q=0}^{\infty} \frac{1}{\lambda_q} w_q(x, \lambda_q x),$$

where each $w_q = u_q - u_{q-1}$ is one such spiral. Ensuring that the final map is C^1 then just requires controlling the amplitudes $\delta_q^{1/2} := \sup_x |w_q|$ so that $\sum_q \delta_q^{1/2} < \infty$. Such control is possible since the amplitude $\sup_x |w_q| \sim \sup_x |a_q|$ only depends on the metric error at step q (and not the frequency λ_q). We refer to the lecture notes [54] for a detailed expository proof.

Recently the construction of Nash (more precisely the construction of Kuiper, where the spiral from (1.5) needs to be replaced by a corrugation) has been visualized for the flat 2-torus in [6], where beautiful pictures showing the fractal nature of the construction have been presented.

On the analytic side the Nash-Kuiper construction has been revisited in [5, 19], where sharper estimates on the approximating sequence have been obtained. In particular, it can be shown that one can additionally ensure

$$\delta_q^{1/2} \lesssim \lambda_q^{-\frac{1}{1+2n_*}}, \tag{1.7}$$

where n_* is a number depending on the dimension n. Such an estimate immediately leads to an improved regularity:

Theorem 1.3 ([5, 19]). *The Nash-Kuiper theorem remains valid for isometric embeddings of class $C^{1,\theta}$ with $\theta < \frac{1}{1+2n_*}$.*

In this note we argue that, although the h-principle contradicts the deterministic behaviour that one expects in partial differential equations arising from classical physics, it nevertheless seems to be of relevance in problems where a statistical description is best suited – and in such cases a suitable notion of weak (i.e. irregular) solution has to be used. For microstructures in solid-solid phase transitions the statistical description is given by gradient Young measures generated by approximate solutions of (1.3). For the incompressible Euler equations, such a statistical description is provided by the Kolmogorov theory of homogeneous turbulence - see Section 3 below.

2. The Euler equations

2.1. Classical solutions. The incompressible Euler equations describe the motion of a perfect incompressible fluid. Written down by L. Euler over 250 years ago, these are the continuum equations corresponding to the conservation of momentum and mass of arbitrary fluid regions. In Eulerian variables they can be written as

$$\begin{aligned} \partial_t v + (v \cdot \nabla) v + \nabla p &= 0, \\ \operatorname{div} v &= 0, \end{aligned} \tag{E}$$

where $v = v(x,t)$ is the velocity and $p = p(x,t)$ is the pressure. In this note we will focus on the 3-dimensional case with periodic boundary conditions; In other words we take the spatial domain to be the flat 3-dimensional torus $\mathbb{T}^3$.

A classical solution on a given time interval $[0,T]$ is defined to be a pair $(v,p) \in C^1(\mathbb{T}^3 \times [0,T])$. It is a well-known fact that, if (v,p) is a classical solution, then the energy is also conserved:

$$\int_{\mathbb{T}^3} |v(x,t)|^2 \, dx = \int_{\mathbb{T}^3} |v(x,0)|^2 \, dx \qquad \text{for all } t > 0. \tag{2.1}$$

This can be obtained by multiplying the first equation in (E) by v itself, integrating in x, and using the fact that

$$\int_{\mathbb{T}^3} v \cdot [(v \cdot \nabla)v] \, dx = - \int_{\mathbb{T}^3} \nabla v : (v \otimes v) \, dx = 0 \tag{2.2}$$

(we use here the common notation $A : B = \sum_{ij} A_{ij} B_{ij}$). Apart from the energy conservation, and despite the rich geometric structure underlying these equations (see e.g. [16] and references therein), little is known about smooth solutions except **(i)** local well-posedness (i.e. existence and uniqueness for short time) in Hölder spaces $C^{1,\alpha}$, $\alpha > 0$ [44] or Sobolev spaces H^s, $s > 5/2$ [31, 41] and **(ii)** the celebrated blow-up criterion of Beale-Kato-Majda [4] and its geometrically refined variants, see e.g. [18].

2.2. Weak solutions. Although distributions were not yet developed in the 1920s, it was certainly recognized already at that time that one needs a notion of solution that allows discontinuities in the vorticity (vortex patches) and in the velocity (vortex sheets). Accordingly, weak solutions of (E) are defined in [44] as a pair $(v,p) \in C(\mathbb{T}^3 \times [0,T])$ such that, for any simply connected region $U \subset \mathbb{T}^3$ with C^1 boundary and any $t \in (0,T)$,

$$\begin{aligned} \int_U v(x,t) \, dx - \int_U v(x,0) \, dx + \int_0^t \int_{\partial U} v(v \cdot \vec{n}) + p \, dA \, ds &= 0, \\ \int_{\partial U} v \cdot \vec{n} \, dA(x) &= 0, \end{aligned} \tag{W}$$

where $\vec{n}$ is the unit outward normal to U. It is easy to see that if $(v,p) \in C^1$ is a solution of (W) then it is a classical solution of (E). Indeed, the derivation of (E) proceeds precisely this way: from the principles of continuum mechanics and the conservation laws of momentum and mass applied to arbitrary fluid regions U one obtains (W), and if in addition $(v,p) \in C^1$, the divergence theorem and a standard localization argument leads to (E).

This definition still includes the pressure. On the other hand it is well known (see e.g. [57]) that the pressure can be recovered (uniquely, upto an additive constant) from (E) via the equation

$$-\Delta p = \operatorname{div} \operatorname{div} (v \otimes v).$$

Therefore one can eliminate the pressure from the equation by projecting the first equation of (E) onto divergence-free fields. One obtains

$$\int_0^T \int_{\mathbb{T}^3} \partial_t \varphi \cdot v + \nabla \varphi : v \otimes v \, dx dt + \int_{\mathbb{T}^3} \varphi(x,0) \cdot v_0(x) \, dx = 0 \tag{D}$$

for all $\varphi \in C^\infty(\mathbb{T}^3 \times [0,T); \mathbb{R}^3)$ with $\operatorname{div} \varphi = 0$. Accordingly, the weakest possible notion of solution of (E) is given by a vectorfield $v \in L^2(\mathbb{T}^3 \times (0,T))$ with $\operatorname{div} v = 0$ in the sense of distributions such that (D) holds.

Solutions of (D) in general exhibit a wild behaviour very different from classical solutions. Here we just list two results and refer to [26] for a more comprehensive survey.

Theorem 2.1.

(i) *[24, 52, 53] There exist infinitely many non-trivial weak solutions $v \in L^\infty(\mathbb{T}^3 \times \mathbb{R})$ of* (D) *which have compact support in time.*

(ii) *[23] Given any continuous strictly positive function $\bar{e} \in C(\mathbb{T}^3 \times [0,T])$ there exist infinitely many weak solutions $v \in L^\infty(\mathbb{T}^3 \times [0,T])$ of* (D) *with*

$$\tfrac{1}{2}|v(x,t)|^2 = \bar{e}(x,t) \text{ for a.e. } (x,t).$$

Part (i) was proved first by V. Scheffer [52] in two dimensions for $v \in L^2_{loc}(\mathbb{R}^2 \times \mathbb{R})$, A. Shnirelman [53] subsequently gave a different proof for $v \in L^2(\mathbb{T}^2 \times \mathbb{R})$. The statement for arbitrary dimension $d \geq 2$ for bounded velocities was obtained in [24] using a reformulation of (D) as a differential inclusion analogous to (1.3) for Lipschitz maps and by applying a modification of Baire category arguments. A refinement of those techniques then leads to the statement in part (ii), see [23]. Note also that in (ii) no statement is made concerning the initial data and thus (ii) is really a statement about non-conservation of the energy rather than about non-uniqueness. For statements concerning non-uniqueness of the initial-value problem we again refer to [23, 26].

2.3. Measure-valued solutions. A stumbling block in obtaining a satisfactory existence theory of weak solutions is the lack of sufficiently strong *a priori* estimates. To overcome this difficulty, two "very weak" notions have been proposed in the literature, both based on considering weakly convergent sequences of Leray solutions of Navier-Stokes with vanishing viscosity: dissipative solutions of P. L. Lions [45] and measure-valued solutions of R. DiPerna and A. Majda [29]. The latter are based on the notion of Young measure and can be described as follows: Given a sequence of velocity fields $v_k(x,t)$, it is known from classical Young measure theory (see e.g. [2, 46, 58]) that there exists a subsequence (not relabeled) and a parametrized probability measure $\nu_{x,t}$ on $\mathbb{R}^3$ such that for all *bounded* continuous functions f,

$$f(v_k(x,t)) \overset{*}{\rightharpoonup} \langle \nu_{x,t}, f \rangle \qquad \text{weakly* in } L^\infty(\mathbb{T}^3 \times (0,T)), \tag{2.3}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality bracket for $C_0^*(\mathbb{R}^3) = \mathcal{M}(\mathbb{R}^3)$. One can interpret the measure $\nu_{x,t}$ as the probability distribution of the velocity field at the point x at time t when the sequence (v_k) exhibits faster and faster oscillations as $k \to \infty$. Since the only known a priori estimate on solutions of the Euler equations is the energy bound, i.e. $v_k \in L^\infty(0,T; L^2(\mathbb{T}^3))$, concentrations could occur for unbounded f, in particular for the energy density $f(v) = \frac{1}{2}|v|^2$. DiPerna and Majda addressed this issue in [29], providing a framework in which both oscillations and concentrations can be described. Following [1] the generalized Young measure can be written as a triple $(\nu, \lambda, \nu^\infty)$, where $\nu = \nu_{x,t}$ is a parametrized probability measure on $\mathbb{R}^3$ as before (the oscillation measure), λ is a Radon measure on $\mathbb{T}^3 \times (0,T)$ (the concentration measure) and $\nu^\infty = \nu^\infty_{x,t}$ is a parametrized probability measure on S^2 defined λ-a.e. (the concentration-angle measure). Then (2.3) can be replaced by

$$f(v_k)dxdt \overset{*}{\rightharpoonup} \langle \nu, f \rangle dxdt + \langle \nu^\infty, f^\infty \rangle \lambda \tag{2.4}$$

in the sense of measures for every $f : \mathbb{R}^3 \to \mathbb{R}$ that possesses an L^2-recession function f^∞ (i.e. such that $f^\infty(\theta) = \lim_{s \to \infty} s^{-2} f(s\theta)$ exists and is continuous). Note that for bounded f the formula in (2.4) reduces to (2.3) because $f^\infty = 0$ in this case.

In particular $(\nu, \lambda, \nu^\infty)$ is able to record oscillations and concentrations in the quadratic term $v \otimes v$ of the Euler equations (D). Denote by id the identity map $\xi \mapsto \xi$ and set $\sigma(\xi) = \xi \otimes \xi$, $\xi \in \mathbb{R}^3$. Noting that $\sigma^\infty = \sigma$, a measure-valued solution of the Euler equations is defined to be a generalized Young measure $(\nu, \lambda, \nu^\infty)$ such that $\operatorname{div} \langle \nu, id \rangle = 0$ in the sense of distributions and

$$\begin{aligned} \int_0^T \int_{\mathbb{T}^3} \partial_t \phi \cdot \langle \nu, id \rangle + \nabla \phi : \langle \nu, \sigma \rangle \, dxdt + \iint_{\mathbb{T}^3 \times (0,T)} \nabla \phi : \langle \nu^\infty, \sigma \rangle \, \lambda(dxdt) \\ = - \int_{\mathbb{T}^3} \phi(x,0) v_0(x) \, dx \end{aligned} \tag{M}$$

for all $\varphi \in C_c^\infty(\mathbb{T}^3 \times [0,T); \mathbb{R}^3)$ with $\operatorname{div} \varphi = 0$.

Observe that (M) is simply a constraint on the first and second moments of the generalized Young measure, i.e. on

$$\overline{v} = \langle \nu_{x,t}, id \rangle, \qquad \overline{v \otimes v} = \langle \nu_{x,t}, \sigma \rangle + \langle \nu^\infty_{x,t}, \sigma \rangle \lambda(dxdt).$$

In particular a measure-valued solution merely gives information on *one-point statistics*, in the sense that there is no information about the correlation between the "statistics" of v_j at different points (x,t) and (x',t'). Moreover there are no *microscopic* constraints, that is, constraints on the distributions of the probability measures. This is very different from other contexts where Young measures have been used, such as conservation laws in one space dimension [28, 56], where the Young measures satisfy additional microscopic constraints in the form of commutativity relations (for instance as a consequence of the div-curl lemma applied to the generating sequence).

Consequently, although the existence of measure-valued solutions for arbitrary initial data is guaranteed [29], there is a huge scope for unnatural non-uniqueness. As it turns out, this same form of non-uniqueness is also present in the a priori stronger notion of distributional solutions:

Theorem 2.2 ([55]). *Given a measure-valued solution of* (M) *there exists a sequence of weak solutions $v_k \in L^2(\mathbb{T}^3 \times (0,T))$ of* (D) *generating this measure-valued solution, in the sense that* (2.4) *holds.*

Thus, solutions of (D) and solutions of (M) are on the same level in terms of their "wild" behaviour. We also emphasize that the construction of solutions as in Theorems 2.1 and 2.2 is based on techniques used to prove Theorem 1.2 (ii): The construction starts by reformulating the Euler equations as a differential inclusion, similar to (1.3), and involves Baire category techniques. We refer to [24] for more details on this comparison.

3. Turbulence and Onsager's conjecture

3.1. Anomalous dissipation and the $5/3$ law. One of the fundamental problems in the theory of turbulence is to find a satisfactory mathematical framework linking the basic continuum equations of fluid motion to the highly chaotic, apparently random behaviour of fully developed turbulent flows. Consider the incompressible Navier-Stokes equations

$$\begin{aligned} \partial_t v + v \cdot \nabla v + \nabla p = \nu \Delta v, \\ \operatorname{div} v = 0, \end{aligned} \tag{3.1}$$

describing the motion of an incompressible viscous fluid, where ν is the viscosity, which, after appropriate non-dimensionalizing, equals $1/\mathrm{Re}$, the reciprocal of the Reynolds number. As ν becomes smaller (equivalently, the Reynolds number becomes larger), the observed motion becomes more and more complex, at some stage becoming chaotic. The statistical theory of turbulence, whose foundations were laid by Kolmogorov in 1941, aims to describe universal patterns in this chaotic, turbulent flow sufficiently far away from the domain boundaries, by postulating that generic flows can be seen as realizations of random fields, and by using the symmetry and scaling properties of the Navier-Stokes equations.

One of the cornerstones of the theory is the famous Kolmogorov-Obukhov $5/3$ law. It states that the energy spectrum $E(k)$, defined to be the kinetic energy per unit mass and unit wavenumber, behaves like a power law

$$E(k) \sim k^{-5/3}. \tag{3.2}$$

This power law, which is supposed to be valid in a certain intermediate range of wavenumbers k - called the inertial range -, away from the large scales (affected by the boundaries of the domain and external forces) and away from the very small scales (affected by dissipation), agrees remarkably well with experiment and numerical simulation. Closely related to the $5/3$ law is the idea of an energy cascade, originally due to Richardson: The energy is introduced at large scales, through nonlinear interaction cascades to smaller and smaller scales until it is dissipated by the viscosity in the very small scales [36]. Indeed, a key hypothesis of the K41 theory is that the mean rate of energy dissipation ϵ is strictly positive and independent of ν in the infinite Reynolds number limit ($\nu \to 0$). This effect in turbulent flows is known as *anomalous dissipation*.

The phenomenological explanation for the energy cascade starts with the decomposition of a "turbulent" velocity field v into a mean value on a given length scale ℓ_q and a turbulent fluctuation:

$$v = v_q + w \tag{3.3}$$

where $v_q = \bar{v}$ is the "coarse-grained", or mean velocity and w is the fluctuation. It is well-known that then v_q satisfies an equation of the form

$$\begin{aligned} \partial_t v_q + \operatorname{div}(v_q \otimes v_q) + \nabla p_q &= \nu \Delta v_q - \operatorname{div} R_q \\ \operatorname{div} v_q &= 0\,, \end{aligned} \tag{3.4}$$

where R_q is the Reynolds stress, equal to

$$R_q = \overline{v \otimes v} - \overline{v} \otimes \overline{v} = \overline{w \otimes w}. \tag{3.5}$$

Assuming that $\nu \ll \ell_q^2$, the term $\nu \Delta v_q$ is negligible and can be dropped, and then by multiplying (3.4) by v_q and integrating, one obtains the (coarse-grained) energy balance

$$\frac{d}{dt} \int |v_q|^2 \, dx = \int \nabla v_q : R_q \, dx.$$

The significance of this is that the energy transfer rate on length scale ℓ_q is proportional to $|\nabla v_q : R_q|$. By interpreting the $5/3$-law (3.2) as a decay rate on the Fourier coefficients of v one obtains

$$|v_q - v| \lesssim \ell_q^{1/3}, \quad |\nabla v_q| \lesssim \ell_q^{-2/3}, \quad \text{and } |R_q| \lesssim \ell_q^{2/3}. \tag{3.6}$$

The latter estimate makes use of the "commutator" structure of the expression (3.5) (see also below). The interpretation is that the $5/3$ laws gives the precise decay rate on the energy spectrum that is consistent with an energy cascade.

3.2. The conjecture of Onsager. Extending the inertial range to infinitely small scales (i.e. $k \to \infty$) corresponds in a certain sense to the limit $\nu \to 0$, when (3.1) becomes the incompressible Euler equations (E). Onsager suggested [50] (see also [34]) the possibility of anomalous dissipation in weak solutions of the Euler equations as a consequence of the energy cascade. We quote from [50]:

> *It is of some interest to note that in principle, turbulent dissipation as described could take place just as readily without the final assistance by viscosity. In the absence of viscosity, the standard proof of the conservation of energy does not apply, because the velocity field does not remain differentiable! In fact it is possible to show that the velocity field in such "ideal" turbulence cannot obey any LIPSCHITZ condition of the form*
>
> $$|\vec{v}(\vec{r'} + \vec{r}) - \vec{v}(\vec{r'})| < (\text{const.})r^n,$$
>
> *for any order n greater than $1/3$; otherwise the energy is conserved.*

Thus, although the K41 theory and the theory of turbulence in general is a statistical theory, concerned with *ensemble averages* of solutions of the Navier-Stokes equations, the suggestion of Onsager turns this into a "pure PDE" question: For weak solutions (v, p) of (W) with

$$|v(x,t) - v(y,t)| \leq C|x-y|^\theta \tag{3.7}$$

(with constant C independent of x, y, t) the conjecture is:

(a) If $\theta > 1/3$, the energy is conserved, i.e. (2.1) holds;

(b) For $\theta < 1/3$ the energy may not be conserved.

Note that, though Onsager's definition of "weak solution" is, strictly speaking, different from the one given above, it can be easily shown that the two concepts are equivalent.

The first part of the conjecture, i.e. assertion (a), has been shown by Eyink in [33] and by Constantin, E and Titi in [17]. The proof amounts to giving a rigorous justification of the formal computation (2.2) and in [17] this is done via a suitable regularization of the equation and a commutator estimate (whereas Onsager's original argument is based on convergence of Fourier series). We sketch the argument. Given a weak solution v satisfying (3.7) with $\theta > 1/3$, mollify v on length scale $\ell > 0$, so that $v_\ell = v * \psi_\ell$. The regularized velocity solves an equation of the type

$$\begin{aligned} \partial_t v_\ell + \operatorname{div}(v_\ell \otimes v_\ell) + \nabla p_\ell &= -\operatorname{div} R_\ell \\ \operatorname{div} v_\ell &= 0\,, \end{aligned} \tag{3.8}$$

where R_ℓ is given by

$$R_\ell = (v \otimes v)_\ell - v_\ell \otimes v_\ell. \tag{3.9}$$

Since v_ℓ is smooth, one readily deduces the energy balance

$$\int_{\mathbb{T}^3} |v_\ell(x,t)|^2\,dx - \int_{\mathbb{T}^3} |v_\ell(x,0)|^2\,dx = 2\int_0^t \int_{\mathbb{T}^3} \nabla v_\ell : R_\ell\,dxds. \tag{3.10}$$

On the other hand, from (3.7) one deduces the estimates

$$\|\nabla v_\ell\|_{C^0} \lesssim \ell^{\theta-1}, \qquad \|R_\ell\|_{C^0} \lesssim \ell^{2\theta},$$

so that, provided $\theta > 1/3$, the right hand side in (3.10) converges to zero as $\ell \to 0$. In fact a more careful estimate (see [17]) allows one to carry out the same argument with the weaker condition

$$\int_0^T \int_{\mathbb{T}^3} |v(x+y,t) - v(x,t)|^3 \, dxdt = o(|y|) \tag{3.11}$$

rather than the uniform-in-time Hölder regularity in (3.7).

Concerning the second part of the conjecture, i.e. statement (b), the first example of a *continuous* solution of the Euler equations dissipating energy was obtained by C. De Lellis and the author in [27]. This was followed by a construction of Hölder continuous solutions as in (3.7) with $\theta < 1/10$ in [25] and subsequently with $\theta < 1/5$ by P. Isett in [40] and shortly after in [9]. The statement is as follows.

Theorem 3.1 ([9, 27, 40]). *Given any smooth strictly positive function $e \in C^\infty([0,1])$ and any $\theta < 1/5$, there exists a solution $(v,p) \in C(\mathbb{T}^3 \times [0,1])$ of* (W) *such that* (3.7) *holds and*

$$\frac{1}{2} \int_{\mathbb{T}^3} |v(x,t)|^2 \, dx = e(t) \quad \text{for all } t \in [0,1].$$

Note that, in comparison with Theorem 2.1, in the statement above only the x-integral of the energy can be specified and there is no statement concerning the non-uniqueness. For a corresponding result concerning the non-uniqueness we refer to [20].

If one relaxes the condition on uniform-in-time Hölder continuity, better results are available. In [7] T. Buckmaster showed that for any $\theta < 1/3$ there exists a solution $(v,p) \in C(\mathbb{T}^3 \times (0,1))$ of (W) with compact support in time with the property that, for almost every time t there exists a number $C = C(t)$ such that

$$|v(x+y,t) - v(x,t)| \leq C(t)|x-y|^\theta \quad \text{for all } x, y \in \mathbb{T}^3.$$

In very recent joint work [10] this was refined so that in addition

$$\int_0^T [v(\cdot,t)]_\theta < \infty,$$

where $[v(\cdot,t)]_\theta$ denotes the Hölder seminorm in x:

$$[f]_\theta = \sup_{x \neq y} \frac{f(x) - f(y)}{|x-y|^\theta}.$$

In these works we are not able to prescribe the energy as in Theorem 3.1, although obviously the energy is not conserved. Hence, comparing with condition (3.11) we see that the gap towards a full proof of Onsager's conjecture remains essentially only at the level of time integrability.

The techniques in all these results are based on the scheme of [27] which aims to adapt the construction of Nash explained above. In the rest of this note we sketch the main ideas.

4. The Nash iteration for Euler

In this section we show the key ideas leading to the proof of Theorem 3.1. Although the basic scheme follows the one introduced in [27], the presentation here uses crucial ideas that were introduced subsequently in the PhD Theses of T. Buckmaster [8] and of P. Isett [40].

The construction of continuous and Hölder-continuous solutions of (W) follows the basic strategy of Nash in the sense that at each step of the iteration, a highly oscillatory correction as the spiral in (1.5) is added. Note that both (E) and the equation of isometries (1.2) is quadratic – the oscillatory perturbation is chosen in such a way as to minimize the linearization, making the quadratic part of leading order. In turn, a finite-dimensional decomposition of the error (c.f. (1.6)) is used to control the quadratic part. There are, however, two important differences:

- The linearization of (1.2) is controlled easily by using the extra codimension(s) in the Nash proof. For Euler, the linearization of (E) leads to a transport equation, which is very difficult to control over long times and leads to a kind of CFL condition; Indeed, the special role of time and advection in the iteration for (E) was only fully understood in [40], see also [9]. This issue is still the main stumbling block in the full resolution of Onsager's conjecture.
- The exponent $1/3$ of Onsager's conjecture requires a sufficiently good correction of the error at each single step, whereas in the Nash iteration several steps (n_* steps) are required – this leads to the exponent $(1+2n_*)^{-1}$ in Theorem 1.3. This, however, means that one-dimensional oscillations, as used in the Nash-Kuiper scheme and, more generally, in convex integration, cannot be used for Euler. Thus, instead of convex integration, we use Beltrami flows, a special family of periodic stationary flows, as the replacement of (1.5) (compare (1.6) with (4.24)).

4.1. The Euler-Reynolds system and characteristic scales. The main idea is to realize that the Nash-Kuiper construction can be viewed as the "reconstruction" of the cascade outlined in Section 3.1 scale-by-scale. At each step $q \in \mathbb{N}$ we construct a triple $(v_q, p_q, \mathring{R}_q)$ solving the Euler-Reynolds system (see [27, Definition 2.1]):

$$\begin{aligned} \partial_t v_q + \operatorname{div}(v_q \otimes v_q) + \nabla p_q &= \operatorname{div} \mathring{R}_q, \\ \operatorname{div} v_q &= 0\,, \end{aligned} \tag{4.1}$$

where (v_q, p_q) is an approximate solution and $\mathring{R}_q$ is a traceless symmetric 3×3 tensor. The *size* of the perturbation

$$w_q := v_q - v_{q-1}$$

will be measured by two parameters:

$$\textit{amplitude: } \delta_q^{1/2}, \qquad \textit{frequency: } \lambda_q,$$

where, along the iteration, we will have $\delta_q \to 0$ and $\lambda_q \to \infty$ at a rate that is (at least) exponential. For the sake of definiteness and for comparison with the Littlewood-Paley approach to turbulence (see [14, 15]) we may think

$$\lambda_q \sim a^q \quad \text{for some } a > 1,$$

(although in the actual proofs a slightly super-exponential growth is required). Here and in what follows, $A \lesssim B$ means that $A \leq cB$ for some universal constant c, and $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. Then, up to controllable errors, w_q will be a function with Fourier-support localized at frequencies comparable to λ_q (in other words a single Littlewood-Paley piece). The more precise formulation is that, denoting the sup-norm by $\|\cdot\|_0$,

$$\|w_q\|_0 \lesssim \delta_q^{1/2}\,, \tag{4.2}$$

$$\|\nabla w_q\|_0 \lesssim \delta_q^{1/2}\lambda_q\,, \tag{4.3}$$

and similarly,

$$\|p_q - p_{q-1}\|_0 \lesssim \delta_q\,, \tag{4.4}$$

$$\|\nabla(p_q - p_{q-1})\|_0 \lesssim \delta_q\lambda_q\,. \tag{4.5}$$

In constructing the iteration, the new perturbation w_{q+1} will be chosen so as to balance the previous Reynolds error $\mathring{R}_q$, in the sense that (cf. equation (3.5)) we have $\|w_{q+1}\otimes w_{q+1}\|_0 \sim \|\mathring{R}_q\|_0$. This is formalized as

$$\|\mathring{R}_q\|_0 \lesssim \delta_{q+1}\,, \tag{4.6}$$

$$\|\nabla\mathring{R}_q\|_0 \lesssim \delta_{q+1}\lambda_q\,, \tag{4.7}$$

We might think of w_q as a mathematical realization of the concept of *eddy* in phenomenological descriptions of turbulence, c.f. [36, Ch 7]. Then, corresponding to eddies of "scale q" we have the following characteristic scales:

- Eddy length scale: $\ell_q \sim \frac{1}{\lambda_q}$;
- Eddy velocity scale: $u_q \sim \delta_q^{1/2}$;
- Eddy time scale: $t_q = \frac{\ell_q}{u_q} \sim \frac{1}{\delta_q^{1/2}\lambda_q}$.

To see that this is consistent with our estimates above, observe that from (4.6)

$$(\partial_t + v_q\cdot\nabla)v_q = \operatorname{div}\mathring{R}_q - \nabla p_q,$$

so that

$$\|(\partial_t + v_q\cdot\nabla)v_q\|_0 \lesssim (\delta_{q+1}+\delta_q)\lambda_q \lesssim \delta_q\lambda_q,$$

which agrees with $\frac{u_q}{t_q}$. Similarly, we will also impose the estimate

$$\|(\partial_t + v_q\cdot\nabla)\mathring{R}_q\|_0 \lesssim \delta_{q+1}\delta_q^{1/2}\lambda_q\,. \tag{4.8}$$

The idea to control the transport derivative $(\partial_t + v_q\cdot\nabla)$ instead of the pure time derivative ∂_t of (v_q, p_q, R_q) was introduced to the scheme by P. Isett in [40].

On the one hand (4.2), (4.4) and (4.6) will imply the convergence of the sequence v_q to a continuous weak solution of the Euler equations. On the other hand the precise dependence of λ_q on δ_q will determine the critical Hölder regularity, similarly to the Nash-Kuiper scheme and Theorem 1.3 above. Finally, control on the energy will be ensured by

$$\left|e(t)(1-\delta_{q+1}) - \int |v_q|^2(x,t)\,dx\right| \le \frac{1}{4}\delta_{q+1}e(t)\,. \tag{4.9}$$

4.2. Conditions on the fluctuation. We define

$$\rho_q(t) = \frac{1}{3(2\pi)^3}\left(e(t)(1-\delta_{q+1}) - \int |v_q|^2(x,t)\,dx\right)$$

and

$$R_q(x,t) = \rho_q(t)\mathrm{Id} - \mathring{R}_q(x,t).$$

It is not difficult to check that (4.9) ensures $\rho_q(t) \sim \delta_{q+1} e(t)$, so that $\|R_q\|_0 \sim \delta_{q+1}$ (c.f. with (4.6)) and

$$\left| \frac{R_q(x,t)}{\delta_{q+1}} - \mathrm{Id} \right| = O(1).$$

Since ρ_q is a function of time only, we can write the Euler-Reynolds system (4.1) as

$$\partial_t v_q + \operatorname{div}(v_q \otimes v_q) + \nabla p_q = -\operatorname{div} R_q,$$

in analogy with (3.4) (with $\nu = 0$). Our aim is to choose the next perturbation w_{q+1} in such a way as to model the fluctuation w in (3.3) leading to the Reynolds stress. Following the idea of Nash and the spiral from (1.5) we make the *ansatz*

$$w_{q+1}(x,t) = W\Big(v_q(x,t), R_q(x,t), \lambda_{q+1}x, \lambda_{q+1}t\Big) + w_{\text{corrector}}(x,t). \tag{4.10}$$

The corrector $w_{\text{corrector}}$ is added to ensure that $\operatorname{div} w_{q+1} = 0$, but for the sake of not overburdening this exposition with technicalities, we will assume it to be negligible subsequently. The key point is how to choose the function $W = W(v, R, \xi, \tau)$.

We make the following assumptions on W:

(H1) $\xi \mapsto W(v, R, \xi, \tau)$ is 2π-periodic with vanishing average, i.e.

$$\langle W \rangle := \frac{1}{(2\pi)^3} \int_{\mathbb{T}^3} W(v, R, \xi, \tau)\, d\xi = 0; \tag{4.11}$$

(H2) The average stress is given by R, i.e.

$$\langle W \otimes W \rangle := R \tag{4.12}$$

for all R in a suitable cone containing the identity matrix;

(H3) The "cell problem" is satisfied:

$$\begin{aligned} \partial_\tau W + v \cdot \nabla_\xi W + \operatorname{div}_\xi (W \otimes W) + \nabla_\xi P = 0, \\ \operatorname{div}_\xi W = 0\,, \end{aligned} \tag{4.13}$$

where $P = P(v, R, \xi, \tau)$ is a suitable pressure;

(H4) W is smooth in all its variables and satisfies the estimates

$$|W| \lesssim |R|^{1/2},\ |\partial_v W| \lesssim |R|^{1/2},\ |\partial_R W| \lesssim |R|^{-1/2}. \tag{4.14}$$

Observe that (4.11)-(4.12) correspond to (3.3)-(3.5), (4.13) arises from plugging the ansatz (4.10) into Euler, and (4.14) are estimates consistent with (4.12).

As a consequence of (H1)-(H2) we obtain

$$\int_{\mathbb{T}^3} |v_{q+1}|^2\, dx \sim \int_{\mathbb{T}^3} |v_q|^2\, dx + \int_{\mathbb{T}^3} \langle |W|^2 \rangle\, dx = \int_{\mathbb{T}^3} |v_q|^2\, dx + 3(2\pi)^3 \rho_q(t),$$

so that (4.9) can be ensured inductively. The main issue is therefore to show that indeed, $\delta_q \to 0$ with $q \to 0$ (so that the scheme converges) and to obtain a relationship between δ_q and λ_q. This requires estimating the new "Reynolds stress" $\mathring{R}_{q+1}$.

4.3. Estimating the new Reynolds stress. Assuming the existence of a function W satisfying (H1)-(H4) above, we can use the ansatz from (4.10) to obtain an estimate on the new "Reynolds stress" $\mathring{R}_{q+1}$. Indeed, since $v_{q+1} = v_q + w_{q+1}$, formally we have

$$\begin{aligned}
\mathring{R}_{q+1} &= \operatorname{div}^{-1}\Big[\partial_t v_{q+1} + \operatorname{div}(v_{q+1}\otimes v_{q+1}) + \nabla p_{q+1}\Big] \\
&= \operatorname{div}^{-1}\Big[\partial_t w_{q+1} + v_q\cdot\nabla w_{q+1}\Big] && (4.15)\\
&+ \operatorname{div}^{-1}\Big[\operatorname{div}(w_{q+1}\otimes w_{q+1} - R_q) + \nabla(p_{q+1}-p_q)\Big] && (4.16)\\
&+ \operatorname{div}^{-1}\Big[w_{q+1}\cdot\nabla v_q\Big] && (4.17)\\
&= \mathring{R}^{(1)}_{q+1} + \mathring{R}^{(2)}_{q+1} + \mathring{R}^{(3)}_{q+1}, && (4.18)
\end{aligned}$$

where div^{-1}, an operator of order -1, is a right inverse of

$$\operatorname{div} : C^\infty(\mathbb{R}^3;\mathcal{S}^{3\times 3}_0) \to C^\infty(\mathbb{R}^3;\mathbb{R}^3).$$

Here $\mathcal{S}^{3\times 3}_0$ is the set of symmetric 3×3 matrices with vanishing trace.

Consider first the term (4.17) (and remember that we ignore the corrector $w_{\text{corrector}}$). Recalling condition (H1) on W, we can expand $\xi \mapsto W(v,R,\xi,\tau)$ in a Fourier-series and write

$$\mathring{R}^{(3)}_{q+1} = \operatorname{div}^{-1}\Big[W\cdot\nabla v_q\Big] = \operatorname{div}^{-1}\sum_{k\in\mathbb{Z}^3,k\neq 0} a_k(x,t)e^{i\lambda_{q+1}k\cdot x}, \tag{4.19}$$

where, using (H2)

$$\|a_k\|_0 \lesssim \|W\|_0\|\nabla v_q\|_0 \lesssim \|R_q\|_0^{1/2}\|\nabla v_q\|_0.$$

Since a_k depends on v_q and R_q, which, owing to our inductive estimates are localized in frequency space to frequencies $\sim \lambda_q$, and since we assume $\lambda_{q+1} \gg \lambda_q$, one may hope for an estimate from (4.19) of the type

$$\big\|\operatorname{div}^{-1}\big[a_k(x,t)e^{i\lambda_{q+1}k\cdot x}\big]\big\|_0 \lesssim \frac{1}{\lambda_{q+1}}\|a_k\|_0, \tag{4.20}$$

provided $k\neq 0$ (since in that case $|k|\geq 1$, there is no issue about small divisors). This estimate can be made rigorous in Hölder spaces using stationary phase arguments, essentially using integration by parts and Schauder estimates (see [27]). For the sake of simplicity in the presentation, let us assume that (4.20) is correct. Using our inductive estimates we then obtain

$$\|\mathring{R}^{(3)}_{q+1}\|_0 \lesssim \frac{\delta^{1/2}_{q+1}\delta^{1/2}_q\lambda_q}{\lambda_{q+1}}.$$

Next, consider (4.15)-(4.16). Here one needs to differentiate W in x and t, and one needs to differentiate between "slow" and "fast" derivatives. For instance

$$\partial_t W = \underbrace{\partial_v W\partial_t v_q + \partial_R W\partial_t R_q}_{\text{slow}} + \underbrace{\lambda_{q+1}\partial_\tau W}_{\text{fast}}.$$

However, owing to condition (H3) (the "cell problem") the fast derivatives in $\mathring{R}^{(1)}_{q+1} + \mathring{R}^{(2)}_{q+1}$ vanish identically. Hence, by some abuse of notation, we may replace (4.15) and (4.16) by

$$\mathring{R}^{(1)}_{q+1} = \operatorname{div}^{-1}\Big[(\partial_t + v_q\cdot\nabla)^{\text{slow}}W\Big], \tag{4.21}$$

$$\mathring{R}^{(2)}_{q+1} \quad = \quad \mathrm{div}^{-1}\Big[\mathrm{div}^{\,\mathrm{slow}}(W\otimes W - R_q)\Big]. \tag{4.22}$$

Observe that the expression in (4.21) is linear in W, hence the same stationary phase argument as above applies. We calculate:

$$(\partial_t + v_q\cdot\nabla)^{\mathrm{slow}} W = \partial_v W\,(\partial_t + v_q\cdot\nabla)v_q + \partial_R W\,(\partial_t + v_q\cdot\nabla)R_q$$

so that, writing as before,

$$\mathring{R}^{(1)}_{q+1} = \mathrm{div}^{-1}\sum_{k\in\mathbb{Z}^3, k\neq 0} b_k(x,t)e^{i\lambda_{q+1}k\cdot x},$$

where, using (H4), we have

$$\|b_k\|_0 \lesssim \|R_q\|_0^{1/2}\|(\partial_t + v_q\cdot\nabla)v_q\|_0 + \|R_q\|_0^{-1/2}\|(\partial_t + v_q\cdot\nabla)R_q\|_0.$$

From the inductive estimates on v_q and R_q in Section 4.1 we then deduce

$$\begin{aligned}\|\mathring{R}^{(1)}_{q+1}\|_0 &\lesssim \frac{1}{\lambda_{q+1}}\left(\delta_{q+1}^{1/2}\delta_q\lambda_q + \delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q\right)\\ &\lesssim \frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}}.\end{aligned}$$

Finally, observe that in (4.22) we have $\langle W\otimes W\rangle = R_q$ because of condition (H2), so that once more, in the expansion of W as a Fourier-series in ξ there is no term $k=0$. Hence the same stationary phase estimate can be applied once more. Writing

$$\mathring{R}^{(2)}_{q+1} = \mathrm{div}^{-1}\sum_{k\in\mathbb{Z}^3, k\neq 0} c_k(x,t)e^{i\lambda_{q+1}k\cdot x}$$

and using (H4) we have the estimate

$$\begin{aligned}\|c_k\|_0 &\lesssim \|W\|_0\|\partial_v W\|_0\|\nabla v_q\|_0 + \|W\|_0\|\partial_R W\|_0\|\nabla R_q\|_0\\ &\lesssim \|R_q\|_0\|\nabla v_q\|_0 + \|\nabla R_q\|_0,\end{aligned}$$

so that

$$\begin{aligned}\|\mathring{R}^{(2)}_{q+1}\|_0 &\lesssim \frac{1}{\lambda_{q+1}}\left(\delta_{q+1}\delta_q^{1/2}\lambda_q + \delta_{q+1}\lambda_q\right)\\ &\lesssim \frac{\delta_{q+1}\lambda_q}{\lambda_{q+1}}.\end{aligned}$$

Summarizing, we obtain

$$\|\mathring{R}_{q+1}\|_0 \lesssim \frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}}. \tag{4.23}$$

Of course, this is just one of the inductive estimates in Section 4.1, similar estimates should be obtained for all the other quantities (4.2)-(4.8). However, this estimate already implies

a relationship between δ_q and λ_q. Indeed, comparing (4.6) and (4.23), the inductive step requires

$$\delta_{q+2} \sim \frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}}.$$

Assuming $\lambda_q \sim a^q$, this would lead to

$$\delta_q^{1/2} \sim a^{-\frac{1}{3}q} \sim \lambda_q^{-\frac{1}{3}},$$

which, comparing with (4.2)-(4.3), *precisely gives exponent* $1/3$ *as the critical Hölder regularity*. Unfortunately, there are several assumptions made in the derivation above. Most importantly, we have assumed the existence of W with properties (H1)-(H4).

4.4. Beltrami flows. Apart from technical issues that we opt to ignore here, the main problem with the calculations above in Subsection 4.3 is that we don't know the existence of a function W satisfying (H1)-(H4).

Our approach starts with a family of Beltrami flows. These are a special class of stationary flows in $\mathbb{T}^3$ based on the identity $\operatorname{div}(W \otimes W) = W \times \operatorname{curl} W - \frac{1}{2}\nabla|W|^2$. In particular any eigenspace of the curl operator, i.e. solution space of the system

$$\begin{aligned} \operatorname{curl} W &= \lambda_0 W \\ \operatorname{div} W &= 0 \end{aligned}$$

leads to a *linear* space of stationary flows. These can be written as

$$\sum_{|k|=\lambda_0} a_k B_k e^{ik\cdot\xi}$$

for normalized complex vectors $B_k \in \mathbb{C}^3$ satisfying $k \cdot B_k = 0$ and $ik \times B_k = \lambda_0 B_k$, and arbitrary coefficients $a_k \in \mathbb{C}$. Choosing $B_{-k} = -\overline{B_k}$ and $a_{-k} = \overline{a_k}$ ensures that W is real-valued. A calculation then shows

$$\langle W \otimes W \rangle = \frac{1}{2}\sum_{|k|=\lambda_0} |a_k|^2 \Big(Id - \frac{k \otimes k}{|k|^2} \Big). \tag{4.24}$$

This identity, which is very similar to the Nash decomposition of the metric error in (1.6), allows one to choose coefficients a_k depending smoothly on R so that condition (H2) is satisfied - this will work at least in a neighbourhood of the identity matrix, and then by scaling in a cone containing Id, see Lemma 3.2 in [27] and also [40] for a geometric proof. In this way we can satisfy conditions (H1)-(H4) *for* $v = 0$, but the transport part of the cell problem (i.e. the term $\partial_\tau + v \cdot \nabla_\xi$) poses problems.

In [25, 27] a "phase function" $\phi_k(v, \tau)$ was introduced to deal with the transport in the cell problem. By considering W of the form

$$\sum_{|k|=\lambda_0} a_k(R)\phi_k(v,\tau) B_k e^{ik\cdot\xi} \tag{4.25}$$

the cell problem in (H3) leads to the equation

$$\partial_\tau \phi_k + i(v \cdot k)\phi_k = 0.$$

However, the exact solution $\phi_k(v,\tau) = e^{-i(v\cdot k)\tau}$ is incompatible with requirement (H4) because $|\partial_v \phi_k| \sim |\tau|$ is unbounded. Instead, an approximation is used such that

$$\partial_\tau \phi_k + i(v\cdot k)\phi_k = O\left(\mu_q^{-1}\right), \qquad |\partial_v \phi_k| \lesssim \mu_q$$

for some parameter μ_q. This leads to the following corrections to (H3) and (H4): (4.13) is only satisfied approximately:

$$\partial_\tau W + v\cdot \nabla_\xi W + \operatorname{div}_\xi (W\otimes W) + \nabla_\xi P = O(\mu_q^{-1})$$

and in (4.14) the second inequality is replaced by

$$|\partial_v W| \lesssim \mu_q |R|^{1/2}.$$

Carrying out the calculations in Section 4.3 with these corrections leads to

$$\|\mathring{R}_{q+1}\|_0 \lesssim \frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}} + \frac{\delta_{q+1}^{1/2}\mu_q}{\lambda_{q+1}} + \frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\mu_q}$$

instead of (4.23). After optimizing in μ_q and comparing with the inductive estimate as in Section 4.3 one arrives at the Hölder exponent $\theta < 1/10$.

One can obtain an improvement on this estimate by realizing that it is better to make an error in the quadratic term of (4.13) rather than the linear transport term. This was one key new idea in [40] and, following [9] leads to the modified *ansatz* (c.f. (4.10))

$$w_{q+1}(x,t) = W\Big(R_q(x,t), \lambda_{q+1}\Phi_q(x,t)\Big) + w_{\text{corrector}}(x,t), \tag{4.26}$$

where $\Phi_q(x,t)$ is the inverse flow of v_q, i.e. the solution of

$$\begin{aligned} \partial_t \Phi_q + v_q\cdot\nabla\Phi_q &= 0,\\ \Phi_q(x,0) &= x. \end{aligned}$$

Notice that such an ansatz automatically satisfies the transport part of the cell problem. On the other hand there will be an error to the Beltrami flow due to the deformation matrix $D\Phi_q$. Furthermore, as v_q converges to a Hölder continuous flow and $\|\nabla v_q\|_0 \to \infty$ as $q\to\infty$, one only has control over $D\Phi_q$ for very short times. To handle this problem we need to apply temporal cut-off functions. Accordingly, the perturbation w_{q+1} takes the form

$$w_{q+1} = \sum_j \chi_j(t) \sum_{|k|=\lambda_0} a_{kj}(R_q) B_k e^{i\lambda_{q+1}k\cdot\Phi_q(x,t)}.$$

Here $(\chi_j)_j$ is a partition of unity of the time interval $[0,1]$ such that for each j the function χ_j is supported on an interval of length $\sim \mu_q^{-1}$ and $|\partial_t \chi| \lesssim \mu_q$ for some parameter μ_q. The calculations of Section 4.3 with this perturbation lead to

$$\|\mathring{R}_{q+1}\|_0 \lesssim \frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}} + \frac{\delta_{q+1}^{1/2}\mu_q}{\lambda_{q+1}} + \frac{\delta_{q+1}\delta_q^{1/2}\lambda_q}{\mu_q}$$

instead of (4.23). Once again optimizing in μ_q and comparing with the inductive estimate as in Section 4.3 one arrives at the Hölder exponent $\theta < 1/5$.

Acknowledgements. The author is supported by ERC Grant Agreement 277993.

References

[1] Alibert, J. J., and Bouchitté, G., *Non-Uniform Integrability and Generalized Young Measure*, J. Convex Analysis 4, **1** (1997), 129–147.

[2] Ball, J. M., *A version of the fundamental theorem for young measures*, In PDEs and continuum models of phase transitions, Springer-Verlag, Berlin/Heidelberg, 1989, pp. 207–215.

[3] Ball, J. M., and James, R. D., *Proposed Experimental Tests of a Theory of Fine Microstructure and the Two-Well Problem*, Phil. Trans. Royal Society A. 338, **1650** (1992), 389–450.

[4] Beale, J. T., Kato, T., and Majda, A., *Remarks on the breakdown of smooth solutions for the* 3*-D Euler equations*, Comm. Math. Phys. 94, **1** (1984), 61–66.

[5] Borisov, Y., *Irregular $C^{1,\beta}$-Surfaces with an Analytic Metric*, Siberian Mathematical Journal 45, **1** (2004), 19–52.

[6] Borrelli, V., Jabrane, S., Lazarus, F., and Thibert, B., *Flat tori in three-dimensional space and convex integration*, Proceedings of the National Academy of Sciences 109, **19** (2012), 7218–7223.

[7] Buckmaster, T., *Onsager's conjecture almost everywhere in time*, http://arxiv.org/abs/1304.1049, 1–24.

[8] ———, *Onsager's conjecture*, PhD thesis, University of Leipzig, 2014.

[9] Buckmaster, T., De Lellis, C., and Székelyhidi, Jr, L., *Transporting microstructure and dissipative Euler flows*, http://arxiv.org/abs/1302.2815, 1–35.

[10] ———, *Dissipative Euler flows with Onsager-critical spatial regularity*, http://arxiv.org/abs/1404.6915, 1–65.

[11] Cellina, A., *On the differential inclusion $x' \in [-1, +1]$*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) 69, 1–2 (1980), 1–6 (1981).

[12] ———, *A view on differential inclusions*, Rend. Semin. Mat. Univ. Politec. Torino 63, **3** (2005).

[13] Cellina, A., and Perrotta, S., *On a problem of potential wells*, J. Convex Analysis, (1995).

[14] Cheskidov, A., and Shvydkoy, R., *Euler equations and turbulence: analytical approach to intermittency*, SIAM J. Math. Anal 46, **1** (2014), 353–374.

[15] Constantin, P., *The Littlewood–Paley spectrum in two-dimensional turbulence*, Theoret. Comp. Fluid Dynamics 9, **3** (1997), 183–189.

[16] ———, *On the Euler equations of incompressible fluids*, Bull. Amer. Math. Soc 44, **4** (2007), 603.

[17] Constantin, P., E, W., and Titi, E. S., *Onsager's conjecture on the energy conservation for solutions of Euler's equation*, Comm. Math. Phys. 165, **1** (1994), 207–209.

[18] Constantin, P., Fefferman, C., and Majda, A. J., *Geometric constraints on potentially singular solutions for the* 3*-D Euler equations*, Comm. PDE 21, 3-4 (1996), 559–571.

[19] Conti, S., De Lellis, C., and Székelyhidi, Jr, L., *h-principle and rigidity for $C^{1,\alpha}$ isometric embeddings*, In Nonlinear Partial Differential Equations: The Abel Symposium 2010, H. Holden and K. H. Karlsen, Eds. Springer, 2012, pp. 83–116.

[20] S. Daneri, *Cauchy Problem for Dissipative Hölder Solutions to the Incompressible Euler Equations*, Comm. Math. Phys., (2014), 1–42.

[21] Dacorogna, B., and Marcellini, P., *General existence theorems for Hamilton-Jacobi equations in the scalar and vectorial cases*, Acta Math. 178, **1** (1997), 1–37.

[22] De Blasi, F. S., and Pianigiani, G., *A Baire category approach to the existence of solutions of multivalued differential equations in Banach spaces*, Funkcial Ekvac, 1982.

[23] De Lellis, C., and Székelyhidi, Jr, L., *On admissibility criteria for weak solutions of the Euler equations*, Arch. Ration. Mech. Anal. 712, 2008.

[24] ———, *The Euler equations as a differential inclusion*, Ann. of Math. (2) 170, **3** (2009), 1417–1436.

[25] ———, *Dissipative Euler flows and Onsager's Conjecture*, http://arxiv.org/abs/1205.3626, 1–40.

[26] ———, *The h-principle and the equations of fluid dynamics*, Bull. Amer. Math. Soc. (N.S.) 49, **3** (2012), 347–375.

[27] ———, *Dissipative continuous Euler flows*, Invent. Math. 193, **2** (2013), 377–407.

[28] DiPerna, R. J., *Compensated compactness and general systems of conservation laws*, Trans. Amer. Math. Soc., (1985), 383–420.

[29] DiPerna, R. J., and Majda, A. J., *Oscillations and concentrations in weak solutions of the incompressible fluid equations*, Comm. Math. Phys. 108, **4** (1987), 667–689.

[30] Dolzmann, G., and Kirchheim, B., *Liquid-like behavior of shape memory alloys*, C. R. Math. Acad. Sci. Paris 336, **5** (2003), 441–446.

[31] Ebin, D., and Marsden, J., *Groups of diffeomorphisms and the motion of an incompressible fluid*, Ann. of Math. (2), (1970), 102–163.

[32] Eliashberg, Y., and Mishachev, N. M., *Introduction to the h-principle*, American Mathematical Society, 2002.

[33] Eyink, G. L., *Energy dissipation without viscosity in ideal hydrodynamics. I. Fourier analysis and local energy transfer*, Phys. D 78, 3-4 (1994), 222–240.

[34] Eyink, G. L., and Sreenivasan, K., *Onsager and the theory of hydrodynamic turbulence*. Rev. Mod. Phys. 78, **1** (2006), 87–135.

[35] Friesecke, G., James, R. D., and Müller, S., *A theorem on geometric rigidity and the derivation of nonlinear plate theory from threeâĂŘ-dimensional elasticity*, Comm. Pure Appl. Math. 55, **11** (2002), 1461–1506.

[36] Frisch, U., *Turbulence*, Cambridge University Press, Cambridge, 1995.

[37] Gromov, M., *Partial differential relations*, vol. 9 of Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer Verlag, Berlin, 1986.

[38] ———, *Local and global in geometry*, IHES preprint (1999), 1–11.

[39] Hilbert, D., and Cohn-Vossen, S., *Geometry and the Imagination*, American Mathematical Society, 1999.

[40] Isett, P., *Hölder continuous Euler flows with compact support in time*, PhD thesis, Princeton University, 2013.

[41] Kato, T., *Nonstationary flows of viscous and ideal fluids in* $\mathbf{R^3}$, J. Functional Analysis 9, (1972), 296–305.

[42] Kirchheim, B., *Rigidity and Geometry of Microstructures*, Habilitation Thesis, Univ. Leipzig, 2003.

[43] Kuiper, N. H., *On C^1-isometric imbeddings. I, II*, Nederl. Akad. Wetensch. Indag. Math. 17, (1955), 545–556, 683–689.

[44] Lichtenstein, L., *Grundlagen der Hydromechanik*, Springer Verlag, 1929.

[45] Lions, P.-L., *Mathematical Topics in Fluid Mechanics: Volume 1: Incompressible Models*, Oxford University Press on Demand, June 1996.

[46] Müller, S., *Variational models for microstructure and phase transitions*, Calculus of Variations and Geometric Evolution Problems, Le ctures given at the 2nd Session of the Centre Internazionale Matematico Estivo, Cetaro, 1996.

[47] Müller, S., and Sverak, V., *Unexpected solutions of first and second order partial differential equations*, In Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998) (1998), pp. 691–702 (electronic).

[48] ———, *Convex integration with constraints and applications to phase transitions and partial differential equations*, J. Eur. Math. Soc. 1, **4** (1999), 393–422.

[49] Nash, J., *C^1 isometric imbeddings*, Ann. of Math. (2) 60, **3** (1954), 383–396.

[50] Onsager, L., *Statistical hydrodynamics*, Nuovo Cim 6, Supplemento, 2 (Convegno Internazionale di Meccanica Statistica) (1949), 279–287.

[51] Reshetnyak, Y. G., *Stability theorems in geometry and analysis*, Springer, 1994.

[52] Scheffer, V., *An inviscid flow with compact support in space-time*, J. Geom. Anal. 3, **4** (1993), 343–401.

[53] Shnirelman, A. I., *On the nonuniqueness of weak solution of the Euler equation*, Comm. Pure Appl. Math. 50, **12** (1997), 1261–1286.

[54] Székelyhidi, Jr, L., *From Isometric Embeddings to Turbulence*, In HCDTE Lecture Notes. Part II. Nonlinear Hyperbolic PDEs, Dispersive and Transport Equations, American Institute of Mathematical Sciences, 2014, pp. 1–66.

[55] Székelyhidi, Jr, L., and Wiedemann, E., *Young Measures Generated by Ideal Incompressible Fluid Flows*, Arch. Ration. Mech. Anal. 206, **1** (2012), 333–366.

[56] Tartar, L., *The compensated compactness method applied to systems of conservation laws*, Systems of nonlinear partial differential equations. Dordrecht, 1977, 263–285.

[57] Temam, R., *Navier-Stokes equations*, vol. 2 of Studies in Mathematics and its Applications, North Holland, Amsterdam, 1984.

[58] Young, L. C., *Lecture on the Calculus of Variations and Optimal Control Theory*, American Mathematical Society, 1980.

Mathematisches Institut, Universität Leipzig, D-04109, Germany
E-mail: szekelyhidi@math.uni-leipzig.de

9. Dynamical Systems and Ordinary Differential Equations

Linear response, or else

Viviane Baladi

Abstract. Consider a smooth one-parameter family $t \mapsto f_t$ of dynamical systems f_t, with $|t| < \epsilon$. Assume that for all t (or for many t close to $t = 0$) the map f_t admits a unique physical invariant probability measure μ_t. We say that *linear response* holds if $t \mapsto \mu_t$ is differentiable at $t = 0$ (possibly in the sense of Whitney), and if its derivative can be expressed as a function of f_0, μ_0, and $\partial_t f_t|_{t=0}$. The goal of this note is to present to a general mathematical audience recent results and open problems in the theory of linear response for chaotic dynamical systems, possibly with bifurcations.

Mathematics Subject Classification (2010). Primary 37C40; Secondary 37D25, 37C30, 37E05.

Keywords. Linear response, transfer operator, Ruelle operator, physical measure, SRB measure, bifurcations, differentiable dynamical system, unimodal maps, expanding interval maps, hyperbolic dynamical systems.

1. Introduction

A discrete-time dynamical system is a self-map $f : M \to M$ on a space M. To any point $x \in M$ is then associated its (future) orbit $\{f^n(x) \mid n \in \mathbb{Z}_+\}$ where $f^0(x) = x$, and $f^n(x) = f^{n-1}(f(x))$, for $n \geq 1$, represents the state of the system at time n, given the "initial condition" x. (If f is invertible, one can also consider the past orbit $\{f^{-n}(x) \mid n \in \mathbb{Z}_+\}$.) In this text, we shall always assume that M is a compact differentiable manifold (possibly with boundary), with the Borel σ-algebra, endowed with a Riemannian structure and thus normalised Lebesgue measure. Many natural dynamical systems are "chaotic" (in particular, a small error in the initial condition will grow exponentially with time) and best understood via ergodic theory. The ergodic approach often starts with finding a "natural" invariant probability measure μ (a probability measure is invariant if $\mu(f^{-1}(E)) = \mu(E)$ for every Borel set). Lebesgue measure is not always invariant, although there are important exceptions such as the angle-doubling map $x \mapsto 2x$ modulo 1 on the circle, hyperbolic linear toral automorphisms such as the "cat map" A_0 defined in (3.2) below, or symplectic diffeomorphisms. However, many interesting dynamical systems which do not preserve Lebesgue admit a "physical" invariant probability measure: The *ergodic basin* of an f-invariant probability measure μ is the set of those initial conditions for which time averages converge to the space average for every continuous function $\varphi : M \to \mathbb{C}$, i.e., the set

$$\{x \in M \mid \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(x)) = \int \varphi \, d\mu \,, \ \forall \varphi \in C^0\}.$$

Proceedings of the International Congress of Mathematicians, Seoul, 2014

An invariant probability measure μ is called *physical* if its ergodic basin has positive Lebesgue measure.

If μ is f-invariant and absolutely continuous with respect to Lebesgue then, if it is in addition ergodic, it is a physical measure because of the Birkhoff ergodic theorem. It was one of the breakthrough discoveries of the 60's, by Anosov and others, that many natural dynamical systems (in particular smooth hyperbolic attractors) admit finitely many physical measures, while in general they do *not* admit any absolutely continuous invariant measure. Physical measures are sometimes called SRB [1] measures after Sinai, Ruelle, and Bowen, who studied them in the sixties [65].

Instead of a single discrete-time dynamical system f, let us now consider a one-parameter family $t \mapsto f_t$ of dynamical systems on the same space M, where $t \in [-, \epsilon, \epsilon]$, for $\epsilon > 0$. We assume that the map $t \mapsto f_t$ is "smooth" (i.e., C^k for some $1 < k \leq \infty$), taking a suitable topology in the image, e.g., that of C^ℓ diffeomorphisms, or (piecewise) C^ℓ endomorphisms of M, for some $\ell > 1$. We can view f_t as a perturbation of the dynamics $f := f_0$. Let us *assume* that there exists a closed set Λ, containing 0 as an accumulation point, such that the map f_t admits a unique physical measure for every $t \in \Lambda$. (We shall give examples where this assumption holds below.) The question we are interested in is: *Does the map $t \mapsto \mu_t$ inherit any of the smoothness of $t \mapsto f_t$ at the point $t = 0$?* In particular, is $t \mapsto \mu_t$ is differentiable at $t = 0$ (possibly by requiring k and ℓ large enough)?

As such, the question is not well defined, because we must be more precise regarding both the *domain* Λ and the *range* $\{\mu_t \mid t \in \Lambda\}$ of the map $t \mapsto \mu_t$. If Λ contains a neighbourhood U of 0, then differentiability is understood in the usual sense, and differentiability properties usually hold throughout U. However, if Λ does not contain [2] any neighbourhood of 0, "differentiability" of $t \mapsto \mu_t$ on Λ should be understood in the sense of the Whitney extension theorem, as was pointed out by Ruelle [49]. In other words, the map $t \mapsto \mu_t$ is called C^m at $0 \in \Lambda$ for a real number $m > 0$ if this map admits a C^m extension from Λ to an open neighbourhood of 0. If $0 \leq m < 1$ this is just continuity or Hölder continuity on a metric set. For $m = 1$, e.g., then "μ_t is C^1 in the sense of Whitney on Λ at $t = 0$" means that there exists a continuous function $\mu_s^{(1)}$, defined for $s \in \Lambda$, so that

$$\mu_s = \mu_0 + s\mu_s^{(1)} + R_s \, , \text{ with } R_s = o(|s|) \, , \quad \forall s \in \Lambda \, .$$

In order to give a precise meaning to $= o(|s|)$, we need to be more specific regarding the topology used in the *range*. Even if μ_t has a density with respect to Lebesgue, the L^1 norm of this density can be too strong to get differentiability. What is often suitable is a distributional norm, i.e., the topology of the dual of C^r for some $r \geq 0$ ($r = 0$ corresponds to viewing μ_t as a Radon measure). In other words, the question is the differentiability of

$$t \mapsto \int \varphi \, d\mu_t \, .$$

where the "observable" φ belongs to $C^r(M)$. In some cases $(C^r(M))^*$ can be replaced by a space of anisotropic distributions (see §3.1).

We emphasize that considering a strict subset $\Lambda_0 \subset \Lambda$ containing 0 as an accumulation point may change the class of Whitney-C^m maps at 0: A given map μ_t defined on Λ could

[1] The notions of SRB and physical measures do not always coincide, see [65]. In the present expository note, we shall ignore this fact.

[2] One could also decide to restrict Λ even if it originally contains a neighbourhood of 0.

be (Whitney) C^m at $0 \in \Lambda_0$, but *not* (Whitney) C^m at $0 \in \Lambda$. It seems fair to take a "large enough" Λ, for example by requiring 0 to be a Lebesgue density point in Λ (i.e., $\lim_{r\to 0} m(\Lambda \cap [-r,r])/(2r) = 1$), or at least 0 *not* to be a point of dispersion in Λ (i.e., $\lim_{r\to 0} m(\Lambda \cap [-r,r])/(2r) > 0$).

We shall focus on $0 < m \leq 1$. (Higher differentiability results, including formulas, can be obtained [47] if one makes stronger smoothness assumptions on the individual dynamical systems $x \mapsto f_t(x)$ and on the map $t \mapsto f_t$.) If we can prove, under some assumptions on the family f_t, on the set Λ, and on k, ℓ, and r, that the map $t \mapsto \mu_t$ is differentiable at $0 \in \Lambda$, then it is natural to ask if there is a formula for

$$\partial_t \int \varphi \, d\mu_t|_{t=0}$$

in terms of f_0, μ_0, φ, and the vector field $v_0 := \partial_t f_t|_{t=0}$. If such a formula exists, it is called the *linear response formula* (it gives the response to first order of the system in terms of the first order of the perturbation). We shall assume that the perturbation takes place in the image point, i.e., there exists vector fields X_s so that

$$v_s := \partial_t f_t|_{t=s} = X_s \circ f_s \,, \quad \forall s,t \in [-\epsilon,\epsilon] \,. \tag{1.1}$$

(If each f_s is invertible, the above is just a definion of X_s.) The mathematical study of linear response has been initiated by Ruelle. In § 3.1, we shall present his pioneering result [44] on smooth hyperbolic systems (Axiom A attractors). Let us just mention now the key *linear response formula* he obtained in [44] for smooth hyperbolic attractors f_t and smoooth observables φ:

$$\partial_t \int \varphi \, \rho_t \, dx|_{t=0} = \sum_{j=0}^{\infty} \int \langle X_0, \mathrm{grad}(\varphi \circ f_0^j) \rangle \, d\mu_0 \,, \tag{1.2}$$

where the sum is exponentially converging. In [46], Ruelle had shown how to derive (1.2) from heuristic arguments, which suggested to consider the following *susceptibility function* associated to f_t and φ:

$$\Phi_t(z) = \sum_{j=0}^{\infty} \int z^j \, \langle X_0, \mathrm{grad}(\varphi \circ f_0^j) \rangle \, d\mu_0 \,. \tag{1.3}$$

Under very weak assumptions, the power series $\Phi_t(z)$ (often denoted $\Phi_t(e^{i\omega})$) has a nonzero radius of convergence. If the radius of convergence is ≤ 1 and the series in the right-hand-side of (1.2) does not converge, Ruelle [48, $(**)$] suggested that the value at $z = 1$ could sometimes be obtained by analytic continuation, possibly giving the linear response formula. However, caution is necessary, as it was discovered since then (see Section 4.2 below) that linear response fails [7] in cases where a meromorphic continuation was known to exist [49], (see also the presentation of the results of [8] in Section 4.1.)

Before we sketch the contents of this note, we make two simple but essential remarks on (1.2). First note that the higher-dimensional version of the Leibniz expression $(X\rho)' = X'\rho + X\rho'$ reads

$$\rho \, \mathrm{div}\, X + \langle X, \mathrm{grad}\, \rho \rangle \,.$$

Second, defining the transfer operator associated to an invertible [3] dynamical system f_t

[3] See (2.1) for the noninvertible version.

(acting, e.g., on L^∞ or L^1) by

$$\mathcal{L}_t\varphi(x) = \frac{\varphi(f_t^{-1}(x))}{|\det Df_t(f_t^{-1}(x))|},$$

we have $\int \mathcal{L}_t(\varphi)\,dx = \int \varphi\,dx$, for all φ (since the dual of $\mathcal{L}_t$ preserves Lebesgue measure, this is the change of variable formula in an integral). If the transfer operator has a nonnegative fixed point $\mathcal{L}_t\rho_t = \rho_t \in L^1$, then $\mu_t = \rho_t\,dx$ is an absolutely continuous invariant probability measure for f_t and thus (if ergodic) a physical measure. In this case, if the eigenvalue 1 for $\mathcal{L}_t$ is simple and isolated, Ruelle's formula (1.2) and integration by parts give,

$$\begin{aligned}
\partial_t \int \varphi\,\rho_t\,dx|_{t=0} &= \sum_{j=0}^{\infty} \int \langle X_0, \mathrm{grad}(\varphi\circ f_0^j)\rangle \rho_0\,dx \\
&= -\sum_{j=0}^{\infty} \int \varphi\circ f_0^j (\rho_0 \,\mathrm{div}\, X_0 + \langle X_0, \mathrm{grad}\,\rho_0\rangle)\,dx \\
&= -\sum_{j=0}^{\infty} \int \varphi \mathcal{L}_0^j (\rho_0 \,\mathrm{div}\, X_0 + \langle X_0, \mathrm{grad}\,\rho_0\rangle)\,dx \\
&= -\int \varphi (1-\mathcal{L}_0)^{-1} (\rho_0 \,\mathrm{div}\, X_0 + \langle X_0, \mathrm{grad}\,\rho_0\rangle)\,dx\,. \qquad (1.4)
\end{aligned}$$

Note that the residue of $(1-z\mathcal{L}_0)^{-1}(\rho_0 \,\mathrm{div}\, X_0 + \langle X_0, \mathrm{grad}\,\rho_0\rangle)\,dx$ at $z=1$ vanishes, because Lebesgue measure is the fixed point of $\mathcal{L}_0^*$, and the manifold is boundaryless, so that $\int(\rho_0 \,\mathrm{div}\, X_0 + \langle X_0, \mathrm{grad}\,\rho_0\rangle)dx = 0$, by integration by parts. The "metaformula" (1.4) for linear response in the last line can be guessed by applying perturbation theory to the fixed point ρ_t of the operator $\mathcal{L}_t$. We shall see in § 3.1 instances where the above is a rigorous argument, *even in cases where μ_t is not absolutely continuous with respect to Lebesgue* (then, μ_t is a distribution, enjoying smoothness along unstable directions), and in Section 4 instances where the computation above is invalid, even in cases where μ_t *is in fact* absolutely continuous with respect to Lebesgue. We emphasize that the tricky point is that the resolvent $(1-z\mathcal{L}_0)^{-1}$ is evaluated at an expression involving differentiation of ρ_0: While ρ_0 itself often belongs to a space on which $\mathcal{L}_0$ has nice spectral properties, this is not always true for its derivative.

The note is organised as follows: In § 2, we give a complete proof of linear response in the baby toy model of smooth locally expanding circle maps. Section 3 contains an account of two nontrivial occurrences of linear response in chaotic dynamics: The breakthrough [44] of Ruelle for smooth hyperbolic systems is presented in § 3.1, while Dolgopyat's result [20] in a (not necessarily structurally stable) partially hyperbolic case is stated in § 3.2. The next section, which contains both recent results and open problems, is devoted to situations where linear response is violated: We consider first the toy model of piecewise expanding interval maps, presenting in § 4.1 our results [9, 10] with Smania, and those with Marmi–Sauzin [8]. Then, we focus on the – more difficult – smooth, nonuniformly expanding, unimodal interval maps, discussing in § 4.2 the work of Ruelle [51], together with our work with Smania [11, 12], and our recent paper with Benedicks and Schnellmann [7]. Finally, § 4.3 contains a brief account of the techniques of proofs in [7].

The survey published by Nonlinearity in 2008 [6] contains a broad viewed account of the results, open problems, and conjectures at the time, with an emphasis on the role played by critical points (or more generally homoclinic tangencies) in the breakdown of linear response. That survey is thus complementary to the present more introductory presentation. (In view of the page limitation for this contribution, we sometimes do not give fully explicit statements and definitions, the reader is invited to consult the quoted references for clarification.)

We refer to Ruelle's articles [46, 48, 52] for motivation, applications to physics, and more conjectures. See also the interesting approach of Hairer and Majda [25], including references of applications to climate-change. In the present note, we do not discuss linear response for continuous time dynamics [17, 50], or for dynamical systems in infinite dimensions (such as coupled map lattices [27, 28]).

2. The toy model of expanding circle maps

In this section we present a proof of linear response in the (baby) toy model of smooth expanding circle maps. The result and proof are well known (and simpler than the analogous arguments in [9, 24]), but we are not aware of any reference.

Let $M = S^1$ be the unit circle, and let $f : S^1 \to S^1$ be a C^2 map which is λ-*locally expanding,* i.e., there exists $\lambda > 1$ so that $|f'(x)| \geq \lambda$ for all x. It is known [38] that such an f admits a unique absolutely continuous invariant probability measure $\mu = \rho\, dx$. This measure is mixing and therefore ergodic. So a C^2 locally expanding map f admits a unique physical measure. In fact, ρ is C^1, and it is everywhere strictly positive. The transfer operator [4]

$$\mathcal{L}\varphi(x) = \sum_{f_0(y)=x} \frac{\varphi(y)}{|f_0'(y)|} \tag{2.1}$$

is bounded on $C^1(S^1)$. It is known (see [4], e.g., for the relevant references to Ruelle and others) that ρ is a fixed point of $\mathcal{L}$, that the eigenvalue 1 of $\mathcal{L}$ (acting on $C^1(S^1)$) has algebraic multiplicity equal to one, and that the rest of the spectrum of $\mathcal{L}$ is contained in a disc of radius strictly smaller than one. (Thus, $\mathcal{L}$ acting on $C^1(S^1)$ has a spectral gap.) Note that the eigenvector of $\mathcal{L}^*$ for the eigenvalue 1 is just normalised Lebesgue measure (by the change of variable formula).

Fix $\lambda > 1$, and consider a C^2 path $t \mapsto f_t$ for $t \in (-\epsilon, \epsilon)$, where each f_t is now C^3 and locally λ-expanding (then, $\mathcal{L}_t$ acts on C^2, and $\rho_t \in C^2$). Assume that $\|f_t - f_s\|_{C^3(S^1,S^1)} = O(|t-s|)$. Then, using the fact that $\mathcal{L}_t$ (acting on $C^2(S^1)$ or $C^1(S^1)$) satisfies the following *Lasota-Yorke* (or Doeblin-Fortet) [5] inequalities

$$\|\mathcal{L}_t^k \varphi\|_{C^j} \leq C\xi^k \|\varphi\|_{C^j} + C^k \|\varphi\|_{C^{j-1}}\,, \quad \forall \varphi\,, \forall k \geq 1\,, j = 1, 2\,, \tag{2.2}$$

(with uniform $0 < \xi < 1$ and $C \geq 1$), together with [6]

$$\|(\mathcal{L}_t - \mathcal{L}_0)\varphi\|_{C^1} = O(|t|)\|\varphi\|_{C^2}\,,$$

one obtains strong stochastic stability:

[4] The number of terms in the sum is a constant finite integer ≥ 2, the *degree* of the map.

[5] What is essential here is the compact embedding of C^j – the strong norm – in C^{j-1} – the weak norm.

[6] See Step 1 in the proof of Theorem 2.2 for a stronger claim.

Theorem 2.1 (Strong stochastic stability, [14]). *There exists $C > 0$ so that*

$$\|\rho_t - \rho_s\|_{C^1} \le C|t-s| \ , \ \forall\, t, s \in (-\epsilon, \epsilon)\,.$$

In addition, for any t there exists $\tau < 1$, so that, for all s close enough to t, the spectrum of $\mathcal{L}_s$, acting on $C^1(S^1)$ or $C^2(S^1)$, outside of the disc of radius τ consists exactly in the simple eigenvalue 1.

The above result implies that $t \mapsto \mu_t$ is Lipschitz, taking the C^1 topology of the density ρ_t of μ_t in the image.

Assume now further (this does not reduce much generality) that $v_t = \partial_s f_s|_{s=t}$ can be written as $v_t = X_t \circ f_t$ with $X_t \in C^2$. Then, we have linear response:

Theorem 2.2 (Linear response formula). *Viewing $\rho_t \in C^2$ as a C^1 function, the map $t \mapsto \rho_t$ is differentiable, and we have*

$$\partial_s \rho_s|_{s=t} = -(1-\mathcal{L}_t)^{-1}((X_t\rho_t)')\,, \quad \forall t \in (-\epsilon, \epsilon)\,.$$

Note that $X_t\rho_t$ is C^2 by assumption. Since integration by parts on the boundaryless manifold S^1 gives $\int (X_t\rho_t)'\,dx = 0$, the residue of the simple pole at $z=1$ of the resolvent $(z-\mathcal{L}_t)^{-1}$ (acting on $C^1(S^1)$) vanishes at $(X_t\rho_t)'$.

We now prove Theorem 2.2, assuming Theorem 2.1:

Proof of Theorem 2.2. The proof consists in three steps, to be proved at the end:

Step 1: Considering $\mathcal{L}_t$ as a bounded operator from $C^2(S^1)$ to $C^1(S^1)$, we claim that the map $t \mapsto \mathcal{L}_t$ is differentiable, and that, for every $t \in (-\epsilon, \epsilon)$, we have

$$\mathcal{M}_t(\varphi) := \partial_s \mathcal{L}_s(\varphi)|_{s=t} = -X_t'\mathcal{L}_t(\varphi) - X_t\mathcal{L}_t\left(\frac{\varphi'}{f'}\right) + X_t\mathcal{L}_t\left(\frac{\varphi f''}{(f')^2}\right).$$

(This step will use $v_t = X_t \circ f_t$.)

Step 2: Let $\Pi_t(\varphi) = \rho_t \cdot \int \varphi\, dx$ be the rank one projector for the eigenvalue 1 of $\mathcal{L}_t$ acting on $C^1(S^1)$. Then, for every $t \in (-\epsilon, \epsilon)$, we have

$$\partial_s \rho_s|_{s=t} = (1-\mathcal{L}_t)^{-1}(1-\Pi_t)\mathcal{M}_t(\rho_t)\,.$$

(Note that $\rho_t \in C^2$, but $\mathcal{M}_t$ is an operator from $C^2(S^1)$ to $C^1(S^1)$.)

Step 3: For every $t \in (-\epsilon, \epsilon)$, we have

$$(1-\mathcal{L}_t)^{-1}[(1-\Pi_t)\mathcal{M}_t(\rho_t)] = -(1-\mathcal{L}_t)^{-1}((X_t\rho_t)')\,.$$

Theorem 2.2 follows from putting together Steps 2 and 3. To conclude, we justify the three steps:

Proof of Step 1. We must show that the operators defined for $s \neq t$ by

$$\mathcal{R}_{t,s} := \frac{\mathcal{L}_t - \mathcal{L}_s}{t-s} - \mathcal{M}_t$$

satisfy $\lim_{s\to t} \|\mathcal{R}_{t,s}\|_{C^2(S^1)\to C^1(S^1)} = 0$. We start by observing that the number of branches of f_s (which is just its degree) does not depend on s. So for any fixed t and any x, each

inverse branch for $f_s^{-1}(x)$, for s close enough to t, can be paired with a well-defined nearby inverse branch $f_t^{-1}(x)$. For two such paired branches, we get, since $\varphi \in C^2$, each f_s is C^3, and $t \mapsto f_t$ is C^2, that

$$\begin{aligned}
&\frac{\varphi(f_t^{-1}(x)}{|f_t'(f_t^{-1}(x))|} - \frac{\varphi(f_s^{-1}(x)}{|f_s'(f_s^{-1}(x))|} \\
&= O(t^2) - (t-s)X_t'(x)\frac{\varphi(f_t^{-1}(x)}{|f_t'(f_t^{-1}(x))|} \\
&\quad - (t-s)X_t(x)\left[\frac{\varphi'(f_t^{-1}(x)}{f_t'(f_t^{-1}(x)|f_t'(f_t^{-1}(x)|} - \frac{\varphi(f_t^{-1}(x)f_t''(f_t^{-1}(x)}{(f_t'(f_t^{-1}(x)))^2|f_t'(f_t^{-1}(x)|}\right].
\end{aligned}$$

□

Proof of Step 2. Fix t. By Theorem 2.1, we can find a positively oriented closed curve γ in the complex plane so that, for any s close to t, the simple eigenvalue 1 of $\mathcal{L}_s$ is contained in the domain bounded by γ, and no other element of the spectrum of $\mathcal{L}_s$ acting on $C^2(S^1)$ lies in this domain. Step 2 then uses classical perturbation theory for isolated simple eigenvalues of bounded linear operators on Banach spaces (see [29], e.g., see also [36] for the use of similar ideas to get spectral stability), which tells us that, for any $\varphi \in C^2$ so that $\Pi_s(\varphi) = \int \varphi\, dx = 1$, we have

$$\rho_s = \frac{1}{2i\pi}\oint_\gamma (z - \mathcal{L}_s)^{-1}\varphi(z)\, dz\,. \tag{2.3}$$

(We used that $\int \rho_s\, dx = 1$ for all s and $\mathcal{L}_s^*(dx) = dx$.) Next, for $z \in \gamma$, we have the identity

$$(z - \mathcal{L}_t)^{-1} - (z - \mathcal{L}_s)^{-1} = (z - \mathcal{L}_t)^{-1}(\mathcal{L}_t - \mathcal{L}_s)(z - \mathcal{L}_s)^{-1}\,,$$

where we view $(z - \mathcal{L}_s)^{-1}$ as acting on $C^2(S^1)$, the difference $(\mathcal{L}_t - \mathcal{L}_s)$ as an operator from $C^2(S^1)$ to C^1, and $(z - \mathcal{L}_t)^{-1}$ as acting on $C^1(S^1)$. Letting s tend to t, and recalling Step 1, we have proved

$$\partial_s(z - \mathcal{L}_s)^{-1}|_{s=t} = (z - \mathcal{L}_t)^{-1}\mathcal{M}_t(z - \mathcal{L}_t)^{-1}\,.$$

Finally, taking (as we may) $\varphi = \rho_t \in C^2$ in (2.3),

$$\begin{aligned}
\partial_s\rho_s|_{s=t} &= \frac{1}{2i\pi}\oint_\gamma (z - \mathcal{L}_t)^{-1}\mathcal{M}_t(z - \mathcal{L}_t)^{-1}\rho_t(z)\, dz \\
&= \frac{1}{2i\pi}\oint_\gamma (z - \mathcal{L}_t)^{-1}\frac{\mathcal{M}_t(\rho_t(z))}{z-1}\, dz\,.
\end{aligned}$$

An easy residue computation completes Step 2. □

Proof of Step 3. It suffices to show $\mathcal{M}_t\rho_t - \Pi_t\mathcal{M}_t\rho_t = -(X_t\rho_t)'$. Step 1 implies

$$\mathcal{M}_t\rho_t = -X_t'\rho_t - X_t\mathcal{L}_t\left(\frac{\rho_t'}{f_t'} - \frac{\rho_t f_t''}{(f_t')^2}\right).$$

Now we use that $\rho_t' = (\mathcal{L}_t\rho_t)' \in C^1$ and

$$(\mathcal{L}_t\varphi)'(x) = \sum_{f_t(y)=x}\frac{\varphi'(y)}{|f_t'(y)|}\frac{1}{f_t'(y)} - \sum_{f_t(y)=x}\frac{\varphi(y)f_t''(y)}{|f_t'(y)|(f_t'(y))^2}\,,$$

to see that

$$\mathcal{L}_t\left(\frac{\rho_t'}{f_t'}-\frac{\rho_t f_t''}{(f_t')^2}\right)=\rho_t'\,.$$

We have shown that $\mathcal{M}_t\rho_t=-(X_t\rho_t)'$, so that $\int \mathcal{M}_t\rho_t\,dx=0$ and $\Pi_t\mathcal{M}_t\rho_t=0$, ending the proof of Step 3, and thus of the theorem. □

3. Linear response

3.1. Smooth hyperbolic dynamics (structural stability). A C^1 diffeomorphism $f:M\to M$ is called *Anosov* if there exist a Df-invariant continuous splitting $TM=E^u\oplus E^s$ of the tangent bundle and constants $C>0$ and $\lambda>1$ so that, for any $x\in M$, all $n\geq 1$, all $v\in E^s(x)$, and all $w\in E^u(x)$,

$$\|Df_x^n(v)\|\leq C\lambda^{-n}\|v\|\,,\ \ \|Df_x^{-n}(w)\|\leq C\lambda^{-n}\|w\|\,. \tag{3.1}$$

Thus, Anosov diffeomorphisms are generalizations of the linear hyperbolic map

$$A_0=\begin{pmatrix}1&1\\1&2\end{pmatrix} \tag{3.2}$$

on the two-torus. Indeed (we refer to [30], e.g., for the basics of hyperbolic dynamics), a small smooth perturbation of A_0 is an Anosov diffeomorphism. Anosov diffeomorphisms f admit (finitely many) SRB measures as soon as they are $C^{1+\epsilon}$, and the SRB measure is unique if the diffeomorphism is transitive.[7] For *Axiom A diffeomorphisms,* hyperbolicity (i.e., the existence of the continuous splitting $E^u\oplus E^s$) is assumed only at T_xM for points x in the nonwandering set Ω; in addition, periodic orbits are assumed to be dense in Ω. Smale's horseshoe is a famous Axiom A diffeomorphism, but SRB measures exist in general only for Axiom A *attractors,* such as the solenoid. (Anosov diffeomorphisms are special cases of Axiom A attractors.) An important property of Axiom A diffeomorphisms is *structural stability:* If f_0 is an Axiom A diffeomorphism, and f_t is close to f_0 (in the C^1 topology), then f_t is also Axiom A, and, in addition f_0 is topologically conjugated to f_t, i.e., there is a one-parameter family [8] of homeomorphisms h_t so that $f_t=h_t\circ f_0\circ h_t^{-1}$.

Linear response holds for smooth hyperbolic systems: After pioneering results of de la Llave et al. [40] and Katok et al. [31], Ruelle proved the following landmark theorem ([44, 45], see also [26]):

Theorem 3.1 (Linear response for smooth hyperbolic systems). *Let M be a compact Riemann manifold. Let $t\mapsto f_t$ be a C^3 map from $(-\epsilon,\epsilon)$ to C^3 diffeomorphisms $f_t:M\to M$. Assume that each f_t is a topologically mixing Axiom A attractor, and let μ_t be its unique SRB probability measure. Then for any $\varphi\in C^2$, the map $t\mapsto\int\varphi\,d\mu_t$ is differentiable on $(-\epsilon,\epsilon)$. In addition, setting $X_t=\partial f_s|_{s=t}\circ f_t^{-1}$, we have*

$$\partial_s\int\varphi\,d\mu_s|_{s=t}=\sum_{j=0}^{\infty}\langle\mathrm{grad}(\varphi\circ f_t^j),X_t\rangle\,d\mu_t\,, \tag{3.3}$$

[7] Transitivity is automatic if f is volume preserving. It is conjectured that all Anosov diffeomorphisms on connected compact manifolds are transitive.

[8] The map $t\mapsto h_t$ is smooth and its derivative α_t solves the twisted cohomological equation (4.4), see also [6] and references therein.

where the series converges (exponentially).

In this situation, one shows that the susceptibility function (1.3) is holomorphic in a disc of radius strictly bigger than one.

Ruelle exploited symbolic dynamics in [44, 45]. For a more modern approach, using anisotropic Banach spaces, see the work of Gouëzel and Liverani ([23, Thm 2.8] for Anosov, and [24, Prop. 8.1] for Axiom A). The modern approach is much simpler, since the transfer operators $\mathcal{L}_t$ of the diffeomorphisms f_t all have a uniform spectral gap on the same Banach space $\mathcal{B}$ of anisotropic distributions, which contains, not only the SRB measure μ_t, but also its "derivative." The "metaformula" (1.4) can then be easily justified rigorously.

3.2. Mild bifurcations. In § 4 we shall see examples where the breakdown of structural stability (the presence of bifurcations in the family f_t) is mirrored by a breakdown of linear response. However, structural stability is *not* necessary to obtain linear response – and neither is the spectral gap [9] of the transfer operator $\mathcal{L}_t$. We briefly describe a result of Dolgopyat [20] on a class of partially hyperbolic maps. We consider partially hyperbolic diffeomorphisms $f : M \to M$ on a smooth compact manifold M, i.e., we assume the tangent bundle is decomposed into invariant bundles $E^c \oplus E^u \oplus E^s$, where E^u and E^s are both nontrivial and enjoy (3.1). A partially hyperbolic diffeomorphism f is called an *Anosov element of a standard abelian Anosov action* if the central bundle E^c of f is tangent to the orbits of a C^∞ action g_t of $\mathbb{R}^d$ so that $fg_t = g_t f$ (see [32, 33]). Assume further that f admits a unique physical (SRB) measure μ, whose basin has total Lebesgue measure. The action is called *rapidly mixing* if for any $m \geq 1$ there exists $C \geq 1$ and a (g_t-admissible) class of smooth functions $\mathcal{F}$ so that, for all subsets S in a suitable class of unstable leaves of f, any $\varphi \in \mathcal{F}$, and for any smooth probability density ψ on S, we have

$$\left| \int_S (\varphi \circ f^n)(x)\psi(x)\, dx - \int \varphi \, d\mu \right| \leq C\|\varphi\|_{\mathcal{F}}\|\psi\| n^{-m} .$$

We refer to [20] for precise definitions of the objects above and of u-Gibbs states, we just recall here that SRB measures are u-Gibbs states. Dolgopyat's result follows:

Theorem 3.2 (Linear response for rapidly mixing abelian Anosov actions [20]). *Let f be a C^∞ Anosov element of a standard abelian Anosov action so that f has a unique SRB measure and is rapidly mixing. Then, for any C^∞ one-parameter family of diffeomorphisms $t \mapsto f_t$ through $f_0 = f$, choosing for each t a u-Gibbs state ν_t for f_t (which can be the SRB measure if it exists), we have that $\int \varphi \, d\nu_t$ is differentiable at $t = 0$ for any $\varphi \in C^\infty$, and the linear response formula* (3.3) *holds. (See [20, p. 405] for the linear response formula.)*

Besides giving a new proof in the Anosov case, applications of Theorem 3.2 include:

- time-one maps f of Anosov flows, which are generically rapidly mixing;
- toral extensions f of Anosov diffeomorphisms F defined by

$$f(x, y) = (F(x), y + \omega(x)) , \quad x \in M , \, y \in \mathbb{T}^d , \, \omega \in C^\infty(M, \mathbb{T}^d) ,$$

  which are generically rapidly mixing (under a diophantine condition).

It seems important here that structural stability may only break down in the central direction. This allows Dolgopyat to use rapid mixing to prove that most orbits can be shadowed, a key feature of his argument.

[9] See the work of Hairer and Majda [25].

4. Or else

The results stated in § 3.1 gave at the time some hope [49] that linear response could hold (at least in the sense of Whitney) for a variety of nonuniformly hyperbolic systems. In the present section we shall state some results obtained since 2007 which indicate that the situation is not so simple. We would like to mention that numerical experiments and physical arguments already gave a hint that something could go wrong (see [21], e.g., for fractal transport, see [35]).

4.1. Piecewise expanding interval maps. Piecewise expanding maps can be viewed as a toy model for the smooth unimodal maps to be discussed in § 4.2. The setting is the following: We let $I = [-1, 1]$ be a compact interval, and consider continuous maps $f : I \to I$ with $f(-1) = f(1) = -1$, and so that $f|_{[-1,0]}$ and $f|_{[0,1]}$ are C^2, with $\inf_{x\neq c} |f'(x)| \geq \lambda > 1$. Such a map is called a *piecewise expanding unimodal map* (for λ). Lasota and Yorke [39] proved in the 70's that such a map posesses a unique absolutely continuous invariant probability measure $\mu = \rho\, dx$, which is always ergodic. In fact, the density ρ is of bounded variation. If μ is mixing, we have exponential decay of correlations for smooth observables, which can be proved by using the spectral gap of the transfer operator $\mathcal{L}_t$ defined by (2.1) acting on the Banach space BV of functions of bounded variation, see e.g. [4]. We set $c = c_0 = 0$, and we put $c_k = f^k(c)$ for $k \geq 1$.

Consider now a C^1 path $t \mapsto f_t$, with each f_t a piecewise expanding unimodal map. Assume in addition that $f_0 = f$ is topologically mixing on $[c_2, c_1]$ (then $\mu = \mu_0$ is mixing), that $c_1 < 1$, and that c is not a periodic point of f_0 (this implies that f_0 is stably mixing, i.e., small perturbations of f_0 remain mixing). Then, applying [39], each f_t admits a unique SRB measure $\mu_t = \rho_t\, dx$ (and each transfer operator $\mathcal{L}_t$ has a spectral gap on BV, the corresponding estimates are in fact uniform). Keller [34] proved that the map

$$t \mapsto \rho_t \in L^1(dx)$$

is Hölder for every exponent $\eta < 1$. In fact, Keller showed

$$\|\rho_t - \rho_s\|_{L^1} \leq C|t - s||\log|t - s||\,. \tag{4.1}$$

From now on, we assume that each f_t is piecewise C^3, that the map $t \mapsto f_t$ is C^2, and that $v = \partial_t f_t|_{t=0} = X \circ f$. An example is given by taking the *tent maps*

$$\begin{aligned} f_t(x) &= a + t - (a + t + 1)x\,, \text{ if } x \in [0, 1]\,,\\ f_t(x) &= a + t + (a + t + 1)x\,, \text{ if } x \in [-1, 0]\,, \end{aligned} \tag{4.2}$$

choosing $0 < a < 1$ so that 0 is not periodic for f_a and so that f_a is mixing (note that $X_0(x) = (a + 1)^{-1}(x + 1)$). Observe that structural stability is strongly violated here: f_t is topologically conjugated to f_s only if $s = t$ [18]. In other words, the family f_t of tent maps undergoes strong bifurcations.

A piecewise expanding map is called *Markov* if c is preperiodic, that is, if there exists $j \geq 2$ so that c_j is a periodic point: $f^p(c_j) = c_j$ for some $p \geq 1$. (In this case, one can show that the invariant density is piecewise smooth, and the susceptibility function is meromorphic.) A Markov map is mixing if its transition matrix is aperiodic, stable mixing then allows to construct easily mixing tent maps.

It turns out that Keller's upper bound (4.1) is optimal, *linear response fails:*

Theorem 4.1 (Mazzolena [42], Baladi [5]). *There exist a Markov piecewise expanding interval map f_0, a path f_t through f_0, with a C^∞ observable φ, a constant $C > 0$, and a sequence $t_n \to 0$, so that*

$$\left| \int \varphi \, d\mu_{t_n} - \int \varphi \, d\mu_0 \right| \geq C|t_n||\log|t_n|| \,, \quad \forall n \,.$$

Setting $v = v_0 = \partial_t f_t|_{t=0}$, and assuming $v = X \circ f$, we introduce

$$\mathcal{J}(f, v) = \sum_{j=0}^{\infty} \frac{v(f^j(c))}{(f^j)'(c_1)} = \sum_{j=0}^{\infty} \frac{X(f^j(c_1))}{(f^j)'(c_1)} \,. \tag{4.3}$$

If $\mathcal{J}(f_0, v_0) = 0$ (a codimension-one condition on the perturbation v or X), we say that the path f_t is *horizontal* (at $t = 0$). This condition was first studied for smooth unimodal maps [3, 60]. In the setting of piecewise expanding unimodal maps, Smania and I proved the following result:

Theorem 4.2 (Horizontality and tangency to the topological class [9, 10]). *A path f_t is called tangent to the topological class of f_0 (at $t = 0$) if there exist a path $\tilde{f}_t$ so that $f_t - \tilde{f}_t = O(t^2)$ and homeomorphisms h_t so that $\tilde{f}_t \circ h_t = h_t \circ f_0$. Then:*

- *The path f_t is horizontal (at $t = 0$) if and only if there is a continuous solution α to the twisted cohomological equation*

$$v(x) = X \circ f(x) = \alpha \circ f(x) - f'(x)\alpha(x) \,, \quad x \neq c \,. \tag{4.4}$$

- *The path f_t is horizontal (at $t = 0$) if and only if it is tangent to the topological class of f_0 (at $t = 0$).*

Note that the family of tent maps given in (4.2) is *not* horizontal.

We already mentioned that $\rho_t \in BV$. Any function g of bounded variation can be decomposed as two functions of bounded variation $g = g^{sing} + g^{reg}$, where the regular component g^{reg} is a continuous function of bounded variation, while the singular component g^{sing} is an at most countable sum of jumps (i.e., Heaviside functions). In the particular case of the invariant density ρ_t of a piecewise expanding unimodal map, we proved [5] that $(\rho_t^{reg})'$ is of bounded variation, while the jumps of ρ_t^{sing} are located along the postcritical orbit c_k, with exponentially decaying weights, so that $(\rho_t^{sing})'$ is an exponentially decaying sum of Dirac masses along the postcritical orbit. *The fact that the derivative of ρ_0 does not belong to a space on which the transfer operator has a spectral gap is the glitch which disrupts the spectral perturbation mechanism described in Section 2* (in Section 3.1 the derivative of the distribution corresponding to the SRB measure *did* belong to a good space of anisotropic distributions). Note also that ρ_0^{sing} is intimately related to the postcritical orbit of f_0, which is itself connected to the bifurcation structure of f_t at f_0. (We refer also to [6].)

Our main result with Smania on piecewise expanding maps reads as follows:

Theorem 4.3 (Horizontality and linear response [9]).

- *If the path f_t is horizontal (at $t = 0$) then the map $t \mapsto \mu_t \in C(I)^*$ is differentiable at $t = 0$ (as a Radon measure), and we have the linear response formula:*

$$\partial_t \mu_t|_{t=0} = -\alpha(\rho^{sing})' - (1 - \mathcal{L}_0)^{-1}(X'\rho^{sing} + (X\rho^{reg})') \, dx \,. \tag{4.5}$$

- *If the path f_t is not horizontal (at $t=0$), then, if in addition $|f'(c_-)| = |f'(c_+)|$ or $\inf_j d(f^j(c), c) > 0$, we have:*

  If the postcritical orbit $\{c_k\}$ is not [10] dense in $[c_2, c_1]$, then there exist $\varphi \in C^\infty$ and $K > 0$ so that for any sequence $t_n \to 0$ so that the postcritical orbit of each f_{t_n} is infinite,

$$\left| \int \varphi \, d\mu_{t_n} - \int \varphi \, d\mu_0 \right| \geq K|t_n||\log|t_n|| \,, \quad \forall n \,. \tag{4.6}$$

  If the postcritical orbit is dense in $[c_2, c_1]$, then there exist $\varphi \in C^\infty$ and sequences $t_n \to 0$ so that

$$\lim_{n\to\infty} \frac{\left| \int \varphi \, d\mu_{t_n} - \int \varphi \, d\mu_0 \right|}{|t_n|} = \infty \,. \tag{4.7}$$

We end this section with some of our results on the susceptibility function (recall (1.3))

$$\Psi_\varphi(z) = \sum_{j=0}^{\infty} \int z^j (\partial_x(\varphi \circ f_0^j)(x)) X_0(x) \, d\mu_0(x)$$

of piecewise expanding unimodal maps (for $\lambda > 1$), the most recent of which were obtained with Marmi and Sauzin (using work of Breuer and Simon [16]):

Theorem 4.4 ([5, 8]). *There exists a nonzero function $\mathcal{U}(z)$, holomorphic in $|z| > \lambda^{-1}$, and, for every non constant $\varphi \in C^0$ so that $\int \varphi \, d\mu_0 = 0$, there exists a nonzero function $\mathcal{V}_\varphi(z)$, holomorphic in $|z| > \lambda^{-1}$, so that the following holds: Put*

$$\sigma_\varphi(z) = \sum_{j=0}^{\infty} \varphi(c_{j+1}) z^j$$

(this function is holomorphic in the open unit disc), and set

$$\Psi^{hol}(z) = -\int \varphi(x)(1 - z\mathcal{L}_0)^{-1}(X'\rho^{sing} + (X\rho^{reg})')(x) \, dx \,.$$

Then:

- *There exists $\tau \in (0,1)$ so that $\Psi^{hol}(z)$ is holomorphic in the disc $|z| < \tau^{-1}$.*
- *The susceptibility function satisfies*

$$\Psi_\varphi(z) = \sigma_\varphi(z)\mathcal{U}(z) + \mathcal{V}_\varphi(z) + \Psi^{hol}(z) \,,$$

  where the function $\mathcal{U}(z)$ vanishes at $z = 1$ if and only if $\mathcal{J}(f, v) = 0$, and in that case, we have

$$\partial_t \int \varphi \, d\mu_t|_{t=0} = \mathcal{V}_\varphi(1) + \Psi^{hol}(1) \,.$$

[10] Generically the postcritical orbit is dense, see the references to Bruin in [56].

- *If $\{c_k\}$ is dense in $[c_2, c_1]$ and $\varphi \neq 0$, then the unit circle is a (strong) natural boundary for $\sigma_\varphi(z)$ (and thus for $\Psi_\varphi(z)$). If* [11] $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \tilde{\varphi}(c_k) = \int \tilde{\varphi}\, d\mu_0$ *for every $\tilde{\varphi} \in C^0$, then for every $\omega \in \mathbb{R}$*

$$\lim_{z \overset{NT}{\to} e^{i\omega}} (z - e^{i\omega})\sigma_\varphi(z) = 0\,,$$

where $z \overset{NT}{\to} e^{i\omega}$ means that $|z| < 1$ tends to $e^{i\omega}$ nontangentially (e.g., radially).

In particular, if the path f_t is horizontal (at $t = 0$) and the postcritical orbit is generic, then

$$\partial_t \int \varphi\, d\mu_t|_{t=0} = \lim_{z \overset{NT}{\to} 1} \Psi_\varphi(z)\,.$$

The law of the iterated logarithm (LIL), a property stronger than Birkhoff genericity, also holds generically for the postcritical orbit of piecewise expanding maps [57]. If the postcritical orbit satisfies (an $e^{i\omega}$ twisted upper bound version of) the LIL, then more can be said about σ_φ and Ψ_φ, see [8, Thm. 5].

Inspired by Breuer–Simon, we introduced in [8] *renacent right-limits,* a simple construction for candidates for a generalised (Borel monogenic [15], e.g.) continuation outside of the unit disc of power series having the unit circle as a natural boundary. In the case of Poincaré simple pole series, Sauzin and Tiozzo [55] showed that this construction gives the (unique) generalised continuation. However, for the susceptibility function of piecewise expanding maps, there are [8] uncountably many such candidates (even in the horizontal case). This may indicate that there is no reasonable way to extend $\Phi_\varphi(z)$ outside of the unit circle. The analogous problem is more delicate for smooth unimodal maps discussed in § 4.2 below, mainly because the natural boundary for the susceptibility function is expected to lie strictly inside the open unit disc — we refer to [8] for open questions and conjectures.

4.2. Smooth unimodal maps. We now consider the more difficult case of *differentiable* maps $f : I \to I$, where $I = [0, 1]$ is again a compact interval, and $c = 1/2$ is now a critical point in the usual sense: $f'(c) = 0$. The map f is still assumed unimodal, with $f(-1) = f(1) = -1$, and $f'(x) > 0$ for $-1 \leq x < c$, while $f'(x) < 0$ for $c < x \leq 1$. We denote $c_k = f^k(c)$ for $k \geq 1$ as before. For convenience, we assume that f is topologically mixing and C^3, with negative Schwarzian derivative (see [18]). Finally, we suppose that $f''(c) < 0$. Of course, f is *not* uniformly expanding since $f'(c) = 0$. One way to guarantee enough (nonuniform) expansion is via the Collet–Eckmann condition: The map f is *Collet–Eckmann* (CE) if there exists $\lambda_c > 1$ and $H_0 \geq 1$ so that

$$|(f^k)'(c_1)| \geq \lambda_c^k\,, \quad \forall k \geq H_0\,.$$

If f is CE, then it admits a (unique) absolutely continuous (SRB) invariant probability measure $\mu = \rho\, dx$ (which is ergodic). We refer to [18] for more about the CE condition, noting here only that the invariant density ρ is not bounded in the current setting — in fact, ρ contains a finite, or infinite exponentially decaying, sum of "spikes"

$$\sqrt{|x - f^k(c)|}^{-1}$$

[11] This assumption of Birkhoff genericity of the postcritical orbit is generic [56].

along the postcritical orbit. Thus, $\rho \in L^p$ for all $1 \le p < 2$, but $\rho \notin L^2$. If f is CE and topologically mixing on $[c_2, c_1]$, then Keller and Nowicki [37] and, independently, Young [63], proved that a spectral gap holds for a suitably defined transfer operator (acting on a "tower"), giving exponential decay of correlations.

We consider again a C^2 path $t \mapsto f_t$, $t \in (-\epsilon, \epsilon)$, say, of C^4 unimodal maps as above, through $f = f_{t_0}$ (with t_0 not necessarily equal to 0) which will be assumed to be (at least) CE. We let $v = v_{t_0} = \partial_t f_t|_{t=t_0}$ and assume that $v = X \circ f$. Noting that $\mathcal{J}(f, v)$ from (4.3) is well defined because of the CE condition, we say that the path f_t is horizontal at $t = t_0$ if $\mathcal{J}(f, v) = 0$.

The fully horizontal case (i.e., $\mathcal{J}(f_t, v_t) = 0$ for all t in a neighbourhood of t_0) happens when f_t is topologically conjugated to f_{t_0} for all t, so that f_t stays in the topological class of f_{t_0}. Then, if f_{t_0} is Collet-Eckmann, all the f_t are Collet-Eckmann (although it is not obvious from the definition, the CE property is a topological invariant [43]) and admit an SRB measure. In this fully horizontal case, viewing ρ_t as a distribution of sufficiently high order, first Ruelle [51] and then Smania and myself [11, 12] obtained linear response, with a linear response formula. (In [11], we even obtain analyticity of the SRB measure.) More precisely, Ruelle [51] considered the analytic case under the Misiurewicz [12] assumption that $\inf_k |f_{t_0}^k(c) - c| > 0$; Smania and myself considered on the one hand [11] a fully holomorphic setting (where the powerful machinery of Mañé-Sad-Sullivan [41] applies), and on the other hand [12] a finitely differentiable setting under a (generic) Benedicks-Carleson-type assumption of topological slow recurrence. The strategy in [12] involves proving the existence of a continuous solution α to the twisted cohomological equation (4.4) if f is Benedicks-Carleson and X corresponds to a horizontal path f_t.

Although the horizontal case is far from trivial (in the present nonuniformly expanding setting, one of the hurdles is to obtain uniform bounds on the constant $\lambda_c(t)$ for CE parameters t close to t_0), it is much more interesting to explore *transversal paths* $t \mapsto f_t$ (undergoing topological bifurcations). The archetypal such situation is given by the so-called *logistic* (or quadratic) family

$$f_t(x) = tx\,(1 - x)\,.$$

A famous theorem of Jacobson says that the set of CE parameters in the logistic family has strictly positive Lebesgue measure (see [18], e.g.). Since the set Λ of CE parameters does not contain any interval, regularity of the map $t \mapsto \mu_t$ for t in Λ can be considered only in the sense of Whitney. Continuity of the map $t \mapsto \mu_t$, for t ranging in some appropriate subset of Λ (and for the weak $*$ topology in the image) was obtained by Tsujii [61] (see also Rychlik-Sorets [54]) in the 90's.

A map f is called *Misiurewicz-Thurston* if there exist $j \ge 2$ and $p \ge 1$ so that $f^p(c_j) = c_j$ and $|(f^p)'(c_j)| > 1$ (in other words, the critical point is *preperiodic,* towards a repelling periodic orbit, this implies that the map has a finite Markov partition). Clearly, Misiurewicz-Thurston implies Misiurewicz and thus Collet-Eckmann. There are only countably many Misiurewicz-Thurston parameters.

For the quadratic family, e.g., Thunberg proved [59, Thm C] that there are superstable parameters s_n of periods p_n, with $s_n \to t$, for a Collet-Eckmann parameter t, so that $\nu_{s_n} \to \nu$, where $\nu_{s_n} = \frac{1}{p_n} \sum_{k=0}^{p_n - 1} \delta_{f_{s_n}^k(c)}$, and ν is the sum of atoms on a repelling periodic orbit of f_t. Other sequences $t_n \to t$ of superstable parameters have the property that $\nu_{t_n} \to \mu_t$, the absolutely continuous invariant measure of f_t. Starting from Thunberg's result, Dobbs

[12] Misiurewicz is nongeneric. It implies Collet-Eckmann.

and Todd [19] have constructed sequences of both renormalisable and non-renormalisable Collet-Eckmann maps $f_{t'_n}$, converging to a Collet-Eckmann map f_t, but such that the SRB measures do not converge. Such counter-examples can be constructed while requiring that f_t and all maps $f_{t'_n}$ are Misiurewicz-Thurston. These examples show that continuity of the SRB measure cannot hold on the set of *all* Collet–Eckmann (or even Misiurewicz-Thurston) parameters: Some uniformity in the constants is needed (already when defining the "appropriate subsets" of [61]).

The main result of our joint work [7] with Benedicks and Schnellmann (which also contains parallel statements on more general transversal familes of smooth unimodal maps) follows:

Theorem 4.5 (Hölder continuity of the SRB measure in the logistic family [7]). *Consider the quadratic family $f_t(x) = tx(1-x)$ on $I = [0,1]$, and let $\Lambda \subset (2,4]$ be the set of Collet-Eckmann parameters t.*

- *There exists $\Delta \subset \Lambda$, of full Lebesgue measure in Λ, so that for every $t_0 \in \Delta$, and for every $\Gamma > 4$, there exists $\Delta_{t_0} \subset \Delta$, with t_0 a Lebesgue density point of Δ_{t_0}, and there exists a constant C so that, for any $\varphi \in C^{1/2}(I)$, for any sequence $t_n \to t_0$, so that $t_n \in \Delta_{t_0}$ for all n, we have*

$$\left| \int \varphi(x) d\mu_{t_n} - \int \varphi(x) d\mu_{t_0} \right| \leq C \, |\varphi|_{C^{1/2}} \, |t_0 - t_n|^{1/2} |\log |t_0 - t_n||^{\Gamma} , \qquad (4.8)$$

  where $\|\varphi\|_{C^{1/2}}$ denotes the $1/2$-Hölder norm of φ.

- *If f_{t_0} is Misiurewicz-Thurston, then there exists $\varphi \in C^\infty$, a constant $C > 1$, and a sequence $t_n \to t_0$, with $t_n \in \Lambda$ for all n, so that*

$$\frac{1}{C} |t_n - t_0|^{1/2} \leq \left| \int \varphi(x) d\mu_{t_n} - \int \varphi(x) d\mu_{t_0} \right| \leq C |t_n - t_0|^{1/2} . \qquad (4.9)$$

The exponent $1/2$ appearing in the theorem is directly related to the nondegeneracy assumption $f''(c) \neq 0$, which of course holds true for the quadratic family. Note also that using a C^∞ (instead of $C^{1/2}$) observable does not seem to allow better upper bounds in (4.8). It is unclear if the logarithmic factor in (4.8) is an artefact of the proof or can be discarded.

The proof of the claim (4.9) of the theorem gives a sequence t_n of Misiurewicz-Thurston parameters, but the continuity result of Tsujii [61] easily yields sequences of non Misiurewicz-Thurston (but CE) parameters t_n. We do *not know* whether t_0 is a Lebesgue density point of the set of sequences giving (4.9). Note that in the toy model from § 4.1, the first analogous construction of counter-examples (Theorem 4.1) was limited to a handful of preperiodic parameters (sequences of maps having preperiodic critical points converging to a map f_{t_0} with a preperiodic critical point), while the currently known set of examples (see (4.6) and (4.7)) are much more general, although not fully satisfactory yet. One important open problem is to describe precisely the set of sequences $t_n \to t_0$ giving rise to violation of linear response for the generic piecewise expanding unimodal maps with dense postcritical orbits in (4.7). This may give useful insight for smooth unimodal maps, both about the largest possible set of sequences giving (4.9), and about relaxing the Misiurewicz-Thurston assumption on f_{t_0}. (Note however that there is a quantitative difference with respect to the piecewise expanding

case [9], where the modulus of continuity in the transversal case was $|\log|t-t_0||\,|t-t_0|$, so that violation of linear response arose from the logarithmic factor alone.)

We suggested in [7] the following weakening of the linear response problem: Consider a one-parameter family f_t of (say, smooth unimodal maps) through f_{t_0} and, for each $\epsilon>0$, a random perturbation of f_t with unique invariant measure μ_t^ϵ like in [58], e.g. Then for each positive ϵ, it should not be very difficult to see that the map $t\to\mu_t^\epsilon$ is differentiable at t_0 (for essentially any topology in the image). Taking a weak topology in the image, like Radon measures, or distributions of positive order, does the limit as $\epsilon\to 0$ of this derivative exist? How is it related with the perturbation? with the susceptibility function or some of its generalised continuations (e.g. in the sense of [8])?

More open questions are listed in [6] and [7, 12]. In particular, the results in [7] give hope that linear response or its breakdown (see [6] and [53]) can be studied for (the two-dimensional) Hénon family, which is transversal, and where continuity of the SRB measure in the weak-$*$ topology was proved by Alves et al. [1, 2] in the sense of Whitney on suitable parameter sets. In [6, (17), (19)], we also give candidates for the notion of horizontality for piecewise expanding maps in higher dimensions and piecewise hyperbolic maps.

4.3. About the proofs. The main tool in the proof of Theorem 4.5 is a *tower construction:* We wish to compare the SRB measure of f_{t_0} to that of f_t for small $t-t_0$. Just like in [12], we use transfer operators $\widehat{\mathcal{L}}_t$ acting on towers, with a projection Π_t from the tower to $L^1(I)$ so that $\Pi_t\widehat{\mathcal{L}}_t=\mathcal{L}_t\Pi_t$, where $\mathcal{L}_t$ is the usual transfer operator, and $\Pi_t\hat\rho_t=\rho_t$ with $\mu_t=\rho_t\,dx$ (here, $\hat\rho_t$ is the fixed point of $\widehat{\mathcal{L}}_t$, and ρ_t is the invariant density of f_t). In [12], we adapted the tower construction from [13] (introduced in [13] to study random perturbations, for which this version is better suited than the otherwise ubiquitous Young towers [64]). This construction allows in particular to work with Banach spaces of continuous functions. Another idea imported from [12] is the use of operators $\widehat{\mathcal{L}}_{t,M}$ acting on truncated towers, where the truncation level M must be chosen carefully depending on $t-t_0$. Roughly speaking, the idea is that f_t is comparable to f_{t_0} for M iterates (corresponding to the M lowest levels of the respective towers), this is the notion of an *admissible pair* (M,t). Denoting by $\hat\rho_{t,M}$ the maximal eigenvector of $\widehat{\mathcal{L}}_{t,M}$, the starting point for both upper and lower bounds is (like in [12]) the decomposition

$$\begin{aligned}\rho_t-\rho_{t_0}=&\left[\Pi_t(\hat\rho_t-\hat\rho_{t,M})+\Pi_{t_0}(\hat\rho_{t_0,M}-\hat\rho_{t_0})\right]\\&+\left[\Pi_t(\hat\rho_{t,M}-\hat\rho_{t_0,M})\right]+\left[(\Pi_t-\Pi_{t_0})(\hat\rho_{t_0,M})\right],\end{aligned}\tag{4.10}$$

for admissible pairs. The idea is then to get upper bounds on the first two terms by using perturbation theory à la Keller-Liverani [36], and to control the last (dominant) term by explicit computations on $\Pi_t-\Pi$ (which represents the "spike displacement," i.e., the effect of the replacement of $1/\sqrt{|x-f_{t_0}^k(c)|}$ by $1/\sqrt{|x-f_t^k(c)|}$ in the invariant density).

We now move to the differences between [12] and [7]: Using a tower with exponentially decaying levels as in [13] or [12] would provide at best an upper modulus of continuity $|t-t_0|^\eta$ for $\eta<1/2$, and would not yield any lower bound. For this reason, we use instead "fat towers" with *polynomially decaying* sizes in [7], working with polynomially recurrent maps. In order to construct the corresponding parameter set, we use recent results of Gao and Shen [22].

Applying directly the results of Keller-Liverani [36] would only bound the contributions of the first and second terms of (4.10) by $|t-t_0|^\eta$ for $\eta<1/2$. In order to estimate the second

term, we prove that $\widehat{\mathcal{L}}_{t,M} - \widehat{\mathcal{L}}_{t_0,M}$ *acting on the maximal eigenvector* is $O(|\log|t-t_0||^{\Gamma}|t-t_0|^{1/2})$ *in the strong* [13] *norm;* in the Misiurewicz-Thurston case we get get a better $O(|t-t_0|^{1/2})$ control). It is usually not possible to obtain strong norm bounds when bifurcations are present [14, 36], and this remarkable feature here is due to our choice of admissible pairs (combined with the fact that the towers for f_t and f_{t_0} are identical up to level M). To estimate the first term, we enhance the Keller-Liverani argument, using again that it suffices to estimate the perturbation for the operators acting on the maximal eigenvector.

The changes just described are already needed to obtain the exponent $1/2$ in the upper bound (4.8). To get lower bound in (4.9), we use that the tower associated to a Misiurewicz-Thurston map f_{t_0} can be required to have levels with sizes bounded from below, and that the truncation level can be chosen to be slightly larger. Finally, working with Banach norms based on L^1 as in [12] would give that the first two terms in (4.10) are $\leq C|t-t_0|^{1/2}$, while the third is $\geq C^{-1}|t-t_0|^{1/2}$ for some large constant $C > 1$. However, *introducing Banach-Sobolev norms based on* L^p *for* $p > 1$ instead, we are able to control the constants and show that the last term dominates the other two.

Acknowledgements. The toy model in Section 2 was presented at a minicourse at the Dynamical Systems Days in Antofagasta, Chile, December 2007.

References

[1] Alves, J.F. and Carvalho, M., Freitas, J.M., *Statistical stability for Hénon maps of the Benedicks-Carleson type*, Ann. Inst. H. Poincaré Anal. Non Linéaire **27** (2010), 595–637.

[2] ———, *Statistical stability and continuity of SRB entropy for systems with Gibbs-Markov structures*, Comm. Math. Phys. **296** (2010), 739–767.

[3] Avila, A., Lyubich, M., de Melo, W., *Regular or stochastic dynamics in real analytic families of unimodal maps*, Invent. Math. **154** (2003), 451–550.

[4] Baladi, V., *Positive Transfer Operators and Decay of Correlations*, Advanced Series in Nonlinear Dynamics 16, World Scientific, Singapore, 2000.

[5] ———, *On the susceptibility function of piecewise expanding interval maps*, Comm. Math. Phys. **275** (2007), 839–859.

[6] ———, *Linear response despite critical points*, Nonlinearity **21** (2008), T81–T90.

[7] Baladi, V., Benedicks, M., and Schnellmann, D., *Whitney–Hölder continuity of the SRB measure for transversal families of smooth unimodal maps*, Preprint, arXiv:1302.3191 (2013).

[8] Baladi, V., Marmi, S., and Sauzin, D., *Natural boundary for the susceptibility function of generic piecewise expanding unimodal maps*, Ergodic Theory Dynam. Systems **34** (2014), 777–800.

[13] The strong norm plays here the role of C^j in (2.2).

[9] Baladi, V. and Smania, D., *Linear response formula for piecewise expanding unimodal maps*, Nonlinearity **21** (2008), 677–711. (Corrigendum, *Nonlinearity* **25** (2012), 2203–2205.)

[10] ———, *Smooth deformations of piecewise expanding unimodal maps*, DCDS Series A **23** (2009), 685–703.

[11] ———, *Analyticity of the SRB measure for holomorphic families of quadratic-like Collet-Eckmann maps*, Proc. Amer. Math. Soc. **137** (2009), 1431–1437.

[12] ———, *Linear response for smooth deformations of generic nonuniformly hyperbolic unimodal maps*, Ann. Sci. Éc. Norm. Sup. **45** (2012), 861–926.

[13] Baladi, V. and Viana, M., *Strong stochastic stability and rate of mixing for unimodal maps*, Ann. Sci. Éc. Norm. Sup. **29** (1996), 483–517.

[14] Baladi, V. and Young, L.-S., *On the spectra of randomly perturbed expanding maps*, Comm. Math. Phys. **156** (1993), 355–385.

[15] E. Borel, *Leçons sur les fonctions monogènes uniformes d'une variable complexe*, Gauthier-Villars, Paris, 1917.

[16] Breuer, J. and Simon, B., *Natural boundaries and spectral theory*, Adv. Math. **226** (2011), 4902–4920.

[17] Butterley, O. and Liverani, C., *Smooth Anosov flows: correlation spectra and stability*, J. Mod. Dyn. **1** (2007), 301–322.

[18] de Melo, W. and van Strien, S., *One-Dimensional Dynamics*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, Berlin, 1993.

[19] Dobbs, N. and Todd, M., *Entropy jumps up, Manuscript* (2014).

[20] Dolgopyat, D., *On differentiability of SRB states for partially hyperbolic systems*, Invent. Math. **155** (2004), 389–449.

[21] Ershov, S.V., *Is a perturbation theory for dynamical chaos possible?*, Physics Letters A **177** (1993), 180–185.

[22] Gao, B. and Shen, W., *Summability implies Collet–Eckmann almost surely*, Ergodic Theory Dynam. Systems, DOI: http://dx.doi.org/10.1017/etds.2012.173.

[23] Gouëzel, S. and Liverani, C., *Banach spaces adapted to Anosov systems*, Ergodic Theory Dynam. Systems **26** (2006), 189–217.

[24] ———, *Compact locally maximal hyperbolic sets for smooth maps: fine statistical properties*, J. Diff. Geom. **79** (2008), 433–477.

[25] Hairer, M. and Majda, A.J., *A simple framework to justify linear response theory*, Nonlinearity **23** (2010), 909–922.

[26] Jiang, M., *Differentiating potential functions of SRB measures on hyperbolic attractors*, Ergodic Theory Dynam. Systems **32** (2012), 1350–1369.

[27] Jiang, M. and de la Llave, R., *Linear response function for coupled hyperbolic attractors*, Comm. Math. Phys. **261** (2006), 379–404.

[28] _____, *Smooth dependence of thermodynamic limits of SRB-measures*, Comm. Math. Phys. **211** (2000), 303–333.

[29] Kato, T., *Perturbation Theory for Linear Operators*, Second corrected printing of the second edition, Springer-Verlag, Berlin, 1984.

[30] Katok, A. and Hasselblatt, B., *Introduction to the modern theory of dynamical systems*, Encyclopedia of Mathematics and its Applications, 54. Cambridge University Press, Cambridge, 1995.

[31] Katok, A., Knieper, G., Pollicott, M., and Weiss, H., *Differentiability and analyticity of topological entropy for Anosov and geodesic flows*, Invent. Math. **98** (1989), 581–597.

[32] Katok, A. and Spatzier, R.J., *First cohomology of Anosov actions of higher rank abelian groups and applications to rigidity*, Publ. Math., Inst. Hautes Étud. Sci. **79** (1994), 131–156.

[33] _____, *Invariant measures for higher-rank hyperbolic abelian actions*, Ergodic Theory Dynam. Systems **16** (1996), 751–778.

[34] Keller, G., *Stochastic stability in some dynamical systems*, Monatshefte Math. **94** (1982), 313–333.

[35] Keller, G., Howard, P.J., and Klages, R., *Continuity properties of transport coefficients in simple maps*, Nonlinearity **21** (2008), 1719–1743.

[36] Keller, G. and Liverani, C., *Stability of the spectrum for transfer operators*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **28** (1999), 141–152.

[37] Keller, G. and Nowicki, T., *Spectral theory, zeta functions and the distribution of periodic points for Collet–Eckmann maps*, Comm. Math. Phys. **149** (1992), 633–680.

[38] Krzyżewski, K. and Szlenk, W., *On invariant measures for expanding differentiable mappings*, Studia Math. **33** (1969), 83–92.

[39] Lasota, A. and Yorke, J.A., *On the existence of invariant measures for piecewise monotonic transformations*, Trans. Amer. Math. Soc. **186** (1973), 481–488.

[40] de la Llave, R., Marco, J.M., and Moriyón, R., *Canonical perturbation theory of Anosov systems and regularity results for the Livšic cohomology equation*, Ann. of Math. **123** (1986), 537–611.

[41] Mañé, R., Sad, P., and Sullivan, D., *On the dynamics of rational maps*, Ann. Sci. École Norm. Sup. **16** (1983), 193–217.

[42] Mazzolena, M., *Dinamiche espansive unidimensionali: dipendenza della misura invariante da un parametro*, Master's Thesis Roma 2, Tor Vergata (2007).

[43] Nowicki, T. and Przytycki, F., *Topological invariance of the Collet-Eckmann property for S-unimodal maps*, Fund. Math. **155** (1998), 33–43.

[44] Ruelle, D., *Differentiation of SRB states*, Comm. Math. Phys. **187** (1997), 227–241.

[45] ———, *Differentiation of SRB states: Corrections and complements*, Comm. Math. Phys. **234** (2003), 185–190.

[46] ———, *General linear response formula in statistical mechanics, and the fluctuation-dissipation theorem far from equilibrium*, Phys. Lett. A **245** (1998), 220–224.

[47] ———, *Nonequilibrium statistical mechanics near equilibrium: computing higher-order terms*, Nonlinearity **11** (1998), 5–18.

[48] ———, *Application of hyperbolic dynamics to physics: some problems and conjectures*, Bull. Amer. Math. Soc. **41** (2004), 275–278.

[49] ———, *Differentiating the absolutely continuous invariant measure of an interval map f with respect to f*, Comm. Math. Phys. **258** (2005), 445–453.

[50] ———, *Differentiation of SRB states for hyperbolic flows*, Ergodic Theory Dynam. Systems **28** (2008), 613–631.

[51] ———, *Structure and f-dependence of the A.C.I.M. for a unimodal map f of Misiurewicz Type*, Comm. Math. Phys. **287** (2009), 1039–1070.

[52] ———, *A review of linear response theory for general differentiable dynamical systems*, Nonlinearity **22** (2009), 855–870.

[53] ———, *Singularities of the susceptibility of a Sinai-Ruelle-Bowen measure in the presence of stable-unstable tangencies*, Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **369** (2011), 482–493.

[54] Rychlik, M. and Sorets, E., *Regularity and other properties of absolutely continuous invariant measures for the quadratic family*, Comm. Math. Phys. **150** (1992), 217–236.

[55] Sauzin, D. and Tiozzo, G., *Generalised continuation by means of right limits*, Preprint, arXiv:1301.1175 (2013).

[56] Schnellmann, D., *Typical points for one-parameter families of piecewise expanding maps of the interval*, Discrete Contin. Dyn. Syst. **31** (2011), 877–911.

[57] ———, *Law of iterated logarithm and invariance principle for one-parameter families of interval maps*, Preprint, arXiv:1309.2116 (2013).

[58] Shen, W., *On stochastic stability of nonuniformly expanding interval maps*, Proc. Lond. Math. Soc. **107** (2013), 1091–1134.

[59] Thunberg, H., *Unfolding of chaotic unimodal maps and parameter dependence of natural measures*, Nonlinearity **14** (2001), 323–338.

[60] Tsujii, M., *Positive Lyapunov exponents in families of one-dimensional dynamical systems*, Invent. Math. **111** (1993), 113–137.

[61] ———, *On continuity of Bowen-Ruelle-Sinai measures in families of one-dimensional maps, Comm*, Math. Phys. **177** (1996), 1–11.

[62] Whitney, H., *Analytic extensions of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc. **36** (1934), 63–89.

[63] Young, L.-S., *Decay of correlations for certain quadratic maps*, Comm. Math. Phys. **146** (1992), 123–138.

[64] Young, L.-S., *Statistical properties of dynamical systems with some hyperbolicity*, Ann. of Math. **147** (1998), 585–650.

[65] ———, *What are SRB measures, and which dynamical systems have them?*, J. Statist. Phys. **108** (2002), 733–754.

UMR 8553 of CNRS, DMA, École Normale Supérieure, 75005 Paris, FRANCE
E-mail: viviane.baladi@ens.fr

Metric stability of the planetary N-body problem

Luigi Chierchia and Gabriella Pinzari

Abstract. The "solution" of the N-body problem (NBP) has challenged astronomers and mathematicians for centuries. In particular, the "metric stability" (i.e., stability in a suitable measure theoretical sense) of the planetary NBP is a formidable achievement in this subject completing an intricate path paved by mathematical milestones (by Newton, Weierstrass, Lindstedt, Poincarè, Birkhoff, Siegel, Kolmogorov, Moser, Arnold, Herman,...). In 1963 V.I. Arnold gave the following formulation of the metric stabiliy of the planetary problem:

If the masses of n planets are sufficiently small in comparison with the mass of the central body, the motion is conditionally periodic for the majority of initial conditions for which the eccentricities and inclinations of the Kepler ellipses are small.

Arnold gave a proof of this statement in a particular case (2 planets in a plane) and outlined a strategy (turned out to be controversial) for the general case. Only in 2004 J. Féjoz, completing work by M.R. Herman, published the first proof of Arnold's statement following a different approach using a "first order KAM theory" (developed by Rüssmann, Herman et al., and based on weaker non-degeneracy conditions) and removing certain secular degeneracies by the aid of an auxiliary fictitious system. Arnold's more direct and powerful strategy – including proof of torsion, Birkhoff normal forms, explicit measure estimates – has been completed in 2011 by the authors introducing new symplectic coordinates, which allow, after a proper symplectic reduction of the phase space, a direct check of classical non–degeneracy conditions.

Mathematics Subject Classification (2010). 70H08, 70K43, 70F10, 70H12, 70K45, 70F15, 70E55, 34C20, 34C27, 34C29, 34D10.

Keywords. Planetary system, N–body problem, metric stability, Quasi–periodic motions, symplectic invariants. Deprit's reduction of the nodes, Birkhoff normal forms, KAM tori.

1. Introduction

On July 5th, 1687 Sir Isaac Newton published his *Philosophiae Naturalis Principia Mathematica*, one of the most influential book in the history of modern science. The main impulse for its publication came from Edmond Halley, who urged Newton to write the mathematical solution of the two–body (Kepler) problem.

In general, the N–body problem (NBP) consists in determining the motion of $N \geq 2$ point–masses (i.e., ideal bodies with no physical dimensions identified with points in the Euclidean three–dimensional space) interacting only through Newton's law of gravitational attraction.

After his complete mathematical description of the general solution for the two body case, Newton immediately turned to the three–body problem (Sun, Earth and Moon) but got

Proceedings of the International Congress of Mathematicians, Seoul, 2014

discouraged, describing it as a "head–aching problem". The immense difficulty in trying to obtain explicitly the general solution of the NBP (something that, later, was proved to be impossible) drove, then, mathematicians to focus on the issue of convergence of formal power series for solutions of the planetary problem, the smallness expansion parameter being the mass ratio between planets and Sun. Many eminent personalities in the mid 1800's, such as Weierstrass and Dirichlet (who claimed to have a proof, which was never found), were convinced that the series were convergent. The question become a major mathematical issue and King Oscar II of Sweden and Norway, enlightened ruler, issued, in 1885, a prize for solving the problem or, in absence of a complete solution, for the best contribution. The prize was finally awarded on the occasion of the king's 60th birthday (21 January, 1889) to Henri Poincaré[1], who came to the belief (albeit not to the proof) that the series were divergent. The convergence problem was exported into a more general (and less degenerate) setting, namely, perturbation theory for non–degenerate nearly–integrable Hamiltonian systems. The breakthrough came in 1954 at the Amsterdam ICM, where N.N. Kolmogorov announced and gave a sketchy proof of his theorem on the preservation of (maximal) quasi–periodic motions[2] in nearly–integrable systems. In his amazing 6–page long article [22] Kolmogorov set the foundation of KAM (Kolmogorov–Arnold–Moser) theory, outlining a (super–exponentially) convergent perturbation theory for real–analytic systems, able to deal with the small divisor problems arising in the formal solutions of quasi–periodic motions: one of the crucial (and ingenious) idea was to fix the frequencies of the final motions rather than initial data.[3] With additions by Moser and Arnold, Kolmogorov's strategy could be used to show, indirectly,[4] convergence of the formal (Lindstedt) series for "general" solutions, where "general" means that the phase space region corresponding to (linearly) stable quasi–periodic motions tends to fill a Cantor set of asymptotic measure density equal to one (as the smallness parameter goes to zero). Thus, a way of rephrasing the main outcome of KAM theory is that analytic nearly–integrable (non–degenerate) Hamiltonian systems are asymptotically metrically stable.

However, in view of the strong degeneracies of the Kepler problem (i.e., of the integrable limit of the planetary NBP), the main hypothesis of Kolmogorov's theorem did not apply to the planetary problem. Besides the real–analyticity assumption, the main hypothesis of Kolmogorov's theorem is that the limit integrable Hamiltonian depends only on d action variables, d being the number of degrees of freedom (:= half of phase–space dimension) and that its gradient map is a local diffeomorphism. In the planetary problem the integrable limit depends only on n actions while the number of degrees of freedom (after reducing the total linear momentum; see below) is $3n$.

In 1963 Arnold, 26, took up the question of extending Kolmogorov's theorem to systems modeling the main features of the planetary problem, namely, Hamiltonian systems with $n + m$ degrees of freedom, whose integrable limit depends only on n action variables[5]

[1] At first Poincarè submitted a contribution containing a serious mistake, which he amended in a feverish effort: the outcome was the famous 270 page memoir [25], by now, regarded as the birth of modern theory of dynamical systems and chaos; compare [3].

[2] In general, a "quasi–periodic" (or "conditionally periodic") orbit with (rationally independent) frequencies $(\omega_1, ..., \omega_d) = \omega \in \mathbb{R}^d$ is a trajectory conjugated to a linear flow, $\theta \to \theta + \omega t$ on a d dimensional torus; if d equals the number of degrees of freedom (i.e., half dimension of the pahse space), the quasi–periodic orbit is called maximal.

[3] For generalities on KAM theory, see, e.g., [2] or [6].

[4] Direct proofs of convergence of Lindstedt series came much later; see [8, 16, 19].

[5] Such systems are sometimes called "properly–degenerate".

(which, in the planetary problem, are the square roots of the semimajor axes of the decoupled 2BP planet–Sun). This implies that the n conjugated angles (the mean anomalies of the 2BP's, in the planetary problem) are *fast angles*, bringing naturally in play averaging theory, according to which the leading dynamics is governed by the average of the Hamiltonian over the fast angles; the resulting Hamiltonian is thus the sum of the integrable limit and the average over the fast angles of the perturbation function (the "secular Hamiltonian"). Now, what happens in the planetary problem is that the secular Hamiltonian has an *elliptic equilibrium* in the origin of the remaining $2m$ symplectic variables, corresponding physically to circular orbits revolving in the same plane. Arnold formulated and gave a detailed proof of a generalization of Kolmogorov's theorem working for properly–degenerate systems with secular Hamiltonian possessing an elliptic equilibrium; he called such theorem the "Fundamental Theorem". The non–degeneracy hypotheses involve, now, not only the integrable limit (which, as in Kolmogorov's theorem, is assumed to define through the gradient map an n–diffeomorphism), but also the *Birkhoff normal form*[6] ("BNF" for short) of the secular $2m$ variables, and in particular the first order Birkhoff invariants (the eigenvalues associated to the elliptic equilibrium) and the second order invariants, which may be viewed as an $(m \times m)$ matrix. The "full" *torsion* (or "twist") hypothesis is guaranteed if such matrix is non–singular. After giving the (long and beautiful) proof of his Fundamental Theorem, Arnold checks the torsion hypothesis in the simpler non–trivial case, namely, 2 planets constrained on a plane. He then discusses how to generalize first to the planar case with n planets, and, from there, to the spacial general case.[7]

However, various serious problems prevented, for long time, to carry over Arnold's strategy. In first place, the standard hypotheses for constructing the BNF is that the first order Birkhoff invariants are non–resonant (i.e., do not have vanishing non–trivial integer coefficient linear combinations) up to a certain order. But indeed, besides a well know resonance related to rotation invariance, which Arnold was aware of, a second rather mysterious resonance was discovered by Herman in the 1990's, namely, that the sum of the first order Birkhoff invariants, in the general spatial case, vanishes identically; such resonance is now known as "Herman resonance". A second and more important problem is related to the torsion hypothesis. Indeed, in the full $6n$ dimensional phase space, the planetary Hamiltonian has an identically vanishing torsion (a fact, proved only recently in [12], ignored by Arnold and only suspected by Herman, compare [20]). Finally, there is a rather vague suggestion by Arnold to check non–degeneracies "bifurcating" from the planar problem, i.e., viewing the planar problem as a limit of the spacial one, which is a fact hard to justify analytically.

Herman's approach is rather different. After convincing himself that in the spatial case there might be a serious torsion problem, he turned to a different KAM technique, based on a different and somewhat weaker non–degeneracy condition, a condition which involves only the first order Birkhoff invariants and the gradient map of the limiting integrable Hamiltonian. Such condition is that the first order Birkhoff invariants – which are parameterized by the semimajor axes – do not lie identically in a fixed plane ("non–planarity" condition). However, as mentioned above, this is not true in the planetary problem since the invariants lie in the intersection of two planes corresponding to the rotational and the Herman's resonances. To overcome this problem, following a trick introduced by Poincaré, Herman modifies the planetary Hamiltonian by adding a term proportional to a function Poisson–

[6]For generalities on Birkhoff normal form theory, see [21]; for a Birkhoff normal form theory adapted to the NBP, see Proposition B.1 below.

[7]In Appendix C we report *verbatim*, some of Arnold's claims and suggestions as given in [1].

commuting with the planetary Hamiltonian; he manages to do that so that the modified Hamiltonian is non–degenerate (i.e., the modified Birkhoff invariants are non–planar). Now, by an abstract argument, two Poisson–commuting Hamiltonians have the same Lagrangian transitive invariant tori, therefore the invariant tori gotten by applying the weaker KAM theory to the modified Hamiltonian are invariant also for the planetary problem.[8] This scheme was worked out, clarified and published by Jacques Féjoz in [17]; see also [18].

Finally, in 2011, the original strategy of Arnold has been reconsidered, from a different point of view, in the paper[9] [11], where, thanks to new symplectic coordinates (called RPS for RegularizedPlanetarySymplectic), it is proven that *in a "partially reduced setting" the planetary problem has indeed non–vanishing torsion.* Recall that the "natural" phase space (after linear momentum reduction) of the planetary $(1+n)$–body problem is $6n$–dimensional and that standard symplectic coordinates are given by Poincaré variables; this setting has been used by Arnold (with minor modifications) and by Herman and Féjoz. In this setting the planetary Hamiltonian is still rotation invariant and admits, therefore, besides energy, other three global analytic integrals, which are the three components of the total angular momentum. Now, while in three dimensions it is customary to use the celebrated Jacobi's classical reduction of the nodes[10] in higher dimensions the reduction of the nodes is not so popular, even though it was knonw since the early 1980's thanks to the work of Deprit [15]. In [11], (an action–angle version of) Deprit variables replace Delaunay variables and, after a Poincaré regularization, one is lead to the new RPS variables. A main feature of these variables is that one symplectic couple of the secular cartesian variables (related to the inclination of the total angular momentum), say (p_n, q_n) are *both* cyclic coordinates (i.e., invariants), which means that the planetary Hamiltonian in such coordinates *does not depend on this couple of variables.* The significance of this fact is that *the phase space is foliated by $(6n-2)$–dimensional symplectic submanifold $\{(p_n, q_n) = \text{const}\}$ on which the planetary Hamiltonian has the same form.* In this partially reduced[11] setting the original Arnold's strategy can be carried out, torsion explicitely checked and all its dynamical consequences drawn: All this will be described below.

2. The classical Hamiltonian of the planetary NBP

In this section (and in Appendix A) we review the classical Hamiltonian description of the planetary NBP due, essentially, to Delaunay and Poincaré.

Newton's equations for $1+n$ bodies (point masses), which interact only through gravitational attraction, are given by:

[8] However, besides not having information about the normal form around the tori of the original Hamiltonian (which is intrinsic in this first order KAM theory), this abstract argument does not allow to read back the KAM structure in the unmodified setting.

[9] This paper is based on the PhD thesis [23].

[10] For a symplectic description of Jacobi's reduction of the nodes, see [4].

[11] Indeed, in these $(6n-2)$–symplectic submanifold, the planetary Hamiltonian still admits an energy–commuting integral, namely the Euclidean length of the total angular momentum. It is possible (and done in [11]) to further reduce to a fully rotationally reduced $(6n-4)$–dimensional phase space, however in such totally reduced setting many symmetries and nice feature shared by Poincaré and RPS variables (such as D'Alembert rules, parities in the secular variables, etc.) are lost and the symplectic description becomes somewhat more clumsy.

$$\ddot{u}^{(i)} = \sum_{\substack{0\le j\le n\\ j\neq i}} \mathrm{m}_j \frac{u^{(j)}-u^{(i)}}{|u^{(i)}-u^{(j)}|^3}\,, \qquad i=0,1,...,n\,, \tag{2.1}$$

where $u^{(i)} = \left(u_1^{(i)}, u_2^{(i)}, u_3^{(i)}\right) \in \mathbb{R}^3$ are the cartesian coordinates of the i^{th} body of mass $\mathrm{m}_i > 0$, $|u| = \sqrt{u\cdot u} = \sqrt{\sum_i u_i^2}$ is the standard Euclidean norm, "dots" over functions denote time derivatives, and the gravitational constant has been set to one (which is possible by rescaling time t). These equations are equivalent to the (standard) Hamilton equations associated to the Hamiltonian function[12]

$$\widehat{\mathcal{H}}_{\mathrm{N}} := \sum_{i=0}^{n} \frac{|U^{(i)}|^2}{2\mathrm{m}_i} - \sum_{0\le i<j\le n} \frac{\mathrm{m}_i\mathrm{m}_j}{|u^{(i)}-u^{(j)}|}\,,$$

where $(U^{(i)}, u^{(i)})$ are standard symplectic variables ($U^{(i)} = \mathrm{m}_i\dot{u}^{(i)}$ is the momentum conjugated to $u^{(i)}$) and the phase space is the "collisionless" open domain in $\mathbb{R}^{6(n+1)}$ given by

$$\widehat{\mathcal{M}} := \{U^{(i)}, u^{(i)} \in \mathbb{R}^3 :\ u^{(i)} \neq u^{(j)}\ ,\ 0\le i\neq j\le n\} \tag{2.2}$$

endowed with the standard symplectic form

$$\sum_{i=0}^{n} dU^{(i)}\wedge du^{(i)} := \sum_{\substack{0\le i\le n\\ 1\le k\le 3}} dU_k^{(i)}\wedge du_k^{(i)}\,. \tag{2.3}$$

Exploiting the invariance of Newton's equation by change of inertial frames, or, equivalently, the existence of the vector–valued integral[13] given by the total linear momentum $\sum_{i=0}^{n} U^{(i)}$, Poincaré showed how to make a "symplectic reduction" lowering by three units the number of degrees of freedom. Indeed, the dynamics generated by $\widehat{\mathcal{H}}_{\mathrm{N}}$ on $\widehat{\mathcal{M}}$ is equivalent to the dynamics on

$$\mathcal{M} := \left\{(X,x) = (X^{(1)},...,X^{(n)},x^{(1)},...,x^{(n)}) \in \mathbb{R}^{6n} : 0\neq x^{(i)}\neq x^{(j)}\ ,\forall\, i\neq j\right\},$$

(endowed with the standard symplectic form $\sum_{i=1}^{n} dX^{(i)}\wedge dx^{(i)}$) by the Hamiltonian

$$\begin{aligned}\mathcal{H}_{\mathrm{plt}}(X,x) &:= \sum_{i=1}^{n}\frac{|X^{(i)}|^2}{2M_i} - \frac{M_i\bar{m}_i}{|x^{(i)}|} + \mu\sum_{1\le i<j\le n}\frac{X^{(i)}\cdot X^{(j)}}{m_0} - \frac{m_im_j}{|x^{(i)}-x^{(j)}|}\\ &=: \mathcal{H}^{(0)}_{\mathrm{plt}}(X,x) + \mu\mathcal{H}^{(1)}_{\mathrm{plt}}(X,x)\,,\end{aligned} \tag{2.4}$$

where the mass of the Sun is[14] $\mathrm{m}_0 = m_0$ and the mass of the planets are $\mathrm{m}_i = \mu m_i$ ($1\le i\le n$), μ being a small parameter, while $M_i := \frac{m_0m_i}{m_0+\mu m_i}$ and $\bar{m}_i := m_0+\mu m_i$. In such description $\mathcal{M}$ corresponds to the (symplectic) submanifold of $\widehat{\mathcal{M}}$ of zero total linear

[12] I.e., the equations $\dot{U}_j^{(i)} = -\partial_{u_j^{(i)}}\widehat{\mathcal{H}}_{\mathrm{N}}$, $\dot{u}_j^{(i)} = \partial_{U_j^{(i)}}\widehat{\mathcal{H}}_{\mathrm{N}}$, $0\le i\le n$, $1\le j\le 3$; for general information on Hamiltonian systems, see, e.g., [2].

[13] Recall that $F(X,x)$ is an integral for $\mathcal{H}(X,x)$ if $\{F,\mathcal{H}\}=0$ where $\{F,G\} = F_X\cdot G_x - F_x\cdot G_X$ denotes the (standard) Poisson bracket; in particular an integral F for $\mathcal{H}$ is constant for the $\mathcal{H}$ flow, i.e., $F\circ\phi^t_{\mathcal{H}}\equiv$ const., where $\phi^t_{\mathcal{H}}$ denotes the Hamiltonian flow generated by $\mathcal{H}$.

[14] Note the different character: upright for unscaled and italic for rescaled masses.

momentum and zero total center of mass and $x^{(i)} = u^{(i)} - u^{(0)}$, for $i \geq 1$, are heliocentric coordinates; full details are given in Appendix A.

Obviously, in such variables, there is no more a conserved total linear momentum,[15] however, the system is *still invariant under rotations* and the total angular momentum

$$\mathrm{C} = (\mathrm{C}_1, \mathrm{C}_2, \mathrm{C}_3) := \sum_{i=1}^{n} \mathrm{C}^{(i)} , \qquad \mathrm{C}^{(i)} := x^{(i)} \times X^{(i)} , \tag{2.5}$$

is still a (vector–valued) integral for $\mathcal{H}_{\rm plt}$. The integrals C_i, however, do not commute (i.e., their Poisson brackets do not vanish[16]) but, for example, $|\mathrm{C}|$ and C_3 are two commuting, independent integrals, a remark that will be crucial in what follows.

Next, by regularizing the Delaunay action–angle coordinates for the n decoupled two–body problems with Hamiltonian $\mathcal{H}^{(0)}_{\rm plt}$ in a neighborhood of co–circular and co–planar motions, Poincaré brings out in a neat way the nearly–integrable structure of planetary NBP. The real–analytic symplectic variables doing the job are usually known as *Poincaré variables*: in such variables the Hamiltonian $\mathcal{H}_{\rm plt}(X, x)$ takes the form

$$\mathcal{H}_{\rm P}(\Lambda, \lambda, \mathrm{z}) = h_{\rm K}(\Lambda) + \mu f_{\rm P}(\Lambda, \lambda, \mathrm{z}) , \ \ (\Lambda, \lambda) \in \mathbb{R}^n_+ \times \mathbb{T}^n , \ \mathrm{z} := (\eta, \mathrm{p}, \xi, \mathrm{q}) \in \mathbb{R}^{4n} \tag{2.6}$$

where the "Kepler" unperturbed term $h_{\rm K}$ is given by

$$h_{\rm K}(\Lambda) := -\sum_{i=1}^{n} \frac{M_i^3 \bar{m}_i^2}{2\Lambda_i^2} , \qquad \Lambda_i := M_i \sqrt{\bar{m}_i a_i}, \tag{2.7}$$

a_i being the semimajor axis of the instantaneous two–body system formed by the $i^{\rm th}$ planet and the Sun; as phase space, we consider a collisionless domain around the "secular origin" $\mathrm{z} = 0$ (which corresponds to co–planar, co–circular motions) of the form

$$(\Lambda, \lambda, \mathrm{z}) = (\Lambda, \lambda, \eta, \mathrm{p}, \xi, \mathrm{q}) \in \mathcal{M}^{6n}_{\rm P} := \mathcal{A} \times \mathbb{T}^n \times B^{4n} \tag{2.8}$$

endowed with the symplectic form $\sum_{i=1}^{n} d\Lambda_i \wedge \lambda_i + \sum_{i=1}^{n} \eta_i \wedge d\xi_i + \sum_{i=1}^{n} d\mathrm{p}_i \wedge d\mathrm{q}_i$; $\mathcal{A}$ is a set of "well separated" semimajor axes

$$\mathcal{A} := \big\{ \Lambda : \ \underline{a}_j < a_j < \overline{a}_j \quad \text{for} \quad 1 \leq j \leq n \big\} \tag{2.9}$$

where $\underline{a}_1, \cdots, \underline{a}_n, \overline{a}_1, \cdots, \overline{a}_n$, are positive numbers verifying $\underline{a}_j < \overline{a}_j < \underline{a}_{j+1}$ for any $1 \leq j \leq n$, $\overline{a}_{n+1} := \infty$, and B^{4n} is a $4n$–dimensional ball around the secular origin $\mathrm{z} = 0$. A complete description of Delaunay and Poincaré variables is given in Appendix A.

Here, let us point out that the Hamiltonian (2.4) retains rotation and reflection invariance and, in particular, invariance by rotation with respect the $k^{(3)}$–axis and invariance by reflection with respect to the coordinate planes. This implies that the perturbation $f_{\rm P}$ in (2.6) satisfies (classical) symmetry relations known as *d'Alembert rules*, which are given by the following transformations:

[15] In particular, $\sum_{i=1}^{n} X^{(i)}$ is *not* an integral for $\mathcal{H}_{\rm plt}$

[16] Indeed, $\{\mathrm{C}_1, \mathrm{C}_2\} = \mathrm{C}_3$, $\{\mathrm{C}_2, \mathrm{C}_3\} = \mathrm{C}_1$ and $\{\mathrm{C}_3, \mathrm{C}_1\} = \mathrm{C}_2$.

$$\begin{cases} (\eta,\xi,\mathrm{p},\mathrm{q}) \to (-\xi,-\eta,\mathrm{q},\mathrm{p})\,, & (\Lambda,\lambda) \to \left(\Lambda, \frac{\pi}{2}-\lambda\right) \\ (\eta,\xi,\mathrm{p},\mathrm{q}) \to (\eta,\xi,-\mathrm{p},-\mathrm{q})\,, & (\Lambda,\lambda) \to (\Lambda,\lambda) \\ (\eta,\xi,\mathrm{p},\mathrm{q}) \to (-\eta,\xi,\mathrm{p},-\mathrm{q})\,, & (\Lambda,\lambda) \to (\Lambda,\pi-\lambda) \\ (\eta,\xi,\mathrm{p},\mathrm{q}) \to (\eta,-\xi,-\mathrm{p},\mathrm{q})\,, & (\Lambda,\lambda) \to (\Lambda,-\lambda) \\ (\Lambda,\lambda,\mathrm{z}) \to (\Lambda,\lambda_1+g,\dots,\lambda_n+g,\mathcal{S}^g\mathrm{z}) \end{cases} \tag{2.10}$$

where, for any $g \in \mathbb{T}$, $\mathcal{S}^g$ acts as synchronous clock–wise rotation by the angle g in the symplectic z_i–planes:

$$\mathcal{S}^g : \mathrm{z} \to \mathcal{S}^g\mathrm{z} = \left(\mathcal{S}_g\mathrm{z}_1, ..., \mathcal{S}_g\mathrm{z}_{2n}\right)\,, \qquad \mathcal{S}_g := \begin{pmatrix} \cos g & \sin g \\ -\sin g & \cos g \end{pmatrix}\,; \tag{2.11}$$

compare (3.26)–(3.31) in [12]. By such symmetries, in particular, the averaged perturbation

$$f_{\mathrm{P}}^{\mathrm{av}}(\Lambda,\mathrm{z}) := \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} f_{\mathrm{P}}(\Lambda,\lambda,\mathrm{z})d\lambda\,, \tag{2.12}$$

which is called the *secular Hamiltonian*, is even in z around the origin $\mathrm{z}=0$ and its expansion in powers of z has the form

$$f_{\mathrm{P}}^{\mathrm{av}} = C_0(\Lambda) + \mathcal{Q}_h(\Lambda)\cdot\frac{\eta^2+\xi^2}{2} + \mathcal{Q}_v(\Lambda)\cdot\frac{\mathrm{p}^2+\mathrm{q}^2}{2} + \mathrm{O}(|\mathrm{z}|^4)\,, \tag{2.13}$$

where $\mathcal{Q}_h$, $\mathcal{Q}_v$ are suitable quadratic forms and $\mathcal{Q}\cdot u^2$ denotes the 2–index contraction $\sum_{i,j}\mathcal{Q}_{ij}u_iu_j$ ($\mathcal{Q}_{ij}$, u_i denoting, respectively, the entries of $\mathcal{Q}$, u). This shows that $\mathrm{z}=0$ *is an elliptic equilibrium for the secular dynamics* (i.e, the dynamics generated by $f_{\mathrm{P}}^{\mathrm{av}}$). The explicit expression of such quadratic forms can be found, *e.g.* , in (36), (37) of [17] (revised version).

The *truncated averaged Hamiltonian*

$$\overline{\mathcal{H}}_{\mathrm{P}}^{\mathrm{av}}(\Lambda,\lambda,\mathrm{z}) := h_{\mathrm{K}} + \mu\Big(C_0(\Lambda) + \mathcal{Q}_h(\Lambda)\cdot\frac{\eta^2+\xi^2}{2} + \mathcal{Q}_v(\Lambda)\cdot\frac{\mathrm{p}^2+\mathrm{q}^2}{2}\Big)$$

is integrable, with $3n$ commuting integrals given by

$$\Lambda_i\,, \qquad \rho_i = \frac{{\eta_i}^2+{\xi_i}^2}{2}\,, \qquad \mathrm{r}_i = \frac{{\mathrm{p}_i}^2+{\mathrm{q}_i}^2}{2}\,, \qquad (1\le i\le n)\,.$$

The general trajectory of this system fills a $3n$–dimensional torus with n fast frequencies $\partial_{\Lambda_i}h_{\mathrm{K}}(\Lambda_i)$ and $2n$ slow frequencies given by

$$\mu\Omega = \mu(\sigma,\varsigma) = \mu(\sigma_1,\cdots,\sigma_n,\varsigma_1,\cdots,\varsigma_n)\,, \tag{2.14}$$

σ_i and ς_i being the real eigenvalues of $\mathcal{Q}_h(\Lambda)$ and $\mathcal{Q}_v(\Lambda)$, respectively. Such tori correspond to n nearly co–planar and co–circular planets rotating around the Sun with Keplerian frequencies $\partial_{\Lambda_i}h_{\mathrm{K}}(\Lambda_i)$ and with small eccentricities and inclinations slightly and slowly oscillating with frequencies $\mu\sigma$ and $\mu\varsigma$.

A fundamental problem in the planetary NBP concerns the perturbative analysis of the integrable dynamics governed by $\overline{\mathcal{H}}_{\mathrm{P}}^{\mathrm{av}}$, when the full planetary Hamiltonian $\mathcal{H}_{\mathrm{P}}$ is considered. The main technical tool is Kolmogorov's 1954 Theorem [22] (which, incidentally, was

clearly motivated by Celestial Mechanics) on the persistence under perturbation of quasi–periodic motions for nearly–integrable system with real–analytic Hamiltonian in *action–angle variables* given by

$$H_\mu(I,\varphi) := h(I) + \mu f(I,\varphi)\,, \qquad (I,\varphi) \in \mathbb{R}^d \times \mathbb{T}^d. \tag{2.15}$$

Kolmogorv's Theorem, however, holds in a neighborhoods of points I_0 where the integrable Hamiltonian is *non–degenerate* in the sense that $\det h''(I_0) \neq 0$, where h'' denotes the Hessian matrix of h (equivalently, the frequency map $I \to h'(I)$ is a local diffeomorphism). This condition is strongly violated by the planetary Hamiltonian since for $\mu = 0$ the integrable (Keplerian) limit depends only on n action variables (the Λ's), while the number of degrees of freedom is $d = 3n$. A nearly–integrable system with Hamiltonian as in (2.15) for which h does not depend upon all the actions I_1,...,I_d is called *properly–degenerate*.[17]

In the next section we recall Arnold's statement on the planetary NBP and outline his strategy of proof based on a generalization of Kolmogorov's theory to properly–degenerate system.

3. Arnold's theorem on the planetary NBP (1963)

In the 1963 paper [1] Arnold – probably in his deeper contribution to KAM theory and Celestial Mechanics – formulated his main result as follows ([1, p. 127]):

Theorem 3.1. *If the masses, eccentricities and inclinations of the planets are sufficiently small, then for the majority of initial conditions the true motion is conditionally periodic and differs little from Lagrangian motion*[18] *with suitable initial conditions throughout an infinite interval of time* $-\infty < t < +\infty$.

Proper degeneracies and Arnold's "Fundamental Theorem". As mentioned above, Kolmogorov opened the route to a rigorous proof of existence of (maximal) quasi–periodic trajectories in Hamiltonian systems, but the planetary system violates drastically the main hypotheses of his theorem. This was a main challenge for his young and brilliant student Vladimir Igorevich Arnold, who at 26 gave a major impulse and draw the path which, eventually, would lead to a complete solution of the metric stability problem for the NBP.
One of the main steps – a result that in [1] Arnold called "The Fundamental Theorem" – is to extend Kolmogorov's Theorem to properly–degenerate systems, and, more specifically, to properly–degenerate systems with "secular" elliptic equilibria (or, more precisely, elliptic lower dimensional tori).
Let us proceed to formulate Arnold's Fundamental Theorem.

Let $\mathcal{M}$ denote the phase space $\mathcal{M} := \big\{(I,\varphi,p,q) :\ (I,\varphi) \in V \times \mathbb{T}^n \text{ and } (p,q) \in B\big\}$

[17] In general, maximal quasi–periodic solutions (i.e., quasi–periodic solutions with d rationally–independent frequencies) for properly–degenerate systems do not exist: trivially, any unperturbed properly–degenerate system on a $2d$ dimensional phase space with $d \geq 2$ will have motions with frequencies not rationally independent over $\mathbb{Z}^d$. But they may exist under further conditions on the perturbation f, as we shall see.

[18] Arnold defines the "Lagrangian motions", at p. 127 as follows: *the Lagrangian motion is conditionally periodic and to the n "rapid" frequencies of the Kepler motion are added n (in the planar problem) or $2n-1$ (in the space problem) "slow" frequencies of the secular motions.* This dynamics corresponds, essentially, to the above "truncated integrable planetary dynamics". The missing frequency in the space problem is because one of the spatial secular frequency, say, ς_n vanishes identically; compare Eq. (3.3) below.

where V is an open bounded region in $\mathbb{R}^n$ and B is a ball around the origin in $\mathbb{R}^{2m}$; $\mathcal{M}$ is equipped with the standard symplectic form

$$dI \wedge d\varphi + dp \wedge dq = \sum_{i=1}^{n} dI_i \wedge d\varphi_i + \sum_{i=1}^{m} dp_i \wedge dq_i \ .$$

Let, also, H_μ be a real–analytic Hamiltonian on $\mathcal{M}$ of the form $H_\mu(I,\varphi,p,q) := h(I) + \mu f(I,\varphi,p,q)$, and denote by $f^{\rm av}$ the average of f over the "fast angles" φ: $f^{\rm av}(I,p,q) := \int_{\mathbb{T}^n} f(I,\varphi,p,q)\frac{d\varphi}{(2\pi)^n}$.

Theorem 3.2 ("The Fundamental Theorem"; [1]). *Assume that $f^{\rm av}$ is of the form*

$$f^{\rm av} = f_0(I) + \sum_{j=1}^{m} \Omega_j(I) r_j + \frac{1}{2}\tau(I) r \cdot r + o_4 \ , \qquad r_j := \frac{p_j^2 + q_j^2}{2} \ , \tag{3.1}$$

where τ is a symmetric $(m \times m)$–matrix and $\lim_{(p,q)\to 0} |o_4|/|(p,q)|^4 = 0$. Assume, also, that $I_0 \in V$ is such that

$$\det h''(I_0) \neq 0 \qquad (*) \ ; \qquad\qquad \det \tau(I_0) \neq 0 \qquad (**) \ . \tag{3.2}$$

Then, in any neighborhood of $\{I_0\} \times \mathbb{T}^d \times \{(0,0)\} \subseteq \mathcal{M}$ there exists a positive measure set of phase points belonging to analytic "KAM tori" spanned by maximal quasi–periodic solutions with $n+m$ rationally–independent (Diophantine[19]) frequencies, provided μ is small enough.

Let us make some remarks.

(i) The function $f^{\rm av}$ in (3.1) is said to be in *Birkhoff normal form* (with respect to the variables p, q) up to order 4 (compare [21] and Appendix B below). Actually, Arnold requires that $f^{\rm av}$ is in Birkhoff normal form up to order 6 (instead of 4); but such condition can be relaxed and (3.1) is sufficient: compare [9], where Arnold's Fundamental Theorem is revisited and various improvements obtained.

(ii) Condition (3.2)–$(*)$ is immediately seen to be satisfied in the general planetary problem; the correspondence with the planetary Hamiltonian in Poincaré variables (2.6) being the following: $m = 2n$, $I = \Lambda$, $\varphi = \lambda$, $z = (p,q)$, $h = h_{\rm K}$, $f = f_{\rm P}$.

(iii) Condition (3.2)–$(**)$ is a "twist" or "torsion" condition on the secular Hamiltonian. It is actually possible to develop a weaker KAM theory where no torsion is required. This theory is due to Rüssmann [27], Herman and Féjoz [17], where $f^{\rm av}$ is assumed to be in Birkhoff normal form up to order 2, $f^{\rm av} = f_0(I) + \sum_{j=1}^{m} \Omega_j(I) r_j + o_2$, and the secular frequency map $I \to \Omega(I)$ is assumed to be *non–planar*, meaning that no neighborhood of I_0 is mapped into an hyperplane.

(iv) The ingenious idea of Arnold in order to remove the proper degeneracy of the system goes roughly as follows. Instead of $h(I)$, consider $\hat{h}(I,r) := h(I) + \mu f_2^{\rm av}(I,r)$ as a new unperturbed part viewed as a function of the actions (I,r), $f_2^{\rm av}(I,r)$ being the

[19] A vector $\omega \in \mathbb{R}^d$ is Diophantine if there exist positive constants γ and c such that $|\omega \cdot k| \geq \gamma/|k|^c$, $\forall\ k \in \mathbb{Z}^d \backslash \{0\}$.

truncation of $f^{\rm av}$ in (3.1) up to degree two in the variables r. By averaging theory, the original Hamiltonian can be symplectically conjugated to a new "effective" nearly–integrable system $\tilde h(I,r)+\mu^a\hat f(I,r,\varphi,\psi)$ ($(\varphi,\psi)\in\mathbb{T}^n\times\mathbb{T}^m$) with $a\in\mathbb{N}$ large enough and $\tilde h$ close to $\hat h$: this is the starting point for constructing Kolmogorov ($n+m$–dimensional) tori (note that the full torsion condition mentioned in the introduction corresponds to the Kolmogorov non–degeneracy of $\hat h$).

(v) The elliptic secular equilibrium $(p,q)=0$ plays a fundamental rôle in this construction. The density of the tori is closer and closer to one as soon as the variables (p,q) (eccentricities and inclinations, in the planetary problem) approach the origin; see also Theorem 5.3 below. Arnold however noticed that, at least in the case of the planar three–body problem, a stronger result holds: $f^{\rm av}$ is *integrable* and one can replace $f_2^{\rm av}$ with $f^{\rm av}$ in the definition of $\hat h$ (see the previous item); this yields a more global and astronomically relevant result. Indeed, the density of the tori depends *only* on μ and *not* on eccentricities and inclinations. The independence of the Kolmogorov tori from eccentricities (in such cases inclinations are not independent quantities[20]) has been proved also for the spatial three–body case and the planar general case [24] (notwithstanding the fact that $f^{\rm av}$ is no longer integrable).

(vi) Actually, the torsion assumption (3.2)–$(**)$ implies stronger results:

– It is possible to give explicit and accurate bounds on the measure of the "Kolmogorov set", i.e., the set covered by the closure of quasi–periodic motions ([9]).

– The quasi–periodic motions found belong to a smooth family of *non–degenerate Kolmogorov tori*, which means, essentially, that the dynamics can be linearized in a neighborhood of each torus.

– The above Kolmogorov tori are cumulation sets for periodic orbits with longer and longer periods. Thus the measure of the closure of periodic orbits tends to fill a set of full measure as the distance from the secular origin $\mathrm{z}=0$ tends to zero, showing that a "metric asymptotic" version of Poincaré's conjecture about the density of periodic orbits in phase space holds in the general planetary NBP around co–planar and co–circular motions; see [7].

On the basis of Theorem 3.2, Arnold's strategy is to compute the Birkhoff normal form (3.1) of the secular Hamiltonian $f_{\rm P}^{\rm av}$ in (2.12) and to check the non–vanishing of the torsion (3.2)–$(**)$, a program which he carried out completely only in the planar three–body case ($n=2$).

The planar three–body case (Arnold, 1963). In the planar case the Poincaré variables become simply $(\Lambda,\lambda,\mathrm{z}):=(\Lambda,\lambda,\eta,\xi)\in\mathbb{R}^n_+\times\mathbb{T}^n\times\mathbb{R}^{2n}$, with the Λ's as in (2.7) and

$$\lambda_i=\ell_i+\mathrm{g}_i\,,\qquad\begin{cases}\eta_i=\sqrt{2(\Lambda_i-\Gamma_i)}\,\cos\mathrm{g}_i\\ \xi_i=-\sqrt{2(\Lambda_i-\Gamma_i)}\,\sin\mathrm{g}_i\end{cases},$$

where, referring to the instantaneous $i^{\rm th}$ two–body system planet–Sun, ℓ_i is the mean anomaly, g_i the argument of the perihelion and Γ_i the absolute value of the $i^{\rm th}$ angular

[20] In the spatial three–body problem completely reduced by rotations, the mutual inclination is a function of eccentricities.

momentum (compare Appendix A for more details). The planetary, planar Hamiltonian, is given by

$$\mathcal{H}_{\text{P,pln}}(\Lambda,\lambda,\mathrm{z}) = h_{\text{K}}(\Lambda) + \mu f_{\text{P,pln}}(\Lambda,\lambda,\mathrm{z})\ ,\quad \mathrm{z} := (\eta,\xi) \in \mathbb{R}^{2n}$$

with $\frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} f_{\text{P,pln}} =: f^{\text{av}}_{\text{P,pln}} = C_0(\Lambda) + \mathcal{Q}_h(\Lambda)\cdot\frac{\eta^2+\xi^2}{2} + \mathrm{O}(|\mathrm{z}|^4)$. In Eq. (3.4.31), p.138 of [1], Arnold computed the first and second order Birkhoff invariants for $n=2$ finding, in the asymptotics $a_1 \ll a_2$:

$$\begin{cases} \Omega_1 = -\dfrac{3}{4} m_1 m_2 \Big(\dfrac{a_1}{a_2}\Big)^2 \dfrac{1}{a_2\Lambda_1}\Big(1+\mathrm{O}\Big(\dfrac{a_1}{a_2}\Big)\Big) \\ \Omega_2 = -\dfrac{3}{4}\, m_2^2\, \dfrac{1}{a_2\Lambda_2}\Big(1+\mathrm{O}\Big(\dfrac{a_1}{a_2}\Big)^2\Big) \end{cases}$$

$$\tau = m_1 m_2 \frac{a_1^2}{a_2^3} \begin{pmatrix} \frac{3}{4\Lambda_1^2} & -\frac{9}{4\Lambda_1\Lambda_2} \\ -\frac{9}{4\Lambda_1\Lambda_2} & -\frac{3}{\Lambda_2^2} \end{pmatrix} (1+\mathrm{O}(a_2^{-5/4}))\ ,$$

which shows that the Ω_j's are non resonant up to any finite order (in a suitable Λ–domain), so that the planetary, planar Hamiltonian can be put in Birkhoff normal form up to order 4 and that the second order Birkhoff invariants are non–degenerate in the sense that[21]

$$\det\tau = -(m_1m_2)^2\,\frac{117}{16}\,\frac{a_1^4}{a_2^6(\Lambda_1\Lambda_2)^2}\,(1+o(1)) = -\frac{117}{16}\frac{1}{m_0^2}\frac{a_1^3}{a_2^7}\,(1+o(1)) \neq 0.$$

This allow to apply Theorem 3.2 and to prove Arnold's planetary theorem in the planar three–body ($n=2$) case.

An extension of this method to the *spatial three–body problem*, exploiting Jacobi's reduction of the nodes and its symplectic realization, is due to P. Robutel [26].

Obstacles to the generalization of Arnold's project: Secular degeneracies. In the general spatial case it is customary to call σ_i the eigenvalues of $\mathcal{Q}_h(\Lambda)$ and ς_i the eigenvalues of and $\mathcal{Q}_v(\Lambda)$, so that $\Omega = (\sigma,\varsigma)$; compare (2.14).

It turns out that such invariants satisfy identically the following two *secular resonances*

$$\varsigma_n = 0\ , \qquad \sum_{i=1}^{n}(\sigma_i+\varsigma_i) = 0 \tag{3.3}$$

and, actually, it can be shown that *these are the only exact resonances identically satisfied by the first order Birkhoff invariants*; compare [17, Prop. 78 at p. 1575].

The first resonance was well known to Arnold, while the second one was apparently discovered by M. Herman in the 1990's and is now known as *Herman resonance*.

Both resonances violate Birkhoff's non–resonance condition (compare Eq. (B.1) below) but *do not violate* a more special Birkhoff condition sufficient for rotational invariant systems, as explained in Appendix B (compare, in particular Eq. (B.3)).

There is, however, a much more serious problem for Arnold's approach, namely, a strong degeneracy of the *second order Birkhoff invariance*, still a reflection of rotational invariance. Indeed, the torsion matrix τ is degenerate, as clarified in [12], where it is proven that τ is

[21] In [1] the τ_{ij} are defined as $1/2$ of the ones defined here.

equivalent to a matrix of the form

$$\begin{pmatrix} \bar\tau & 0 \\ 0 & 0 \end{pmatrix} \tag{3.4}$$

$\bar\tau$ being a matrix of order $(2n-1)$.

4. Proofs of Arnold's theorem

Herman-Fejóz proof (2004). In 2004 J. Fejóz [17] published the first complete proof of a general version of Arnold's planetary theorem: this proof completed a long project carried out by M. Herman. In order to avoid fourth order computations, Herman (also because seemed to suspect the degeneracy of the matrix of the second order Birkhoff invariant; compare the Remark towards the end of p. 24 of [20]), turned to a weaker KAM theory, which makes use of a "first order KAM condition" based on the non–planarity of the frequency map. But, the resonances (3.3) show that the frequency map lies in the intersection of two planes, violating the non–planarity condition. To overcome this problem Herman and Féjoz use a trick by Poincarè, consisting in modifying the Hamiltonian by adding a commuting Hamiltonian, so as to remove the degeneracy. By a Lagrangian intersection theory argument, if two Hamiltonian commute and $\mathcal{T}$ is a Lagrangian invariant transitive torus for one of them, then $\mathcal{T}$ is invariant (but not necessarly transitive) also for the other Hamiltonian; compare [17, Lemma 82, p. 1578]. Thus, the KAM tori constructed for the modified Hamiltonian are indeed invariant tori also for the original system. Now, the expression of the vertical component of the total angular momentum C_3 has a particular simple expression in Poincaré variables: indeed, $\mathrm{C}_3 = \sum_{j=1}^n \Big(\Lambda_j - \frac{1}{2}(\eta_j^2 + \xi_j^2 + \mathrm{p}_j^2 + \mathrm{q}_j^2)\Big)$, so that the modified Hamiltonian $\mathcal{H}_\delta := \mathcal{H}_{\text{P}}(\Lambda, \lambda, \mathrm{z}) + \delta \mathrm{C}_3$ is easily seen to have a non–planar frequency map (first order Birlhoff invariants), and the above abstract remark applies.

Herman's KAM theory (as given in [17]) works in the C^∞ category, so that the tori obtained in [17] are proven to be C^∞, on the other hand, since the planetary Hamiltonian flow is real–analytic, it is natural to expect that also their maximal quasi–periodic solutions (and the tori they span) are real–analytic. This is proven in [13], where Rüßmann first–order KAM theory [27] is extended to properly–degenerate systems.

Completion of Arnold's project (2011). In [11] Arnold's original strategy is reconsidered and full torsion of the planetary problem is proved by introducing new symplectic variables (called RPS–variables standing for Regularized Planetary Symplectic variables), which allow for a symplectic partial reduction of rotations eliminating one degree of freedom (i.e., lowering by two units the dimension of the phase space). In such reduced setting the first resonance in (3.3) disappears (but not the second one) and the question about the torsion is reduced to study the determinant of $\bar\tau$ in (3.4), which, in fact, is shown to be non–singular; compare [11, §8] and [12] (where a precise connection is made between the Poincaré and the RPS–variables compare also Theorem 5.1 below).

In the next section we shall review the main ideas and techniques discussed in [11].

5. A new symplectic view of the planetary phase space and completion of Arnold's project

We start by describing the new set of symplectic variables, which allow to have a new insight on the symplectic structure of the phase space of the planetary model, or, more in general, of any rotational invariant model.

The idea is to start with action–angle variables having, among the actions, two independent commuting integrals related to rotations, for example, the Euclidean length of the total angular momentum C and its vertical component C_3, and then (imitating Poincaré) to regularize around co–circular and co–planar configurations.

The variables that do the job are a "planetary" action–angle version of certain variables introduced by A. Deprit in[22] 1983 [15].

The Regularized planetary symplectic (RPS) variables. Let $n \geq 2$, $1 \leq i \leq n$, and consider the "partial angular momenta" $S^{(i)} := \sum_{j=1}^{i} \mathrm{C}^{(j)}$, (note that $S^{(n)} = \sum_{j=1}^{n} \mathrm{C}^{(j)} =: \mathrm{C}$) and define the "Deprit nodes"

$$\begin{cases} \nu_i := S^{(i)} \times \mathrm{C}^{(i)} \,, & 2 \leq i \leq n \\ \nu_1 := \nu_2 \\ \nu_{n+1} := k^{(3)} \times \mathrm{C} =: \bar{\nu} \,; \end{cases}$$

(recall the definition of the "individual" and total angular momenta in (2.5)).

The *Deprit action–angle variables* $(\Lambda, \Gamma, \Psi, \ell, \gamma, \psi)$ are defined as follows. Let P_i denote the coordinates of the i^{th} instantaneous perihelion (relatively to the instantaneous planet–Sun 2–body system), let $(k^{(1)}, k^{(2)}, k^{(3)})$ be the standard orthonormal basis in $\mathbb{R}^3$, and, for $u, v \in \mathbb{R}^3$ lying in the plane orthogonal to a non–vanishing vector w, denote by $\alpha_w(u, v)$ the positively oriented angle (mod 2π) between u and v (orientation follows the "right hand rule", the thumb being w).

The Deprit variables Λ, Γ and ℓ are in common with the Delaunay variables (compare (A.4) in Appendix A), while

$$\gamma_i := \alpha_{\mathrm{C}^{(i)}}(\nu_i, P_i), \qquad \Psi_i := \begin{cases} |S^{(i+1)}| \,, & 1 \leq i \leq n-1 \\ \mathrm{C}_3 := \mathrm{C} \cdot k^{(3)} & i = n, \end{cases}$$

$$\psi_i := \begin{cases} \alpha_{S^{(i+1)}}(\nu_{i+2}, \nu_{i+1}) & 1 \leq i \leq n-1 \\ \zeta := \alpha_{k^{(3)}}(k^{(1)}, \bar{\nu}) & i = n. \end{cases}$$

Define also $G := |\mathrm{C}| = |S^{(n)}|$.

The "Deprit inclinations" ι_i are defined through the relations

$$\cos \iota_i := \begin{cases} \dfrac{\mathrm{C}^{(i+1)} \cdot S^{(i+1)}}{|\mathrm{C}^{(i+1)}||S^{(i+1)}|} \,, & 1 \leq i \leq n-1 \,, \\ \dfrac{\mathrm{C} \cdot k^{(3)}}{|\mathrm{C}|} \,, & i = n \,. \end{cases}$$

Similarly to the case of the Delaunay variables, the Deprit action–angle variables are not defined when the Deprit nodes ν_i vanish or the eccentricitiy $e_i \notin (0, 1)$, but on the do-

[22] See also [10] and [11].

main where they are well defined they yield a real–analytic set of symplectic variables, i.e., $\sum_{i=1}^n dX^{(i)} \wedge dx^{(i)} = \sum_{i=1}^n d\Lambda_i \wedge d\ell_i + d\Gamma_i \wedge d\gamma_i + d\Psi_i \wedge d\psi_i$; for a proof, see [10] or §3 of [11].

The RPS variables are given by[23] $(\Lambda, \lambda, z) := (\Lambda, \lambda, \eta, \xi, p, q)$ with (again) the Λ's as in (2.7) and, for $1 \le i \le n$,

$$\lambda_i = \ell_i + \gamma_i + \psi^n_{i-1}\ , \qquad \begin{cases} \eta_i = \sqrt{2(\Lambda_i - \Gamma_i)}\ \cos\left(\gamma_i + \psi^n_{i-1}\right) \\ \xi_i = -\sqrt{2(\Lambda_i - \Gamma_i)}\ \sin\left(\gamma_i + \psi^n_{i-1}\right) \end{cases}$$

$$\begin{cases} p_i = \sqrt{2(\Gamma_{i+1} + \Psi_{i-1} - \Psi_i)}\ \cos\psi^n_i \\ q_i = -\sqrt{2(\Gamma_{i+1} + \Psi_{i-1} - \Psi_i)}\ \sin\psi^n_i \end{cases}$$

where $\Psi_0 := \Gamma_1$, $\Gamma_{n+1} := 0$, $\psi_0 := 0$, $\psi^n_i := \sum_{i \le j \le n} \psi_j$. On the domain of definition, the RPS variables are symplectic:

$$\sum_{i=1}^n d\Lambda_i \wedge d\ell_i + d\Gamma_i \wedge d\gamma_i + d\Psi_i \wedge d\psi_i = \sum_{i=1}^n d\Lambda_i \wedge d\lambda_i + d\eta_i \wedge d\xi_i + dp_i \wedge dq_i\ ;$$

for a proof, see [23] or [11, §4].

As phase space, consider a set of the same form as in (2.8), (2.9), namely

$$(\Lambda, \lambda, z) \in \mathcal{M}^{6n}_{\text{RPS}} := \mathcal{A} \times \mathbb{T}^n \times B^{4n} \tag{5.1}$$

with B a $4n$–dimensional ball around the origin (origin, which corresponds, as in Poincaré variables, to planar co–circular motions).

Poincaré and RPS variables are intimately connected: If we denote by

$$\phi^{\text{RPS}}_{\text{P}} : \quad (\Lambda, \lambda, z) \to (\Lambda, \lambda, \mathrm{z}) \tag{5.2}$$

the symplectic trasformation between RPS and Poincaré variables, then the following result holds.

Theorem 5.1 ([12]). *The symplectic map ϕ^{RPS}_P in* (5.2) *has the form*

$$\lambda = \lambda + \varphi(\Lambda, z) \qquad \mathrm{z} = \mathcal{Z}(\Lambda, z)$$

where $\varphi(\Lambda, 0) = 0$ and, for any fixed Λ, the map $\mathcal{Z}(\Lambda, \cdot)$ is 1:1, symplectic (i.e., it preserves the two form $d\eta \wedge d\xi + dp \wedge dq$) and its projections verify, for a suitable $\mathcal{V} = \mathcal{V}(\Lambda) \in \mathrm{SO}(n)$,

$$\Pi_\eta \mathcal{Z} = \eta + O_3\ ,\ \Pi_\xi \mathcal{Z} = \xi + O_3\ ,\ \Pi_{\mathrm{p}} \mathcal{Z} = \mathcal{V} p + O_3\ ,\ \Pi_{\mathrm{q}} \mathcal{Z} = \mathcal{V} q + O_3\ .$$

where $O_3 = O(|z|^3)$.

Partial reduction of rotations. Recalling that $\Gamma_{n+1} = 0$, $\Psi_{n-1} = |S^{(n)}| = |\mathrm{C}|$, $\Psi_n = \mathrm{C}_3$, $\psi_n = \alpha_{k^{(3)}}(k^{(1)}, k_3 \times \mathrm{C})$ one sees that

$$\begin{cases} p_n = \sqrt{2(|\mathrm{C}| - \mathrm{C}_3)}\ \cos\psi_n \\ q_n = -\sqrt{2(|\mathrm{C}| - \mathrm{C}_3)}\sin\psi_n\ , \end{cases}$$

[23] Beware of notations: we use upright characters for Poincaré variables $(\Lambda, \lambda, \mathrm{z}) := (\Lambda, \lambda, \eta, \mathrm{p}, \xi, \mathrm{q})$ and standard italic for RPS variables $(\Lambda, \lambda, z) := (\Lambda, \lambda, \eta, \xi, p, q)$.

showing that *the conjugated variables p_n and q_n are both integrals and hence both cyclic* for the planetary Hamiltonian, which, therefore, in such variables, will have the form

$$\mathcal{H}_{\text{RPS}}(\Lambda, \lambda, \bar{z}) = h_{\text{K}}(\Lambda) + \mu f_{\text{RPS}}(\Lambda, \lambda, \bar{z}) \,, \tag{5.3}$$

where $\bar{z}$ denotes the set of variables

$$\bar{z} := (\eta, \xi, \bar{p}, \bar{q}) := \big((\eta_1, \dots, \eta_n), (\xi_1, \dots, \xi_n), (p_1, \dots, p_{n-1}), (q_1, \dots, q_{n-1})\big) \,.$$

In other words, *the phase space $\mathcal{M}^{6n}_{\text{rps}}$ in* (5.1) *is foliated by $(6n-2)$–dimensional invariant manifolds*

$$\mathcal{M}^{6n-2}_{p_n,q_n} := \mathcal{M}^{6n}_{\text{RPS}}\,|_{p_n,q_n=\text{const}} \,, \tag{5.4}$$

and since the restriction of the standard symplectic form on such manifolds is simply $d\Lambda \wedge d\lambda + d\eta \wedge d\xi + d\bar{p} \wedge d\bar{q}$, such submanifolds are symplectic and the planetary flow is the standard Hamiltonian flow generated by $\mathcal{H}_{\text{RPS}}$ in (5.3). The submanifolds depend upon a particular orientation of the total angular momentum: in particular, $\mathcal{M}^{6n-2}_0$ correspond to the total angular momentum parallel to the vertical k_3–axis. Notice, also, that *the analytic expression of the planetary Hamiltonian $\mathcal{H}_{\text{rps}}$ is the same on each submanifold.*

In view of these observations, it is enough to study the planetary flow of $\mathcal{H}_{\text{RPS}}$ on, say, the vertical submanifold $\mathcal{M}^{6n-2}_0$.

Planetary Birkhoff normal forms and torsion. The RPS variables share with Poincaré variables classical *D'Alembert symmetries*, i.e., $\mathcal{H}_{\text{RPS}}$ is invariant under the transformations (2.10), $\mathcal{S}$ being as in (2.11); compare also Remark 3.3 of [12].

This implies that the averaged perturbation $f^{\text{av}}_{\text{RPS}} := \dfrac{1}{(2\pi)^n}\displaystyle\int_{\mathbb{T}^n} f_{\text{RPS}}\; d\lambda$ also enjoys D'Alembert rules and thus has an expansion analogue to (2.13), but independent of (p_n, q_n):

$$f^{\text{av}}_{\text{RPS}}(\Lambda, \bar{z}) = C_0(\Lambda) + \mathcal{Q}_h(\Lambda) \cdot \frac{\eta^2 + \xi^2}{2} + \bar{\text{Q}}_{\text{v}}(\Lambda) \cdot \frac{\bar{p}^2 + \bar{q}^2}{2} + \text{O}(|\bar{z}|^4) \tag{5.5}$$

with $\mathcal{Q}_h$ of order n and $\bar{\text{Q}}_{\text{v}}$ of order $(n-1)$. Notice that the matrix $\mathcal{Q}_h$ in (5.5) is the same as in (2.13), since, when $p = (\bar{p}, p_n) = 0$ and $q = (\bar{q}, q_n) = 0$, Poincaré and RPS variables coincide.

Using Theorem 5.1, one can also show that $\text{Q}_{\text{v}} := \begin{pmatrix} \bar{\text{Q}}_{\text{v}} & 0 \\ 0 & 0 \end{pmatrix}$ is conjugated (by a unitary matrix) to $\mathcal{Q}_v$ in (2.13), so that the eigenvalues $\bar{\varsigma}_i$ of $\bar{\text{Q}}_{\text{v}}$ coincide with $(\varsigma_1, ..., \varsigma_{n-1})$, as one naively would expect.

In view of the remark after (3.3), and of rotation–invariant Birkhoff theory,[24] one sees that one can construct, in an open neighborhood of co–planar and co–circular motions, the Birkhoff normal form of $f^{\text{av}}_{\text{RPS}}$ at any finite order.

More precisely, for $\epsilon > 0$ small enough, denoting

$$\mathcal{P}_\epsilon := \mathcal{A} \times \mathbb{T}^n \times B^{4n-2}_\epsilon \,, \qquad B^{4n-2}_\epsilon := \{\bar{z} \in \mathbb{R}^{4n-2} : |\bar{z}| < \epsilon\} \,,$$

an ϵ–neighborhood of the co–circular, co–planar region, one can find a real–analytic symplectic transformation $\phi_\mu : (\Lambda, \breve{\lambda}, \breve{z}) \in \mathcal{P}_\epsilon \to (\Lambda, \lambda, \bar{z}) \in \mathcal{P}_\epsilon$ such that $\breve{\mathcal{H}} := \mathcal{H}_{\text{RPS}} \circ \phi_\mu =$

[24] According to which the only forbidden frequencies for constructing the Birkhoff normal form are generated by those integer vectors k such that $\sum k_i = 0$; compare Proposition B.2, Appendix B below.

$h_{\kappa}(\Lambda) + \mu f(\Lambda, \breve{\lambda}, \breve{z})$ with

$$\breve{f}_{\rm av}(\Lambda, \breve{z}) := \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} f \, d\breve{\lambda} = C_0(\Lambda) + \Omega \cdot \breve{\rm R} + \frac{1}{2} \, \bar{\tau} \, \breve{\rm R} \cdot \breve{\rm R} + \breve{\mathcal{P}}(\Lambda, \breve{z})$$

where

$$\begin{cases} \Omega = (\sigma, \bar{\varsigma}) \\ \breve{z} := (\breve{\eta}, \breve{\xi}, \breve{p}, \breve{q}) \, , \quad \breve{\rm R} = (\breve{\rho}, \breve{r}) \, , \quad \breve{\mathcal{P}}(\Lambda, \breve{z}) = O(|\breve{z}|^6) \, , \\ \breve{\rho} = (\breve{\rho}_1, \cdots, \breve{\rho}_n) \, , \quad \breve{r} = (\breve{r}_1, \cdots, \breve{r}_{n-1}) \, , \\ \breve{\rho}_i := \frac{\breve{\eta}_i^2 + \breve{\xi}_i^2}{2} \, , \quad \breve{r}_i = \frac{\breve{p}_i^2 + \breve{q}_i^2}{2} \end{cases}$$

With straightforward (but not trivial!) computations, one can then show full torsion for the planetary problem.
More precisely, one finds (compare Proposition 8.1 of [11]):

Theorem 5.2. *For $n \geq 2$ and $0 < \delta_\star < 1$ there exist $\bar{\mu} > 0$, $0 < \underline{a}_1 < \overline{a}_1 < \cdots < \underline{a}_n < \overline{a}_n$ such that, on the set $\mathcal{A}$ defined in* (2.9) *and for $0 < \mu < \bar{\mu}$, the matrix $\bar{\tau}$ is non–singular:* [*i.e., double point should be added*] $\det \bar{\tau} = d_n(1 + \delta_n)$, *where $|\delta_n| < \delta_\star$ and*

$$d_n := (-1)^{n-1} \frac{3}{5} \Big(\frac{45}{16} \, \frac{1}{m_0^2} \Big)^{n-1} \frac{m_2}{m_1 m_0} a_1 \Big(\frac{a_1}{a_n} \Big)^3 \prod_{2 \leq k \leq n} \Big(\frac{1}{a_k} \Big)^4 .$$

Kolmogorov tori for the planetary problem. At this point one can apply to the planetary Hamiltonian in normalized variables $\breve{\mathcal{H}}(\Lambda, \breve{\lambda}, \breve{z})$ Arnold's Theorem 3.2 above completing Arnold's project on the planetary N–body problem.

Indeed, by using the refinements of Theorem 3.2 as given in [9], from Theorem 5.2 there follows

Theorem 5.3. *There exists positive constants ϵ_*, c_* and C_* such that the following holds. If $0 < \epsilon < \epsilon_*$ and $0 < \mu < \epsilon^6 / (\log \epsilon^{-1})^{c_*}$ then each symplectic submanifold $\mathcal{M}^{6n-2}_{p_n, q_n}$* (5.4) *contains a positive measure $\mathcal{H}_{RPS}$–invariant Kolmogorov set $\mathcal{K}_{p_n, q_n}$, which is actually the suspension of the same Kolmogorov set $\mathcal{K} \subseteq \mathcal{P}_\epsilon$, which is $\breve{\mathcal{H}}$–invariant.*
Furthermore, $\mathcal{K}$ is formed by the union of $(3n-1)$–dimensional Lagrangian, real–analytic tori on which the $\breve{\mathcal{H}}$–motion is analytically conjugated to linear Diophantine quasi–periodic motions with frequencies $(\omega_1, \omega_2) \in \mathbb{R}^n \times \mathbb{R}^{2n-1}$ with $\omega_1 = O(1)$ and $\omega_2 = O(\mu)$.
Finally, $\mathcal{K}$ satisfies the bound[25] $\operatorname{meas} \mathcal{P}_\epsilon \geq \operatorname{meas} \mathcal{K} \geq \Big(1 - C_* \sqrt{\epsilon} \Big) \operatorname{meas} \mathcal{P}_\epsilon$.

Conley-Zehnder stable periodic orbits. The tori $\mathcal{T} \in \mathcal{K}$ form a (Whitney) smooth family of *non–degenerate Kolmogorov tori*, which means the following. The tori in $\mathcal{K}$ can be parameterized by their frequency $\omega \in \mathbb{R}^{3n-1}$ (i.e., $\mathcal{T} = \mathcal{T}_\omega$) and there exists a real–analytic symplectic diffeomorphism $\nu : (y, x) \in B^m \times \mathbb{T}^m \to \nu(y, x; \omega) \in \mathcal{P}_\epsilon$, $m := 3n - 1$, uniformly Lipschitz in ω (actually C^∞ in the sense of Whitney) such that, for each ω

- $\breve{\mathcal{H}} \circ \nu = E + \omega \cdot y + Q$; (Kolmogorov's normal form)
- $E \in \mathbb{R}$ (the energy of the torus); $\omega \in \mathbb{R}^m$ is a Diophantine vector;
- $Q = O(|y|^2)$ and $\det \displaystyle\int_{\mathbb{T}^m} \partial_{yy} Q(0, x) \, dx \neq 0$, (non–degeneracy)

[25] In particular, $\operatorname{meas} \mathcal{K} \simeq \epsilon^{4n-2} \simeq \operatorname{meas} \mathcal{P}_\epsilon$.

- $\mathcal{T}_\omega = \nu(0, \mathbb{T}^m)$.

Now, in the first paragraph of [14] Conley and Zehnder, putting together KAM theory (and in particular exploiting Kolmogorv's normal form for KAM tori) together with Birkhoff–Lewis fixed–point theorem show that long–period periodic orbits cumulate densely on Kolmogorov tori so that, in particular, *the Lebesgue measure of the closure of the periodic orbits can be bounded below by the measure of the Kolmogorov set.* Notwithstanding the proper degeneracy, this remark applies also in the present situation and as a consequence of Theorem 5.3 and of the fact that the tori in $\mathcal{K}$ are non–degenerate Kolmogorov tori it follows ([7]) that *in the planetary model the measure of the closure of the periodic orbits in $\mathcal{P}_\epsilon$ can be bounded below by a constant times ϵ^{4n-2}.*

A. Details on the classical Hamiltonian structure

Inertial manifold. Equations (2.1) are invariant by change of "inertial frames", i.e., by change of variables of the form $u^{(i)} \to u^{(i)} - (a + ct)$ with fixed $a, c \in \mathbb{R}^3$. This allows to restrict the attention to the manifold of "initial data" given by

$$\sum_{i=0}^{n} \mathrm{m}_i u^{(i)}(0) = 0 \,, \qquad \sum_{i=0}^{n} \mathrm{m}_i \dot{u}^{(i)}(0) = 0 \,; \tag{A.1}$$

indeed, just replace the coordinates $u^{(i)}$ by $u^{(i)} - (a + ct)$ with

$$a := \mathrm{m}_{\mathrm{tot}}^{-1} \sum_{i=0}^{n} \mathrm{m}_i u^{(i)}(0) \qquad \text{and} \qquad c := \mathrm{m}_{\mathrm{tot}}^{-1} \sum_{i=0}^{n} \mathrm{m}_i \dot{u}^{(i)}(0) \,, \qquad \mathrm{m}_{\mathrm{tot}} := \sum_{i=0}^{n} \mathrm{m}_i \,.$$

The total linear momentum $\mathrm{M}_{\mathrm{tot}} := \sum_{i=0}^{n} \mathrm{m}_i \dot{u}^{(i)}$ does not change along the flow of (2.1), i.e., $\dot{\mathrm{M}}_{\mathrm{tot}} = 0$ along trajectories; therefore, by (A.1), $\mathrm{M}_{\mathrm{tot}}(t)$ vanishes for all times. But, then, also the position of the total center of mass $B(t) := \sum_{i=0}^{n} \mathrm{m}_i u^{(i)}(t)$ is constant ($\dot{B} = 0$) and, again by (A.1), $B(t) \equiv 0$. In other words, the manifold of initial data (A.1) is invariant under the flow generated by (2.1).

The Linear momentum reduction. In view of the invariance properties discussed above, in the variables $(U^{(i)}, u^{(i)}) \in \widehat{\mathcal{M}}$, (recall (2.2) and that $U^{(i)} := \mathrm{m}_i \dot{u}^{(i)}$), it is enough to consider the submanifold $\widehat{\mathcal{M}}_0 := \{(U, u) \in \widehat{\mathcal{M}} : \sum_{i=0}^{n} \mathrm{m}_i u^{(i)} = 0 = \sum_{i=0}^{n} U^{(i)}\}$, which corresponds to the manifold described in (A.1).

The submanifold $\widehat{\mathcal{M}}_0$ is symplectic, i.e., the restriction of the form (2.3) to $\widehat{\mathcal{M}}_0$ is again a symplectic form; indeed: $\big(\sum_{i=0}^{n} dU^{(i)} \wedge du^{(i)}\big)\Big|_{\widehat{\mathcal{M}}_0} = \sum_{i=1}^{n} \frac{\mathrm{m}_0 + \mathrm{m}_i}{\mathrm{m}_0}\, dU^{(i)} \wedge du^{(i)}$.

Poincaré's *symplectic reduction* ("reduction of the linear momentum") goes as follows. Let $\phi_{\mathrm{he}} : (R, r) \to (U, u)$ be the linear transformation given by

$$\phi_{\mathrm{he}} : \begin{cases} u^{(0)} = r^{(0)} \,, & u^{(i)} = r^{(0)} + r^{(i)} \,, \quad (i = 1, ..., n) \\ U^{(0)} = R^{(0)} - \sum_{i=1}^{n} R^{(i)} \,, & U^{(i)} = R^{(i)} \,, \quad (i = 1, ..., n) \,; \end{cases} \tag{A.2}$$

such transformation is symplectic, i.e., $\sum_{i=0}^n dU^{(i)}\wedge du^{(i)} = \sum_{i=0}^n dR^{(i)}\wedge dr^{(i)}$. recall that this means, in particular, that in the new variables the Hamiltonian flow is again standard: more precisely, one has that $\phi^t_{\widehat{\mathcal{H}}_{\rm N}}\circ\phi_{\rm he} = \phi_{\rm he}\circ\phi^t_{\widehat{\mathcal{H}}_{\rm N}\circ\phi}$.

Letting $\mathrm{m}_{\rm tot} := \sum_{i=0}^n \mathrm{m}_i$ one sees that, in the new variables, $\widehat{\mathcal{M}}_0$ reads

$$\Big\{(R,r)\in\mathbb{R}^{6(n+1)}:\ R^{(0)}=0, r^{(0)} = -\mathrm{m}_{\rm tot}^{-1}\sum_{i=1}^n \mathrm{m}_i r^{(i)}\ ,\ 0\neq r^{(i)}\neq r^{(j)}\ \forall\, 1\le i\neq j\le n\Big\}.$$

The restriction of the 2–form (2.3) to $\widehat{\mathcal{M}}_0$ is simply $\displaystyle\sum_{i=1}^n dR^{(i)}\wedge dr^{(i)}$ and

$$\mathcal{H}_{\rm N} := \widehat{\mathcal{H}}_{\rm N}\circ\phi_{\rm he}|_{\mathcal{M}_0} = \sum_{i=1}^n \frac{|R^{(i)}|^2}{2\frac{\mathrm{m}_0\mathrm{m}_i}{\mathrm{m}_0+\mathrm{m}_i}} - \frac{\mathrm{m}_0\mathrm{m}_i}{|r^{(i)}|} + \sum_{1\le i<j\le n}\frac{R^{(i)}\cdot R^{(j)}}{\mathrm{m}_0} - \frac{\mathrm{m}_i\mathrm{m}_j}{|r^{(i)}-r^{(j)}|}\ .$$

The dynamics generated by $\widehat{\mathcal{H}}_{\rm N}$ on $\widehat{\mathcal{M}}_0$ is equivalent to the dynamics generated by the Hamiltonian $(R,r)\in\mathbb{R}^{6n}\to\mathcal{H}_{\rm N}(R,r)$ on

$$\mathcal{M}_0 := \Big\{(R,r) = (R^{(1)},...,R^{(n)},r^{(1)},...,r^{(n)})\in\mathbb{R}^{6n} : 0\neq r^{(i)}\neq r^{(j)}\ ,\forall i\neq j\Big\}$$

with respect to the standard symplectic form $\sum_{i=1}^n dR^{(i)}\wedge dr^{(i)}$; to recover the full dynamics on $\widehat{\mathcal{M}}_0$ from the dynamics on $\mathcal{M}_0$ one will simply set $R^{(0)}(t)\equiv 0$ and $r^{(0)}(t) := -\mathrm{m}_{\rm tot}^{-1}\displaystyle\sum_{i=1}^n \mathrm{m}_i r^{(i)}(t)$.

Since we are interested in the planetary case, we perform the trivial rescaling by a small positive parameter μ:

$$m_0 := \mathrm{m}_0\ ,\ \mathrm{m}_i = \mu m_i\ (i\ge 1)\ ,\quad X^{(i)} := \frac{R^{(i)}}{\mu}\ ,\ x^{(i)} := r^{(i)}\ ,$$
$$\mathcal{H}_{\rm plt}(X,x) := \frac{1}{\mu}\mathcal{H}_{\rm N}(\mu X,x)\ ,$$

a transformation which leaves unchanged Hamilton's equations.

Delaunay and Poincaré variables. The Hamiltonian $\mathcal{H}^{(0)}_{\rm plt}$ in (2.4) governes the motion of n decoupled two–body problems with Hamiltonian

$$h^{(i)}_{\rm 2B} = \frac{|X^{(i)}|^2}{2M_i} - \frac{M_i\bar m_i}{|x^{(i)}|}\ ,\qquad (X^{(i)},x^{(i)})\in\mathbb{R}^3\times\mathbb{R}^3_* := \mathbb{R}^3\times(\mathbb{R}^3\backslash\{0\})\ .$$

Such two–body sytems are, as well known, integrable. The explicit "symplectic integration" is done by means of the *Delaunay variables*, whose construction we, now, briefly, recall (for full details and proofs, see, e.g., [5]).

Assume that $h^{(i)}_{\rm 2B}(X^{(i)},x^{(i)})<0$ so that the Hamiltonian flow $\phi^t_{h^{(i)}_{\rm 2B}}(X^{(i)},x^{(i)})$ evolves on a Keplerian ellipse $\mathfrak{E}_i$ and assume that the eccentricity $e_i\in(0,1)$.

Let a_i, P_i denote, respectively, the *semimajor axis* and the *perihelion* of $\mathfrak{E}_i$.

Let $\mathrm{C}^{(i)}$ denote the i^{th} angular momentum $\mathrm{C}^{(i)} := x^{(i)} \times y^{(i)}$.

Let us, also, introduce the "Delaunay nodes"

$$\bar\nu_i := k^{(3)} \times \mathrm{C}^{(i)} \quad 1 \le i \le n \,, \tag{A.3}$$

where $(k^{(1)}, k^{(2)}, k^{(3)})$ is the standard orthonormal basis in $\mathbb{R}^3$. Finally, for $u, v \in \mathbb{R}^3$ lying in the plane orthogonal to a non–vanishing vector w, let $\alpha_w(u, v)$ denote the positively oriented angle (mod 2π) between u and v (orientation follows the "right hand rule").

The *Delaunay action–angle variables* $(\Lambda_i, \Gamma_i, \Theta_i, \ell_i, \mathrm{g}_i, \theta_i)$ are, then, defined as

$$\begin{cases} \Lambda_i := M_i\sqrt{\bar m_i a_i} \\ \ell_i := \text{mean anomaly of } x^{(i)} \text{ on } \mathfrak{E}_i \end{cases} , \quad \begin{cases} \Gamma_i := |\mathrm{C}^{(i)}| = \Lambda_i\sqrt{1-e_i^2} \\ \mathrm{g}_i := \alpha_{\mathrm{C}^{(i)}}(\bar\nu_i, P_i) \end{cases}$$
$$\begin{cases} \Theta_i := \mathrm{C}^{(i)} \cdot k^{(3)} \\ \theta_i := \alpha_{k^{(3)}}(k^{(1)}, \bar\nu_i) \end{cases} \tag{A.4}$$

Notice that the Delaunay variables are defined on an open set of full measure of the Cartesian phase space $\mathbb{R}^{3n} \times \mathbb{R}^{3n}_*$, namely, on the set where $e_i \in (0, 1)$ and the nodes $\bar\nu_i$ in (A.3) are well defined; on such set the "Delaunay inclinations" i_i defined through the relations

$$\cos \mathrm{i}_i := \frac{\mathrm{C}^{(i)} \cdot k^{(3)}}{|\mathrm{C}^{(i)}|} = \frac{\Theta_i}{\Gamma_i} \,, \tag{A.5}$$

are well defined and we choose the branch of $\cos^{-1}$ so that $\mathrm{i}_i \in (0, \pi)$.

The Delaunay variables become singular when $\mathrm{C}^{(i)}$ is vertical (the Delaunay node is no more defined) and in the circular limit (the perihelion is not unique). In these cases different variables have to been used (see below).

On the set where the Delaunay variables are well posed, they define a *symplectic set of action–angle variables*, i.e., $\sum_{i=1}^n dX^{(i)} \wedge dx^{(i)} = \sum_{i=1}^n d\Lambda_i \wedge d\ell_i + d\Gamma_i \wedge d\mathrm{g}_i + d\Theta_i \wedge d\theta_i$, for a proof, see §3.2 of [5].

In Delaunay action–angle variables $((\Lambda, \Gamma, \Theta), (\ell, \mathrm{g}, \theta))$ the Hamiltonian $\mathcal{H}^{(0)}_{\mathrm{plt}}$ takes the form (2.7). We shall restrict our attention to the collisionless phase space

$$\mathcal{M}_{\mathrm{plt}} := \left\{ \Lambda_i > \Gamma_i > \Theta_i > 0 \,, \quad \frac{\Lambda_i}{M_i\sqrt{\bar m_i}} \neq \frac{\Lambda_j}{M_j\sqrt{\bar m_j}} \,, \ \forall\, i \neq j \right\} \times \mathbb{T}^{3n} \,,$$

endowed with the standard symplectic form $\sum_{i=1}^n d\Lambda_i \wedge d\ell_i + d\Gamma_i \wedge d\mathrm{g}_i + d\Theta_i \wedge d\theta_i$.

Notice that the $6n$–dimensional phase space $\mathcal{M}_{\mathrm{plt}}$ is foliated by $3n$–dimensional $\mathcal{H}^{(0)}_{\mathrm{plt}}$–invariant tori $\{\Lambda, \Gamma, \Theta\} \times \mathbb{T}^3$, which, in turn, are foliated by n–dimensional tori $\{\Lambda\} \times \mathbb{T}^n$, expressing geometrically the degeneracy of the integrable Keplerian limit of the $(1 + n)$–body problem.

A regularization of the Delaunay variables in their singular limit was introduced by Poincaré, in such a way that the set of action–angle variables $((\Gamma, \Theta), (\mathrm{g}, \theta))$ is mapped onto cartesian variables regular near the origin, which corresponds to co–circular and co–planar motions, while the angles conjugated to Λ_i, which remains invariant, are suitably shifted.

More precisely, the *Poincaré variables* are given by $(\Lambda, \lambda, \mathrm{z}) := (\Lambda, \lambda, \eta, \xi, \mathrm{p}, \mathrm{q}) \in \mathbb{R}^n_+ \times \mathbb{T}^n \times \mathbb{R}^{4n}$, with the Λ's as in (A.4) and

$$\lambda_i = \ell_i + \mathrm{g}_i + \theta_i, \quad \begin{cases} \eta_i = \sqrt{2(\Lambda_i - \Gamma_i)}\, \cos(\theta_i + \mathrm{g}_i) \\ \xi_i = -\sqrt{2(\Lambda_i - \Gamma_i)}\, \sin(\theta_i + \mathrm{g}_i) \end{cases} , \quad \begin{cases} \mathrm{p}_i = \sqrt{2(\Gamma_i - \Theta_i)}\, \cos\theta_i \\ \mathrm{q}_i = -\sqrt{2(\Gamma_i - \Theta_i)}\, \sin\theta_i \end{cases}$$

Notice that $e_i = 0$ corresponds to $\eta_i = 0 = \xi_i$, while $\mathrm{i}_i = 0$ corresponds to $\mathrm{p}_i = 0 = \mathrm{q}_i$; compare (A.4) and (A.5).

On the domain of definition, the Poincaré variables are symplectic

$$\sum_{i=1}^{n} d\Lambda_i \wedge d\ell_i + d\Gamma_i \wedge d\mathrm{g}_i + d\Theta_i \wedge d\theta_i = \sum_{i=1}^{n} d\Lambda_i \wedge d\lambda_i + d\eta_i \wedge d\xi_i + d\mathrm{p}_i \wedge d\mathrm{q}_i \; ;$$

for a proof, see Appendix C of [4].

B. Birkhoff normal forms

In this appendix we recall a few known and less known facts about the general theory of Birkhoff normal forms.

Consider as phase space a $2m$ ball B_δ^{2m} around the origin in $\mathbb{R}^{2m}$ and a real–analytic Hamiltonian of the form $H(w) = c_0 + \Omega \cdot \mathrm{r} + o(|w|^2)$ where

$$\begin{cases} w = (u_1, \dots, u_m, v_1, \dots, v_m) \in \mathbb{R}^{2m} \, , \\ \mathrm{r} = (\mathrm{r}_1, \dots, \mathrm{r}_m) \, , \quad \mathrm{r}_j = \frac{u_j^2 + v_j^2}{2} \, . \end{cases}$$

the symplectic form being $\sum du_i \wedge dv_i$. The components Ω_j of Ω are called the first order Birkhoff invariants. The following is a classical result due to G.D. Birkhoff.

Proposition B.1. *Assume that the first order Birkhoff invariants Ω_j verify, for some $a > 0$ and integer s,*

$$|\Omega \cdot k| \geq a > 0, \quad \forall \, k \in \mathbb{Z}^m : \; 0 < |k|_1 := \sum_{j=1}^{m} |k_j| \leq 2s \, . \tag{B.1}$$

Then, there exists $0 < \delta' \leq \delta$ and a symplectic transformation $\breve{\phi} : \breve{w} \in B_{\delta'}^{2m} \to w \in B_\delta^{2m}$ which puts H into Birkhoff normal form up to the order $2s$, i.e.,

$$H \circ \breve{\phi} = c_0 + \Omega \cdot \breve{\mathrm{r}} + \sum_{2 \leq h \leq s} P_h(\breve{\mathrm{r}}) + o(|\breve{w}|^{2s})] \, , \tag{B.2}$$

where P_h are homogeneous polynomials in $\breve{\mathrm{r}}_j = |\breve{w}_j|^2/2 := (\breve{u}_j^2 + \breve{v}_j^2)/2$ of degree h.

Less known is that the hypotheses of this proposition may be loosened in the case of *rotation invariant Hamiltonians*: this fact, for example, has been used neither in [1] nor in [17].

First, let us generalize the class of Hamiltonian functions so as to include the secular Hamiltonian (2.13): let us consider an open, bounded, connected set $U \subseteq \mathbb{R}^n$ and consider the phase space $\mathcal{D} := U \times \mathbb{T}^n \times B_\delta^{2m}$, endowed with the standard symplectic form $dI \wedge d\varphi + du \wedge dv$.

We say that a *Hamiltonian $H(I, \varphi, w)$ on $\mathcal{D}$ is rotation invariant* if $H \circ \mathcal{R}^g = H$ for any $g \in \mathbb{T}$, where $\mathcal{R}^g$ is a *symplectic rotation by an angle* $g \in \mathbb{T}$ on $\mathcal{D}$, i.e., a symplectic map of the form $\mathcal{R}^g : (I, \varphi, w) \to (I', \varphi', w')$ with $I_i' = I_i, \; \varphi_i' = \varphi_i + g, \; w' = \mathcal{S}^g w$, with $\mathcal{S}^g$ definined in (2.11).

Now, consider a φ–independent real–analytic Hamiltonian $H : (I, \varphi, w) \in \mathcal{D} \to H(I, w) \in \mathbb{R}$ of the form $H(I, w) = c_0(I) + \Omega(I) \cdot \mathrm{r} + o(|w|^2; I)$, by $f = o(|w|^2; I)$ we mean that $f = f(I, w)$ and $|f|/|w|^2 \to 0$ as $w \to 0$.

Then, it can be proven the following

Proposition B.2. *Assume that H is rotation–invariant and that the first order Birkhoff invariants Ω_j verify, for all $I \in U$, for some $a > 0$ and integer s*

$$|\Omega \cdot k| \geq a > 0, \quad \forall\, 0 \neq k \in \mathbb{Z}^m : \sum_{i=1}^{n} k_i = 0 \text{ and } |k|_1 \leq 2s\,. \tag{B.3}$$

Then, there exists $0 < \delta' \leq \delta$ and a symplectic transformation $\breve{\phi} : (I, \breve{\varphi}, \breve{w}) \in \breve{\mathcal{D}} := U \times \mathbb{T}^n \times B^{2m}_{\delta'} \to (I, \varphi, w) \in \mathcal{D}$ which puts H into Birkhoff normal form up to the order $2s$ as in (B.2) *with the coefficients of P_h and the reminder depending also on I. Furthermore, $\breve{\phi}$ leaves the I–variables fixed, acts as a $\breve{\varphi}$–independent shift on $\breve{\varphi}$, is $\breve{\varphi}$–independent on the remaining variables and is such that*

$$\breve{\phi} \circ \mathcal{R}^g = \mathcal{R}^g \circ \breve{\phi}\,. \tag{B.4}$$

The proof of Proposition B.2 may be found in §7.2 in [11].

C. Arnold's statements (from [1])

- *Conditionally periodic motions in the many–body problem have been found. If the masses of n "planets" are sufficiently small in comparison with the mass of the central body, the motion is conditionally periodic for the majority of initial conditions for which the eccentricities and inclinations of the Kepler ellipses are small. Further, the major semiaxis perpetually remain close to their original values and the eccentricities and inclinations remain small.* [1, p. 87]
- *With the help of the fundamental theorem*[26] *of Chapter IV, we investigate in this chapter the class of "planetary" motions in the three–body and many–body problems. We show that, for the majority of initial conditions under which the instantaneous orbits of the planets are close to circles lying in a single plane, perturbation of the planets on one another produces, in the course of an infinite interval of time, little change on these orbits provided the masses of the planets are sufficiently small.*
  In particular, it follows from our results that *in the n-body problem there exists a set of initial conditions having a positive Lebesgue measure and such that, if the initial positions and velocities of the bodies belong to this set, the distances of the bodies from each other will remain perpetually bounded.* [1, p.125]
- At p. 127 one finds Theorem 3.1 reported at the beginning of § 3 above.
- As mentioned in the introduction, Arnold provides a full detailed proof, checking the non–degeneracy conditions of his fundamental theorem, only for the two–planet model ($n = 2$) in the planar regime. As for generalizations, he states:

[26] I.e., Theorem 3.2 above.

- **The plane problem of $n > 2$ planets.** *The arguments of §2 and 3 easily carry over to the case of more than two planets. [· · ·] We shall not dwell on the details of the calculations which lead to the results of §1, 4.* [1, p. 139]
- Finally, for the spatial general case:

  The rather lengthy calculations involved in the solution of (3.5.9), the construction of variables satisfying conditions 1)–4), and the verification of non–degeneracy conditions analogous to the arguments of § 4 will not be discussed here. [1, p. 142]

Acknowledgements. L.C. is partially supported by Prin 2012 Italian national project "Critical Point Theory and Perturbative Methods for Nonlinear Differential Equations"; G.P. is partially supported by 2012 European Research Council FP7 project and by 2013 STAR project of University of Naples, Federico II.

References

[1] V. I. Arnold, *Small denominators and problems of stability of motion in classical and celestial mechanics*, Uspehi Mat. Nauk **18**(6(114)) (1963), 91–192. English translation in: Russian Math. Surveys, **18**(6) (1963), 85–191.

[2] V. I. Arnold, V. V. Kozlov, and A. I. Neishtadt, *Mathematical aspects of classical and celestial mechanics*, volume 3 of Encyclopaedia of Mathematical Sciences, Springer-Verlag, Berlin, third edition, 2006. [Dynamical systems. III], Translated from the Russian original by E. Khukhro.

[3] J. Barrow–Green. *Poincaré and the three body problem History of Mathematics*, American Mathematical Society. Providence, RI. **11**, xvi+272 p., 1997.

[4] L. Biasco, L. Chierchia, and E. Valdinoci, *Elliptic two-dimensional invariant tori for the planetary three-body problem*, Arch. Rational Mech. Anal. **170**:91–135, 2003. See also: Corrigendum. Arch. Ration. Mech. Anal. **180** (507–509), 2006.

[5] A. Celletti and L. Chierchia, *KAM stability and celestial mechanics*, Mem. Amer. Math. Soc., **187**(878):viii+134, 2007.

[6] L. Chierchia, *Kolmogorov-Arnold-Moser (KAM) Theory*, In Encyclopedia of Complexity and Systems Science, Editor-in-chief: Meyers, Robert A. Springer, 2009.

[7] ———, *Periodic solutions of the planetary N-body problem*, in "XVIIth International Mathematical Congress on Mathematical Physics", Edited by A. Jensen, 269-280, World Scientific (2014).

[8] L. Chierchia and C. Falcolini, *A Direct Proof of a Theorem by Kolmogorov in Hamiltonian Systems*, Annali Scuola Normale Sup. Pisa, Scienze Fisiche e Matematiche, XXI Fasc. **4** (1994), 541–593.

[9] L. Chierchia and G. Pinzari, *Properly–degenerate KAM theory following V.I. Arnold)*, Discrete Contin. Dyn. Syst. Ser. S **3**(4) (2010), 545–578.

[10] ———, *Deprit's reduction of the nodes revisited*, Celest. Mech. Dyn. Astr. **109**(3) (2011), 285–301.

[11] ———, *The planetary N-body problem: symplectic foliation, reductions and invariant tori*, Invent. Math. **186**(1) (2011), 1–77.

[12] ———, *Planetary Birkhoff normal forms*, Journal of Modern Dynamics **5**(4) (2011), 623–664.

[13] L. Chierchia and F. Pusateri, *Analytic Lagrangian tori for the planetary many–body problem*, Ergod. Th. & Dynam. Sys. **29**, 3, 849–873 , June 2009.

[14] C. Conley and E. Zehnder, *An index theory for periodic solutions of a Hamiltonian system*, In Geometric dynamics (Rio de Janeiro, 1981), volume **1007** of Lecture Notes in Math. pp. 132–145. Springer, Berlin, 1983.

[15] A. Deprit, *Elimination of the nodes in problems of n bodies*, Celestial Mech. **30**(2) (1983), 181–195.

[16] H. Eliasson, *Absolutely convergent series expansions for quasi periodic motions*, Math. Phys. Electron. J. **2** (1996), Paper 4, p. 33 (electronic). [This is the publication, with minor revisions, of a 1988 ETH preprint]

[17] J. Féjoz, *Démonstration du 'théorème d'Arnold' sur la stabilité du système planétaire (d'après Herman)*, Ergodic Theory Dynam. Systems **24**(5) (2004), 1521–1582. Revised version (2007) available at http://people.math.jussieu.fr/~fejoz/articles.html

[18] J. Féjoz, *On "Arnold's theorem" in celestial mechanics – a summary with an appendix on the Poincaré coordinates*, Discrete and Continuous Dynamical Systems **33** (2013), 3555–3565.

[19] G. Gallavotti, *Twistless KAM tori, quasi flat homoclinic intersections, and other cancellations in the perturbation series of certain completely integrable Hamiltonian systems*, A review. Rev. Math. Phys. **6** (1994), no. 3, 343–411.

[20] M.R. Herman, *Torsion du problème planètaire*, edited by J. Fejóz in 2009.
Available in the electronic 'Archives Michel Herman' at http://www.college-de-france.fr/default/EN/all/equ_dif/archives_michel_herman.htm.

[21] H. Hofer and E. Zehnder. *Symplectic Invariants and Hamiltonian Dynamics*, Birkhäuser Verlag, Basel, 1994.

[22] A. N. Kolmogorov, *On the conservation of conditionally periodic motions under small perturbation of the Hamiltonian*, Dokl. Akad. Nauk. SSR **9**8 (1954), 527–530.

[23] G. Pinzari. *On the Kolmogorov set for many–body problems*, PhD thesis, Università Roma Tre, April 2009. http://ricerca.mat.uniroma3.it/Dottorato/TESI/pinzari/pinzari.pdf

[24] ———, *Aspects of the planetary Birkhoff normal form*, Regular and Chaotic Dynamics **18**(6) (2013), 860–906.

[25] H. Poincaré, *Sur le probléme des trois corps et les équations de la dynamique*, Acta Mathematica **13**, Issue 1, December 1890, A3–A270.

[26] P. Robutel, *Stability of the planetary three-body problem. II*, KAM theory and existence of quasiperiodic motions, Celestial Mech. Dynam. Astronom. **62**(3) (1995), 219–261. See also: *Erratum*, Celestial Mech. Dynam. Astronom **84**(3) (2002), 317.

[27] H. Rüßmann, *Invariant Tori in Non-Degenerate Nearly Integrable Hamiltonian Systems*, R. & C. Dynamics **2**(6) (March 2001), 119–203.

Dipartimento di Matematica, Università "Roma Tre", Largo S.L. Murialdo 1, 00146 Roma (Italy)
E-mail: luigi@mat.uniroma3.it

Dipartimento di Matematica ed Applicazioni, Università di Napoli "Federico II", Monte Sant'Angelo – Via Cinthia I-80126 Napoli (Italy)
E-mail: gabriella.pinzari@unina.it

Dynamics of C^1-diffeomorphisms: global description and prospects for classification

Sylvain Crovisier

Abstract. We are interested in finding a dense part of the space of C^1-diffeomorphisms which decomposes into open subsets corresponding to different dynamical behaviors: we discuss results and questions in this direction.

In particular we present recent results towards a conjecture by J. Palis: any system can be approximated either by one which is hyperbolic (and whose dynamics is well understood) or by one which exhibits a homoclinic bifurcation (a simple local configuration involving one or two periodic orbits).

Mathematics Subject Classification (2010). Primary 37C20, 37C50, 37D25, 37D30; Secondary 37C29, 37D05, 37G25.

Keywords. Differentiable dynamical systems, closing lemma, homoclinic tangency, heterodimensional cycle, partial hyperbolicity, generic dynamics.

1. Introduction

A differentiable transformation - a diffeomorphism or a flow - on a manifold defines a dynamical systems: our goal is to describe the long time behavior of its orbits. In some cases, the dynamics, though rich, can be satisfactorily understood: the hyperbolic systems introduced by Anosov and Smale [5, 78] break down into finitely many transitive pieces, can be coded by a finite alphabet, admit physical measures which represent the orbit of Lebesgue-almost every point, are structurally stable...

The dynamics of a particular system may be quite particular and too complicated. One will instead consider a large class of systems on a fixed compact connected smooth manifold M without boundary. For instance:

- the spaces of C^r diffeomorphisms $\mathrm{Diff}^r(M)$ or vector fields $\mathcal{X}^r(M)$, for $r \geq 1$,
- the subspace $\mathrm{Diff}^r_\omega(M)$ of those preserving a volume or symplectic form ω,
- the spaces of C^r Hamiltonians $H\colon M \to \mathbb{R}$ (when M is symplectic), and of C^{r+1} Riemannian metrics on M (defining the geodesic flows on TM), etc.

This approach (present in [79]) allows to study typical dynamics in the class, but also their stability, i.e. how properties change when the system is replaced by a system nearby. For finite-dimensional classes of systems (like polynomial automorphisms of $\mathbb{C}^2$ with fixed degree, or directional flows on flat surfaces with fixed genus) one can consider sets of parameters with full Lebesgue measure; for larger classes, one can introduce (non-degenerate)

Proceedings of the International Congress of Mathematicians, Seoul, 2014

parametrized families of systems, as in [64]. Working on a Baire space (mainly $\mathrm{Diff}^1(M)$) we intend here to describe dense subsets of systems that are G_δ (i.e. Baire-generic) or ultimately even open.

The main difficulty is to *perturb the system while controlling the dynamics*. Weaker topologies offer more flexibility under perturbations, but less control on the dynamics. In practice one works in the C^1-topology: for smoother systems, new dynamical properties appear as Pesin theory [18, 65], KAM (see [87] and section 2.4), robust homoclinic tangencies on surfaces [54, 57], ... but few is known about perturbations in higher topology (even about the existence of periodic orbits), see for instance [70]. However, producing C^1-open sets, one also describes part of the smoother systems and presumably gives insights for more regular dynamics.

After initial works focused on hyperbolicity, three main problems have emerged.

1.1. Density of hyperbolicity. Smale has explicitly stated [80] the following problem for the class of C^r-endomorphisms of the interval (which has been solved affirmatively [47]) and for the class of one-dimensional complex polynomials with fixed degree (still unknown).

Problem 1 (Smale). *In which class of systems is hyperbolicity dense?*

In the space of diffeomorphisms $\mathrm{Diff}^r(M)$, $r \geq 1$, the (open) subset of hyperbolic systems is dense when $\dim(M) = 1$, but this is not the case for any manifold. Open sets $\mathcal{U}$ of non-hyperbolic diffeomorphisms have been obtained as follow:

- When $\dim(M) \geq 3$, Abraham and Smale have built [4] in a non-empty open set $\mathcal{U}$ a dense family of diffeomorphisms with a heterodimensional cycle.
- When $\dim(M) = 2$ and $r \geq 2$, Newhouse has built [57] in a non-empty open set $\mathcal{U}$ a dense family of diffeomorphisms with a homoclinic tangency.

These notions are defined below. Surprisingly the case $r = 1$ and $\dim(M) = 2$ is still unknown and has a particular importance for our study.

For $f \in \mathrm{Diff}^r(M)$ and any point p in a hyperbolic periodic orbit (or more generally in a hyperbolic set), the stable and unstable sets $W^s(p)$ and $W^u(p)$ (i.e. the sets of $z \in M$ such that $d(f^n(p), f^n(z)) \to 0$ as n goes to $+\infty$ and $-\infty$ respectively) are immersed submanifolds with transversal intersection at p.

Definition 1.1 (Homoclinic bifurcation). A *homoclinic tangency* is a non-transverse intersection $z \in W^u(p) \cap W^s(p)$ associated to a hyperbolic periodic point p. A *heterodimensional cycle* is a pair of intersections $z \in W^u(p) \cap W^s(q)$ and $z' \in W^u(q) \cap W^s(p)$ associated to hyperbolic points p, q such that the dimension of $W^s(p)$ is strictly smaller than the one of $W^s(q)$. See figure 1.1.

In both of these configurations the point z is non-wandering and admits a unit tangent vector whose norm decreases to 0 under forward *and* backward iterations.

1.2. Obstructions to hyperbolicity. Palis has conjectured [59–62] a positive answer to the following problem in the class $\mathrm{Diff}^r(M)$, $r \geq 1$.

Problem 2 (Palis' conjecture). *Approximate any system in a class by one which is hyperbolic or which exhibits a homoclinic tangency or a heterodimensional cycle.*

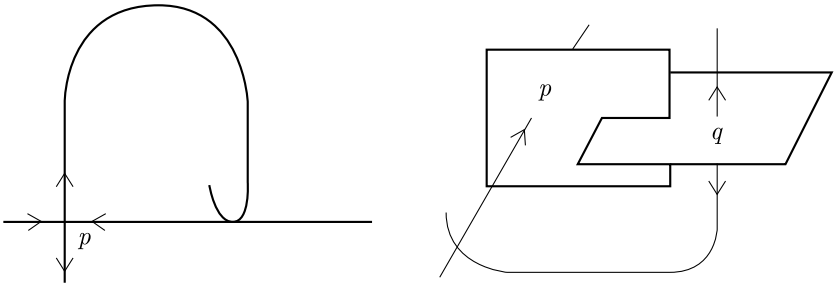


Figure 1.1. Homoclinic tangency and heterodimensional cycle.

The question is to obtain a complete list of simple obstructions to the hyperbolicity. With Mañé's work on stability [53], one knows [6, 43] that a diffeomorphism is non-hyperbolic if and only if it can be C^1-approximated by a diffeomorphism having a non-hyperbolic periodic point. Two reasons justify that people now look for homoclinic bifurcations rather than weak periodic orbits.

Cascade of bifurcations, robustness. The existence of non-hyperbolic periodic points or homoclinic bifurcations associated to periodic orbits are one-codimensional configurations and do not occur for open sets of systems. Replacing the periodic orbits by transitive hyperbolic sets in definition 1.1, one may obtain open sets of homoclinic bifurcations and get robust obstructions to hyperbolicity: this happens for homoclinic tangencies of C^2 diffeomorphisms of surface [57] and to some extend in higher dimension [63]; this also happens in some cases for homoclinic bifurcations in the space of C^1 diffeomorphisms when $\dim(M) \geq 3$ [21, 22]. Indeed for a hyperbolic set, the "dimension" of its stable set can be larger than the dimension of its stable leaves.

These homoclinic bifurcations are thus in general not isolated and as pointed out by Bonatti and Díaz, one can strengthen problem 2 by requiring the homoclinic tangencies and heterodimensional cycles to be robust.

Dynamical consequences. The unfolding of these bifurcations involve rich dynamics: homoclinic tangencies generate locally generic sets of diffeomorphisms displaying infinitely many attracting or repelling periodic orbits [57] (which is known as the *Newhouse phenomenon*). Heterodimensional cycles generate robustly isolated transitive and non-hyperbolic sets [19, 37].

Pujals and Sambarino have solved Palis conjecture for C^1 diffeomorphisms on surfaces [72]. In higher dimensions some partial results have been obtained, for instance [32, 68, 69, 84]. The following one has been proved in [34].

Theorem 1.2. *In* $\mathrm{Diff}^1(M)$ *any diffeomorphism can be approximated by one which*

- *either exhibits a homoclinic tangency or a heterodimensional cycle,*
- *or is* essentially hyperbolic*: there exist finitely many hyperbolic attractors* (*respectively repellers*) *whose basin is (open and) dense in* M.

In some cases, the dynamics break down into only finitely many pieces, even after perturbation: these systems, called *tame* are easier to study and may help to test some conjectures. Using Mañé's work, a non-hyperbolic tame dynamics can be perturbed to create two close periodic points with different stable dimensions. The tameness implies that they belong to a

same piece, hence may be connected in a heterodimensional cycle. Consequently the Palis conjecture holds in this case for the C^1-topology. This includes in particular the conservative dynamics (see section 2.4).

1.3. Decomposition of the dynamical space. [13, 34, 79] propose to generalize problem 2 by decomposing (an open and dense subset of) the considered space of systems into regions which display different dynamical properties.

Problem 3. *Identify new dynamics which allow to split a class of systems.*

With Pujals, we suggest [34] to focus on two kinds of dynamical properties:

- *Mechanisms.* We mean simple dynamical configurations which are non isolated (maybe even robust) and which generate rich dynamical behaviors.
- *Phenomena.* That is any dynamical property which provides a global description of the system and holds on a large subset of systems.

A mechanism may generate a phenomenon: for instance the homoclinic tangencies generate the Newhouse phenomenon for C^2 surface diffeomorphisms. It may also be an obstruction: one of the first dichotomy was obtained by Newhouse for symplectomorphisms (hyperbolicity or existence of an elliptic periodic orbit, see theorem 2.13). This mechanism - the existence of an elliptic periodic point - is robust, hence provides an obstruction to hyperbolicity also for higher topologies.

Another example of dichotomy mechanism/phenomenon is in the following result (which answers a weak version of Palis conjecture). It is proved in [31]; the surface and 3-dimensional cases were obtained before in [24] and [72]:

Theorem 1.3. *The space* $\mathrm{Diff}^1(M)$ *contains a dense subset* $\mathcal{MS} \cup \mathcal{I}$ *which is the union of two disjoint open sets:*

- $\mathcal{MS}$ *is the set of Morse-Smale diffeomorphisms, i.e. whose dynamics is hyperbolic and has only finitely many periodic orbits. Any other orbit accumulates in the future (resp. in the past) towards one of these periodic orbits.*
- $\mathcal{I}$ *is the set of diffeomorphisms* f *which have a transverse homoclinic orbit: there exists a hyperbolic periodic point* p *whose stable and unstable manifolds have a transverse intersection point different from* p*.*
  In particular, there exists a compact set $A \subset M$ *and an iterate* f^n *such that* $f^n(A) = A$ *and the restriction of* f^n *to* A *is topologically conjugate to the shift on* $\{0,1\}^{\mathbb{Z}}$*. Hence there exists infinitely many periodic orbits.*

The global dynamics for $f \in \mathcal{MS}$ is very simple, robust under perturbation, and similar to the time-one map of the gradient flow of a Morse function. Moreover the topological entropy (which measures the "complexity" of the system) vanishes.

In the second case, the transverse homoclinic intersection, which is a very simple and robust configuration, implies a very rich behavior, as discovered by Poincaré and Birkhoff, and the topological entropy is non zero. The dynamics however is not described outside a local region of M.

Contents. We first discuss generic properties that are consequences of connecting lemmas for pseudo-orbits. We then present results which led to theorems 1.2 and 1.3 above and

to other dichotomies inside $\mathrm{Diff}^1(M)$, see also [33]. We present several questions which emerged during the last years, and among them some conjectures by Bonatti [13]. All these results and questions together constitute a panorama of the main dynamics which appear in the space of C^1-diffeomorphisms.

Many of the following results were obtained in collaboration with colleagues (and friends), and in particular with C. Bonatti and E. Pujals. This subject would have been very different without your viewpoints, thank you!

I am also grateful to F. Béguin, R. Potrie, E. Pujals, M. Sambarino, A. Wilkinson and X. Wang for their comments on the text.

2. Decomposition of the dynamics

We say that a dynamical property is C^1-generic if it holds on a dense G_δ subset of $\mathrm{Diff}^1(M)$. These properties have been studied in many works, see [33]. We describe here properties shared either by *all* or by *all C^1-generic* diffeomorphisms f and that are related to the connecting lemma for pseudo-orbits stated below.

2.1. Chain-recurrence. An open set $U \subset M$ is *attracting* if $f(\overline{U}) \subset U$. It decomposes the dynamics into the invariant disjoint compact sets $A^+ = \cap_n f^n(U)$ and $A^- = \cap_n f^n(M \setminus U)$. The orbits in the complement $M \setminus (A^+ \cup A^-)$ are strongly non-recurrent. By repeating this process one decomposes [28] the dynamics into pieces which can be also obtained with the notion of pseudo-orbits as follows.

For $\varepsilon > 0$, a *ε-pseudo-orbit* is a sequence $(x_n)_{n\in\mathbb{Z}}$ such that $d(f(x_n), x_{n+1}) < \varepsilon$ for each n. We denote $x \dashv y$ if for each $\varepsilon > 0$ there exists a ε-pseudo-orbit (x_n) and $m \geq 1$ such that $x_0 = x$ and $x_m = y$.

Definition 2.1. The *chain-recurrent set* is the (invariant) set $\mathcal{R}(f) = \{x,\ x \dashv x\}$.

The chain-recurrent set is compact and contains the set of periodic point $\mathrm{Per}(f)$. The other classical notions of recurrence - the non-wandering set $\Omega(f)$, the limit set $L(f)$, the recurrent set $\mathrm{Rec}(f)$ - are all contained in $\mathcal{R}(f)$ and contain $\mathrm{Per}(f)$.

On $\mathcal{R}(f)$ we define the equivalence relation $x \vdash\dashv y$ whenever there exists a periodic ε-pseudo-orbit which contains x, y for each $\varepsilon > 0$.

Definition 2.2. The chain-recurrent classes of f are the equivalence classes of $\vdash\dashv$.

They are pairwise disjoint invariant compact subsets of the chain-recurrent set.

A chain-recurrence class Λ is a *quasi-attractor* if it admits a basis of attracting neighborhoods. (It is an *attractor* if $\Lambda = \cap_{n\in\mathbb{N}} f^n(U)$ for some neighborhood U.)

Definition 2.3. Let K be an invariant compact set.

- K is *chain-transitive* if for any $x, y \in K$ and $\varepsilon > 0$ there exists a periodic ε-pseudo-orbit in K which contains x, y.
- K is *transitive* if for any non-empty open sets U, V of K, there exists $n \geq 1$ such that $f^n(U) \cap V \neq \emptyset$.
- K is *topologically mixing* if for any non-empty open sets U, V of K, there exists $n_0 \geq 1$ such that $f^n(U) \cap V \neq \emptyset$ for each $n \geq n_0$.

The chain-recurrence classes are the chain-transitive sets which are maximal for the inclusion.

Definition 2.4. The *homoclinic class* $H(\mathcal{O})$ of a hyperbolic periodic orbit $\mathcal{O}$ is the closure of the transverse intersections between $W^s(\mathcal{O})$ and $W^u(\mathcal{O})$.

The homoclinic classes satisfy three interesting properties (see [3, 55]):

- $H(\mathcal{O})$ contains a dense set of periodic orbits $\mathcal{O}'$ that are *homoclinically related* to $\mathcal{O}$, i.e. such that $W^u(\mathcal{O})$ and $W^s(\mathcal{O}')$ (resp. $W^u(\mathcal{O}')$ and $W^s(\mathcal{O})$) have a transverse intersection point.
- $H(\mathcal{O})$ is transitive (hence contained in a chain-recurrence class): there exists a unique $\ell \geq 1$ (called the *period* of $H(\mathcal{O})$) and a subset $A \subset H(\mathcal{O})$ such that $f^\ell(A) = A$, $H(\mathcal{O}) = A \cup f(A) \cdots \cup f^{\ell-1}(A)$ and $f^i(A) \cap A$ has empty interior in $H(\mathcal{O})$ when $0 < i < \ell$; moreover A is topologically mixing for f^ℓ.
- For any diffeomorphism g that is C^1-close to f, the orbit $\mathcal{O}$ has a continuation $\mathcal{O}_g$ (given by the implicit function theorem), which gives a notion of *continuation* $H(\mathcal{O}_g)$ of a homoclinic class.

For general diffeomorphisms, two homoclinic classes may intersect and not coincide.

2.2. Closing and connecting lemmas in the C^1-topology. In $\mathrm{Diff}^1(M)$ it is possible to perturb one orbit in order to create periodic points (Pugh's closing lemma [67]) or to connect invariant manifolds of hyperbolic periodic points (Hayashi's connecting lemma [44]). With Bonatti, we have extended [14] these technics to pseudo-orbits and obtained:

Theorem 2.5 (Connecting lemma for pseudo-orbits). *Let us consider $x, y \in M$ and assume the following non-resonance condition:*

$$\forall n \geq 1,\ \forall v \in TM \setminus \{0\}, \quad Df^n.v \neq v.$$

If $x \dashv y$, there exists g, C^1-close to f, such that $g^n(x) = y$ for some $n \geq 1$.
If $x \in \mathcal{R}(f)$, there exists g, C^1-close to f, such that x is periodic for g.

The perturbation can not be local: one needs to "close all the jumps" of a pseudo-orbit. For that purpose we had to build a section of the dynamics:

Lemma 2.6 (Topological towers). *There exists $C > 0$ (which only depends on $\dim(M)$) such that for any $f \in \mathrm{Diff}^1(M)$, any $N \geq 1$ and any (not necessarily invariant) compact set K that does not contain any i-periodic point, $1 \leq i \leq C.N$, there exists $U \subset M$ open such that*

- *U is disjoint from $f^i(U)$ for $1 \leq i \leq N$,*
- *K is contained in the union of the $C.N$ first iterates of U.*

The perturbations in the closing and connecting lemmas may introduce shortcuts in the pseudo-orbits: for instance theorem 2.5 does not describe the regions which are visited by the orbit $x, g(x), \ldots, g^n(x)$. Mañé [51] has shown that one can control the distribution of the periodic orbits in the closing lemma:

Theorem 2.7 (Ergodic closing lemma). *There exists a dense G_δ subset $\mathcal{G}$ of $\mathrm{Diff}^1(M)$ such that for any $f \in \mathcal{G}$ and any ergodic probability μ, there exists a sequence of periodic orbits which converge to μ for the weak-$*$ topology.*

The next result [30] gives a topological control on the support of the orbits.

Theorem 2.8 (Global connecting lemma). *There exists a dense G_δ subset $\mathcal{G}$ of $\mathrm{Diff}^1(M)$ such that any $f \in \mathcal{G}$ has the following properties:*

- *For any points $x_1, \dots, x_k$ satisfying $x_i \dashv x_{i+1}$ for each $1 \le i < k$, and for any $\delta > 0$ there exists an orbit of f which intersects each ball $B(x_i, \delta)$.*
- *For any chain-transitive set K and any $\delta > 0$, there exists a periodic orbit $\mathcal{O}$ which is δ-close to K for the Hausdorff topology.*

Extension to other classes of systems. We stress that the perturbations are supported in a union of small disjoint balls. This makes difficult the extension of these methods to classes of systems which do not allow local perturbation. For the geodesic flow, Contreras has shown [29] that one can modify the tangent dynamics above periodic orbits by C^2-perturbations, but the following problem is still open:

Problem 4. *Prove a closing lemma for the geodesic flow (space of C^2 metrics).*

2.3. Decomposition of C^1-generic diffeomorphisms. For C^1-generic diffeomorphisms we have better information on the chain-recurrence classes.

a- Chain-recurrence classes. As a consequence of the connecting and closing lemmas, we obtain for any C^1-generic diffeomorphism f:

- The periodic points are (hyperbolic and) dense in the chain-recurrent set:
$$\overline{\mathrm{Per}(f)} = \overline{L(f)} = \overline{\mathrm{Rec}(f)} = \Omega(f) = \mathcal{R}(f).$$
- Any chain-recurrence class is limit of a sequence of periodic orbits for the Hausdorff topology.
- Each chain-recurrence class containing a periodic orbit $\mathcal{O}$ coincides with the homoclinic class $H(\mathcal{O})$. Any two homoclinic classes are thus disjoint or equal.

b- Periodic orbits. For C^1-generic diffeomorphisms, the periodic orbits inside a homoclinic class $H(\mathcal{O})$ have a nice structure (see [2, 21, 27]):

- Any two periodic orbits in $H(\mathcal{O})$ with same stable dimension are homoclinically related.
- Any two periodic orbits in $H(\mathcal{O})$ with different stable dimension belong to a robust heterodimensional cycle.
- The set of stable dimensions of periodic points of $H(\mathcal{O})$ is an interval of $\mathbb{N}$.
- For any two periodic orbits $\mathcal{O}_1, \mathcal{O}_2$ in $H(\mathcal{O})$ and any $\theta \in [0,1]$, there exist periodic orbits in $H(\mathcal{O})$ which are arbitrarily close (for the weak-$*$ topology on finite Borel measures) to the barycenter $\theta \cdot \mathcal{O}_1 + (1-\theta) \cdot \mathcal{O}_2$.

More about the tangent dynamics above periodic orbits appear in [12, 16, 42, 77].

c- Isolated and tame classes. It is equivalent for a chain-recurrence class Λ to be isolated (i.e. Λ is open in $\mathcal{R}(f)$) and to coincide with the maximal invariant set $\cap_{n\in\mathbb{Z}} f^n(U)$ in one of its neighborhoods U. A stronger property is:

Definition 2.9. A chain-recurrence class Λ is *tame* if the maximal invariant set $\Lambda_g := \cap_{n\in\mathbb{Z}} g^n(U)$ in a neighborhood U is a chain-recurrent class for any g C^1-close to f. A diffeomorphism is tame if all its chain-recurrence classes are tame.

If f is C^1-generic, an isolated chain-recurrence class is a tame homoclinic class. Consequently f is tame if and only if it has finitely many chain-recurrence classes.

The chain-recurrence classes of hyperbolic diffeomorphisms are always isolated (this is part of Smale's spectral theorem). There are robust examples of isolated chain-recurrence classes containing periodic points of different stable dimensions (hence not hyperbolic), see section 3.3. As already noticed at the end of section 1.2, the tame classes of C^1-generic diffeomorphisms are easier to study. See [23].

Robust transitivity. Let Λ be a tame class of f, C^1-generic. One may wonder if Λ_g is still transitive or even a homoclinic class for g C^1-close to f. A counter example appears in [17], but it uses the fact that it is not a quasi-attractor for f nor for f^{-1}. One can thus ask:
Is any perturbation of C^1-generic transitive diffeomorphism still transitive ?
With Abdenur we have answered this question affirmatively assuming the diffeomorphism is partially hyperbolic with a one-dimensional center bundle.

Topological mixing. A variation of the connecting lemma gives [3] for C^1-generic diffeomorphisms that any isolated homoclinic class $H(\mathcal{O})$ decomposes into *disjoint* compact sets $A \cup f(A) \cup \cdots \cup f^{\ell-1}(A)$ where ℓ is the period of the class.

In particular one gets the following dichotomy:

Corollary 2.10. *There exist two disjoint open sets $\mathcal{U}_1, \mathcal{U}_2$ whose union is dense in $\mathrm{Diff}^1(M)$ and which satisfy:*

- *$\mathcal{U}_1$ is the set of diffeomorphisms having a non-empty attracting set $U \neq M$.*
- *the diffeomorphisms in a dense G_δ subset of $\mathcal{U}_2$ are topologically mixing.*

d- Non-isolated classes, aperiodic classes. Homoclinic classes with robust homoclinic tangencies may create non-isolated homoclinic classes (accumulated by sinks) for generic diffeomorphisms: Newhouse has proved that this occurs on surfaces for the C^2-topology, and in higher dimension for the C^1-topology [58].

Bonatti and Díaz have shown [20] that in some cases these sinks may be turned into non trivial classes: hence a non-isolated homoclinic class ejects, after perturbation, new homoclinic classes with similar properties. This phenomenon is further studied in [16] and called *virality*. Such a C^1-generic diffeomorphism present infinite sequences of distinct non-isolated homoclinic classes, whose limit is a chain-recurrence class disjoint from $\mathrm{Per}(f)$.

Definition 2.11. The chain-recurrence classes which do not contain any periodic point are called *aperiodic classes*.

Few is known about the dynamics of aperiodic classes: the aperiodic classes obtained in [20] are odometers, but Bonatti and Shinohara are developing a perturbation tool which would allow to build non transitive or non uniquely ergodic aperiodic classes.

Some questions remain about non-isolated classes (see also conjectures in [13]):

Is any aperiodic class accumulated by non-isolated (viral) homoclinic classes?
Is any non-isolated homoclinic class accumulated by aperiodic classes?
One may answer negatively to the second question with examples of C^1-generic diffeomorphisms having no aperiodic classes and infinitely many homoclinic classes. Indeed, Potrie [66] has built a non-isolated homoclinic class admitting a neighborhood where the other chain-recurrence classes are contained in countably many surfaces. These are homoclinic classes if hyperbolicity is C^1-dense on surfaces.

e- Quasi-attractors. Theorem 2.5 gives C^1-generically:

- A chain-recurrence class is a quasi-attractor, once it is *Lyapunov stable*: there exists a basis of neighborhoods U such that $f(U) \subset U$.
- There exists a dense G_δ subset $\mathcal{X} \subset M$ such that for any $x \in \mathcal{X}$, the limit set of the forward orbit $(f^n(x))_{n\geq 0}$ is a quasi-attractor.

Attractors may not exist: [25] gives an example of a C^1-generic diffeomorphism with a quasi-attractor Λ which is unique and non-isolated. This quasi-attractor is *essential*: its basin, i.e. the set of points x such that $f^n(x)$ accumulate on a subset of Λ as $n \to +\infty$, is dense in a non-empty open set. Also the basin of a quasi-attractor may be small: the aperiodic classes described in [20] are quasi-attractors; each basin is reduced to the class itself and has empty interior.

One may ask the following for C^1-generic diffeomorphisms (see also [13]):
Is the union of the basins of essential attractors dense in M?
For quasi-attractors is it equivalent to be essential and to be a homoclinic class?
On attractors, does there exist a physical measure*?* (an ergodic probability where the forward orbit of Lebesgue-almost every point in the basin equidistributes.)

2.4. Conservative dynamics - ergodicity. Conservative systems are chain-transitive. The connecting lemma for pseudo-orbits gives (see [3, 8, 14]):

Theorem 2.12. *There exists a dense G_δ subset $\mathcal{G} \subset \mathrm{Diff}^1_\omega(M)$ such that any diffeomorphism $f \in \mathcal{G}$ is topologically mixing.*

The same statement is false in $\mathrm{Diff}^r_\omega(M)$ when ω is a volume form and r is large: by KAM, there may exist a robust one-codimensional invariant torus (see [87]).

As already noticed, the C^1 Palis conjecture holds in this setting (see [31, 56]):

Theorem 2.13. *In $\mathrm{Diff}^1_\omega(M)$, any diffeomorphism can be approximated by f which is hyperbolic or which satisfies the following robust property:*

– (symplectic case) f has a periodic point with a simple eigenvalue of modulus 1.

– (volume case, $\dim(M) \geq 3$) there exists a robust heterodimensional cycle.

One can compare to the following [14, 38, 46, 76] (see definitions in section 3.1):

Theorem 2.14. *In $\mathrm{Diff}^1_\omega(M)$, any diffeomorphism can be approximated by one with a completely elliptic periodic point (eigenvalues are simple, of modulus* 1*), or:*

– (symplectic case) by one which is partially hyperbolic and robustly transitive,

– (volume case) by one which has a (non-trivial) dominated splitting.

The existence of a completely elliptic periodic point is an obstruction to robust transitivity [7] and in the symplectic case, the robust transitivity is characterized by partial hyperbolicity. In the volume-preserving case, Dolgopyat and Wilkinson conjectured [38]:

Conjecture 2.15. *In the volume preserving case, the sets of robustly transitive diffeomorphisms and of those having a dominated splitting have the same closure in* $\mathrm{Diff}^1_\omega(M)$.

A stronger notion of undecomposability involves the ergodicity of the volume:

Problem 5. *Is ergodicity dense (hence Baire-generic) in* $\mathrm{Diff}^1_\omega(M)$*?*

The C^1-generic systems in $\mathrm{Diff}^1_\omega(M)$ with positive metric entropy are ergodic (this is proved in [9] for the symplectic and in [10] for the volume preserving cases). However C^1-generic systems with zero metric entropy also occur [11].

There exists (non-empty) C^1-open sets of ergodic diffeomorphisms in $\mathrm{Diff}^r_\omega(M)$ when $r > 1$: these diffeomorphisms (which include the hyperbolic systems) are called *stably ergodic* and were studied intensively (see [85]). Note that it is not known if they exist also in $\mathrm{Diff}^1_\omega(M)$. Pugh and Shub have conjectured that stable ergodicity is dense in the space of C^r partially hyperbolic diffeomorphisms.

In parallel to conjecture 2.15, with Avila and Wilkinson we proposed [10]:

Conjecture 2.16. *For* $r > 1$*, the sets of stably ergodic diffeomorphisms and of those having a dominated splitting have the same* C^1*-closure in* $\mathrm{Diff}^r_\omega(M)$.

In this direction we obtained [10]:

Theorem 2.17. *In the space of volume-preserving diffeomorphisms* $\mathrm{Diff}^r_\omega(M)$*,* $r > 1$*, those having a partially hyperbolic splitting* $E^s \oplus E^c \oplus E^u$ *into non-trivial bundles are contained in the closure of the set of stably ergodic diffeomorphisms.*

3. Notions of weak hyperbolicity

Let K be an invariant set for $f \in \mathrm{Diff}^1(M)$. We recall the classical notion:

Definition 3.1. K is (uniformly) *hyperbolic* if there exists an invariant continuous splitting $T_K = E^s \oplus E^u$ and $N \geq 1$ such that $\|Df^N_{|E^s}\| \leq 1/2$ and $\|Df^{-N}_{|E^u}\| \leq 1/2$ (i.e., E^s and E^u are uniformly contracted by f and f^{-1} respectively on K).
A diffeomorphism is *hyperbolic* if each chain-recurrence class is hyperbolic.

It is well-known that hyperbolic sets satisfy several important properties: they can be continued for diffeomorphisms C^1-close, each of their points has stable and unstable manifolds, their pseudo-orbits are shadowed by orbits,... We present now several weaker notions of hyperbolicity which sometimes keep these properties and will appear in the next sections.

3.1. Tangent dynamics - partial hyperbolicity. Pesin theory describes systems where the uniformity in definition 3.1 is relaxed: Oseledets theorem associates to any ergodic probability μ its Lyapunov exponents $\lambda_1 \leq \cdots \leq \lambda_{\dim(M)}$ which are the possible limits of $\log(\|Df^n(x).u\|)/n$ as $n \to \infty$ for any $u \in T_xM$ and a.e. $x \in M$. When each λ_i is non-zero, μ is *non-uniformly hyperbolic*.

Here is another way to relax hyperbolicity which allows vanishing exponents.

Definition 3.2. An invariant splitting $T_K M = E \oplus F$ by linear sub-bundles above K is *dominated* if there is $N \geq 1$ such that $\|Df^N(x).u\| \leq 1/2\|Df^N(x).v\|$ for each $x \in K$ and each unit vectors $u \in E_x$ and $v \in F_x$.

This definition extends to splittings into a larger number of invariant bundles. The *finest dominated splitting* is the (unique) one which maximizes this number.

Definition 3.3. A dominated splitting $T_K M = E^s \oplus E^c \oplus E^u$ is *partially hyperbolic* if E^s (resp. E^u) is uniformly contracted by f (resp. f^{-1}) and if one of the bundles E^s, E^u is non-trivial.

Dominated splittings and partial hyperbolicity extend to the closure of K and to invariant sets in a neighborhood of K for diffeomorphisms C^1-close. Moreover any point of a partially hyperbolic set has unique (strong) stable and unstable manifolds tangent to E^s and E^u, that we denote by $W^{ss}(x)$ and $W^{uu}(x)$.

Hirsch, Pugh and Shub have built [45] a weak notion of center manifold:

Theorem 3.4. *If K has a dominated splitting $T_K M = E_1 \oplus F \oplus E_2$, there exists a* locally invariant plaque family tangent to F, *i.e. a map $\mathcal{W}\colon F \to M$ satisfying:*

- *Each induced map $\mathcal{W}_x\colon F_x \to M$ is an embedding, depends continuously on $x \in K$ for the C^1-topology, $\mathcal{W}_x(0) = x$, and the image is tangent to F_x at x.*
- *There exists $\rho > 0$ such that $\mathcal{W}_x(B(0,\rho))$ is sent by f in $\mathcal{W}_{f(x)}$ for each x.*

The image of $\mathcal{W}_x$ (still denoted $\mathcal{W}_x$) is a *plaque*. The plaque family is in general not unique; the union of two different plaques may not be a sub manifold.

One can sometimes obtain the following stability along the plaques (see [34]):

Definition 3.5. The plaque family $\mathcal{W}$ is *trapped* if $f(\overline{\mathcal{W}_x}) \subset \mathcal{W}_{f(x)}$ for each $x \in K$.

F is *thin-trapped* if "inside a plaque family $\mathcal{W}$", there exist plaque families which are trapped and whose plaques have arbitrarily small diameters.

3.2. Dynamics along one-dimensional center directions. Let K be a chain-transitive set with a dominated splitting $E_1 \oplus F \oplus E_2$, $\dim(F) = 1$. Pesin theory does not describe the local dynamics along the direction F when the Lyapunov exponents along F vanish. These dynamics are studied in [72] when E_1 or E_2 is degenerated. When they are not, we introduced the next notion [31, 32]:

Definition 3.6. A *center model* for K is a compact metric space $\widehat{K}$ and continuous maps $\widehat{f}\colon \widehat{K} \times [0,1] \to \widehat{K} \times [0,+\infty)$, $\pi\colon \widehat{K} \times [0,+\infty) \to M$ such that:

- $\widehat{f}$ is a local homeomorphism near $\widehat{K} \times \{0\}$,
- for each x, there is x' satisfying $\widehat{f}(\{x\} \times [0,1]) \subset \{x'\} \times [0,+\infty)$,
- $\pi(\widehat{K} \times \{0\}) = K$ and π semi conjugates $\widehat{f}$ and f,
- each $t \mapsto \pi(x,t)$ is a C^1-embedding which depends continuously on x for the C^1-topology and the image is tangent to $F_{\pi(x,t)}$.

From theorem 3.4, K admits a center model: $\widehat{K}$ is the unit tangent bundle associated to E^c, hence π is two-to-one on $\widehat{K} \times \{0\}$.

Using pseudo-orbits, the local dynamics of center models can be classified into four different types: $\widehat{K} \times \{0\}$ is a quasi-attractor or not, for $\widehat{f}$ or for $\widehat{f}^{-1}$. This allows us to prove that one of the following (not exclusive) cases occurs for the local dynamics along a locally invariant plaque family $\mathcal{W}$ tangent to F:

Thin-trapped. *The bundle F is thin-trapped.*

If E_2 is uniformly contracted by f^{-1}, a weak shadowing lemma implies that the unstable set of K meets the stable set of a periodic orbit whose chain-recurrence class is non-trivial. See [33, Prop. 10.20], [32, Prop. 4.5].

Chain-recurrent. *In any neighborhood U of K, there exist $x \in K$, a non trivial curve I with $f^n(I) \subset \mathcal{W}_{f^n(x)}$, for $n \in \mathbb{Z}$, and a chain-transitive set $\Lambda \supset K \cup I$.*

If U is small, any periodic orbit $\mathcal{O} \subset U$, whose exponent along F is close to 0 and having a point close to the middle of I, belongs to the chain-recurrence class of K. See [36, corollary 4.4].

Semi chain-unstable. *There exists a locally invariant* half *plaque family $\mathcal{W}^+$. It is thin-trapped by f^{-1}; for any $x \in K$ and $z \in \mathcal{W}^+_x$ we have $x \dashv z$.*

If E_1, E_2 are uniformly contracted by f, f^{-1} respectively, f is C^1-generic and K is not twisted, it is contained in a homoclinic class. See [32, Prop. 4.4].

The twisted geometry above is very particular. For the definition, one extends continuously F in a neighborhood of K. Locally, it is trivial, hence orientable.

Definition 3.7. A partially hyperbolic set K with a one-dimensional center bundle is *twisted* if for any $x, y \in K$ close, one can connect $W^{uu}_{loc}(x)$ to $W^{ss}_{loc}(y)$ and $W^{uu}_{loc}(y)$ to $W^{ss}_{loc}(x)$ by two curves tangent to F having the same orientation. (Figure 3.1.)

Using Pugh's and Mañé's closing lemma arguments, when K does not contain periodic point and is twisted, one can find [32, prop. 3.2] a C^1-perturbation g having a periodic orbit $\mathcal{O}$ close to K such that $W^{ss}(\mathcal{O})$ and $W^{uu}(\mathcal{O})$ intersect.

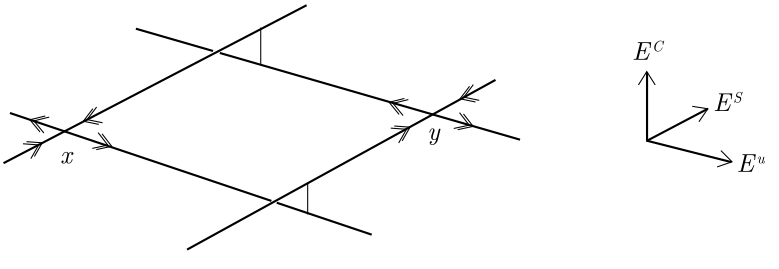


Figure 3.1. Two close points in a twisted set.

3.3. Chain hyperbolicity. The following notion is introduced in [34]:

Definition 3.8. A homoclinic class $H(\mathcal{O})$ is *chain-hyperbolic* if it has a dominated splitting $T_{H(\mathcal{O})}M = E^{cs} \oplus E^{cu}$ and plaque families $\mathcal{W}^{cs}, \mathcal{W}^{cu}$ such that:

(i) $\mathcal{W}^{cs}$ (resp. $\mathcal{W}^{cu}$) is tangent to E^{cs} (resp. E^{cu}) and trapped by f (resp. f^{-1});

(ii) there exists $p \in \mathcal{O}$ such that $\mathcal{W}^{cs}_p \subset W^s(p)$ and $\mathcal{W}^{cu}_p \subset W^u(p)$.

$\mathcal{W}^{cs}$ (resp. $\mathcal{W}^{cu}$) are called *center-stable* (resp. *center-unstable*) plaque families.

Properties. At large scale, chain-hyperbolicity looks similar to hyperbolicity and both notions share several properties:

- *Robustness.* If $H(\mathcal{O})$ is a chain-recurrence class and E^{cs}, E^{cu} are thin-trapped by f and f^{-1}, its continuation $H(\mathcal{O}_g)$ is chain-hyperbolic for g C^1-close.
- *Strong periodic points.* $H(\mathcal{O})$ contains a dense set $\mathcal{P}$ of periodic points p satisfying definition 3.8(ii) and whose exponents are bounded away from 0. They have a hyperbolic continuation for g in a uniform neighborhood of f.
- *Invariant manifolds.* For any $x \in H(\mathcal{O})$ and $y \in \mathcal{W}^{cs}_x$, one has $y \dashv x$. A transverse intersection of center stable and center unstable plaques still belongs to $H(\mathcal{O})$.

Examples. By deforming a hyperbolic diffeomorphism near a periodic point, one can build robust examples of non-hyperbolic isolated chain-hyperbolic classes, see [26]. One can also build examples from skew product maps.
Remark. Other constructions of robustly transitive sets exist. The center bundle may be parabolic [17], or tangent to a foliation with compact or non-compact leaves [21].

4. Dynamics far from homoclinic tangencies...

If x is a homoclinic tangency for a hyperbolic periodic orbit with stable dimension d^s, there is no dominated splitting $E \oplus F$ with $\dim(E) = d^s$ on the orbit of x. We discuss now the converse and consider the diffeomorphisms f that can not be approximated by homoclinic tangencies: $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathrm{Tang}}$ where Tang denotes the collection of diffeomorphisms which exhibit a homoclinic tangency.

Theorems 4.3 and 4.4 below imply that these dynamics are partially hyperbolic:

Theorem 4.1. *There exists an open and dense subset $\mathcal{U} \subset \mathrm{Diff}^1(M) \setminus \overline{\mathrm{Tang}}$ such that any $f \in \mathcal{U}$ has at most finitely many sinks and sources and any of its other chain-recurrence classes Λ has a partially hyperbolic splitting:*

$$T_\Lambda M = E^s \oplus E^c_1 \oplus \cdots \oplus E^c_\ell \oplus E^u,$$

where E^s (resp. E^u) is non trivial, uniformly contracted (resp. expanded) and where each E^c_i is one-dimensional.

The decomposition of the center into one-dimensional sub-bundles limits the pathological behaviors. In particular, these systems admit symbolic extensions and any continuous map $\varphi\colon M \to \mathbb{R}$ has an equilibrium state (see [36, 50]).

4.1. Existence of weak periodic points inside the class. Improving a technics of [72], Wen [83] and Gourmelon [41] have shown:

Theorem 4.2. *For any diffeomorphism $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathrm{Tang}}$ and any $d^s \geq 1$, the decomposition into stable and unstable spaces, above the hyperbolic periodic orbit with stable dimension d^s, is a dominated splitting.*

With results of section 2.3(a) this implies that for C^1-generic diffeomorphisms far from the homoclinic tangencies, any chain-recurrence class Λ (but maybe finitely many sinks and sources) has a non-trivial dominated splitting $E \oplus F$.

Assuming that E is not uniformly contracted, we have to build a sequence of periodic orbit with smaller stable dimension which accumulates on Λ, so that E can be further decomposed. Such periodic orbits exist in a neighborhood of Λ (essentially from Mañé's ergodic closing lemma), but the difficulty is to approximate the whole class Λ in Hausdorff topology. There is an easy case: Λ contains periodic points with at least one Lyapunov exponent along E which is positive or close to zero. Indeed in this case Λ is a homoclinic class which contains a dense set of such periodic points and E can be decomposed.

These periodic orbits in Λ are obtained under weak hyperbolicity by shadowing: either from non-uniform hyperbolicity (Liao's "selecting lemma" [48, 49, 84]) or from topological hyperbolicity when the Lyapunov exponents vanish (using center models). Such weak hyperbolicity fails only when Λ has a partially hyperbolic splitting $E^s \oplus E^c \oplus E^u$ with $\dim(E^c) = 1$, $E^s \oplus E^c \subset E$ and such that the Lyapunov exponent along E^c of any invariant probability on Λ vanishes (Λ is aperiodic).

The stable dimensions of periodic points of Λ is an interval in $\mathbb{N}$ and one gets the following statement proved in [36] (already obtained in [40] for minimal sets):

Theorem 4.3. *There is a dense G_δ set $\mathcal{G} \subset \mathrm{Diff}^1(M) \setminus \overline{\mathrm{Tang}}$ such that for $f \in \mathcal{G}$:*
Any aperiodic class Λ is partially hyperbolic: $T_\Lambda M = E^s \oplus E^c \oplus E^u$, $\dim(E^c) = 1$, and the center Lyapunov exponent of any invariant probability on Λ vanishes.
Any homoclinic classes $H(\mathcal{O})$ is partially hyperbolic:

- *$T_{H(\mathcal{O})}M = E^s \oplus E^c_1 \oplus \cdots \oplus E^c_\ell \oplus E^u$,*
- *each E^c_i is one-dimensional and $H(\mathcal{O})$ contains (weak) periodic orbits whose Lyapunov exponent along E^c_i is arbitrarily close to 0;*
- *either $E^s \oplus E^c_1$ (resp. $E^c_\ell \oplus E^u$) is thin-trapped by f (resp. f^{-1}) or there is $p \in H(\mathcal{O})$ periodic whose stable (resp. unstable) space is E^s_p (resp. E^u_p).*

The last item follows from section 3.2: there exist periodic points whose stable space contains E^c_1, so the semi chain-unstable case does not occur. If the chain-recurrent one holds, there is a (weak) periodic point p whose stable space is E^s_p.

4.2. Extreme bundles. The fact that the uniforms bundles E^s, E^u in theorem 4.3 are non trivial comes from the next result obtained with Pujals and Sambarino [35]. It generalizes Mañé's argument for interval endomorphisms [52] and previous works [34, 72, 73]. This completes the statement of theorem 4.1.

Theorem 4.4. *For any $f \in \mathrm{Diff}^2(M)$ and any invariant compact set K with a dominated splitting $T_K M = E \oplus F$, $\dim(F) = 1$ such that:*

- *each periodic point in K has an unstable space containing F,*
- *there is no periodic closed curve in $K \setminus \mathrm{Per}(f)$ tangent to F,*

then F is uniformly expanded.

An important tool of the proof is the construction of "semi-geometrical" Markov rectangles, that are laminated charts by curves tangent to F.

4.3. Dichotomy Morse-Smale / homoclinic intersections. For proving theorem 1.3, it is enough to take a C^1-generic diffeomorphism f far from homoclinic tangencies whose homoclinic classes are reduced to isolated periodic orbits (even after perturbation). One has to consider an aperiodic class.

Any aperiodic class is partially hyperbolic with one-dimensional center and section 3.2 applies. It can not be twisted since a transverse homoclinic intersection would appear after perturbation. The three types can be ruled out since they would give the existence of a non-trivial homoclinic class. Hence there is no aperiodic class and theorem 1.3 follows.

4.4. Quasi-attractors. Using the technics of the section 3.2, one gets more information on quasi-attractors for generic diffeomorphisms in $\mathrm{Diff}^1(M) \setminus \overline{\mathrm{Tang}}$:

- Quasi-attractors are homoclinic classes $H(\mathcal{O})$, see [86].
- Considering the splitting $T_{H(\mathcal{O})}M = E^s \oplus E^c_1 \oplus \cdots \oplus E^c_\ell \oplus E^u$, there exists a periodic orbit in $H(\mathcal{O})$ whose unstable dimension is equal to $\dim(E^u)$, see [34, theorem 4].
- Quasi-attractors are essential attractors (proved by Bonatti, Gan, Li, D.Yang).
- If moreover f can not be approximated by diffeomorphisms with a heterodimensional cycle, the number of quasi-attractors is finite, see [34].

By studying the geometry of invariant compact sets saturated by the strong unstable leaves, we proved recently with Sambarino and Potrie the finiteness of the quasi-attractors for C^1-generic systems in the class of diffeomorphisms whose chain-recurrences classes are partially hyperbolic with a one-dimensional center bundle. This class offers an ideal setting to study the uniqueness of physical measures and equilibrium states (see [75, 82] for smooth diffeomorphisms and [74] for C^1-generic hyperbolic diffeomorphisms).

4.5. Obstruction to the Newhouse phenomenon. A hyperbolic periodic orbit $\mathcal{O}$ is *sectionally dissipative* if its two largest Lyapunov exponents λ_1, λ_2 (counted with multiplicity) satisfy $\lambda_1 + \lambda_2 < 0$. If such an orbit has a homoclinic tangency, one can obtain a sink by C^1-small perturbation. Theorem 4.4 implies that the converse holds C^1-generically:

Corollary 4.5. *For any open set $\mathcal{V} \subset \mathrm{Diff}^1(M)$, the next properties are equivalent:*

- *Baire-generic diffeomorphisms in $\mathcal{V}$ have infinitely many sinks,*
- *densely in $\mathcal{V}$ there exist diffeomorphisms exhibiting homoclinic tangencies associated to sectionally dissipative periodic points.*

One can expect to characterize the absence of Newhouse phenomenon:

Conjecture 4.6. *There exist two disjoint open sets $\mathcal{U}_1, \mathcal{U}_2$ whose union is dense in $\mathrm{Diff}^1(M)$ and which satisfy the following properties:*

- *Baire-generic diffeomorphisms in $\mathcal{U}_1$ have infinitely many sinks;*
- *the diffeomorphisms $f \in \mathcal{U}_2$ are volume hyperbolic: each chain-recurrence class Λ, which is not a sink, has a dominated splitting $T_\Lambda = E \oplus F$ where F is non-trivial and $|\det(Df^N_{|F})| > 1$ on Λ for some $N \geq 1$.*

5. ... and far from heterodimensional cycles

Theorem 1.2 is now a consequence of the following (from [34]):

Theorem 5.1. *There exists a dense G_δ subset $\mathcal{G} \subset \mathrm{Diff}^1(M)$ such that for any $f \in \mathcal{G}$, any quasi-attractor which is partially hyperbolic with a one-dimensional center bundle is either hyperbolic or contains a (robust) heterodimensional cycle.*

In this section one considers a quasi-attractor with a partially hyperbolic splitting $E^s \oplus E^c \oplus E^u$, $\dim(E^c) = 1$, for f C^1-generic. By section 4.4 it is a homoclinic class $H(\mathcal{O})$. One can assume that all the periodic points in $H(\mathcal{O})$ have stable dimension equal to $\dim(E^s)+1$ and, from theorem 4.3, that $E^s \oplus E^c$ is thin-trapped.

Indeed, by the results of section 2.3(b), the conclusion of the theorem 5.1 holds if $H(\mathcal{O})$ contains two periodic points with different stable dimension and by section 4.4 it always contains periodic points of stable dimension $\dim(E^s) + 1$.

5.1. Strong homoclinic intersections. Let us assume that there exist diffeomorphisms g that are C^1-close to f and satisfy the following property.

Definition 5.2. $H(\mathcal{O}_g)$ has a *strong homoclinic intersection* if there exist periodic points p, q homoclinically related to $\mathcal{O}_g$ such that $(W^{ss}(p) \setminus \{p\}) \cap W^u(q) \neq \emptyset$.

An invariant set K with a partially hyperbolic splitting $E^{ss} \oplus F$ has a *strong connection* if it contains a point x such that $(W^{ss}(x) \setminus \{x\}) \cap K \neq \emptyset$.

By theorem 4.3, if $H(\mathcal{O})$ is not hyperbolic it contains weak periodic points. A strong homoclinic intersection, for a diffeomorphism close, can be moved on these weak points. This gives a robust heterodimensional cycle by C^1-perturbation.

The non-existence of strong connection allows to reduce the dimension of the ambient manifold: the following is a consequence [15] of Whitney's extension theorem and of a graph transform argument.

Theorem 5.3. *Any invariant set K with a partially hyperbolic splitting $E^{ss} \oplus F$ and no strong connection is contained in a C^1 submanifold Σ tangent to F which is locally invariant: $\Sigma \cap f(\Sigma)$ is a neighborhood of K in Σ.*

When $H(\mathcal{O})$ is contained in a locally invariant submanifold Σ tangent to $E^c \oplus E^u$, theorem 4.4 implies that E^c is uniformly contracted. Hence we are reduced to the case where a strong connection exists, i.e. $H(\mathcal{O})$ contains $x \neq y$ such that $W^{ss}(x) = W^{ss}(y)$.

In a homoclinic class the periodic points are dense, hence one can consider p_x, p_y periodic close to x, y respectively so that $W^u_{loc}(p_x)$ and $W^u_{loc}(p_y)$ are close to the local unstable manifolds of x, y. One can hope that the projections of the unstable manifolds of x, y by strong stable holonomy are "topologically transverse". This implies that there exists $x' \in W^u_{loc}(p_x)$, $y' \in W^u_{loc}(p_y)$ such that $W^{ss}(x') = W^{ss}(y')$. Since $H(\mathcal{O})$ is a quasi-attractor, x', y' are still in the class. Other more degenerated cases may occur and have to be handled by other arguments.

We now have to deal with the following problem:

Reduced problem. Assume that $H(\mathcal{O})$ contains periodic points p_x, p_y homoclinically related to $\mathcal{O}$ and $x \neq y$ such that $x \in W^{uu}(p_x)$, $y \in W^{uu}(p_y)$, $W^{ss}(x) = W^{ss}(y)$. Does there exist g near f such that $H(\mathcal{O}_g)$ has a strong homoclinic intersection?

5.2. Pointwise continuation of chain-hyperbolic classes. The class $H(\mathcal{O})$ is chain-hyperbolic and the results of section 3.3 apply. The points p_x, p_y may be taken in the set of strong periodic points $\mathcal{P}$ and for any diffeomorphism g in a neighborhood $\mathcal{U}$ of f, the continuations of the points of $\mathcal{P}$ are well-defined and dense in $H(\mathcal{O}_g)$. This allows to introduce the following notion, similar to the branched holomorphic motion considered by Dujardin and Lyubich in [39] for holomorphic families of polynomial automorphisms of $\mathbb{C}^2$.

Definition 5.4. For any $g, g' \in \mathcal{U}$, one says that $x \in H(\mathcal{O}_g)$ and $x' \in H(\mathcal{O}_{g'})$ have the *same continuation* if there exists a sequence (p_n) in $\mathcal{P}$ such that $(p_{n,g})$ converges to x and $(p_{n,g'})$ converges to x'.

A point of $H(\mathcal{O}_g)$ may have several continuations in $H(\mathcal{O}_{g'})$. However in our setting the center-unstable bundle of the chain-hyperbolic structure is uniformly expanded. Consequently for any point x in the unstable manifold of a point $p_x \in \mathcal{P}$, its continuation - denoted by x_g - is unique and depends continuously in g.

In the setting of the reduced problem above, the continuations x_g, y_g belong to the same center stable plaques for any $g \in \mathcal{U}$ and we are led to ask:

Does there exist g near f such that $W^{ss}(x_g) \neq W^{ss}(y_g)$?

Indeed $\mathcal{W}^{cs}_x \setminus W^{ss}(x)$ has two connected components; if there exists g^+, g^- such that y_{g^+} and y_{g^-} belong to the continuations of different components of $\mathcal{W}^{cs}_x \setminus W^{ss}(x)$, by considering $q_x, q_y \in \mathcal{P}$ close enough to x, y and an arc (g_t) in $\mathcal{U}$ between g^+ and g^-, one finds a diffeomorphism g such that $W^{ss}(q_{x,g})$ and $W^u(q_{y,g})$ intersect. This gives a strong homoclinic intersection as required.

5.3. How to remove a strong connection. We are still in the (simplest) setting of the reduced problem above and look for $g \in \mathcal{U}$ such that the strong connection between x and y is broken.

The idea is to modify f in a ball $B(f^{-1}(x), r)$ so that $W^u(p_{x,g})$ intersects a given component of $\mathcal{W}^{cs}_{x_f} \setminus W^{ss}(x_f)$. The distance $d(x_g, g(f^{-1}(x)))$ is arbitrarily small with respect to the size r of the support of the perturbation. If the positive orbit of y does not return "too fast" in the support of the perturbation, using the weak hyperbolicity, one shows that the distances $d(y, y_g)$ and $d(W^{ss}_{loc}(y), W^{ss}_{loc}(y_g))$ are small also and the connection is broken (figure 5.1). A different argument is performed when the returns of the positive orbit of y near x are fast. See [33, 34].

6. Panorama of the dynamics in $\mathrm{Diff}^1(M)$

We end this text by summing up several questions and conjectures which allow to structure the space of C^1-diffeomorphisms. Most of them already appear in [13, 33].

6.1. Global dynamics. As noticed in [21, section 1.3], all the examples of C^1-generic non-hyperbolic systems involve heterodimensional cycles (this becomes false in higher topologies). This justifies:

Conjecture 6.1 (Bonatti-Díaz hyperbolicity conjecture). *Any diffeomorphism can be approximated in* $\mathrm{Diff}^1(M)$ *by one which is hyperbolic or exhibits a robust heterodimensional cycle.*

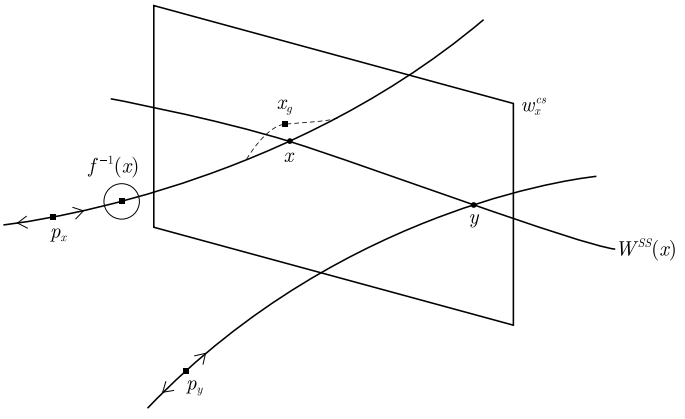


Figure 5.1. A broken strong connection.

Motivated by the results of sections 4.5 and 4.4 we expect a positive answer to following conjecture made by Bonatti in [13].

Conjecture 6.2 (Bonatti's finiteness conjecture). *In* $\mathrm{Diff}^1(M) \setminus \overline{\mathrm{Tang}}$*, there exists an (open and) dense subset of tame diffeomorphisms.*

However there exists robust examples of transitive dynamics in $\overline{\mathrm{Tang}}$ and a positive answer to the previous conjecture would not give a dichotomy.

Problem 6. *Characterize non-tame dynamics: find a robust mechanism which generates non-tame dynamics and whose union with tame systems is dense in* $\mathrm{Diff}^1(M)$.

Note that both conjectures imply the C^1 Palis conjecture and that the first one implies a positive answer to Smale problem for C^1-diffeomorphisms on surfaces.

A last example: the universal dynamics. Let us mention that there exists a non-empty open set $\mathcal{U}$ of diffeomorphisms having a homoclinic class with no dominated splitting, such that the volume is contracted above one periodic orbit and is expanded above another one. This implies [20] that the dynamics of the C^1-generic diffeomorphisms f in $\mathcal{U}$ are *universal*: any diffeomorphism g of the unit ball $B(0,1) \subset \mathbb{R}^{\dim(M)}$ may be approximated by the restriction of some iterates f^n to some balls $B \subset M$. A similar property holds for C^∞-diffeomorphisms on surfaces [81].

Assuming that the two conjectures above hold, one can decompose the space $\mathrm{Diff}^1(M)$ into disjoint regions with increasing complexity, as pictured on figure 6.1.

6.2. Local dynamics. The previous conjectures do not control where the homoclinic bifurcations occur. We state now more precise questions which allow to break the conjectures into three steps.

Let f be a C^1-generic and non-hyperbolic diffeomorphism.

I. Localization. One knows that one of the chain-recurrence classes is non-hyperbolic. We expect that this is the case for at least one homoclinic class:

Ia. *Does f exhibit a non-hyperbolic homoclinic class?*

Ib. *If f is not tame, does it have a non-isolated homoclinic class?*

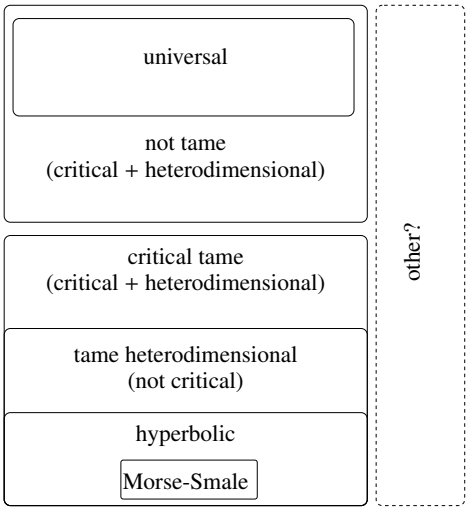


Figure 6.1. Structure of the dynamical space $\mathrm{Diff}^1(M)$.

II. Local dichotomies. Let $H(\mathcal{O})$ be a non-hyperbolic homoclinic class $H(\mathcal{O})$. Strengthening Palis conjecture, one may look for homoclinic bifurcations inside the class $H(\mathcal{O})$ (rather than in a neighborhood).

IIa. *Does $\mathcal{O}_g$ belong to a heterodimensional cycle for some g close?*

IIb. *If $H(\mathcal{O})$ is not tame, does $\mathcal{O}_g$ has a homoclinic tangency for some g close?*

III. Robustness. At last, we are aimed to stabilize the homoclinic bifurcation. This is possible for heterodimensional cycles as shown in [21]. Let us assume that $\mathcal{O}_g$ has a homoclinic tangency for some g close to f.

Does $\mathcal{O}$ belong to a hyperbolic set with robust homoclinic tangencies?

(The same question for heterodimensional cycles has been answered in [21].)

These intermediate questions have been discussed in the case of Smale's problem for surface diffeomorphisms [1]. Moreira has shown [54] that robust tangencies do not occur for C^1-diffeomorphisms on surface, solving the step III in this case. A possible approach for the two first steps is to control the lack of dominated splitting by considering the critical set introduced by Pujals and F. Rodriguez-Hertz [71].

6.3. Tangent dynamics. Some of the previous questions may be addressed by a better understanding of the weak hyperbolicity on each chain-recurrence class. Considering the known examples, the case of tame diffeomorphisms or of diffeomorphisms far from the homoclinic tangencies [23, 36], and the results [15, 16, 35], we formulate the following conjectures. Recent discussions with X. Wang seem to bring a partial answer towards the first one.

Conjecture 6.3. *Let $H(\mathcal{O})$ be a non-hyperbolic homoclinic class for a C^1-generic f, and $E^s \oplus E^c_1 \oplus \cdots \oplus E^c_\ell \oplus E^u$ the finest dominated splitting such that E^s (resp. E^u) is the maximal uniformly contracted (resp. expanded) sub-bundle.*

Then, the minimal stable dimension k of the periodic points satisfies

$$\dim(E^s) \leq k < \dim(E^s \oplus E^c_1).$$

Moreover, when $\dim(E^c_1) \geq 2$ *and* $\dim(E^s) < k$ *two cases are possible:*

- *On a periodic orbit, the sum of all the Lyapunov exponents inside E^c_1 is positive. Then $k = \dim(E^s)+1$ and the class is contained in a locally invariant submanifold tangent to $E^c_1 \oplus \cdots \oplus E^c_\ell \oplus E^u$ (and is not isolated).*
- *For any ergodic probability, the sum of the $\dim(E^s \oplus E^c_1) - k + 1$ larger Lyapunov exponents inside E^c_1 is negative. (The volume along E^c_1 is contracted.)*

The next conjectures implies that for an aperiodic class with a dominated splitting $E \oplus F$, either E is uniformly contracted or F is uniformly expanded.

Conjecture 6.4. *Let Λ be an aperiodic class for a C^1-generic f, and $E^s \oplus E^c \oplus E^u$ the dominated splitting such that E^s (resp. E^u) is the maximal uniformly contracted (resp. expanded) sub-bundle. Then E^c has dimension larger or equal to 2 and does not admit a finer dominated splitting.*

References

[1] F. Abdenur, C. Bonatti, S. Crovisier, and L. Díaz, *Generic diffeomorphisms on compact surfaces*, Fundamenta Mathematicae **187** (2005), 127–159.

[2] F. Abdenur, C. Bonatti, S. Crovisier, L. Díaz, and L. Wen, *Periodic points and homoclinic classes*, Ergodic Theory & Dynam. Systems **27** (2007), 1–22.

[3] F. Abdenur and S. Crovisier, *Transitivity and topological mixing for C^1 diffeomorphisms*, Essays in mathematics and its applications, Springer (2012), 1–16.

[4] R. Abraham and S. Smale, *Nongenericity of Ω-stability*, Global analysis I, Proc. Symp. Pure Math. AMS **14** (1970), 5–8.

[5] D. Anosov, *Geodesic flows on closed Riemannian manifolds of negative curvature*, Trudy Mat. Inst. Steklov **90** (1967).

[6] N. Aoki, *The set of Axiom A diffeomorphisms with no cycles*, Bol. Soc. Brasil. Mat. **23** (1992), 21–65.

[7] A. Arbieto and C. Matheus, *A pasting lemma and some applications for conservative systems*, Ergodic Theory & Dynamical Systems **27** (2007), 1399–1417.

[8] M.-C. Arnaud, C. Bonatti, and S. Crovisier, *Dynamiques symplectiques génériques*, Ergodic Theory & Dynamical Systems **25** (2005), 1401–1436.

[9] A. Avila, J. Bochi, and A. Wilkinson, *Nonuniform center bunching and the genericity of ergodicity among C^1 partially hyperbolic symplectomorphisms*, Ann. Sci. Éc. Norm. Supér. **42** (2009), 931–979.

[10] A. Avila, S. Crovisier, and A. Wilkinson, *Diffeomorphisms with positive metric entropy*, In preparation.

[11] J. Bochi, *Genericity of zero Lyapunov exponents*, Ergodic Theory & Dynamical Systems **22** (2002), 1667–1696.

[12] J. Bochi and C. Bonatti, *Perturbation of the Lyapunov spectra of periodic orbits*, Proc. Lond. Math. Soc. **105** (2012), 1–48.

[13] C. Bonatti, *Towards a global view of dynamical systems, for the C^1-topology*, Ergodic Theory & Dynam. Systems **31** (2011), 959–993.

[14] C. Bonatti and S. Crovisier, *Récurrence et généricité*, Invent. Math. **158** (2004), 33–104.

[15] _____, *Center manifolds for partially hyperbolic set without strong unstable connections*. arXiv:1401.2452.

[16] C. Bonatti, S. Crovisier, L. Díaz, and N. Gourmelon, *Internal perturbations of homoclinic classes: non-domination, cycles, and self-replication*, Ergodic Theory & Dynam. Systems **33** (2013), 739-776.

[17] C. Bonatti, S. Crovisier, N. Gourmelon, and R. Potrie, *Tame dynamics and robust transitivity.* arXiv:1112.1002. To appear at Transaction of the AMS.

[18] C. Bonatti, S. Crovisier, and K. Shinohara, *The $C^{1+\alpha}$ hypothesis in Pesin theory revisited*, Journal of Modern Dynamics **7** (2013), 605–618.

[19] C. Bonatti and L. Díaz, *Persistence of transitive diffeomorphims*, Ann. of Math. **143** (1995), 367–396.

[20] _____, *On maximal transitive sets of generic diffeomorphisms*, Publ. Math. I.H.E.S. **96** (2003), 171–197.

[21] _____, Robust heterodimensional cycles and C^1-generic dynamics. *J. Inst. Math. Jussieu* **7** (2008), 469–525.

[22] _____, *Abundance of C^1-robust tangencies*, Trans. AMS **364** (2012), 5111–5148.

[23] C. Bonatti, L. Díaz, and E. Pujals, *A C^1-generic dichotomy for diffeomorphisms: weak forms of hyperbolicity or infinitely many sinks or sources*, Ann. of Math. **158** (2003), 355–418.

[24] C. Bonatti, S. Gan, and L. Wen, *On the existence of non-trivial homoclinic classes*, Ergodic Theory & Dynam. Systems **27** (2007), 1473–1508.

[25] C. Bonatti, M. Li, and D. Yang, *On the existence of attractors*, Trans. AMS **365** (2013), 1369–1391.

[26] C. Bonatti and M. Viana, *SRB measures for partially hyperbolic systems whose central direction is mostly contracting*, Israel J. Math. **115** (2000) 157–193.

[27] C. Carballo, C. Morales, and M.-J. Pacifico, *Homoclinic classes for generic C^1 vector fields*, Ergodic Theory & Dynam. Systems **23** (2003), 403–415.

[28] C. Conley, *Isolated Invariant Sets and the Morse Index*, CBMS Regional Conference Series in Mathematics **38**, American Mathematical Society, Providence. RI (1978).

[29] G. Contreras, *Geodesic flows with positive topological entropy, twist maps and hyperbolicity*, Ann. of Math. **172** (2010), 761–808.

[30] S. Crovisier, *Periodic orbits and chain-transitive sets of C^1-diffeomorphisms*, Publ. Math. Inst. Hautes Études Sci. **104** (2006), 87–141.

[31] _____, *Birth of homoclinic intersections: a model for the central dynamics of partially hyperbolic systems*, Ann. of Math. **172** (2010), 1641–1677.

[32] _____, *Partial hyperbolicity far from homoclinic bifurcations*, Advances in Math. **226** (2011), 673–726.

[33] _____, *Perturbation de la dynamique de difféomorphismes en petite régularité*, Astérisque **354** (2013).

[34] S. Crovisier and E. Pujals, *Essential hyperbolicity versus homoclinic bifurcations.* arXiv:1011.3836.

[35] S. Crovisier, E. Pujals, and M. Sambarino, *Hyperbolicity of the extreme bundles.* In preparation.

[36] S. Crovisier, M. Sambarino, and D. Yang, *Partial Hyperbolicity and Homoclinic Tangencies.* To appear at Journal of the EMS. arXiv:1103.0869.

[37] L. Díaz, *Robust nonhyperbolic dynamics and heterodimensional cycles*, Ergodic Theory & Dynam. Systems **15** (1995), 291–315.

[38] D. Dolgopyat and A. Wilkinson, *Stable accessibility is C^1 dense. Geometric methods in dynamics. II*, Astérisque **287** (2003), 33–60.

[39] R. Dujardin and M. Lyubich, *Stability and bifurcations for dissipative polynomial automorphisms of C^2*, arXiv:1305.2898.

[40] S. Gan, L. Wen, and D. Yang, *Minimal non-hyperbolicity and index completeness*, Discrete Contin. Dyn. Syst. **25** (2009), 1349–1366.

[41] N. Gourmelon, *Generation of homoclinic tangencies by C^1-perturbations*, Discrete Contin. Dyn. Syst. **26** (2010), 1–42.

[42] _____, *A Franks' lemma that preserves invariant manifolds*. arXiv: 0912.1121 and arXiv:1212.6638.

[43] S. Hayashi, *Diffeomorphisms in $\mathcal{F}^1(M)$ satisfy Axiom A*, Ergodic Theory & Dynam. Systems **12** (1992), 233–253.

[44] _____, *Connecting invariant manifolds and the solution of the C^1-stability and Ω-stability conjectures for flows*, Ann. of Math. **145** (1997), 81–137.

[45] M. Hirsch, C. Pugh, and M. Shub, *Invariant Manifolds*, Lecture Notes in Mathematics **583**, Springer Verlag, Berlin, 1977.

[46] V. Horita and A. Tahzibi, *Partial hyperbolicity for symplectic diffeomorphisms*, Ann. Inst. H. Poincaré Anal. Non Linéaire **23** (2006), 641–661.

[47] O. Kozlovski, W. Shen, and S. Van Strien, *Density of hyperbolicity in dimension one*, Ann. of Math. **166** (2007), 145–182.

[48] S. Liao, *On the stability conjecture*, Chinese Ann. Math. **1** (1980), 9–30.

[49] _____, *Obstruction sets. II*, Beijing Daxue Xuebao **2** (1981), 1–36.

[50] G. Liao, M. Viana, and J. Yang, *The Entropy Conjecture for Diffeomorphisms away from Tangencies*, Journal of the EMS **15** (2013), 2043–2060.

[51] R. Mañé, *An ergodic closing lemma*, Ann. of Math. **116** (1982), 503–540.

[52] _____, *Hyperbolicity, sinks and measure in one dimensional dynamics*, Commun. Math. Phys. **100** (1985), 495–524 and Comm. Math. Phys. **112** (1987), 721–724.

[53] _____, *A proof of the C^1 stability conjecture*, Inst. Hautes études Sci. Publ. Math. **66** (1988), 161–210.

[54] C. Moreira, *There are no C^1-stable intersections of regular Cantor sets*, Acta Math. **206** (2011), 311–323.

[55] S. Newhouse, *Hyperbolic limit sets*, Trans. Amer. Math. Soc. **167** (1972), 125–150.

[56] _____, *Quasi-elliptic periodic points in conservative dynamical systems*, Amer. J. Math. **99** (1977), 1061–1087.

[57] _____, *The abundance of wild hyperbolic sets and nonsmooth stable sets for diffeomorphisms*, Publ. Math. I.H.E.S. **50** (1979), 101–151.

[58] _____, *Lectures on dynamical systems*, In Dynamical systems, CIME. Lect., Bressanone 1978, Progress in Mathematics **8**, Birkäuser (1980), 1–114.

[59] J. Palis, *Homoclinic bifurcations, sensitive-chaotic dynamics and strange attractors*, Adv. Ser. Dynam. Systems **9** (1991), 466–472.

[60] J. Palis, A global view of dynamics and a conjecture on the denseness of finitude of attractors. In *Géométrie complexe et systèmes dynamiques*, *Astérisque* **261** (2000), 335–347.

[61] _____, *A global perspective for non-conservative dynamics*, Ann. I. H. Poincaré **22** (2005), 485–507.

[62] _____, Open questions leading to a global perspective in dynamics. *Nonlinearity* **21** (2008), T37–T43.

[63] J. Palis and M. Viana, *High dimension diffeomorphisms displaying infinitely many periodic attractors*, Ann. of Math. **140** (1994), 207–250.

[64] J. Palis and J.-C. Yoccoz, *Non-uniformly hyperbolic horseshoes arising from bifurcations of Poincaré heteroclinic cycles*, Publ. Math. I.H.E.S. **110** (2009), 1–217.

[65] Y. Pesin, *Families of invariant manifolds that correspond to nonzero characteristic exponents*, Math. USSR-Izv. **40** (1976), 1261–1305.

[66] R. Potrie, *Wild Milnor attractors accumulated by lower-dimensional dynamics*, Ergodic Theory & Dynam. Systems **34** (2014), 236–262.

[67] C. Pugh, *The closing lemma*, Amer. J. Math. **89** (1967), 956–1009.

[68] E. Pujals, *On the density of hyperbolicity and homoclinic bifurcations for 3D- diffeomorphisms in attracting regions*, Discrete Contin. Dyn. Syst. **16** (2006), 179–226.

[69] ———, *Density of hyperbolicity and homoclinic bifurcations for attracting topologically hyperbolic sets*, Discrete Contin. Dyn. Syst. **20** (2008), 335–405.

[70] ———, *Some simple questions related to the C^r-stability conjecture*, Nonlinearity **21** (2008), T233–T237.

[71] E. Pujals and F. Rodriguez-Hertz, *Critical points for surface diffeomorphisms*, J. Mod. Dyn. **1** (2007), 615–648.

[72] E. Pujals and M. Sambarino, *Homoclinic tangencies and hyperbolicity for surface diffeomorphisms*, Ann. of Math. **151** (2000), 961–1023.

[73] ———, *Density of hyperbolicity and tangencies in sectional dissipative regions*, Ann. Inst. H. Poincaré Anal. Non Linéaire **26** (2009), 1971–2000.

[74] H. Qiu, *Existence and uniqueness of SRB measure on C^1 generic hyperbolic attractors*, Comm. Math. Phys. **302** (2011), 345–357.

[75] F. Rodriguez Hertz, M.-A. Rodriguez Hertz, A. Tahzibi, and R. Ures, *Maximizing measures for partially hyperbolic systems with compact center leaves*, Ergodic Theory & Dynam. Systems **32** (2012), 825–839.

[76] R. Sagin and Z. Xia, *Partial hyperbolicity or dense elliptic periodic points for C^1-generic symplectic diffeomorphisms*, Trans. Amer. Math. Soc. **358** (2006), 5119–5138.

[77] K. Shinohara, *On the indices of periodic points in C^1-generic wild homoclinic classes in dimension three*, Discrete Contin. Dyn. Syst. **31** (2011), 913–940.

[78] S. Smale, *Differentiable dynamical systems*, Bull. AMS **73** (1967), 747–817.

[79] ———, *Stability and genericity in dynamical systems. Séminaire Bourbaki 374*, Lecture Notes in Mathematics **180** (1971).

[80] ———, *Mathematical problems for the next century*, Math. Intel. **20** (1998), 7–15.

[81] D. Turaev, *Maps close to the identity and universal maps in the Newhouse domain.* arXiv:1009.0858.

[82] M. Viana and J. Yang, *Physical measures and absolute continuity for one-dimensional center direction*, Ann. Inst. H. Poincaré Anal. Non Linéaire **30** (2013), 845–877.

[83] L. Wen, *Homoclinic tangencies and dominated splittings*, Nonlinearity **15** (2002), 1445–1469.

[84] ———, *Generic Diffeomorphisms away from homoclinic tangencies and heterodimensional cycles*, Bull. Braz. Math. Soc. **35** (2004), 419–452.

[85] A. Wilkinson, *Conservative partially hyperbolic dynamics*, Proceedings of the International Congress of Mathematicians, volume III (2010), 1816–1836.

[86] J. Yang, *Lyapunov stable chain recurrent classes.* arXiv:0712.0514.

[87] J.-C. Yoccoz, *Travaux de Herman sur les tores invariants*, Séminaire Bourbaki 754, Astérisque **206** (1992), 311–344.

Laboratoire de Mathématiques d'Orsay, CNRS - UMR 8628, Université Paris-Sud 11, Orsay 91405, France.

E-mail: Sylvain.Crovisier@math.u-psud.fr

Weak KAM Theory: the connection between Aubry-Mather theory and viscosity solutions of the Hamilton-Jacobi equation

Albert Fathi

Abstract. The goal of this lecture is to explain to the general mathematical audience the connection that was discovered in the last 20 or so years between the Aubry-Mather theory of Lagrangian systems, due independently to Aubry and Mather in low dimension, and to Mather in higher dimension, and the theory of viscosity solutions of the Hamilton-Jacobi equation, due to Crandall and Lions, and more precisely the existence of global viscosity solutions due to Lions, Papanicolaou, and Varhadan.

Mathematics Subject Classification (2010). Primary 37J50, 35F21; Secondary 70H20.

Keywords. Lagrangian, Hamiltonian, Hamilton-Jacobi, Aubry-Mather, weak KAM.

1. Introduction

This lecture is not intended for specialists, but rather for the general mathematical audience. Lagrangian Dynamical Systems have their origin in classical physics, especially in celestial mechanics. The Hamilton-Jacobi method is a way to obtain trajectories of a Lagrangian system through solutions of the Hamilton-Jacobi equation. However, solutions of this equation easily develop singularities. Therefore for a long time, only local results were obtained. Since the 1950's, several major developments both on the dynamical side, and the PDE side have taken place. In the 1980's, on the dynamical side there was the famous Aubry-Mather theory for twist maps, discovered independently by S. Aubry [2] and J.N. Mather [20], and its generalization to higher dimension by J.N. Mather [21, 22] in the framework of classical Lagrangian systems. On the PDE side, there was the viscosity theory of the Hamilton-Jacobi equation, due to M. Crandall and P.L. Lions [8], which introduces weak solutions for this equation, together with the existence of global solutions for the stationary Hamilton-Jacobi equation on the torus obtained by P.L. Lions, G. Papanicolaou, and S.R.S. Varadhan [18]. In 1996, the author found the connection between these apparently unrelated results: the Aubry and the Mather sets can be obtained from the global weak (=viscosity) solutions. Moreover, these sets serve as natural uniqueness sets for the stationary Hamilton-Jacobi equation, see [13]. Independently, a little bit later Weinan E [11] found the connection for twist maps, with some partial ideas for higher dimensions, and L.C. Evans and D. Gomes [12] showed how to obtain Mather measures from the PDE point of view.

In this introduction, we quickly explain some of these results.

Proceedings of the International Congress of Mathematicians, Seoul, 2014

Although all results are valid for Tonelli Hamiltonians defined on the cotangent space T^*M of a compact manifold, in this introduction, we will stick to the case where $M = \mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k$.

A Tonelli Hamiltonian H on $\mathbb{T}^k$ is a function $H : \mathbb{T}^k \times \mathbb{R}^k \to \mathbb{R}, (x,p) \mapsto H(x,p)$, where $x \in \mathbb{T}^k$ and $p \in \mathbb{R}^k$, which satisfies the following conditions:

(i) The Hamiltonian H is C^r, with $r \geq 2$.

(ii) (Strict convexity in the momentum) The second derivative $\partial^2 H/\partial p^2(x,p)$ is positive definite, as a quadratic form, for every $(x,p) \in \mathbb{T}^k \times \mathbb{R}^k$.

(iii) (Superlinearity) $H(x,p)/\|p\|$ tends to $+\infty$, uniformly in $x \in \mathbb{T}^k$, as $\|p\|$ tends to $+\infty$, where $\|\cdot\|$ is the Euclidean norm.

In fact, it is more accurate to consider H as a function on the *cotangent* bundle $\mathbb{T}^k \times (\mathbb{R}^k)^*$, where $(\mathbb{R}^k)^*$ is the vector space dual to $\mathbb{R}^k$. To avoid complications, in this introduction, we identify $(\mathbb{R}^k)^*$ to $\mathbb{R}^k$ in the usual way (using the canonical scalar product).

There is a flow ϕ_t^* associated to the Hamiltonian. This flow is given by the ODE

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p}(x,p) \\ \dot{p} &= -\frac{\partial H}{\partial x}(x,p). \end{aligned} \tag{1.1}$$

It is easy to see that the Hamiltonian H is constant along solutions of the ODE. Since the level sets of the function H are compact by the superlinearity condition (iii), this flow ϕ_t^* is defined for all $t \in \mathbb{R}$, and therefore is a genuine dynamical system.

The following theorem is due to John Mather [21, 22] with a contribution by Mário Jorge Dias Carneiro [9].

Theorem 1.1. *There exists a convex superlinear function $\alpha : \mathbb{R}^k \to \mathbb{R}$ such that for every $P \in \mathbb{R}^k$, we can find a non-empty compact subset $\tilde{\mathcal{A}}^*(P) \subset \mathbb{T}^k \times \mathbb{R}^k$, called the Aubry set of H for P, satisfying:*

1) *The set $\tilde{\mathcal{A}}^*(P)$ is non-empty and compact.*
2) *The set $\tilde{\mathcal{A}}^*(P)$ is invariant by the flow ϕ_t^*.*
3) *The set $\tilde{\mathcal{A}}^*(P)$ is a graph on the base $\mathbb{T}^k$, i.e. the restriction of the projection $\pi : \mathbb{T}^k \times \mathbb{R}^k \to \mathbb{T}^k$ to $\tilde{\mathcal{A}}^*(P)$ is injective.*
4) *The set $\tilde{\mathcal{A}}^*(P)$ is included in the level set $\{(x,p) \in \mathbb{T}^k \times \mathbb{R}^k \mid H(x,p) = \alpha(P)\}$.*

John Mather [21, 22] gave also a characterization of the probability measures invariant by ϕ_t^* whose support is included in the Aubry set $\tilde{\mathcal{A}}^*(P)$.

Theorem 1.2. *For every $P \in \mathbb{R}^k$, and every Borel probability measure $\tilde{\mu}$ on $\mathbb{T}^k \times \mathbb{R}^k$ which is invariant by the flow ϕ_t^*, we have*

$$-\alpha(P) \leq \int_{\mathbb{T}^k \times \mathbb{R}^k} \frac{\partial H}{\partial p}(x,p)[p-P] - H(x,p)\, d\tilde{\mu}(x,p),$$

with equality if and only if $\tilde{\mu}(\tilde{\mathcal{A}}^*(P)) = 1$, i.e. the support of $\tilde{\mu}$ is contained in $\tilde{\mathcal{A}}^*(P)$.

Part 4) Theorem 1.1 is the contribution of Mário Jorge Dias Carneiro. It leads us to a connection with the Hamilton-Jacobi equation. In fact, there is a well-known way to obtain invariant sets which are both graphs on the base and contained in a level set of H. It is given by the Hamilton-Jacobi theorem.

Theorem 1.3 (Hamilton-Jacobi). *Let $u : \mathbb{T}^k \to \mathbb{R}$ be a C^2function. If $P \in \mathbb{R}^k$, the graph*

$$\mathrm{Graph}(P + \nabla u) = \{(x, P + \nabla u(x)) \mid x \in \mathbb{T}^k\}$$

is invariant under the Hamiltonian flow of H if and only if H is constant on $\mathrm{Graph}(P+\nabla u)$. i.e. if and only if u is a solution of the (stationary) Hamilton-Jacobi equation

$$H(x, P + \nabla u(x)) = c, \text{ for every } x \in \mathbb{T}^k,$$

where c is a constant independent of x.

Therefore, it is tempting to try to obtain the Aubry-Mather sets from invariant graphs. This cannot be done with u smooth, since this would give too many invariant tori in general Hamiltonian systems, see the explanations in the next section.

In fact, Crandall and Lions [8] developed a notion of weak PDE solution for the Hamilton-Jacobi equation, called viscosity solution. The following global existence theorem was obtained by Lions, Papanicolaou, and Varadhan [18].

Theorem 1.4. *Suppose that $H : \mathbb{T}^k \times \mathbb{R}^k \to \mathbb{R}$ is continuous and satisfies the superlinearity condition* (iii) *above. For every P, there exists a unique constant $\bar{H}(P)$ such that the Hamilton-Jacobi equation*

$$H(x, P + \nabla u(x)) = \bar{H}(P) \tag{1.2}$$

admits a global weak (viscosity) solution $u : \mathbb{T}^k \to \mathbb{R}$.

The solutions obtained in this last theorem are automatically Lipschitz due to the superlinearity of H. Of course, if we add a constant to a solution of equation (1.2) we still obtain a solution. However, it should be emphasized that there may be a pair of solutions whose difference is not a constant.

Theorem 1.4 above was obtained in 1987, and Mather's work [21] was essentially completed by 1990. John Mather visited the author at the University of Florida in Gainesville in the fall of 1988, and explained that he had obtained some results on existence of Aubry sets for Lagrangians in higher dimension, i.e. beyond twist maps.

In 1996, the author obtained the following result, see [13].

Theorem 1.5 (Weak Hamilton-Jacobi). *The function α of Mather, and the function $\bar{H}$ of Lions-Papanicolaou-Varadhan are equal. Moreover, if $u : \mathbb{T}^k \to \mathbb{R}$ is a weak (=viscosity) solution of*

$$H(x, P + \nabla u(x)) = \bar{H}(P) = \alpha(P), \tag{1.3}$$

then the graph

$$\mathrm{Graph}(P + \nabla u) = \{(x, \nabla u(x)) \mid \text{ for } x \in \mathbb{T}^k, \text{ such that } u \text{ has a derivative at } x\}$$

satisfies the following properties:

1) *Its closure $\overline{\mathrm{Graph}(P + \nabla u)}$ is compact, and projects onto the whole of $\mathbb{T}^k$.*

2) *For every $t > 0$, we have $\phi^*_{-t}\big(\overline{\mathrm{Graph}(P + \nabla u)}\big) \subset \mathrm{Graph}(P + \nabla u)$.*

Therefore, the subset $\tilde{\mathcal{I}}^*(P+u) = \bigcap_{t\geq 0} \phi^*_{-t}\big(\overline{\mathrm{Graph}(P + \nabla u)}\big)$ is compact non-empty and invariant under ϕ^*_t, for every $t \in \mathbb{R}$, and the closure $\overline{\mathrm{Graph}(P + \nabla u)}$ is contained in the unstable subset $W^u(P + \tilde{\mathcal{I}}^*(u))$ defined by

$$W^u(\tilde{\mathcal{I}}^*(P + u)) = \{(x, p) \in \mathbb{T}^k \times \mathbb{R}^k \mid \phi^*_t(x, p) \to \tilde{\mathcal{I}}^*(u),\ \text{as } t \to -\infty\}.$$

Moreover, the Aubry set $\tilde{\mathcal{A}}^*(P)$ for H is equal to the intersection of the sets $\tilde{\mathcal{I}}^*(P + u)$, where the intersection is taken on all weak (viscosity) solutions of equation (1.3)*.*

In fact, denoting by $\tilde{\mathcal{A}}^*_H(P)$, and α_H, the Aubry sets and the α function for the Hamiltonian H, it is not difficult to see that $\alpha_H(P) = \alpha_{H_P}(0)$, and also that $\tilde{\mathcal{A}}^*_H(P)$ can be obtained from $\tilde{\mathcal{A}}^*_{H_P}(0)$, where H_P is the Tonelli Hamiltonian defined by $H_P(x, p) = H(x, P + p)$. Therefore we will later on only give the proof of Theorem 1.5 for the case $P = 0$.

It is the author's strong belief that the real discoverer of the above theorem should have been Ricardo Mañé. His untimely death in 1995 prevented him from discovering this theorem as can be attested by his last work [19].

The reader should also be aware that what we are covering is just the beginning of weak KAM theory. It is 18 years old. It does not do justice to the marvelous contributions done by others in this subject since 1996. The author strongly apologizes to all these mathematicians who have carried the theory way beyond the author's imagination or wildest dream.

2. Motivation

Some motivation for Aubry-Mather, and hence for weak KAM theory, came from celestial mechanics, and problems related to more general classical mechanical systems studied by Lagrangian or Hamiltonian methods.

We will give a (very partial) description of this motivation. There are also some historical comments. The reader should not take them seriously. They are here for the sake of a good story. The author does not claim that this historical account is accurate.

Although celestial mechanics is about the motion of several bodies in $\mathbb{R}^3$ with different masses, we will use a simplified model, and start with the motion of a free particle of mass m in the Euclidean space $\mathbb{R}^k$ (if $k = 3n$, this is also the motion of n particles in $\mathbb{R}^3$, all with same mass m). The trajectory $\gamma : \mathbb{R} \to \mathbb{R}^k$ of such a particle satisfies $\ddot{\gamma}(t) = 0$, for all $t \in \mathbb{R}$. Therefore $\gamma(t) = x + tv$, where $x = \gamma(0)$ is the initial position and v is the initial speed. The speed of the trajectory is the time derivative $\dot{\gamma}$, in particular $v = \dot{\gamma}(0)$. It is better to convert the second order ODE given by $\ddot{\gamma}(t) = 0$ to a first order ODE on the configuration space $\mathbb{R}^k \times \mathbb{R}^k$ taking into account both position and speed. A point in $\mathbb{R}^k \times \mathbb{R}^k$ will be denoted by (x, v), where $x \in \mathbb{R}^k$ is the position component and $v \in \mathbb{R}^k$ is the speed component. The speed curve of γ is $\Gamma(t) = (\gamma(t), \dot{\gamma}(t))$. This curve takes values in $\mathbb{R}^k \times \mathbb{R}^k$ and satisfies the first order ODE

$$\dot{\Gamma}(t) = X_0(\Gamma(t)),$$

where the vector field X_0 on $\mathbb{R}^k \times \mathbb{R}^k$ is given by $X_0(x, v) = (v, 0)$. Conversely, any solution of this ODE is a possible speed curve of a free particle of mass m. The solutions of the ODE yield a flow ϕ^0_t on $\mathbb{R}^k \times \mathbb{R}^k$, defined by

$$\phi^0_t(x, v) = (x + tv, v).$$

Observe that the sets $\mathbb{R}^k \times \{v\}, v \in \mathbb{R}^k$ give a decomposition of $\mathbb{R}^k \times \mathbb{R}^k$ into subsets which are invariant by the flow ϕ_t^0. We will address the following problem: if we perturb this system a little bit can we still see such a pattern, i.e. a (partial) decomposition, into invariant subsets?

To make things more precise, we add a smooth (at least C^2) potential $V : \mathbb{R}^k \to \mathbb{R}$ to our mechanical system. To avoid problems caused by non-compactness, we will assume that V is $\mathbb{Z}^k$ periodic, i.e. it satisfies $V(x + z) = V(x)$, for all $x \in \mathbb{R}^k$, and all $z \in \mathbb{Z}^k$. Therefore V is defined on $\mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k$. The equation of motion is now given by the Newton equation

$$m\ddot{\gamma}(t) = -\nabla V(\gamma(t)).$$

Again this defines a first order ODE on $\mathbb{T}^k \times \mathbb{R}^k$ using the vector field

$$X(x, v) = \big(v, -\frac{1}{m}\nabla V[\gamma(t)]\big).$$

This ODE has a flow on $\mathbb{T}^k \times \mathbb{R}^k$ which we will denote by ϕ_t. The orbits of our flow are precisely the speed curves of possible motions of a particle in the potential V.

Before proceeding further, it is convenient to recall the Lagrangian and Hamiltonian aspects of a classical mechanical system since they will play a major role in the theory. The Lagrangian $L : \mathbb{T}^k \times \mathbb{R}^k$ is defined by

$$L(x, v) = \frac{1}{2}m\|v\|^2 - V(x),$$

where $\|\cdot\|$ is the usual Euclidean norm on $\mathbb{R}^k$. Using this Lagrangian, the Newton equation becomes

$$\frac{d}{dt}\left[\frac{\partial L}{\partial v}(\gamma(t), \dot{\gamma}(t))\right] = \frac{\partial L}{\partial x}(\gamma(t), \dot{\gamma}(t)). \tag{2.1}$$

The equation above is the Euler-Lagrange equation associated to the Lagrangian L. It shows that the trajectories are extremal curves for the Lagrangian, as we now explain. A Lagrangian like L is used to define the action $\mathbb{L}(\gamma)$ of the curve $\gamma : [a, b] \to \mathbb{T}^k$ by

$$\mathbb{L}(\gamma) = \int_a^b L(\gamma(s), \dot{\gamma}(s))\, ds.$$

A curve $\gamma : [a, b] \to \mathbb{T}^k$ is called a minimizer (for L) if for every curve $\delta : [a, b] \to \mathbb{T}^k$, with $\delta(a) = \gamma(a), \delta(b) = \gamma(b)$, we have $\mathbb{L}(\delta) \geq \mathbb{L}(\gamma)$. These curves play a particular role in Aubry-Mather theory. They have to be found among the curves which are critical points for the action functional $\mathbb{L}$. These critical points are called extremals. More precisely, a curve $\gamma : [a, b] \to \mathbb{T}^k$ is called an extremal for L, if the functional $\mathbb{L}$ on the space of curves $\delta : [a, b] \to \mathbb{T}^k$, with $\delta(a) = \gamma(a), \delta(b) = \gamma(b)$, has a vanishing derivative $D_\gamma \mathbb{L}$ at γ. By the classical theory of Calculus of Variations, this is the case if and only if γ satisfies the Euler-Lagrange equation (2.1). Therefore the possible trajectories of our particle for the potential V are precisely the extremals for L.

For the Hamiltonian aspects, one has to introduce the dual variable $p = mv$. In fact, this dual variable should be understood as an element of the dual space $(\mathbb{R}^k)^*$, which means that p should be considered as the linear form $\langle p, \cdot\rangle$ on $\mathbb{R}^k$. A better way to think of p is to define it by $p = \partial L/\partial v(x, v)$. The Hamiltonian $H : \mathbb{T}^k \times (\mathbb{R}^k)^*$ is then defined by

$$H(x, p) = \frac{1}{2m}\|p\|^2 + V(x).$$

It is not difficult to see that H is also given by

$$H(x,p) = \max_{v\in\mathbb{R}^k} p(v) - L(x,v).$$

The Legendre transform $\mathcal{L} : \mathbb{T}^k \times \mathbb{R}^k \to \mathbb{T}^k \times (\mathbb{R}^k)^*$ is a diffeomorphism defined by

$$\mathcal{L}(x,v) = (x, \frac{\partial L}{\partial v}(x,v)).$$

If one uses the Legendre transform to transport the flow ϕ_t to the flow $\phi_t^* = \mathcal{L}\phi_t\mathcal{L}^{-1}$ on $\mathbb{T}^k \times (\mathbb{R}^k)^*$, using the Euler-Lagrange equation and the definition of H, it is not difficult to see that ϕ_t^* is the flow of the ODE (1.1).

$$\dot{x} = \frac{\partial H}{\partial p}(x,p)$$
$$\dot{p} = -\frac{\partial H}{\partial x}(x,p).$$

This means that ϕ_t^* is the Hamiltonian flow associated to H.

Since we are now interested in perturbing the motion of the free particle, we will denote by $\phi_t^V, L_V, H_V, \dots$ the objects associated to the potential V. Of course, for $V = 0$, we get back the flow ϕ_t^0, or rather the induced flow on the quotient $\mathbb{T}^k \times \mathbb{R}^k$. In that case $L_0(x,v) = \|v\|^2/2$, and $H_0(x,p) = \|p\|^2/2$, the flow ϕ_t^0 is the geodesic flow of the flat canonical metric on $\mathbb{T}^k$, and ϕ_t^{*0} is the geodesic flow on the cotangent bundle. The decomposition into invariant sets for the flow ϕ_t^{*0} is given by $\{(x,p) \mid p = P\}, P \in \mathbb{R}^k$. Notice that this is the graph of the solution $u = 0$ of the Hamilton-Jacobi equation

$$H_0(x, P + d_x u) = \frac{1}{2}\|P\|^2.$$

One could try to understand the persistence or non-persistence of the invariant sets by trying to solve for V small the Hamilton-Jacobi equation

$$H_V(x, P + d_x u) = c(P).$$

Unfortunately, it is almost impossible to find a C^1 solution of such an equation for a given V, and all P. In fact, as we now see in the simple example of a pendulum, there must be some condition on P to be able to do that.

Example 2.1. We consider the function $V_\epsilon(t) = \epsilon \cos 2\pi t$ on the 1-dimensional torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. The Hamiltonian $H_\epsilon(x,p) = 1/2p^2 + \epsilon \cos 2\pi t$ has levels which are 1-dimensional. The Hamiltonian flow ϕ_t^ϵ is the flow of the ODE

$$\dot{x} = p$$
$$\dot{p} = -2\pi\epsilon \sin(2\pi x).$$

Therefore the flow ϕ_t^ϵ has exactly two fixed points $(0,0)$ and $(1/2,0)$. There are also two orbits homoclinic to the fixed point $(0,0)$ (i.e. converging to the fixed point when $t \to \pm\infty$). The union of the fixed point $(0,0)$ and its two homoclinic orbits is the level $H_\epsilon = \epsilon$, see the figure below. The other orbits of the Hamiltonian flow are periodic. A level set $H_\epsilon = c$ is

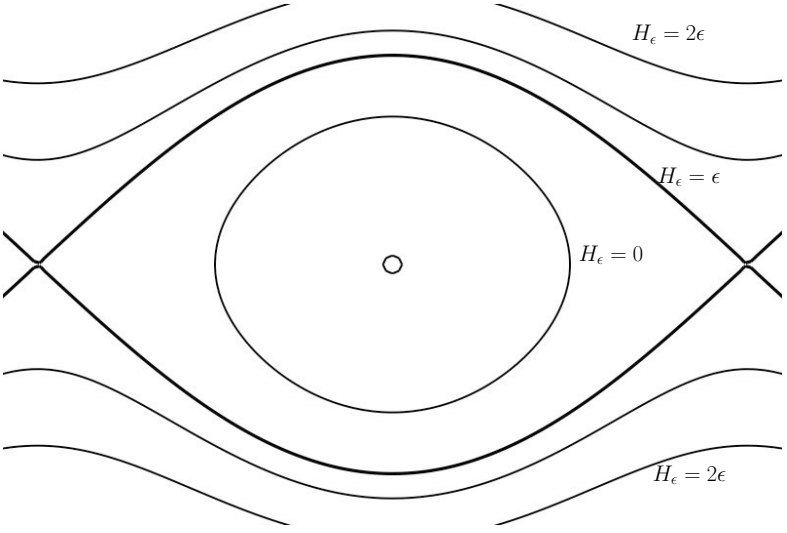


Figure 1. The pendulum.

just one orbit if $c < \epsilon$, and a pair of orbits if $c > \epsilon$. The level set for $c = \epsilon$ is given by the equation

$$\frac{1}{2}p^2 + \epsilon \cos 2\pi x = \epsilon.$$

Hence the region $\{(x, p) \mid H_\epsilon(x, p) \leq \epsilon\}$ is enclosed between the two graphs $p = \pm\sqrt{\epsilon}(2 - 2\cos 2\pi x)^{1/2}$. The area A_ϵ of $\{(x, p) \mid H_\epsilon(x, p) \leq \epsilon\}$ rescales as $A_\epsilon = \sqrt{\epsilon}A_1$, where $A_1 > 0$ is the area of $\{(x, p) \mid p^2 + 2\cos 2\pi x \leq 2\}$. Suppose that for a given $P \in \mathbb{R}$, we can find, for some $c \in \mathbb{R}$, a C^1 solution $u : \mathbb{T} \to \mathbb{R}$ of

$$H_\epsilon(x, P + u'(x)) = c. \tag{2.2}$$

This implies that the level set $H_\epsilon = c$, which is a subset of $\mathbb{T} \times \mathbb{R}$, projects onto the whole of $\mathbb{T}$. Therefore $c \geq \epsilon$, and the area between the curve $p = P + u'(x)$, and the curve $p = 0$ is an absolute value larger than $A_\epsilon/2.$, i.e. $|\int_{\mathbb{T}} P + u'(x)\, dx| \geq A_\epsilon/2$. But $\int_{\mathbb{T}} u'(x)\, dx = 0$, since u' is the derivative of a C^1 function on $\mathbb{T}$. If follows that $|P| \geq \sqrt{\epsilon}A_1/2$. In particular, the set $p = 0$ does not deform to an invariant set for ϵ as small as we want.

Note that $\phi_t^{*V_\epsilon}$ still remembers part of the set $p = 0$. In fact, the points in $p = 0$ are fixed points for ϕ_t^{*0}, and the flow $\phi_t^{*V_\epsilon}$ must also have fixed points, because the fixed points of $\phi_t^{*V_\epsilon}$ are precisely the critical points of H_{V_ϵ}, and by superlinearity the function H_{V_ϵ}must have critical points (at least a minimum) on $\mathbb{T} \times \mathbb{R}$.

Another fact that can be readily seen on Figure 1, is that for $|P| \geq \epsilon$, the Hamilton-Jacobi equation (2.2) has a solution. This solution is C^∞ for $|P| > \epsilon$. But it is only C^1 for $|P| = \epsilon$, in which case the derivative u' is only piecewise C^1 because its graph has a corner at $x = 0$.

In fact, there are always problems with resonances. This goes back to the work of Henri Poincaré [24] on the three body problem. To explain this in our case, we come back to the flow ϕ_t^0 defined on the tangent bundle of $\mathbb{T}^k$. The invariant sets are given by $T_v = \{(x, v) \mid x \in \mathbb{T}^k\}, v \in \mathbb{R}^k$. If the coordinates of v are all rational, then the motion on T_v

is periodic. It can be shown that perturbing the system destroys most of the T_v's. However necessarily some periodic orbits must still exist. In fact, the periodic orbits on T_v are all in the same homotopy class, and they minimize, in that homotopy class, the action for the Lagrangian $L_0(x,v) = \|v\|^2/2$. If we perturb the Lagrangian L_0 to a Tonelli Lagrangian L, by the direct method in the Calculus of Variations, there are minimizers of the L-action in this homotopy class. For a long time, it was believed that most of the T_v's would disappear under a general perturbation except maybe for some periodic orbits. It came as a surprise, when A.N. Kolmogorov [17] announced the stability property for T_v, for v far away from the rational vectors, at least for analytic perturbations. This was extended by V.I. Arnold [1], and also by J. Moser [23] to cover differentiable perturbations in the C^r topology. This is the now famous KAM theory. In fact, not only do the T_v persist for some $v's$, but they persist for more and more v's as the size of the perturbation becomes smaller and smaller, for example in the C^∞ topology. The set of v's for which this is possible tends to a set of full Lebesgue measure as the perturbation vanishes.

It turns out that the KAM method proves more than what we just said. Fix a $v_0 \in \mathbb{R}^k$ to which the KAM theorem applies. The invariant set T_{v_0} for ϕ_t^0 is a torus and on that torus ϕ_t^0 is the linear flow $(t,x) \mapsto (x + tv_0)$. For V small enough, KAM theory finds a smooth imbedding map $i_{V,v_0} : T_{v_0} \to \mathbb{T}^k \times \mathbb{R}^k$ such that:

1) the image $i_{V,v_0}(T_{v_0})$ is invariant under ϕ_t^V;
2) the imbedding i_{V,v_0} is a conjugation between the linear flow $\phi_t^0|T_{v_0}$ and the restriction of ϕ_t^V to the image $i_{V,v_0}(T_{v_0})$.

So not only does the set persist (with a deformation) but the dynamics remain the same. The imbedding i_{V,v_0} is the identity for $V = 0$, and it depends continuously on V. Moreover, we have:

3) the image $\mathcal{L}_V(i_{V,v_0}(T_{v_0}))$ of $i_{V,v_0}(T_{v_0})$ by the Legendre transform, which is invariant under ${\phi_t^*}^V$, is a Lagrangian graph on the base. This means that we can find $P_{V,v_0} \in \mathbb{R}^k$, and a smooth function $u_{V,v_0} : \mathbb{T}^k \to \mathbb{R}$ such that $\mathcal{L}_V(i_{V,v_0}(T_{v_0})) = \mathrm{Graph}(P_{V,v_0} + du_{V,v_0}) = \{(x, P_{V,v_0} + d_x u_{V,v_0}) \mid x \in \mathbb{T}^k\}$.

Since this graph $\mathrm{Graph}(P_{V,v_0} + du_{V,v_0})$ is invariant by ${\phi_t^*}^V$, by the Hamilton-Jacobi theorem the function u_{V,v_0} solves the equation $H_V(x, P_{V,v_0} + d_x u_{V,v_0}) = c_{V,v_0}$, for some constant c_{V,v_0}. Hence, although the KAM theory is rooted in Dynamical Systems and tries to find a part of the dynamics that is conjugated to a simple linear dynamic on the torus, it nevertheless produces smooth solutions to the Hamilton-Jacobi equation.

Of course, it remained to understand what happens to the invariant tori when they disappear. In 1982, independently, Aubry [2] and Mather [20] studied twist maps on the annulus (they can be thought as giving examples of a discretization of Tonelli Hamiltonians on $\mathbb{T}^2$). They showed that the invariant circles of the standard twist diffeomorphism $(x,r) \mapsto (x + r, r)$ of $\mathbb{T} \times [0,1]$ never completely disappear. In fact, periodic orbits persist for r rational, and for r irrational there exists either an invariant Cantor subset or an invariant curve. In all cases, the invariant sets are Lipschitz graphs on (part of) the base $\mathbb{T}$. It is important to note that these results are not only perturbative, but that they also hold for all area preserving twist maps of $\mathbb{T} \times [0,1]$.

Around 1989, John Mather extended the existence of these sets to Tonelli Hamiltonians in higher dimension [21, 22].

3. The general setting

We will consider the more general setting of a Hamiltonian $H : T^*M \to \mathbb{R}$ defined on the cotangent space T^*M of the compact connected manifold without boundary M. We will denote by (x, p) a point in T^*M, where $x \in M$, and $p \in T^*_x M$.

The Hamiltonian H is said to be Tonelli, if it satisfies conditions (i), (ii), and (iii) of the Introduction. Only condition (iii) needs an explanation. We replace the Euclidean norm on $\mathbb{R}^k$, by any family of norms $\|\cdot\|_x, x \in M$, on the fibers of $TM \to M$, coming from a Riemannian metric on M. Note that, by the compactness of M, any two such families are uniformly equivalent, i.e. their ratio is uniformly bounded away from 0 and from $+\infty$. Therefore condition (iii) is

(iii) (Superlinearity) $H(x, p)/\|p\|_x \to +\infty$, uniformly in $x \in M$, as $\|p\|_x \to +\infty$.

The Hamiltonian flow ϕ^*_t is still defined on T^*M. In local coordinates in M it is still the flow of the ODE (1.1). The flow is complete because H is constant on orbits, and has compact level sets by superlinearity.

We introduce the Lagrangian $L : TM \to \mathbb{R}$, defined on the tangent bundle TM of M, by

$$L(x, v) = \sup_{p \in T^*_x M} p(v) - H(x, p). \tag{3.1}$$

Since H is superlinear, this sup is always attained. Moreover, since the function $p \mapsto p(v) - H(x, p)$ is C^1 and strictly convex, this sup is achieved at the only p at which its derivative vanishes, namely the only p, where $v = \partial H/\partial p(x, p)$.

The Lagrangian L is as differentiable as H is, and it satisfies the Tonelli properties (i), (ii), (iii) of the introduction. The Legendre transform $\mathcal{L} : TM \to T^*M$ is defined by

$$\mathcal{L}(x, v) = \big(x, \frac{\partial L}{\partial v}(x, v)\big). \tag{3.2}$$

Using the Tonelli properties, it can be shown that $\mathcal{L} : TM \to T^*M$ is a global C^{r-1} diffeomorphism. Moreover, its inverse is given by

$$\mathcal{L}^{-1}(x, p) = \big(x, \frac{\partial H}{\partial p}(x, p)\big). \tag{3.3}$$

Definition (3.1) of the Lagrangian yields the Fenchel inequality

$$p(v) \leq L(x, v) + H(x, p). \tag{3.4}$$

Furthermore, there is equality in the Fenchel inequality if and only if $(x, p) = \mathcal{L}(x, v)$, which is equivalent to $p = \partial L/\partial v(x, v)$, and also to $v = \partial H/\partial p(x, p)$.

Since for any given $p \in T^*_x M$, we can find a $v \in T_x M$, for which the Fenchel inequality is an equality, we obtain

$$H(x, p) = \sup_{v \in T_x M} p(v) - L(x, v). \tag{3.5}$$

The Lagrangian L is used to define the action $\mathbb{L}(\gamma)$ of the curve $\gamma : [a, b] \to M$ by

$$\mathbb{L}(\gamma) = \int_a^b L(\gamma(s), \dot\gamma(s))\, ds.$$

The notion of minimizer and extremal for L are the same as in §**2** above. By the classical theory of Calculus of Variations, the curve $\gamma : [a, b] \to M$ is an extremal if and only if it satisfies, in local coordinates on M, the Euler-Lagrange equation

$$\frac{d}{dt}\left[\frac{\partial L}{\partial v}(\gamma(t), \dot\gamma(t))\right] = \frac{\partial L}{\partial x}(\gamma(t), \dot\gamma(t)). \tag{3.6}$$

If we carry out the derivation with respect to t in this last equation, we get

$$\frac{\partial^2 L}{\partial v^2}(\gamma(t), \dot\gamma(t))(\ddot\gamma(t), \cdot) = \frac{\partial L}{\partial x}(\gamma(t), \dot\gamma(t))(\cdot) - \frac{\partial^2 L}{\partial x \partial v}(\gamma(t), \dot\gamma(t))(\dot\gamma(t), \cdot),$$

where the dot $\cdot$ means that we consider maps on the linear space $T_x M$. Since L is Tonelli, the bilinear form $\partial^2 L/\partial v^2(\gamma(t), \dot\gamma(t))$ is invertible. Therefore we can solve for $\ddot\gamma(t)$, and obtain $\ddot\gamma(t) = X(\gamma(t), \dot\gamma(t))$, where X is a vector field $U \to \mathbb{R}^k$, with U is a coordinate patch in M, and $k = \dim(M)$. The solutions of this second order ODE are exactly the extremals, a concept which does not depend on the choice of a coordinate system. Hence, these local second order ODE's define a global second order ODE on M. Taking into account not only position, but also speed, it becomes a first order ODE on TM, which is called the Euler-Lagrange ODE, and its flow ϕ_t is called the Euler Lagrange flow. We give in the next proposition the well-known properties of the Euler Lagrange flow.

Proposition 3.1. *If $\gamma : [a, b] \to M$ is an extremal for the Lagrangian L, then its speed curve $t \mapsto (\gamma(t), \dot\gamma(t))$ is a piece of an orbit of the Euler-Lagrange flow ϕ_t, i.e., we have $(\gamma(t), \dot\gamma(t)) = \phi_{t-t_0}(\gamma(t_0), \dot\gamma(t_0))$, for all $t_0, t \in [a, b]$.*

Conversely, denoting by $\pi : TM \to M$ is the canonical projection, for every $(x, v) \in TM$, the curve $\gamma_{(x,v)}(t) = \pi\phi_t(x, v)$ is an extremal, whose speed curve satisfies $(\gamma_{(x,v)}(t), \dot\gamma_{(x,v)}(t)) = \phi_t(x, v)$.

We now come to the relation between the Euler-Lagrange flow and the Hamiltonian flow.

Proposition 3.2. *The Legendre transform $\mathcal{L} : TM \to T^*M$ is a conjugacy between the Euler-Lagrange flow ϕ_t and the Hamiltonian flow ϕ_t^*. This means that $\phi_t^* = \mathcal{L}\phi_t\mathcal{L}^{-1}$. In particular, the flow ϕ_t is complete, since this is the case for ϕ_t^*.*

An important property of Tonelli Lagrangians is existence and regularity of minimizers.

Theorem 3.3 (Tonelli [5, 7, 14, 21]). *Suppose that L is a C^r Tonelli Lagrangian on the compact manifold M. For every $x, y \in M$, for every $a, b \in \mathbb{R}$, with $a < b$, we can find a minimizer $\gamma : [a, b] \to M$, with $\gamma(a) = x$, and $\gamma(b) = y$. Moreover, any minimizer is automatically a C^r extremal. In particular, its speed curve is a piece of an orbit of the Euler-Lagrange flow.*

We now define $h_t(x, y)$ as the minimal action of a curve from x to y in the time $t > 0$.

$$h_t(x, y) = \inf_\gamma \int_a^b L(\gamma(s), \dot\gamma(s))\, ds, \tag{3.7}$$

where the infimum is taken over all piecewise C^1 curves $\gamma : [a, b] \to M$, with $b - a = t$, $\gamma(a) = x$, and $\gamma(b) = y$. Since L, in our setting, does not depend on time, the action of a curve $\gamma : [a, b] \to M$ is the same as the action of any of its shifted in time curves

$\gamma_\sigma : [a+\sigma, b+\sigma] \to M, \gamma_\sigma(s) = \gamma(s-\sigma)$, with $\sigma \in \mathbb{R}$. Therefore, if a_0, b_0 are fixed with $b_0 - a_0 = t$, to define $h_t(x,y)$ we could have taken the infimum over all curves $\gamma : [a_0, b_0] \to M$, with $\gamma(a_0) = x$, and $\gamma(b_0) = y$. Moreover, by Tonelli's theorem the infimum is always achieved on a C^r curve, if L is $C^r, r \geq 2$. Hence we could have restricted the curves to obtain the infimum to C^r curves (even to C^∞ by density, although the minimizer may not be C^∞ if the Lagrangian L is not C^∞).

Here are the elementary properties of h_t

Lemma 3.4. *If L is a C^r Tonelli Lagrangian, and d is a distance on M obtained from a Riemannian metric, we have:*

1) *For every x, y, and every $a, b \in \mathbb{R}$ with $b - a = t$, there exists a C^r minimizer $\gamma : [a,b] \to M$, with $\gamma(a) = x, \gamma(b) = y$ such that $h_t(x,y) = \int_a^b L(\gamma(s), \dot\gamma(s))\, ds$.*
2) *There exists a finite constant A such that $h_t(x,x) \leq At$, for every $x \in M$.*
3) *There exists a finite constant B such that $h_{d(x,y)}(x,y) \leq Bd(x,y)$, for every $x, y \in M$, with $x \neq y$.*
4) *For every $x, y \in M$, and every $t, t' > 0$, we have*

$$h_{t+t'}(x,y) = \inf_{z \in M} h_t(x,z) + h_{t'}(z,y).$$

5) *For every $K \geq 0$, we can find a finite constant $C(K)$ such that*

$$h_t(x,y) \geq Kd(x,y) + C(K)t, \text{ for all } x, y \in M.$$

Proof. Part 1) is a consequence of Tonelli's theorem 3.3 above.
To prove Part 2), we use the constant path $s \mapsto x$, to obtain

$$h_t(x,x) \leq tL(x,0) \leq At, \text{ with } A = \max_{x \in M} L(x,0).$$

We now prove part 3). By the compactness of M, we can find a geodesic $\gamma : [0, d(x,y)] \to M$ parametrized by arc-length (i.e. $\|\dot\gamma(s)\|_{\gamma(s)} = 1$ everywhere), with $\gamma(0) = x$, and $\gamma(d(x,y)) = y$. If we set

$$B = \sup\{L(x,v) \mid x \in M, v \in T_xM, \|v\|_x \leq 1\},$$

we see that $h_{d(x,y)}(x,y) \leq \mathbb{L}(\gamma) \leq Bd(x,y)$.
Part 4) follows from the fact that to go from x to y in time $t + t'$, we have first to go in time t to some point $z \in M$ then we go from z to y in time t', and, moreover, the action for the concatenated path is the sum of the action of the path from x to z and of the action of the path from z to y.
For part 5), fix $K \geq 0$. We first prove that there exists a finite constant $C(K)$ such that

$$L(x,v) \geq K\|v\|_x + C(K), \tag{3.8}$$

By the superlinearity of L, we know that $(L(x,v) - K\|v\|_x)/\|v\|_x$ tends uniformly to $+\infty$ as $\|v\|_x \to +\infty$. By the compactness of M, it follows that the constant $C(K) = \inf_{TM} L(x,v) - K\|v\|_x$ is finite. Therefore, the inequality (3.8) holds with this $C(K)$.

Given a curve $\gamma : [a, b] \to M$, if we apply (3.8) with $x = \gamma(s)$, and $v = \dot\gamma(s)$, and integrate on $[a, b]$, we obtain

$$\mathbb{L}(\gamma) \geq K \int_a^b \|\dot\gamma(s)\|_{\gamma(s)}\, ds + C(K)(b - a).$$

But the Riemannian length $\int_a^b \|\dot\gamma(s)\|_{\gamma(s)}\, ds$ of γ is $\geq d(\gamma(a), \gamma(b))$. Hence

$$\mathbb{L}(\gamma) \geq K d(\gamma(a), \gamma(b)) + C(K)(b - a).$$

Taking the infimum on all paths $\gamma : [0, t] \to M$, with $\gamma(0) = x, \gamma(t) = y$ finishes the proof of part 5). □

We now come to the most important property of h_t. This is what started weak KAM theory. This property has been discovered independently by many people. When the author himself discovered it back in 1996, he explained it to Michel Herman in his office in Paris. After hearing it, Michel Herman opened the drawer of his desk, got out a copy of the paper of W.H. Fleming [16] published in 1969, which contained an equivalent form of this statement. This is the oldest instance that the author knows of the following lemma.

Lemma 3.5 (Fleming, [16]). *For every $t_0 > 0$, the family of functions $h_t : M \times M \to \mathbb{R}$, $t \geq t_0$ is equi-Lipschitzian.*

For the proof of Fleming's lemma see §**8** below. In fact, more is true: the family $h_t : M \times M \to \mathbb{R}, t \geq t_0$ is equi-semiconcave. Again this fact has been well-known for sometime now in the theory of viscosity solutions [6]. For a proof, in our setting, of this extension of Fleming's lemma see the appendices of [15].

4. The Lax-Oleinik semi-group, and its fixed points

Rather than introducing the theory of viscosity solutions, we are going to give its evolution semi-group, i.e. the semi-group obtained by solving (in the viscosity sense) the equation

$$\frac{\partial U}{\partial t}(t, x) + H\big(x, \frac{\partial U}{\partial x}(t, x)\big) = 0,$$

on $[0, +\infty[\times M$ with given initial condition $u : M \to \mathbb{R}$, for $t = 0$. This semi-group T_t^-, called the Lax-Oleinik semi-group, acts on the space $\mathcal{C}^0(M, \mathbb{R})$ of real-valued continuous functions on M. It can be expressed directly in that case using the functions $h_t, t > 0$, by

$$T_t^- u(x) = \inf_{y \in M} u(y) + h_t(y, x). \tag{4.1}$$

This definition is valid for $t > 0$, of course T_0^- is the identity. Notice that by Fleming's lemma 3.5, not only is $T_t^- u$ continuous for $t > 0$, but for every $t_0 > 0$ the whole family $T_t^- u$, for $t \geq t_0, u \in \mathcal{C}^0(M, \mathbb{R})$ is equi-Lipschitzian. By the Ascoli-Arzelá theorem, this suggests that the image of T_t^- is "almost" relatively compact. In fact, to be able to apply the Ascoli-Arzelá theorem, we would also need uniform boundedness. This is not the case but as we will see, we can easily overcome this small difficulty.

We first give the properties of T_t^-.

Proposition 4.1. *The Lax-Oleinik satisfies the following properties:*

1) *For every $u \in \mathcal{C}^0(M,\mathbb{R})$, and every $c \in \mathbb{R}$, we have $T_t^-(c+u) = c + T_t^-(u)$, for all $t \in [0,+\infty[$.*
2) *For every $u, v \in \mathcal{C}^0(M,\mathbb{R})$, with $u \leq v$, we have $T_t^- u \leq T_t^- v$, for all $t \in [0,+\infty[$.*
3) *For every $u, v \in \mathcal{C}^0(M,\mathbb{R})$, we have $\|T_t^- u - T_t^- v\|_0 \leq \|u-v\|_0$, for all $t \in [0,+\infty[$, where $\|\cdot\|_0$ is the sup (or C^0) norm on $\mathcal{C}^0(M,\mathbb{R})$.*
4) *The family $T_t^-, t \geq 0$ is a semi-group, i.e. for every $u \in \mathcal{C}^0(M,\mathbb{R})$, we have $T_{t+t'}^- u = T_t^-[T_{t'}^-(u)]$.*
5) *For every given $u \in \mathcal{C}^0(M,\mathbb{R})$, the curve $t \mapsto T_t^- u$ is continuous for the sup norm topology on $\mathcal{C}^0(M,\mathbb{R})$.*
6) *For every $t_0 > 0$, the family $T_t^- u, t \geq t_0, u \in \mathcal{C}^0(M,\mathbb{R})$ is equi-Lipschitz.*

Proof. Parts 1) and 2) are obvious from the definition of the semi-group T_t^-.

To show part 3), we observe that $-\|u-v\|_0 + v \leq u \leq v + \|u-v\|_0$. Therefore using 2) and 1), we obtain $-\|u-v\|_0 + T_t^- v \leq T_t^- u \leq T_t^- v + \|u-v\|_0$, which implies part 3). Part 4) is a consequence of part 4) of Lemma 3.4.

It remains to prove part 5). We first consider the case where $u : M \to \mathbb{R}$ is Lipschitz. We prove that $\|T_t^- u - u\|_0 \to 0$, when $t \to 0$. By part 2) of Lemma 3.4 and the definition of T_t^-, we obtain

$$T_t^- u(x) \leq u(x) + h_t(x,x) \leq u(x) + At.$$

Hence $T_t^- u - u \leq At$. If we denote by K a Lipschitz constant for u, we have $u(y) + Kd(y,x) \geq u(x)$, combining with part 5) of Lemma 3.4, we get

$$u(y) + h_t(y,x) \geq u(y) + Kd(y,x) + C(K)t \geq u(x) + C(K)t.$$

Taking the infimum over $y \in M$, we conclude that $T_t^- u(x) \geq u(x) + C(K)t$. Hence $u - T_t^- u \leq -C(K)t$. Combining the two inequalities yields

$$\|T_t^- u - u\|_0 \leq t \max(A, -C(k)).$$

Therefore $\|T_t^- u - u\|_0 \to 0$, when $t \to 0$.

If $u \in \mathcal{C}^0(M,\mathbb{R})$, we can find a sequence of C^1 functions $u_n : M \to \mathbb{R}$ such that $\|u_n - u\|_0 \to 0$, as $n \to +\infty$. Since a C^1 function on the compact manifold M is Lipschitz, we have $\|T_t^- u_n - u_n\|_0 \to 0$, as $t \to 0$, for every n. Since $\|T_t^- u_n - T_t^- u\|_0 \leq \|u_n - u\|_0$, for every $t > 0$, it is not difficult to conclude that $\|T_t^- u - u\|_0 \to 0$, when $t \to 0$. To show the continuity of $t \to T_t^- u$ on $[0,+\infty[$, we use the semi-group property 4), and 2), to obtain that for $t' \geq t$, we have $\|T_{t'}^- u - T_t^- u\|_0 = \|T_t^-(T_{t'-t}^- u) - T_t^- u\|_0 \leq \|T_{t'-t}^- u - u\|_0$. This can be rewritten as $\|T_{t'}^- u - T_t^- u\|_0 \leq \|T_{|t'-t|}^- u - u\|_0$, which is valid also in the case $t \geq t'$. Therefore the continuity of $t \to T_t^- u$ at 0 implies the continuity on $[0,+\infty[$. □

We are now in a position to prove the existence of global weak solutions of the Hamilton-Jacobi equation.

Theorem 4.2 (Weak KAM Solution). *We can find $c \in \mathbb{R}$, and a function $u \in \mathcal{C}^0(M,\mathbb{R})$ such that $u = T_t^- u + ct$, for every $t > 0$. Necessarily u is Lipschitz, and $c = -\lim_{t\to+\infty} T_t^- v/t$, for every $v \in \mathcal{C}^0(M,\mathbb{R})$.*

Proof. We define $\mathrm{Lip}_K(M,\mathbb{R})$ as the subset of Lipschitz functions in $\mathcal{C}^0(M,\mathbb{R})$ with Lipschitz constant $\leq K$. This subset is closed and convex in $\mathcal{C}^0(M,\mathbb{R})$. Moreover, if we fix a base point $x_0 \in M$, by the Arzelà-Ascoli theorem, the closed convex subset $\mathrm{Lip}_K^{x_0}(M,\mathbb{R}) = \{u \in \mathrm{Lip}_K(M,\mathbb{R}) \mid u(x_0) = 0\}$ is compact. Fix $t_0 > 0$, by part 6) of Proposition 4.1, there exists a constant $K(t_0)$ such that for every $t \geq t_0$, the image of T_t^- is contained in $\mathrm{Lip}_{K(t_0)}(M,\mathbb{R})$. Therefore, for $t \geq t_0$, we can define the continuous non-linear operator $\hat{T}_t^- : \mathcal{C}^0(M,\mathbb{R}) \to \mathrm{Lip}_{K(t_0)}^{x_0}(M,\mathbb{R})$ by $u \mapsto T_t^- u - T_t^- u(x_0)$. Since $\hat{T}_t^-$ sends the compact convex subset $\mathrm{Lip}_{K(t_0)}^{x_0}(M,\mathbb{R})$ to itself, by the Schauder-Tykhonov theorem [10, Theorem 2.2, pages 414-415], the map $\hat{T}_t^-$ has a fixed point. We now show that we can find a common fixed point for the family $\hat{T}_t^-, t > 0$. We first note that $\hat{T}_t^-, t > 0$, is a semi-group. In fact $\hat{T}_{t'}^-(\hat{T}_t^- u) = T_{t'}^- \hat{T}_t^- u - T_{t'}^- \hat{T}_t^- u(x_0)$. Since

$$\begin{aligned} T_{t'}^- \hat{T}_t^- u &= T_{t'}^-(T_t^- u - T_t^- u(x_0)) = T_{t'}^-(T_t^- u) - T_t^- u(x_0) \\ &= T_{t'+t}^- u - T_t^- u(x_0), \end{aligned}$$

we obtain

$$\begin{aligned} \hat{T}_{t'}^- \hat{T}_t^- u &= T_{t'+t}^- u - T_t^- u(x_0) - [T_{t'+t}^- u(x_0) - T_t^- u(x_0)] = T_{t'+t}^- u - T_{t'+t}^- u(x_0) \\ &= \hat{T}_{t'+t}^- u. \end{aligned}$$

This semi-group property implies that $\mathrm{Fix}(\hat{T}_{1/2^{n+1}}^-) \subset \mathrm{Fix}(\hat{T}_{1/2^n}^-)$, for every integer $n \geq 1$, where $\mathrm{Fix}(\hat{T}_t^-)$ is the set of fixed points of $\hat{T}_t^-$ in $\mathcal{C}^0(M,\mathbb{R})$. Since, for $t > 0$, the non-empty set $\mathrm{Fix}(\hat{T}_t^-)$ is closed and contained in $\mathrm{Lip}_{K(t)}^{x_0}(M,\mathbb{R})$, it is compact. Therefore the non-increasing sequence $\mathrm{Fix}(\hat{T}_{1/2^n}^-), n \geq 1$, has a non-empty intersection. If u is in this intersection, it is fixed by every $\hat{T}_{1/2^n}^-$. By the semi-group property we obtain $u = \hat{T}_t^- u$, for every t in the dense set of rational numbers of the form $p/2^n, p \in \mathbb{N}, n \geq 1$. But $t \mapsto \hat{T}_t^- u$ is continuous by part 5) of Proposition 4.1. Hence $u = \hat{T}_t^- u$, for every $t \geq 0$. Therefore, we obtained a $u \in \mathcal{C}^0(M,\mathbb{R})$ such that $u = T_t^- u + c_t$, for every $t \geq 0$, where $c_t = -T_t^- u(x_0) \in \mathbb{R}$. Since

$$T_{t'}^- u = T_{t'}^-[T_t^- u + c_t] = T_{t'}^- T_t^- u + c_t = T_{t'+t}^- u + c_t,$$

we infer

$$u = T_{t'}^- u + c_{t'} = T_{t'+t}^- u + c_t + c_{t'}.$$

This implies that $c_{t'+t} = c_{t'} + c_t$. The continuity of $t \mapsto c_t = u - T_t^- u$ implies $c_t = tc$, where $c = c_1$. This finishes the proof of the existence of u and c.

Note that u is necessarily Lipschitz, since $T_t^- u$ is Lipschitz for $t > 0$. To prove the last claim of the theorem on c, we first observe that $T_t^- u/t = u/t - c$. Since the function u is bounded on the compact set M, we do get $\lim_{t\to+\infty} T_t^- u/t = -c$. By part 3) of Proposition 4.1, for any $v \in \mathcal{C}^0(M,\mathbb{R})$, we have $\|T_t^- v/t - T_t^- u/t\|_0 \leq \|v - u\|_0/t \to 0$, as $t \to +\infty$. □

Definition 4.3 (Critical value). We will denote by $c(H)$, or $c(L)$ the only constant c for which we can find a weak KAM solution, i.e. the only constant c for which we can find a function $u : M \to \mathbb{R}$, with $u = T_t^- u + c$, for every $t > 0$. This constant is called the Mañé critical value.

5. Domination and calibration

The proof of the following proposition is straightforward from the definitions.

Proposition 5.1 (Characterization of subsolutions). *Let $u : M \to \mathbb{R}$ be a function, and $c \in \mathbb{R}$. The following are equivalent*

1) *for every $t > 0$, we have $u \leq T_t u + ct$;*
2) *for every $t > 0$, and every $x, y \in M$, we have $u(y) - u(x) \leq h_t(x, y) + ct$;*
3) *for every continuous, piecewise C^1 curve $\gamma : [a, b] \to M$, we have*

$$u(\gamma(b)) - u(\gamma(a)) \leq \int_a^b L(\gamma(s), \dot\gamma(s))\, ds + c(b - a). \tag{5.1}$$

It is convenient to introduce the following definition.

Definition 5.2 (Domination). If $u : M \to \mathbb{R}$ is a function, and $c \in \mathbb{R}$, we say that u is dominated by $L + c$, which we denote by $u \prec L + c$, if it satisfies inequality (5.1) above, for every piecewise C^1 curve $\gamma : [a, b] \to M$.

Lemma 5.3. *There is a constant B such that any function $u : M \to \mathbb{R}$, dominated by $L + c$, is Lipschitz with Lipschitz constant $\leq B + c$.*

Proof. By the domination condition $u(y) - u(x) \leq h_{d(x,y)}(x, y) + cd(x, y)$. Therefore, we obtain $u(y) - u(x) \leq (B + c)d(x, y)$, with B given by part 3) of Lemma 3.4. □

Recall that by Rademacher's theorem, Lipschitz functions are differentiable a.e.

Lemma 5.4. *Let $u : M \to \mathbb{R}$ be dominated by $L + c$. If the derivative $d_x u$ exists at some given $x \in M$, then $H(x, d_x u) \leq c$. In particular, the function u is an almost everywhere subsolution of the Hamilton-Jacobi equation $H(x, d_x u) = c$.*

Proof. Suppose $d_x u$ exists at $x \in M$. For a given $v \in T_x M$, let $\gamma : [0, 1] \to M$ be a C^1 curve with $\gamma(0) = x, \dot\gamma(0) = v$. Applying (5.1) to the curve $\gamma|[0, t]$, for every $t \in [0, 1]$, we obtain

$$u(\gamma(t)) - u(\gamma(0)) \leq \int_0^t L(\gamma(s), \dot\gamma(s))\, ds + ct.$$

Dividing by $t > 0$ and letting $t \to 0$, we get $d_{\gamma(0)} u(\dot\gamma(0)) \leq L(\gamma(0), \dot\gamma(0)) + c$. By the choice of γ, we conclude that $d_x u(v) - L(x, v) \leq c$. But $H(x, d_x u) = \sup_{v \in T_x M} d_x u(v) - L(x, v)$. Hence $H(x, d_x u) \leq c$. □

It should not come as a surprise that curves satisfying the equality in (5.1) enjoy special properties. It is convenient to give them a name.

Definition 5.5 (Calibrated curve). Suppose that $u : M \to \mathbb{R}$ is dominated by $L + c$. A curve $\gamma : [a, b] \to M$ is said to be (u, L, c)-calibrated if

$$u(\gamma(b)) - u(\gamma(a)) = \int_a^b L(\gamma(s), \dot\gamma(s))\, ds + c(b - a).$$

Recall that for $t \in \mathbb{R}$, and $\gamma : [a, b] \to M$, the curve $\gamma_t : [a + t, b + t] \to M$ is defined by $\gamma_t(s) = \gamma(s - t)$. Since L does not depend on time, we have $\mathbb{L}(\gamma_t) = \mathbb{L}(\gamma)$.

Proposition 5.6. *If $u \prec L + c$, then any (u, L, c)-calibrated curve $\gamma : [a, b] \to M$ is a minimizer. In particular, it is as smooth as L. Moreover, for every $[a', b'] \subset [a, b]$, the restriction $\gamma|[a', b']$ is also (u, L, c)-calibrated, and so is the curve γ_t for all $t \in \mathbb{R}$.*

Proof. If $\delta : [a, b] \to M$ is a curve with $\delta(a) = \gamma(a), \delta(b) = \gamma(b)$, we have

$$\mathbb{L}(\gamma) + c(b - a) = u(\gamma(b)) - u(\gamma(a)) = u(\delta(b)) - u(\delta(a)) \leq \mathbb{L}(\delta) + c(b - a).$$

Therefore $\mathbb{L}(\gamma) \leq \mathbb{L}(\delta)$, and γ is a minimizer. The regularity of γ is given by Tonelli's theorem.

We next use the domination $u \prec L + c$ to obtain

$$\begin{aligned} u(\gamma(a')) - u(\gamma(a)) &\leq \int_a^{a'} L(\gamma(s), \dot\gamma(s))\, ds + c(a' - a) \\ u(\gamma(b')) - u(\gamma(a')) &\leq \int_{a'}^{b'} L(\gamma(s), \dot\gamma(s))\, ds + c(b' - a') \\ u(\gamma(b)) - u(\gamma(b')) &\leq \int_{b'}^{b} L(\gamma(s), \dot\gamma(s))\, ds + c(b - b'). \end{aligned} \tag{5.2}$$

If we add these three inequalities, we obtain

$$u(\gamma(b)) - u(\gamma(a)) \leq \int_a^b L(\gamma(s), \dot\gamma(s))\, ds + c(b - a),$$

which is an equality. Therefore the three inequalities in (5.2) are equalities. The middle equality means that $\gamma|[a', b']$ is (u, L, c)-calibrated. The last part follows from $\gamma_t(a + t) = \gamma(a)$, $\gamma_t(b + t) = \gamma(b)$, and $\mathbb{L}(\gamma_t) = \mathbb{L}(\gamma)$. □

We now extend the notion of calibration to non-compact curves. For a curve $\gamma : I \to M$ defined on the not-necessarily compact interval $I \subset \mathbb{R}$, we say that γ is (u, L, c)-calibrated if the restriction $\gamma|[a, b]$ is (u, L, c)-calibrated for every compact subinterval $[a, b] \subset \mathbb{R}$. By Proposition 5.6 above this definition coincides with Definition 5.5 when I is compact.

Although a dominated function is differentiable almost everywhere, it might not be obvious to explicitly find a point where the derivative exists. The following lemma provides such points.

Lemma 5.7. *Assume that $u : M \to \mathbb{R}$ is $L + c$ dominated, and let $\gamma : [a, b] \to M$ be (u, L, c)-calibrated. We have:*

1) *If $d_{\gamma(t)}u$ exists at some $t \in [a, b]$, then*

$$H(\gamma(t), d_{\gamma(t)}u) = c, \text{ and } d_{\gamma(t)}u = \partial L/\partial v(\gamma(t), \dot\gamma(t)).$$

2) *If $t \in]a, b[$, then the derivative $d_{\gamma(t)}u$ does indeed exist.*

Proof. We prove 1) for $t \in [a, b[$. The argument can be slightly modified to obtain the proof for $t \in]a, b]$. By Proposition 5.6, for $t + \epsilon \leq b$, we have

$$u(\gamma(t+\epsilon)) - u(\gamma(t)) = \int_t^{t+\epsilon} L(\gamma(s), \dot{\gamma}(s))\, ds + c\epsilon.$$

Dividing by $\epsilon > 0$ and letting $\epsilon \to 0$, we obtain $d_{\gamma(t)}u(\dot{\gamma}(t)) = L(\gamma(t), \dot{\gamma}(t)) + c$. By (3.5), this implies $H(\gamma(t), d_{\gamma(t)}u) \geq d_{\gamma(t)}u(\dot{\gamma}(t)) - L(\gamma(t), \dot{\gamma}(t)) = c$. But by Lemma 5.4, we also know that $H(\gamma(t), d_{\gamma(t)}u) \leq c$. Therefore, we get $c = H(\gamma(t), d_{\gamma(t)}u) = d_{\gamma(t)}u(\dot{\gamma}(t)) - L(\gamma(t), \dot{\gamma}(t))$. This proves the first part of 1), but also the second one because the last equality shows that we have equality in the Fenchel inequality $H(\gamma(t), d_{\gamma(t)}u) + L(\gamma(t), \dot{\gamma}(t)) \geq d_{\gamma(t)}u(\dot{\gamma}(t))$.

To prove part 2), we will construct two C^1 functions $\psi, \theta : V \to \mathbb{R}$, defined on the neighborhood V of $x = \gamma(t)$, and such that

$$\psi(y) \leq u(y) - u(x) \leq \theta(y),$$

on V, with equality at x. We leave it to the reader to show that $d_x\theta = d_x\psi$, and that this common derivative is also the derivative of u at x. We will construct θ, since the argument for ψ is analogous. Let us first choose a domain U of a smooth chart $\varphi : U \to \mathbb{R}^k$ of the manifold M, with $x = \gamma(t) \in U$, we can find $a' < t < b'$ such that $\gamma([a', b']) \subset U$. To simplify notations we use φ to identify U with its image in $\mathbb{R}^k$. For y close enough to x the path $\gamma_y : [a', t] \to \mathbb{R}^k$ defined by

$$\gamma_y(s) = \gamma(s) + \frac{s - a'}{t - a'}(y - x),$$

will have an image contained in U, and therefore can be considered as a path in M. Note that γ_y starts at $\gamma(a')$, and ends at y. Hence by $u \prec L + c$, we obtain

$$u(y) - u(\gamma(a')) \leq \int_{a'}^t L(\gamma_y(s), \dot{\gamma}_y(s))\, ds + c(t - a')$$

Moreover for $y = x$, we have $\gamma_x = \gamma$, and the inequality above is an equality. Therefore subtracting the equality at x from the inequality at y, we get

$$u(y) - u(x) \leq \int_{a'}^t L(\gamma_y(s), \dot{\gamma}_y(s)) - L(\gamma(s), \dot{\gamma}(s))\, ds.$$

We can now define $\theta(y)$ for y close to x by

$$\begin{aligned}\theta(y) &= \int_{a'}^t L(\gamma_y(s), \dot{\gamma}_y(s)) - L(\gamma(s), \dot{\gamma}(s))\, ds \\ &= \int_{a'}^t L\left(\gamma(s) + \frac{s - a}{t - a}(y - x), \dot{\gamma}(s) + \frac{1}{t - a}(y - x)\right) - L(\gamma(s), \dot{\gamma}(s))\, ds.\end{aligned}$$

From the last expression, it is clear that θ is as smooth as L. Moreover, we have $u(y) - u(x) \leq \theta(y)$, and $\theta(x) = 0$ as required. □

For $(x,v) \in TM$, let us recall that $\gamma_{(x,v)}$ is the curve defined by $\gamma_{(x,v)}(t) = \pi\phi_t(x,v)$, see Proposition 3.1. It satisfies $(\gamma_{(x,v)}(t), \dot\gamma_{(x,v)}(t)) = \phi_t(x,v)$.

If $u : M \to \mathbb{R}$ is dominated by $L + c$, for $a, b \in \mathbb{R}$, with $a < b$, we define the sets $\tilde{\mathcal{G}}_{a,b}(u)$, and $\tilde{\mathcal{G}}_b(u)$ by

$$\begin{aligned} \tilde{\mathcal{G}}_{a,b}(u) &= \{(x,v) \in TM \mid \gamma_{(x,v)} \text{ is } (u,L,c)\text{-calibrated on } [a,b]\}, \\ \tilde{\mathcal{G}}_b(u) &= \{(x,v) \in TM \mid \gamma_{(x,v)} \text{ is } (u,L,c)\text{-calibrated on }]-\infty,b]\}. \end{aligned} \tag{5.3}$$

Of course, the sets $\tilde{\mathcal{G}}_{a,b}(u)$ depend not only on u, but also on c, and a better notation would be $\tilde{\mathcal{G}}_{a,b}(u,c)$. However, we will only use them later with u a weak KAM solution, and $c = c(H)$.

Proposition 5.8. *Suppose $u : M \to \mathbb{R}$ is dominated by $L+c$. Given $a < b$, the set $\tilde{\mathcal{G}}_{a,b}(u)$ is compact. Moreover, any (u,L,c)-calibrated curve $\gamma : [a,b] \to M$ is of the form $\gamma_{(x,v)}|[a,b]$, for some $(x,v) \in \tilde{\mathcal{G}}_{a,b}(u)$.*

Proof. We first observe that $\tilde{\mathcal{G}}_{a,b}(u)$ is closed in TM. We have $(x,v) \in \tilde{\mathcal{G}}_{a,b}(u)$ if and only if

$$u \circ \pi(\phi_b(x,v)) - u \circ \pi(\phi_a(x,v)) = \int_a^b L\phi_s(x,v)\, ds + c(b-a).$$

It follows that $\tilde{\mathcal{G}}_{a,b}(u)$ is closed in TM, since both sides of the equality above are continuous as functions of $(x,v) \in TM$.

We now prove the compactness of $\tilde{\mathcal{G}}_{a,b}(u)$. By part 1) of Proposition 5.6, we know that $\gamma_{(x,v)}|[a,b]$ is a minimizer. Therefore by Lemma 8.1, we can find a finite constant κ_{b-a} such that $\|\dot\gamma_{(x,v)}(a)\|_{\gamma_{(x,v)}(a)} \leq \kappa_{b-a}$, for every $(x,v) \in \tilde{\mathcal{G}}_{a,b}(u)$. Since $(x,v) = \phi_{-a}(\gamma_{(x,v)}(a), \dot\gamma_{(x,v)}(a))$, the compactness of $\tilde{\mathcal{G}}_{a,b}(u)$ follows.

If $\gamma : [a,b] \to M$ is (u,L,c)-calibrated it is a minimizer. Hence its speed curve satisfies $(\gamma(t), \dot\gamma(t)) = \phi_{t-a}(\gamma(a), \dot\gamma(a))$. Therefore $\gamma = \gamma_{(x,v)}|[a,b]$, where $(x,v) = \phi_{-a}(\gamma(a), \dot\gamma(a))$. □

Proposition 5.9. *Suppose $u \prec L + c$, for $a' \leq a \leq b \leq b'$, and $t \in \mathbb{R}$, we have*

1) $\tilde{\mathcal{G}}_b(u) = \cap_{a<b} \tilde{\mathcal{G}}_{a,b}(u)$, $\tilde{\mathcal{G}}_{a',b'}(u) \subset \tilde{\mathcal{G}}_{a,b}(u)$, *and* $\tilde{\mathcal{G}}_{b'}(u) \subset \tilde{\mathcal{G}}_b(u)$.
2) $\phi_{-t}\tilde{\mathcal{G}}_{a,b}(u) = \tilde{\mathcal{G}}_{a+t,b+t}(u)$, *and* $\phi_{-t}\tilde{\mathcal{G}}_b(u) = \tilde{\mathcal{G}}_{b+t}(u)$.
3) $\mathcal{L}[\tilde{\mathcal{G}}_b(u)] \subset \mathrm{Graph}\,(du)$, *for* $b > 0$.
4) $H \circ \mathcal{L}[\tilde{\mathcal{G}}_0(u)] = c$, *and* $\mathcal{L}[\tilde{\mathcal{G}}_0(u)] \subset \overline{\mathrm{Graph}\,(du)}$, *where*

$$\mathrm{Graph}\,(du) = \{(x, d_x u) \mid \textit{for } x \in M \textit{ at which } d_x u \textit{ exists}\}. \tag{5.4}$$

Proof. A curve $\gamma :]-\infty, b] \to M$ is (u,L,c)-calibrated if and only if its restriction to any compact interval $[a,b], a < b$ is (u,L,c)-calibrated. This proves the first equality. The inclusions follow from Proposition 5.6. We prove the equality $\phi_{-t}\tilde{\mathcal{G}}_{a,b}(u) = \tilde{\mathcal{G}}_{a+t,b+t}(u)$. The equality $\phi_{-t}\tilde{\mathcal{G}}_b(u) = \tilde{\mathcal{G}}_{b+t}(u)$ follows from this last one by taking intersections over $a < b$. We have $(x,v) \in \phi_{-t}\tilde{\mathcal{G}}_{a,b}(u)$ if and only if $\phi_t(x,v) \in \tilde{\mathcal{G}}_{a,b}(u)$. This is equivalent to $\gamma_{\phi_t(x,v)}$ is (u,L,c)-calibrated on $[a,b]$. This last condition is equivalent to $\gamma_{(x,v)}$ is (u,L,c)-calibrated on $[a+t, b+t]$, since $\gamma_{\phi_t(x,v)}(s) = \gamma_{(x,v)}(s+t)$. Hence $(x,v) \in \phi_{-t}\tilde{\mathcal{G}}_{a,b}(u)$ if and only if $(x,v) \in \tilde{\mathcal{G}}_{a+t,b+t}(u)$.

We now prove parts 3) and 4). If $(x,v) \in \tilde{\mathcal{G}}_b(u), b > 0$, by Lemma 5.7, since $(x,v) = (\gamma_{(x,v)}(0), \dot{\gamma}_{(x,v)}(0))$, the function u has a derivative at $\gamma_{(x,v)}(0) = x$, which satisfies

$$H(x, d_x u) = c, \text{ and } d_x u = \partial L/\partial v(x,v).$$

Hence $\mathcal{L}(x,v) = (x, d_x u) \in \text{Graph}\,(du)$, and $H \circ \mathcal{L}(x,v) = c$. To finish the proof, we note that $\phi_{-b}\tilde{\mathcal{G}}_0(u) = \tilde{\mathcal{G}}_b(u)$, for $b > 0$. Therefore $\mathcal{L}[\phi_{-b}\tilde{\mathcal{G}}_0] \subset \text{Graph}\,(du)$, and $H\mathcal{L}[\phi_{-b}\tilde{\mathcal{G}}_0] = c$. If we let $b \to 0$, we obtain $\mathcal{L}[\tilde{\mathcal{G}}_0(u)] \subset \overline{\text{Graph}\,(du)}$. □

Proposition 5.10. *Let $u : M \to \mathbb{R}$ be a weak KAM solution, then*

$$\pi(\tilde{\mathcal{G}}_0(u)) = M, \text{ and } \mathcal{L}[\tilde{\mathcal{G}}_0(u)] = \overline{\text{Graph}\,(du)}.$$

Moreover $H(x, d_x u) = c(H)$ at every point $x \in M$ where $d_x u$ exists.

Proof. We first prove that $\pi^{-1}(x) \cap \tilde{\mathcal{G}}_0(u)$ is not empty, for every $x \in M$. Since $\tilde{\mathcal{G}}_0(u)$ is the decreasing intersection of the compact sets $\tilde{\mathcal{G}}_{[-t,0]}(u), t > 0$, see Proposition 5.9, it suffices to show that for a given $x \in M$, and a given $t > 0$, we have $\pi^{-1}(x) \cap \tilde{\mathcal{G}}_{[-t,0]}(u) \neq \emptyset$. Since, the function u is a weak KAM solution, we have $u(x) = \inf_{y \in M} u(y) + h_t(y,x) + c(H)t$. By the compactness of M and the continuity of both u and h_t, we can find $y \in M$ such that $u(x) = u(y) + h_t(y,x) + c(H)t$. By part 1) of Lemma 3.4, we can find $\gamma : [-t, 0] \to M$, with $\gamma(-t) = y, \gamma(0) = x$, and

$$h_t(y,x) = \int_{-t}^{0} L(\gamma(s), \dot{\gamma}(s))\, ds.$$

Hence

$$u(\gamma(0)) - u(\gamma(-t)) = \int_{-t}^{0} L(\gamma(s), \dot{\gamma}(s))\, ds + c(H)t,$$

and γ is $(u, L, c(H))$-calibrated. This implies $(x, \dot{\gamma}(0)) = (\gamma(0), \dot{\gamma}(0)) \in \tilde{\mathcal{G}}_{[-t,0]}(u)$. Therefore $\pi^{-1}(x) \cap \tilde{\mathcal{G}}_{[-t,0]}(u) \neq \emptyset$, as was to be shown.

From the previous Proposition 5.9, we already know that the compact set $\mathcal{L}[\tilde{\mathcal{G}}_0(u)]$ is contained in the closure $\overline{\text{Graph}\,(du)}$. To finish the proof, it suffices to show that $(x, d_x u) \in \mathcal{L}[\tilde{\mathcal{G}}_0(u)]$, for every x at which $d_x u$ exists. Fix such an x. By the first part of the proposition, we can find $v \in T_x M$ with $(x,v) \in \tilde{\mathcal{G}}_0(u)$. Therefore, the curve $\gamma_{(x,v)}$ is (u, L, c)-calibrated on $]-\infty, 0]$, with $(\gamma_{(x,v)}(0), \dot{\gamma}_{(x,v)}(0)) = (x,v)$. By Lemma 5.7, we have $(x, d_x u) = \mathcal{L}(x,v)$. □

Corollary 5.11. *A C^1 weak KAM solution is a solution of the Hamilton-Jacobi equation $H(x, d_x u) = c(H)$.*

We will prove the converse of Corollary 5.11 in §**9**.

6. The weak Hamilton-Jacobi theorem

Theorem 6.1 (Weak Hamilton-Jacobi theorem). *If $u : M \to \mathbb{R}$ is a weak KAM solution, then*

$$\phi_{-t}^*(\overline{\text{Graph}(du)}) \subset \text{Graph}\,(du),$$

for every $t > 0$. Therefore the intersection

$$\tilde{\mathcal{I}}^*(u) = \cap_{t\geq 0}\phi^*_{-t}[\mathrm{Graph}(du)] = \cap_{t\geq 0}\phi^*_{-t}[\overline{\mathrm{Graph}(du)}] \tag{6.1}$$

is a non-empty compact ϕ^*_t-invariant set, contained in $\mathrm{Graph}(du)$*. This implies that $\tilde{\mathcal{I}}^*(u)$ is a (partial) graph on the base M.*

If we set $\tilde{\mathcal{I}}(u) = \mathcal{L}^{-1}[\tilde{\mathcal{I}}^*(u)]$, then this last set is non-empty, compact, ϕ_t-invariant, and is also a graph on the base M. Moreover, we have

$$\tilde{\mathcal{I}}(u) = \{(x,v) \in TM \mid \gamma_{(x,v)} \text{ is } (u, L, c(H))\text{-calibrated on }]-\infty, +\infty[\}. \tag{6.2}$$

Both sets $\tilde{\mathcal{I}}(u), \tilde{\mathcal{I}}^*(u)$ are called the Aubry set of the weak KAM solution u.

Proof. By Propositions 5.9 and 5.10, for $t > 0$, we know that $\phi_{-t}\tilde{\mathcal{G}}_0 = \tilde{\mathcal{G}}_t$ is decreasing, $\mathcal{L}[\tilde{\mathcal{G}}_0(u)] = \overline{\mathrm{Graph}\,(du)}$, and $\tilde{\mathcal{G}}_t \subset \mathrm{Graph}(du)$. Since the diffeomorphism $\mathcal{L}$ conjugates ϕ_t and ϕ^*_t, we obtain $\phi^*_{-t}(\overline{\mathrm{Graph}(du)}) \subset \mathrm{Graph}\,(du)$, for every $t > 0$. This implies (6.1). The non-emptiness follows from the fact that one of the intersections in (6.1) is a decreasing intersection of compact sets. The graph property follows from the inclusion $\tilde{\mathcal{I}}^*(u) \subset \mathrm{Graph}\,(du)$.

To prove (6.2), using again the conjugacy property of $\mathcal{L}$, we obtain that $\tilde{\mathcal{I}}(u)$ is the decreasing intersection of $\phi_{-t}(\mathcal{L}^{-1}[\overline{\mathrm{Graph}(du)}]) = \phi_{-t}\tilde{\mathcal{G}}_0 = \tilde{\mathcal{G}}_t, t > 0$. Hence a point (x,v) is in $\tilde{\mathcal{I}}(u)$ if and only if it is in $\tilde{\mathcal{G}}_t(u)$, for every $t > 0$. By definition of $\tilde{\mathcal{G}}_t(u)$, this means that $\gamma_{(x,v)}$ is $(u, L, c(H))$-calibrated on $]-\infty, t]$, for every $t > 0$, or equivalently $\gamma_{(x,v)}$ is $(u, L, c(H))$-calibrated on $]-\infty, +\infty[$. □

7. Mather measures, Aubry and Mather sets

Let $\tilde{\mu}$ be a Borel probability measure on TM. Since L is bounded below the integral $\int_{TM} L\, d\tilde{\mu} \in \mathbb{R} \cup \{+\infty\}$ always makes sense. Moreover, if $u : M \to \mathbb{R}$ is a continuous function then $u \circ \pi$ is continuous bounded on TM, therefore $u \circ \pi$ is $\tilde{\mu}$-integrable.

Theorem 7.1. *Suppose that $\tilde{\mu}$ is a Borel probability measure on TM which is invariant under the Euler-Lagrange flow ϕ_t, then*

$$\int_{TM} L\, d\tilde{\mu} \geq -c(H).$$

Moreover, there are such invariant measures $\tilde{\mu}$ which realize the equality.

In fact, if $u : M \to \mathbb{R}$ is a weak KAM solution then an invariant measure $\tilde{\mu}$ satisfies $\int_{TM} L\, d\tilde{\mu} = -c(H)$ if and only if the support $\mathrm{supp}(\tilde{\mu})$ *of $\tilde{\mu}$ is contained in the Aubry set $\tilde{\mathcal{I}}(u)$ of u.*

Proof. If L is not $\tilde{\mu}$ integrable then $\int_{TM} L\, d\tilde{\mu} = +\infty$, and there is nothing to prove. Therefore we can assume that L is integrable for $\tilde{\mu}$.

Fix a weak KAM solution u. For $(x,v) \in TM$, expressing the domination condition $u \prec L + c(H)$ along the curve $\gamma_{(x,v)}(s) = \pi\phi_s(x,v)$ yields

$$u \circ \pi(\phi_{t'}(x,v)) - u \circ \pi(\phi_t(x,v)) \leq \int_t^{t'} L(\phi_s(x,v))\, ds + c(H)(t'-t), \tag{7.1}$$

for all $t, t' \in \mathbb{R}$, with $t \leq t'$, and all $(x, v) \in TM$. If we integrate this inequality with respect to the measure $\tilde{\mu}$, we obtain

$$\int_{TM} u\pi\phi_{t'}\, d\tilde{\mu} - \int_{TM} u\pi\phi_t\, d\tilde{\mu} \leq \int_{TM} \int_t^{t'} L\phi_s\, ds\, d\tilde{\mu} + c(H)(t' - t).$$

By the ϕ_s-invariance of $\tilde{\mu}$, the left hand side above is 0. Moreover, using Fubini theorem together with the ϕ_s-invariance on the right hand side, we find that the inequality above is

$$0 \leq (t' - t) \int_{TM} L\, d\tilde{\mu} + c(H)(t' - t). \tag{7.2}$$

This of course implies $\int_{TM} L\, d\tilde{\mu} \geq -c(H)$.

We have $\int_{TM} L\, d\tilde{\mu} = -c(H)$, if and only if (7.2) is an equality. But this last inequality was obtained by integration of (7.1), therefore (7.2) is an equality if and only if (7.1) is an equality for $\tilde{\mu}$-almost every $(x, v) \in TM$. Since both sides of (7.1) are continuous in (x, v), we conclude that $\int_{TM} L\, d\tilde{\mu} = -c(H)$ if and only if (7.1) is an equality on the support on $\mathrm{supp}(\tilde{\mu})$. By (6.2) this last condition is equivalent to $\mathrm{supp}(\tilde{\mu}) \subset \tilde{\mathcal{I}}(u)$.

Since the compact set $\tilde{\mathcal{I}}(u)$ is non-empty and invariant by the flow, we can find an invariant measure $\tilde{\mu}$ with $\mathrm{supp}(\tilde{\mu}) \subset \tilde{\mathcal{I}}(u)$. Therefore $\int_{TM} L\, d\tilde{\mu} = -c(H)$. □

Definition 7.2 (Mather measures, Mather set). A Mather measure (for the Lagrangian L) is a Borel probability ϕ_s-invariant measure $\tilde{\mu}$ satisfying $\int_{TM} L\, d\tilde{\mu} = -c(H)$. The Mather set $\tilde{\mathcal{M}}$ (of the Lagrangian L) is the closure of $\bigcup_{\tilde{\mu}} \mathrm{supp}\, \tilde{\mu}$, where the union is taken over all Mather measures $\tilde{\mu}$.

By Theorem 7.1, the Mather set is not empty. The Aubry set $\tilde{\mathcal{I}}(u)$ depends on the choice of the weak KAM solution. The way to make it independent of choice is the following definition.

Definition 7.3 (Aubry set). The Aubry set $\tilde{\mathcal{A}}$ of the Lagrangian L (resp. $\tilde{\mathcal{A}}^*$ of the Hamiltonian H) is $\bigcap_u \tilde{\mathcal{I}}(u)$ (resp. $\bigcap_u \tilde{\mathcal{I}}^*(u)$), where the intersection is taken over all weak KAM solutions $u : M \to \mathbb{R}$.

Note that we use here the notation $\tilde{\mathcal{A}}^*$ instead of the notation $\tilde{\mathcal{A}}^*(0)$ used in the Introduction §**1**.

Corollary 7.4. *The Aubry sets $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{A}}^*$ are not empty. In fact, we have $\tilde{\mathcal{M}} \subset \tilde{\mathcal{A}}$, and $\mathcal{L}(\tilde{\mathcal{A}}) = \tilde{\mathcal{A}}^*$. Both the Mather set and the Aubry sets are graphs on the base M, since $\tilde{\mathcal{A}}^* \subset \mathrm{Graph}(du)$, for any weak KAM solution $u : M \to \mathbb{R}$.*

The results obtained in this section finishes the proof of Theorem 1.5 for the case $P = 0$. As explained in the introduction, the case for a a general $P \in \mathbb{R}^k$ follows from this one.

8. Proof of Fleming's lemma

It will be helpful to consider the energy $E : TM \to \mathbb{R}$, defined by $E = H \circ \mathcal{L}$. Since H is superlinear, and $\mathcal{L}$ is a homeomorphism, for every $K \in \mathbb{R}$, the set $\{(x, v) \mid E(x, v) \leq K\}$ is compact. Moreover, since $\mathcal{L}$ conjugates the Lagrangian flow ϕ_t to the Hamiltonian ϕ_t^*, the energy is constant along speed curves of extremals.

Lemma 8.1. *Given $t_0 > 0$, there exists a finite constant κ_{t_0}, such that every minimizer $\gamma : [a, b] \to M$, with $b - a \geq t_0$, satisfies $\|\dot\gamma(s)\|_{\gamma(s)} \leq \kappa_{t_0}$, for every $s \in [a, b]$.*

Proof. Call $\delta : [a, b] \to M$ a geodesic, parametrized proportionally to arc-length, with $\delta(a) = \gamma(a), \delta(b) = \gamma(b)$, and whose length is $d(\gamma(a), \gamma(b))$. The speed $\|\dot\delta(s)\|_{\delta(s)}$ of the geodesic is constant for $s \in [a, b]$. The length of γ is therefore $(b - a)\|\dot\delta(s)\|_{\delta(s)}$, for any $s \in [a, b]$. This implies

$$(b - a)\|\dot\delta(s)\|_{\delta(s)} = d(\gamma(a), \gamma(b)) \leq \operatorname{diam}(M).$$

Hence $\|\dot\delta(s)\|_{\delta(s)} \leq \operatorname{diam}(M)/t_0$. If we set $C^1_{t_0} = \sup\{L(x, v) \mid \|v\|_x \leq \operatorname{diam}(M)/t_0\}$, we see that the action of δ is bounded by $(b - a)C^1_{t_0}$. Since γ is a minimizer, we obtain $\int_a^b L(\gamma(s), \dot\gamma(s))\, ds \leq (b - a)C^1_{t_0}$. This implies that there exists $s_0 \in [a, b]$ such that $L(\gamma(s_0), \dot\gamma(s_0)) \leq C^1_{t_0}$. By the superlinearity of L, the constant

$$C^2_{t_0} = \sup\{\|v\|_x \mid L(x, v) \leq C^1_{t_0}\}$$

is finite. Since the energy is constant along the speed curve of an extremal, we get

$$E(\gamma(s), \dot\gamma(s)) = E(\gamma(s_0), \dot\gamma(s_0)) \leq C^3_{t_0},$$

where $C^3_{t_0} = \sup\{E(x, v) \mid \|v\|_x \leq C^2_{t_0}\}$. Hence, for every $s \in [a, b]$, we have

$$\|\dot\gamma(s_0)\|_{\gamma(s_0)} \leq \kappa_{t_0},$$

where $\kappa_{t_0} = \sup\{\|v\|_x \mid E(x, v) \leq C^3_{r_0}\}$. □

Lemma 8.2. *If $t_0 > 0$, and a finite $\beta \geq 1$ are given, we can find a constant $K_{t_0,\beta}$ such that*

$$|h_t(x, y) - h_{t'}(x, y)| \leq K_{t_0,\beta}|t - t'|,$$

for every $x, y \in M$, and $t, t' \geq t_0$, with $\max(t/t', t'/t) \leq \beta$.

Proof. By Tonelli's theorem, we can find a minimizer $\gamma : [0, t'] \to M$, with $\gamma(0) = x$, and $\gamma(t') = y$. Since γ is a minimizer

$$h_{t'}(x, y) = \int_0^{t'} L(\gamma(s), \dot\gamma(s))\, ds.$$

Note that by Lemma 8.1 above we have $\|\dot\gamma(s)\|_{\gamma(s)} \leq \kappa_{t_0}$, for every $s \in [0, t']$. If we define $\tilde\gamma : [0, t] \to M$ by $\tilde\gamma(s) = \gamma(t't^{-1}s)$. Since $\tilde\gamma(0) = x$, and $\tilde\gamma(t) = y$, we get

$$\begin{aligned} h_t(x, y) &\leq \int_0^t L(\tilde\gamma(s), \dot{\tilde\gamma}(s))\, ds \\ &\leq \int_0^t L(\gamma(t't^{-1}s), t't^{-1}\dot\gamma(t't^{-1}s))\, ds \\ &= \int_0^{t'} L(\gamma(s'), t't^{-1}\dot\gamma(s'))tt'^{-1}\, ds'. \end{aligned}$$

where the last line was obtained by the change of variable $s' = t't^{-1}s$. Therefore, we have

$$h_t(x,y) - h_{t'}(x,y) \leq \int_0^{t'} L(\gamma(s), t't^{-1}\dot\gamma(s))tt'^{-1} - L(\gamma(s),\dot\gamma(s))\,ds. \tag{8.1}$$

Since L is at least C^1, we can find a Lipschitz constant $K_{t_0,\beta}$ of the map $(x,v,\alpha) \mapsto L(x,\alpha^{-1}v)\alpha$, on the compact set $\{(x,v,\alpha) \mid (x,v) \in TM, \|v\|_x \leq \kappa_{t_0}, \beta^{-1} \leq \alpha \leq \beta\}$. This fact together with inequality (8.1) yield

$$h_t(x,y) - h_{t'}(x,y) \leq \int_0^{t'} K_{t_0,\beta}|tt'^{-1} - 1|\,ds = K_{t_0,\beta}|t - t'|.$$

By symmetry this finishes the proof. □

Proof of Fleming's lemma 3.5. Assume $t \geq t_0$. By part 3) and 4) of Lemma 3.4, we have

$$\begin{aligned} h_{t+d(x,x')+d(y,y')}(x',y') &\leq h_{d(x',x)}(x',x) + h_t(x,y) + h_{d(y,y')}(y,y') \\ &\leq h_t(x,y) + B(d(x',x) + d(y,y')). \end{aligned} \tag{8.2}$$

If we set $t' = t + d(x,x') + d(y,y')$, we have $t, t' \geq t_0, t/t' \leq 1$, and $t'/t = 1 + (d(x',x) + d(y,y'))/t \leq 1 + 2\operatorname{diam}(M)t_0^{-1}$. By Lemma 8.2 with $\beta = 1 + 2\operatorname{diam}(M)t_0^{-1}$, we get

$$h_t(x',y') \leq h_{t+d(x,x')+d(y,y')}(x',y') + K_{t_0,\beta}(d(x',x) + d(y,y')).$$

Combining with the inequality (8.2), we obtain

$$h_t(x',y') - h_t(x,y) \leq (B + K_{t_0,\beta})(d(x',x) + d(y,y')).$$

By symmetry this finishes the proof of Fleming's Lemma. □

9. A C^1 solution of the Hamilton-Jacobi equation is a weak KAM solution

In this section, we assume that $u : M \to \mathbb{R}$ is a C^1 solution of the Hamilton-Jacobi equation $H(x, d_x u) = c$ (for every $x \in M$). We first prove that $u \prec L + c$. Assume $\gamma : [a,b] \to M$ is a C^1 curve, together with the Hamilton-Jacobi equation, the Fenchel inequality gives $d_{\gamma(s)}u(\dot\gamma(s)) \leq L(\gamma(s),\dot\gamma(s)) + H(\gamma(s), d_{\gamma(s)}u) = L(\gamma(s),\dot\gamma(s)) + c$. Integrating on $[a,b]$ yields $u(\gamma(b)) - u(\gamma(b)) \leq \mathbb{L}(\gamma) + c(b-a)$. Therefore, by Proposition 5.1, we have $u \leq T_t^- u + ct$, for every $t \geq 0$. To show the opposite inequality, we prove the following lemma.

Lemma 9.1. *For every $x \in M$, there exists a curve $\gamma_x :]-\infty, +\infty[\to M$, which is (u, L, c)-calibrated.*

Proof. Since the Legendre transform $\mathcal{L}$ is a homeomorphism, we can define a continuous vector field X_u on M by $X_u(x) = \partial H/\partial p(x, d_x u)$. By the equality case in Fenchel equality, and the fact that $H(x, d_x u) = c$, we have

$$d_x u(X_u(x)) = L(x, X_u(x)) + c, \text{ for every } x \in M. \tag{9.1}$$

Since X_u is continuous, we can apply the Cauchy-Peano theorem to find a solution γ_x of X_u with $\gamma_x(0) = X$. Moreover, by compactness of M, we can assume that the solution γ_x of X_u is defined on the whole of $\mathbb{R}$. Using (9.1) along γ_x, we obtain

$$d_{\gamma_x(s)}u(\dot\gamma_x(s)) = L(\gamma_x(s), \dot\gamma_x(s)) + c, \text{ for every } x \in M.$$

It remains now to integrate this equality on an arbitrary compact interval $[a, b]$ to see that γ_x is (u, L, c)-calibrated on $\mathbb{R}$. □

We now show that $u \geq T_t^- u + ct$, for every $t \geq 0$. Fix $x \in M$, and pick γ_x given by Lemma 9.1. For $t > 0$, we have

$$u(x) - u(\gamma_x(-t)) = \int_{-t}^{0} L(\gamma_x(s), \dot\gamma_x(s))\, ds + ct \geq h_t(\gamma_x(-t), x) + ct.$$

Therefore $u(x) \geq u(\gamma_x(-t)) + h_t(\gamma_x(-t), x) + ct \geq T_t^- u(x) + ct$.

Acknowledgements. The author benefited from support from Institut Universitaire de France, and from ANR ANR-07-BLAN-0361-02 KAM &ANR-12-BS01-0020 WKBHJ. He also enjoyed the hospitality of both the Korean IBS Center For Geometry and Physics in Pohang, and CIMAT in Guanajuato, while writing this paper.

References

[1] Arnold, V.I., *Proof of a theorem of A. N. Kolmogorov on the preservation of conditionally periodic motions under a small perturbation of the Hamiltonian* (Russian), Uspehi Mat. Nauk **18** (1963), 13–40.

[2] Aubry, S., *The devil's staircase transformation in incommensurate lattices*, in The Riemann problem, complete integrability and arithmetic applications, Bures-sur-Yvette/New York, 1979/1980. Lecture Notes in Math. 925, Springer, Berlin-New York, 1982. 221–245.

[3] Bardi, M. and Cappuzzo-Dolceta, I., *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*, Systems & Control: Foundations and Applications, Birkhäuser Boston Inc., Boston, MA, 1997.

[4] Barles, G., *Solutions de viscosité des équations de Hamilton-Jacobi*, Mathématiques et Applications 17, Springer-Verlag, Paris, 1994.

[5] Buttazzo, G., Giaquinta, M., and Hildebrandt, S., *One-dimensional variational problems*, Oxford Lecture Series in Mathematics and its Applications 15, Oxford Univ. Press, New York, 1998.

[6] Cannarsa, P. and Sinestrari, C., *Semiconcave Functions, Hamilton-Jacobi Equations, and Optimal Control*, Progress in Nonlinear Differential Equations and Their Applications 58, Birkhäuser, Boston, 2004.

[7] Clarke, F.H., *Methods of dynamic and nonsmooth optimization*, CBMS-NSF Reg. Conf. Series in App. Math. 57, SIAM, Philadelphia, PA, 1989.

[8] Crandall, M.G. and Lions, P.L., *Viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc. **277**(1983), 1–42.

[9] Dias Carneiro, M.J., *On minimizing measures of the action of autonomous Lagrangians*, Nonlinearity **8** (1995), 1077–1085.

[10] Dugundji, J., *Topology*, Allyn and Bacon Inc., Boston, 1970.

[11] E, Weinan, *Aubry-Mather theory and periodic solutions of the forced Burgers equation*, Comm. Pure Appl. Math. **52** (1999), 811–828.

[12] Evans, L. C. and Gomes, D. *Effective Hamiltonians and averaging for Hamiltonian dynamics. I-II*, Arch. Ration. Mech. Anal. **157** (2001), 1–33 & **161** (2002), 271–305.

[13] Fathi, A., *Théorème KAM faible et théorie de Mather sur les systèmes lagrangiens*, C. R. Acad. Sci. Paris Sér. I Math. **324**(9) (1997), 1043–1046.

[14] _____, *Weak KAM Theorem and Lagrangian Dynamics*, Preprint, http://tinyurl.com/AFnotesWeakKam, http://tinyurl.com/oh4qjv3.

[15] Fathi, A. and A. Figalli, *Optimal transportation on non-compact manifolds*, Israel J. Math. **175** (2010), 1–59.

[16] Fleming, W.H., *The Cauchy problem for a nonlinear first order partial differential equation*, J. Differential Equations **5** (1969), 515–530.

[17] Kolmogorov, A.N., *Théorie générale des systèmes dynamiques et mécanique classique*, in Proceedings of the International Congress of Mathematicians, Amsterdam, 1954, Vol. 1 315–333. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co 1957.

[18] Lions, P.-L., Papanicolau, G., and Varadhan, S.R.S., *Homogenization of Hamilton-Jacobi equation*, Unpublished preprint, (1987).

[19] Mañé, R., *Lagrangian flows: the dynamics of globally minimizing orbits*, Bol. Soc. Brasil. Mat. (N.S.) **28** (1997), 141–153.

[20] Mather, J.N., *Existence of quasiperiodic orbits for twist homeomorphisms of the annulus*, Topology **21** (1982), 457-467.

[21] _____, *Action minimizing measures for positive definite Lagrangian systems*, Math. Z. **207** (1991), 169–207.

[22] _____, *Variational construction of connecting orbits*, Ann. Inst. Fourier **43** (1993), 1349–1386.

[23] Moser, J., *A new technique for the construction of solutions of nonlinear differential equations*, Proc. Nat. Acad. Sci. U.S.A. **47** (1961), 1824–1831.

[24] Poincaré, H., *Les Méthodes nouvelles de la Mécanique Céleste*, 3 tomes, Paris, Grauthier-Viltars, 1892–1899.

46 allée d'Italie, 69364 Lyon, Cedex 07, France

E-mail: albert.fathi@ens-lyon.fr

The low-density limit of the Lorentz gas: periodic, aperiodic and random

Jens Marklof

Abstract. The Lorentz gas is one of the simplest, most widely used models to study the transport properties of rarified gases in matter. It describes the dynamics of a cloud of non-interacting point particles in an infinite array of fixed spherical scatterers. More than one hundred years after its conception, it is still a major challenge to understand the nature of the kinetic transport equation that governs the macroscopic particle dynamics in the limit of low scatterer density (the Boltzmann-Grad limit). Lorentz suggested that this equation should be the linear Boltzmann equation. This was confirmed in three celebrated papers by Gallavotti, Spohn, and Boldrighini, Bunimovich and Sinai, under the assumption that the distribution of scatterers is sufficiently disordered. In the case of strongly correlated scatterer configurations (such as crystals or quasicrystals), we now understand why the linear Boltzmann equation fails and what to substitute it with. A particularly striking feature of the periodic Lorentz gas is a heavy tail for the distribution of free path lengths, with a diverging second moment, and superdiffusive transport in the limit of large times.

Mathematics Subject Classification (2010). Primary 82C40; Secondary 35Q20, 37A17, 37D50, 60G55, 52C23.

Keywords. Boltzmann equation, Boltzmann-Grad limit, homogeneous flow, Lorentz gas, quasicrystal, Ratner's theorem, superdiffusion.

1. Introduction

The Lorentz gas describes the time evolution of a cloud of non-interacting point particles in an infinitely extended array of fixed scatterers. In the simplest setting of zero external force fields, each particle moves with constant velocity along a straight line until it hits a sphere of radius r, where it is scattered elastically. Besides specular reflection (as in Lorentz' original setting), we will also allow more general spherically symmetric scattering maps, for example those resulting from muffin-tin Coulomb potentials. The scatterers are centered at the points of a locally finite subset $\mathcal{P} \subset \mathbb{R}^d$, which is fixed once and for all. The configuration space of the Lorentz gas is thus $\mathcal{K}_r = \mathbb{R}^d \setminus (\mathcal{P} + \mathcal{B}_r^d)$ where $\mathcal{B}_r^d$ is the open ball in $\mathbb{R}^d$ of radius r, centered at the origin. The phase space of the Lorentz gas is $\mathrm{T}(\mathcal{K}_r)$, the tangent bundle of $\mathcal{K}_r$. We use the convention that, for $\boldsymbol{q} \in \partial\mathcal{K}_r$, the tangent vector $\boldsymbol{v}$ points away from the scatterer.[1] Given initial data $(\boldsymbol{q}, \boldsymbol{v}) \in \mathrm{T}(\mathcal{K}_r)$ at time $t = 0$, we denote position and velocity at time $t \in \mathbb{R}$ by $(\boldsymbol{q}(t), \boldsymbol{v}(t))$. For notational reasons it is convenient to also define the dynamics inside the scatterer by $(\boldsymbol{q}(t), \boldsymbol{v}(t)) = (\boldsymbol{q}, \boldsymbol{v})$ for every $(\boldsymbol{q}, \boldsymbol{v}) \in \mathrm{T}(\mathbb{R}^d) \setminus \mathrm{T}(\mathcal{K}_r)$. With this, the phase space is $\mathrm{T}(\mathbb{R}^d) = \mathbb{R}^d \times \mathbb{R}^d$. The Liouville measure of our dynamics is

Proceedings of the International Congress of Mathematicians, Seoul, 2014

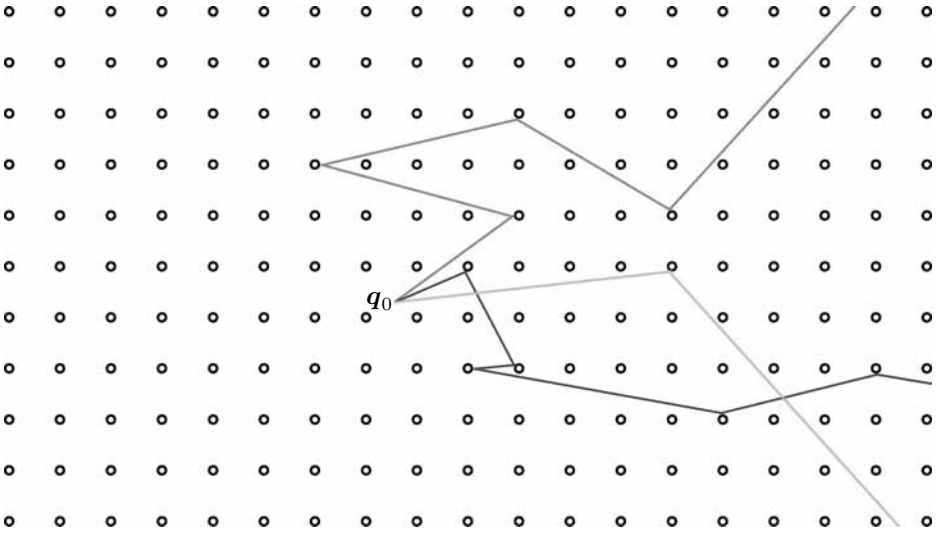


Figure 1.1. The Lorentz gas for a periodic scatterer configuration $\mathcal{P} = \mathbb{Z}^2$, with three distinct particle trajectories starting at the point $\boldsymbol{q}_0$.

the Lebesgue measure $d\boldsymbol{q}\, d\boldsymbol{v}$. Since we have assumed that the scattering map is elastic, the particle speed $\|\boldsymbol{v}\|$ is a constant of motion. We may therefore restrict the dynamics, without loss of generality, to the unit tangent bundle $\mathrm{T}^1(\mathbb{R}^d) = \mathbb{R}^d \times \mathrm{S}_1^{d-1}$, where the Liouville measure is now the Lebesgue measure restricted to $\|\boldsymbol{v}\| = 1$. We assume that $\mathcal{P}$ has constant density $\overline{n} > 0$, i.e. for any bounded $\mathcal{D} \subset \mathbb{R}^d$ with $\mathrm{vol}_{\mathbb{R}^d}(\mathcal{D}) > 0$ and $\mathrm{vol}_{\mathbb{R}^d}(\partial\mathcal{D}) = 0$ ($\mathrm{vol}_{\mathbb{R}^d}$ denotes the Lebesgue measure in $\mathbb{R}^d$ and $\partial\mathcal{D}$ the boundary of $\mathcal{D}$) we have

$$\lim_{R\to\infty} \frac{\#(\mathcal{P} \cap R\mathcal{D})}{\mathrm{vol}_{\mathbb{R}^d}(R\mathcal{D})} = \overline{n}. \tag{1.1}$$

By a trivial rescaling of length units, we may assume in the following that $\overline{n} = 1$.

In the present setting, the Boltzmann-Grad limit is defined as the limit of low scatterer density. *Density* refers here to the *volume* density, i.e., the relative volume $v_d r^d$ occupied by the scatterers, rather than their *number* density $\overline{n} = 1$. The constant $v_d = \mathrm{vol}_{\mathbb{R}^d}(\mathcal{B}_1^d) = \pi^{d/2}/\Gamma(\frac{d+2}{2})$ is the volume of the d-dimensional unit ball. For a fixed scatterer configuration $\mathcal{P}$ the Boltzmann-Grad limit corresponds therefore to taking $r \to 0$. To capture the dynamics of the Lorentz gas in this limit, we measure length and time in units of the mean free path length,[2] which is asymptotic to $v_{d-1}^{-1} r^{d-1}$ (as $r \to 0$). To this end we introduce the *macroscopic* coordinates

$$(\boldsymbol{Q}(t), \boldsymbol{V}(t)) = (r^{d-1}\boldsymbol{q}(r^{-(d-1)}t), \boldsymbol{v}(r^{-(d-1)}t)) \in \mathrm{T}^1(\mathbb{R}^d). \tag{1.2}$$

The mean free path length is now given by the r-independent quantity $\overline{\xi} = v_{d-1}^{-1}$. The

[1] We ignore the case when scatterers overlap. This configuration will be statistically insignificant in the limit $r \to 0$ for $\mathcal{P}$ with constant density.

[2] The mean free path length is defined as the average distance travelled between collisions.

evolution of an initial macroscopic particle density $f \in \mathrm{T}^1(\mathbb{R}^d)$ is defined by the linear operator

$$[L_r^t f](\boldsymbol{Q}, \boldsymbol{V}) := f(\boldsymbol{Q}(-t), \boldsymbol{V}(-t)) \tag{1.3}$$

where $(\boldsymbol{Q}(-t), \boldsymbol{V}(-t))$ are the macroscopic particle coordinates corresponding to the data $(\boldsymbol{Q}(0), \boldsymbol{V}(0)) = (\boldsymbol{Q}, \boldsymbol{V})$ at time $t = 0$.

The question is: For a given scatterer configuration $\mathcal{P}$, does L_r^t have a (weak) limit as $r \to 0$? That is, for every $t > 0$ is there

$$L^t : \mathrm{L}^1(\mathrm{T}^1(\mathbb{R}^d)) \to \mathrm{L}^1(\mathrm{T}^1(\mathbb{R}^d)) \tag{1.4}$$

such that, for every $f \in \mathrm{L}^1(\mathrm{T}^1(\mathbb{R}^d))$ and bounded $\mathcal{A} \subset \mathrm{T}^1(\mathbb{R}^d)$ with boundary of zero Lebesgue measure,

$$\lim_{r\to 0} \int_{\mathcal{A}} L_r^t f(\boldsymbol{Q}, \boldsymbol{V})\, d\boldsymbol{Q}\, d\boldsymbol{V} = \int_{\mathcal{A}} L^t f(\boldsymbol{Q}, \boldsymbol{V})\, d\boldsymbol{Q}\, d\boldsymbol{V} \ ? \tag{1.5}$$

Using Boltzmann's heuristics, Lorentz [25] predicted in 1905 that the answer to this question should be "yes" and that the particle density $f_t := L^t f$ at time t satisfies the *linear* Boltzmann equation (also referred to as the *kinetic Lorentz equation*)

$$(\partial_t + \boldsymbol{V} \cdot \partial_{\boldsymbol{Q}}) f_t(\boldsymbol{Q}, \boldsymbol{V}) = \int_{\mathbb{R}^d} [f_t(\boldsymbol{Q}, \boldsymbol{V}') - f_t(\boldsymbol{Q}, \boldsymbol{V})]\, \sigma(\boldsymbol{V}, \boldsymbol{V}')\, d\boldsymbol{V}', \tag{1.6}$$

where $\sigma(\boldsymbol{V}, \boldsymbol{V}')$ is the differential cross section of a single scatterer (see Section 2). Lorentz' heuristic derivation was, over sixty years later, confirmed rigorously for random scatterer configurations $\mathcal{P}$ by Gallavotti [20] and Spohn [41], where the convergence in (1.5) is established for the ensemble average. Boldrighini, Bunimovich and Sinai [10] proved a stronger result by showing that for a *fixed* realisation of a Poisson process the limit (1.5) exists almost surely (cf. Section 5). One can in fact show that, for initial data $(\boldsymbol{Q}_0, \boldsymbol{V}_0)$ randomly distributed in $\mathrm{T}^1(\mathbb{R}^d)$ according to an absolutely continuous probability measure Λ, the curve $t \mapsto (\boldsymbol{Q}(t), \boldsymbol{V}(t))$ converges in distribution to a random flight process, where the free flight times are independent identically distributed random variables with an exponential distribution. Eq. (1.6) is precisely the Fokker-Planck-Kolmogorov equation of the limit process (cf. Section 5).

In his 2006 ICM address [22] (cf. also [23]), Golse pointed out that, due to the heavy tail of the free path length distribution [11, 21, 30], the linear Boltzmann equation fails in the case $\mathcal{P} = \mathbb{Z}^d$. The main objective of this paper is to illustrate the deeper reason behind this failure not only for general *periodic* scatterer configurations, see Section 6 and [14, 27–30], but as well for *aperiodic* point sets with strong long-range correlations, cf. Sections 7, 8 and [31, 32, 45]. We will uncover a new class of random flight processes that emerge in the Boltzmann-Grad limit (Sections 2, 3) and whose transport equations generalise the linear Boltzmann equation (1.6) in a natural way (Section 4).

A major open question in the field is whether the dynamics in the Lorentz gas converges, in the limit of large times t, to Brownian motion. The first seminal result in this direction was the proof of a central limit theorem for the two-dimensional *periodic* Lorentz gas with finite horizon[3] by Bunimovich and Sinai [12]. For general invariance principles, see Melbourne and Nicol [35] and references therein. In the case of the infinite-horizon periodic

[3] *Finite horizon* means that the free path length has an upper bound. This requires a suitable choice of scatterer configuration $\mathcal{P}$ (e.g. a triangular lattice) and sufficiently large scatterer radius r.

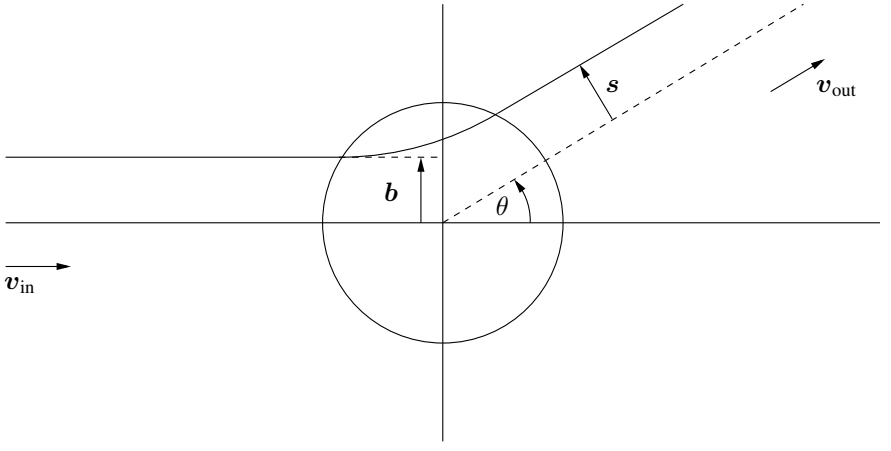


Figure 2.1. Scattering in the unit ball.

Lorentz gas, again in dimension $d = 2$ and with fixed radius $r > 0$, Bleher [6] conjectured a superdiffusive central limit theorem with a $\sqrt{t \log t}$ normalisation, rather than the standard $\sqrt{t}$ in the finite horizon case. Bleher's conjecture was first proved by Szász and Varjú [42] for the discrete-time billiard map, and by Dolgopyat and Chernov [19] for the billiard flow.[4] It is currently unknown how to extend these results to higher dimensions $d \geq 3$ or to aperiodic scatterer configurations [4, 5, 15, 17, 18, 24, 36, 43]. The problem becomes tractable, however, if we pass to the low-density limit $r \to 0$: If $\mathcal{P}$ is a typical realisation of a Poisson process, then the limiting random flight process satisfies a central limit theorem with $\sqrt{t}$ scaling, in any dimension $d \geq 2$. This follows from standard techniques in the theory of Markov processes [37] as pointed out by Spohn [41]. If $\mathcal{P}$ is a Euclidean lattice, then the limiting random flight process satisfies a *superdiffusive* central limit theorem with $\sqrt{t \log t}$ normalisation, again in any dimension $d \geq 2$. See Section 9 and [34] for further details.

2. Intercollision flights

We begin by defining the scattering map, which we assume is spherically symmetric, preserves angular momentum and is the same for each scatterer. Let us choose a coordinate frame so that the incoming velocity is aligned with the first coordinate axis (cf. Figure 2.1),

$$\boldsymbol{v}_{\text{in}} = \boldsymbol{e}_1 := (1, 0, \ldots, 0). \tag{2.1}$$

(All vectors are represented as row vectors.) The *impact parameter* $\boldsymbol{b}$ is the orthogonal projection of the point of impact onto the plane orthogonal to $\boldsymbol{v}_{\text{in}}$, measured in units of r. In the present frame, $\boldsymbol{b} = (0, \boldsymbol{w})$ with $\boldsymbol{w} \in \mathcal{B}_1^{d-1}$. (We will also refer to $\boldsymbol{w}$ as impact parameter.) When $\boldsymbol{w} \neq \boldsymbol{0}$, the outgoing velocity is

$$\boldsymbol{v}_{\text{out}} = \boldsymbol{v}_{\text{in}} \cos\theta + (0, \widehat{\boldsymbol{w}}) \sin\theta, \tag{2.2}$$

[4] Superdiffusive central limit theorems have also been established for compact planar billiards, such as the stadium [3] and billiards with cusps [2].

where the angle θ is called the *scattering angle* and $\widehat{\boldsymbol{w}} := w^{-1}\boldsymbol{w}$ with $w := \|\boldsymbol{w}\|$. For $\boldsymbol{w} = \boldsymbol{0}$ we simply assume $\boldsymbol{v}_{\text{out}} = -\boldsymbol{v}_{\text{in}}$. By the assumed spherical symmetry, $\theta = \theta(w)$ is only a function of the length $w \in [0, 1[$ of the impact parameter $\boldsymbol{w}$. Equation (2.2) can be expressed as

$$\boldsymbol{v}_{\text{out}} = \boldsymbol{v}_{\text{in}} S(\boldsymbol{w})^{-1}, \tag{2.3}$$

with the matrix

$$S(\boldsymbol{w}) = \exp \begin{pmatrix} 0 & -\theta(w)\widehat{\boldsymbol{w}} \\ \theta(w)\, {}^{\text{t}}\widehat{\boldsymbol{w}} & 0_{d-1} \end{pmatrix} \in \mathrm{SO}(d). \tag{2.4}$$

The *exit parameter* is defined as the orthogonal projection of the point of exit onto the plane orthogonal to $\boldsymbol{v}_{\text{out}}$, and is given by

$$\boldsymbol{s} = -w\boldsymbol{v}_{\text{in}} \sin\theta + (0, \boldsymbol{w}) \cos\theta = (0, \boldsymbol{w}) S(\boldsymbol{w})^{-1}. \tag{2.5}$$

The differential scattering cross section $\sigma(\boldsymbol{v}_{\text{in}}, \boldsymbol{v}_{\text{out}})$ is defined by the relation

$$\sigma(\boldsymbol{v}_{\text{in}}, \boldsymbol{v}_{\text{out}})\, d\boldsymbol{v}_{\text{out}} = d\boldsymbol{w}. \tag{2.6}$$

Note that in the present setting $\sigma(\boldsymbol{v}_{\text{in}}, \boldsymbol{v}_{\text{out}}) = \sigma(\boldsymbol{v}_{\text{out}}, \boldsymbol{v}_{\text{in}})$.

For simplicity, we assume throughout this paper that one of the following conditions holds:[5]

(A) $\theta \in \mathrm{C}^1([0, 1[\,)$ is strictly decreasing with $\theta(0) = \pi$ and $\theta(w) > 0$.

(B) $\theta \in \mathrm{C}^1([0, 1[\,)$ is strictly increasing with $\theta(0) = -\pi$ and $\theta(w) < 0$.

This hypothesis is satisfied for many scattering maps, e.g. specular reflection[6] or the scattering in the muffin-tin Coulomb potential $V(\boldsymbol{q}) = \alpha \max(\|\boldsymbol{q}\|^{-1} - 1, 0)$ with $\alpha \notin \{-2E, 0\}$, where E denotes the total energy, cf. [29].

An inductive argument shows that there is a sequence $(\boldsymbol{w}_n)_{n\in\mathbb{N}}$ in $\mathcal{B}_1^{d-1}$, so that the impact parameter $\boldsymbol{b}_n$, exit velocity $\boldsymbol{v}_n$ and exit parameter $\boldsymbol{s}_n$ at the nth collision are given by the frame-independent formulas

$$\boldsymbol{v}_n = \boldsymbol{e}_1 R_n^{-1}, \qquad \boldsymbol{b}_n = (0, \boldsymbol{w}_n) R_{n-1}^{-1} \qquad \boldsymbol{s}_n = (0, \boldsymbol{w}_n) R_n^{-1} \tag{2.7}$$

where

$$R_n := R(\boldsymbol{v}_0) S(\boldsymbol{w}_1) \cdots S(\boldsymbol{w}_n). \tag{2.8}$$

Here $R : \mathrm{S}_1^{d-1} \to \mathrm{SO}(d)$ is smooth up to finitely many singular points, such that $\boldsymbol{v}R(\boldsymbol{v}) = \boldsymbol{e}_1$ for all $\boldsymbol{v} \in \mathrm{S}_1^{d-1}$. For an example see footnote 3 on p. 1968 of [28].

We can now express position and velocity at time $t > 0$ as[7]

$$\boldsymbol{q}(t) = \boldsymbol{q}_{\nu(t)} + (t - \tau_{\nu(t)})\boldsymbol{v}(t) + O(r\nu(t)), \qquad \boldsymbol{v}(t) = \boldsymbol{v}_{\nu(t)}, \tag{2.9}$$

where

$$\tau_n := \sum_{j=1}^{n} t_j, \quad \tau_0 := 0, \tag{2.10}$$

[5] All results extend in fact to more general scattering maps, see [29] for details.

[6] Here $\theta(w) = \pi - 2\arcsin(w)$ and thus condition (A) holds.

[7] The $O(r\nu(t))$-error is simply due to the fact that we have not included the jumps of position at each scattering. In the case of specular reflections, all formulas are exact.

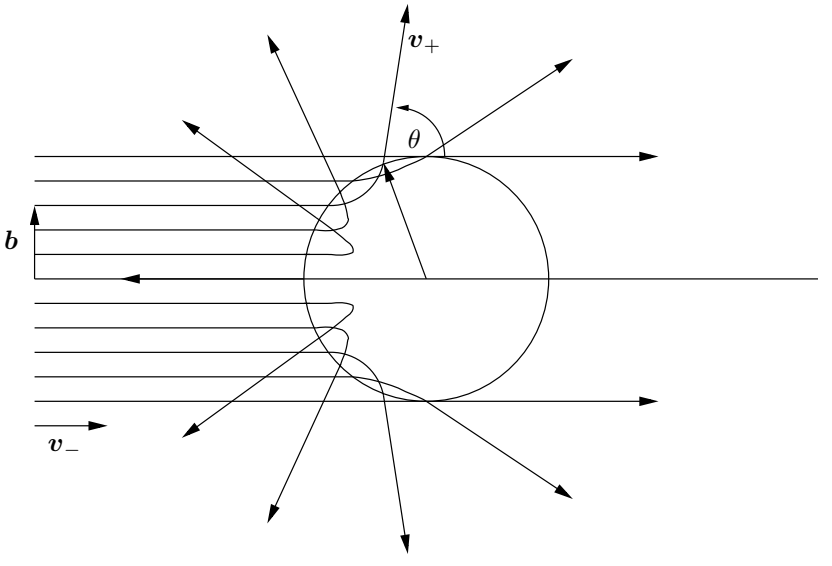


Figure 2.2. Illustration of a scattering map satisfying Hypothesis (A).

is the time to the nth collision, t_j is the jth intercollision time,

$$\nu(t) := \max\{n \in \mathbb{Z}_{\geq 0} : \tau_n \leq t\} \tag{2.11}$$

is the number of collisions within time t,

$$\boldsymbol{q}_n := \sum_{j=1}^{n} t_j \boldsymbol{v}_{j-1} \tag{2.12}$$

is the particle location at the nth collision[8] and

$$\boldsymbol{v}_n = R(\boldsymbol{v}_0)S(\boldsymbol{w}_1)\cdots S(\boldsymbol{w}_n)\boldsymbol{e}_1 \tag{2.13}$$

is the velocity after the nth collision as calculated in (2.7).

In the macroscopic coordinates (1.2), the above translates to

$$\boldsymbol{Q}(t) = \boldsymbol{Q}_{\mathcal{V}(t)} + (t - \mathcal{T}_{\mathcal{V}(t)})\boldsymbol{V}(t) + O(r^d \mathcal{V}(t)), \qquad \boldsymbol{V}(t) = \boldsymbol{V}_{\mathcal{V}(t)} \tag{2.14}$$

where $\boldsymbol{Q}_n = r^{d-1}\boldsymbol{q}_n$, $\boldsymbol{V}_n = \boldsymbol{v}_n$, $\mathcal{T}_n = r^{d-1}\tau_n$ and

$$\mathcal{V}(t) := \nu(r^{-(d-1)}t) = \max\{n \in \mathbb{Z}_{\geq 0} : \mathcal{T}_n \leq t\}. \tag{2.15}$$

3. A refined Stosszahlansatz

We will now investigate the particle trajectory corresponding to random initial conditions $(\boldsymbol{Q}_0, \boldsymbol{V}_0)$ and outline a strategy to establish the convergence to a random flight process in

[8] Again, this is up to an error of order $O(rn)$.

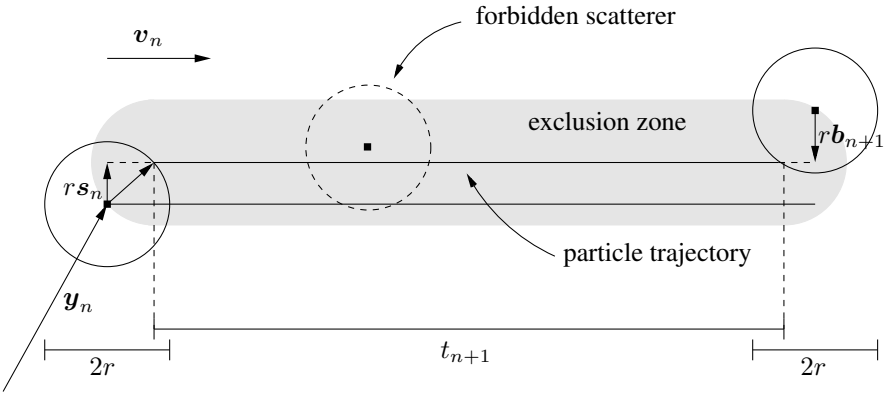


Figure 3.1. Intercollision flight in the Lorentz gas between the nth and $(n+1)$st collision. The exclusion zone is a cylinder of radius r with spherical caps.

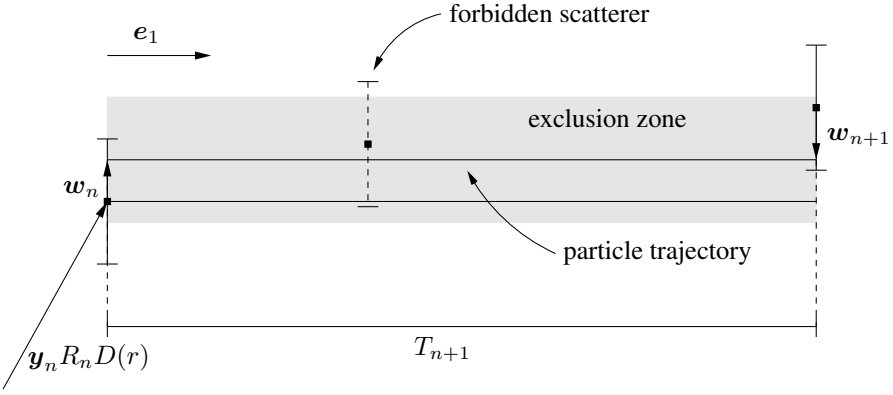


Figure 3.2. The intercollision flight in Fig. 3.1 after applying the linear map $R_n D(r)$ with r very small. The exclusion zone is now approximately a cylinder with flat caps.

the Boltzmann-Grad limit.[9] We will, for now, keep the scatterer configuration $\mathcal{P}$ general, and discuss in later sections examples of $\mathcal{P}$ which allow a rigorous treatment.

Let us focus on the nth and $(n+1)$st collision and consider a parallel beam of particles with given velocity $\boldsymbol{v}_{n-1}$ that hit a scatterer located at $\boldsymbol{y}_n$ with a certain intensity distribution λ in the impact parameter $\boldsymbol{w}_n$ (Figure 3.1).[10] The task is now to calculate the probability of hitting the next scatterer in a small time interval around t_{n+1} with impact parameter

[9] We assume here that $(\boldsymbol{Q}_0, \boldsymbol{V}_0)$ is distributed according to a fixed, absolutely continuous probability measure Λ on $\mathrm{T}^1(\mathbb{R}^d)$. One can, of course also prepare the initial particle cloud on smaller scales. For example take $(\boldsymbol{q}_0, \boldsymbol{v}_0) = (r^{-(d-1)}\boldsymbol{Q}_0, \boldsymbol{V}_0)$ random with respect to a fixed absolutely continuous Λ. In the case of the periodic and the quasicrystal Lorentz gas [28, 29, 32] we are even able to consider more singular Λ: Fix $\boldsymbol{q}_0$ and only take $\boldsymbol{v}_0$ random according to an absolutely continuous measure on the unit sphere. In this case, we have convergence for *every* $\boldsymbol{q}_0$, with the same limit distribution for *almost every* $\boldsymbol{q}_0$.

[10] The measure λ will of course depend on the history of the particle beam, and in particular on r, but let us assume here for the sake of argument that λ is a *fixed* Borel probability measure on $\mathcal{B}_1^{d-1}$. A key part of the paper [29] deals with the problem of r-dependent measures in the setting of the periodic Lorentz gas, by obtaining uniform estimates over families of λ.

near $\boldsymbol{w}_{n+1}$. Recall that we expect t_{n+1} to be of order $r^{-(d-1)}$, and it is natural to set $T_n = r^{d-1}t_n$. We now first shift our coordinate system by $-\boldsymbol{y}_n - r(\boldsymbol{s}_n + \boldsymbol{v}_n\sqrt{1-\|\boldsymbol{s}_n\|^2})$ so the left center of the cylinder is now at the origin, then rotate our coordinate system by $R_n \in \mathrm{SO}(d)$, so that the outgoing velocity $\boldsymbol{v}_n$ becomes $\boldsymbol{e}_1$, cf. (2.7), and finally apply the linear transformation given by the matrix

$$D(r) = \begin{pmatrix} r^{d-1} & \boldsymbol{0} \\ {}^{\mathrm{t}}\boldsymbol{0} & r^{-1}1_{d-1} \end{pmatrix} \tag{3.1}$$

which rescales the length units along and perpendicular to the cylinder. Note that the caps of the cylinder become flat as $r \to 0$, cf. Fig. 3.2. In particular

$$\begin{aligned} r\big(\boldsymbol{s}_n + \boldsymbol{v}_n\sqrt{1-\|\boldsymbol{s}_n\|^2}\big)R_nD(r) &= r\big((0,\boldsymbol{w}_n) + (1,\boldsymbol{0})\sqrt{1-\|\boldsymbol{w}_n\|^2}\big)D(r) \\ &= (0,\boldsymbol{w}_n) + O(r^d). \end{aligned} \tag{3.2}$$

The rotation matrix R_n is, by (2.8), given by $R_n = R_{n-1}S(\boldsymbol{w}_n)$ where R_{n-1} is fixed (since $\boldsymbol{v}_{n-1}$ is assumed fixed in this discussion). For $\boldsymbol{w}_n$ random according to λ, we are interested in the probability that the particle hits the next scatterer at a time T_{n+1} in the interval $A =]\xi, \xi + d\xi[$ and with impact parameter $\boldsymbol{w}_{n+1}$ in some box $B \subset \mathcal{B}_1^{d-1}$. This probability is, for small r, approximately[11] equal to the probability that the random point set

$$\widetilde{\Theta}_r(\boldsymbol{y}_n) = (\mathcal{P} - \boldsymbol{y}_n)R_{n-1}S(\boldsymbol{w}_n)D(r) - (0,\boldsymbol{w}_n) \tag{3.3}$$

does not intersect the cylinder $\mathfrak{Z}(\xi) =]0,\xi[\times\mathcal{B}_1^{d-1}$ and has (at least[12]) one point in the box $A\times B$. Our general objective is therefore to try to prove that there is a random point process[13] $\Theta(\boldsymbol{y})$ in $\mathbb{R}^d$ and a random variable $\boldsymbol{h} \in \mathcal{B}_1^{d-1}$ distributed according to λ such that, for every fixed $\boldsymbol{y} \in \mathcal{P}$,

$$\widetilde{\Theta}_r(\boldsymbol{y}) \xrightarrow[r\to 0]{} \widetilde{\Theta}(\boldsymbol{y}) := \Theta(\boldsymbol{y}) - (0,\boldsymbol{h}) \tag{3.4}$$

in finite-dimensional distribution. This means that for any $k \in \mathbb{N}$, $\mathcal{A}_1,\dots,\mathcal{A}_k \subset \mathbb{R}^d$ bounded with boundary of measure zero and $n_1,\dots,n_k \in \mathbb{Z}_{\geq 0}$, we have

$$\lim_{r\to 0} \mathbb{P}\big(\#(\widetilde{\Theta}_r(\boldsymbol{y}) \cap \mathcal{A}_i) = n_i\ \forall i\big) = \mathbb{P}\big(\#(\widetilde{\Theta}(\boldsymbol{y}) \cap \mathcal{A}_i) = n_i\ \forall i\big). \tag{3.5}$$

It is crucial that $\Theta(\boldsymbol{y})$ and $\boldsymbol{h}$ are independent, and that $\Theta(\boldsymbol{y})$ is independent of the choice of λ and R_n. We conclude that, if the convergence in (3.4) indeed holds in finite-dimensional distribution (as we are dealing with only two test sets, $\mathfrak{Z}(\xi)$ and $A \times B$, convergence in two-dimensional distribution is in fact sufficient) then the probability that the particle hits the next scatterer at a time $T_{n+1} \in A$ and with impact parameter $\boldsymbol{w}_{n+1} \in B$, is in the limit $r \to 0$ given by

$$\mathbb{P}\big(\widetilde{\Theta}(\boldsymbol{y}_n) \cap \mathfrak{Z}(\xi) = \emptyset,\ \#(\widetilde{\Theta}(\boldsymbol{y}_n) \cap (A\times B)) = 1\big). \tag{3.6}$$

In some instances, $\Theta(\boldsymbol{y})$ will not depend on the scatterer location $\boldsymbol{y}$, for example when $\mathcal{P}$ is a realisation of a Poisson process or a Euclidean lattice, as we shall see below. If $\Theta(\boldsymbol{y})$ does

[11] This approximation is justified, if the limit distribution is continuous in ξ.

[12] We assume that, in the limit $r \to 0$, the probability of having one point in a small set is approximately the same as the probability of having one ore more points. As in footnote 11, this is justified, if the limit distribution is continuous in ξ.

[13] Throughout this paper, we will represent random point processes as random point sets.

depend on the scatterer location, the hope is that this dependence is "mild," in the sense that there exists a probability space $(\Sigma, \mathcal{F}, \mathrm{m})$ and a map

$$\iota : \mathcal{P} \to \Sigma, \qquad \boldsymbol{y} \mapsto \iota(\boldsymbol{y}), \tag{3.7}$$

so that $\Theta(\boldsymbol{y})$ depends only on the value of $\iota(\boldsymbol{y})$. We will call $\iota(\boldsymbol{y})$ the *colour* of $\boldsymbol{y}$, and consider the colourised scatterer configuration,

$$\{(\boldsymbol{y}, \iota(\boldsymbol{y})) : \boldsymbol{y} \in \mathcal{P}\} \subset \mathbb{R}^d \times \Sigma. \tag{3.8}$$

We assume furthermore that the colour in (3.8) is distributed according to the probability measure m on Σ, in the sense that (cf. (1.1)) for any bounded $\mathcal{D} \subset \mathbb{R}^d$ with $\mathrm{vol}_{\mathbb{R}^d}(\mathcal{D}) > 0$, $\mathrm{vol}_{\mathbb{R}^d}(\partial\mathcal{D}) = 0$ and any measurable set $B \subset \Sigma$ with $\mathrm{m}(\partial B) = 0$,

$$\lim_{R\to\infty} \frac{\#\{\boldsymbol{y} \in \mathcal{P} \cap R\mathcal{D} : \iota(\boldsymbol{y}) \in B\}}{\mathrm{vol}_{\mathbb{R}^d}(R\mathcal{D})} = \mathrm{m}(B). \tag{3.9}$$

Let us define $\Omega := \Sigma \times \mathcal{B}_1^{d-1}$ as the product space of colour and impact parameters, with probability measure $\mathrm{p} = \mathrm{m} \times v_{d-1}^{-1} \mathrm{vol}_{\mathbb{R}^{d-1}}$. Instead of (3.4), we must now consider the convergence for the corresponding colourised point processes. Once we understand the colourised limit, we can compute the limit distribution for the probability of emerging from a scatterer with a given colour and exit parameter ω_n, and hitting the next scatterer at time $T_n \in \,]\xi, \xi + d\xi[$ with colour and impact parameter $\omega_{n+1} \in B \subset \Omega$. We denote this probability by

$$\int_B k(\omega_n, \xi, \omega)\, d\xi\, \mathrm{p}(d\omega), \tag{3.10}$$

which defines the *transition kernel* $k(\omega', \xi, \omega)$. The conclusion of the above heuristics is now that the particle trajectory

$$\Xi_r : t \mapsto (\boldsymbol{Q}(t), \boldsymbol{V}(t)), \tag{3.11}$$

with random initial condition $(\boldsymbol{Q}_0, \boldsymbol{V}_0)$ distributed according to some absolutely continuous measure Λ on $\mathrm{T}^1(\mathbb{R}^d)$, converges in the Boltzmann-Grad limit to the continuous-time random flight process $\Xi(t)$ in $\mathrm{T}^1(\mathbb{R}^d)$ defined as follows.

Consider the sequences of random variables $\underline{\xi} = (\xi_n)_{n\in\mathbb{N}}$ and $\underline{\eta} = (\eta_n)_{n\in\mathbb{N}}$ defined by the Markov chain

$$n \mapsto (\xi_n, \eta_n) \tag{3.12}$$

with state space $\mathbb{R}_{>0} \times \Omega$ and transition probability $(n \geq 2)$

$$\mathbb{P}((\xi_n, \eta_n) \in A \mid \xi_{n-1}, \eta_{n-1}) = \int_A k(\eta_{n-1}, \xi, \omega)\, d\xi\, \mathrm{p}(d\omega), \tag{3.13}$$

where the transition kernel $k(\omega', \xi, \omega)$ is defined by (3.10). The initial distribution is

$$\mathbb{P}((\xi_1, \eta_1) \in A) = \int_A K(\xi, \omega)\, d\xi\, \mathrm{p}(d\omega), \tag{3.14}$$

where

$$K(\xi, \omega) := \frac{1}{\bar{\xi}} \int_\xi^\infty \int_\Omega k(\omega', \xi', \omega)\, \mathrm{p}(d\omega')\, d\xi'. \tag{3.15}$$

The time-reversibility of the underlying microscopic dynamics (for every fixed $r > 0$) implies that the transition kernel k is symmetric, i.e.

$$k(\omega, \xi, \omega') = k(\omega', \xi, \omega). \tag{3.16}$$

Because the transition probability (3.13) is independent of ξ_{n-1}, the chain $n \mapsto \eta_n$ is also Markovian, with transition probability

$$\mathbb{P}(\eta_n \in A \mid \eta_{n-1}) = \int_A \int_0^\infty k(\eta_{n-1}, \xi, \omega)\, d\xi\, \mathbb{p}(d\omega). \tag{3.17}$$

The stationary measure for this Markov chain is $\mathbb{p}$, and the distribution of free path lengths with respect to this measure is defined as

$$\Psi_0(\xi) := \int_\Omega \int_\Omega k(\omega', \xi, \omega)\, \mathbb{p}(d\omega)\, \mathbb{p}(d\omega'). \tag{3.18}$$

Let us write $\eta_n = (\chi_n, \boldsymbol{h}_n)$, where $\chi_n \in \Sigma$ is the colour and $\boldsymbol{h}_n \in \mathcal{B}_1^{d-1}$ the impact parameter. In analogy with the deterministic setting (2.9)–(2.15), we define the random variables

$$\mathcal{T}_n^{\mathrm{BG}} := \sum_{j=1}^n \xi_j, \quad \mathcal{T}_0^{\mathrm{BG}} := 0, \tag{3.19}$$

$$\mathcal{V}^{\mathrm{BG}}(t) := \max\{n \in \mathbb{Z}_{\geq 0} : \mathcal{T}_n^{\mathrm{BG}} \leq t\}, \tag{3.20}$$

$$\boldsymbol{Q}_n^{\mathrm{BG}} := \boldsymbol{Q}_0 + \sum_{j=1}^n \xi_j \boldsymbol{V}_{j-1}^{\mathrm{BG}}, \qquad \boldsymbol{V}_n^{\mathrm{BG}} := R(\boldsymbol{V}_0) S(\boldsymbol{h}_1) \cdots S(\boldsymbol{h}_n) \boldsymbol{e}_1, \tag{3.21}$$

$$\boldsymbol{Q}^{\mathrm{BG}}(t) := \boldsymbol{Q}_{\mathcal{V}^{\mathrm{BG}}(t)}^{\mathrm{BG}} + (t - \mathcal{T}_{\mathcal{V}^{\mathrm{BG}}(t)}^{\mathrm{BG}}) \boldsymbol{V}^{\mathrm{BG}}(t), \qquad \boldsymbol{V}^{\mathrm{BG}}(t) := \boldsymbol{V}_{\mathcal{V}^{\mathrm{BG}}(t)}^{\mathrm{BG}}. \tag{3.22}$$

The notation "BG" stands for *Boltzmann-Grad limit* and is used to differentiate from the deterministic counterparts (2.9)–(2.15). Note that none of the above depend explicitly on colour. The hidden variable "colour" is needed to make (3.12) a Markov chain. The random flight process Ξ is thus defined as

$$t \mapsto \Xi(t) := \big(\boldsymbol{Q}^{\mathrm{BG}}(t), \boldsymbol{V}^{\mathrm{BG}}(t)\big). \tag{3.23}$$

The convergence of the random process Ξ_r in (3.11) to Ξ answers in particular our question (1.5), since the former implies the convergence in (1.5) with L^t defined by

$$\int_{\mathcal{A}} L^t f(\boldsymbol{Q}, \boldsymbol{V})\, d\boldsymbol{Q}\, d\boldsymbol{V} = \mathbb{P}(\Xi(t) \in \mathcal{A}). \tag{3.24}$$

Here $f = \Lambda'$ is the Radon–Nikodym derivative of Λ.

4. A generalised Boltzmann equation

This limiting process $\Xi(t)$ defined in (3.23) is in general not a continuous-time Markov process,[14] but can be turned into one by extending the state space as follows. We define the

[14] A consequence of this fact is that the family of operators L^t in (3.24) does not form a semigroup, i.e., $L^t L^s = L^{t+s}$ does not hold for all $s, t > 0$.

time until the next scattering by

$$T^{\mathrm{BG}}(t) := \mathcal{T}^{\mathrm{BG}}_{\mathcal{V}^{\mathrm{BG}}(t)+1} - t, \tag{4.1}$$

the colour of the next scatterer by

$$\chi^{\mathrm{BG}}(t) := \chi_{\mathcal{V}^{\mathrm{BG}}(t)+1}, \tag{4.2}$$

and the exit velocity of the next scattering by

$$\boldsymbol{V}^{\mathrm{BG}}_{+}(t) := \boldsymbol{V}^{\mathrm{BG}}_{\mathcal{V}^{\mathrm{BG}}(t)+1}. \tag{4.3}$$

The process

$$t \mapsto \widetilde{\Xi}(t) := \big(\boldsymbol{Q}^{\mathrm{BG}}(t), \boldsymbol{V}^{\mathrm{BG}}(t), T^{\mathrm{BG}}(t), \chi^{\mathrm{BG}}(t), \boldsymbol{V}^{\mathrm{BG}}_{+}(t)\big) \tag{4.4}$$

is now a Markov process with state space $\mathrm{T}^1(\mathbb{R}^d)\times\mathbb{R}_{>0}\times\Sigma\times\mathrm{S}_1^{d-1}$ and backward equation[15]

$$\begin{cases} (\partial_t + \boldsymbol{V}\cdot\partial_{\boldsymbol{Q}} - \partial_\xi) f_t(\boldsymbol{Q},\boldsymbol{V},\xi,\chi,\boldsymbol{V}_+) = [\mathcal{C}f_t](\boldsymbol{Q},\boldsymbol{V},\xi,\chi,\boldsymbol{V}_+) \\ \lim\limits_{t\to 0} f_t(\boldsymbol{Q},\boldsymbol{V},\xi,\chi,\boldsymbol{V}_+) = \Lambda'(\boldsymbol{Q},\boldsymbol{V})\, K(\xi,\omega)\, \sigma(\boldsymbol{V},\boldsymbol{V}_+), \end{cases} \tag{4.5}$$

with $K(\xi,\omega)$ as in (3.15) and the collision operator $\mathcal{C}$ is defined by

$$\begin{aligned} &[\mathcal{C}f](\boldsymbol{Q},\boldsymbol{V},\xi,\chi,\boldsymbol{V}_+) \\ &\qquad = \sigma(\boldsymbol{V},\boldsymbol{V}_+)\int_{\mathrm{S}_1^{d-1}}\int_\Sigma f(\boldsymbol{Q},\boldsymbol{V}',0,\chi',\boldsymbol{V})\, k(\omega',\xi,\omega)\, d\mathrm{m}(\chi')\, d\boldsymbol{V}', \end{aligned} \tag{4.6}$$

where

$$\omega' := (\chi', \boldsymbol{s}(\boldsymbol{V}',\boldsymbol{V})R(\boldsymbol{V})), \qquad \omega := (\chi, \boldsymbol{b}(\boldsymbol{V},\boldsymbol{V}_+)R(\boldsymbol{V})). \tag{4.7}$$

A stationary solution of eq. (4.5) is given by

$$f_t(\boldsymbol{Q},\boldsymbol{V},\xi,\chi,\boldsymbol{V}_+) = K(\xi,\omega)\,\sigma(\boldsymbol{V},\boldsymbol{V}_+), \tag{4.8}$$

which corresponds to $\Lambda =$ Liouville measure. To see this, note that the left hand side of the first line in (4.5) is

$$\sigma(\boldsymbol{V},\boldsymbol{V}_+)\,\overline{\xi}^{-1}\int_\Omega k(\omega',\xi,\omega)\, d\mathbb{p}(\omega'). \tag{4.9}$$

Furthermore, we have

$$\overline{\xi}K(0,\omega') = \int_0^\infty\int_\Omega k(\omega'',\xi,\omega')\, d\xi\, d\mathbb{p}(\omega'') = \int_0^\infty\int_\Omega k(\omega',\xi,\omega'')\, d\xi\, d\mathbb{p}(\omega'') = 1.$$

The right hand side of the first line in (4.5) therefore equals, in view of (2.6),

$$\sigma(\boldsymbol{V},\boldsymbol{V}_+)\int_{\mathrm{S}_1^{d-1}}\int_\Sigma \sigma(\boldsymbol{V}',\boldsymbol{V})\, K(0,\omega')\, k(\omega',\xi,\omega)\, d\mathrm{m}(\chi')\, d\boldsymbol{V}'$$

[15] This equation is also known as *Fokker-Planck-Kolmogorov equation*.

$$= \sigma(\boldsymbol{V},\boldsymbol{V}_+)\,\overline{\xi}^{-1}\int_{\mathrm{S}_1^{d-1}}\int_\Sigma k(\omega',\xi,\omega)\,d\mathbb{p}(\omega'), \quad (4.10)$$

which equals (4.9) This shows that (4.8) is indeed a stationary solution of (4.5).

Let us now illustrate the above programme with a number of examples, where all or part of the heuristics can be made rigorous. The principal questions we would like to answer, for a given scatterer configuration $\mathcal{P}$, are: *Does the limit* (1.5) *exist? What is the limit process* $\Theta(\boldsymbol{y})$*? What is the transition kernel* $k(\omega',\xi,\omega)$*?*

We begin with the classic setting where $\mathcal{P}$ is a typical realisation of a Poisson process and will show how the generalised linear Boltzmann equation (4.5) reduces to the original.

5. Random scatterer configuration

The Poisson process $\Theta = \Theta_{\text{Poisson}}$ in $\mathbb{R}^d$ with intensity $\overline{n} = 1$ is characterised by the property that for any collection of bounded, pairwise disjoint Borel sets $\mathcal{A}_1,\dots,\mathcal{A}_k$ and integers $n_1,\dots,n_k \ge 0$,

$$\mathbb{P}(\#(\Theta\cap\mathcal{A}_i) = n_i\ \forall i) = \prod_{i=1}^k \frac{(\mathrm{vol}_{\mathbb{R}^d}(\mathcal{A}_i))^{n_i}}{n_i!}\mathrm{e}^{-\mathrm{vol}_{\mathbb{R}^d}(\mathcal{A}_i)}. \quad (5.1)$$

We will assume in this section that $\mathcal{P}$ is a fixed realisation of a Poisson process. In a seminal paper, Boldrighini, Bunimovich and Sinai [10] have shown that the limit (1.5) exists almost surely and is given by the linear Boltzmann equation (1.6).

Theorem 5.1 (Boldrighini, Bunimovich and Sinai, 1983 [10]). *The convergence in* (1.5) *holds for a typical realisation* $\mathcal{P}$ *of a Poisson process, and* $f_t = L^t f$ *satisfies the linear Boltzmann equation* (1.6)*.*

This result was previously proved by Gallavotti [20] on average for random $\mathcal{P} = \Theta_{\text{Poisson}}$, and by Spohn [41] for more general random scatterer configurations and scattering potentials.

In the present setting, the limit process $\Xi(t)$ is in fact already a continuous time Markov process and the extension to $\widetilde{\Xi}(t)$ is not necessary. Nevertheless it is instructive to see how the backward equation (4.5) reduces to the linear Boltzmann equation (1.6).

A review of the arguments used in [10] shows that the convergence (3.4) holds in finite-dimensional distribution for almost all $\mathcal{P}$ with limit $\Theta(\boldsymbol{y}) = \Theta_{\text{Poisson}}$ and thus, by the translation invariance of the Poisson process, $\widetilde{\Theta}(\boldsymbol{y}) = \Theta_{\text{Poisson}}$. The limiting point process is evidently independent of $\boldsymbol{y}$, and we may paint all scatterers in the same colour. That is, Σ is the space of one element. We can thus identify Ω with $\mathcal{B}_1^{d-1}$ and set $\mathbb{p}(d\boldsymbol{w}) = v_{d-1}^{-1}d\boldsymbol{w}$. The Poisson distribution yields in (3.6) the transition kernel

$$k(\omega',\xi,\omega) = \overline{\xi}^{-1}\,\mathrm{e}^{-\xi/\overline{\xi}}, \qquad K(\xi,\omega) = \overline{\xi}^{-1}\,\mathrm{e}^{-\xi/\overline{\xi}}. \quad (5.2)$$

The ansatz

$$f_t(\boldsymbol{Q},\boldsymbol{V},\xi,\chi,\boldsymbol{V}_+) = g_t(\boldsymbol{Q},\boldsymbol{V})\,\sigma(\boldsymbol{V},\boldsymbol{V}_+)\,\overline{\xi}^{-1}\,\mathrm{e}^{-\xi/\overline{\xi}} \quad (5.3)$$

in the backward equation (4.5) of $\widetilde{\Xi}(t)$ shows that, after a separation of variables, the function $g_t(\boldsymbol{Q},\boldsymbol{V})$ is a solution of the linear Boltzmann equation (1.6). More directly, one can show that $\Xi(t)$ is Markov, and that the linear Boltzmann equation is the backward equation of $\Xi(t)$.

6. Periodic scatterer configuration

The opposite extreme of a random scatterer configuration is a perfectly periodic point set $\mathcal{P}$. We assume in this section that $\mathcal{P}$ is a Euclidean lattice $\mathcal{L}$ of covolume one. More general periodic scatterer configurations are considered as a special case in the framework of quasicrystals, cf. Section 8.

Theorem 6.1 (Marklof and Strömbergsson, 2008 [29]). *The convergence in* (1.5) *holds for every Euclidean lattice $\mathcal{P} = \mathcal{L}$ of covolume one, where L^t is independent of the choice of $\mathcal{L}$.*

The main result of [29] is in fact more general: It extends to the convergence in distribution of the random process Ξ_r in (3.11) to Ξ. The proof of Theorem 6.1 turns the heuristics of Section 3 into a rigorous argument. Let us describe some of the key objects.

Every Euclidean lattice of covolume one can be written as $\mathcal{L} = \mathbb{Z}^d M$ for some $M \in \mathrm{SL}(d,\mathbb{R})$. Since the stabiliser of $\mathbb{Z}^d$ under right multiplication by $G = \mathrm{SL}(d,\mathbb{R})$ is the subgroup $\Gamma = \mathrm{SL}(d,\mathbb{Z})$, one can show that there is a bijection

$$\begin{aligned} \Gamma\backslash G &\xrightarrow{\sim} \{\text{Euclidean lattices of covolume one}\} \\ \Gamma M &\mapsto \mathbb{Z}^d M. \end{aligned} \tag{6.1}$$

It is a well known fact that any fundamental domain of $\Gamma = \mathrm{SL}(d,\mathbb{Z})$ has finite Haar measure in $G = \mathrm{SL}(d,\mathbb{R})$. This implies that there is a unique probability measure μ on $\Gamma\backslash G$ invariant under the natural G-action (which is multiplication from the right). We define a random point process in $\mathbb{R}^d$ by setting $\Theta_{\text{lattice}} = \mathbb{Z}^d M$ with M random in $\Gamma\backslash G$ according to μ and the above identification (6.1) of $\Gamma\backslash G$ and the space of lattices. We will call Θ_{lattice} a *random lattice*.

The following theorem says that, for any fixed $\mathcal{P} = \mathcal{L}$ the convergence in (3.4) holds with $\Theta = \Theta_{\text{lattice}}$. Note that by translational invariance of $\mathcal{L}$, all point processes in (3.4) are independent of $\boldsymbol{y}$, and we will write in the following $\widetilde{\Theta}_r$ instead of $\widetilde{\Theta}_r(\boldsymbol{y})$.

Theorem 6.2 ([28]). *Let λ be an absolutely continuous probability measure on $\mathcal{B}_1^{d-1}$, let $\mathcal{A}_1,\dots,\mathcal{A}_k \subset \mathbb{R}^d$ bounded with boundary of measure zero and $n_1,\dots,n_k \in \mathbb{Z}_{\geq 0}$. Then*

$$\lim_{r\to 0} \mathbb{P}\big(\#(\widetilde{\Theta}_r \cap \mathcal{A}_i) = n_i\,\forall i\big) = \mathbb{P}\big(\#((\Theta_{\text{lattice}} - (0,\boldsymbol{h})) \cap \mathcal{A}_i) = n_i\,\forall i\big). \tag{6.2}$$

This theorem is a consequence of equidistribution of large spheres on $\Gamma\backslash G$:

Theorem 6.3 ([28]). *For any $M \in \Gamma\backslash G$, any bounded continuous $f : \mathcal{B}_1^{d-1} \times \Gamma\backslash G \to \mathbb{R}$ and any absolutely continuous probability measure λ on $\mathcal{B}_1^{d-1}$,*

$$\lim_{r\to 0} \int_{\mathcal{B}_1^{d-1}} f(\boldsymbol{w}, MS(\boldsymbol{w})D(r))\,d\lambda(\boldsymbol{w}) = \int_{\mathcal{B}_1^{d-1}} \int_{\Gamma\backslash G} f(\boldsymbol{w}, M)\,d\mu(M)\,d\lambda(\boldsymbol{w}). \tag{6.3}$$

Theorem 6.2 is derived from Theorem 6.3 by choosing in (6.3) as test function f the characteristic function of the set

$$\big\{(\boldsymbol{w}, M) \in \mathcal{B}_1^{d-1} \times \Gamma\backslash G : \#\big((\mathbb{Z}^d M - (0,\boldsymbol{w}) \cap \mathcal{A}_i\big) = n_i\,\forall i\big\}. \tag{6.4}$$

This choice does of course not produce a continuous f, but one can show that (6.4) has boundary of measure zero in $\mathcal{B}_1^{d-1} \times \Gamma\backslash G$, and thus the characteristic function can be approximated sufficiently well by continuous functions. Details of this technical argument can be found in [28], Sections 5 and 6.

Since the limit process Θ_{lattice} is independent of $\boldsymbol{y}$ there is no need for colour (as in the Poisson setting), and we again identify Ω with $\mathcal{B}_1^{d-1}$, and set $\mathbb{p}(d\boldsymbol{w}) = v_{d-1}^{-1} d\boldsymbol{w}$. In order to work out the transition kernel $k(\boldsymbol{w}', \xi, \boldsymbol{w})$ in (3.10), set $X = \Gamma\backslash G$ and define the subspace

$$X(\boldsymbol{y}) = \{M \in X : \boldsymbol{y} \in \mathbb{Z}^d M\} \tag{6.5}$$

of those lattices (of covolume one) that contain a given $\boldsymbol{y} \in \mathbb{R}^d$. In [28] we construct a probability measure $\nu_{\boldsymbol{y}}$ on $X(\boldsymbol{y})$ so that

$$d\mu(M) = d\nu_{\boldsymbol{y}}(M)\, d\boldsymbol{y}. \tag{6.6}$$

With this, we can infer that

$$k(\boldsymbol{w}', \xi, \boldsymbol{w}) = \overline{\xi}^{-1} \nu_{\boldsymbol{y}}\big(\big\{M \in X(\boldsymbol{y}) : \mathbb{Z}^d M \cap (\mathfrak{Z}(\xi) + (0, \boldsymbol{w}')) = \emptyset\big\}\big) \tag{6.7}$$

where $\boldsymbol{y} = (\xi, \boldsymbol{w}' - \boldsymbol{w})$. For an explicit description of the $\nu_{\boldsymbol{y}}$-measure of the above set, see [30], Section 2.2. In dimension $d = 2$, when $\mathcal{B}_1^1 =]-1, 1[$, eq. (6.7) can be used to calculate an explicit formula for the transition kernel. We have [27]

$$k(\boldsymbol{w}', \xi, \boldsymbol{w}) = \frac{12}{\pi^2}\, \Upsilon\left(1 + \frac{\xi^{-1} - \max(|\boldsymbol{w}|, |\boldsymbol{w}'|) - 1}{|\boldsymbol{w} - \boldsymbol{w}'|}\right) \tag{6.8}$$

with

$$\Upsilon(x) = \begin{cases} 0 & \text{if } x \le 0 \\ x & \text{if } 0 < x < 1 \\ 1 & \text{if } x \ge 1. \end{cases} \tag{6.9}$$

For independent derivations of Formula (6.8) that do not employ eq. (6.7) but a more direct approach based on Farey dissections, see Bykovskii and Ustinov [13] and Caglioti and Golse [14].

There are no such formulas in higher dimension, although (6.7) can be used to extract information to obtain asymptotics for $\xi \to 0$ and $\xi \to \infty$, cf. [30]. We have in particular

$$\frac{1 - 2^{d-1}\overline{\xi}^{-1}\xi}{\zeta(d)\overline{\xi}} \le k(\boldsymbol{w}', \xi, \boldsymbol{w}) \le \frac{1}{\zeta(d)\overline{\xi}}, \tag{6.10}$$

and so for small ξ this implies $k(\boldsymbol{w}', \xi, \boldsymbol{w}) = (\zeta(d)\overline{\xi})^{-1} + O(\xi)$. Here $\zeta(d)$ is the Riemann zeta function and $\zeta(d)^{-1}$ is the relative density of primitive lattice points in $\mathbb{Z}^d$. Compare (6.10) with the result for the Poisson process (Section 5):

$$k_{\text{Poisson}}(\boldsymbol{w}', \xi, \boldsymbol{w}) = \overline{\xi}^{-1} \mathrm{e}^{-\xi/\overline{\xi}} = \overline{\xi}^{-1} - \overline{\xi}^{-2}\xi + O(\xi^2). \tag{6.11}$$

The asymptotics of $k(\boldsymbol{w}', \xi, \boldsymbol{w})$ for large ξ is more complicated to state, see [30]. We will here focus on tail asymptotics for the distribution of free path lengths [30]. For any $\xi > 0$, we have

$$\Psi_0(\xi) = \frac{1}{\overline{\xi}\zeta(d)} + O(\xi), \tag{6.12}$$

and for $\xi \to \infty$

$$\Psi_0(\xi) = \frac{A_d}{\xi^3} + O\big(\xi^{-3-\frac{2}{d}}\big) \begin{cases} 1 & \text{if } d = 2 \\ \log \xi & \text{if } d = 3 \\ 1 & \text{if } d \ge 4 \end{cases} \tag{6.13}$$

with the constant

$$A_d = \frac{2^{2-d}}{d(d+1)\zeta(d)}. \tag{6.14}$$

These asymptotics sharpen earlier upper and lower bounds by Bourgain, Golse and Wennberg [11, 21]. Note that (6.13) implies that the density $\Psi_0(\xi)$ has no second moment. In dimension $d = 2$ there is an explicit formula for $\Psi_0(\xi)$ conjectured by Dahlqvist [16], and proved by Boca and Zaharescu [7]. This formula of course also follows directly from the expression for the transition kernel (6.8), cf. [27].

7. Several lattices

The previous two examples, random and periodic, could be analysed without the need to introduce colour. We will now describe a first example where the extension of $\Xi(t)$ to a Markov process $\widetilde{\Xi}(t)$ (as outlined in Section 4) requires finitely many colours.

We consider a scattering configuration given by the union of N distinct affine Euclidean lattices,

$$\mathcal{P} = \bigcup_{i=1}^{N} \mathcal{L}_i \tag{7.1}$$

where each $\mathcal{L}_i$ has covolume $\overline{n}_i^{-1}$. We will assume that the lattices are *pairwise incommensurable* in the sense that for any $i \neq j$, $c > 0$ and $\boldsymbol{a} \in \mathbb{R}^d$, the intersection $\mathcal{L}_i \cap (c\mathcal{L}_j + \boldsymbol{a})$ is contained in some affine linear subspace of dimension strictly less than d.[16] This ensures in particular that the density of $\mathcal{P}$ is $\overline{n} = \overline{n}_1 + \ldots + \overline{n}_N$. As before, we stipulate without loss of generality that $\overline{n} = 1$.

To describe the random point processes and corresponding collision kernels, we require, in addition to a random lattice Θ_{lattice} in the previous section, the notion of a random *affine* lattice. This is defined as $\Theta_{\text{affine}} = (\mathbb{Z}^d + \boldsymbol{\alpha})M$ where $\boldsymbol{\alpha}$ is a random variable uniformly distributed in $\mathbb{T}^d = \mathbb{Z}^d\backslash\mathbb{R}^d$ and M is distributed with respect to Haar measure μ on $\Gamma\backslash G$ as before. Note that Θ_{affine} is well defined, since $\mathbb{T}^d$ and the Lebesgue measure on $\mathbb{T}^d$ are invariant under the natural Γ action (by right multiplication). We denote by $\Theta^{(1)}_{\text{affine}}, \ldots, \Theta^{(N)}_{\text{affine}}$ independent copies of Θ_{affine}, which are furthermore independent of Θ_{lattice}.

For $\boldsymbol{y} \in \mathcal{L}_j$ for some j, and $\boldsymbol{y} \notin \mathcal{L}_i$ for all $i \neq j$, we define the point process $\Theta_{\text{union}}(\boldsymbol{y})$ by

$$\Theta_{\text{union}}(\boldsymbol{y}) = \overline{n}_j^{-1/d}\Theta_{\text{lattice}} \cup \bigcup_{i\neq j} \big(\overline{n}_i^{-1/d}\Theta^{(i)}_{\text{affine}}\big). \tag{7.2}$$

In the following theorem, we say $\boldsymbol{y} \in \mathcal{P}$ is *generic*, if $\boldsymbol{y} \in \mathcal{L}_j$ is not rationally related to the other lattices $\mathcal{L}_i$ ($i \neq j$) in a sense made precise in [31] (see the discussion after [31, Thm. 1]). The set of non-generic $\boldsymbol{y}$ in $\mathcal{P}$ is contained in a finite union of affine subspaces of dimension $< d$, and hence has zero relative density.

Theorem 7.1 ([31]). *Let λ be an absolutely continuous probability measure on $\mathcal{B}_1^{d-1}$, let $\mathcal{A}_1, \ldots, \mathcal{A}_k \subset \mathbb{R}^d$ bounded with boundary of measure zero and $n_1, \ldots, n_k \in \mathbb{Z}_{\geq 0}$. Then,*

[16] This condition is not essential in the proof of convergence, but ensures that the limit distributions have a particularly simple form. The case when all N lattices are commensurable is a special case of the setting discussed in Section 8.

for generic $\boldsymbol{y} \in \mathcal{P}$,

$$\lim_{r\to 0} \mathbb{P}(\#(\widetilde{\Theta}_r(\boldsymbol{y}) \cap \mathcal{A}_i) = n_i \,\forall i) = \mathbb{P}\big(\#((\Theta_{\text{union}}(\boldsymbol{y}) - (0,\boldsymbol{h})) \cap \mathcal{A}_i) = n_i \,\forall i\big). \tag{7.3}$$

The current setting requires N colours. In the notation of Section 3, we set $\Sigma = \{1,\dots,N\}$, $\iota(\boldsymbol{y}) = i$ if $\boldsymbol{y} \in \mathcal{L}_i$, and define $\mathfrak{m}$ as the probability measure on Σ so that $\mathfrak{m}(\{i\}) = \overline{n}_i$. We prove in [31] that the probability of emerging from a generic (as defined above) scatterer with a given colour j' and random exit parameter $\boldsymbol{w}'$ (distributed according to a fixed, absolutely continuous Borel probability measure λ on $\mathcal{B}_1^{d-1}$), and hitting the next scatterer at time $T_n \in \,]\xi, \xi + d\xi[$ with colour j and impact parameter $\boldsymbol{w} \in B \subset \mathcal{B}_1^{d-1}$ converges in the Boltzmann-Grad limit to (3.10). If the lattices are incommensurable as assumed above, the transition kernel in (3.10) is given by

$$k((\boldsymbol{w}',j),\xi,(\boldsymbol{w},j)) = k^{(1)}(\boldsymbol{w}',\overline{n}_j\xi,\boldsymbol{w}) \prod_{\substack{i=1\\ i\neq j}}^{N} \int_{\overline{n}_i\xi}^{\infty} \Psi(\xi')\,d\xi', \tag{7.4}$$

and for $j' \neq j$,

$$k((\boldsymbol{w}',j'),\xi,(\boldsymbol{w},j)) = \overline{\xi}\, K^{(1)}(\overline{n}_{j'}\xi,\boldsymbol{w}')\, K^{(1)}(\overline{n}_j\xi,\boldsymbol{w}) \prod_{\substack{i=1\\ i\neq j',j}}^{N} \int_{\overline{n}_i\xi}^{\infty} \Psi(\xi')\,d\xi', \tag{7.5}$$

where $k^{(1)}(\boldsymbol{w}',\xi,\boldsymbol{w})$ is the transition kernel for a single lattice in (6.7), $K^{(1)}(\xi,\boldsymbol{w})$ the corresponding integrated kernel in (3.15) for a single lattice, and

$$\Psi(\xi) := \frac{1}{v_{d-1}} \int_{\mathcal{B}_1^{d-1}} K^{(1)}(\xi,\boldsymbol{w})\,d\boldsymbol{w}. \tag{7.6}$$

The above formulas and (6.13) imply the following tail estimate for the distribution of free path lengths:

$$\Psi_0(\xi) = \frac{N(N+1)A_d^N\overline{\sigma}^{N-1}}{2^N\overline{n}_1\cdots\overline{n}_N}\,\xi^{-(N+2)} \times \begin{cases} \big(1+O(\xi^{-1})\big) & \text{if } d=2\\ \big(1+O(\xi^{-\frac{2}{3}}\log\xi)\big) & \text{if } d=3\\ \big(1+O(\xi^{-\frac{2}{d}})\big) & \text{if } d\geq 4. \end{cases} \tag{7.7}$$

The proof of the above results follows the same strategy as in the single-lattice case studied in Section 6. The principal difference is that the equidistribution in the space of lattices stated in Theorem 6.3 has to be generalised to the equidistribution in products: Consider the subgroup $\widehat{\Gamma} = \Gamma_1 \times \cdots \times \Gamma_N$ in $\mathrm{SL}(d,\mathbb{R})^N$, where each Γ_i is a lattice in $\mathrm{SL}(d,\mathbb{R})$. We denote by $\mu_{\widehat{\Gamma}}$ the unique $\mathrm{SL}(d,\mathbb{R})^N$ invariant probability measure on $\widehat{\Gamma}\backslash\mathrm{SL}(d,\mathbb{R})^N$, and by φ the diagonal embedding of $\mathrm{SL}(d,\mathbb{R})$ in $\mathrm{SL}(d,\mathbb{R})^N$, i.e. $\varphi(M) = (M,\dots,M)$. Recall that two lattices Γ and Γ' in $\mathrm{SL}(d,\mathbb{R})$ are said to be *commensurable* if their intersection $\Gamma\cap\Gamma'$ is also a lattice; otherwise Γ and Γ' are *incommensurable*.

Theorem 7.2 ([31]). *Let $\Gamma_1,\dots,\Gamma_N \in \mathrm{SL}(d,\mathbb{R})$ be pairwise incommensurable lattices, and $M \in \mathrm{SL}(d,\mathbb{R})$. Let λ be a Borel probability measure on $\mathcal{B}_1^{d-1}$, absolutely continuous with respect to Lebesgue measure, and let $f : \mathcal{B}_1^{d-1} \times \widehat{\Gamma}\backslash\mathrm{SL}(d,\mathbb{R})^N \to \mathbb{R}$ be bounded continuous. Then*

$$\lim_{r\to 0}\int_{\mathcal{B}_1^{d-1}} f\big(\boldsymbol{w},\varphi(MS(\boldsymbol{w})D(r))\big)\,d\lambda(\boldsymbol{w}) = \int_{\mathcal{B}_1^{d-1}\times\widehat{\Gamma}\backslash\,\mathrm{SL}(d,\mathbb{R})^N} f(\boldsymbol{w},g)\,d\lambda(\boldsymbol{w})\,d\mu_{\widehat{\Gamma}}(g). \quad (7.8)$$

The key ingredient in the proof of this statement is Ratner's measure classification theorem [38] via a theorem of Shah on the equidistribution of translates of unipotent orbits [40, Thm. 1.4]. Theorem 6.3 corresponds to the special case $N = 1$. For $N = 2$ the proof is simpler than for $N \geq 3$, see [26]. Theorem 7.2 is in fact an oversimplification—the proof of convergence to the transition kernel $k(\omega', \xi, \omega)$ in fact requires a variant of Theorem 7.2 for products of spaces of affine lattices, cf. [31, Thm. 10].

The paper [31] proves the convergence to $k(\omega', \xi, \omega)$ for a random exit parameter with fixed probability measure λ. What is still missing is a proof of the analogue of Theorem 5.1 (for random scatterer configurations $\mathcal{P}$) or Theorem 6.1 (where $\mathcal{P}$ is a single lattice). It is likely that the proof will follow the same line of arguments as in the periodic setting [29].

8. Quasicrystals

The third class of examples for scattering configurations $\mathcal{P}$ that lead to a generalised Boltzmann equation—and the second that requires colour—are *quasicrystals*. We restrict our attention to quasicrystals constructed by the *cut-and-project method*, following closely the presentation in [32]. Examples include many classic quasicrystals (such as the vertex set of a Penrose tiling) as well as locally finite periodic point sets. In contrast to the previous section, cut-and-project scatterer configurations generally require a continuous spectrum of colours.

A cut-and-project set $\mathcal{P} \subset \mathbb{R}^d$ is defined as follows, cf. [1]. For $m \geq 0$, $n = d + m$, let

$$\pi : \mathbb{R}^n \to \mathbb{R}^d, \qquad \pi_{\mathrm{int}} : \mathbb{R}^n \to \mathbb{R}^m \quad (8.1)$$

be the orthogonal projections of $\mathbb{R}^n = \mathbb{R}^d \times \mathbb{R}^m$ onto the first and second factor, respectively. $\mathbb{R}^d$ will be called the *physical space*, and $\mathbb{R}^m$ the *internal space*. Let $\mathcal{L} \subset \mathbb{R}^n$ be a lattice of full rank. The closure

$$\mathcal{A} := \overline{\pi_{\mathrm{int}}(\mathcal{L})} \subset \mathbb{R}^m \quad (8.2)$$

is an abelian subgroup, and we denote by $\mathcal{A}^0$ the connected component of $\mathcal{A}$ containing 0. $\mathcal{A}^0$ is a linear subspace of $\mathbb{R}^m$ of dimension m_1. We find vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{m_2}$ ($m = m_1 + m_2$) so that

$$\mathcal{A} = \mathcal{A}^0 \oplus \mathbb{Z}\pi(\boldsymbol{a}_1) \oplus \ldots \oplus \mathbb{Z}\pi(\boldsymbol{a}_{m_2}). \quad (8.3)$$

The Haar measure of $\mathcal{A}$ is denoted by $\mu_{\mathcal{A}}$ and normalised so that $\mu_{\mathcal{A}}\big|_{\mathcal{A}^0}$ is the standard Lebesgue measure on $\mathcal{A}^0$. For $\mathcal{V} := \mathbb{R}^d \times \mathcal{A}^0$, we note that $\mathcal{L} \cap \mathcal{V}$ is a full rank lattice in $\mathcal{V}$. For $\mathcal{W} \subset \mathcal{A}$ with non-empty interior, we call

$$\mathcal{P} = \mathcal{P}(\mathcal{W}, \mathcal{L}) = \{\pi(\boldsymbol{\ell}) : \boldsymbol{\ell} \in \mathcal{L},\ \pi_{\mathrm{int}}(\boldsymbol{\ell}) \in \mathcal{W}\} \quad (8.4)$$

a *cut-and-project set*. $\mathcal{W}$ is called the *window set*. If the boundary of the window set has $\mu_{\mathcal{A}}$-measure zero, we say $\mathcal{P}(\mathcal{W}, \mathcal{L})$ is *regular*. We will furthermore assume that $\mathcal{W}$ and $\mathcal{L}$ are chosen so that the map

$$\pi_{\mathcal{W}} : \{\boldsymbol{\ell} \in \mathcal{L} : \pi_{\mathrm{int}}(\boldsymbol{\ell}) \in \mathcal{W}\} \to \mathcal{P} \quad (8.5)$$

is bijective. This is to avoid coincidences in $\mathcal{P}$. It follows from Weyl equidistribution that such $\mathcal{P}$ have density

$$\overline{n} = \frac{\mu_{\mathcal{A}}(\mathcal{W})}{\mathrm{vol}_{\mathbb{R}^d}(\mathcal{V}/(\mathcal{L}\cap\mathcal{V}))}. \tag{8.6}$$

Furthermore, for $\boldsymbol{y}\in\mathcal{P}$ there is $\boldsymbol{\ell}\in\mathcal{L}$ such that $\boldsymbol{\ell}=\pi(\boldsymbol{y})$ and

$$\mathcal{P}(\mathcal{W},\mathcal{L})-\boldsymbol{y}=\mathcal{P}(\mathcal{W}-\boldsymbol{y}_{\rm int},\mathcal{L}),\qquad \boldsymbol{y}_{\rm int}:=\pi_{\rm int}(\boldsymbol{\ell}). \tag{8.7}$$

This suggests to define the colour chart $\iota:\mathcal{P}\to\Sigma:=\mathcal{W}$ with $\iota(\boldsymbol{y})=\boldsymbol{y}_{\rm int}$. The aim is now to describe the "closure" (in a suitable sense) of the orbit of $\mathcal{P}$ under the $\mathrm{SL}(d,\mathbb{R})$-action and construct a probability measure on it. This will yield, as we shall see, our limit random process $\Theta(\boldsymbol{y})$ in (3.4).

Set $G=\mathrm{SL}(n,\mathbb{R})$, $\Gamma=\mathrm{SL}(n,\mathbb{Z})$ and define the embedding (for any $g\in G$)

$$\varphi_g:\mathrm{SL}(d,\mathbb{R})\hookrightarrow G,\qquad A\mapsto g\begin{pmatrix}A & 0_{d\times m}\\ 0_{m\times d} & 1_m\end{pmatrix}g^{-1}. \tag{8.8}$$

Since $\mathrm{SL}(d,\mathbb{R})$ is generated by unipotent subgroups, Ratner's theorems [38, 39] imply that there is a (unique) closed connected subgroup $H_g\le G$ such that:

(i) $\Gamma\cap H_g$ is a lattice in H_g.

(ii) $\varphi_g(\mathrm{SL}(d,\mathbb{R}))\subset H_g$.

(iii) The closure of $\Gamma\backslash\Gamma\varphi_g(\mathrm{SL}(d,\mathbb{R}))$ is $\Gamma\backslash\Gamma H_g$.

We will call H_g a *Ratner subgroup*. We denote the unique H_g-invariant probability measure on $\Gamma\backslash\Gamma H_g$ by $\mu_{H_g}=\mu_g$. Note that $\Gamma\backslash\Gamma H_g$ is isomorphic to the homogeneous space $(\Gamma\cap H_g)\backslash H_g$.

Pick $g\in G$, $\delta>0$ such that $\mathcal{L}=\delta^{1/n}\mathbb{Z}^n g$. Then one can show [32, Prop. 3.5] that $\pi_{\rm int}(\delta^{1/n}\mathbb{Z}^n hg)\subset\mathcal{A}$ for all $h\in H_g$, and $\pi_{\rm int}(\delta^{1/n}\mathbb{Z}^n hg)=\mathcal{A}$ for μ_g-almost all $h\in H_g$. The image of the map

$$\Gamma\backslash\Gamma H_g\to\{\text{point sets in }\mathbb{R}^d\},\qquad h\mapsto\mathcal{P}(\mathcal{W}-\boldsymbol{y}_{\rm int},\delta^{1/n}\mathbb{Z}^n hg) \tag{8.9}$$

defines a space of cut-and-project sets, and the push-forward of μ_g equips it with a probability measure. We have thus defined a random point process $\Theta_{\text{quasi}}(\boldsymbol{y})$ in $\mathbb{R}^d$, which is $\mathrm{SL}(d,\mathbb{R})$ invariant, and whose typical realisation is a cut-and-project set with window $\mathcal{W}-\boldsymbol{y}_{\rm int}$ and internal space $\mathcal{A}$. This process is precisely the limit process we are looking for:

Theorem 8.1 ([32]). *Let λ be an absolutely continuous probability measure on $\mathcal{B}_1^{d-1}$, let $\mathcal{A}_1,\dots,\mathcal{A}_k\subset\mathbb{R}^d$ bounded with boundary of measure zero and $n_1,\dots,n_k\in\mathbb{Z}_{\ge0}$. Then, for every $\boldsymbol{y}\in\mathcal{P}(\mathcal{W},\mathcal{L})$,*

$$\lim_{r\to0}\mathbb{P}(\#(\widetilde{\Theta}_r(\boldsymbol{y})\cap\mathcal{A}_i)=n_i\,\forall i)=\mathbb{P}\big(\#((\Theta_{\text{quasi}}(\boldsymbol{y})-(0,\boldsymbol{h}))\cap\mathcal{A}_i)=n_i\,\forall i\big). \tag{8.10}$$

This statement is (as in previous sections) a consequence of equidistribution. The following equidistribution theorems generalise Theorem 6.3 stated earlier, and are used in the proof of Theorem 8.1. As in the case of Theorem 7.2, they are a consequence of Ratner's measure classification theorems [38], and in particular follow from a theorem of Shah [40, Thm. 1.4] on the equidistribution of translates of unipotent orbits.

Theorem 8.2 ([32]). *Fix $g \in G$, $M \in \mathrm{SL}(d,\mathbb{R})$. For any bounded continuous $f : \mathcal{B}_1^{d-1} \times \Gamma\backslash\Gamma H_g \to \mathbb{R}$ and any absolutely continuous probability measure λ on $\mathcal{B}_1^{d-1}$,*

$$\lim_{r\to 0} \int_{\mathcal{B}_1^{d-1}} f(\boldsymbol{w}, \varphi_g(MS(\boldsymbol{w})D(r)))\, d\lambda(\boldsymbol{w}) = \int_{\mathcal{B}_1^{d-1}} \int_{\Gamma\backslash\Gamma H_g} f(\boldsymbol{w}, h)\, d\mu_g(h)\, d\lambda(\boldsymbol{w}). \tag{8.11}$$

What are the subgroups H_g that can arise in the above construction? For almost every lattice $\mathcal{L}$ in the space of lattices, we have $H_g = G$. Furthermore, if $m < d$, then for *every* $\mathcal{L}$ with the property that $\pi|_{\mathcal{L}}$ is injective, we have $H_g = G$ [32, Prop. 2.1]. A interesting class of examples when $m \geq d$ and $H_g \neq G$ are cut-and-project sets constructed from algebraic number fields. The Penrose tilings fall into this class. Let us briefly sketch how such quasicrystals can be obtained as cut-and-project sets. Let K be a totally real number field of degree $N \geq 2$ over $\mathbb{Q}$, $\mathfrak{O}_K$ the ring of integers of K, and $\pi_1 = \mathrm{id}, \pi_2, \ldots, \pi_N$ the distinct embeddings $K \hookrightarrow \mathbb{R}$. We also use π_i to denote the component-wise embeddings

$$\pi_i : K^d \hookrightarrow \mathbb{R}^d, \qquad \boldsymbol{x} \mapsto (\pi_i(x_1), \ldots, \pi_i(x_d)), \tag{8.12}$$

and similarly for the entry-wise embeddings of $d \times d$ matrices,

$$\pi_i : \mathrm{M}_d(K) \hookrightarrow \mathrm{M}_d(\mathbb{R}). \tag{8.13}$$

Now consider the lattice

$$\mathcal{L} = \{(\boldsymbol{x}, \pi_2(\boldsymbol{x}), \ldots, \pi_N(\boldsymbol{x})) : \boldsymbol{x} \in \mathfrak{O}_K^d\} \tag{8.14}$$

in $\mathbb{R}^{Nd}$. This is a lattice of full rank. The dimension of the internal space is $m = (N-1)d$. It is a fact of "basic" number theory [44] that $\mathcal{A} := \overline{\pi_{\mathrm{int}}(\mathcal{L})} = \mathbb{R}^m$, so that $\mathcal{V} = \mathbb{R}^{Nd}$. Choose $g \in G$ and $\delta > 0$ so that $\mathcal{L} = \delta^{1/Nd}\mathbb{Z}^{Nd} g$. Then [32, Sect. 2.2.1.] shows that

$$H_g = g\,\mathrm{SL}(d,\mathbb{R})^N g^{-1}, \qquad \Gamma \cap H_g = g\,\mathrm{SL}(d,\mathfrak{O}_K) g^{-1}, \tag{8.15}$$

where $\mathrm{SL}(d,\mathfrak{O}_K)$ is a Hilbert modular group.

A further example of a cut-and-project set is to take the union of finite translates of a given cut-and-project set. This is explained in [32, Sect. 2.3]. Let us here discuss the special case of periodic Delone sets, i.e., the union finite translates of a given lattice $\mathcal{L}_0$ of full rank in $\mathbb{R}^d$. An example of such a set is the honeycomb lattice, which in the context of the Boltzmann-Grad limit of the Lorentz gas was recently studied by Boca et al. [8, 9] with different techniques. The scatterer configuration $\mathcal{P}$ we are now interested in is the union of m copies of the same lattice $\mathcal{L}_0$ translated by $\boldsymbol{t}_1, \ldots, \boldsymbol{t}_m \in \mathbb{R}^d$,

$$\mathcal{P} = \bigcup_{j=1}^{m} (\boldsymbol{t}_j + \mathcal{L}_0). \tag{8.16}$$

We assume that the $\boldsymbol{t}_j$ are chosen in such a way that the above union is disjoint. Let us now show that $\mathcal{P}$ can be realised as a cut-and-project set $\mathcal{P}(\mathcal{L}, \mathcal{W})$. Let

$$\mathcal{L} = (\mathcal{L}_0 \times \{\boldsymbol{0}\}) + \sum_{j=1}^{m} \mathbb{Z}\,(\boldsymbol{t}_j, \boldsymbol{e}_j) \subset \mathbb{R}^n, \tag{8.17}$$

where $\mathbf{0} \in \mathbb{R}^m$ and $\boldsymbol{e}_1, \dots, \boldsymbol{e}_m$ are the standard basis vectors in $\mathbb{R}^m$. The set $\mathcal{L}$ is evidently a lattice of full rank in $\mathbb{R}^n$. Note that

$$\pi_{\mathrm{int}}(\mathcal{L}) = \sum_{j=1}^{m} \mathbb{Z}\, \boldsymbol{e}_j = \mathbb{Z}^m, \tag{8.18}$$

and therefore the closure of this set is $\mathcal{A} = \mathbb{Z}^m$ with connected component $\mathcal{A}^0 = \{\mathbf{0}\}$. It follows that for the window set

$$\mathcal{W} = \bigcup_{j=1}^{m} \{\boldsymbol{e}_j\} \subset \mathcal{A} \tag{8.19}$$

we indeed have

$$\mathcal{P}(\mathcal{L}, \mathcal{W}) = \bigcup_{j=1}^{m} (\boldsymbol{t}_j + \mathcal{L}_0). \tag{8.20}$$

Let us now determine H_g in this setting. Take $g_0 \in \mathrm{SL}(d, \mathbb{R})$ so that $\mathcal{L}_0 = \overline{n}_0^{-1/d} \mathbb{Z}^d g_0$, where $\overline{n}_0$ is the density of $\mathcal{L}_0$. Set

$$T = \begin{pmatrix} \boldsymbol{t}_1 \\ \vdots \\ \boldsymbol{t}_m \end{pmatrix} \in \mathrm{M}_{m\times d}(\mathbb{R}). \tag{8.21}$$

We then have $\mathcal{L} = \overline{n}_0^{-1/n} \mathbb{Z}^n g$, for

$$g = \overline{n}_0^{1/n} \begin{pmatrix} \overline{n}_0^{-1/d} g_0 & 0 \\ T & 1_m \end{pmatrix} \in \mathrm{SL}(n, \mathbb{R}). \tag{8.22}$$

Suppose $\boldsymbol{a}_1, \dots, \boldsymbol{a}_d$ is a basis of $\mathcal{L}_0$ so that the vectors $\boldsymbol{a}_1, \dots, \boldsymbol{a}_d, \boldsymbol{t}_1, \dots, \boldsymbol{t}_m$ are linearly independent over $\mathbb{Q}$. Then

$$H_g = \left\{ \begin{pmatrix} h & 0 \\ u & 1_m \end{pmatrix} : h \in \mathrm{SL}(d, \mathbb{R}),\ u \in \mathrm{M}_{m\times d}(\mathbb{R}) \right\}. \tag{8.23}$$

The Ratner subgroups that appear in the case of rational translates $\boldsymbol{t}_j$ are discussed in [32, Sect. 2.3.1].

Theorem 8.1 gives a complete description of the limit processes $\Theta(\boldsymbol{y})$ that may arise in the case of cut-and-project sets (as defined above). This answers in particular a question on the distribution of free path lengths raised by Wennberg [45], see [32] for details. We do not have a comprehensive solution to the remaining *"Does the limit* (1.5) *exist?"* and *"What is the transition kernel* $k(\omega', \xi, \omega)$*?"* yet, but plan to address these in a forthcoming paper [33].

9. Superdiffusion

One of the central challenges in non-equilibrium statistical mechanics is to establish whether the dynamics of a test particle converges, in the limit of large times and after a suitable rescaling of length units, to Brownian motion. The first important step in the proof of such

an invariance principle is the central limit theorem for the displacement $\boldsymbol{Q}(t) - \boldsymbol{Q}_0$, suitably normalised by a factor $\sigma(t)$. If $\sigma(t) \asymp \sqrt{t}$, we say the dynamics is *diffusive*. If $\sigma(t)/\sqrt{t} \to 0$ or $\sigma(t)/\sqrt{t} \to \infty$ as $t \to \infty$, the dynamics is called *subdiffusive* or *superdiffusive*, respectively. In the case of fixed scatterer radius r, most results are restricted to the periodic setting and dimension $d = 2$, recall Section 1. In the case of the Boltzmann-Grad limit with a random scatterer configuration, we have a central limit theorem with standard $\sqrt{t}$ normalisation:

Theorem 9.1. *Let $\boldsymbol{Q}^{\mathrm{BG}}(t)$ denote the position variable of the random flight process $\Xi(t)$ for a Poisson scatterer configuration (cf. Section 5). Then there exists a constant $\sigma_d > 0$ such that, for any bounded continuous $f : \mathbb{R}^d \to \mathbb{R}$ and any*[17] *$(\boldsymbol{Q}_0, \boldsymbol{V}_0) \in \mathrm{T}^1(\mathbb{R}^d)$,*

$$\lim_{t\to\infty} \mathbf{E}\, f\left(\frac{\boldsymbol{Q}^{\mathrm{BG}}(t) - \boldsymbol{Q}_0}{\sigma_d\sqrt{t}}\right) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(\boldsymbol{x})\, \mathrm{e}^{-\frac{1}{2}\|\boldsymbol{x}\|^2} d\boldsymbol{x}. \tag{9.1}$$

This theorem follows from standard techniques in the theory of Markov processes [37], as pointed out by Spohn [41]. On the other hand, the Boltzmann-Grad limit of a periodic Lorentz gas satisfies a superdiffusive central limit theorem with $\sqrt{t \log t}$ normalisation:

Theorem 9.2 (Marklof and Tóth, 2014 [34]). *Let $\boldsymbol{Q}^{\mathrm{BG}}(t)$ denote the position variable of the random flight process $\Xi(t)$ for a periodic scatterer configuration (cf. Section 6). Then, for any bounded continuous $f : \mathbb{R}^d \to \mathbb{R}$ and any*[18] *$(\boldsymbol{Q}_0, \boldsymbol{V}_0) \in \mathrm{T}^1(\mathbb{R}^d)$,*

$$\lim_{t\to\infty} \mathbf{E}\, f\left(\frac{\boldsymbol{Q}^{\mathrm{BG}}(t) - \boldsymbol{Q}_0}{\Sigma_d\sqrt{t\log t}}\right) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(\boldsymbol{x})\, \mathrm{e}^{-\frac{1}{2}\|\boldsymbol{x}\|^2} d\boldsymbol{x} \tag{9.2}$$

with $\Sigma_d^2 := \frac{A_d}{2d\xi}$.

Recall that A_d is the constant in the tail asymptotics of the free path lengths (6.13). This means in particular that Σ_d is independent of the choice of scattering map (within the admissible class). Although the superdiffusive scaling is intimately related to the fact that the second moment of the distribution of free path lengths diverges, the proof of Theorem 9.2 requires further information on the transition kernel $k(\omega', \xi, \omega)$. The main ingredients of our proof are (a) exponential decay of correlations in the sequence of random variables $(\eta_n, \boldsymbol{V}_n)_{n\in\mathbb{N}}$ and (b) the Lindeberg central limit theorem for the independent random variables $(\xi_n|\underline{\eta})_{n\in\mathbb{N}}$ conditioned on $\underline{\eta} = (\eta_n)_{n\in\mathbb{N}}$. For full details, see [34].

Acknowledgments. The research leading to the results presented here has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP/2007-2013) / ERC Grant Agreement n. 291147. The author is furthermore supported by a Royal Society Wolfson Research Merit Award. Much of the material presented in this paper is based on joint work with Andreas Strömbergsson and Bálint Tóth. I would like to thank Andreas and Bálint for the fruitful collaboration. I am grateful to Daniel El-Baz, Jory Griffin, Andreas Strömbergsson, Bálint Tóth, Jim Tseng and Ilya Vinogradov for their comments on the first draft of this paper.

[17] Because we have already passed to the Boltzmann-Grad limit, we may here consider the random process $\boldsymbol{Q}^{\mathrm{BG}}(t)$ either with *fixed* initial data (as stated) or with random initial data distributed according to Λ (as assumed in all previous sections).

[18] Cf. footnote 17.

References

[1] M. Baake and U. Grimm, *Aperiodic order*, Vol. 1, Encyclopedia Math. Appl. 149, Cambridge University Press, Cambridge, 2013.

[2] P. Bálint, N. Chernov, and D. Dolgopyat, *Limit theorems for dispersing billiards with cusps*, Comm. Math. Phy. **308** (2011), 479–510.

[3] P. Bálint and S. Gouëzel, *Limit theorems in the stadium billiard*, Comm. Math. Phys. **263** (2006), 461–512.

[4] P. Bálint and I.P. Tóth, *Exponential decay of correlations in multi-dimensional dispersing billiards*, Ann. Henri Poincaré **9** (2008), 1309–1369.

[5] ———, *Example for exponential growth of complexity in a finite horizon multi-dimensional dispersing billiard*, Nonlinearity **25** (2012), 1275–1297.

[6] P.M. Bleher, *Statistical properties of two-dimensional periodic Lorentz gas with infinite horizon*, J. Stat. Phys. **66** (1992), 315–373.

[7] F.P. Boca and A. Zaharescu, *The distribution of the free path lengths in the periodic two-dimensional Lorentz gas in the small-scatterer limit*, Comm. Math. Phys. **269** (2007), 425–471.

[8] F.P. Boca and R.N. Gologan, *On the distribution of the free path length of the linear flow in a honeycomb*, Ann. Inst. Fourier (Grenoble) **59** (2009), 1043–1075.

[9] F.P. Boca, *Distribution of the linear flow length in a honeycomb in the small-scatterer limit*, New York J. Math. **16** (2010), 651–735.

[10] C. Boldrighini, *L.A. Bunimovich and Y.G. Sinai, On the Boltzmann equation for the Lorentz gas*, J. Stat. Phys. **32** (1983), 477–501.

[11] J. Bourgain, F. Golse and B. Wennberg, *On the distribution of free path lengths for the periodic Lorentz gas*, Comm. Math. Phys. **190** (1998), 491–508.

[12] L.A. Bunimovich and Y.G. Sinai, *Statistical properties of Lorentz gas with periodic configuration of scatterers*, Comm. Math. Phys. **78** (1980), 479–497.

[13] V.A. Bykovskii and A.V. Ustinov, *Trajectory statistics in inhomogeneous Sinai problem for 2-dimensional lattice*, Izv. Ran. Ser. Mat. **73** (2009), 17–36.

[14] E. Caglioti and F. Golse, *On the Boltzmann-Grad limit for the two dimensional periodic Lorentz gas*, J. Stat. Phys. **141** (2010), 264–317.

[15] N.I. Chernov, *Statistical properties of the periodic Lorentz gas, Multidimensional case*, Journal of Statistical Physics, **74** (1994), 11–53.

[16] P. Dahlqvist, *The Lyapunov exponent in the Sinai billiard in the small scatterer limit*, Nonlinearity, **10** (1997), 159–173.

[17] C.P. Dettmann, *New horizons in multidimensional diffusion: the Lorentz gas and the Riemann hypothesis*, J. Stat. Phys. **146** (2012), 181–204.

[18] ———, *Diffusion in the Lorentz gas*, arXiv:1402.7010.

[19] D.I. Dolgopyat and N.I. Chernov, *Anomalous current in periodic Lorentz gases with an infinite horizon*, Russian Math. Surveys **64** (2009), 73–124.

[20] G. Gallavotti, *Divergences and approach to equilibrium in the Lorentz and the wind-tree-models*, Phys. Rev. **185** (1969), 308–322.

[21] F. Golse and B. Wennberg, *On the distribution of free path lengths for the periodic Lorentz gas. II*, M2AN Math. Model. Numer. Anal. **34** (2000), 1151–1163.

[22] F. Golse, *The periodic Lorentz gas in the Boltzmann-Grad limit*, Proc. ICM 2006, Madrid, Spain, 183–201.

[23] ———, *On the periodic Lorentz gas and the Lorentz kinetic equation*, Ann. Fac. Sci. Toulouse Math. **17** (2008), 735–749.

[24] M. Lenci and S. Troubetzkoy, *Infinite-horizon Lorentz tubes and gases: recurrence and ergodic properties*, Physica D **240** (2011), 1510–1515.

[25] H. Lorentz, *Le mouvement des électrons dans les métaux*, Arch. Néerl. **10** (1905), 336–371.

[26] J. Marklof, *Horospheres and Farey fractions. Dynamical numbers—interplay between dynamical systems and number theory*, 97–106, Contemp. Math. **532**, Amer. Math. Soc., Providence, RI, 2010.

[27] J. Marklof and A. Strömbergsson, *Kinetic transport in the two-dimensional periodic Lorentz gas*, Nonlinearity **21** (2008) 1413–1422.

[28] ———, *The distribution of free path lengths in the periodic Lorentz gas and related lattice point problems*, Annals of Math. **172** (2010), 1949–2033.

[29] ———, *The Boltzmann-Grad limit of the periodic Lorentz gas*, Annals of Math. **174** (2011), 225–298.

[30] ———, *The periodic Lorentz gas in the Boltzmann-Grad limit: Asymptotic estimates*, GAFA **21** (2011), 560–647.

[31] ———, *Power-law distribution for the free path length in Lorentz gases*, J. Stat. Phys. **155** (2014), 1072–1086.

[32] ———, *Free path lengths in quasicrystals*, Comm. Math. Phys., DOI 10.1007/s00220-014-2011-3.

[33] ———, *Kinetic transport in quasicrystals*, in preparation.

[34] J. Marklof and B. Tóth, *Superdiffusion in the periodic Lorentz gas*, arXiv:1403.6024.

[35] I. Melbourne and M. Nicol, *A vector-valued almost sure invariance principle for hyperbolic dynamical systems*, Annals of Prob. **37** (2009), 478–505.

[36] P. Nandori, D. Szász, and T. Varjú, *Tail asymptotics of free path lengths for the periodic Lorentz process*, On Dettmann's geometric conjectures, arXiv:1210.2231.

[37] G.C. Papanicolaou, *Asymptotic analysis of transport processes*, Bull. AMS **81** (1975), 330–392.

[38] M. Ratner, *On Raghunathan's measure conjecture*, Annals of Math. **134** (1991), 545–607.

[39] ———, *Raghunathan's topological conjecture and distributions of unipotent flows*, Duke Math. J. **63** (1991), 235–280.

[40] N.A. Shah, *Limit distributions of expanding translates of certain orbits on homogeneous spaces*, Proc. Indian Acad. Sci. Math. Sci. **106** (1996), 105–125.

[41] H. Spohn, *The Lorentz process converges to a random flight process*, Comm. Math. Phys. **60** (1978), 277–290.

[42] D. Szász and T. Varjú, *Limit laws and recurrence for the planar Lorentz process with infinite horizon*, J. Stat. Phys. **129** (2007) 59–80.

[43] D. Szász, *Some challenges in the theory of (semi)-dispersing billiards*, Nonlinearity **21** (2008), T187–T193.

[44] A. Weil, *Basic Number Theory*, 3rd ed., Springer-Verlag, New York, 1974.

[45] B. Wennberg, *Free path lengths in quasi crystals*, J. Stat. Phys. **147** (2012), 981–990.

School of Mathematics, University of Bristol, Bristol BS8 1TW, U.K.
E-mail: j.marklof@bristol.ac.uk

Fractal geometry and dynamical bifurcations

Carlos Gustavo T. de A. Moreira

Abstract. In this survey we will describe results that relate qualitative properties of dynamical systems (and bifurcations of dynamical systems) to geometrical properties of invariant sets of these systems; fractal dimensions of hyperbolic invariant sets have a key role in such results. We conclude with a discussion of some results on geometrical properties of the classical Markov and Lagrange spectra of diophantine approximations and of dynamical variations of them, which can be proved using techniques of fractal geometry developed primarily in the context of dynamical bifurcations mentioned above.

Mathematics Subject Classification (2010). Primary 37C29, 28A80; Secondary 28A78, 11J06.

Keywords. Fractal geometry, homoclinic bifurcations, diophantine approximations.

1. Introduction

Homoclinic bifurcations are perhaps the most important mechanism that creates complicated dynamical systems from simple ones. This phenomenon, studied since Poincaré, happens when an element of a family of diffeomorphisms presents a hyperbolic periodic point whose stable and unstable manifolds have a non-transverse intersection.

The first interesting case is the study of *homoclinic explosions* in surfaces: We consider one-parameter families (φ_μ), $\mu \in (-1, 1)$ of diffeomorphisms of a surface for which φ_μ is uniformly hyperbolic for $\mu < 0$, and φ_0 presents a quadratic homoclinic tangency associated to a hyperbolic periodic point (which may belong to a *horseshoe* - a compact, locally maximal, hyperbolic invariant set of saddle type). It unfolds for $\mu > 0$ creating locally two transverse intersections between the stable and unstable manifolds of (the continuation of) the periodic point. A main question is what happens for (most) positive values of μ. The following figure depicts such a situation for $\mu = 0$.

Regular Cantor sets on the line play a fundamental role in dynamical systems and notably also in some problems in number theory. They are defined by expansive maps and have some kind of self-similarity property: small parts of them are diffeomorphic to big parts with uniformly bounded distortion (see a precise definition in the next section). In both settings, dynamics and number theory, a key question is whether the arithmetic difference of two such sets has non-empty interior.

A horseshoe Λ in a surface is locally diffeomorphic to the cartesian product of two regular Cantor sets: the so-called *stable* and *unstable* Cantor sets K^s and K^u of Λ, given by intersections of Λ with local stable and unstable manifolds of some points of the horseshoe. The Hausdorff dimension of Λ, which is equal to the sum of the Hausdorff dimensions of K^s

Proceedings of the International Congress of Mathematicians, Seoul, 2014

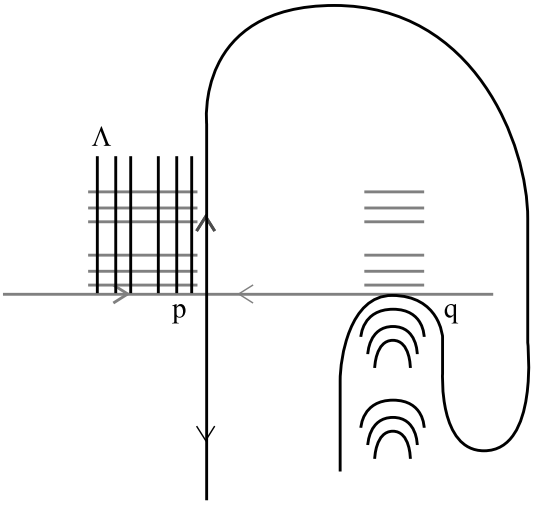


Figure 1.1.

and K^u, plays a fundamental role in several results on homoclinic bifurcations associated to Λ. In what follows, we will denote Hausdorff dimensions by HD.

>From the dynamics side, in 1983, J. Palis and F. Takens ([26], [27]) proved the following theorem about homoclinic bifurcations associated to a hyperbolic set:

Theorem 1.1. *Let (φ_μ), $\mu \in (-1,1)$ be a family of diffeomorphisms of a surface presenting a homoclinic explosion at $\mu = 0$ associated to a periodic point belonging to a horseshoe Λ. Assume that $HD(\Lambda) < 1$. Then*

$$\lim_{\delta \to 0} \frac{m(H \cap [0,\delta])}{\delta} = 1,$$

where $H := \{\mu > 0 \mid \varphi_\mu$ is persistently hyperbolic$\}$.

A central fact used in the proof of this result is that if K_1 and K_2 are regular Cantor sets on the real line such that the sum of their Hausdorff dimensions is smaller than one, then $K_1 - K_2 = \{x - y \mid x \in K_1, y \in K_2\}$ (the *arithmetic difference* between K_1 and K_2) is a set of zero Lebesgue measure (indeed of Hausdorff dimension smaller than 1). In the same year, looking for some kind of characterization property for this phenomenon, Palis conjectured (see [24], [25]) that for generic pairs of regular Cantor sets (K_1, K_2) of the real line either $K_1 - K_2$ has zero measure or else it contains an interval (the last statement should correspond in homoclinic bifurcations to open sets of tangencies). A slightly stronger statement is that, if K_1 and K_2 are generic regular Cantor sets and the sum of their Hausdorff dimensions is bigger than 1, then $K_1 - K_2$ contains intervals.

Another motivation for the conjecture was Newhouse's work in the seventies, when he introduced the concept of thickness of a regular Cantor set, another fractal invariant associated to Cantor sets on the real line. It was used in [21] to exhibit open sets of diffeomorphisms with persistent homoclinic tangencies, therefore without hyperbolicity. It is possible ([22]) to prove that, under a dissipativity hypothesis, in such an open set there is a residual set of diffeomorphisms which present infinitely many coexisting sinks. In [23], it is proved that

under generic hypotheses every family of surface diffeomorphisms that unfold a homoclinic tangency goes throught such an open set. Is to be noted that in general (as in the case studied in [27] and described above) these sets can have zero density. See [28] for a detailed presentation of these results. An important related question by Palis is whether the sets of parameter values corresponding to infinitely many coexisting sinks have typically zero Lebesgue measure.

An earlier and totally independent development had taken place in number theory. In 1947, M. Hall ([4]) proved that any real number can be written as the sum of two numbers whose continued fractions coefficients (of positive index) are at most 4. More precisely, if $C(4)$ is the regular subset formed of such numbers in $[0,1]$, then one has $C(4)+C(4) = [\sqrt{2}-1, 4(\sqrt{2}-1)]$. We will see a consequence of this result on the classical Markov and Lagrange spectra related to diophantine approximations in section 4, where we will also present and study dynamical generalizations of these spectra and discuss some of their geometrical properties, related to regular Cantor sets.

In section 2 we will discuss the positive solution of Palis' conjecture in the C^k-topology, $k > 1$, by Moreira and Yoccoz, and some dynamical consequences of it on the study of homoclinic bifurcations. We will also discuss how the situation is considerably different in the C^1-topology. In section 3 we will discuss some counterparts of these results in higher dimensions.

2. Regular Cantor sets and homoclinic bifurcations

A Cantor set K is a *C^k-regular Cantor set*, $k \geq 1$, if:

i) there are disjoint compact intervals $I_1, I_2, \ldots, I_r$ such that $K \subset I_1 \cup \cdots \cup I_r$ and the boundary of each I_j is contained in K;

ii) there is a C^k expanding map ψ defined in a neighbourhood of $I_1 \cup I_2 \cup \cdots \cup I_r$ such that, for each j, $\psi(I_j)$ is the convex hull of a finite union of some intervals I_s satisfying:

ii.1) for each j, $1 \leq j \leq r$ and n sufficiently big, $\psi^n(K \cap I_j) = K$;
ii.2) $K = \bigcap_{n\in\mathbb{N}} \psi^{-n}(I_1 \cup I_2 \cup \cdots \cup I_r)$.

Remark 2.1. If k is not integer, say $k = m + \alpha$, with $m \geq 1$ integer and $0 < \alpha < 1$ we assume that ψ is C^m and $\psi^{(m)}$ is α-Hölder.

We say that $\{I_1, I_2, \ldots, I_r\}$ is a *Markov partition* for K and that K is *defined* by ψ.

Remark 2.2. In general, we say that a set $X \subset \mathbb{R}$ is a Cantor set if X is compact, without isolated points and with empty interior. Cantor sets in $\mathbb{R}$ are homeomorphic to the classical ternary Cantor set $K_{1/3}$ of the elements of $[0,1]$ which can be written in base 3 using only digits 0 and 2. The set $K_{1/3}$ is itself a regular Cantor set, defined by the map $\psi : [0,1/3] \cup [2/3,1] \to \mathbb{R}$ given by $\psi[x] = 3x - \lfloor 3x \rfloor$.

An *interval of the construction of the regular Cantor set* K is a connected component of $\psi^{-n}(I_j)$ for some $n \in \mathbb{N}$, $j \leq r$.

Given $s \in [1,k]$ and another regular Cantor set $\tilde{K}$, we say that $\tilde{K}$ is close to K in the C^s topology if $\tilde{K}$ has a Markov partition $\{\tilde{I}_1, \tilde{I}_2, \ldots, \tilde{I}_r\}$ such that the interval $\tilde{I}_j$ has endpoints close to the endpoints of I_j, for $1 \leq j \leq r$ and $\tilde{K}$ is defined by a C^s map $\tilde{\psi}$ which is close to ψ in the C^s topology.

The C^{1+}-topology is such that a sequence ψ_n converges to ψ if there is some $\alpha > 0$ such that ψ_n is $C^{1+\alpha}$ for every $n \geq 1$ and ψ_n converges to ψ in the $C^{1+\alpha}$-topology.

The concept of stable intersection of two regular Cantor sets was introduced in [12]: two Cantor sets K_1 and K_2 have stable intersection if there is a neighbourhood V of (K_1, K_2) in the set of pairs of C^{1+}-regular Cantor sets such that $(\widetilde{K}_1, \widetilde{K}_2) \in V \Rightarrow \widetilde{K}_1 \cap \widetilde{K}_2 \neq \emptyset$.

In the same paper conditions based on renormalizations were introduced to ensure stable intersections, and applications of stable intersections to homoclinic bifurcations were obtained: roughly speaking, if some translations of the stable and unstable regular Cantor sets associated to the horseshoe at the initial bifurcation parameter $\mu = 0$ have stable intersection then the set $\{\mu > 0 \mid \varphi_\mu \text{ presents persistent homoclinic tangencies}\}$ has positive Lebesgue density at $\mu = 0$. It was also shown that this last phenomenon can coexist with positive density of hyperbolicity in a persistent way.

Besides, the following question was posed in [12]: Does there exist a dense (and automatically open) subset $\mathcal{U}$ of

$$\Omega^\infty = \{(K_1, K_2); K_1, K_2 \text{ are } C^\infty - \text{regular Cantor sets and } HD(K_1) + HD(K_2) > 1\}$$

such that $(K_1, K_2) \in \mathcal{U} \Rightarrow \exists\, t \in \mathbb{R}$ such that $(K_1, K_2 + t)$ has stable intersection? A positive answer to this question implies a strong version of Palis' conjecture. Indeed, $K_1 - K_2 = \{t \in \mathbb{R} \mid K_1 \cap (K_2 + t) \neq \emptyset\}$, so, if $(K_1, K_2 + t)$ has stable intersection then t belongs persistently to the interior of $K_1 - K_2$.

The results of [18] gave an affirmative answer to this question, proving the following

Theorem 2.3. *There is an open and dense set $\mathcal{U} \subset \Omega^\infty$ such that if $(K_1, K_2) \in \mathcal{U}$, then $I_s(K_1, K_2)$ is dense in $K_1 - K_2$ and $HD((K_1 - K_2) \backslash I_s(K_1, K_2)) < 1$, where*

$$I_s(K_1, K_2) := \{t \in \mathbb{R} \mid K_1 \text{ and } (K_2 + t) \text{ have stable intersection}\}.$$

The same result works if we replace stable intersection by d-stable intersection, which is defined by asking that any pair $(\widetilde{K}_1, \widetilde{K}_2)$ in some neighbourhood of (K_1, K_2) satisfies $HD(\widetilde{K}_1 \cap \widetilde{K}_2) \geq d$: most pairs of Cantor sets $(K_1, K_2) \in \Omega^\infty$ have d-stable intersection for any $d < HD(K_1) + HD(K_2) - 1$.

The open set $\mathcal{U}$ mentioned in the above theorem is very large in Ω^∞ in the sense that generic n-parameter families in Ω^∞ are actually contained in $\mathcal{U}$.

The proof of this theorem depends on a sufficient condition for the existence of stable intersections of two Cantor sets, related to a notion of renormalization, based on the fact that small parts of regular Cantor sets are diffeomorphic to the whole set: the existence of a *compact recurrent set* of relative positions of limit geometries of them. Roughly speaking, it is a compact set of relative positions of regular Cantor sets such that, for any relative position in such a set, there is a pair of (small) intervals of the construction of the Cantor sets such that the renormalizations of the Cantor sets associated to these intervals belong to the interior of the same compact set of relative positions.

The main result is reduced to prove the existence of recurrent compact sets of relative positions for most pairs of regular Cantor sets whose sum of Hausdorff dimensions is larger

than one. A central argument in the proof of this fact is a probabilistic argument à la Erdős: we construct a family of perturbations with a large number of parameters and show the existence of such a compact recurrent set with large probability in the parameter space (without exhibiting a specific perturbation which works). See [18] for more details.

An important result in fractal geometry which is used in the proof is the famous Marstrand's theorem ([11]), according to which, given a Borel set $X \subset \mathbb{R}^2$ with $HD(X) > 1$ then, for almost every $\lambda \in \mathbb{R}$, $\pi_\lambda(X)$ has positive Lebesgue measure, where $\pi_\lambda : \mathbb{R}^2 \to \mathbb{R}$ is given by $\pi_\lambda(x, y) = x - \lambda y$. In particular, if K_1 and K_2 are regular Cantor sets with $HD(K_1) + HD(K_2) > 1$ then, for almost every $\lambda \in \mathbb{R}$, $K_1 - \lambda K_2$ has positive Lebesgue measure. Lima and Moreira gave combinatorial alternative proofs of Marstrand's theorem, first in the case of cartesian products of regular Cantor sets ([6]) and then in the general case ([7]).

In [19], we prove the following fact concerning generic homoclinic bifurcations associated to two dimensional saddle-type hyperbolic sets (horseshoes) with Hausdorff dimension bigger than one: typically there are translations of the stable and unstable Cantor sets having stable intersection, and so it yields open sets of stable tangencies in the parameter line with positive density at the initial bifurcation value. Moreover, the union of such a set with the hyperbolicity set in the parameter line generically has full density at the initial bifurcation value. This extends a theorem by Palis and Yoccoz ([29]).

The situation is quite different in the C^1-topology, in which stable intersections do not exist:

Theorem 2.4 ([13]). *Given any pair (K, K') of regular Cantor sets, we can find, arbitrarily close to it in the C^1 topology, pairs $(\tilde{K}, \tilde{K}')$ of regular Cantor sets with $\tilde{K} \cap \tilde{K}' = \emptyset$.*

Moreover, for generic pairs (K, K') of C^1-regular Cantor sets, the arithmetic difference $K - K'$ has empty interior (and so is a Cantor set).

The main technical difference between the C^1 case and the C^2 (or even $C^{1+\alpha}$) cases is the lack of bounded distortion of the iterates of ψ in the C^1 case, and this fact will be fundamental for the proof of the previous result.

The previous result may be used to show that there are no C^1 robust tangencies between leaves of the stable and unstable foliations of respectively two given hyperbolic horseshoes Λ_1, Λ_2 of a diffeomorphism of a surface. This is also very different from the situation in the C^∞ topology - for instance, in [19] it is proved that, in the unfolding of a homoclinic or heteroclinic tangency associated to two horseshoes, when the sum of the correspondent stable and unstable Hausdorff dimensions is larger than one, there are generically stable tangencies associated to these two horseshoes. This result is done in the following

Theorem 2.5 ([13]). *Given a C^1 diffeomorphism ψ of a surface M having two (non necessarily disjoint) horseshoes Λ_1, Λ_2, we can find, arbitrarily close to it in the C^1 topology, a diffeomorphism $\tilde{\psi}$ of the surface for which the horseshoes Λ_1, Λ_2 have hyperbolic continuations $\tilde{\Lambda}_1, \tilde{\Lambda}_2$, and there are no tangencies between leaves of the stable and unstable foliations of $\tilde{\Lambda}_1$ and $\tilde{\Lambda}_2$, respectively. Moreover, there is a generic set $\mathcal{R}$ of C^1 diffeomorphisms of M such that, for every $\check{\psi} \in \mathcal{R}$, there are no tangencies between leaves of the stable and unstable foliations of Λ_1, Λ_2, for any horseshoes Λ_1, Λ_2 of $\check{\psi}$.*

Since stable intersections of Cantor sets are the main known obstructions to density of hyperbolicity for diffeomorphisms of surfaces, the previous result gives some hope of proving density of hyperbolicity in the C^1 topology for diffeomorphisms of surfaces, a well-known

question by Smale. In particular in the work [14] on a family of two-dimensional maps (the so-called Benedicks-Carleson *toy model* for Hénon dynamics) in which we prove that in this family there are diffeomorphisms which present stable homoclinic tangencies (Newhouse's phenomenon) in the C^2-topology but their elements can be arbitrarily well approximated in the C^1-topology by hyperbolic maps.

3. Results in higher dimensions

In [20], Moreira, Palis and Viana consider 1-parameter families of diffeomorphisms $\varphi_\mu : M^n \to M^n$, $\mu \in (-1,1)$ of class C^2, unfolding a generic homoclinic tangency associated to a periodic point p_0 contained in a horseshoe Λ_0 of φ_0. We suppose that φ_μ is hyperbolic (Axiom A) for $\mu < 0$, which implies that the weakest contracting and weakest expanding eigenvalues of p_0 are real numbers.

Let d_s and d_u be the Hausdorff dimensions of $W^s(p_0) \cap \Lambda_0$ and $W^u(p_0) \cap \Lambda_0$, respectively. We show the following result:

Theorem 3.1. *There are two open sets $\mathcal{U}$ and $\mathcal{V}$ of families (φ_μ) as before whose union is dense such that:*

i) *If $(\varphi_\mu) \in \mathcal{U}$ then $d_s + d_u < 1$ and $H = \{\mu \in (-1,1) \mid \varphi_\mu$ is hyperbolic$\}$ has full Lebesgue density at $\mu = 0$.*

ii) *If $(\varphi_\mu) \in \mathcal{V}$ then $d_s + d_u > 1$ and $T_s = \{\mu > 0 \mid \varphi_\mu$ presents persistent homoclinic tangencies associated to the hyperbolic continuation Λ_μ of $\Lambda_0\}$ is an open set with positive lower density at $\mu = 0$.*

An important technical notion defined in this paper is that of *upper dimension*, which is defined as follows.

Let Λ be a horseshoe for a diffeomorphism $\varphi : M \to M$, and P be some Markov partition of Λ. We call *vertical n-cylinder* any subset of Λ defined by prescribing the first n symbols in the backward itinerary with respect to P. Let $\mathcal{V}_n$ be the set of vertical n-cylinders. There is a dual notion of *horizontal n-cylinder*, where one considers forward itinerary.

Fix $\varepsilon > 0$ small. For each $V \in \mathcal{V}_n$ let $D_s(V) = \sup\{\mathrm{diam}(W^s_\varepsilon(x) \cap V) \mid x \in \Lambda \cap V\}$. We define $\widetilde{\lambda}_n$ by the relation

$$\sum_{V \in \mathcal{V}_n} D_s(V)^{\widetilde{\lambda}_n} = 1,$$

and we let the *upper stable dimension* be given by $\overline{d}_s(\Lambda) = \lim_{n\to\infty} \widetilde{\lambda}_n$. There is a dual notion of *upper unstable dimension* $\overline{d}_u(\Lambda)$, dealing with W^u_ε instead of W^s_ε.

It is not difficult to show that the limit always exists, and $\overline{d}_s$ is an upper-semicontinuous function of φ. Moreover,

$$\mathrm{HD}(W^s_\varepsilon(x) \cap \Lambda) \le \overline{d}_s(\Lambda) \quad \text{for each } x \in \Lambda.$$

It is possible to show that the equality holds in most cases when $\overline{d}_s < 1$.

One of our main lemmas is the construction (perhaps after perturbation of φ_0) of strong-stable and strong-unstable foliations of codimension 1 for hyperbolic subsets of Λ_0 with almost the same upper dimensions $\overline{d}_s$ and $\overline{d}_u$. These foliations are used to (essentially) reduce

the study of the geometries of the stable and unstable foliations near the initial homoclinic tangency to the bidimensional case.

In [1], Bonatti and Diaz introduced the concept of *blenders*, which became very important in the study of partially hyperbolic diffeomorphisms and of C^1-robustly non-hyperbolic systems:

Definition 3.2 (cs-blender). Let $\varphi : M \to M$ be a diffeomorphism. A transitive hyperbolic set Γ of φ whose stable bundle has dimension $k \geq 2$ is a *cs-blender* if there are a C^1-neighborhood $\mathcal{U}$ of φ and a C^1-open set $\mathcal{D}$ of embeddings of $(k-1)$-dimensional disks D into M such that, for every diffeomorphism $\psi \in U$, every disk $D \in \mathcal{D}$ intersects the local stable manifold $W^s_{loc}(\Gamma_\psi)$ of the continuation Γ_ψ of Γ for ψ.

In [17] Moreira and Silva prove the following:

Theorem 3.3. *Typical horseshoes (in arbitrary dimension) Λ with $\overline{d}_s(\Lambda) > 1$ are cs-blenders. Moreover, for any $q \in \Lambda$ and typical C^1 real maps $g : W^s_{loc}(q) \to \mathbb{R}$, $g(W^s_{loc}(q) \cap \Lambda)$ has persistently non-empty interior.*

Notice that if $HD(W^s_{loc}(q) \cap \Lambda) > 1$ for some $q \in \Lambda$ then the hypothesis $\overline{d}_s(\Lambda) > 1$ is automatically satisfied.

The proof uses the above results of [20] in order to reduce the problem to the context where the horseshoe has a strong-stable foliation of codimension one. The main result in this context is that projections of stable Cantor sets of the type $W^s_{loc}(q) \cap \Lambda$ along strong stable leaves persistently contain intervals. The proof of this result is inspired by the techniques of [18]: renormalization operators are defined, and a recurrent compact set criterion which is a sufficient condition for the desired conclusion is established. Then the proof follows a probabilistic method: the existence of a recurrent compact set is proved for most perturbations in a suitable family of small perturbations with a large number of parameters. An important step in the preparation of this last argument is a Marstrand-like result inspired in the paper [31] by Simon, Solomyak and Urbański.

Another natural generalization of [18] is related to the following question:

Question. *Let $\pi : \mathbb{R}^n \to \mathbb{R}^k$ be a surjective linear map. Under which conditions on $K_1, \ldots, K_n$ regular Cantor sets, the set $\pi(K_1 \times \ldots \times K_n)$ contains a non-empty open set of $\mathbb{R}^k$?*

The results of [18] imply that, in the case $(n, k) = (2, 1)$, the condition $HD(K_1) + HD(K_2) > 1$ is typically sufficient.

Some natural conditions related to $HD(K_1), \ldots, HD(K_n)$ are needed, indeed: let $e_1, \ldots, e_n$ be the canonical basis of $\mathbb{R}^n$. Then for all $I \subset \{1, \ldots, n\}$

$$HD(\pi(K_1 \times \ldots \times K_n)) \leq \sum_{i \in I} HD(K_i) + \dim \Big(\text{span} \{\pi(e_i), i \in I^c\} \Big).$$

We say that $t \in \mathbb{R}^k$ is a stable projection value for $K_1, \ldots, K_n$ if $t \in \pi(\widetilde{K}_1 \times \ldots \times \widetilde{K}_n)$ for any $(\widetilde{K}_1, \ldots, \widetilde{K}_n)$ perturbation of $(K_1, \ldots, K_n)$ in C^{1+}-topology of regular Cantor sets. $P_s(K_1, \ldots, K_n)$ denotes the set of such stable projection values t.

In [9], López and Moreira give the following answer to this question:

Theorem 3.4. *There is an open and dense subset $\mathcal{U}$ of the set*

$$\Big\{(K_1,\dots,K_n) \mid K_1,\dots,K_n \text{ are } C^\infty\text{-regular Cantor sets with } \sum_{i\in I} HD(K_i) + \dim\Big(\text{span}\,\{\pi(e_i), i\in I^c\}\Big) > k, \text{ for all } I\subset\{1,\dots,n\}, I\neq\emptyset\Big\},$$

such that, if $(K_1,\dots,K_n)\in\mathcal{U}$, then $P_s(K_1,\dots,K_n)$ is dense in $\pi(K_1\times\ldots\times K_n)$ and

$$HD(\pi(K_1\times\ldots\times K_n)\backslash P_s(K_1,\dots,K_n)) < k.$$

An important technical tool in the proof is the following generalization of Marstrand's theorem, proved in [8]:

Let $K_1,\dots,K_n$ be Borel subsets of $\mathbb{R}^{m_1},\dots,\mathbb{R}^{m_n}$ respectively, and $\pi:\mathbb{R}^{m_1}\times\ldots\times\mathbb{R}^{m_n}\to\mathbb{R}^k$ be a surjective linear map. We set:

$$\mathfrak{m} := \min\left\{\sum_{i\in I} HD(K_i) + \dim\pi(\bigoplus_{i\in I^c}\mathbb{R}^{m_i}), I\subset\{1,\dots,n\}, I\neq\emptyset\right\}.$$

Consider the space $\Lambda_m = \{(t,O), t\in\mathbb{R}, O\in SO(m)\}$ with the natural measure and set $\Lambda=\Lambda_{m_1}\times\ldots\times\Lambda_{m_n}$. For every $\lambda=(t_1,O_1,\dots,t_n,O_n)\in\Lambda$ and every $x=(x^1,\dots,x^n)\in\mathbb{R}^{m_1}\times\ldots\times\mathbb{R}^{m_n}$ we define $\pi_\lambda(x)=\pi(t_1O_1x^1,\dots,t_nO_nx^n)$. Then we have

Theorem 3.5.

(i) *If $\mathfrak{m}>k$, then $\pi_\lambda(K_1\times\ldots\times K_n)$ has positive k-dimensional Lebesgue measure for almost every $\lambda\in\Lambda$.*

(ii) *If $\mathfrak{m}\le k$ and $HD(K_1\times\ldots\times K_n)=HD(K_1)+\ldots+HD(K_n)$, then*

$$HD(\pi_\lambda(K_1\times\ldots\times K_n)) = \mathfrak{m}$$

for almost every $\lambda\in\Lambda$.

4. The Markov and Lagrange spectra and generalizations

Let α be an irrational number. According to Dirichlet's theorem, the inequality $|\alpha-\frac{p}{q}|<\frac{1}{q^2}$ has infinitely many rational solutions $\frac{p}{q}$. Hurwitz improved this result by proving that $|\alpha-\frac{p}{q}|<\frac{1}{\sqrt{5}q^2}$ also has infinitely many rational solutions $\frac{p}{q}$ for any irrational α, and that $\sqrt{5}$ is the largest constant that works for any irrational α. However, for particular values of α we can improve this constant.

More precisely, we define $k(\alpha) := \sup\{k>0 \mid |\alpha-\frac{p}{q}|<\frac{1}{kq^2}$ has infinitely many rational solutions $\frac{p}{q}\} = \limsup_{p,q\to+\infty}(q|q\alpha-p|)^{-1}$. We have $k(\alpha)\ge\sqrt{5}$, $\forall\alpha\in\mathbb{R}\setminus\mathbb{Q}$ and $k\left(\frac{1+\sqrt{5}}{2}\right)=\sqrt{5}$. We will consider the set $L=\{k(\alpha)\mid\alpha\in\mathbb{R}\setminus\mathbb{Q}, k(\alpha)<+\infty\}$.

This set is called the Lagrange spectrum. Hurwitz's theorem determines the smallest element of L, which is $\sqrt{5}$. This set L encodes many diophantine properties of real numbers.

It is a classical subject the study of the geometric structure of L. Markov proved in 1879 ([10]) that

$$L \cap (-\infty, 3) = \left\{ k_1 = \sqrt{5} < k_2 = 2\sqrt{2} < k_3 = \frac{\sqrt{221}}{5} < \dots \right\}$$

where k_n is a sequence (of irrational numbers whose squares are rational) converging to 3, which means that the "beginning" of the set L is discrete. This is not true for the whole set L. As we mentioned in the introduction, M. Hall proved in 1947 ([4]) that if $C(4)$ is the regular Cantor set formed by the numbers in $[0, 1]$ whose coefficients in the continued fractions expansion are bounded by 4, then one has $C(4)+C(4) = [\sqrt{2}-1, 4(\sqrt{2}-1)]$. This implies that L contains a whole half line (for instance $[6, +\infty)$), and G. Freiman determined in 1975 ([3]) the biggest half line that is contained in L, which is $[c, +\infty)$, with

$$c = \frac{2221564096 + 283748\sqrt{462}}{491993569} \cong 4,52782956616\dots .$$

These last two results are based on the study of sums of regular Cantor sets, whose relationship with the Lagrange spectrum will be explained below.

If the continued fraction of α is $\alpha = [a_0; a_1, a_2, \dots,]$ then we have the following formula $k(\alpha) = \limsup_{n\to\infty}(\alpha_n + \beta_n)$, where $\alpha_n = [a_n; a_{n+1}, a_{n+2}, \dots]$ and $\beta_n = [0; a_{n-1}, a_{n-2}, \dots, a_1]$. This follows from the equality

$$\left| \alpha - \frac{p_n}{q_n} \right| = \frac{1}{(\alpha_{n+1} + \beta_{n+1})q_n^2}, \quad \forall n \in \mathbb{N},$$

where $p_n/q_n, n \in \mathbb{N}$ are the convergents of the continued fraction of α.

This formula for $k(\alpha)$ implies that we have the following alternative definition of the Lagrange spectrum L:

Let $\Sigma = (\mathbb{N}^*)^{\mathbb{Z}}$ be the set of all bi-infinite sequences of positive integers. If $\underline{\theta} = (a_n)_{n\in\mathbb{Z}} \in \Sigma$, let $\alpha_n = [a_n; a_{n+1}, a_{n+2}, \dots]$ and $\beta_n = [0; a_{n-1}, a_{n-2}, \dots], \forall n \in \mathbb{Z}$. We define $f(\underline{\theta}) = \alpha_0 + \beta_0 = [a_0; a_1, a_2, \dots] + [0; a_{-1}, a_{-2}, \dots]$. We have

$$L = \left\{ \limsup_{n\to\infty} f(\sigma^n \underline{\theta}), \underline{\theta} \in \Sigma \right\},$$

where $\sigma\colon \Sigma \to \Sigma$ is the shift defined by $\sigma((a_n)_{n\in\mathbb{Z}}) = (a_{n+1})_{n\in\mathbb{Z}}$.

Let us define the Markov spectrum M by $M = \{\sup_{n\in\mathbb{Z}} f(\sigma^n \underline{\theta}), \underline{\theta} \in \Sigma\}$. It also has an arithmetical interpretation, namely

$$M = \left\{ \left(\inf_{(x,y)\in\mathbb{Z}^2\setminus(0,0)} |f(x, y)| \right)^{-1} : f(x, y) = ax^2 + bxy + cy^2,\ b^2 - 4ac = 1 \right\}.$$

It is well-known (see [2]) that M and L are closed sets of the real line and $L \subset M$.

We have the following result about the Markov and Lagrange spectra:

Theorem 4.1. *Given $t \in \mathbb{R}$ we have*

$$HD(L \cap (-\infty, t)) = HD(M \cap (-\infty, t)) =: d(t)$$

and $d(t)$ is a continuous surjective function from $\mathbb{R}$ to $[0, 1]$. Moreover:

i) $d(t) = \min\{1, 2D(t)\}$, *where* $D(t) := HD(k^{-1}(-\infty, t)) = HD(k^{-1}(-\infty, t])$ *is a continuous function from* $\mathbb{R}$ *to* $[0, 1)$.

ii) $\max\{t \in \mathbb{R} \mid d(t) = 0\} = 3$

iii) $d(\sqrt{12}) = 1$.

A fundamental tool in the proof of this result is the theorem below.

We say that a C^2-regular Cantor set on the real line is *essentially affine* if there is a C^2 change of coordinates for which the dynamics that defines the corresponding Cantor set has zero second derivative on all points of that Cantor set. Typical C^2-regular Cantor sets are not essentially affine.

The *scale recurrence lemma*, which is the main technical lemma of [18], can be used in order to prove the following

Theorem 4.2. *If* K *and* K' *are regular Cantor sets of class* C^2 *and* K *is non essentially affine, then* $HD(K + K') = \min\{HD(K) + HD(K'), 1\}$.

There is a presentation of a version of this result (with a slightly different hypothesis) in [30]. That version is also proved by Hochman and Shmerkin in [5].

As we have seen, the sets M and L can be defined in terms of symbolic dynamics. Inspired by these characterizations, we may associate to a dynamical system together with a real function generalizations of the Markov and Lagrange spectra as follows:

Definition 4.3. Given a map $\psi : X \to X$ and a function $f : X \to \mathbb{R}$, we define the associated dynamical Markov and Lagrange spectra as $M(f, \psi) = \{\sup_{n\in\mathbb{N}} f(\psi^n(x)), x \in X\}$ and $L(f, \psi) = \{\text{limsup}_{n\to\infty} f(\psi^n(x)), x \in X\}$, respectively. Given a flow $(\varphi^t)_{t\in\mathbb{R}}$ in a manifold X, we define the associated dynamical Markov and Lagrange spectra as $M(f, (\varphi^t)) = \{\sup_{t\in\mathbb{R}} f(\varphi^t(x)), x \in X\}$ and $L(f, (\varphi^t)) = \{\text{limsup}_{t\to\infty} f(\varphi^t(x)), x \in X\}$, respectively.

In an ongoing work in collaboration with A. Cerqueira, we prove the following result, which generalizes a corresponding fact in the context of the classical Markov and Lagrange spectra:

Lemma 4.4. *Let* (φ, f) *be a generic pair, where* $\varphi\colon M^2 \to M^2$ *is a diffeomorphism with* $\Lambda \subset M^2$ *a hyperbolic set for* φ *and* $f : M \to \mathbb{R}$ *is* C^2. *Let* π_s, π_u *be the projections of the horseshoe* Λ *to the stable and unstable regular Cantor sets* K^s, K^u *associated to it (along the unstable and stable foliations of* Λ*). Given* $t \in \mathbb{R}$*, we define*

$$\Lambda_t = \bigcap_{m\in\mathbb{Z}} \varphi^m(\{p \in \Lambda | f(p) \le t\}),\ K^s_t = \pi_s(\Lambda_t),\ K^u_t = \pi_u(\Lambda_t).$$

Then the functions $d_s(t) = HD(K^s_t)$ *and* $d_u(t) = HD(K^u_t)$ *are continuous and coincide with the corresponding box dimensions.*

The following result is a consequence of the scale recurrence lemma of [18] (its first part also follows from [5]):

Lemma 4.5. *Let* (φ, f) *be a generic pair, where* $\varphi\colon M^2 \to M^2$ *is a diffeomorphism with* $\Lambda \subset M^2$ *a hyperbolic set for* φ *and* $f : M \to \mathbb{R}$ *is* C^2. *Then*

$$HD(f(\Lambda)) = \min(HD(\Lambda), 1).$$

Moreover, if $HD(\Lambda) > 1$ *then* $f(\Lambda)$ *has persistently non-empty interior.*

Using the previous lemmas we prove a generalization of the results on dimensions of the dynamical spectra:

Theorem 4.6. *Let (φ, f) be a generic pair, where $\varphi\colon M^2 \to M^2$ is a* **conservative** *diffeomorphism with $\Lambda \subset M^2$ a hyperbolic set for φ and $f : M \to \mathbb{R}$ is C^2. Then*

$$HD(L(f,\Lambda) \cap (-\infty, t)) = HD(M(f,\Lambda) \cap (-\infty, t)) =: d(t)$$

is a continuous real function whose image is $[0, \min(HD(\Lambda), 1)]$.

Finally we will describe some results obtained in collaboration with S. Romaña.

Theorem 4.7 ([15]). *Let Λ be a horseshoe associated to a C^2-diffeomorphism φ such that $HD(\Lambda) > 1$. Then there is, arbitrarily close to φ a diffeomorphism φ_0 and a C^2-neighborhood W of φ_0 such that, if Λ_ψ denotes the continuation of Λ associated to $\psi \in W$, there is an open and dense set $H_\psi \subset C^1(M, \mathbb{R})$ such that for all $f \in H_\psi$, we have*

$$int\ L(f, \Lambda_\psi) \neq \emptyset \text{ and } int\ M(f, \Lambda_\psi) \neq \emptyset,$$

where $int\ A$ denotes the interior of A.

The classical Markov and Lagrange spectra can also be characterized as sets of maximum heights and asymptotic maximum heights, respectively, of geodesics in the modular surface $N = \mathbb{H}^2/PSL(2,\mathbb{Z})$. We extend in [16] the fact that these spectra have non-empty interior to the context of negative, non necessarily constant curvature as follows:

Theorem 4.8. *Let M provided with a metric g_0 be a complete noncompact surface M with finite Gaussian volume and Gaussian curvature bounded between two negative constants,* i.e.*, if K_M denotes the Gaussian curvature, then there are constants $a, b > 0$ such that*

$$-a^2 \leq K_M \leq -b^2 < 0.$$

Denote by SM its unitary tangent bundle and by ϕ its geodesic flow.

Then there is a metric g close to g_0 and a dense and C^2-open subset $\mathcal{H} \subset C^2(SM, \mathbb{R})$ such that

$$int\ M(f, \phi_g) \neq \emptyset \text{ and } int\ L(f, \phi_g) \neq \emptyset$$

for any $f \in \mathcal{H}$, where ϕ_g is the vector field defining the geodesic flow of the metric g.

Moreover, if X is a vector field sufficiently close to ϕ_g then

$$int\ M(f, X) \neq \emptyset \text{ and } int\ L(f, X) \neq \emptyset$$

for any $f \in \mathcal{H}$.

Acknowledgements. The author is partially supported by CNPq and by the Palis-Balzan project.

References

[1] C. Bonatti and L. Diaz, *Persistent Transitive Diffeomorphisms*, Annals of Math., **143**(2) (1996), 357-396.

[2] T.W. Cusick and M.E. Flahive, *The Markoff and Lagrange spectra*, Math. Surveys and Monographs, no. 30, A.M.S. (1989).

[3] G.A. Freiman, *Diophantine approximation and geometry of numbers* (The Markoff spectrum), Kalininskii Gosudarstvennyi Universitet, Moscow, 1975.

[4] M. Hall, *On the sum and product of continued fractions*, Annals of Math., **48** (1947), 966–993.

[5] M. Hochman and P. Shmerkin, *Local entropy averages and projections of fractal measures*, Annals of Math. **175**(3) (2012), 1001-1059.

[6] Y. Lima and C.G. Moreira, *A combinatorial proof of Marstrand's theorem for products of regular Cantor sets*, Expo. Math. **29** (2011), no. 2, 231–239.

[7] ———, *Yet another proof of Marstrand's theorem*, Bulletin of the Brazilian Mathematical Society **42** (2011), no. 2, 331–345.

[8] C.G. Moreira and J.E. López Velázquez, *A generalization of Marstrand's theorem for projections of cartesian products*, Accepted for publication in Annales de L'Institut Henri Poincaré-Analyse Non Linéaire., 2014.

[9] ———, *Stable projections of cartesian products of regular Cantor sets*, To appear. See also http://www.impa.br/opencms/en/ensino/downloads/teses_de_doutorado/teses_2012/jorge_erick_lopez_velazquez.pdf.

[10] A. Markov, *Sur les formes quadratiques binaires indfinies*, Math. Ann. **15** (1879), 381-406.

[11] J.M. Marstrand, *Some fundamental geometrical properties of plane sets of fractional dimensions*, Proceedings of the London Mathematical Society **4** (1954), 257-302.

[12] C.G. Moreira, *Stable intersections of Cantor sets and homoclinic bifurcations*, Ann. Inst. H. Poincaré Anal. Non Linéaire **13** (1996), no. 6, 741–781.

[13] ———, *There are no C^1-stable intersections of regular Cantor sets*, Acta Mathematica **206** (2011), no. 2, 311-323 .

[14] C. Matheus, C.G. Moreira, E. Pujals, *Axiom A versus Newhouse phenomena for Benedicks-Carleson toy models*, Annales Scientifiques de l'École Normale Supérieure **46** (2013), fascicule **6** (2013), 857-878.

[15] C.G. Moreira and S. Romaña, *On the Lagrange and Markov Dynamical Spectra*, arXiv:1310.3903.

[16] ———, *Markov and Lagrange Dynamical Spectra for Geodesic Flows in Surfaces with Negative Curvature*, To appear. See also http://www.impa.br/opencms/en/ensino/downloads/teses_de_doutorado/teses_2013/Sergio_Augusto_Romana_Ibarra.pdf.

[17] C.G. Moreira and W. Silva, *On the geometry of horseshoes in higher dimensions,* http://arxiv.org/abs/1210.2623.

[18] C.G. Moreira and J.-C. Yoccoz, *Stable intersections of regular Cantor sets with large Hausdorff dimensions* Annals Of Mathematics, **154** (2001), n. 1, 45-96.

[19] ———, *Tangences homoclines stables pour des ensembles hyperboliques de grande dimension fractale*, Annales Scientifiques de l'École Normale Supérieure, **43**, fascicule **1** (2010), 1-68.

[20] C.G. Moreira, J. Palis, and M. Viana *Homoclinic tangencies and fractal invariants in arbitrary dimension/Tangences homoclines et invariants fractaux en dimension arbitraire*, Comptes Rendus de l'Académie des Sciences. Série 1, Mathématique, **333** (2001), 475-480.

[21] S. Newhouse, *Non density of Axiom $A(a)$ on S^2* , Proc. A.M.S. Symp. Pure Math., **14** (1970), 191–202.

[22] ———, *Diffeomorphisms with infinitely many sinks*, Topology, **13**, (1974), 9–18.

[23] ———, *The abundance of wild hyperbolic sets and nonsmooth stable sets for diffeomorphisms*, Publ. Math. IHES, **50** (1979), 101–151.

[24] J. Palis, *Homoclinic orbits, hyperbolic dynamics and fractional dimension of Cantor sets*, Contemporary Mathematics **58** (1987), 203–216.

[25] ———, *Homoclinic bifurcations, sensitive chaotic dynamics and strange attractors*, Dynamical Syst. and Related Topics. World Scientific (1991), 466-473.

[26] J. Palis and F. Takens, *Cycles and measure of bifurcation sets for two-dimensional diffeomorphisms*, Invent. Math., **82** (1985),379–442.

[27] ———, *Hyperbolicity and the creation of homoclinic orbits*, Annals of Math. **125** (1987), 337–374.

[28] ———, *Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations: fractal dimensions and infinitely many attractors*, Cambridge Univ. Press, 1992.

[29] J. Palis and J.C. Yoccoz, *Homoclinic Tangencies for Hyperbolic sets of large Hausdorff Dimension Bifurcations*, Acta Mathematica, **172** (1994), 91–136.

[30] P. Shmerkin, *Moreira's Theorem on the arithmetic sum of dynamically defined Cantor sets*, http://arxiv.org/abs/0807.3709.

[31] Simon, K., Solomyak, B., and Urbański, M., *Hausdorff Dimension of limit sets for parabolic IFS with overlaps*, Pacific Journal of Mathematics, **201**, No. 2 (2001).

IMPA, Estrada D. Castorina, 110, 22460-320, Rio de Janeiro, RJ, Brazil
E-mail: gugu@impa.br

Zeta functions for Anosov flows

Mark Pollicott

Abstract. Dynamical zeta functions, by analogy with their more famous counterparts in number theory, are a useful tool to study certain types of dynamical systems. An important application is to the geodesic flow on a negatively curved surface. For surfaces of constant negative curvature the properties of the Selberg zeta function have been well understood for over half a century. However, understanding the properties of the corresponding zeta function for the more general setting of surfaces of variable negative curvature benefits from this more dynamical viewpoint.

Mathematics Subject Classification (2010). Primary 37C30; Secondary 11M36.

Keywords. Dynamical zeta functions, thermodynamical formalism.

1. Introduction

The best known setting for zeta functions is undoubtably that of analytic number theory, and so perhaps this is a good starting place to motivate the study of zeta functions for Anosov flows. We therefore begin with the best known zeta function, namely the *Riemann zeta function*, which is the complex function defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s \in \mathbb{C},$$

which converges for $Re(s) > 1$. The Riemann zeta function was actually studied in 1737 by Euler. Indeed, it was Euler who proved the equivalent presentation

$$\zeta(s) = \prod_{p} (1 - p^{-s})^{-1}$$

in terms of what is now called an Euler product over the prime numbers p. However, when in 1859 Riemann was elected a member of the Berlin Academy of Sciences he reported "On the number of primes less than a given magnitude" [65] in a departure from his previous, and subsequent, research. In particular, he established the following basic properties of this zeta function.

Theorem 1.1 (Riemann). *The zeta function $\zeta(s)$ converges to a non-zero analytic function for $Re(s) > 1$. Moreover,*

(1) *$\zeta(s)$ has a single (simple) pole at $s = 1$; and*

Proceedings of the International Congress of Mathematicians, Seoul, 2014

(2) $\zeta(s)$ *extends to all complex numbers* $s \in \mathbb{C}$.

In particular, $\zeta(s)$ has an analytic extension to the entire complex plane, except for a simple pole at $s = 1$.

One of the main applications of the Riemann zeta function was to prove the prime number theorem (shown independently by Hadamard and de la Vallee Poussin in 1896). Let $\pi(x)$ denote the number of primes p which are less than x, for $x > 0$.

Theorem 1.2 (Prime Number Theorem). $\pi(x) \sim \frac{x}{\log x}$ *as* $x \to +\infty$.

The asymptotic formula in the theorem means that $\lim_{x\to+\infty} \frac{\pi(x)}{x/\log x} = 1$. The proof depends on the additional knowledge that $\zeta(s)$ has no zeros on the line $Re(s) = 1$.

It remains a major problem (famously posed in Hilbert's 8th problem from his list of 23 problems from the 1900 International Congress of Mathematicians) to find the optimal asymptotic formulae. This can be formulated in terms of the zeros of the zeta function in the following slightly nonstandard form.

Conjecture 1.3 (Riemann Hypothesis). [1] *The Riemann zeta function* $\zeta(s)$ *is analytic and non-zero on the half-plane* $Re(s) > \frac{1}{2}$*, except for a simple pole at* $s = 1$.

The consequences of the validity of this conjecture for the behaviour of $\pi(x)$ are well known. In particular, the Riemann hypothesis would improve the Prime Number Theorem (Theorem 1.2) by giving a very strong error term, i.e., we would know that

$$\pi(x) = \mathrm{Li}(x) + O\left(x^{1/2}\log x\right)$$

where $\mathrm{Li}(x) = \int_2^x \frac{1}{\log u} du$ (with, of course, $\mathrm{Li}(x) \sim \frac{x}{\log x}$ as $x \to +\infty$).

We will describe the analogous zeta functions in both geometric and dynamical settings (the Selberg and Ruelle zeta functions, respectively). In each case, we will be interested in understanding how far they can be extended analytically or meromorphically (the analogue of Theorem 1.1 and Conjecture 1.3, respectively).

2. The zeta function for geodesics

It is very striking that many of the features of the prime numbers and the Riemann zeta function $\zeta(s)$ have counterparts in the geometry of compact surfaces of constant curvature. Let V denote a compact surface with constant negative Gaussian curvature $\kappa = -1$. Instead of prime numbers we can consider closed geodesics γ, of which there are a countable infinity on V since there is exactly one in every conjugacy class of the fundamental group $\pi_1(V)$. We will adopt the convention that closed geodesics are oriented (i.e., we count the two orientations of the same curve as two distinct closed geodesics), Let $l(\gamma)$ denote the length of the closed geodesic γ, say, then we recall the original definition of the Selberg zeta function.

[1] According to Littlewood [41] this topped Hardy's famous wish list from the 1920s: (1) Prove the Riemann Hypothesis; (2) Make 211 not out in the fourth innings of the last test match at the Oval; (3) Find an argument for the nonexistence of God which shall convince the general public; (4) Be the first man at the top of Mount Everest; (5) Be proclaimed the first president of the U.S.S.R., Great Britain, and Germany; and (6) Murder Mussolini.

Definition 2.1. The *Selberg zeta function* is defined by

$$Z(s) = \prod_{\gamma} \prod_{n=0}^{\infty} \left(1 - e^{-(s+n)l(\gamma)}\right), \quad s \in \mathbb{C}.$$

This converges to an analytic function for $Re(s) > 1$. However, for later convenience we prefer to consider an alternative version of this zeta function (due to Ruelle) of the form

$$\zeta_S(s) = \prod_{\gamma} \left(1 - e^{-sl(\gamma)}\right)^{-1}, \quad s \in \mathbb{C},$$

which again converges to a non-zero analytic function for $Re(s) > 1$. This has a reassuringly similar form to that of the Riemann zeta function, by formally replacing the prime numbers p by the weights $e^{l(\gamma)}$ associated to each closed geodesic γ. Clearly, we can write $\zeta_S(s) = Z(s+1)/Z(s)$. The analogue of Theorem 1.1 is then the following [30, 73].

Theorem 2.2. *The zeta function $\zeta_S(s)$ converges to a non-zero analytic function for $Re(s) > 1$. Moreover,*

(1) *$\zeta_S(s)$ has a simple pole at $s = 1$; and*

(2) *$\zeta_S(s)$ extends to all complex numbers $s \in \mathbb{C}$.*

In this case the extension of $\zeta_S(s)$ to $\mathbb{C}$ is as a meromorphic function.

Pursuing the analogy between prime numbers and closed geodesics, the similar properties of the two zeta functions leads to a result on counting closed geodesics corresponding to that of the Prime Number Theorem (Theorem 1.2). Let $\Pi(x)$ denote the number of prime closed geodesics γ with length $l(\gamma)$ less than x.

Theorem 2.3 (Prime Geodesic Theorem). $\Pi(x) \sim \frac{e^x}{x}$ *as* $x \to +\infty$.

This geometric zeta function $\zeta_S(s)$ has some advantages over that of the classical Riemann zeta function $\zeta(s)$. In particular, the poles and zeros of $\zeta_S(s)$ can be explicitly characterized, and an analogue of the Riemann hypothesis holds. We will return to this later in §6.

However, if one takes the broader view of Riemannian geometry it is natural to ask if these results generalise to geodesics on surfaces with variable negative curvature (or, more generally, higher dimensional manifolds with negative sectional curvatures). In fact, at about the same time that the above ideas were taking root, Anosov developed a completely different dynamical framework which would ultimately help address these questions [1].

3. Anosov flows

In order to formulate a dynamical analogue of the previous zeta functions we want to replace the prime numbers in $\zeta(s)$ (or closed geodesics in $\zeta_S(s)$) by closed orbits for appropriate flows which, in particular, we require to have a countable infinity of prime closed orbits. A particularly important class of such flows is that of Anosov flows.

Definition 3.1. We say that a C^∞ flow $\phi_t : M \to M$ on a compact manifold M is *Anosov* if the following hold.

(1) There is a $D\phi_t$-invariant splitting $TM = E^0 \oplus E^s \oplus E^u$ such that

   (a) E^0 is one dimensional and tangent to the flow direction;
   (b) There exist $C, \lambda > 0$ such that $\|D\phi_t|E^s\| \leq Ce^{-\lambda t}$ and $\|D\phi_{-t}|E^u\| \leq Ce^{-\lambda t}$ for $t > 0$,

(2) The flow is transitive (i.e., there exists a dense orbit).

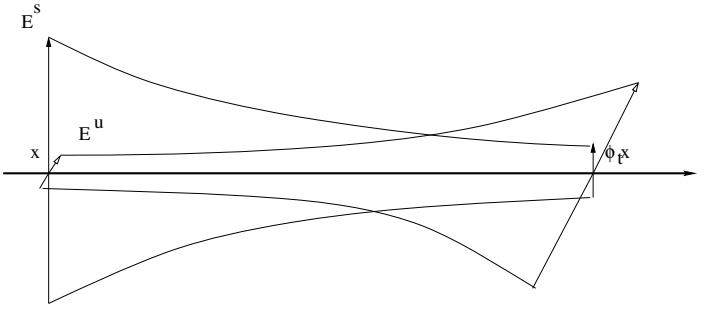


Figure 3.1. An Anosov flow. The flow contracts the bundle E^s and expands the bundle E^u

A crucial feature of Anosov flows is that they have a countable number of closed orbits τ whose least periods $\lambda(\tau)$ tend to infinity. Moreover, for our purposes an important fact is that they can be used to study geodesics on surfaces of variable negative curvature via the associated geodesic flow.

3.1. Geodesic flows. Let V be a compact surface with (variable) negative curvature $\kappa(x) < 0$, for $x \in V$, and let $M = SV := \{v \in TV : \|v\| = 1\}$ be the unit tangent bundle. Let ϕ_t be the associated geodesic flow, i.e., $\phi_t(v) = \dot{\gamma}_v(t)$ where $\gamma_v : \mathbb{R} \to V$ is the unit speed geodesic with $\dot{\gamma}_v(0) = v$. The closed orbits for the geodesic flow then correspond to closed geodesics on V.

Remark 3.2. Geodesic flows on surfaces of negative curvature, their dynamical properties and their analysis via symbolic coding were studied in a fundamental paper by Hadamard [29], only two years after his proof of The Prime Number Theorem (Theorem 1.2). This work was popularized in a 1906 book by the French physicist Duhen, and subsequently translated into German by Adler. In 1909, Adler's family shared a house with Einstein and his translation may (or may not) have influenced Einstein's work on general relativity. However, Adler is better known for assassinating the prime minister of Austria, Count Karl von Stürgkh on 21st October, 1916 [77].

More generally, a geodesic flow on a compact surface with some positive curvature may be an Anosov flow providing there is sufficient negative curvature, in an appropriate sense. We can illustrate this by two particularly simple examples of Anosov geodesic flows.

Example 3.3. Consider an idealized linkage, by which we mean a mechanical system consisting of a series of rigid rods where each rod has two either fixed pivots or movable joints connecting their ends. Furthermore, we can assume that all of the mass is concentrated on the joints (and friction, inertia, gravity, etc. can be neglected) and consider the time evolution

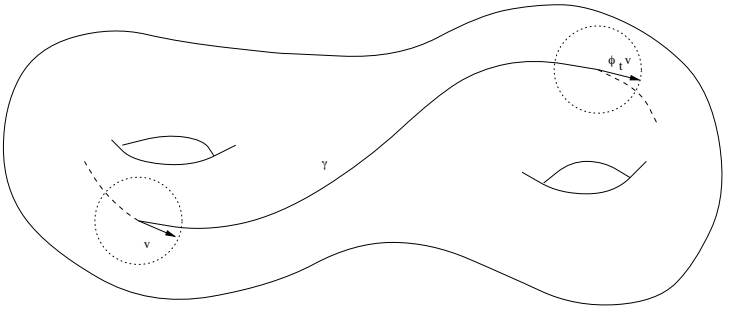


Figure 3.2. The geodesic flow on a negatively curved compact surface. This is an Anosov flow, with the bundles E^s, E^u being associated to the horocycles.

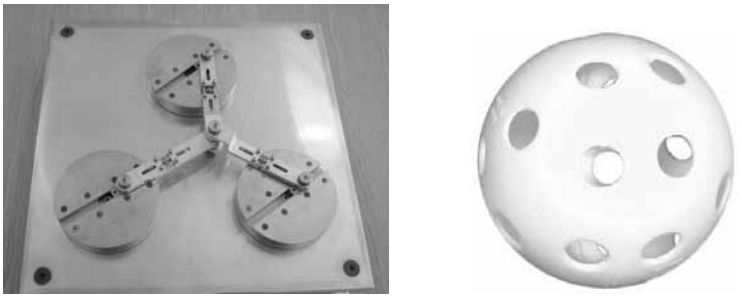


Figure 3.3. Two examples of Anosov geodesic flows: (i) The linkage example of Mackay and Hunt; (ii) A practice golf ball which resembles the surface of Donnay and Pugh.

of this idealised mechanical linkage in its phase space [33]. Its behaviour is described by a geodesic flow on its two dimensional configuration space. In particular, Hunt and MacKay constructed examples of triple linkages (based on the topological examples of Thurston and Weeks) for which the flow is Anosov. Other examples were investigated in [46].

Example 3.4. Donnay and Pugh showed how to construct a surface which can be embedded into three dimensional Euclidean space, and for which the geodesic flow is Anosov [16]. The construction begins with two concentric spheres (with mild positive curvature) which are then connected by a large number of judiciously placed small tubes (with strong negative curvature).

One of the basic properties of Anosov flows is structural stability, by which any sufficiently small perturbation of the flow still results in an Anosov flow and thus gives a wealth of related examples. However, we next recall a second basic class of examples which are fundamentally different to geodesic flows.

3.2. Suspensions of Anosov diffeomorphisms. Given any homeomorphism $f : X \to X$ of a compact metric space X and a strictly positive continuous function $r : X \to \mathbb{R}^+$ we can associate a new space

$$\widehat{X} = \{(x, u) : 0 \leq u \leq r(x)\}/(x, r(x)) \sim (f(x), 0).$$

We can then define a flow $f_t : \widehat{X} \to \widehat{X}$ by $f_t(x,u) = (x, u+t)$, subject to the equivalence relation.

We now specialise to the special case that the homeomorphism is a C^∞ Anosov diffeomorphism $f : X \to X$ on a compact manifold. The general definition can be found in [8]. However, the simplest example to have in mind is that of an Arnol'd CAT map (standing for Continuous Automorphism on a Torus), e.g., $f : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ defined by $f(x,y) = (2x+y, x+y)$. In the case that $f : X \to X$ is an Anosov diffeomorphism and $r : X \to \mathbb{R}^+$ is C^∞ the flow $f_t : \widehat{X} \to \widehat{X}$ is a C^∞ Anosov flow.

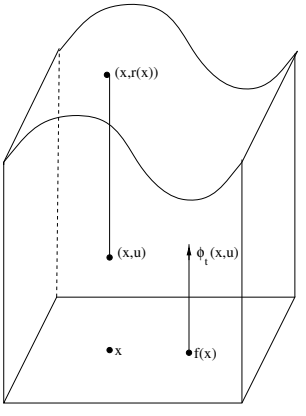


Figure 3.4. A suspension flow $f_t : \widehat{X} \to \widehat{X}$ over a transformation $f : X \to X$.

We say that an Anosov flow $\phi_t : M \to M$ is *topologically weak mixing* if there is no non-trivial solution to $F \circ \phi_t = e^{iat} F$ with $a \in \mathbb{R}$ and $F \in C^0(M, \mathbb{C})$. Whereas Anosov geodesic flows are always topologically weak mixing, the suspension Anosov flows may not always be so (for example, when r is a constant function). However, topologically weak mixing is a generic assumption and we will henceforth assume it to simplify the exposition.

4. The Ruelle zeta function for Anosov flows

It should now be fairly clear what the natural generalisation of the zeta function $\zeta_S(s)$ to Anosov flows $\phi_t : M \to M$ in general, and geodesic flows in particular, should be. We will denote by τ a primitive closed orbit and let $\lambda(\tau) > 0$ denote its period (i.e., for any $x_\tau \in \tau$ the period is the smallest value $t > 0$ for which $\phi_t x_\tau = x_\tau$). However, to specify the domain of convergence of the zeta function we still need to introduce the notion of the topological entropy $h(\phi)$ of the flow. The definition in the general case can be found in [55], for example, but there is a particularly simple equivalent formulation in the specific context of geodesic flows (on negatively curved surfaces) which we now recall.

Example 4.1 (Topological entropy of geodesic flows). Let V be a surface with negative curvature. We can lift the Riemannian metric on V to the universal cover $\widetilde{V}$ and consider the rate of growth of volume of balls $B_{\tilde{V}}(x,R)$ of radius $R > 0$ in $\widetilde{V}$. Then the topological

entropy for the geodesic flow $\phi_t : M \to M$ is given by

$$h(\phi) := \lim_{T\to+\infty} \frac{1}{T} \log \mathrm{Vol}(B_{\tilde{V}}(x, R))$$

for any $x \in \widetilde{V}$ [44]. In the special case of a surface of constant curvature $\kappa = -1$ we easily see that the geodesic flow has topological entropy $h(\phi) = 1$.

By analogy with the product form of the Riemann zeta function $\zeta(s)$ (and the geometric zeta function $\zeta_S(s)$) we can now define the corresponding zeta function for an Anosov flow as follows.

Definition 4.2. The *Ruelle zeta function* for an Anosov flow is defined by

$$\zeta_R(s) = \prod_{\tau} \left(1 - e^{-s\lambda(\tau)}\right)^{-1}, \quad s \in \mathbb{C},$$

which converges to a non-zero analytic function for $Re(s) > h(\phi)$.

There was also a version proposed by Smale [76] closer to original definition $Z(s)$ of Selberg, but we will use the formulation of Ruelle.

Riemann	Selberg	Ruelle
Number Theory	Geometry	Dynamical Systems
primes p	closed geodesics γ	closed orbits τ
p	$e^{l(\gamma)}$ for lengths $l(\gamma)$	$e^{\lambda(\tau)}$ for periods $\lambda(\tau)$
$\zeta(s) = \prod_p (1 - p^{-s})^{-1}$	$\zeta_S(s) = \prod_\gamma (1 - e^{-sl(\gamma)})^{-1}$	$\zeta_R(s) = \prod_\tau (1 - e^{-s\lambda(\tau)})^{-1}$

Table 4.1. Comparing the definitions of three zeta functions: Riemann zeta function, the geometric zeta functions and the Ruelle zeta function.

Example 4.3. In the case of the geodesic flow on a compact negatively curved surface the closed orbits τ of period $\lambda(\tau)$ correspond to closed geodesics γ of length $l(\gamma) = \lambda(\tau)$. In particular, we then have that $\zeta_S(s) = \zeta_R(s)$.

The following theorem shows that Theorem 2.2 for constant curvature surfaces generalises to Anosov flows. In particular, it shows that the theorem generalises to surfaces of variable negative curvature. Let $\phi_t : M \to M$ be a C^∞ Anosov flow.

Theorem 4.4. *The zeta function $\zeta_R(s)$ converges to a non-zero analytic function for $Re(s) > h(\phi)$. Moreover,*

(1) *$\zeta_R(s)$ has a simple pole at $s = h(\phi)$; and*

(2) *$\zeta_R(s)$ extends to all complex numbers $s \in \mathbb{C}$ as a meromorphic function.*

The first part of this theorem was proved by Ruelle in [68].[2] The second part was proved in [26], with an alternative proof being given in [17]. If we have only have finite regularity

[2] An amusing reminiscence appears in Ruelle's article [67].

(i.e., a C^k Anosov flow $\phi_t : M \to M$ with $1 \leq k < +\infty$) then we still get an extension, albeit to a half plane. In particular, there exists $\lambda > 0$ (from the definition of the Anosov flow) such that $\zeta_R(s)$ has an extension to $Re(s) > h(\phi) - \lambda[\frac{k}{2}]$ [26].

In the general case of topologically weak mixing Anosov flows (which, we recall, includes geodesic flows on negatively curved surfaces) the zeta function $\zeta_R(s)$ has no zeros or poles on the line $Re(s) = h(\phi)$, other than $s = h(\phi)$, by analogy with the corresponding property for the Riemann zeta function $\zeta(s)$. Given a topologically weak mixing Anosov flow, we reuse the notation $\Pi(x) = \{\tau : \lambda(\tau) \leq x\}$, this time to denote the number of closed orbits τ with least period $\lambda(\tau)$ less than x. The following theorem was originally due to Margulis [47, 48] although the proof using zeta functions appears in [54, 55].

Theorem 4.5 (Prime Orbit Theorem). *For a topologically weak-mixing Anosov flow we have that* $\Pi(x) \sim \frac{e^{h(\phi)}}{h(\phi)x}$ *as* $x \to +\infty$.

Restricting to the case of geodesic flows on compact negatively curved surfaces, we have the following generalisation of Theorem 2.3.

Corollary 4.6. *For compact surfaces of variable negative curvature we have that* $\Pi(x) \sim \frac{e^{h(\phi)}}{h(\phi)x}$ *as* $x \to +\infty$.

In particular, this gives the extension of Theorem 2.3 to the case of variable negative curvature.

Remark 4.7 (Zeta functions for Anosov diffeomorphisms). We have omitted a detailed discussion of the case of zeta functions for Anosov diffeomorphisms $f : X \to X$, despite their mathematical and historical importance. In this context, the natural definition of the zeta function is that of Artin and Mazur [2]. Let $N(n)$, $n \geq 1$, denote the number of fixed points $f^n x = x$ for $f^n : X \to X$ and define

$$\zeta_{AM}(z) = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} N(n)\right), \quad z \in \mathbb{C},$$

which converges for $|z|$ sufficiently small. This zeta function can also be written in Euler product form as $\zeta(z) = \prod_\tau (1 - z^{|\tau|})^{-1}$ with the product over prime closed orbits $\tau = \{x, fx, \cdots, f^{n-1}x\}$ of least period $|\tau| = n$. For Anosov diffeomorphisms, or even more generally for Axiom A diffeomorphisms, this is a rational function, (i.e., a quotient of two polynomials in z) [20, 28, 44]. This is perhaps reminiscent of the results for the Lefschetz zeta function, the Weil zeta functions for finite fields, and the Ihara zeta functions for finite graphs [6, 80]. There are also closely related results for interval maps [31].

More generally, one might weight τ by taking the values along the points in the orbit of a suitable function $F : X \to \mathbb{C}$ and then associate a more general version of the Artin-Mazur zeta function

$$\zeta_{R,F}(z) = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{T^n x = x} \exp\left(\sum_{i=0}^{n-1} F(T^i x)\right)\right).$$

Of course, this reduces to the Artin-Mazur zeta function above when F is identically zero. On the other hand, this gives the zeta function for the suspension Anosov flow when $F = -sr$ (where $s \in \mathbb{C}$ and $r : X \to \mathbb{R}^+$). Much is known about the domain of $\zeta_{R,F}(z)$ through the work of many authors [3, 43, 55, 66, 68, 69].

5. Techniques for C^∞ Anosov flows

We briefly recall the two principle approaches to studying zeta functions for Anosov flows. Broadly speaking, these share the common strategy of using suitable operators on appropriate Banach spaces. In both approaches, the extension of the zeta function is related to the spectral properties of the corresponding operator and the poles and zeros are related to spectra of the operators. The earlier approach used symbolic dynamics and a reduction to the study of transfer operators on classical Banach spaces of Hölder functions for subshifts of finite type [55]. The more recent approaches avoid this somewhat non-canonical reduction and involve similar, but technically different, operators on specially tailored Banach spaces of distributions [26].

5.1. Symbolic dynamics. The classical approach to studying the Ruelle zeta function $\zeta_R(s)$ was based on the use of symbolic dynamics, whereby the Anosov system was modelled by the suspension of a two sided subshift of finite type $\sigma : X \to X$ by a Hölder continuous function $r : X \to \mathbb{R}$. This is similar in spirit to the suspension construction described in §3.2, although in the general case the subshift arises from considering the Poincaré map on a finite number of judiciously chosen codimension one transverse sections. We refer to [64] and [9] for more details.

In this approach one replaces the two sided subshift of finite type (which is a homeomorphism) by the one sided subshift of finite type (which is a finite-to-one local homeomorphism). This corresponds to artificially suppressing the effect of the contracting direction for the flow. One can then considers the Banach space of Hölder continuous functions on the one sided shift space and an associated transfer operators (parameterised by complex numbers $s \in \mathbb{C}$) which averages over the preimages [55, 68].

Remark 5.1. This approach has the advantage that it applies in the even more general setting of Smale's Axiom A flows [76] and which, as we will see later in § 7.3.1, has applications to geodesic flows on infinite volume surfaces. However, in this more general context of Axiom A flows the zeta function $\zeta_R(s)$ may not extend meromorphically to $\mathbb{C}$, and there are examples of Axiom A flows for which the zeta function has an essential singularity [25, 68].

5.2. Anisotropic spaces. Despite its early success, the previous approach has the distinct disadvantage that one cannot make use of the smoothness of the flow. A more recent approach has been to work with simpler operators, but to consider more sophisticated Banach spaces of distributions. The origins for these ideas lie in the papers of Rugh [72] and Fried [21] for real analytic Anosov diffeomorphisms and flows, respectively, and by Kitaev [40] for Anosov diffeomorphisms of finite differentiability. Two different approaches to defining these Banach spaces were developed for diffeomorphisms by Baladi-Tsujii [3] and Goëuzel-Liverani [27]. The approach in the latter paper was then extended to flows in [11] and [12]. However, in order to apply this method to the Ruelle zeta function $\zeta_R(s)$ it is necessary to generalise it to Banach spaces of forms [26], as was already anticipated in the work of Ruelle [66]. In particular, the extension of the zeta function is via closely related complex functions called determinants which, as the name suggests, are extended using spectral properties of the associated transfer operators. The determinants associated to forms can then be combined to recover the zeta function.

6. Error terms for counting closed orbits

In the introduction we recalled the importance of the Riemann Hypothesis (Conjecture 1.3) for the Riemann zeta function. We now consider its analogues in the context of the zeta functions for closed geodesics on constant and variable negatively curved manifolds, respectively. We begin with the classical results in the case of the geometric zeta function $\zeta_S(s)$ for a compact surface V of constant curvature $\kappa = -1$.

6.1. The surface case. We begin with the case that V is a compact surface of constant negative curvature $\kappa = -1$. Let $\Delta : L^2(V) \to L^2(V)$ be the usual self-adjoint Laplace-Beltrami operator with eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$ satisfying $\Delta\psi_n + \lambda_n\psi_n = 0$. In particular, we have the following partial analogue of the Riemann Hypothesis (Conjecture 1.3) [30].

Theorem 6.1. *For a compact surface V of constant curvature $\kappa = -1$ the zeta function $\zeta_S(s)$ has a non-zero analytic extension to a half plane, $Re(s) > 1 - \epsilon$ where*

$$\epsilon = \begin{cases} \frac{1}{2} - \sqrt{\frac{1}{4} - \lambda_1} & \text{if } 0 < \lambda_1 < \frac{1}{4} \\ \frac{1}{2} & \text{if } \lambda_1 \geq \frac{1}{4}, \end{cases}$$

except for a simple pole at $s = 1$.

Combining this with additional bounds on the modulus of the zeta function in this region gives an error term estimate for number $\Pi(x)$ of geodesics γ with length $l(\gamma)$ less than x of the following form.

Corollary 6.2. *For $0 < \epsilon' < \epsilon$, $\Pi(x) = \mathrm{Li}(e^x) + O\left(e^{(1-\epsilon')x}\right)$.*

The error term has a simple geometric interpretation in terms of the geometry of the surface V. More precisely, the value λ_1 is proportional to the length of the shortest closed geodesic dividing the surface into two parts [74].

Remark 6.3. Hilbert and Polya proposed the idea of tying to understand the location of the zeros of the Riemann zeta function in terms of eigenvalues of some (as of yet) undiscovered self-adjoint operator whose necessarily real eigenvalues are related to the zeros. This idea has yet to reach fruition for the Riemann zeta function, but the approach works particularly well for the Selberg Zeta function where the associated operator is the Laplacian. Interestingly, it was Selberg who presented an alternative elementary proof (i.e., not using $\zeta(s)$, but significantly harder) of the Prime Number Theorem (Theorem 1.2) [73].

The corresponding results for surfaces of variable negative curvature and the Ruelle zeta function $\zeta_R(s)$ require a more dynamical proof which, unfortunately, gives a less quantifiable estimate on the size of the extension (and consequently the error term) [14, 62].

Theorem 6.4. *For a compact surface of (variable) negative curvature $\kappa < 0$ there exists $\epsilon > 0$ such that $\zeta_R(s)$ has an analytic zero-free extension to $Re(s) > h(\phi) - \epsilon$, except for a simple pole at $s = h(\phi)$.*

The same argument as in the case of the Selberg zeta function then leads to an exponential error term when counting closed geodesics on a surface of variable curvature [62]. Let $\Pi(x)$ again denote the number of closed geodesics γ with length $l(\gamma) \leq x$.

Corollary 6.5. *For* $0 < \epsilon' < \epsilon$, $\Pi(x) = \mathrm{Li}(e^{h(\phi)x}) + O(e^{(h(\phi)-\epsilon')x})$.

We do not know, for example, if such error terms can be achieved for closed orbits of general weak mixing Anosov flows. In the wider context of Axiom A flows the zeta function may have poles arbitrarily close to the line $Re(s) = h(\phi)$, and consequently no such exponential error term could be expected [60, 70]. However, weaker error terms can sometimes be obtained under quite modest assumptions [15, 63].

6.2. The higher dimensional case. There is a partial generalisation of Theorem 6.4 to higher dimensional manifolds [26]. We say that the variable negative sectional curvatures of a compact manifold V are *a-pinched* if they all lie in the range $[-1, -a]$.

Theorem 6.6. *Let V be a compact manifold with variable negative sectional curvatures that are $\frac{1}{9}$-pinched. There exists $\epsilon > 0$ such that $\zeta_R(s)$ has an analytic zero-free extension to $Re(s) > h(\phi) - \epsilon$, except for a simple pole at $s = h(\phi)$.*

Again, by complete analogy with the derivations of Corollaries 6.2 and 6.5, this has the following corollary under the same hypotheses as the theorem.

Corollary 6.7. *For* $0 < \epsilon' < \epsilon$, $\Pi(x) = \mathrm{Li}(e^{h(\phi)x}) + O(e^{(h(\phi)-\epsilon')x})$

These can be viewed as a generalisation of previous results. Firstly, this generalizes Corollary 6.2 in the particular case of *constant* negative curvature [32], [30]. Secondly, this partly generalizes Corollary 4.6 for the case of geodesic flow on variable negative curvature manifolds by adding an exponential error estimate. However, we do not know whether the conclusions of Theorem 6.6 and Corollary 6.7 remain true without any pinching condition on the sectional curvatures.

7. Applications

The theory of dynamical zeta functions has a surprisingly wide range of applications, from which we present a small selection.

7.1. Decay of correlations and resonances. There is a complementary problem to counting closed orbits for Anosov flows in which one considers mixing (or decay of correlations) with respect to Gibbs measures. The problem of estimating the error terms on counting functions for closed orbits naturally corresponds to estimates on error terms in decay of correlations.

Let $\phi_t : M \to M$ be an Anosov flow and let $F : M \to \mathbb{R}$ and $G : M \to \mathbb{R}$ be two smooth functions.

Definition 7.1. We define the *correlation function* for a ϕ-invariant probability measure μ by

$$\rho(t) := \int F\phi_t G d\mu - \int F d\mu \int G d\mu.$$

We say that the flow $\phi_t : M \to M$ is *(strong) mixing* relative to the measure μ if $\rho(t) \to 0$ for any $F, G \in C^\infty(M)$.

A basic question to ask is about the speed at which $\rho(t) \to 0$ as $t \to +\infty$. For this problem, a natural class of measures μ to study are Gibbs measures for Hölder continuous function $A : M \to \mathbb{R}$ which includes, for example, the Sinai-Ruelle-Bowen measure (which is precisely the normalised Liouville measure in the case of geodesic flows) and the Bowen-Margulis measure of maximal entropy. In particular, when $A = 0$ the Gibbs measure is the Bowen-Margulis measure, and when A is the infinitesimal expansion along the unstable manifolds then the Gibbs measure is the Sinai-Ruelle-Bowen measure. In either case, we can conveniently characterize the associated Gibbs measure μ_A in terms of weighted closed orbits:

$$\int B d\mu_A = \lim_{T\to+\infty} \frac{\sum_{\lambda(\tau)\leq T} \lambda_B(\tau) e^{-\lambda_A(\tau)}}{\sum_{\lambda(\tau)\leq T} \lambda(\tau) e^{-\lambda_A(\tau)}}$$

for any $B \in C^0(M)$, where $\lambda_A(\tau) = \int_0^{\lambda(\tau)} A(\phi_t x_\tau) dt$ and $\lambda_B(\tau) = \int_0^{\lambda(\tau)} B(\phi_t x_\tau) dt$, for any $x_\tau \in \tau$ [9, 52].

7.1.1. Geodesic flows on surfaces. If has been known since the work of Fomin and Gelfand [19] that the geodesic flow on compact surfaces with constant curvature $\kappa = -1$ has exponential decay of correlations with respect to the normalized Liouville measure. Their proof used representation theory and the associated decay of matrix coefficients. However, these methods do not extend to the geometric setting of manifolds with variable negative curvature and a different approach is required [14].

Theorem 7.2. *Let $\phi_t : M \to M$ be the geodesic flow for a compact surface with variable negative curvature and let μ be a Gibbs measure for a Hölder continuous function. Then the correlation function $\rho(t)$ tends to zero exponentially fast, i.e., there exist constants $C, \epsilon > 0$ such that $|\rho(t)| \leq Ce^{-\epsilon t}$ for all $t > 0$.*

In particular, this result applies to the important examples of the Bowen-Margulis and normalized Liouville measures described above.

7.1.2. Geodesic flows in higher dimensions. It was also shown in [19] that the geodesic flow on a three dimensional manifold with constant curvature $\kappa = -1$ has exponential decay of correlations with respect to the Liouville measure, and the basic method generalises to arbitrary dimensions [50]. Moreover, it also applies to frame flows for three dimensional manifolds, which has been useful, for example, in the recent work of Kahn and Markovic [38]. However, for manifolds with variable negative sectional curvatures a dynamical viewpoint is again necessary [14].

Theorem 7.3. *Let $\phi_t : M \to M$ be the geodesic flow for a compact manifold with variable negative sectional curvatures that are $\frac{1}{4}$-pinched and let μ be a Gibbs measure for a Hölder continuous function. Then the correlation function $\rho(t)$ tends to zero exponentially fast.*

The proof of Dolgopyat of the above theorem is stated for normalized Liouville measure, but there is additional property required for more general Gibbs measures, which can apparently be deduced for geodesic flows using a Shadowing Lemma of Mohsan [57] (see also [78, 79]). In particular, the result applies to both the Bowen-Margulis and normalized Liouville measures, although in these particular cases the following stronger results are known [26, 42].

Theorem 7.4. *Let $\phi_t : M \to M$ be the geodesic flow for a compact manifold with variable negative sectional curvatures.*

(1) *If μ is the normalized Liouville measure, then the correlation function $\rho(t)$ tends to zero exponentially fast.*

(2) *If the sectional curvatures are $\frac{1}{9}$-pinched and μ is the Bowen-Margulis measure then the correlation function $\rho(t)$ tends to zero exponentially fast.*

By contrast, in the more general setting of Axiom A flows it is possible to give examples there the flow mixes arbitrarily slowly [60, 70].

7.1.3. Fourier transforms and resonances. A standard approach to understanding the asymptotic behaviour of such functions $\rho(t)$ is by considering the Fourier transform. More precisely, we write

$$\widehat{\rho}(z) = \int_{-\infty}^{\infty} e^{izt}\rho(t)dt, \quad z \in \mathbb{C},$$

where it is defined. The following well known classical result gives explicit connection between the domain of $\widehat{\rho}(z)$ and the asymptotic behaviour of $\rho(t)$.

Theorem 7.5 (Paley-Wiener). *For a function $\rho(t)$ the following are equivalent.*

(1) *The Fourier transform $\widehat{\rho}(z)$ has an analytic extension to some strip $|Im(z)| < \epsilon$, and is integrable along lines parallel to the real axis.*

(2) *$\rho(t)$ tends to zero exponentially fast.*

An early result on the domain of the Fourier transform $\widehat{\rho}(z)$ of the correlation function was the following [60, 71].

Theorem 7.6. *Let $\phi_t : M \to M$ be a C^∞ Anosov flow and let μ be a Gibbs measure for a Hölder continuous function. Then there exists $\epsilon > 0$ so that the function $\widehat{\rho}(z)$ has a meromorphic extension to $|Im(z)| < \epsilon$.*

The poles in the meromorphic extension given in Theorem 7.6 are sometimes called *resonances*. The original proof of Theorem 7.6 used the method of symbolic dynamics and thus even applies in the more general context of C^1 Axiom A flows. In the particular context of the normalized Liouville measure [11], [12] and the Bowen-Margulis measure [26] there are stronger results.

Perhaps somewhat surprisingly, the proof of Theorem 7.6 shows in the case that the Gibbs measure is the Bowen-Margulis measure μ_0 that the poles of $\widehat{\rho}(z)$ are intimately related to the poles of the Ruelle zeta function $\zeta_R(s)$.

Theorem 7.7. *The poles $z = \pm b \pm ia$ for $\widehat{\rho}(z)$ (with $a, b \in \mathbb{R}^+$) give rise to poles $s = h(\phi) + a \pm ib$ for $\zeta_R(s)$.*

7.2. Computation of numerical values. Dynamical zeta functions can sometimes be used to give alternative expressions for certain numerical dynamical characteristics, such as the Hausdorff dimension of invariant sets or the Lyapunov exponents, and thus provide an alternative method for their computation which is often quite efficient. This basic method, based on what are now known as cycle expansions, was pioneered by Cvitanović and his coauthors [13]. We briefly describe two applications of this approach.

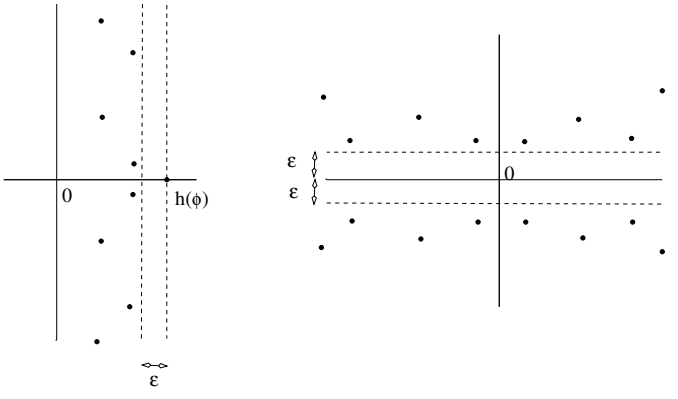


Figure 7.1. (a) The poles for $\zeta_R(s)$; and (b) The poles for $\widehat{\rho}(z)$. We can informally think of this as translating by $-h(\phi)$, turning the picture through 90 degrees, and adding its reflection.

7.2.1. Hausdorff dimension. We can consider a rational map $T : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with a hyperbolic Julia set $\mathcal{J}$ (i.e., $\mathcal{J}$ is the closure of the union of the periodic points $T^n z = z$ for $|(T^n)'(z)| > 1$, and we require that $\sup_{z \in \mathcal{J}} |(T^n)'(z)| > 1$). The Hausdorff Dimension $\dim_H(\mathcal{J})$ of the Julia set can then be approximated using the values $|(T^n)'(z)|$ of the derivatives at period points $T^n z = z$ with periods less than N. McMullen showed how these values could be used to compute approximations d_N to the Hausdorff dimension satisfying $\dim_H(\mathcal{J}) = d_N + O(\theta^N)$, for $N \geq 1$ and a fixed value $0 < \theta < 1$, using approximations based on eigenvalues of matrices [49]. Using an approach based on the Ruelle zeta function shows that precisely the same values for periodic points, but used in different combinations, leads to a faster approximation $\mathcal{D}_N$ with $\dim_H(\mathcal{J}) = \mathcal{D}_N + O(\theta^{N^{3/2}})$, for $N \geq 1$ and a fixed value $0 < \theta < 1$. Analogous results hold for the limit sets of Schottky groups and certain related Kleinian groups [34], as well as for continued fractions with deleted digits [35].

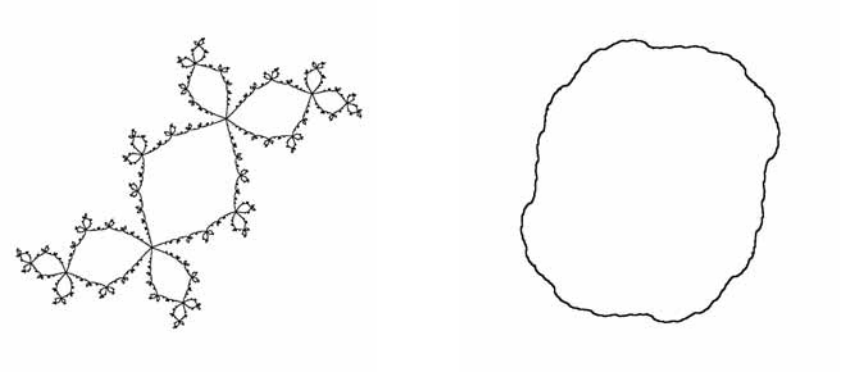


Figure 7.2. Two hyperbolic Julia sets: (i) a Douady Rabbit; and (ii) a quasi-circle.

7.2.2. Lyapunov exponents. Given a finite set of $d \times d$ matrices $A_1, \cdots, A_k$, with $d \geq 2$ and $k \geq 2$, and a probability vector $(p_1, \cdots, p_k)$ one can associate the (largest) Lyapunov

exponent [24] defined by

$$\lambda = \lim_{n\to+\infty} \frac{1}{n} \sum_{i_1,\cdots,i_n} p_{i_1} \cdots p_{i_n} \log \|A_{i_1} \cdots A_{i_n}\|.$$

In the particular case that the matrices are strictly positive then it is possible to approximate λ using the maximal eigenvalues $\lambda_{i_1\cdots i_n}$ of the finite products $A_{i_1} \cdots A_{i_n}$ with $1 \leq n \leq N$ [58]. Using the Ruelle zeta function these values can be used to get approximations λ_N to the Lyapunov exponent satisfying $\lambda = \lambda_N + O(\theta^{N^{1+\frac{1}{d-1}}})$, for $N \geq 1$ and a fixed value $0 < \theta < 1$ [61]. This has applications, for example, to computing entropy rates for binary symmetric processes. There are similar types of estimates for the Lyapunov exponents for C^ω Markov expanding maps [36].

7.3. Variations on the theme of the geometric zeta function. The dynamical viewpoint sometimes provides a useful tool for extending the zeta function $\zeta_S(s)$ for geodesic flows to related settings. We briefly illustrate this viewpoint with the examples of infinite volume surfaces and the semi-classical zeta function.

7.3.1. Zeta functions for infinite volume surfaces. The setting of geodesic flows on infinite area surfaces associated to convex cocompact Fuchsian groups is one in which the dynamical viewpoint proves particularly useful. Although the associated geodesic flow is not Anosov the restriction to the recurrent part of the flow, which contains all of the closed orbits, is essentially a real analytic Axiom A flow and so the associated zeta function $\zeta_S(s)$ can be studied by adapting the dynamical approach of [66] and, in particular, it can be shown to have a meromorphic extension to $\mathbb{C}$.

However, in this setting the zeros and poles of the zeta function $\zeta_S(s)$ are now more difficult to describe than in the case of compact surfaces, associated to cocompact Fuchsian groups. As a consequence of the dynamical approach and estimates on the zeta function some results are known on the distribution of zeros and poles [4, 56]. Moreover, it is known that a weak analogue of Theorem 6.1 still holds, in as much as there is a non-zero analytic extension to a half-plane $Re(s) > h(\phi) - \epsilon$ in the spirit of Theorem 6.4 [51]. But the empirical behaviour of the resonances appears to be very different from that of the zeta function for compact surfaces [5].

7.3.2. Zeros for semi-classical zeta functions. Recently, Faure and Tsujii proved new results on the semi-classical zeta function

$$\zeta_{SC}(s) = \exp\left(-\sum_{\tau}\sum_{m=1}^{\infty} \frac{e^{-sm\lambda(\tau)}}{m|\det(I - D(\tau)^m)|^{\frac{1}{2}}}\right)$$

where $D(\tau)$ is the Jacobian for the Poincaré map for τ intersecting a small transverse section to the flow [18]. They have shown that some of the results on the locations of zeros for the original Selberg zeta function $\zeta_S(s)$ have analogs for $\zeta_{SC}(s)$. In particular, they show that the zeros, with only finitely many exceptions, lie in vertical strips.

7.4. Variations on the theme of the Ruelle zeta function. There are also interesting variations on the Ruelle zeta function $\zeta_R(s)$ for Anosov flows. We illustrate this with a more general weighted Ruelle zeta function and L-functions.

7.4.1. The generalized Ruelle zeta function. A more general version of the Ruelle zeta function takes the form:

$$\zeta_R^A(s) = \prod_\tau \left(1 - e^{\lambda_A(\tau) - s\lambda(\tau)}\right)^{-1}, \quad s \in \mathbb{C},$$

where $A : M \to \mathbb{C}$ is a C^∞ function and as before $\lambda_A(\tau) = \int_0^{\lambda(\tau)} A(\phi_t x_\tau) dt$, for any $x_\tau \in \tau$ This converges to a non-zero analytic function for $Re(s) > P(A)$, where $P(A)$ is the pressure of the function A. (In the special case that $A = 0$ is identically zero then $P(0) = h(\phi)$ and the zeta function reduces to the original Ruelle zeta function, i.e., $\zeta_R^0(s) = \zeta_R(s)$.) This was studied in [69] and was shown to have a simple pole at $P(s)$ and a meromorphic extension to a larger domain. This leads to corresponding asymptotic and equidistribution results for weighted closed orbits [52]. Finally, a closely related zeta function is the differential zeta function introduced in [53].

7.4.2. L-functions. In prime number theory, the Riemann zeta function $\zeta(s)$ has a useful generalization to L-functions which are used in the study of the distribution of primes in congruence classes.

The analogue of these complex functions for Anosov flows are given by

$$L_{R_\chi}(s) = \prod_\tau \det\left(1 - e^{-s\lambda(\tau)} R_\chi([\tau])\right)^{-1}, \quad s \in \mathbb{C},$$

where $R_\chi : \pi_1(M) \to U(n)$ is an irreducible unitary representation, and which again converges for $Re(s) > h(\phi)$ [55]. In the particular case that $n = 1$ and $R_\chi = I$ is trivial then the L-function reduces to the original Ruelle zeta function, i.e., $L_I(s) = \zeta_R(s)$.

A particularly elegant variant of this approach is where one considers the special case of the geodesic flow $\phi_t : M \to M$ on the unit tangent bundle $M = SV$ of a compact negatively curved surface V and $\chi : H_1(V, \mathbb{Z}) \to \mathbb{C}$ is a character. This associates to each closed geodesic γ with homology class $[\gamma] \in H_1(V, \mathbb{Z})$ the weight $\chi([\gamma])$. Using properties of these L-functions one can show that for any $\alpha \in H_1(V, \mathbb{Z})$ the number $\Pi(x, \alpha)$ of closed geodesics with length $l(\gamma) \leq T$ and $[\gamma] = \alpha$ satisfies

$$\Pi(x, \alpha) \sim C \frac{e^{h(\phi)x}}{T^{b/2+1}}, \text{ as } x \to +\infty,$$

where b is the first Betti number of M [39, 59].

Remark 7.8. Among the many topics we have not discussed, are the Patterson conjecture [10], the Lefschetz theorem for flows [23, 37], and results on the closely related Poincaré series. For many other topics related to counting problems, we refer the reader to [75].

References

[1] Anosov, D.V., *Geodesic flows on closed Riemannian manifolds with negative curvature*, Proc. Steklov Inst. Math. **90** (1967), 1–235.

[2] Artin, M. and Mazur, B., *On periodic points*, Ann. of Math. **81** (1965), 82–99.

[3] Baladi, V. and Tsujii, M., *Dynamical determinants and spectrum for hyperbolic diffeomorphisms*, Contemp. Math. **469** (2008), 21–68.

[4] Borthwick, D., *Spectral theory of infinite-area hyperbolic surfaces*, Progress in Mathematic 256, Birkhauser, Boston, 2007.

[5] ———, *Distribution of Resonances for Hyperbolic Surfaces*, Exp. Math. **23** (2014), 25–45.

[6] Bowen, R., Lanford, and O. E. III, *Zeta functions of restrictions of the shift transformation*, Proc. Sympos. Pure Math. **XIV** (1970), 43–49.

[7] Bowen, R., *Symbolic dynamics for hyperbolic flows*, Amer. J. Math. **95** (1973), 429–460.

[8] ———, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Lecture Notes in Mathematics 470, Springer-Verlag, Berlin, 1975.

[9] Bowen, R. and Ruelle, D., *The ergodic theory of Axiom A flows*, Invent. Math. **29** (1975), 181–202.

[10] Bunke, U. and Olbrich, M., *Group cohomology and the singularities of the Selberg zeta function associated to a Kleinian group*, Ann. of Math. **149** (1999), 627–689.

[11] Butterley, O. and Liverani, C., *Smooth Anosov flows: correlation spectra and stability*, J. Mod. Dyn. **1** (2007), 301–322.

[12] ———, *Robustly invariant sets in fiber contracting bundle flows*, J. Mod. Dyn. **7** (2013), 255–267.

[13] Cvitanović, P., Artuso, R., Mainieri, R., Tanner, G., and Vattay, G., *Chaos: Classical and Quantum*, Niels Bohr Institute, Copenhagen, 2005.

[14] Dolgopyat, D., *On decay of correlations in Anosov flows*, Ann. of Math. **147** (1998), 357–390.

[15] ———, *Prevalence of rapid mixing in hyperbolic flows*, Ergod. Th. Dynam. Sys. **18** (1998), 1097–1114.

[16] Donnay, V. J., Pugh, and C. C., *Anosov geodesic flows for embedded surfaces*, Astérisque **287** (2003), 61–69.

[17] Dyatlov, S., Zworski, M., *Dynamical zeta functions for Anosov flows via microlocal analysis*, preprint.

[18] Faure, F. and Tsujii, M., *Band structure of the Ruelle spectrum of contact Anosov flows*, C. R. Math. Acad. Sci. Paris **351** (2013), 385–391.

[19] Fomin, S. V. and Gelfand, I.M., *Geodesic flows on manifolds of constant negative curvature*, Amer. Math. Soc. Transl. **1** (1955), 49–65.

[20] Fried, D., *Rationality for isolated expansive sets*, Adv. in Math. **65** (1987), 35–38.

[21] ———, *Meromorphic zeta functions for analytic flows*, Comm. Math. Phys. **174** (1995), 161–190.

[22] ———, *The zeta functions of Ruelle and Selberg. I.*, Ann. Sci. École Norm. Sup. **19** (1986), 491–517.

[23] ———, *Lefschetz formulas for flows*, Contemp. Math. **58** (1987), 19–69.

[24] Furstenberg, H. and Kesten, H., *Products of random matrices*, Ann. Math. Statist. **31** (1960), 457–469.

[25] Gallavotti, G., *Funzioni zeta ed insiemi basilari*, Atti. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. **61** (1976), 309–317.

[26] Giulietti, P., Liverani, C., and Pollicott, M., *Anosov flows and dynamical zeta functions*, Ann. of Math. **178** (2013), 687–773.

[27] Gouëzel, S. and Liverani, C., *Banach spaces adapted to Anosov systems*, Ergod. Th. Dynam. Sys. **26** (2006), 189–217.

[28] Guckenheimer, J., *Axiom A + nocycles* $\Rightarrow$ $\zeta_f(t)$ *rational*, Bull. Amer. Math. Soc. **76** (1970), 592–594.

[29] Hadamard, *J., Les surfaces à courbures opposées et leurs lignes géodésiques*, J. Math. Pures Appl. **IV** (1898), 27–73.

[30] Hejhal, D. A., *The Selberg trace formula for* $PSL(2,\mathbb{R})$, Lecture Notes in Mathematics 548, Springer-Verlag, Berlin, 1976.

[31] Hofbauer, F. and Keller, G., *Zeta-functions and transfer-operators for piecewise linear transformations*, J. Reine Angew. Math. **352** (1984), 100–113.

[32] Huber, H., *Zur analytischen Theorie hyperbolischen Raumformen und Bewegungsgruppen*, Math. Ann. **138** (1959) 1–26.

[33] Hunt, T. J. and MacKay, R. S., *Anosov parameter values for the triple linkage and a physical system with a uniformly chaotic attractor*, Nonlinearity **16** (2003), 1499–1510.

[34] Jenkinson, O. and Pollicott, M., *Calculating Hausdorff dimensions of Julia sets and Kleinian limit sets*, Amer. J. Math. **124** (2002), 495–545.

[35] ———, *Computing the dimension of dynamically defined sets:* E_2 *and bounded continued fractions*, Ergod. Th. Dynam. Sys. **21** (2001) 1429–1445.

[36] ———, *Computing invariant densities and metric entropy*, Comm. Math. Phys. **211** (2000), 687–703.

[37] Juhl, A., *Cohomological theory of dynamical zeta functions*, Progress in Mathematics 194, Birkhüser Verlag, Basel, 2001.

[38] Kahn, J. and Markovic, V., *Immersing almost geodesic surfaces in a closed hyperbolic three manifold*, Ann. of Math., **175** (2012), 1127–1190.

[39] Katsuda, A. and Sunada, T., *Closed orbits in homology classes*, Inst. Hautes Études Sci. Publ. Math. **71** (1990), 5–32.

[40] Kitaev, A. Yu., *Fredholm determinants for hyperbolic diffeomorphisms of finite smoothness*, Nonlinearity **12** (1999), 141–179.

[41] Littlewood, J. E., *Littlewood's miscellany*, Cambridge University Press, Cambridge, 1986.

[42] Liverani, C., *On contact Anosov flows*, Ann. of Math. **159** (2004), 1275–1312.

[43] Liverani, C. and Tsujii, M., *Zeta functions and dynamical systems*, Nonlinearity **19** (2006), 2467–2473.

[44] Manning, A., *Topological entropy for geodesic flows*, Ann. of Math. **110** (1979), 567–573.

[45] _____, *Axiom A diffeomorphisms have rational zeta functions*, Bull. London Math. Soc. **3** (1971), 215–220.

[46] Magalhães, M. L. S., *Pollicott, M., Geometry and dynamics of planar linkages*, Comm. Math. Phys. **317** (2013), 615–634.

[47] Margulis, G. A., *On some applications of ergodic theory to the study of manifolds on negative curvature*, Functional Anal. Appl. **3** (1969), 89–90.

[48] _____, *On some aspects of the theory of Anosov systems*, Springer-Verlag, Berlin, 2004.

[49] McMullen, C. T., *Hausdorff dimension and conformal dynamics. III. Computation of dimension*, Amer. J. Math. **120** (1998), 691–721.

[50] Moore, C. C., *Exponential decay of correlation coefficients for geodesic flows*, Math. Sci. Res. Inst. Publ. **6** (1987), 163–181.

[51] F. Naud, *Expanding Maps on Cantor Sets and Analytic Continuation of Zeta Functions*, Ann. Sci. École Norm. Sup. **38** (2005), 116–153.

[52] Parry, W., *Equilibrium states and weighted uniform distribution of closed orbits*, Lecture Notes in Math. **1342** (1988), 617–625.

[53] _____, *Synchronisation of canonical measures for hyperbolic attractors*, Comm. Math. Phys. **106** (1986), 267–275.

[54] Parry, W. and Pollicott, M., *An analogue of the prime number theorem for closed orbits of Axiom A flows*, Ann. of Math. **118** (1983), 573–591.

[55] _____, *Zeta functions and the periodic orbit structure of hyperbolic dynamics*, Astérisque **187–188** (1990), 1–268.

[56] Patterson, S. J. and Perry, P. A., *The divisor of Selberg's zeta function for Kleinian groups*, Duke Math. J. **106** (2001), 321–390.

[57] Paulin, F., Pollicott, M., and Shapira, B., *Equilibrium states in negative curvature*, Astérisque, to appear.

[58] Peres, Y., *Domains of analytic continuation for the top Lyapunov exponent*, Ann. Inst. H. Poincaré Probab. Statist. **28** (1992), 131–148.

[59] Phillips, R. and Sarnak, P., *Geodesics in homology classes*, Duke Math. J. **55** (1987), 287–297.

[60] Pollicott, M., *On the rate of mixing of Axiom A flows*, Invent. Math. **81** (1985), 413–426.

[61] ———, *Maximal Lyapunov exponents for random matrix products*, Invent. Math. **181** (2010), 209–226.

[62] Pollicott, M. and Sharp, R., *Exponential error terms for growth functions on negatively curved surfaces*, Amer. J. Math. **120** (1998), 1019–1042.

[63] ———, *Error terms for closed orbits of hyperbolic flows*, Ergod. Th. Dynam. Sys. **21** (2001), 545–562.

[64] Ratner, M., *Markov partitions for Anosov flows on n-dimensional manifolds*, Israel J. Math. **15** (1973), 92–114.

[65] Riemann, B., *Über die Anzahl der Primzahlen unter einer gegebenen Grösse*, Monatsberichte der Berliner Akademie, Leipzig, 1892.

[66] Ruelle, D., *Zeta-functions for expanding maps and Anosov flows*, Invent. Math. **34** (1976), 231–242.

[67] ———, *Dynamical zeta functions and transfer operators*, Notices Amer. Math. Soc. **49** (2002), 887–895.

[68] ———, *Thermodynamic formalism*, Cambridge University Press, Cambridge, 2004.

[69] ———, *Generalized zeta-functions for Axiom A basic sets*, Bull. Amer. Math. Soc. **82** (1976), 153–156.

[70] ———, *Flots qui ne mélangent pas exponentiellement*, C. R. Acad. Sci. Paris Sér. I Math. **296** (1983), 191–193.

[71] ———, *Resonances for Axiom A flows*, J. Differential Geom. **25** (1987), 99–116.

[72] Rugh, H. H., *Generalized Fredholm determinants and Selberg zeta functions for Axiom A dynamical systems*, Ergod. Th. Dynam. Sys. **16** (1996), 805–819.

[73] Selberg, A., *Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series*, J. Indian Math. Soc. B **20** (1956), 47–87.

[74] Schoen, R., Wolpert, S., and Yau, S. T., *Geometric bounds on the low eigenvalues of a compact surface, In Geometry of the Laplace operator*, Proc. Sympos. Pure Math. **XXXVI** (1980), 279–285.

[75] Sharp, R., *Periodic orbits of hyperbolic flows*, 73–138, in [48].

[76] Smale, S., *Differentiable dynamical systems*, Bull. Amer. Math. Soc. **73** (1967), 747–817.

[77] Steiner, F., *Quantum Chaos*, Festschrift Universität Hamburg 1994: Schlaglichter der Forschung zum **75** Jahrestag (1994), 542–564.

[78] Stoyanov, L., *Spectra of Ruelle transfer operators for Axiom A rows*, Nonlinearity 24 (2011), 1089–1120.

[79] _____, *Pinching conditions, linearization and regularity of Axiom A flows*, Disc. & Cont. Dyn. Sys. 33 (2013), 391–412.

[80] Terras, A., *Zeta functions of graphs*, Cambridge Studies in Advanced Mathematics, 128. Cambridge University Press, Cambridge, 2011.

Department of Mathematics, Zeeman Building, Warwick University, Coventry, CV4 7AL, England
E-mail: masdbl@warwick.ac.uk

Resonances for geodesic flows on negatively curved manifolds

Frédéric Faure and Masato Tsujii

Abstract. We report some recent progress in the study of geodesic flows on negatively curved manifolds (or more generally contact Anosov flows). We consider one-parameter groups of transfer operators associated to the flows and investigate the spectra of their generators. The main ingredients are the recent results about a *band structure* of the discete spectrum, which are obtained in the authors' joint works.

Mathematics Subject Classification (2010). Primary 37C30; Secondary 37D40.

Keywords. geodesic flow, Ruelle-Pollicott resonance, dynamical zeta function.

1. Introduction

Let (N, g) be a closed Riemann manifold. The geodesic flow $f^t : TN \to TN$ is the flow that describes the motion of a free particle on N. As usual, we consider its restriction $f^t : T_1N \to T_1N$ to the unit tangent bundle T_1N. It is well known that qualitative properties of the geodesic flow depend strongly on the sectional curvature of (N, g). If the sectional curvature is *negative* everywhere, the geodesic flow is unstable in the sense that the orbits depend sensitively on the initial points and exhibits typically chaotic behavior as a consequence. Indeed the geodesic flows on negatively curved manifolds are regarded as types of uniformly hyperbolic (continuous) dynamical systems and studied extensively since the work[19] of Hadamard.

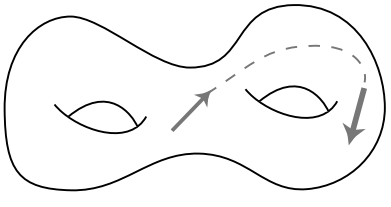


Figure 1.1. Geodesic flow on a negatively curved manifold

In uniformly hyperbolic (discrete or continuous) dynamical systems, long-time statistical properties of the orbits are rather independent of their initial points if we ignore the sets of initial points of measure zero and is robust under perturbations of the systems. Such

Proceedings of the International Congress of Mathematicians, Seoul, 2014

statistical properties of hyperbolic (or more general) dynamical systems have been among the main subjects in ergodic theory. For geodesic flows on negatively curved manifolds, they have been studied by Hopf, Anosov, Sinai, Pollicott and more recently by Dolgopyat[5] and Liverani[21] among many others.

The transfer operator, which describes the action of a dynamical system on functions (or densities), is a powerful tool in the study of statistical properties of the orbits. Since the pioneering work[23] of Ruelle on expanding maps, it has been gradually understood that the transfer operators (or, their generators in the case of flows) for uniformly hyperbolic dynamical systems have discrete spectra if we set up an appropriate function spaces for them to act on. The discrete spectra thus appear are called Ruelle-Pollicott resonances and known to be intrinsic to the transfer operators, that is, independent of the choice of the function spaces. In the recent works[9, 16], it is established that the generators for the transfer operators associated to general smooth Anosov flows have discrete spectra.

For the next step, we consider the structure of the discrete spectra thus appears. How are they distributed on the complex plane? In the case of geodesic flows on negatively curved manifolds (or a little more general type of flows), the main results presented in the next section show that the discrete spectrum of the generator has a *band structure*, that is, it is contained in several bands parallel to the imaginary axis, as illustrated in Figure 2.1. Also some information on the distributions of the discrete spectrum inside the bands will be given. Interestingly, these results are obtained by *semi-classical analysis* of the transfer operators. This is not so surprising as it sounds. As we look into the structure of the transfer operator, we find that the most important part is its action on the components of functions having high frequency in the direction of the flow. (Roughly speaking, most of the other components are scattered in the phase space by the hyperbolicity of the flow.) Thus we are lead to study the situation where such components of functions are transferred by the flow. After a long time, some of such components come close to each other by hyperbolicity of the flow and we have to analyze the interference between them. For this analysis, the ideas and techniques from semiclassical analysis work naturally and effectively.

The dynamical zeta functions are functions of one complex variable which are defined in term of periodic orbits of dynamical systems. Analytic properties of dynamical zeta function is closely related to the spectral properties of the transfer operators through Atiyah-Bott trace formula. Indeed the zeros and poles of dynamical zeta functions are related to the discrete eigenvalues of the generators of some transfer operators. In relation to the band structure of the discrete spectrum mentioned above, we find a particularly interesting case of dynamical zeta function, which is called the semi-classical or Gutzwiller-Voros zeta function. We present a result on the distribution of its zeros, which is reminiscent of the famous result of Selberg on the zeta function named after him. (See [22] for instance.)

2. Spectrum of transfer operators

The geodesic flows on negatively curved manifolds are typical examples of the so-called *contact Anosov flow*. To begin with, let us recall a few basic definitions. A C^∞ flow $f^t : M \to M$ on a closed manifold M is called an *Anosov flow* if there is a Df^t-invariant continuous decomposition $TM = E_0 \oplus E_s \oplus E_u$ of the tangent bundle such that E_0 is a one-dimensional subbundle spanned by the generating vector field of the flow and that the actions of Df^t on the subbundles E_s and E_u are exponentially contracting and expanding

respectively, that is,

$$\|Df^t|_{E_s}\| < Ce^{-\chi_0 t}, \quad \|Df^{-t}|_{E_u}\| < Ce^{-\chi_0 t} \quad \text{for } t \geq 0$$

with some constants $C > 0$ and $\chi_0 > 0$. Suppose in addition that M is of odd dimension, say $(2d+1)$-dimension. A *contact form* α on M is a differential 1-form satisfying the complete non-integrability condition: $\alpha \wedge (d\alpha)^d(x) \neq 0$ for all $x \in M$. A *contact Anosov* flow $f^t : M \to M$ is an Anosov flow that preserves a contact form α on M. The geodesic flow on a negatively curved manifold is a contact Anosov flow because it preserves the contact form given as the restriction of the canonical one form to T_1N (identified with T_1^*N by the Riemann metric).

We consider the one-parameter group of transfer operators

$$\mathcal{L}^t : C^\infty(M) \to C^\infty(M), \quad \mathcal{L}^t u(x) = g^t(f^{-t}(x)) \cdot u(f^{-t}(x)) \tag{2.1}$$

associated to a contact Anosov flow $f^t : M \to M$, where $g^t : M \to \mathbb{C} \setminus \{0\}$ is a multiplicative cocycle, that is, satisfies

$$g^{t+s}(x) = g^t(f^s(x)) \cdot g^s(x) \quad \forall s, t \in \mathbb{R}.$$

For the spectrum of the generator for $\mathcal{L}^t$, we have the following results:

Theorem 2.1 ([12]). *There exists a scale of Hilbert spaces $\mathcal{H}^r(M)$ for $r > 0$, with $C^r(M) \subset \mathcal{H}^r(M) \subset (C^r(M))'$, such that*

(I) *The transfer operators $\mathcal{L}^t$ for $t \geq 0$ extend to a strongly continuous semi-group of bounded operators on $\mathcal{H}^r(M)$ and the spectrum of the generator A in the region $\Re(s) > -\chi_0 + c$ consists of discrete eigenvalues with finite multiplicities, where $c > 0$ is a constant depending on f^t and g^t.*

(II) *The discrete eigenvalues in* (I) *have "band structure" in the following sense. For any $\epsilon > 0$, the discrete eigenvalues of the generator A are contained in the ϵ-neighborhood of the region*

$$B = \bigcup_{k=0}^{\infty} B_k, \qquad B_k = \{z \in \mathbb{C} \mid \gamma_k^- \leq \Re(z) \leq \gamma_k^+\}$$

up to finitely many exceptions[1] *, where*

$$\begin{aligned}\gamma_k^- &:= \lim_{t\to\infty} \frac{1}{t} \log \min_{x\in N} \left(|g^t(x)| \cdot (\det |(Df_x^t|_{E_u})|)^{-1/2} \cdot \|Df_x^t|_{E_u}\|_{\max}^{-k} \right) \\ &\leq \gamma_k^+ := \lim_{t\to\infty} \frac{1}{t} \log \max_{x\in N} \left(|g^t(x)| \cdot \det |(Df_x^t|_{E_u})|^{-1/2} \cdot \|Df_x^t|_{E_u}\|_{\min}^{-k} \right),\end{aligned}$$

and $\|L\|_{\max}$ ($\|L\|_{\min}$) is the maximum (minimum) singular value of L.

The first claim (I) has already been given in the previous results[9, 16] though we consider different function spaces. The second claim (II) is obtained by detailed analysis of the structure of the transfer operators.

[1] The number of exceptional eigenvalues may increase if we consider smaller $\epsilon > 0$.

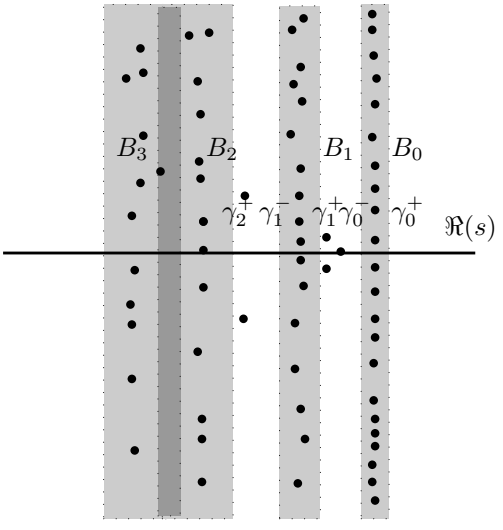


Figure 2.1. The band structure of the discrete spectrum of the generator$A$

Remark 2.2. In the following cases, we will observe the band structure clearly:

(a) The case where the expansion of Df_x^t on the subbundles E_u is almost uniform and $g^t(x)$ is almost constant. This will be the case for geodesic flows if the sectional curvature is almost constant (and negative).

(b) The case where the coefficient $g^t(x)$ is C^0-close to

$$g_0^t(x) := (1/2)\log|\det Df^t|_{E_u}(x)|.$$

In this case, we have $\gamma_1^+ < \gamma_0^- \fallingdotseq \gamma_0^+ \fallingdotseq 0$, so that, the rightmost band B_0 is a thin strip around the imaginary axis while the other bands B_k, $k \geq 1$, is separated from B_0 to the left.

In general the bands B_k, $k = 0, 1, \cdots$, may intersect each other and the union B of the bands may be just a half plane. In such case, the claim (II) of the theorem is somewhat vacuous. But, since the band structure is a consequence of a more fundamental structure of the transfer operator as we will explain later, we expect some related structure of the discrete eigenvalues (such as the claim (IV) in the next theorem) even in such cases.

Remark 2.3. The bounds $\gamma_k^\pm$ for the bands B_k are not (and maybe far from) optimal. For instance, the bound γ_0^+ may exceed the obvious bound given by the topological pressure. There will be better bounds for the bands B_k. (But this improvement will require new ideas and methods.)

Under an additional assumption on disjointness of the bands B_k, we have

Theorem 2.4 ([12]). (*Continued from Theorem 2.1*) *If the rightmost band B_0 is disjoint from the other bands B_k, $k \geq 1$, the following hold true.*

(III) *A (weak) analgoue of the Weyl law for the density of eigenvalues holds. Precisely, for any $\epsilon > 0$ and any $\delta > 0$, there exists a constant $C > 1$ such that $|\nu|$ is sufficiently large, the following holds for the density of eigenvalues :*

$$C^{-1}|\nu|^d \leq \frac{\#\{\textit{eigenvalues of } A \textit{ in } [\gamma_0^- - \epsilon, \gamma_0^+ + \epsilon] \times i[\nu, \nu + |\nu|^\delta]\}}{|\nu|^\delta} \leq C|\nu|^d.$$

(IV) *Most of the eigenvalues in B_0 concentrate along the line*

$$\Re(s) = \bar{\gamma}_0 := (1/t) \int \log(|g^t| \cdot |\det(Df^t|_{E_u})|^{-1/2}) d\mu \in [\gamma_0^-, \gamma_0^+]$$

in the limit $\mathrm{Im}(s) \to \pm\infty$, where $\mu = \alpha \wedge (d\alpha)^d$ is the contact volume. (Note that the quantity $\bar{\gamma}_0$ actually does not depend on $t > 0$.) More precisely, for any $0 < \epsilon' < \epsilon$, the number of eigenvalues of A in the strips on the both sides

$$([\gamma_0^- - \epsilon, \gamma_0^+ + \epsilon] \setminus [\bar{\gamma}_0 - \epsilon', \bar{\gamma}_0 + \epsilon']) \times i[\nu - 1, \nu + 1]$$

is of smaller order than $|\nu|^d$ as $\nu \to \pm\infty$.

(V) *The resolvent $(s - A)^{-1}$ is meromorphic on the region $\Re(s) > r - \chi_0 + c$, where $c > 0$ is the constant given in Claim* (I)*. Further, for any $\epsilon > 0$, there exists $C > 0$ such that $(s - A)^{-1}$ is uniformly bounded on the intersection of the complement of the ϵ-neighborhood of B with the region $|\Im(s)| > C$.*

Remark 2.5. We expect that the asymptotic formula in Claim (III) holds in more precise form with $\delta = 0$. But we have difficulties (which may be technical ones) in proving the lower bound in such precision.

For the spectrum of the transfer operators $\mathcal{L}^t$, the theorems above yield the following corollary, which is obtained by the second author previously.

Corollary ([25, 26]). *The essential spectral radius of the transfer operator $\mathcal{L}^t$ acting on $\mathcal{H}^r(M)$ is bounded by $e^{\gamma_0^+ t}$.*

3. The semi-classical zeta functions

Smale[24] introduced a dynamical zeta function (in the case of flows) by

$$Z(s) = \prod_{k=0}^{\infty} \prod_{\gamma \in \Gamma} (1 - e^{-(s+k)|\gamma|}) = \exp\left(- \sum_{k=0}^{\infty} \sum_{\gamma \in \Gamma} \sum_{m=1}^{\infty} \frac{e^{-(s+k)m|\gamma|}}{m} \right)$$

where Γ denotes the set of prime periodic orbits for the flow and $|\gamma|$ denotes the prime period of $\gamma \in \Gamma$. Later on, its variants are considered in dynamical system theory and also in related fields of physics. Below we consider the semi-classical (or Gutzwiller-Voros) zeta function

$$Z_{sc}(s) = \exp\left(- \sum_{\gamma \in \Gamma} \sum_{m=1}^{\infty} \frac{1}{m} \frac{e^{-sm|\gamma|}}{\sqrt{\det(1 - D_\gamma^m)}} \right) \tag{3.1}$$

as a particularly interesting one in relation to the results presented in the last section. (Here D_γ denotes the differential of the Poincaré map along a prime periodic orbit $\gamma \in \Gamma$.) Note that, if we consider the geodesic flow on a closed surface (N, g) with curvature $\equiv -1$, both of the dynamical zeta functions above coincide with the Selberg zeta function[22] associated to (N, g).

Analytic properties of dynamical zeta function and spectral properties of the transfer operators is related through the so-called Atiyah-Bott trace formula. We recall the relation (very) briefly. Usually the transfer operator $\mathcal{L}^t$ is not compact and it is not possible to calculate its trace in a legal manner. However we can calculate its Atiyah-Bott trace $\mathrm{Tr}_{AB}\mathcal{L}^t$, which is defined as the integration of the (Schwartz) kernel $K(x, y; t)$ of $\mathcal{L}^t$ along the diagonal set, and get the formula

$$\mathrm{Tr}_{AB}\mathcal{L}^t := \int_M K(x, x; t)dx = \sum_{\gamma\in\Gamma}\sum_{m=1}^{\infty} \frac{|\gamma| \cdot g^{m|\gamma|}(p_\gamma)}{|\det(1 - D_\gamma^m)|} \cdot \delta(t - m|\gamma|), \tag{3.2}$$

where p_γ is any point on $\gamma \in \Gamma$. This is a distribution (measure) as a function of t. For simplicity, let us suppose that the hyperbolic decomposition $TM = E_0 \oplus E_s \oplus E_u$ is smooth (this is of course not true in most of the cases) and consider the vector-valued transfer operators $\hat{\mathcal{L}}_k^t : C^\infty(E_u^{\wedge k}) \to C^\infty(E_u^{\wedge k})$, acting on the sections of the vector bundle $E_u^{\wedge k}$ and defined by

$$\hat{\mathcal{L}}_k^t u(x) = |\det Df^t|_{E_u}(f^{-t}(x))|^{-1/2} \cdot (Df^t)^{\wedge k}(u(f^{-t}(x))). \tag{3.3}$$

Regarding these operators as generalizations of the transfer operators $\mathcal{L}^t$, we compute their Atiyah-Bott trace as

$$\mathrm{Tr}_{AB}\,\hat{\mathcal{L}}_k^t = \sum_{\gamma\in\Gamma}\sum_{m=1}^{\infty} \frac{|\gamma| \cdot |\det D_\gamma^u|^{-m/2} \cdot \mathrm{Tr}\,(((D_\gamma^u)^m)^{\wedge k})}{|\det(\mathrm{Id} - D_\gamma^m)|} \cdot \delta(t - m \cdot |\gamma|)$$

where D_γ^u denotes the restriction of D_γ to E_u. We assume that E_u is orientable for simplicity. Then formal calculation gives the following expression of the semi-classical zeta function:

$$Z_{sc}(s) = \exp\left(-\int_{+0}^{\infty} \frac{e^{-st}}{t} \sum_{k=0}^{d} (-1)^{d-k}\,\mathrm{Tr}_{AB}\,\hat{\mathcal{L}}_k^t dt\right) \tag{3.4}$$

From this expression, the discrete spectrum of the generator A_k of $\hat{\mathcal{L}}_k^t$ is expected to appear as zeros or poles of $Z_{sc}(s)$ depending on the parity of k. Notice that the operators $\hat{\mathcal{L}}_k^t$ for the case $k = d$ is scalar-valued and that this is equivalent to the transfer operator $\mathcal{L}^t$ with the coefficient g_0^t in Remark 2.2 (b). This observation gives us intuition to the next theorem.

Theorem 3.1 ([13]). *The zeros of the semi-classical zeta function $Z_{sc}(s)$ concentrate along the imaginary axis with gaps on the both sides. More precisely, for any $\epsilon > 0$, the zeros are contained in the region*

$$R = R_0 \cup R_1, \quad R_0 = \{|\Re(s)| < \epsilon\}, \quad R_1 = \{\Re(s) < -\chi_0 + \epsilon\}$$

where $\chi_0 > 0$ is that in the definition of Anosov flow, up to finitely many exceptions. The poles of $Z_{sc}(s)$ is contained in R_1 up to finitely many exceptions. The density of the zeros in the thin strip R_0 satisfies the analogue of the Weyl law given in Claim (III) *of Theorem 2.4.*

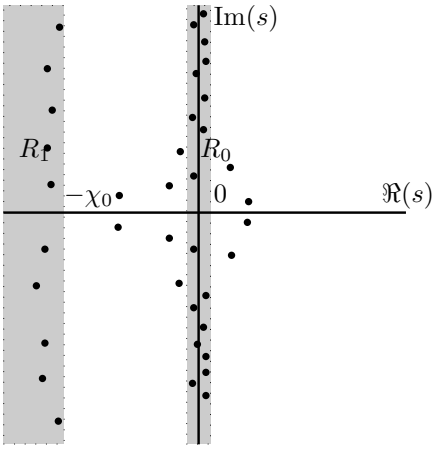


Figure 3.1. The zeros of $Z_{sc}(s)$

Remark 3.2. One technical difficulty in the proof of Theorem 3.1 is caused by the fact that the hyperbolic decomposition $TM = E_0 \oplus E_s \oplus E_u$ is not smooth and only Hölder continuous in general. This invalidates analytic treatment of the transfer operators $\hat{\mathcal{L}}_k^t$. We avoid this difficulty by considering the natural extension of the flow to the Grassmann bundle of M. This kind of idea was originally used in the paper [4]. See [18] also.

4. Wave packet transform

We sketch the ideas behind the results presented in the previous sections and also some other related results. Notice that we do not intend to give precise argument. (Also note that the author is distorting things in his flavor.) We refer [14] for more detailed account.

Our basic idea is to regard functions as superpositions of wave packets, which are simple functions localized both in the real and frequency space, and analyze the transfer operators by observing how they transfer a wave packet to another. Below we briefly discuss how we can go along this idea in reality.

Decomposition into wave packets. Decomposition of functions into wave packets is a rather common idea in analysis and applied mathematics. It is called wave packet transform (or different names in different fields). Below we introduce it for explanation in the following. (See [15, Ch. 3] for more details.) First we consider a smooth function $\phi : \mathbb{R}^D \to \mathbb{C}$ on $\mathbb{R}^D$ localized around the origin. For instance, we can take the Gauss function $\phi(x) = e^{-|x|^2}$. For $(x, \xi) = T^*\mathbb{R}^D = \mathbb{R}^{2D}$, we set

$$\phi_{x,\xi}(y) = e^{i\xi(y-x)} \cdot \phi(\delta^{-1}(y - x))$$

where $\delta > 0$ is a scaling parameter. This is a wave packet which is localized around the point $x \in \mathbb{R}^D$ and whose Fourier transform $\hat{\phi}_{x,\xi}(\cdot)$ is localized around the point $\xi \in \mathbb{R}^D$.

The wave packet transform $B : \mathcal{S}(\mathbb{R}^D) \to \mathcal{S}(\mathbb{R}^{2D})$ is defined by

$$Bu(x,\xi) = \int \overline{\phi_{x,\xi}(y)} \cdot u(y) dy.$$

We define $B^* : \mathcal{S}(\mathbb{R}^{2D}) \to \mathcal{S}(\mathbb{R}^D)$ by

$$B^* v(y) = (2\pi\delta)^{-D} \|\phi\|_{L^2}^{-2} \cdot \int \phi_{x,\xi}(y) \cdot u(y) dx d\xi$$

so that, by simple calculation,

$$B^* Bu(y) = (2\pi\delta)^{-D} \|\phi\|_{L^2}^{-2} \int \phi_{x,\xi}(y) \cdot \overline{\phi_{x,\xi}(y')} \cdot u(y') dx d\xi dy' = u(y).$$

This implies that each function $u \in \mathcal{S}(\mathbb{R}^D)$ is expressed as an integration of wave packets $\phi_{x,\xi}(\cdot)$:

$$u(y) = \int \phi_{x,\xi}(y) \cdot ((2\pi\delta)^{-D} \|\phi\|_{L^2}^{-2} \cdot Bu(x,\xi)) dx d\xi$$

The operator B extends to an embedding $B : L^2(\mathbb{R}^D) \to L^2(\mathbb{R}^{2D})$, which is isometric up to multiplication by a constant.

Induced transfer operators on the cotangent bundle. We next look at the transformation between wave packets induced by a transfer operator. Let $f : \mathbb{R}^D \to \mathbb{R}^D$ be a C^∞ diffeomorphism, $g : \mathbb{R}^D \to \mathbb{C}$ a C^∞ smooth function and $\mathcal{L}$ the transfer operator defined by

$$\mathcal{L}u(x) = g(x) \cdot u(f^{-1}(x)).$$

The transformation between wave packets is described by the operator

$$\mathcal{L}^{\mathrm{lift}} : L^2(\mathbb{R}^D) \to L^2(\mathbb{R}^{2D}), \qquad \mathcal{L}^{\mathrm{lift}} = B \circ \mathcal{L} \circ B^*$$

which makes the following diagram commutes:

$$\begin{array}{ccc} L^2(\mathbb{R}^{2D}) & \xrightarrow{\mathcal{L}^{\mathrm{lift}}} & L^2(\mathbb{R}^{2D}) \\ {\scriptstyle B}\uparrow & & {\scriptstyle B}\uparrow \\ L^2(\mathbb{R}^D) & \xrightarrow{\mathcal{L}} & L^2(\mathbb{R}^D) \end{array}$$

The operator $\mathcal{L}^{\mathrm{lift}}$ is an integral operator with smooth kernel. If the diffeomorphism f is an affine map and if $\phi(\cdot)$ is the Gauss function, we can give an explicit expression of this operator. (See [11, Lemma 4.8].) For a general diffeomorphism f, the operator $\mathcal{L}^{\mathrm{lift}}$ is not very simple. But it should be intuitively rather obvious that the image of the wave packet $\phi_{x,\xi}(\cdot)$ by $\mathcal{L}^t$ will localize around $f^t(x)$ and its Fourier transform will localize around $(Df^{-t})^*_x(\xi)$. Indeed the transformation $\mathcal{L}^{\mathrm{lift}}$ is closely related to the induced mapping on the cotangent bundle $T^*\mathbb{R}^D = \mathbb{R}^{2D}$:

$$D^* f^{-1} : T^*\mathbb{R}^D \to T^*\mathbb{R}^D, \quad (D^* f)^{-1}(x,\xi) = (f(x), (Df^{-t})^*_x(\xi)) \tag{4.1}$$

This relation leads us to an idea (or a viewpoint) to consider the properties of the transfer operator $\mathcal{L}$ by looking at the transformation (4.1).

Remark 4.1. The idea mentioned above gives us nice intuition for the analysis of transfer operators as we will explain below. However we have to be cautious about some technical problems in making such intuition into rigorous argument. For instance, it is more natural (and necessary) to consider smaller wave packets to look the parts of functions with higher frequency. So actually we have to vary the scaling factor δ depending on the norm of ξ and sometimes also on the direction of ξ. Below we ignore such technical problems and so the following explanation is rather simplistic.

Anosov diffeomorphisms. Now let us consider the transfer operator associated to an C^∞ Anosov diffeomorphism $f : M \to M$ from the viewpoint introduced above. (We extend the argument above to diffeomorphisms on manifolds in an obvious manner using local charts.) We look into the induced mapping $D^*f^{-1} : T^*M \to T^*M$ on the cotangent bundle T^*M in (4.1). When we study a dynamical system, one of the first things to do is to identify its non-wandering set. In this case, the non-wandering set of the dynamics of D^*f^{-1} is the zero section $M \times \{0\}$ and the dynamics is non-recurrent outside of its small neighborhood $\mathcal{U}$. (See Figure 4.1.)

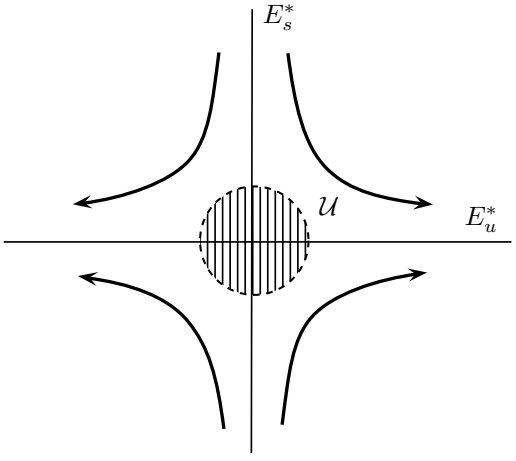


Figure 4.1. A schematic picture of the action of D^*f^{-1} for an Anosov diffeomorphism f.

Thus it is natural to decompose the transfer operator $\mathcal{L}$ (or $\mathcal{L}^{\mathrm{lift}}$) into two parts

$$\mathcal{L} = \mathcal{L}_{\mathrm{cpt}} + \mathcal{L}_{\mathrm{non-rec}}$$

where the former is the action of $\mathcal{L}$ on the wave packets corresponding to points in $\mathcal{U}$ and the latter is the action on the remaining. The former part $\mathcal{L}_{\mathrm{cpt}}$ concerns only wave packets with low frequency and hence is compact (and moreover belongs to the trace class). The latter is dissipative in a sense because of the fact that D^*f^{-1} is not recurrent on the outside of $\mathcal{U}$ and basically negligible when we consider the spectrum and trace. This explains the reason why the transfer operators for Anosov diffeomorphisms have discrete spectrum and also that we can consider their traces. In fact, we have

Theorem 4.2 ([1–3, 10, 17, 20]). *There exist a scale of Hilbert (or Banach) spaces $\mathcal{H}^r(M)$ for $r \geq 0$ with $C^r(M) \subset \mathcal{H}^r(M) \subset (C^r(M))'$ such that the transfer operator $\mathcal{L}$ extends to*

a bounded operator $\mathcal{L} : \mathcal{H}^r(M) \to \mathcal{H}^r(M)$ and its essential spectral radius is bounded by $C\|g\|_\infty \cdot e^{-r\chi}$ where $\chi > 0$ is the hyperbolicity exponent of the Anosov diffeomorphism f and $C > 0$ is a constant that may depend on f. In particular the spectrum of the operator $\mathcal{L} : \mathcal{H}^r(M) \to \mathcal{H}^r(M)$ on the outside of the disk of radius $C\|g\|_\infty \cdot e^{-r\chi}$ consists of discrete eigenvalues with finite multiplicities. The dynamical zeta function

$$\zeta(z) = \exp\left(- \sum_{m=1}^{\infty} \frac{1}{m} \sum_{x \in Fix(F^m)} g^{(m)}(x) \right)$$

extends to a meromorphic function on $\mathbb{C} \setminus \{0\}$.

Remark 4.3. In [1, 2, 10], the Hilbert space $\mathcal{H}^r(M)$ in the theorem above is constructed essentially in the following manner. From non-recurrence of D^*f^{-1} on the outside of $\mathcal{U}$, it is possible to define a smooth function $\mathcal{W}^r : T^*M \to \mathbb{R}$ for any $r > 0$ so that

$$\mathcal{W}^r(D^*f^{-1}(x,\xi)) \le e^{-r\chi} \cdot \mathcal{W}^r(x,\xi) \quad \text{when } (x,\xi) \notin \mathcal{U} \text{ and } D^*f^{-1}(x,\xi) \notin \mathcal{U}.$$

We define $\mathcal{H}^r(M)$ as the completion of $C^\infty(M)$ with respect to the norm $\|u\|_{\mathcal{H}^r} := \|\mathcal{W}^r \cdot B\|_{L^2}$ where B is the wave packet transform discussed previously. Then we see that the operator norm of the latter part $\mathcal{L}_{\mathrm{non-rec}}$ is bounded by $C\|g\|_\infty \cdot e^{-r\chi}$ and consequently the essential spectral radius of $\mathcal{L}$ on $\mathcal{H}^r(M)$ is bounded by this factor.

Anosov flows. We next consider the case of Anosov flows. In this case, the argument is more subtle. Let $f^t : M \to M$ be a C^∞ Anosov flow. The induced flow on the cotangent bundle is just (4.1) with f replaced by f^t. In this case, the non-wandering set for such flow on the cotangent bundle is

$$\Omega = (E_s \oplus E_u)^\perp \subset T^*M.$$

Notice that Ω is not compact and also not a smooth subset of T^*M in general. These are sources of subtleness in the argument for Anosov flows. But we can follow the argument in the case of Anosov diffeomorphisms partly as follows. Let $\mathcal{C}$ be a smooth conical neighborhood of $\Omega \subset T^*M$ with smooth boundary so that the induced flow $(Df^{-t})^* : T^*M \to T^*M$ is non-recurrent on the outside of $\mathcal{C}$. (See Figure 4.2.) Let $T^*M = E_0^* \oplus E_s^* \oplus E_u^*$ be the decomposition of the cotangent bundle which is dual to that of the tangent bundle in the definition of Anosov flows.

Notice that the action of the flow f^t does not change the frequency of functions in the flow direction and hence it is natural to restrict the action of the transfer operator $\mathcal{L}^t$ to functions whose frequency in the flow direction is in some fixed range, say $[a,b] \subset \mathbb{R}$. We therefore consider the restriction of the flow $(Df^{-t})^* : T^*M \to T^*M$ to the subset

$$X(a,b) = \{(x,\xi) \in T^*M \mid \xi(V(x)) \in [a,b]\}$$

where $V(x)$ is the generating vector field of the flow f^t. This means that we look at the spectrum of the generator of $\mathcal{L}^t$ in the region $\mathbb{R} \times i[a,b] \subset \mathbb{C}$. The dynamics of $(Df^{-t})^*$ restricted to $X(a,b)$ is similar to the case of Anosov diffeomorphism: the intersection $\mathcal{C} \cap X(a,b)$ is relatively compact and $(Df^{-t})^*$ is non-recurrent on the outside of $\mathcal{C}$. This explains the reason why the generator of $\mathcal{L}^t$ have discrete spectrum. Further, from the argument in the proof of the Weyl law, we expect that the number of eigenvalues for such restriction is proportional to (or at least bounded by) the symplectic volume of $\mathcal{C} \cap X(a,b) \subset T^*M$. In fact, we have

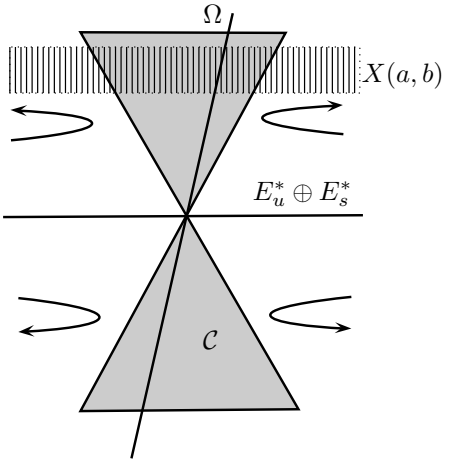


Figure 4.2. A schematic picture for the action of $(Df^{-t})^*$ for an Anosov flow f^t. Horizontal hyperplanes parallel to $E_u^* \oplus E_s^*$ are preserved and the dynamics on them will look like that illustrated in Figure 4.1.

Theorem 4.4 ([7, 9, 16]). *There exists a Hilbert space $\mathcal{H}^r(M)$ such that the transfer operator $\mathcal{L}^t$ extends to a strongly continuous one-parameter semigroup on $\mathcal{H}^r(M)$ and the spectral set of the generator A in the region $\Re(s) \geq -r\chi_0 + c$ consists of discrete eigenvalues with finite multiplicities, where c is a constant depending on f^t and g^t. Further*

(a) *For any given $\beta > 0$. the number of eigenvalues in the region $[-\beta, \infty) \times i[\nu, \nu + \sqrt{|\nu|}]$ is of smaller order than $\langle \nu \rangle^{2d+(1/2)}$.* ([9])

(b) *The dynamical zeta function $Z(s)$ has meromorphic extensions to the whole complex plane $\mathbb{C}$.* ([7, 16])

5. Microlocal analysis of contact Anosov flows

We now consider the case of contact Anosov flows (or the geodesic flows on negatively curved manifolds), to which the main results concern. In this case the non-wandering set Ω is smooth and coincides with the one-dimensional subbundle $\langle \alpha \rangle$ of T^*M spanned by the contact one form α. This is quite different from the case of general Anosov flows mentioned in the last section. By virtue of this fact, we may perform more detailed analysis of the structure of the lifted transfer operators $(\mathcal{L}^t)^{\mathrm{lift}}$ in a smaller neighborhood of the non-wandering set Ω. For instance, the claim (a) on the number of resonances in Theorem 4.4 above has been made precise as follows.

Theorem 5.1 ([6]). *Let $f^t : M \to M$ be a contact Anosov flow and let $\dim M = 2d + 1$. Let $\mathcal{L}^t$ be the transfer operator as in* (2.1)*. For any $\beta > 0$, there exists $C > 0$ such that the number of eigenvalues for the generator of $\mathcal{L}^t$ in the region $[-\beta, \infty) \times i[\nu, \nu + \sqrt{|\nu|}]$ is bounded by $C\langle \nu \rangle^{d+(1/2)}$.*

Our main results, Theorem 2.1 and Theorem 2.4, are obtained also by analyzing the action of the lifted operator $(\mathcal{L}^t)^{\mathrm{lift}}$ in a small neighborhood of the subset Ω. Compared with

the argument in the last section, the novelty is that we consider interference between the images of wave packets.

Remark 5.2. The analysis of the interference between the images of wave packets is basically in the same spirit as the so-called Dolgopyat argument[5, 21]. In contact Anosov flows, cancellation between the images of wave packets occurs in the most effective manner in a sense. This is observed and made use of in the argument in [25, 26]. In the following, this cancellation between wave packets is formulated in a more sophisticated manner in terms of the symplectic structure on the cotangent bundle T^*M, which allows us to obtain more detailed information of the transfer operators.

Suppose that $g^t \equiv 1$ for simplicity, so that $\mathcal{L}^t$ is a unitary operator with respect to the L^2-norm. The key fact is that the subset $\Omega = \langle \alpha \rangle$ is a symplectic submanifold of T^*M on the outside of the zero section. This is a consequence of the fact that α is a contact form. Hence the tangent space $T_p(T^*M)$ at $p \in \Omega$ is decomposed as

$$T_p(T^*M) = T_p\Omega \oplus (T_p\Omega)^\perp \tag{5.1}$$

where $T_p\Omega$ is the tangent space of Ω at p and $(T_p\Omega)^\perp$ is its symplectic orthogonal subspace. Further the induced maps $(Df^{-t})^* : T^*M \to T^*M$ preserve this decomposition, because it is a symplectomorphism. This implies that "micro-locally" the transfer operator $\mathcal{L}^t$ is decomposed as a tensor product of two unitary operators (with respect to the L^2 norm),

$$(\mathcal{L}^t)^{\rm lift} \simeq (\mathcal{L}^t)^{\rm lift}_{\parallel} \otimes (\mathcal{L}^t)^{\rm lift}_{\perp} \tag{5.2}$$

where $\mathcal{L}^t_{\parallel}$ (resp. $\mathcal{L}^t_{\perp}$) describes the action of $\mathcal{L}^t$ on the wave packets in the directions in $T_p\Omega$ (resp. $(T_p\Omega)^\perp$) from p.

Remark 5.3. The structure explained above is found in [8] for linear prequantum Anosov maps, a simplified model of contact Anosov flows. If the flow f^t were linear, the decomposition (5.2) is exact and we can write each of $\mathcal{L}^t_{\parallel}$ and $\mathcal{L}^t_{\perp}$ explicitly. (See [13, Section 4].) In reality, the flow f^t is non-linear of course. Still, when we consider components of functions with higher frequency in the flow direction, we can look them in higher resolution and we may localize the action of f^t in smaller regions where f^t is better-approximated by its linearization. Hence the decomposition (5.2) become true asymptotically (and locally) in the high frequency limit.

The latter operator $(\mathcal{L}^t)^{\rm lift}_{\perp}$ is closely related to the action of the induced map $(Df^{-t})^*$ restricted to the subspace $(T_p\Omega)^\perp$. This linear map $(Df^{-t})^*|_{(T_p\Omega)^\perp}$ is conjugated by a symplectic linear isomorphism to the linear map

$$(DS^{-1})^* = S \oplus (S^{-1})^* : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$$

on the cotangent bundle $T^*\mathbb{R}^d = \mathbb{R}^{2d}$ induced by an expanding linear map $S : \mathbb{R}^d \to \mathbb{R}^d$ corresponding to $Df^t_p|_{E_u}$. Indeed we see that the operator $(\mathcal{L}^t)^{\rm lift}_{\perp}$ is modeled by the lift of the (L^2-normalized) transfer operator

$$L_S : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d), \quad L_S u(x) = |\det S|^{-1/2} u \circ S^{-1}(x).$$

Thus we are lead to consider the structure of the operator L_S, keeping in mind that it as a model for the latter component $(\mathcal{L}^t)^{\rm lift}_{\perp}$ in the decomposition (5.2). Recall that we are interested in the action of $\mathcal{L}^t$ on the wave packets corresponding to points in a small neighborhood

of Ω. Hence we focus on the action of L_S on wave packets that corresponds to points in a small neighborhood of the origin $0 \in T^*\mathbb{R}^d = \mathbb{R}^{2d}$ (which corresponds to $p \in \Omega$). Observe that the operator L_S preserves the decomposition of functions u,

$$u = [\text{Polynomial of order} \le k] + [\text{Remainder}],$$

given by the Taylor expansion at the origin. And, by setting up an appropriate function space (see Remark 4.3), we can show that the action of L_S on the former component is dominating. That is to say, the main part of the operator L_S is its action on the polynomial up to some order, say $k \ge 1$, and the reminder part is dominated by that action in the operator norm. Now it is easy to observe that the space of polynomials on $\mathbb{R}^d$ of order $\le k$ is decomposed into the subspaces P_ℓ of homogeneous polynomials of order $0 \le \ell \le k$. These subspaces are preserved by L_S and we have the estimate

$$C^{-1}|\det S|^{-1/2}\|S\|_{\max}^{-\ell} \cdot \|u\| \le \|L_S u\| \le C|\det S|^{-1/2}\|S\|_{\min}^{-\ell} \cdot \|u\| \quad \text{for } u \in P_\ell.$$

This implies that the spectral set of L_S is contained in the union

$$D = \bigcup_{0 \le \ell \le k} \{r_\ell^- \le |z| \le r_\ell^+\} \cup \{|z| \le r_{k+1}^+\}$$

of annuli (and a disk) where $r_\ell^- = |\det S|^{-1/2}\|S\|_{\max}^{-\ell}$ and $r_\ell^+ = |\det S|^{-1/2}\|S\|_{\min}^{-\ell}$. This is the origin of the band structure given in Theorem 2.1.

References

[1] Viviane Baladi and Masato Tsujii, *Anisotropic Hölder and Sobolev spaces for hyperbolic diffeomorphisms*, Ann. Inst. Fourier (Grenoble) **57** (2007), no. 1, 127–154.

[2] ———, *Dynamical determinants and spectrum for hyperbolic diffeomorphisms*, Geometric and probabilistic structures in dynamics, Contemp. Math., vol. 469, Amer. Math. Soc., Providence, RI, 2008, pp. 29–68.

[3] Michael Blank, Gerhard Keller, and Carlangelo Liverani, *Ruelle-Perron-Frobenius spectrum for Anosov maps*, Nonlinearity **15** (2002), no. 6, 1905–1973.

[4] Predrag Cvitanović, Per E. Rosenqvist, Gábor Vattay, and Hans Henrik Rugh, *A Fredholm determinant for semiclassical quantization*, Chaos, **3**, (1993), no. 4, 619–636.

[5] Dmitry Dolgopyat, *On decay of correlations in Anosov flows*, Ann. of Math. (2) **147** (1998), no. 2, 357–390.

[6] Kiril Datchev, Semyon Dyatlov, and Maciej Zworski, *Sharp polynomial bounds on the number of Pollicott-Ruelle resonances*, arXiv:1208.4330, to appear in Ergodic Theory Dynamical Systems.

[7] Semyon Dyatlov and Maciej Zworski, *Dynamical zeta functions for Anosov flows via microlocal analysis*, arXiv:1306.4203.

[8] Frédéric Faure, *Prequantum chaos: resonances of the prequantum cat map*, J. Mod. Dyn. **1** (2007), no. 2, 255–285

[9] Frédéric Faure and Johannes Sjöstrand, *Upper bound on the density of Ruelle resonances for Anosov flows*, Comm. Math. Phys. **308** (2011), no. 2, 325–364.

[10] Frédéric Faure, Nicolas Roy and Johannes Sjöstrand, *Semi-classical approach for Anosov diffeomorphisms and Ruelle resonances*, Open Math. J. **1** (2008), 35–81

[11] Frédéric Faure and Masato Tsujii, *Prequantum transfer operator for symplectic Anosov diffeomorphism*, arXiv:1206.0282.

[12] ———, *Band structure of the Ruelle spectrum of contact Anosov flows*, C. R. Math. Acad. Sci. Paris **351** (2013), no. 9-10, 385–391.

[13] ———, *The semiclassical zeta function for geodesic flows on negatively curved manifolds*, arXiv:1311.4932.

[14] ———, *Semi classical Approach for the Ruelle-Pollicott Spectrum of Hyperbolic Dynamics*, in Analytic and Probabilistic Approaches to Dynamics in Negative Curvature, Springer INdAM Series, Vol. 9, 2014

[15] Gerald B. Folland, Harmonic analysis in phase space, Annals of Mathematics Studies, **122**, Princeton University Press, 1989, x+277.

[16] P. Giulietti, C. Liverani, and M. Pollicott, *Anosov flows and dynamical zeta functions*, Ann. of Math. (2) **178** (2013), no. 2, 687–773.

[17] Sébastien Gouëzel and Carlangelo Liverani, *Banach spaces adapted to Anosov systems*, Ergodic Theory Dynam. Systems **26** (2006), no. 1, 189–217.

[18] ———, *Compact locally maximal hyperbolic sets for smooth maps: fine statistical properties*, J. Differential Geom. **79** (2008), no. 3, 433–477.

[19] J. Hadamard, *Sur la forme des lignes géodésiques à l'infini et sur les géodésiques des surfaces réglées du second ordre*, Bull. Soc. Math. France **26** (1898), 195–216.

[20] A. Yu. Kitaev, *Fredholm determinants for hyperbolic diffeomorphisms of finite smoothness*, Nonlinearity **12**(1999), 141–179.

[21] Carlangelo Liverani, *On contact Anosov flows*, Ann. of Math. (2) **159** (2004), no. 3, 1275–1312.

[22] H. P. McKean, *Selberg's trace formula as applied to a compact Riemann surface*, Comm. Pure Appl. Math. **25** (1972), 225–246,

[23] David Ruelle, *Zeta-functions for expanding maps and Anosov flows*, Invent. Math. **34** (1976), no. 3, 231–242.

[24] Stephen Smale, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. **73** (1967), 747–817.

[25] Masato Tsujii, *Quasi-compactness of transfer operators for contact Anosov flows*, Nonlinearity **23** (2010), no. 7, 1495–1545.

[26] ———, *Contact Anosov flows and the Fourier-Bros-Iagolnitzer transform*, Ergodic Theory Dynam. Systems **32** (2012), no. 6, 2083–2118.

Institut Fourier, UMR 5582 100 rue des Maths, BP74 FR-38402 St Martin d'Hères, FRANCE
E-mail: frederic.faure@ujf-grenoble.fr

Faculty of Mathematics, Kyushu University, 744, Motooka, Nishi-ku, Fukuoka, 819-0395, JAPAN
E-mail: tsujii@math.kyushu-u.ac.jp

Recent developments in interval dynamics

Weixiao Shen and Sebastian van Strien

Abstract. Dynamics in dimension-one has been an extremely active research area over the last decades. In this note we will describe some of the new developments of the recent years.

Mathematics Subject Classification (2010). Primary 37F10; Secondary 30D05.

Keywords. One-dimensional dynamics.

1. Density of hyperbolicity

Interval maps $f\colon [0,1] \to [0,1]$ can have a surprisingly rich and complicated dynamics. In this paper we will describe results which show that in spite of this one can describe the metric orbit structure of 'most' maps extremely well.

The dynamics of *hyperbolic* maps can be described most easily: for these maps, Lebesgue almost every point in the interval is attracted to some hyperbolic periodic orbit (with multiplier between -1 and 1). By a result by Mañé [65] (for a simpler proof see [98]) it is equivalent to say that a map is hyperbolic if (i) each critical point of f is in the basin of a periodic attractor and (ii) each periodic orbit is hyperbolic. Since the period of periodic attractors is bounded, see [66], it follows that hyperbolic maps have at most finitely many periodic attractors.

As mentioned, hyperbolic maps are very well-understood. The following theorem (which was obtained by the authors, jointly with Kozlovski, see [50]) shows that 'most' maps are hyperbolic.

Theorem 1.1 (Density of hyperbolicity for real polynomials). *Any real polynomial can be approximated by hyperbolic real polynomials of the same degree.*

The above theorem allows us to prove the analogue of the Fatou conjecture in the smooth case, see [51], thus solving the 2nd part of Smale's eleventh problem for the 21st century [91]:

Theorem 1.2 (Density of hyperbolicity for smooth one-dimensional maps). *Hyperbolic maps are dense in the space of C^k maps of the compact interval or the circle, $k = 1, 2, \ldots, \infty, \omega$.*

For quadratic maps $f_a = ax(1-x)$, the above theorems assert that the periodic windows (corresponding to hyperbolic maps with attracting periodic orbits) are dense in the bifurcation diagram. The quadratic case turns out to be special, because in this case certain return maps become almost linear. This special behaviour does not even hold for maps of the form $x \mapsto x^4 + c$.

Proceedings of the International Congress of Mathematicians, Seoul, 2014

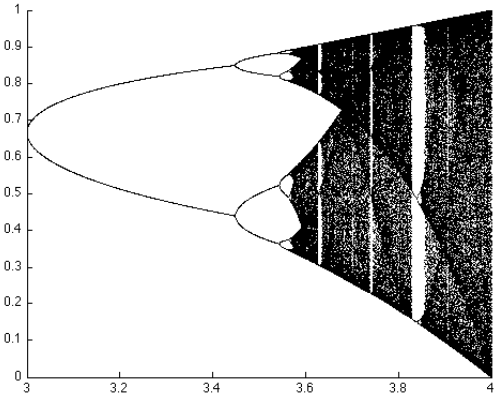


Figure 1.1. The Feigenbaum diagram

The problem of density of hyperbolicity in dimension-one has been considered since the 1920's. Indeed:

- Fatou stated the analogue of this problem in the context of rational maps on the Riemann sphere as a conjecture in the 1920's, see [33, page 73] and also [67, Section 4.1].
- Smale gave this problem 'naively' as a thesis problem in the 1960's, see [90].
- In 1971, Jakobson proved that the set of hyperbolic maps is dense in the C^1 topology, see [43].
- In the mid 1990's, the conjecture was solved in the quadratic case $x \mapsto ax(1-x)$ in a major breakthrough by Lyubich [61] and independently also by Graczyk and Świątek, [36] and [37].
- In 2000, Blokh and Misiurewicz [15] considered the problem of density of hyperbolicity in the C^2 topology, and were able to obtain a partial result.
- A few years later, Shen [87] proved C^2 density of hyperbolic maps.

Note that every hyperbolic map satisfying a mild transversality condition, namely that no critical point is eventually mapped onto another critical point, is *structurally stable*. So density of hyperbolicity implies that structural stable maps are dense.

1.1. Density of hyperbolicity within a large space of real transcendental map. Density of hyperbolicity also holds within classes of much more general maps, for example within the famous Arnol'd family and within the space of trigonometric polynomials. Indeed it was shown by the second author in a joint paper with Rempe, see [79], that

Theorem 1.3. *Density of hyperbolicity holds within the following spaces:*

1. *real transcendental entire functions, bounded on the real line, whose singular set is finite and real;*
2. *transcendental functions* $f\colon \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ *that preserve the circle and whose singular set (apart from* $0, \infty$*) is contained in the circle.*

In [79] also a number of other open problems are solved, including a number of conjectures of behaviour de Melo, Salomão and Vargas [29]. In this paper density of (real) hyperbolicity is also established replacing in assumption (1) the boundedness condition by a sector condition.

1.2. Hyperbolicity is dense within generic one-parameter families of one-dimensional maps.

Theorem 1.4 (Hyperbolicity is dense within generic families). *For any generic family $\{g_t\}_{t\in[0,1]}$ of smooth intervals maps (generic, in the sense of Baire), the following properties hold:*

- *the number of critical points of each of the maps g_t is bounded;*
- *the set of parameters t for which all critical points of g_t are in basins of periodic attractors, is open dense.*

The proof of this result follows easily from the theorems in the previous subsection, see [99]. On the other hand, as is shown in the same paper, it is easy to construct a real analytic one-parameter family f_t, $t \in [0,1]$ of polynomials so that none of the polynomials in this family are hyperbolic:

Theorem 1.5 (A family of cubic maps with robust chaos). *There exists a real analytic one-parameter family $\{f_t\}$ of interval maps (consisting of cubic polynomials) so that f_t has no periodic attractor for any $t \in [0,1]$, and so that not all maps within this family are topologically conjugate.*

1.3. Density of hyperbolicity for more general maps. Density of hyperbolicity is false in dimension ≥ 2. For a list of related interesting questions concerning the higher dimensional case, see [77].

The situation for rational maps on the Riemann sphere may well be more hopeful. In that context one has the following well-known conjecture, going back to Fatou:

Conjecture 1.6 (Density of hyperbolicity for rational maps). *Hyperbolic maps are dense within this space of rational maps of degree d on the Riemann sphere.*

In [64] it was shown that this conjecture follows from

Conjecture 1.7. *If a rational map carries a measurable invariant line field on its Julia set, then it is a Lattès map.*

More about this conjecture and related results can be found in [67]. In [50, 86] and finally [52] it was shown that real polynomials (acting on $\mathbb{C}$) do not carry such invariant line fields. Moreover, real polynomials have Julia sets which are locally connected, see [26, 50, 52, 55]. In [80] it was shown that, under some mild assumptions, real transcendental maps also do not carry invariant line fields.

Interestingly, any rational map on the Riemann sphere such that the multiplier of each periodic orbit is real, either has a Julia set which is contained in a circle (or line) or is a Lattès map, see [32].

1.4. Strategy of the proof: local versus global perturbations. Density of hyperbolicity means that given a map f one can find a map g so that g is hyperbolic and so that $g -$

f is 'small' in the C^k topology. It is tempting to consider the setting where g is a local perturbation of f. The purpose would then be to find a small 'bump' function h so that $g = f + h$ becomes hyperbolic. The difficulty with this approach is that orbits will pass many times through the support of the bump function. Pugh's approach in his proof of the C^1 closing lemma, is to find a suitable neighbourhood U of x so that the first return of x to U is not too close to the boundary of x. In this way he is able to construct a function h whose support is in U, which creates a new fixed point of the first return of $g = f + h$ to U, in such a manner that h is C^1 close to zero. A related approach was used successfully in [43] to prove density of hyperbolicity in the C^1 topology, and in [15] for the C^2 topology, but with added assumptions on the dynamics of f. In [87], this approach was used in the case when one has a 'lot of Koebe space' while in the 'essentially bounded geometry' the proof relied on rigidity (in the sense described below). This rigidity approach also is the key ingredient in the proof of Theorem 1.1. As there is a great deal of evidence that local perturbations cannot be used to prove density of hyperbolicity in general, we discuss rigidity extensively in the next section.

1.5. Strategy of the proof: quasi-symmetric rigidity. Consider the following situation. Take a family of real quadratic maps $f_c(z) = z^2 + c$. To prove density of hyperbolicity we need to prove that there exists no interval of parameters $[c', c'']$ so that each map f_c with $c \in [c', c'']$ is non-hyperbolic. Sullivan showed that this follows from quasi-symmetric rigidity of any non-hyperbolic map f_c. Here f_c is called *quasi-symmetrically rigid* if the following property holds:

> If $f_{\tilde{c}}$, $f_{\hat{c}}$ are topologically conjugate to f_c, then $f_{\tilde{c}}$, $f_{\hat{c}}$ are quasi-symmetrically conjugate.

Here, as usual, a homeomorphism $h\colon [0,1] \to [0,1]$ is called *quasi-symmetric* (often abbreviated as *qs*) if there exists $K < \infty$ so that

$$\frac{1}{K} \leq \frac{h(x+t) - h(x)}{h(x) - h(x-t)} \leq K$$

for all $x - t, x, x + t \in [0,1]$. By results about quasi-conformal maps (specifically the Measurable Riemann Mapping Theorem) it follows that the set of parameters $\tilde{c}$ so that $f_{\tilde{c}}$ is topologically conjugate to f_c is either a single point or an open interval $I(f_c)$. Since $I(f_c)$ is also a closed set (this follows from some basic kneading theory), the fact that $I(f_c)$ and its complement are both non-empty gives a contradiction unless $I(f_c)$ is a single point.

This argument does not go through directly for real polynomial maps with more than one critical point, but using related arguments, one still obtains that quasi-symmetric rigidity implies density of hyperbolicity, see [50, Section 2]. In the case of real analytic maps the argument to prove density of hyperbolicity is more subtle, see [51].

2. Quasi-symmetric rigidity

As remarked in the previous section, all current proofs of density of hyperbolicity rely on quasi-symmetric rigidity. The most general form can be found in [25], and states:

Theorem 2.1 (Quasi-symmetric rigidity). *Assume that $f, g\colon [0,1] \to [0,1]$ are real analytic and topologically conjugate. Alternatively, assume that $f, g\colon S^1 \to S^1$ are topologically conjugate and that f and g each have at least one critical point or at least one periodic point. Moreover, assume that the topologically conjugacy is a bijection between*

(1) *the set of critical points and the order of corresponding critical points is the same;*

(2) *the set of parabolic periodic points.*

Then the conjugacy between f and g is quasi-symmetric.

The proof of this theorem builds on the machinery developed in [50]. This paper was written jointly by the authors and Kozlovski; it developed many of the key ingredients required to prove density of hyperbolicity, see [51]. Theorem 2.1 is an extension of these results, and was obtained jointly by Clark and the 2nd author, and uses all of the technology from[51], but also extends ideas from [56].

Indeed, when f, g are real analytic, then we will use the fact that these maps have holomorphic extensions to small neighbourhoods of $[0,1]$. Nevertheless, in [25] we prove the analogous result when f and g are merely C^3 maps, under some weak additional assumptions; in this case we will use that f, g have asymptotically holomorphic extensions near $[0,1]$, but will need to deal with the fact that high iterates of f and g are not necessarily close to holomorphic.

It is not hard to see that if conditions (1) or (2) in the previous theorem are not satisfied, then the maps are not even necessarily Hölder conjugate.

Special cases of this theorem we known before: Lyubich [61] and Graczyk & Świątek [37] proved this result for real quadratic maps. As we will see their method of proof in the quadratic case does not work if the degree of the map is > 2. For the case of real polynomials with only real critical points (of even order), this theorem was proved in [50]. For maps which are real analytic, it was shown in [87, Theorem 2, page 345] that there exists a qs-conjugacy *restricted to* $\omega(c)$ under the additional assumptions that the maps have no neutral periodic points, only non-degenerate critical points and have *'essentially bounded geometry'*. For covering maps of the circle (of degree ≥ 2) with one-critical point a global qs-conjugacy was constructed under the additional assumption that $\omega(c)$ is non-minimal and have no neutral periodic points, see [56]. When $\omega(c)$ is minimal, a qs-conjugacy restricted to $\omega(c)$ was constructed in [56].

For circle maps without periodic points, it is known that any two analytic critical circle homeomorphisms with one critical point, with the same irrational rotation number and the same order of the critical points are C^1-smoothly conjugate, see [47] (their work builds on earlier work of de Faria, de Melo and Yampolsky on renormalisation and in a recent paper was generalised to the smooth case, [39]). In ongoing work, Clark and the 2nd author are aiming to show that the methods in 2.1 can be extended to the case of circle homeomorphisms with several critical points. Note that the presence of critical points is necessary for circle homeomorphisms, because for circle diffeomorphisms the analogous statement is false. Indeed, otherwise one can construct maps for which some sequence of iterates has *almost a saddle-node fixed point*, resulting in larger and larger passing times near these points. This phenomenon is also referred to as *a sequence of saddle-cascades*. It was used by Arnol'd and Herman to construct examples of diffeomorphisms of the circle which are conjugate to irrational rotations, but where the conjugacy is neither absolutely continuous, nor qs and for which the map has no σ-finite measures, see [40] and also Section I.5 in [30]. In the diffeo-

morphic case, to get qs or C^1 one needs assumptions on the rotation number (to avoid these sequences of longer and longer saddle-cascades).

In general, one cannot expect C^1, because having a C^1 conjugacy implies that corresponding periodic orbits have the same multiplier.

We should also remark that there are also analogues of these theorems for polynomials in $\mathbb{C}$, but then one must assume that f is only finitely renormalizable, see for example [52], but also see [24].

2.1. Applications of quasi-symmetric rigidity. Quasi-symmetric rigidity is a crucial step towards proving the following types of results:

(1) hyperbolicity is dense, see subsection 1.5.

(2) within certain families of maps, conjugacy classes are connected, see Theorems A and 2.2 in [21].

(3) monotonicity of entropy; for families such as $[0,1] \ni x \mapsto a\sin(\pi x)$, see Section 3.

2.2. Complex box mappings. It turns out to be rather convenient to show quasi-symmetric rigidity by using extensions to the complex plane. This approach is rather natural, as a quasi-symmetric homeomorphism on the real line is always the restriction of a quasi-conformal homeomorphism on the complex plane. More precisely, the idea is to construct an extension of the first return map to some interval, to the complex plane as a 'complex box mapping', see Figure 2.1 in the multimodal case. Roughly speaking, this is a map $F\colon U \to V$ so that each component of U is mapped as a branched covering onto a component of V, and components of U are either compactly contained in a component of V or they are equal to such a component. Components of $F^{-n}(V)$ are called *puzzle pieces*. We also require (roughly speaking) that F is unbranched near the boundary of U (slightly more precisely, that there exists an annulus neighbourhood A of ∂V so that $F\colon F^{-1}(A) \to A$ is an unbranched covering and so that $\mathrm{mod}\,(A)$ is universally bounded from below). If one has such numerical bounds, then F is said to have *complex bounds*. The existence of these complex bounds was first proved by Sullivan for certain unimodal maps. The general unimodal case was dealt with in [55] and somewhat later in [35] and [60]. Later this was extended to the multimodal case for certain maps in [92] and more generally in [87]. The most general result appears in a joint paper of the 2nd author with Clark and Trejo [26]. In that paper complex bounds are associated to any real analytic interval map. In fact, even in the C^3 case complex bounds are constructed in that paper, but in the smooth case the map F is only asymptotically holomorphic.

We should note that in the non-renormalisable real-analytic case one obtains complex bounds at arbitrary deep levels, as soon as one has a complex box mapping. That this is the case follows from the construction of the enhanced nest (discussed in the next subsection) and an interesting lemma due to Kahn and Lyubich, see [45]. This tool is about pulling back a thin annulus, and shows that the modulus of the pullback of this annulus is much better than one might expect. In the real case, one can simplify and strengthen the statement and proof of Kahn and Lyubich's result as follows, see [52, Lemma 9.1]:

Lemma 2.2 (Small Distortion of Thin Annuli). *For every $K \in (0,1)$ there exists $\kappa > 0$ such that if $A \subset U$, $B \subset V$ are simply connected domains symmetric with respect to the real line, $F : U \to V$ is a real holomorphic branched covering map of degree D with all critical points real which can be decomposed as a composition of maps $F = f_1 \circ \cdots \circ f_n$ with all maps f_i real and either real univalent or real branched covering maps with just one critical*

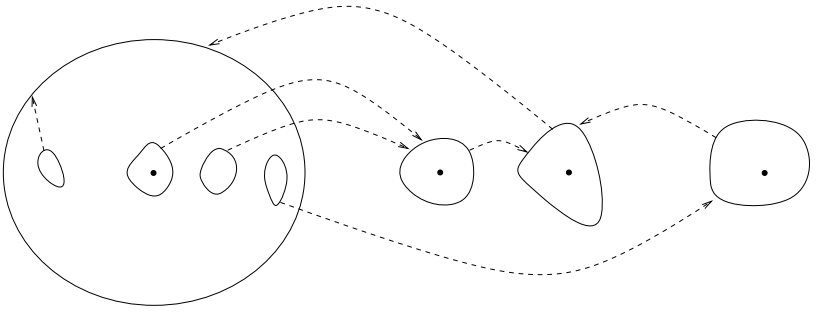


Figure 2.1. A box mapping.

point, the domain A is a connected component of $f^{-1}(B)$ symmetric with respect to the real line and the degree of $F|_A$ is d, then

$$\operatorname{mod}(U - A) \geq \frac{K^D}{2d} \min\{\kappa,\ \operatorname{mod}(V - B)\}.$$

2.3. How to prove quasi-symmetric rigidity? Consider the complex box mappings associated to two conjugate maps. To show that the conjugacy is quasi-symmetric one proceeds as follows:

(1) Define a sequence of puzzle pieces U_{n_i} called the **enhanced nest**, so that there exists k_i for which $F^{k(i)}: U_{n_{i+1}} \to U_{n(i)}$ is a branched covering map with degree bounded by some universal number N. This enhanced nest is chosen so that it transfers geometric information rather efficiently from small scale to large scale, but so that the degree of $F^{k(i)}: U_{n_{i+1}} \to U_{n(i)}$ remains universally bounded. This enhanced nest was one of the main new ingredients in [50]. It turns out that the post-critical sets do not come close to the boundary of the puzzle pieces in the enhanced nest, which implies that the puzzle pieces have uniformly bounded shape. Another important property of the enhanced nest is that decaying geometry and bounded geometry alternate quite regularly in the nest, which was used in [58] to study the Hausdorff dimension of Cantor attractors. The enhanced nest construction is also used for example in [26, 52, 75, 78, 94].

(2) In fact, if the interval maps extend to a holomorphic map on a neighbourhood of the real line, then one can partially define a quasi-conformal conjugacy near critical points, and then spread the definition to the whole complex plane fairly easily. This method was called the *spreading principle* in [50].

(3) Because of the spreading principle mentioned above, it then suffices to construct a partial-conjugacy on a puzzle piece in the enhanced nest which is 'natural on the boundary'. Given the above, this can easily be done using the *QC-criterion* from the appendix of [50]. and bounded shape of the puzzle pieces (bounded shape is very easy to derive from complex bounds, see [52, Section 10]). One can also proceed as in [5]. Our QC criterion was a variation of Heinonen-Koskela's theorem [42]. This theorem and its variations were used to prove rigidity result previously in [34, 41, 56, 83**?**], where in the last work, the author explicitly stated that a bounded shape property of puzzle pieces implies rigidity for non-renormalizable unicritical maps.

It is of course conceivable that one can prove quasi-symmetric rigidity using entirely real methods. This hinges on questions of the following type:

Question 2.3. *Consider the space $\mathcal{A}$ of maps of the form $z \mapsto |z|^d + c$ where $d > 1$ is not necessarily an integer and where c is real. Does one have quasi-symmetric rigidity for maps within the space $\mathcal{A}$? Are two topologically conjugate maps in $\mathcal{A}$ without periodic attractors (or both critically finite) necessarily the same?*

One of the difficulties with such a real approach is that it is not so easy to know how to use the information that the exponent d is fixed within the family $\mathcal{A}$: the exponent is not 'visible' in the real line. On the other hand, if d is an even integer, and $z \mapsto z^d + c$, then of course the local degree of the map at 0 is different for different values of d. Without fixing the degree d the answer to the question above is definitely negative. An affirmative answer to the above question would imply density of hyperbolicity and monotonicity of entropy in this family.

3. Monotonicity of entropy

In the late 70's, the following question attracted a lot of interest: does the topological entropy of the interval map $x \mapsto ax(1-x)$ depend monotonically on $a \in [0, 4]$? In the mid 80's this question was solved in the affirmative:

Theorem 3.1. *The topological entropy of the interval map $x \mapsto ax(1-x)$ depends monotonically on $a \in [0, 4]$.*

In the 80's several proofs of this appeared. One of these uses Thurston's rigidity theorem, see [70]. Another proof relies on Douady-Hubbard's univalent parametrisation of hyperbolic components, see [31], and a third proof is due to Sullivan; for a description of these proofs see [30]. All these proofs consider the map $x \mapsto ax(1-x)$ as a polynomial acting on the complex plane. A rather different method was used by Tsujii, [97]. He showed that periodic orbits bifurcate in the 'right' direction using a calculation on how the multiplier depends on the parameter. Unfortunately, Tsujii's proof also does not work for maps of the form $z \mapsto |z|^a + c$ with a not an integer.

In the early 90's, Milnor (see [69]) posed the more general

Conjecture 3.2 (Monotonicity Conjecture). *The set of parameters within a family of real polynomial interval maps, for which the topological entropy is constant, is connected.*

Milnor and Tresser proved this conjecture for cubic polynomials, see [71] (see also [28]). Their ingredients are planar topology (in the cubic case the parameter space is two-dimensional) and density of hyperbolicity for real quadratic maps.

A few years ago, Bruin and the 2nd author were able to give a proof of the general case of this conjecture. More precisely, given $d \geq 1$ and $\epsilon \in \{-1, 1\}$, consider the space P^d_ϵ of real polynomials $f\colon [0,1] \to [0,1]$ of fixed degree d with $f(\{0,1\}) \subset \{0,1\}$, with all critical points in $(0,1)$ and with the first lap orientation preserving if $\epsilon = 1$ and orientation reversing if $\epsilon = -1$. We call ϵ the *shape* of f. In [21] we proved the general case:

Theorem 3.3 (Monotonicity of Entropy). *For each integer $d \geq 1$, each $\epsilon \in \{-1, 1\}$ and each $c \geq 0$,*

$$\{f \in P^d_\epsilon; h_{top}(f) = c\}$$

is connected.

The proof in [21] also shows that the set of maps in P_ϵ^d with the same kneading sequence is connected and gives a precise description of the bifurcations that occur when one of the periodic attractors loses hyperbolicity. The main ingredient in the proof is quasi-symmetric rigidity. Recently, Kozlovski announced a simplification of the proof in [21] of this theorem (using semi-conjugacies to maps with constant absolute value of the slopes, rather than stunted sawtooth maps).

3.1. Non-local connectivity of isentropes and non-monotonicity in separate variables. It is possible to parametrize the family P^d by critical values. The following example shows that it is not true that topological entropy depends monotonically on each of these parameters. Define $f_{a,b}(x) = 2ax^3 - 3ax^2 + b$ for $a = b + 0.515$. This cubic map has critical points 0 and 1 and critical values $f(0) = b$, and $f(1) = b - a = 0.515$. It is shown in [21] that there are values of b such that the map $a \mapsto h_{top}(f_{a,b})$ is not monotone.

Related to this, it is shown in [22] that isentropes in P^d, when $d \geq 5$ are not locally connected. It is not known whether isentropes in P^3 or in P^4 are locally connected. For related results and questions, see [100].

4. Measure-theoretical dynamics

We shall now discuss the dynamics of a map $f : N \to N$, where $N = [0, 1]$ or S^1 from measure-theoretical point of view. Recall that a Borel probability measure μ is *invariant* for f if for each Borel set $A \subset N$ we have $\mu(f^{-1}A) = \mu(A)$. We say that μ is *ergodic* if a Borel set A with $f^{-1}(A) = A$ satisfies either $\mu(A) = 0$ or $\mu(A) = 1$. The basin $B(\mu)$ of μ is the set of points $x \in N$ for which

$$\frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i(x)} \to \mu \text{ as } n \to \infty, \tag{4.1}$$

where the convergence is with respect to the weak star topology. If $B(\mu)$ has positive Lebesgue measure, then we say that μ is a *physical measure*. Clearly, if O is an attracting periodic orbit, then the averaged Dirac measure $\mu_O = \frac{1}{\#O}\sum_{p\in O} \delta_p$ is a physical measure. An ergodic *acip*, i.e., an invariant probability measure which is absolutely continuous with respect to the Lebesgue measure, is also a physical measure, by Birkhorff's ergodic theorem.

4.1. Typical physical measures. Conjecturally these are the only two types of physical measures for typical interval maps, from measure-theoretical point of view. Indeed, in the major breakthrough [63], Lyubich proved that *within the quadratic family* $f_a(x) = ax(1 - x)$, $1 \leq a \leq 4$, *for almost every* a, *either* f_a *is hyperbolic or* f_a *has an ergodic acip*. In an earlier celebrated work [44], Jakobson showed that the set of a for which f_a has an ergodic acip has positive Lebesgue measure.

An analogue of Lyubich's theorem in the multi-critical case is widely open at the moment, due to the multi-dimensional feature of the corresponding parameter space. However, a generalization to the case of unimodal polynomials of even degree $d \geq 2$ is nearly completed. The work [6] extends the result of [62], showing that for any even integer $d \geq 2$, and almost every $a \in [1, 4]$, $f_a(x) = \frac{a}{4}(1 - (1 - 2x)^d)$ either is hyperbolic, or has an ergodic

acip, or is infinitely renormalizable. Moreover, Avila and Lyubich [4] developed a novel way to obtain exponential convergence along hybrid classes for infinitely renormalizable maps. One can expect a complete proof of the generalization of Lyubich's theorem for unimodal maps of a given degree will be available soon. Nevertheless, let us mention in a joint work with Bruin, the authors of this paper proved that for all even integer d, and almost every $1 \le a \le 4$, $\frac{a}{4}(1-(1-2x)^d)$ has a unique physical measure which might be supported on a Cantor set.

4.2. Existence of acip. We shall now discuss some recent advances on existence of acip for smooth interval maps. In order to apply some version of the real Koebe distortion to control distortion, we often assume f lies in the class $\mathcal{A}_3$ defined below. A map $f : [0,1] \to [0,1]$ is in the class $\mathcal{A}_k$ if the following holds: f is C^1 and C^k outside the critical set $\mathrm{Crit}(f) = \{c : f'(c) = 0\}$; moreover, for each $c \in \mathrm{Crit}(f)$, there exists $\ell_c > 1$ and C^k diffeomorphismsms φ_c, ψ_c of $\mathbb{R}$ such that $\varphi_c(c) = \psi_c(f(c)) = 0$ and $|\psi_c(f(x))| = |\varphi_c(x)|^{\ell_c}$ holds in a neighborhood of c.

The following theorem was obtained by the authors in joint with Bruin and Rivera-Letelier.

Theorem 4.1 (Existence of acip [19]). *Let $f \in \mathcal{A}_3$ be an interval map with all periodic points hyperbolic repelling. Assume that the following* large derivatives condition *holds: for each $c \in \mathrm{Crit}(f)$,*

$$|Df^n(f(c))| \to \infty \text{ as } n \to \infty.$$

Then f has an acip μ with density $\frac{d\mu}{dLeb} \in L^p$ for each $p < \ell_{\max}/(\ell_{\max}-1)$ where $\ell_{\max} = \sup_{c\in\mathrm{Crit}(f)} \ell_c$.

The unimodal case was done earlier by the authors in joint with Bruin [20]. The existence of acip for interval maps has been proved previously in more restrictive settings, including

- in [72], for maps satisfying the *Misiurewicz* condition: $\omega(c) \cap \mathrm{Crit}(f) = \emptyset$ for each $c \in \mathrm{Crit}(f)$;
- in [27] for unimodal maps satisfying the *Collet-Eckmann* condition (together with other conditions): for the critical point c, $\liminf_{n\to\infty} \frac{1}{n} \log|Df^n(f(c))| > 0$;
- in [74] for unimodal maps satisfying the following summability condition: if c is the critical point and ℓ is the order, then $\sum_{n=0}^{\infty} |Df^n(f(c))|^{-1/\ell} < \infty$,

among others. All of these results assume that f has negative Schwarizian outside $\mathrm{Crit}(f)$ in order to apply the real Koebe principle to control distortion, but now we know that the required distortion control is also valid for maps $f \in \mathcal{A}_3$, after [48] and [101, Theorem C].

We should however note that the large derivatives condition is not a necessary condition for the existence of an acip, even though an acip necessarily has positive metric entropy: there exists a unimodal map in the class $\mathcal{A}_3$ with $\liminf |Df^n(f(c))| = 0$ and with an acip [16]. It is also known (not surprisingly) that existence of acip is not a topological (or quasisymmetric) condition [17].

Question 4.2. *Determine topological (or quasisymmetric) conjugacy classes in $\mathcal{A}_3$ such that each map in the class has an acip.*

4.2.1. Ingredients of the proof of Theorem 4.1. An intermediate step of the proof is to show that the large derivatives condition implies *backward contraction* in the sense of

Rivera-Letelier [84], which means the following: if $\widetilde{B}_c(\delta)$ denotes the component of $f^{-1}(f(c)-\delta, f(c)+\delta)$ which contains c and

$$\Gamma(\delta) = \inf \left\{ \frac{\delta}{|U|} : \begin{array}{c} U \text{ is a component of } f^{-n}(\widetilde{B}_c(\delta)) \text{ containing } f(c') \\ \text{for some } c, c' \in \mathrm{Crit}(f) \text{ and } n \geq 0 \end{array} \right\}$$

then $\Gamma(\delta) \to \infty$ as $\delta \to 0$. It turns out that the backward contraction property is equivalent to the large derivatives condition [57].

It is well-known that for any Borel probability measure ν, any accumulation point of the following sequence

$$\frac{1}{n} \sum_{i=0}^{n-1} (f^i)_*(\nu)$$

in the weak star topology is an invariant probability measure of f, where $(f^i)_*\nu(A) = \nu(f^{-i}(A))$. Thus it suffices to prove the following statement: for each $0 < \kappa < 1$ there exists $C = C(\kappa)$ such that

$$(f^n)_*(\mathrm{Leb})(A) = |f^{-n}(A)| \leq C|f(A)|^{\kappa/\ell_{\max}},$$

holds for all Borel $A \subset [0,1]$ and all $n \geq 0$. The backward contraction property makes it possible to obtain the estimate when A is an interval close to the critical set. For general A, the paper uses a sliding argument from [74], and Mãné's theorem [65].

4.3. Decay of correlation. A different way to obtain existence of acip is via *inducing*. Let us say a map $F : \mathcal{U} \to \mathcal{V}$, where $\mathcal{U} \subset \mathcal{V}$ are open subsets of $[0,1]$, is a *Markov* map, if for each component U of $\mathcal{U}$, $F|U$ is a C^1 diffeomorphism onto a component of $\mathcal{V}$. A Markov map F is *induced* by a map f if there is a continuous function $s : \mathcal{U} \to \{1, 2, \ldots\}$ such that $F(x) = f^{s(x)}(x)$. (So $s(\cdot)$ takes constant value in each U.) We shall often consider Markov maps with extra properties:

(i) $\mathcal{V}$ is an interval;

(i′) $\mathcal{V}$ consists of finitely many intervals;

(ii) (Bounded distortion) There exist $C > 0$ and $\alpha \in (0,1)$ such that

$$\frac{|DF^n(x)|}{|DF^n(y)|} \leq C|F^n(x) - F^n(y)|^\alpha,$$

whenever $F^i(x)$ and $F^i(y)$ belong to the same component of $\mathcal{U}$ for each $i = 0, 1, \ldots, n-1$.

A Markov map $F : \mathcal{U} \to \mathcal{V}$ with the properties (i′) and (ii) has an absolutelty continuous invariant propbability measure ν such that $d\nu/d\mathrm{Leb}$ is bounded away from 0 and ∞. If we can construct an induced Markov map F for a map f such that (i′), (ii) and the following hold:

$$a_s := |\{s(x) \geq s\}| \to 0 \text{ as } s \to \infty,$$

then the original system f has an acip

$$\mu := \frac{1}{\sum_{s=1}^{\infty} a_s} \sum_U \sum_{j=0}^{s|U-1} (f^j)_*(\nu|U),$$

where the sum runs over all components of $\mathcal{U}$. One advantage of inducing is that through estimating the speed of convergence of $a_s \to 0$, one can obtain finer statistical properties of the system.

The following theorem was proved by the 1st author in joint with Rivera-Letelier, improving an earlier result [18] considerably.

Theorem 4.3 (Decay of correlation [85]). *Assume that $f \in \mathcal{A}_3$ is topologically exact and satisfies the large derivatives condition. Then there is an induced Markov map $F : \mathcal{U} \to \mathcal{V}$ such that (i) and (ii) and the following tail estimate hold:*

$$a_s = O(s^{-p}) \text{ for each } p > 0, \text{ as } s \to \infty.$$

In particular, the unique acip μ of f is super-polynomially mixing: for each essentially bounded $\varphi : [0,1] \to \mathbb{R}$ and each Hölder continuous $\psi : [0,1] \to \mathbb{R}$,

$$C_n(\varphi, \psi) := \int_0^1 \varphi \circ f^n \psi d\mu - \int_0^1 \varphi d\mu \int_0^1 \psi d\mu$$

converges to 0 superpolynomially fast as $n \to \infty$.

Here we say that f is topologically exact if for each non-empty open subset U of $[0,1]$, there exists a positive integer n such that $f^n(U) = [0,1]$. This is a necessary condition for f to have a mixing acip. The last statement was deduced from the tail estimate via Young's tower [102]. Note that the tail estimate also implies finer statistical properties of the sequence $\{\psi \circ f^n\}_{n=0}^{\infty}$ (considered as a sequence of random variables with identical distribution), such as the Central Limit Theorem [102], Almost Sure Invariance Principle [68], etc, for ψ Hölder. The paper [85] also dealt with existence and mixing properties of invariant probablity measures with respect to conformal measures (supported on Julia sets) of maximal dimension for a large class of complex rational maps. This paper used the induced Markov map to study the geomtery of the Julia set.

Much recent progress on theomodynamical formalism for one-dimensional maps also used inducing to construct invariant probablity measures with respect to various conformal measures, see for example [23, 76, 82].

For the proof of Theorem 4.3, an adaptation is used of the inducing scheme, called *canonical inducing*, developed in [81, 82]. A crucial new estimate is the following backward shrinking estimate for maps with large derivatives (Theorem B): *there exists $\rho > 0$ such that*

$$\theta_n := \{|J|\colon J \text{ is an interval such that } |f^n(J)| \leq \rho\}$$

converges to zero super-polynomially fast. Theorem C relates the quantity θ_n to the tail estimate of a suitably constructed induced Markov map, provided the map has *badness exponent* 0 which was the statement of Theorem A.

It is known that $\theta_n \to 0$ exponentially fast (the topological Collet-Eckmann condition, equivalent to the Collet-Eckmann condition in the unimodal case) is equivalent to having an exponentially mixing acip [73, 81]. It would be interesting to know

Question 4.4. *For a topologically exact interval map $f \in \mathcal{A}_3$, is $\theta_n \to 0$ superpolynomially fast equivalent to having a unique acip which is superpolynomially mixing?*

An affirmative solution to Question 2.11 in [85] implies an affirmative answer to the question above.

4.4. Stochastic stability. An interval map with an acip is not hyperbolic and hence not structurally stable. The notion of stochastic stability, posed by Kolmogrov and Sinai, asks for stability of statistical properties under random perturbations. Given a map $f : [0,1] \to [0,1]$, an *ε-random (pseudo) orbit* is by definition a sequence $\{x_n\}_{n=0}^{\infty}$ such that $|f(x_n) - x_{n+1}| \leq \varepsilon$. Roughly speaking, stochastic stability means when $\varepsilon > 0$ is small, for most of the ε-random orbits $\{x_n\}_{n=0}^{\infty}$, the asymptotic distribution, $\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \delta_{x_i}$, is close to a physical measure of f. Note that if $f([0,1]) \subset (0,1)$ and $\varepsilon > 0$ small enough, then the space of all ε-random orbits can be identified with $[0,1] \times [-\varepsilon, \varepsilon]^{\mathbb{N}}$ by the following formula:

$$\{x_n\}_{n=0}^{\infty} \mapsto (x_0, x_1 - f(x_0), x_2 - f(x_1), \ldots).$$

So the space of sequences $\{x_n\}_{n=0}^{\infty}$ can be endowed with a probability measure $\mathbb{P}_\varepsilon$ which corresponds to $m \times m_\varepsilon^{\mathbb{N}}$, where m denotes the Lebesgue measure on $[0,1]$ and m_ε denotes the normalised Lebesgue measure on $[-\varepsilon, \varepsilon]$. In the literature, reference measures other than $\mathbb{P}_\varepsilon$ have also been considered on the space of ε-random orbits, corresponding to different types of random perturbations. The measure $\mathbb{P}_\varepsilon$ corresponds to the so-called *additive noise* model.

Recently the 1st author proved the following theorem.

Theorem 4.5 (Stochastic Stability [88]). *Suppose $f \in \mathcal{A}_3$ is ergodic with respect to the Lebsgue measure and that the following summability condition holds: for each $c \in \mathrm{Crit}(f)$,*

$$\sum_{n=0}^{\infty} |Df^n(f(c))|^{-1} < \infty. \tag{4.2}$$

Then the unique acip of f is stochastic stable in the strong sense: For each $\varepsilon > 0$ there exists a unique probability measure μ_ε absolutely continuous with respect to the Lebesgue measure, such that for $\mathbb{P}_\varepsilon$-a.e. ε-random orbits $\{x_n\}_{n=0}^{\infty}$,

$$\frac{1}{n} \sum_{i=0}^{n-1} \delta_{x_i} \to \mu_\varepsilon$$

as $n \to \infty$ in the weak star topology. Moreover, the density $\frac{d\mu_\varepsilon}{dLeb}$ converges in L^1 to the density of the unique acip of f as $\varepsilon \to 0$.

See the Main Theorem of [88] for a more general statement, which covers a very general type of random perturbation. Previously, stochastic stability was studied for interval maps with a Benedicks-Carleson type condition [12, 13] (or even stronger) which thus has exponential decay of correlation, see [11, 14, 46, 95]. It is surprising that the stochastic stability of the Manneville-Pomeau map $x \mapsto x + x^{1+\alpha} \mod 1$, which is probably the simplest non-uniformly expanding dynamical system, was only established very recently by the authors in [89].

Li and Wang [59] proved stochastic stability for unimodal maps f with a wild attractor where the physical measure is supported on the Cantor attractor. It raises a curious question whether there exists an interval map with a stochastically unstable physical measure.

The crucial step in the proof of Theorem 4.5 was to establish lower bounds of the derivative of the first return maps to critical neighborhoods: Let $\varepsilon > 0$ be small and let $\widetilde{B}_c(\varepsilon)$ be

defined as in § 4.2.1. Then for *all* ε-random orbits $\{x_i\}_{i=0}^n$ with $x_0 \in \widetilde{B}_{c_1}(\varepsilon)$, $x_n \in \widetilde{B}_{c_2}(\varepsilon)$ for some $c_1, c_2 \in \mathrm{Crit}(f)$ and $x_1, x_2, \dots, x_{n-1} \notin \bigcup_{c\in\mathrm{Crit}(f)} \widetilde{B}_c$, we have

$$\prod_{i=1}^{n-1} |Df(x_i)| \geq \frac{\Lambda(\varepsilon)}{\varepsilon^{1-\ell_{c_2}^{-1}}} \exp(\varepsilon^{\alpha(\varepsilon)} n),$$

where $\Lambda(\varepsilon) \to \infty$ and $\alpha(\varepsilon) \to 0$ as $\varepsilon \to 0$. The measure μ_ε was constructed using a random inducing scheme initiated in [8]. See also [1, 2].

4.5. Jakobson's theorem. The lower bound for derivative plays a crucial role in a generalization of Jakobson's theorem by B. Gao and the 1st author [38]. Among a huge number of works in generalizing Jakobson's theorem, our approach is close to that of [96]. While the paper worked with general one-parameter families, the following is the main result obtained for polynomial maps.

Theorem 4.6 (Summability implies Collet-Eckmann alomost surely [38]). *Fix an integer $n \geq 2$. For each $\boldsymbol{a} = (a_0, a_1, \dots, a_n) \in \mathbb{R}^{n+1}$ write $P_{\boldsymbol{a}}(x) = \sum_{i=0}^n a_i x^i$. Let Λ_n denote the collection of $\boldsymbol{a} \in \mathbb{R}^{n+1} \setminus \{\boldsymbol{0}\}$ for which the following hold: (i) $P_{\boldsymbol{a}}([0,1]) \subset [0,1]$ and (ii) $P_{\boldsymbol{a}} : [0,1] \to [0,1]$ satisfies the summability condition (4.2). Then Λ_n has positive measures and almost every $\boldsymbol{a} \in \Lambda_n$ satisfies the Collet-Eckmann condition, and the following polynomial recurrence conditions: for each $\beta > 1$, and any critical points c, c' of $P_{\boldsymbol{a}}|[0,1]$, we have $|P_{\boldsymbol{a}}^k(c) - c'| \geq k^{-\beta}$ for all k sufficiently large.*

The proof is done by purely real analytic method, except we had to use a recent tranversality result due to Levin [54] which was based on complex methods. For the case $n = 2$, the transversality result was known before in [3, 53]. For the quadratic family, the Collet-Eckmann and polynomial recurrence conditions are satisfied by almost every non-hyperbolic map [7]. It would be interesting to push the real analytic method further, for instance, to see whether the summability condition can be replaced by the large derivatives condition in Theorems 4.5 and 4.6.

Finally we would like to draw the reader's attention to the works [9, 10] where the "modulus of continuity" of $t \mapsto \mu_t$ over "good" non-uniformly expanding maps is studied for families f_t of unimodal maps, where μ_t is the acip for f_t.

References

[1] José F. Alves and Helder Vilarinho, *Strong stochastic stability for non-uniformly expanding maps*, Ergodic Theory Dynam. Systems **33** (2013), no. 3, 647–692. MR3062897.

[2] José F. Alves and Vítor Araújo, *Random perturbations of nonuniformly expanding maps*, Geometric methods in dynamics. I. Astérisque No. **286** (2003), xvii, 25–62. MR2052296(2005e:37058).

[3] Artur Avila, *Infinitesimal perturbations of rational maps*, Nonlinearity **15** (2002), no. 3, 695–704. MR1901100.

[4] Artur Avila and Mikhail Lyubich, *The full renormalization horseshoe for unimodal maps of higher degree: exponential contraction along hybrid classes*, Publ. Math. Inst. Hautes études Sci. No. **114** (2011), 171-223. MR2854860.

[5] Artur Avila, Jeremy Kahn, Mikhail Lyubich, and Weixiao Shen, *Combinatorial rigidity for unicritical polynomials*, Ann. of Math. (2) **170** (2009), no. 2, 783–797. MR2552107.

[6] Artur Avila, Mikhail Lyubich, and Weixiao Shen, *Parapuzzle of the Multibrot set and typical dynamics of unimodal maps*, J. Eur. Math. Soc. (JEMS) **13** (2011), no. 1, 27–56. MR2735075.

[7] Artur Avila and Carlos Gustavo Moreira, *Statistical properties of unimodal maps: the quadratic family*, Ann. of Math. (2) **161** (2005), no. 2, 831–881. MR2153401.

[8] Viviane Baladi, Michael Benedicks, and Véronique Maume-Deschamps, *Almost sure rates of mixing for i.i.d. unimodal maps*, Ann. Sci. École Norm. Sup. (4) **35** (2002), no. 1, 77–126. MR1886006.

[9] Viviane Baladi, Michael Benedicks, and Daniel Schnellmann, *Whitney-Holder continuity of the SRB measure for transversal families of smooth unimodal maps*, arXiv:1302.3191v2.

[10] Viviane Baladi and Daniel Smania, *Linear response for smooth deformations of generic nonuniformly hyperbolic unimodal maps*, Ann. Sci. Éc. Norm. Supér. (4) **45** (2012), no. 6, 861–926. MR3075107.

[11] Viviane Baladi and Marcelo Viana, *Strong stochastic stability and rate of mixing for unimodal maps*, Ann. Sci. École Norm. Sup. (4) **29** (1996), no. 4, 483–517. MR1386223(97d:58170).

[12] Michael Benedicks and Lennart Carleson, *On iterations of* $1 - ax^2$ *on* $(-1, 1)$, Ann. of Math. (2) **122** (1985), no. 1, 1–25. MR0799250.

[13] _____, *The dynamics of the Hénon map*, Ann. of Math. (2) **133** (1991), no. 1, 73–169. MR1087346.

[14] Michael Benedicks and Lai-Sang Young, *Absolutely continuous invariant measures and random perturbations for certain one-dimensional maps*, Ergodic Theory Dynam. Systems **12** (1992), no. 1, 13–37. MR1162396.

[15] Alexander Blokh and Michał Misiurewicz, *Typical limit sets of critical points for smooth interval maps*, Ergodic Theory Dynam. Systems **20** (2000), no. 1, 15–45. MR2001d:37041.

[16] Henk Bruin, *Invariant measures of interval maps*, Ph.D. Thesis, University of Delft (1994).

[17] _____, *The existence of absolutely continuous invariant measures is not a topological invariant for unimodal maps*, Ergodic Theory Dyn. Syst. **18** (3), (1998), 555-565. MR1631716.

[18] Henk Bruin, Stefano Luzzatto, and Sebastian van Strien, *Decay of correlations in one-dimensional dynamics*, Ann. Sci. école Norm. Sup. (4) **36** (2003), no. 4, 621–646. MR2013929.

[19] Henk Bruin, Juan Rivera-Letelier, Weixiao Shen, and Sebastian van Strien, *Large derivatives, backward contraction and invariant densities for interval maps*, Invent. Math. **172** (2008). MR2393079.

[20] Henk Bruin, Weixiao Shen, and Sebastian van Strien, *Invariant measures exist without a growth condition*, Comm. Math. Phys. **241** (2003), no. 2-3, 287–306. MR2013801.

[21] Henk Bruin and Sebastian van Strien, *Monotonicity of entropy for real multimodal maps (2009)*, To appear in Journal of the AMS.

[22] _____, *On the structure of isentropes of polynomial maps*, Dyn. Syst. **28** (2013), no. 3, 381–392. MR3170622.

[23] Henk Bruin and Mike Todd, *Equilibrium states for interval maps: the potential* $\log|Df|$, Ann. Sci. Éc. Norm. Supér. (4) **42** (2009), no. 4, 559–600. MR2568876.

[24] Davoud Cheraghi, *Combinatorial rigidity for some infinitely renormalizable unicritical polynomials*, Conform. Geom. Dyn., An electronic Journal of AMS **14** (2010), 219–255.

[25] Trevor Clark and Sebastian van Strien, *Quasi-symmetric rigidity*, In preparation, 2009/2014.

[26] Trevor Clark, Sebastian van Strien, and Sofia Trejo, *Complex box bounds for real maps*, http://arxiv.org/abs/1310.8338 (2012).

[27] P. Collet, and J.-P. Eckmann, *Positive Liapunov exponents and absolute continuity for maps of the interval*, Ergodic Theory Dynam. Systems **3** (1983), no. 1, 13–46. MR0743027.

[28] Silvina P. Dawson, Roza Galeeva, John Milnor, and Charles Tresser, *A monotonicity conjecture for real cubic maps*, Real and complex dynamical systems (Hillerød, 1993), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 464, Kluwer Acad. Publ., Dordrecht, 1995, pp. 165–183. MR1351522 (96g:58132).

[29] Welington de Melo, Pedro A. S. Salomão, and Edson Vargas, *A full family of multimodal maps on the circle*, Ergodic Theory Dynam. Systems **31** (2011), no. 5, 1325–1344. MR2832248 (2012i:37062).

[30] Welington de Melo and Sebastian van Strien, *One-dimensional dynamics*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 25, Springer-Verlag, Berlin, 1993. MR95a:58035.

[31] Adrien Douady, *Topological entropy of unimodal maps: monotonicity for quadratic polynomials*, Real and complex dynamical systems (Hillerød, 1993), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 464, Kluwer Acad. Publ., Dordrecht, 1995, pp. 65–87. MR1351519 (96g:58106).

[32] Alexandre Eremenko and Sebastian van Strien, *Rational maps with real multipliers*, Trans. Amer. Math. Soc. **363** (2011), no. 12, 6453–6463. MR2833563.

[33] P. Fatou, *Sur les equations fonctionnelles, II*, Bull. Soc. Math. France **48** (1920), 33–94.

[34] Jacek Graczyk and and Stanislav Smirnov, *Non-uniform hyperbolicity in complex dynamics*, Invent. Math. **175** (2009), 334–415.

[35] Jacek Graczyk and Grzegorz Świątek, *Box mappings and polynomial dynamics*, Dynamical systems and chaos, Vol. 1 (Hachioji, 1994), World Sci. Publ., River Edge, NJ, 1995, pp. 53–59. MR1479905 (98g:58044).

[36] _____, *Generic hyperbolicity in the logistic family*, Ann. of Math. (2) **146** (1997), no. 1, 1–52. MR1469316 (99b:58079).

[37] _____, *The real Fatou conjecture*, Annals of Mathematics Studies, vol. 144, Princeton University Press, Princeton, NJ, 1998. MR1657075 (2000a:37020).

[38] Bing Gao and Weixiao Shen, *Summability implies Collet-Eckmann almost surely*, Ergod. Theo. Dynam. Syst. doi:10.1017/etds.2012.173.

[39] Pablo Guarino and Welington de Melo, *Rigidity of smooth critical circle maps*, (2013).

[40] M.R. Herman, *Sur la conjugaison differentiable des difféomorphismes du cercle a des rotations*. Publ. Math. I.H.E.S. **49** (1979), 5–234.

[41] Peter Haïssinsky, *Chirurgie parabolique*, C. R. Acad. Sci. Paris Sér. I Math., **327** (1998), 195–198.

[42] Juha Heinonen and Pekka Koskela, *Definitions of quasiconformality*, Invent. Math., **120** (1995), 61–79.

[43] M. V. Jakobson, *Smooth mappings of the circle into itself*, Mat. Sb. (N.S.) **85 (127)** (1971), 163–188. MR44 #7587.

[44] _____, *Absolutely continuous invariant measures for one-parameter families of one-dimensional maps*, Comm. Math. Phys. **81** (1981), no. 1, 39–88. MR0630331.

[45] Jeremy Kahn and Mikahil Lyubich, *The quasi-additivity law in conformal geometry*, Ann. of Math. (2) **169** (2009), no. 2, 561–593. MR2480612.

[46] A. Katok and Y. Kifer, Y., *Random perturbations of transformations of an interval*, J. Analyse Math. **47** (1986), 193–237. MR0874051 (88c:58053).

[47] K. Khanin and A. Teplinsky, *Robust rigidity for circle diffeomorphisms with singularities*, Invent. Math. **169** (2007), no. 1, 193–218. MR2308853 (2008i:37079).

[48] O. Kozlovski, *Getting rid of the negative Schwarzian derivative condition*, Ann. of Math. (2) **152** (2000), no. 3, 743–762.

[49] _____, *Cyclicity in families of circle maps*, Ergodic Theory Dynam. Systems **33** (2013), no. 5, 1502–1518. MR3103092.

[50] O. Kozlovski, W. Shen, and S. van Strien, *Rigidity for real polynomials*, Ann. of Math. (2) **165** (2007), no. 3, 749–841. MR2335796 (2008m:37063).

[51] _____, *Density of hyperbolicity in dimension one*, Ann. of Math. (2) **166** (2007), no. 1, 145–182. MR2342693 (2008j:37081).

[52] Oleg Kozlovski and Sebastian van Strien, *Local connectivity and quasi-conformal rigidity of non-renormalizable polynomials*, Proc. Lond. Math. Soc. (3) **99** (2009), no. 2, 275–296. MR2533666.

[53] Genadi Levin, *On an analytic approach to the Fatou conjecture*, Fund. Math. **171** (2002), no. 2, 177–196. MR1880383.

[54] _____, *Perturbations of weakly expanding critical orbits*, in 'Frontiers in complex dynamics: in Celebration of John Milnor's 80th birthday', Princeton University Press (2014), 163–196.

[55] Genadi Levin and Sebastian van Strien, *Local connectivity of the Julia set of real polynomials*, Ann. of Math. (2) **147** (1998), no. 3, 471–541. MR99e:58143.

[56] _____, *Bounds for maps of an interval with one critical point of inflection type. II*, Invent. Math. **141** (2000), no. 2, 399–465. MR2001i:37061.

[57] Huaibin Li and Weixiao Shen, *On non-uniform hyperbolicity assumptions in one-dimensional dynamics*, Sci. China Math. **53** (2010), no. 7, 1663–1677. MR2665501.

[58] Simin Li and Weixiao Shen, *Hausdorff dimension of Cantor attractors in one-dimensional dynamics*, Invent. Math. **171** (2008), no. 2, 345–387. MR2367023.

[59] Simin Li and Qihan Wang, *The slow recurrence and stochastic stability of unimodal interval maps with wild attractors*, Nonlinearity **26** (2013), no. 6, 1623–1637. MR3053435.

[60] M. Lyubich and M. Yampolsky, *Dynamics of quadratic polynomials: complex bounds for real maps*, Ann. Inst. Fourier (Grenoble) **47** (1997), no. 4, 1219–1255. MR98m:58113.

[61] Mikhail Lyubich, *Dynamics of quadratic polynomials. I, II*, Acta Math. **178** (1997), no. 2, 185–247, 247–297. MR98e:58145.

[62] _____, *Dynamics of quadratic polynomials. III. Parapuzzle and SBR measures.* Géométrie complexe et systémes dynamiques (Orsay, 1995). Astérisque No. 261 (2000), xii–xiii, 173–200. MR1755441.

[63] _____, *Almost every real quadratic map is either regular or stochastic.* Ann. of Math. (2) **156** (2002), no. 1, 1-78.MR1935840.

[64] R. Mañé, P. Sad, and D. Sullivan, *On the dynamics of rational maps*, Ann. Sci. École Norm. Sup. (4) **16** (1983), no. 2, 193–217. MR732343 (85j:58089).

[65] Ricardo Mañé, *Hyperbolicity, sinks and measure in one-dimensional dynamics*, Comm. Math. Phys. **100** (1985), no. 4, 495–524. MR87f:58131.

[66] M. Martens, W. de Melo, and S. van Strien, *Julia-Fatou-Sullivan theory for real one-dimensional dynamics*, Acta Math. **168** (1992), no. 3-4, 273–318. MR93d:58137.

[67] Curtis T. McMullen, *Complex dynamics and renormalization*, Annals of Mathematics Studies, vol. 135, Princeton University Press, Princeton, NJ, 1994. MR96b:58097.

[68] Ian Melbourne and Matthew Nicol, *Almost sure invariance principle for nonuniformly hyperbolic systems*, Comm. Math. Phys. **260** (2005), no. 1, 131–146. MR2175992.

[69] John Milnor, *Remarks on iterated cubic maps*, Experiment. Math. **1** (1992), no. 1, 5–24. MR1181083 (94c:58096).

[70] John Milnor and William Thurston, *On iterated maps of the interval*, Dynamical systems (College Park, MD, 1986–87), Lecture Notes in Math., vol. 1342, Springer, Berlin, 1988, pp. 465–563. MR970571 (90a:58083).

[71] John Milnor and Charles Tresser, *On entropy and monotonicity for real cubic maps*, Comm. Math. Phys. **209** (2000), no. 1, 123–178, With an appendix by Adrien Douady and Pierrette Sentenac. MR1736945 (2001e:37048).

[72] Michael Misiurewicz, *Absolutely continuous measures for certain maps of an interval*, Publ. Math., Inst. Hautes Etud. Sci. **53** (1981), 17–51. MR0623553 (83j:58072).

[73] Tomasz Nowicki and Duncan Sands, *Non-uniform hyperbolicity and universal bounds for S-unimodal maps*, Invent. Math. **132** (1998), no. 3, 633–680. MR1625708.

[74] Tomasz Nowicki and Sebastian van Strien, *Invariant measures exist under a summability condition for unimodal maps*, Invent. Math. **105** (1991), no. 1, 123–136. MR1109621.

[75] Wenjuan Peng, Weiyuan Qiu, Pascale Roesch, Lei Tan, and Yongcheng Yin, *A tableau approach of the KSS nest*, Conform. Geom. Dyn. **14** (2010), 35–67. MR2600535 (2011d:37079).

[76] Yakov Pesin and Samuel Senti, *Thermodynamical formalism associated with inducing schemes for one-dimensional maps*, Mosc. Math. J. **5** (2005), no. 3, 669–678, 743–744. MR2241816.

[77] Enrique R. Pujals, *Some simple questions related to the C^r stability conjecture*, Nonlinearity **21** (2008), no. 11, T233–T237. MR2448223 (2009m:37052).

[78] WeiYuan Qiu and YongCheng Yin, *Proof of the Branner-Hubbard conjecture on Cantor Julia sets*, Sci. China Ser. A **52** (2009), no. 1, 45–65. MR2471515 (2009j:37074).

[79] Lasse Rempe and Sebastian van Strien, *Density of hyperbolicity for classes of real transcendental entire functions and circle maps*, http://arxiv.org/abs/1005.4627 (2010).

[80] ———, *Absence of line fields and Mañé's theorem for nonrecurrent transcendental functions*, Trans. Amer. Math. Soc. **363** (2011), no. 1, 203–228. MR2719679 (2011m:37073).

[81] Feliks Przytycki and Juan Rivera-Letelier, *Statistical properties of topological Collet-Eckmann maps.* Ann. Sci. École Norm. Sup. (4) 40 (2007), no. 1, 135–178. MR2332354.

[82] _____, *Nice inducing schemes and the thermodynamics of rational maps*, Comm. Math. Phys. **301** (2011), no. 3, 661–707. MR2784276.

[83] Feliks Przytycki and Steffen Rohde, *Rigidity of holomorphic Collet-Eckmann repellers* Ark. Mat. **37** (1999), 357–371. MR1714763.

[84] Juan Rivera-Letelier, *A connecting lemma for rational maps satisfying a no-growth condition*, Ergodic Theory Dynam. Systems **27** (2007), no. 2, 595–636. MR2308147.

[85] Juan Rivera-Letelier and Weixiao Shen, *Statistical properties of one-dimensional maps under weak hyperbolicity assumptions*, Preprint 2010 arXiv:1004.0230v2.

[86] Weixiao Shen, *On the measurable dynamics of real rational functions*, Ergodic Theory Dynam. Systems **23** (2003), no. 3, 957–983. MR1992673 (2004e:37069).

[87] _____, *On the metric properties of multimodal interval maps and C^2 density of Axiom A*, Invent. Math **156** (2004), no. 2, 301–403.

[88] _____, *On stochastic stability of non-uniformly expanding interval maps*, Proc. Lond. Math. Soc. (3) **107** (2013), no. 5, 1091–1134. MR3126392.

[89] Weixiao Shen and Sebastian van Strien, *On stochastic stability of expanding circle maps with neutral fixed points*, Dyn. Syst. **28** (2013), no. 3, 423–452. MR3170624.

[90] Steve Smale, *The mathematics of time*, Springer-Verlag, New York, 1980, Essays on dynamical systems, economic processes, and related topics. MR607330 (83a:01068).

[91] _____, *Mathematical problems for the next century*, Mathematics: frontiers and perspectives, Amer. Math. Soc., Providence, RI, 2000, pp. 271–294. MR2001i:00003.

[92] Daniel Smania, *Complex bounds for multimodal maps: bounded combinatorics*, Nonlinearity **14** (2001), no. 5, 1311–1330. MR1862823 (2003j:37057).

[93] _____, *Puzzle geometry and rigidity: the Fibonacci cycle is hyperbolic*, J. Amer. Math. Soc. **20** (2007), no. 3, 629–673. MR2291915.

[94] Lei Tan and Yongcheng Yin, *The unicritical Branner-Hubbard conjecture*, Complex dynamics, A. K. Peters, Wellesley, MA, 2009, pp. 215–227. MR2508258.

[95] Masato Tsujii, *Small random perturbations of one-dimensional dynamical systems and Margulis-Pesin entropy formula*, Random Comput. Dynam. **1** (1992/93), no. 1, 59–89.

[96] _____, *Positive Lyapunov exponents in families of one-dimensional dynamical systems*, Invent. Math. **111** (1993), no. 1, 113–137. MR1193600.

[97] _____, *A simple proof for monotonicity of entropy in the quadratic family*, Ergodic Theory Dynam. Systems **20** (2000), no. 3, 925–933. MR1764936 (2001g:37048).

[98] Sebastian van Strien, *Hyperbolicity and invariant measures for general C^2 interval maps satisfying the Misiurewicz condition*, Comm. Math. Phys. **128** (1990), no. 3, 437–495. MR91g:58161.

[99] _____, *One-parameter families of smooth interval maps: density of hyperbolicity and robust chaos*, Proc. Amer. Math. Soc. **138** (2010), no. 12, 4443–4446. MR2680068 (2012b:37114).

[100] _____, *Milnor's conjecture on monotonicity of topological entropy: Results and questions*, in 'Frontiers in complex dynamics: in Celebration of John Milnor's 80th birthday', Princeton University Press (2014), 323–338.

[101] Sebastian van Strien and Edson Vargas, *Real bounds, ergodicity and negative Schwarzian for multimodal maps*, J. Amer. Math. Soc. **17** (2004), no. 4, 749–782.

[102] Lai-Sang Young, *Recurrence times and rates of mixing*, Israel J. Math. **110** (1999), 153–188.

National University of Singapore and Imperial College, London
E-mail: matsw@nus.edu.sg; s.van-strien@imperial.ac.uk

10. Partial Differential Equations

From molecular dynamics to kinetic theory and hydrodynamics

Thierry Bodineau, Isabelle Gallagher, and Laure Saint-Raymond

Abstract. In these notes we present the main ingredients of the proof of the convergence of the distribution function of a tagged particle in a background initially at equilibrium, towards the solution to the heat equation. We also show how the process associated with the tagged particle converges in law towards a Brownian motion.

Mathematics Subject Classification (2010). 76P05 ; 82B21 ; 82B40

Keywords. Boltzmann equation, BBGKY hierarchy, heat equation, Brownian motion, low density limit, branching process.

1. Introduction

1.1. Microscopic and macroscopic models for rarefied gases. At the second International Congress of Mathematicians held in Paris in 1900, D. Hilbert presented ten of his famous list of twenty-three open questions [25]. Some of those questions have since been solved, and some remain open to this day. Among these, we are interested here in the sixth problem related to the axiomatization of physics. The challenge is to understand whether or not the different models describing the dynamics of fluids are consistent, and more precisely *to develop "mathematically the limiting processes [...] which lead from the atomistic view to the laws of motion of continua".*

1.1.1. The particle description. At the atomistic scale, a system of N particles in a domain $\mathbb{D} \subset \mathbb{R}^d$ can be described by their N positions $X_N := (x_1, \dots, x_N)$ in $\mathbb{D}^N$ and N velocities $V_N := (v_1, \dots, v_N)$ in $\mathbb{R}^{dN}$, where $d \geq 2$ denotes the dimension. These evolve according to Newton's laws. For instance assuming that they are identical and interact via a pairwise potential at some scale $\varepsilon > 0$, the positions and velocities are related by the following system of ODEs: for $1 \leq i \leq N$,

$$\frac{dx_i}{dt} = v_i\,, \qquad m\frac{dv_i}{dt} = -\frac{1}{\varepsilon}\sum_{j\neq i} \nabla\Phi\Big(\frac{x_i - x_j}{\varepsilon}\Big)\,, \tag{1.1}$$

where m is the mass of the particles (which we shall assume from now on equal to 1 to simplify) and the force exerted by particle j on particle i is $-\frac{1}{\varepsilon}\nabla\Phi\Big(\frac{x_i - x_j}{\varepsilon}\Big)$. Note that

Proceedings of the International Congress of Mathematicians, Seoul, 2014

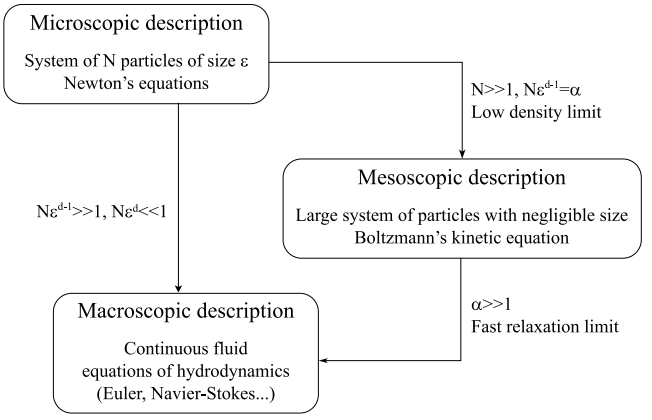


Figure 1. The different levels of description of a perfect gas

these equations are nothing else than the Hamiltonian system associated with the energy

$$H_N(X_N, V_N) := \sum_{i=1}^{N} \frac{1}{2}|v_i|^2 + \sum_{i\neq j} \Phi\Big(\frac{x_i - x_j}{\varepsilon}\Big)\,.$$

To avoid complicated billiard free dynamics on $\mathbb{D}$, we shall focus here on the case of the unit torus $\mathbb{D} = \mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$. For the sake of simplicity, we shall further assume that the interaction is pointwise: the particles are N hard spheres of diameter $\varepsilon > 0$ and centers $X_N := (x_1, \dots, x_N)$, interacting via elastic collisions: namely if there exists $j \neq i$ such that $|x_i - x_j| = \varepsilon$, then the incoming velocities (v_i^{in}, v_j^{in}) are related to the outgoing velocities (v_i^{out}, v_j^{out}) by

$$\begin{aligned} v_i^{in} &= v_i^{out} - \nu^{i,j} \cdot (v_i^{out} - v_j^{out})\, \nu^{i,j} \\ v_j^{in} &= v_j^{out} + \nu^{i,j} \cdot (v_i^{out} - v_j^{out})\, \nu^{i,j}\,, \end{aligned} \tag{1.2}$$

where $\nu^{i,j} := (x_i - x_j)/|x_i - x_j|$. The wellposedness of this system of ODEs is not an obvious fact, due to the possible clustering of collision times between particles which could lead to a finite-time blow-up, or to the possibility that three or more particles collide at the same time. However it can be proved (see [1, 2] in the case of an infinite number of particles, or [20] for instance in the easier situation under study) that the set of initial configurations leading to such pathologies is of measure zero, hence it will be neglected from now on.

In the following to simplify notation, we shall denote, for $1 \leq i \leq N$, $z_i := (x_i, v_i)$ and $Z_N := (z_1, \dots, z_N)$. The distribution function $f_N(t, Z_N)$ associated with the system (1.1) satisfies the Liouville equation

$$\partial_t f_N + \sum_{i=1}^{N} v_i \cdot \nabla_{x_i} f_N - \frac{1}{\varepsilon} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} \nabla_x \Phi\Big(\frac{x_i - x_j}{\varepsilon}\Big) \cdot \nabla_{v_i} f_N = 0 \quad \text{in } \mathbb{T}^{dN} \times \mathbb{R}^{dN}\,.$$

In the case of hard-spheres (1.2) this equation becomes

$$\partial_t f_N + \sum_{i=1}^{N} v_i \cdot \nabla_{x_i} f_N = 0\,, \tag{1.3}$$

and it is set in $\mathcal{D}_N^\varepsilon \times \mathbb{R}^{dN}$ with $\mathcal{D}_N^\varepsilon := \left\{ X_N \in \mathbb{T}^{dN}\,, \forall i \neq j\,, |x_i - x_j| > \varepsilon \right\}$ with a specular reflection on the boundary. We now distinguish pre-collisional configurations from post-collisional ones by defining for indexes $1 \leq i \neq j \leq N$

$$\partial\mathcal{D}_{N,\pm}^\varepsilon(i,j) := \Big\{ Z_N \in \mathbb{T}^{dN} \times \mathbb{R}^{dN} \,/\, |x_i - x_j| = \varepsilon\,, \quad \pm(v_i - v_j)\cdot(x_i - x_j) > 0 \\ \text{and} \quad \forall (k,\ell) \in \left\{[1,N] \setminus \{i,j\}\right\}^2, |x_k - x_\ell| > \mathcal{D}_N^\varepsilon \Big\}\,.$$

Given Z_N on $\partial\mathcal{D}_{N,+}^\varepsilon(i,j)$, we define $Z_N^{(i,j)} \in \partial\mathcal{D}_{N,-}^\varepsilon(i,j)$ as the configuration having the same positions $(x_k)_{1\leq k\leq N}$, the same velocities $(v_k)_{k\neq i,j}$ for non interacting particles, and the following pre-collisional velocities for particles i and j

$$\begin{aligned} v_i^{(i,j)} &:= v_i - \frac{1}{\varepsilon^2}(v_i - v_j)\cdot(x_i - x_j)(x_i - x_j) \\ v_j^{(i,j)} &:= v_j + \frac{1}{\varepsilon^2}(v_i - v_j)\cdot(x_i - x_j)(x_i - x_j)\,. \end{aligned} \tag{1.4}$$

Then on $\partial D_N^\varepsilon(i,j)$ the following boundary condition holds:

$$f_N(t, Z_N) = f_N(t, Z_N^{(i,j)}). \tag{1.5}$$

1.1.2. From particles to fluids. From the knowledge of $Z_N(t)$, one can define *observable* quantities such as the empirical density, momentum and energy:

$$\begin{aligned} &\rho_N(t,x) := \frac{1}{N}\sum_{i=1}^{N} \delta\big(x - x_i(t)\big)\,, \quad \rho_N u_N(t,x) := \frac{1}{N}\sum_{i=1}^{N} v_i(t)\delta\big(x - x_i(t)\big)\,, \\ &e_N(t,x) := \frac{1}{2}\rho_N(u_N^2 + d\,\theta_N)(t,x) := \frac{1}{2N}\sum_{i=1}^{N} |v_i(t)|^2\delta\big(x - x_i(t)\big)\,. \end{aligned} \tag{1.6}$$

To obtain laws of motion of continua one starts from those observables and one takes the limit $N \to \infty$ with $\varepsilon \to 0$. By definition, rarefied gases are those for which there is no excluded volume in the state relation, meaning that $N\varepsilon^d \ll 1$ for the hydrodynamic limit (see Figure 1). Fluid equations are the asymptotic form of the conservations of empirical density, momentum and energy. In order to get a closed system of equations we need to show that the microscopic fluxes converge to some macroscopic fluxes depending on the macroscopic density ρ, momentum ρu and internal energy e, in the limit $N \to \infty$. This convergence has to be understood in the sense of the law of large numbers with respect to the density f_N (solution to the Liouville equation). The point is therefore to establish that "locally" $f_N(t)$ is close to an equilibrium measure. This fact is not known in the case of the deterministic dynamics of hard spheres.

By adding a small noise term which exchanges the momenta of nearby particles, Olla, Varadhan and Yau [35] proved the almost sure convergence of the empirical density, velocity

and energy to the solution of the Euler equation

$$\begin{aligned}\partial_t\rho + \nabla_x\cdot(\rho u) &= 0\\ \partial_t(\rho u) + \nabla_x\cdot(\rho u\otimes u + \rho\theta\mathrm{Id}) &= 0\\ \partial_t(\rho e) + \nabla_x\cdot(\rho u e + \rho\theta u) &= 0\end{aligned}$$

as long as it has a smooth solution. The result [35] follows from the ergodicity of the infinite system of interacting particles: the translation invariant stationary measures of the dynamics minimizing the entropy production are the Gibbs measures.

The Navier-Stokes equations are the next order corrections to the Euler equations. In order to derive them one needs to show that the microscopic current is well approximated by the sum of the macroscopic current and a much smaller viscosity term. The mathematical interpretation of this viscous term is given by some fluctuation-dissipation equation. In order to avoid the difficulties of the multiscale asymptotics, we may consider the case when the leading order (compressible) approximation is just a constant and turn to the incompressible Navier-Stokes equations. The rigorous derivation of the incompressible Navier-Stokes equations from particle systems has then been obtained in the framework of stochastic lattice models, first by Esposito, Marra and Yau [19], under some regularity assumption which was later removed by Quastel and Yau [38].

Remark 1.1. Note that this approach also provides convergence results for fluids with excluded volume, i.e. when $N\varepsilon^d = O(1)$.

Remark 1.2. The complexity of the problem is such that there is still no complete derivation of any fluid model starting from the full deterministic Hamiltonian dynamics, regardless of the regime.

1.1.3. The Boltzmann equation. In his statement of the sixth problem, Hilbert actually suggested that an intermediate step between the atomistic and the continuous points of view could be the "mesoscopic" scale, governed by the Boltzmann equation obtained in the low density limit $N\to\infty$, $N\varepsilon^{d-1}\alpha^{-1}=1$ (see Figure 1).

More precisely the idea is to start with the description of the particle system via its distribution function f_N, satisfying the Liouville equation (1.3). Then one aims at deriving a closed equation on the probability distribution $f(t,x,v)$ of one particle (describing the probability for a particle to be at time t at position x with velocity v). As we shall see in the formal derivation in Section 1.2 below, the one-particle density distribution f is the limit (as $N\to\infty$) of

$$f_N^{(1)}(t,z_1) := \int f_N(t,Z_N)\,dz_2\dots dz_N\,, \tag{1.7}$$

assuming that f_N is unchanged under the relabeling of particles, namely

$$f_N(t,Z_{\sigma(N)}) = f_N(t,Z_N)\,,\quad \forall\sigma\in\mathfrak{S}_N\,.$$

Under the *chaos assumption*, i.e. assuming that the particles are independent and identically distributed, one obtains heuristically that the function f satisfies the Boltzmann equation

$$\partial_t f + v\cdot\nabla_x f = \alpha Q(f,f) \tag{B}$$

with Q, a local operator in x and t, defined by

$$Q(f,f) := \int_{\mathbb{S}^{d-1}\times\mathbb{R}^{d-1}} [f(v')f(v_1') - f(v)f(v_1)]\,b(v-v_1,\omega)\,dv_1 d\omega$$

and (v', v_1') are given by $v' = v + \omega \cdot (v_1 - v)\,\omega$ and $v_1' = v_1 - \omega \cdot (v_1 - v)\,\omega$. The function $b(v - v_1, \omega)$ is the collision kernel. In the case of hard-spheres interacting elastically as in (1.2), one has

$$b(v - v_1, \omega) = \big(\omega \cdot (v_1 - v)\big)_+ .$$

Note that the Boltzmann collision operator $Q(f, f)$ can be split into a gain term and a loss term: the loss term counts all collisions in which a given particle of velocity v will encounter another particle, of velocity v_1, and thus will change its velocity leading to a loss of particles of velocity v; on the other hand, the gain term measures the number of particles of velocity v which are created due to a collision between particles of velocities v' and v_1'.

Because particles are indistinguishable, v and v_1 play symmetric roles in the collision integral. The reversibility of the elementary collision process implies moreover that the change of variables $(v', v_1', \omega) \to (v, v_1, \omega)$ has unit jacobian, so that for any smooth function φ defined on $\mathbb{R}^d$, one has formally (under suitable decay and smoothness assumptions on f)

$$\int Q(f,f)\varphi(v)dv = \frac{1}{4}\int [f(v')f(v_1') - f(v)f(v_1)]\big(\varphi(v) + \varphi(v_1) - \varphi(v') - \varphi(v_1')\big) \times \big((v - v_1)\cdot\omega\big)_+ \, dv dv_1 d\omega \,.$$

In particular, choosing $\varphi(v) = 1$, then $\varphi(v) = v$ and $\varphi(v) = |v|^2$, we formally obtain the conservation of mass, momentum and energy

$$\partial_t \rho + \nabla_x \cdot (\rho u) = 0\,, \quad \partial_t(\rho u) + \nabla_x \cdot \int f v \otimes v dv = 0\,,$$
$$\partial_t(\rho u^2 + d\rho\theta) + \nabla_x \cdot \int f|v|^2 v dv = 0\,,$$

where

$$\rho(t,x) := \int f(t,x,v)\,dv, \quad \rho u(t,x) := \int f(t,x,v)\,v dv,$$
$$\frac{1}{2}\rho(t,x)\big(|u(t,x)|^2 + d\,\theta(t,x)\big) = \frac{1}{2}\int f(t,x,v)\,|v|^2 dv\,.$$

On the other hand, taking $\varphi = \log f$ in the previous identity, we also get

$$D(f) := -\int Q(f,f)\log f(v) dv \geq 0\,,$$

from which we deduce the entropy inequality, referred to as Boltzmann's H theorem,

$$\int f\log f(t,x,v)dxdv + \alpha\int_0^t \int D(f)(s,x)dsdx \leq \int f^0 \log f^0(x,v)dxdv\,,$$

where f^0 is the initial data of f. This means in particular that the Boltzmann equation describes irreversible dynamics. More precisely, we expect the Boltzmann equation to predict a relaxation towards thermodynamic equilibria, which are minimizers of the entropy for fixed mass, momentum and energy. This is in apparent contradiction with the fact that the Liouville equation and Newton's laws are reversible, and satisfy the Poincaré recurrence principle. We shall comment more on that later on (see Remark 1.6).

Remark 1.3. Note that, in general, the collision integral does not make sense under the only physical estimates. Formally, the conservations of mass and energy indeed provide

$$\iint f(t,x,v)(1+|v|^2)dvdx = \iint f^0(x,v)(1+|v|^2)dvdx,$$

whereas Boltzmann's H-theorem gives the decay of entropy $\iint f\log f(t,x,v)dxdv$. In other words, the collision operator involves the product of two functions of x which are only known to be in some $L\log L$ Orlicz space.

1.1.4. From Boltzmann to fluids. From the works of Hilbert [26] and Chapman-Enskog [12, 16], it is known that most fluid equations can be formally obtained from the Boltzmann equation (B). In the fast relaxation limit $\alpha\to\infty$, i.e. when the mean free path $1/\alpha$ is very small compared to the typical observation length, we indeed expect the collision process to be dominating and the solution to the Boltzmann equation to be close to local thermodynamic equilibrium. The evolution of the gas should therefore be well approximated by fluid equations.

Let us define M_f, the local Maxwellian of same moments as f, by

$$M_f(t,x,v) := \frac{\rho(t,x)}{(2\pi\theta(t,x))^{\frac{d}{2}}} e^{-\frac{|v-u(t,x)|^2}{2\theta(t,x)}}.$$

At leading order, replacing f by M_f in the conservation laws, we get the compressible Euler equations. Collecting all contributions to the local thermodynamic equilibrium at leading order, we then introduce the following Ansatz to describe the purely kinetic part of f

$$f = M_f\Big(1+\sum_{j=1}^{+\infty}\frac{1}{\alpha^j}g_j\Big).$$

The crucial point is that the collision operator linearized around M_f, denoted by $-\mathcal{L}_{M_f}$, is a Fredholm operator on $L^2(M_f dv)$ with kernel spanned by the collision invariants 1, v and $|v|^2$. Denoting by $\Pi^\perp$ the projection onto the orthogonal of the kernel of $-\mathcal{L}_{M_f}$ we get at the next order

$$\mathcal{L}_{M_f} g_1 = -\frac{1}{\alpha}\Pi^\perp\left(\frac{v\cdot\nabla_x M_f}{M_f}\right).$$

Inverting $\mathcal{L}_{M_f}$ on the orthogonal of its kernel, one obtains as first correction to the compressible Euler equations the weakly dissipative, compressible Navier-Stokes system with $O(1/\alpha)$ dissipation terms

$$\begin{aligned}
\partial_t\rho + \nabla_x\cdot(\rho u) &= 0\\
\partial_t(\rho u) + \nabla_x\cdot(\rho u\otimes u + \rho\theta\,\mathrm{Id}) &= \frac{1}{\alpha}\nabla_x\cdot(\kappa_1(\rho,\theta)\nabla_x u)\\
\partial_t\big(\rho(|u|^2+d\,\theta)\big) + \nabla_x\cdot\big(\rho(|u|^2+(d+2)\theta)u\big) &= \frac{1}{\alpha}\nabla_x\cdot(\kappa_2(\rho,\theta)\nabla_x\theta)\\
&\quad + \frac{1}{\alpha}\nabla_x\cdot(\kappa_1(\rho,\theta)\nabla_x u\cdot u).
\end{aligned}$$

For a more detailed presentation of formal asymptotic expansions, we refer to [39].

Since the solutions of the first order hydrodynamic approximation exhibit singularities such as shocks or discontinuities, the question of their stability after blow-up time seems out of reach at the present time (see [33] before the blow-up time). A natural idea to avoid these complicated questions about the compressible Euler equations is to consider fluctuations around some special solutions, the simplest ones being global equilibria

$$M_\beta(v) := \left(\frac{\beta}{2\pi}\right)^{\frac{d}{2}} e^{-\beta|v|^2/2}\,. \tag{1.8}$$

At present time, this perturbative framework leading to viscous incompressible fluid models is essentially the only one in which unconditional results are available [4, 22, 44], describing the fast relaxation limit. From the formal expansion, we know that the diffusion terms will be of order 1 if time is rescaled by factor α. We denote by $\tau = t/\alpha$ the macroscopic time variable. Then, in order for the nonlinear convection term to remain bounded, we need the fluctuation to be at most of order α^{-1}. This corresponds to having the Mach and Knudsen numbers of the same order of magnitude, which is in agreement with the Von Karman relation for perfect gases giving the Reynolds number as the ratio of the Mach and Knudsen numbers.

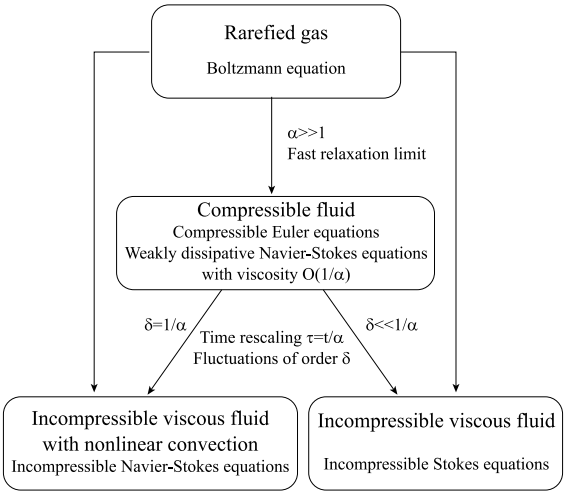


Figure 2. Hydrodynamic limits of the Boltzmann equation

Remark 1.4. It is important to realize that considering the fast relaxation limit is only possible if the solution f of (B) is known to exist for a time independent of α. Therefore the mathematical study of hydrodynamic limits requires either additional (regularity and smallness) assumptions on the initial distribution f^0, or to consider a very weak notion of solution (namely the renormalized solutions introduced by DiPerna and Lions [15]).

1.1.5. From particles to Boltzmann. In order to use the Boltzmann equation as an intermediate step between particles and fluids, the remaining task consists in justifying the limit from $f_N^{(1)}$ defined in (1.7) to f, for a large enough time interval so that one can follow with the (known) limit from (B) to fluid equations. The precise setting (in particular the choice of

the scaling $N\varepsilon^{d-1}\alpha^{-1} = 1$ mentioned above) in which to carry out that limit was identified by Grad in [23]. Lanford presented in [29] a detailed scheme of proof, which was completed by a number of authors (see [13, 14, 42] for important contributions in the hard sphere case, and [20, 37] for a complete proof in the hard sphere case as well as the case of a compactly supported, repulsive potential).

However those results only hold for a microscopic time of order $1/\alpha$, and therefore it is impossible to this day to carry out sequentially the particle-to-Boltzmann limit followed by the Boltzmann-to-fluid limit. The difficulty is to find a suitable functional framework to prove the propagation of chaos, and more generally to obtain a good control of correlations for long enough times. We indeed do not expect to get better estimates than for the limiting Boltzmann equation (see Remarks 1.3 and 1.4).

In these notes, we show how in a *linear* setting, the full program can go through: the Lanford proof can be made to hold for a long enough time in order to carry out the hydrodynamic limit. The limit equation obtained in our setting is the heat equation: a precise statement is given in Paragraph 1.4 below.

1.2. The Boltzmann-Grad limit for a system of hard spheres.

1.2.1. The setting. From now on to simplify we shall restrict our attention to the case of hard-spheres interactions (1.2), although everything would work in the same way for an adequate, compactly supported repulsive potential (see [20] or [37] for the precise assumptions required on the potential). As explained in Paragraph 1.1.3, the solution to the Boltzmann equation is obtained by taking the limit on the first marginal defined in (1.7). Let us integrate the Liouville equation (1.3) over the variables $(z_2, \dots, z_N)$. Using Green's formula to handle the contribution of the boundary, one comes up formally with the following equation on $f_N^{(1)}$:

$$\partial_t f_N^{(1)} + v_1 \cdot \nabla_{x_1} f_N^{(1)} = (N-1)\varepsilon^{d-1} \int_{\mathbb{S}^{d-1}\times\mathbb{R}^d} f_N^{(2)}(t, x_1, v_1, x_1 + \varepsilon\omega, v_2) \times \big((v_2 - v_1)\cdot\omega\big)\, dv_2 d\omega\,,$$

where for $1 \le s \le N$ one denotes

$$f_N^{(s)}(t, Z_s) := \int f_N(t, Z_N) \mathbf{1}_{X_N \in \mathcal{D}_N^\varepsilon} dz_{s+1} \dots dz_N\,.$$

The right-hand side can be modified as follows: we split the integral according to

$$\begin{aligned}\int f_N^{(2)}(t, x_1, x_2 + \varepsilon\omega, v_1, v_2)(v_2 - v_1)\cdot\omega\, dv_2 d\omega \\ = \int_{(v_2-v_1)\cdot\omega>0} f_N^{(2)}(t, x_1, v_1, x_2 + \varepsilon\omega, v_2)(v_2 - v_1)\cdot\omega\, dv_2 d\omega \\ + \int_{(v_2-v_1)\cdot\omega<0} f_N^{(2)}(t, x_1, v_1, x_2 - \varepsilon\omega, v_2)(v_2 - v_1)\cdot\omega\, dv_2 d\omega\,,\end{aligned}$$

and in the case when $(v_2 - v_1)\cdot\omega > 0$ (which corresponds to post-collisional configurations), one can use boundary condition (1.5) on f_N to replace the (outgoing) velocities (v_1, v_2) by (incoming) velocities (v_1', v_2') with according to (1.2),

$$v_1' = v_1 + \omega\cdot(v_2 - v_1)\,\omega\,, \quad v_2' = v_2 - \omega\cdot(v_2 - v_1)\,\omega\,.$$

One then obtains the following equation:

$$\partial_t f_N^{(1)} + v_1 \cdot \nabla_{x_1} f_N^{(1)} = \alpha C_{1,2} f_N^{(2)} \tag{1.9}$$

with

$$\begin{aligned}(C_{1,2} f_N^{(2)})(t, x_1, v_1) := (N-1)\varepsilon^{d-1}\alpha^{-1} \int_{\mathbb{S}^{d-1}\times\mathbb{R}^d} \Big(f_N^{(2)}(t, x_1, v_1', x_2 + \varepsilon\omega, v_2') \\ - f_N^{(2)}(t, x_1, v_1, x_2 - \varepsilon\omega, v_2) \Big) \big((v_2 - v_1)\cdot\omega\big)_+ \, dv_2 d\omega \,.\end{aligned} \tag{1.10}$$

This equation is reminiscent of the Boltzmann equation (B): we recall that taking the limit $N \to \infty$ we assume that the factor $(N-1)\varepsilon^{d-1}\alpha^{-1}$ in (1.10) converges to 1, and moreover if the function $f_N^{(2)}$ is continuous, then

$$f_N^{(2)}(t, x_1, v_1, x_2 + \varepsilon\omega, v_2) \sim f_N^{(2)}(t, x_1, v_1, x_2, v_2) \,.$$

The main, crucial difference between the equation on $f_N^{(1)}$ and (B) lies in the fact that there is no reason in general for $f_N^{(2)}$ to be a product of $f_N^{(1)}$. Assuming nevertheless that when N goes to infinity, the following asymptotics hold:

$$f_N^{(1)}(t, z_1) \sim f(t, z_1) \quad \text{and} \quad f_N^{(2)}(t, z_1, z_2) \sim f(t, z_1) f(t, z_2) \tag{1.11}$$

then plugging this Ansatz into (1.9,1.10) the function f does satisfy formally the Boltzmann equation (B).

Assumption (1.11) is wrong for a fixed N because of the interactions between particles. However as N goes to infinity, the chaos property (1.11) can be shown to hold asymptotically. To make the above argument rigorous, the main difficulty is to prove the propagation of chaos, namely that the almost factorized structure (1.11) is preserved at time $t > 0$. Actually the strategy of Lanford consists in proving much more, since the actual hierarchy of equations satisfied by the collection of marginals $(f_N^{(s)})_{1\le s\le N}$ is shown to converge, as N goes to infinity under the scaling $N\varepsilon^{d-1}\alpha^{-1} = 1$, to a limit (infinite) hierarchy known as the Boltzmann hierarchy. The wellposedness of both hierachies (a prequisite to the convergence) ensures that if the initial data looks like a tensor product, meaning

$$f_{N|t=0}(Z_N) = \frac{1}{\mathcal{Z}_N} \prod_{i=1}^N f^0(z_i) \mathbf{1}_{X_N \in \mathcal{D}_N^\varepsilon} \,, \quad \mathcal{Z}_N := \int \prod_{i=1}^N f^0(z_i) \mathbf{1}_{X_N \in \mathcal{D}_N^\varepsilon} \, dZ_N \,,$$

then so does the solution asymptotically, meaning that as N goes to infinity, in a sense to be made precise one has

$$f_N^{(s)}(t, Z_s) \sim \prod_{i=1}^s f(t, z_i) \mathbf{1}_{X_s \in \mathcal{D}_s^\varepsilon}$$

and f must satisfy the Boltzmann equation. In the following to simplify notation we set for $1 \le s$,

$$f^{\otimes s}(t, Z_s) := \prod_{i=1}^s f(t, z_i) \,.$$

1.2.2. Statement of the result. Lanford's theorem may be stated as follows; a sketch of proof is presented in Section 2 (for a complete proof see [20]).

Theorem 1.5 (From Particles to Boltzmann equation). *Let $d \geq 2$ be given, and consider a nonnegative continuous function f^0 defined on $\mathbb{T}^d \times \mathbb{R}^d$. Assume that for some μ_0 in $\mathbb{R}$ and $\beta_0 > 0$,*

$$f^0(x,v) \leq e^{-\mu_0} M_{\beta_0}(v) \quad \text{and} \quad \int_{\mathbb{T}^d \times \mathbb{R}^d} f^0(x,v)\, dxdv = 1\,.$$

There exists a time $T^* > 0$ depending only on μ_0 and β_0 such that the following holds: if f_N solves (1.3) *with initial data*

$$\begin{aligned} f_{N|t=0}(Z_N) &:= \mathcal{Z}_N^{-1} (f^0)^{\otimes N}(Z_N) \mathbf{1}_{X_N \in \mathcal{D}_N^\varepsilon}\,, \\ \mathcal{Z}_N &:= \int (f^0)^{\otimes N}(Z_N) \mathbf{1}_{X_N \in \mathcal{D}_N^\varepsilon}\, dZ_N \end{aligned} \tag{1.12}$$

then for all $1 \leq s$, one has

$$f_N^{(s)}(t, Z_s) \to f^{\otimes s}(t, Z_s) \quad \text{as} \quad N \to \infty \quad \text{with} \quad N\varepsilon^{d-1}\alpha^{-1} = 1\,,$$

locally uniformly in $[0, \alpha^{-1}T^*[\times \Omega_s$, where Ω_s is given by

$$\Omega_s = \{ Z_s \in \mathbb{T}^{sd} \times \mathbb{R}^{sd} \,/\, \forall t \in \mathbb{R}, \forall i \neq j, \quad x_i - x_j - t(v_i - v_j) \neq 0 \}$$

and f solves (B) *with initial data f^0.*

In particular the first marginal does converge, almost everywhere, to the solution of the Boltzmann equation (B).

Remark 1.6. The limiting process entails a loss of information which causes irreversibility: the exact position of the particles is indeed lost, and the deflection angle becomes a random parameter.

In the case when the particles are not initially independent, the convergence still holds as proved in [20], but the asymptotics is generally not described by a closed equation on the first marginal. Under suitable assumptions (bounds and convergence) on the initial marginals $(f^{(s)}_{N|t=0})_{1 \leq s \leq N}$, the limiting marginals $(f^{(s)})_{s \geq 1}$ satisfy an infinite hierarchy of equations, referred to as Boltzmann's hierarchy. Particular solutions of this hierarchy are

- the chaotic solutions already mentioned $f^{(s)} = f^{\otimes s}$ with f solution to the full nonlinear Boltzmann equation (B);
- fluctuations describing the dynamics of a tagged particle in a background at equilibrium
$$f^{(s)}(t, Z_s) = M_\beta^{\otimes s}(V_s) \varphi_\alpha(t, z_1)$$
where φ_α is a solution to the *linear Boltzmann equation*:
$$\partial_t \varphi_\alpha + v \cdot \nabla_x \varphi_\alpha = -\alpha L(\varphi_\alpha) \tag{LB}$$
with
$$L(\varphi_\alpha) := \iint [\varphi_\alpha(v) - \varphi_\alpha(v')] M_\beta(v_1)\, b(v - v_1, \omega)\, dv_1 d\omega\,.$$

Note that in both cases the closure of the hierarchy is encoded in the particular form of the initial data.

1.3. From the linear Boltzmann equation to the heat equation. As noticed above, it is difficult to go from particles to fluids via the Boltzmann equation, because Lanford's theorem is only true for times which are a priori not uniformly bounded from below with α (see Theorem 1.5 above). However in the linear setting (LB), global solutions for the limit equation exists. It has been known for a long time that the hydrodynamic limit of (LB) is the heat equation. As stated in [24, 26], L is a Fredholm operator of domain $L^2(\mathbb{R}^d, aM_\beta dv)$ with

$$a(v) := \int_{\mathbb{S}^{d-1}\times\mathbb{R}^d} M_\beta(v_1)\big((v-v_1)\cdot\omega\big)_+ \, d\omega dv_1 \,,$$

and its kernel reduces to the constant functions. We can then define the vector $b(v) = \big(b_k(v)\big)_{k\le d} \in (\mathrm{Ker}\, L)^\perp$ by $Lb_k(v) = v_k$ for all $k \le d$, and the diffusion coefficient

$$\kappa_\beta := \int_{\mathbb{R}^d} v\cdot b(v)\; M_\beta(v) dv \,. \tag{1.13}$$

The following result holds (see for instance [3, 36]).

Theorem 1.7 (From Linear Boltzmann to the heat equation). *Let ρ^0 be a function in $C^4(\mathbb{T}^d)$ and let ρ be the unique, bounded solution to*

$$\partial_\tau \rho - \kappa_\beta \Delta_x \rho = 0 \quad \textit{in} \quad \mathbb{T}^d \,, \quad \rho_{|\tau=0} = \rho^0 \,. \tag{1.14}$$

Let φ_α be the unique solution to (LB) *with initial data $\varphi_{\alpha|t=0} = \rho^0$. Then for all $T > 0$ there is a constant $C_T > 0$ such that*

$$\sup_{\tau\in[0,T]} \sup_{(x,v)\in\mathbb{T}^d\times\mathbb{R}^d} \Big|\big(\varphi_\alpha(\alpha\tau,x,v) - \rho(\tau,x)\big) M_\beta(v)\Big| \le C_T \alpha^{-1} \,.$$

Remark 1.8. The same result holds in the more general case when the initial data $\varphi_{\alpha|t=0}$ to (LB) depends on both variables x and v. In the whole of this text we choose to simplify the presentation by considering only the well-prepared case when $\varphi_{\alpha|t=0}(x,v) = \rho^0(x)$, although the proofs to follow may be adapted to a more general situation $\varphi_{\alpha|t=0} = \varphi^0(x,v)$.

1.4. Statement of the result. In these notes we present a convergence result from an interacting particle system to the heat equation (and the Brownian motion), using the linear Boltzmann equation as an intermediate step. As mentioned in Paragraph 1.2.2, the linear Boltzmann equation can be understood as the limit of the one-particle distribution corresponding to one (or a few) tagged particle in a background of particles initially at equilibrium. The heat equation should therefore be the equation satisfied by the limit of that one-particle distribution, after an adequate rescaling of the time and the density of the background particles. The result proved in [6] is the following.

Theorem 1.9 (From particles to the heat equation). *Consider N hard spheres on the space $\mathbb{T}^d \times \mathbb{R}^d$, initially distributed according to the distribution*

$$\begin{aligned} f_N^0(Z_N) := \mathcal{Z}_N^{-1} \mathbf{1}_{\mathcal{D}_\varepsilon^N}(X_N) \rho^0(x_1) M_\beta^{\otimes N}(V_N) \,, \\ \textit{with} \quad \mathcal{Z}_N := \int \mathbf{1}_{\mathcal{D}_\varepsilon^N}(X_N)\, dX_N \,, \end{aligned} \tag{1.15}$$

where $\rho^0 \leq C_0$ is a continuous, of integral one, function on $\mathbb{T}^d$. Then the distribution $f_N^{(1)}(\alpha\tau, x, v)$ remains close for the L^∞-norm to the solution $\rho(\tau, x)M_\beta(v)$ of the linear heat equation (1.14)*:*

$$\sup_{\tau\in[0,T]}\sup_{(x,v)\in\mathbb{T}^d\times\mathbb{R}^d}\left|f_N^{(1)}(\alpha\tau, x, v) - \rho(\tau, x)M_\beta(v)\right| \to 0$$

in the limit $N \to \infty$, with α going to infinity much slower than $\sqrt{\log\log N}$, and with $N\varepsilon^{d-1}\alpha^{-1} = 1$. In the same asymptotic regime, the process $\Xi(\tau) = x_1(\alpha\tau)$ associated with the tagged particle converges in law towards a Brownian motion of variance κ_β, initially distributed under measure ρ^0.

The long time behavior of a particle in a medium (Lorentz gas, weak interactions...) has been widely studied and we refer to [41] for a survey of the models and results.

In the framework described in this paper, the convergence of $f_N^{(1)}(t, x, v)$ to the solution of the linear Boltzmann equation has been shown to hold in the Boltzmann-Grad limit for any time $t > 0$ in [5, 30]. The convergence in Theorem 1.9 however is quantitative and therefore allows us to obtain controls of the distribution for times $t = \alpha\tau$ diverging with N.

The case of a Lorentz gas is somewhat different in nature, since the tagged particle moves in a frozen background (see [32] for a survey). Many results have been obtained in that direction : see for instance [8, 21] for the convergence of the distribution of the tagged particle to the solution of (LB) and [11] for the convergence to the brownian motion. In the quantum counterpart of the Lorentz gas, the convergence to the quantum Brownian motion has been derived in [17, 18] and these approaches use a truncation of series which is reminiscent of the method explained in Section 3.3 (see also [31]).

2. Proof of Lanford's theorem

In this section we shall give the main steps of the proof of Theorem 1.5. We refer to [20] for all the details.

2.1. The BBGKY hierarchy. We recall that the equation satisfied by the first marginal $f_N^{(1)}$ given in (1.9) involves the second marginal $f_N^{(2)}$. In order to analyze this equation, it is therefore necessary to write the equation satisfied by $f_N^{(2)}$, which involves $f_N^{(3)}$..., and so we are finally naturally led to studying the full hierarchy of equations given formally by

$$\partial_t f_N^{(s)} + V_s \cdot \nabla_{X_s} f_N^{(s)} = \alpha C_{s,s+1} f_N^{(s+1)} \tag{2.1}$$

with

$$\begin{aligned} C_{s,s+1} f_N^{(s+1)}(t, Z_s) := (N-s)\varepsilon^{d-1}\alpha^{-1}\sum_{i=1}^{s}\int_{\mathbb{S}^{d-1}\times\mathbb{R}^d} \omega\cdot(v_{s+1}-v_i) \\ \times f_N^{(s+1)}(t, Z_s, x_i + \varepsilon\omega, v_{s+1}) d\omega dv_{s+1}\,. \end{aligned} \tag{2.2}$$

This operator can be split into a gain and a loss term, depending on the sign of $\omega\cdot(v_{s+1}-v_i)$: we write $C_{s,s+1} = C_{s,s+1}^+ - C_{s,s+1}^-$, where

$$C_{s,s+1}^{\pm} f_N^{(s+1)} = \sum_{i=1}^{s} C_{s,s+1}^{\pm,i} f_N^{(s+1)} \tag{2.3}$$

the index i referring to the index of the interaction particle among the s "fixed" particles, with the notation

$$\begin{aligned}\big(C^{\pm,i}_{s,s+1}f^{(s+1)}_N\big)(Z_s) := (N-s)\varepsilon^{d-1}\alpha^{-1}\int_{\mathbb{S}^{d-1}\times\mathbb{R}^d}(\omega\cdot(v_{s+1}-v_i))_\pm \\ \times f^{(s+1)}_N(Z_s, x_i+\varepsilon\omega, v_{s+1})\,d\omega dv_{s+1}\,,\end{aligned} \tag{2.4}$$

the index $+$ corresponding to post-collisional configurations and the index $-$ to pre-collisional configurations. This hierarchy of equations is known as the BBGKY hierarchy, after N. Bogoliubov [7], M. Born, and H. S. Green [9], J. G. Kirkwood [27] and J. Yvon [45].

2.2. The Boltzmann hierarchy. From the BBGKY hierarchy presented in the previous paragraph, we can formally derive the limiting hierarchy, referred to as Boltzmann's hierarchy. Consider a set of particles $Z_{s+1} = (Z_s, x_i+\varepsilon\omega, v_{s+1})$ such that (x_i, v_i) and $(x_i+\varepsilon\omega, v_{s+1})$ are post-collisional: $(x_{s+1}-x_i)\cdot(v_{s+1}-v_i) > 0$. We recall the boundary condition (1.5)

$$f^{(s+1)}_N(t, Z_s, x_i+\varepsilon\omega, v_{s+1}) = f^{(s+1)}_N(t, Z^*_s, x_i+\varepsilon\omega, v^*_{s+1})$$

where $Z^*_s = (z_1,\dots,z^*_i,\dots z_s)$, $x^*_i := x_i$ and (v^*_i, v^*_{s+1}) are the pre-collisional velocities:

$$v^*_i := v_i - \omega\cdot(v_i-v_{s+1})\,\omega\,,\quad v^*_{s+1} := v_{s+1} + \omega\cdot(v_i-v_{s+1})\,\omega\,.$$

Then neglecting the spatial micro-translations in the arguments of $f^{(s+1)}_N$ we formally obtain from (2.4) the following asymptotic expression for the collision operator at the limit:

$$C^0_{s,s+1} = C^{0+}_{s,s+1} - C^{0-}_{s,s+1}\,,$$

with

$$\begin{aligned}C^{0+}_{s,s+1}f^{(s+1)}(t,Z_s) &:= \sum_{i=1}^s\int\big(\omega\cdot(v_{s+1}-v_i)\big)_+ \\ &\quad\times f^{(s+1)}(t,x_1,v_1,\dots,x_i,v^*_i,\dots,x_s,v_s,x_i,v^*_{s+1})d\omega dv_{s+1}\,,\\ C^{0-}_{s,s+1}f^{(s+1)}(t,Z_s) &:= \sum_{i=1}^s\int\big(\omega\cdot(v_{s+1}-v_i)\big)_+ f^{(s+1)}(t,Z_s,x_i,v_{s+1})d\omega dv_{s+1}\,.\end{aligned}$$

At this stage, the Boltzmann hierarchy is introduced as the formal limit of the BBGKY hierarchy and the core of Lanford's strategy is to justify the convergence. Note also that the Boltzmann hierarchy involves an infinite number of recursive equations for the functions $\{f^{(s)}\}_{s\geq 1}$, as opposed to the BBGKY hierarchy which couples only the density marginals up to N.

2.3. The iterated Duhamel formula. In order to prove the convergence of $f^{(s+1)}_N$ to $f^{(s+1)}$ for a fixed s let us write the solutions $f^{(s+1)}_N$ and $f^{(s+1)}$ by Duhamel's formula. Denoting by $\mathbf{\Psi}_s(t)$ the s-particle flow associated with the hard-sphere system, and by $\mathbf{T}_s$ the associated solution operator, we have formally

$$f^{(s)}_N(t) = \mathbf{T}_s(t)f^{(s)}_{N|t=0} + \int_0^t \mathbf{T}_s(t-\tau)C_{s,s+1}f^{(s+1)}_N(\tau)\,d\tau\,.$$

Since we distinguish between pre-collisional and post-collisional configurations, we expect the initial data to play a special role. We therefore iterate the previous Duhamel formula to express the solution to the BBGKY hierarchy as an operator acting on the initial data :

$$f_N^{(s)}(t) = \sum_{k=0}^{N-s} \alpha^k \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} \mathbf{T}_s(t-t_1) C_{s,s+1} \mathbf{T}_{s+1}(t_1-t_2) \dots \\ \dots \mathbf{T}_{s+k}(t_k) f^{(s+k)}_{N|t=0} \, dt_k \dots dt_1 \, . \tag{2.5}$$

Similarly the solution to the Boltzmann hierarchy can be recast as

$$f^{(s)}(t) = \sum_{k=0}^{\infty} \alpha^k \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} \mathbf{S}_s(t-t_1) C^0_{s,s+1} \mathbf{S}_{s+1}(t_1-t_2) C^0_{s+1,s+2} \cdots \\ \dots \mathbf{S}_{s+k}(t_k) f^{(s+k)}_{|t=0} \, dt_k \dots dt_1 \, , \tag{2.6}$$

where $\mathbf{S}_s(t)$ denotes the s-particle free-flow.

The goal is now to prove the convergence from (2.5) to (2.6) as N goes to infinity with $N\varepsilon^{d-1}\alpha^{-1} = 1$ and $1 \leq s$ is fixed. Several points need to be adressed:

1. the convergence of both series (2.5) and (2.6) over k (uniformly in N);
2. the convergence of the initial data $f^{(s+k)}_{N|t=0}(Z_{s+k})$ to $f^{(s+k)}_{|t=0}(Z_{s+k})$;
3. the convergence of the collision operators $C_{s,s+1}$ to $C^0_{s,s+1}$;
4. the convergence of the transport operators $\mathbf{T}_s$ to $\mathbf{S}_s$.

Point 2 is not totally obvious due to the singularities induced by the conditioning associated to the exclusion in $f^{(s+k)}_{N|t=0}$. However, defining

$$\mathcal{Z}_N := \int_{\mathbb{R}^{2dN}} \mathbf{1}_{X_N \in \mathcal{D}_N^\varepsilon} (f^0)^{\otimes N}(Z_N) \, dZ_N \, ,$$

standard arguments lead to

$$1 \leq \mathcal{Z}_N^{-1} \mathcal{Z}_{N-s} \leq \left(1 - C\varepsilon \|f^0\|_{L^\infty(\mathbb{R}^d_x, L^1(\mathbb{R}^d_v))}\right)^{-s} , \tag{2.7}$$

and this estimate leads to the expected convergence outside the diagonals.

Point 3 was formally studied in the Paragraph 2.2 and we shall not detail this argument further. Note that the continuity along the normal vector to the boundary (and hence the definition of the trace at the boundary) is obtained recursively by construction of the elementary terms of the series as combinations of collision and transport operators applied to the initial data. Continuity of the initial data is f^0 required in order to prove that the effects of the spatial micro-translations in the collisions will be negligible. In the next two paragraphs we shall concentrate on the more difficult points 1. (Paragraph 2.4) and 4. (Paragraph 2.5).

2.4. Uniform bounds. In order to obtain uniform a priori bounds for solutions to the BBGKY and Boltzmann hierarchies, we need to introduce some norms on the space of sequences $(g^{(s)})_{s\geq 1}$. These norms, although not exactly equivalent, are inspired from the

ensemble formalism in statistical physics. At the canonical level, given $\varepsilon > 0$, $\beta > 0$, an integer $s \geq 1$, and a measurable function $g_s : \mathcal{D}_s^\varepsilon \times \mathbb{R}^{ds} \to \mathbb{R}$, we let

$$|g_s|_{\varepsilon,s,\beta} := \text{supess}_{Z_s \in \mathcal{D}_s^\varepsilon \times \mathbb{R}^{ds}} \left(|g_s(Z_s)| \exp\left(\frac{\beta}{2}|V_s|^2\right) \right) . \tag{2.8}$$

We also define, for a continuous function $g_s : \mathbb{T}^{ds} \times \mathbb{R}^{ds} \to \mathbb{R}$,

$$|g_s|_{0,s,\beta} := \sup_{Z_s \in \mathbb{T}^{ds} \times \mathbb{R}^{ds}} \left(|g_s(Z_s)| \exp\left(\frac{\beta}{2}|V_s|^2\right) \right) .$$

Next we denote by $X_{\varepsilon,s,\beta}$ the Banach space of measurable functions from $\mathcal{D}_s^\varepsilon \times \mathbb{R}^{ds}$ to $\mathbb{R}$ with finite $|\cdot|_{\varepsilon,s,\beta}$ norm, and similarly $X_{0,s,\beta}$ denotes the Banach space of continuous functions from $\mathbb{T}^{ds} \times \mathbb{R}^{ds}$ to $\mathbb{R}$ with finite $|\cdot|_{0,s,\beta}$ norm. At the grand-canonical level, for sequences of functions $G = (g_s)_{s\geq 1}$, with $g_s : \mathcal{D}_s^\varepsilon \times \mathbb{R}^{ds} \to \mathbb{R}$, we let for $\varepsilon > 0$, $\beta > 0$, and $\mu \in \mathbb{R}$,

$$\|G\|_{\varepsilon,\beta,\mu} := \sup_{s\geq 1} \left(|g_s|_{\varepsilon,s,\beta} \exp(\mu s) \right) .$$

We define similarly for $G = (g_s)_{s\geq 1}$, with $g_s : \mathbb{T}^{ds} \times \mathbb{R}^{ds} \to \mathbb{R}$ continuous,

$$\|G\|_{0,\beta,\mu} := \sup_{s\geq 1} \left(|g_s|_{0,s,\beta} \exp(\mu s) \right) .$$

Finally we denote $\mathbf{X}_{\varepsilon,\beta,\mu}$ the Banach space of sequences of functions $G = (g_s)_{s\geq 1}$, with $g_s \in X_{\varepsilon,s,\beta}$ and $\|G\|_{\varepsilon,\beta,\mu} < \infty$ and similarly $\mathbf{X}_{0,\beta,\mu}$ is the Banach space of sequences of continuous functions $G = (g_s)_{s\geq 1}$, with $g_s \in X_{0,s,\beta}$ and $\|G\|_{0,\beta,\mu} < \infty$.

The conservation of energy for the s-particle flow is reflected in identities

$$|\mathbf{T}_s(t) g_s|_{\varepsilon,s,\beta} = |g_s|_{\varepsilon,s,\beta} \quad \text{and} \quad |\mathbf{S}_s(t) h_s|_{0,s,\beta} = |h_s|_{0,s,\beta} ,$$

for all parameters $\beta > 0$, $\mu \in \mathbb{R}$, and for all $g_s \in X_{\varepsilon,s,\beta}$ and $h_s \in X_{0,s,\beta}$.

The collision operators $C_{s,s+1}$ and $C^0_{s,s+1}$ on the other hand involve a linear loss in s and in the velocity variable, since one can check that for almost every $t > 0$, and almost everywhere in Z_s,

$$\begin{aligned} &\left| \left(\mathbf{T}_s(-t) C_{s,s+1} \mathbf{T}_{s+1}(t) g_{s+1} \right)(Z_s) \right| \\ &\qquad \leq C \beta^{-\frac{d}{2}} \Big(s\beta^{-\frac{1}{2}} + \sum_{1\leq i\leq s} |v_i| \Big) e^{-\frac{\beta}{2}|V_s|^2} |g_{s+1}|_{\varepsilon,s+1,\beta} , \end{aligned} \tag{2.9}$$

and

$$\left| \left(C^0_{s,s+1} g_{s+1} \right)(Z_s) \right| \leq C \beta^{-\frac{d}{2}} \Big(s\beta^{-\frac{1}{2}} + \sum_{1\leq i\leq s} |v_i| \Big) e^{-\frac{\beta}{2}|V_s|^2} |g_{s+1}|_{0,s+1,\beta} . \tag{2.10}$$

As pointed out above, in order to make sense of the trace at the boundary it is necessary to study $\mathbf{T}_s(-t) C_{s,s+1} \mathbf{T}_{s+1}(t)$ and not the operator $C_{s,s+1}$ alone (see [40] for a detailed discussion).

The idea behind analytical type results is to compensate the loss of continuity in (2.9) and (2.10) (giving rise typically to a factor $s(s+1)\cdots(s+k-1)$ in the elementary terms

of the Duhamel expansions (2.5) and (2.6)) by the successive time integrations (leading to a factor $t^k/k!$). We then expect the series in k to be convergent for small values of αt. More precisely, it follows from rather standard arguments of the Cauchy-Kowalewski type (see [34] or [43]) that for an initial data bounded in $\mathbf{X}_{\varepsilon,\beta_0,\mu_0}$ then the solution to the BBGKY hierarchy at time t is bounded in $\mathbf{X}_{\varepsilon,\beta_0-c\alpha t,\mu_0-c\alpha t}$ for some fixed $c>0$, as long as $\beta_0 - c\alpha t > 0$. A similar result holds for the Boltzmann hierarchy: if the initial data is bounded in $\mathbf{X}_{0,\beta,\mu}$ then the solution to the Boltzmann hierarchy at time t is bounded in $\mathbf{X}_{0,\beta_0-c\alpha t,\mu_0-c\alpha t}$, as long as $\beta_0 - c\alpha t > 0$. This explains why the Lanford theorem only holds for a short time in general: it is the time for which one can guarantee a uniform bound for all the terms in the hierarchy. We shall call $[0, T^*/\alpha]$ this life span from now on (where T^* depends only on β_0 and μ_0).

Remark 2.1. Actually the precise estimates of [20] show that T^* is essentially proportional to $\exp(\mu_0)$, which controls the weighted norm $|f^0|_{0,1,\beta}$. This corresponds typically to the life span we would obtain for the quadratic Boltzmann equation (B) developing a simple L^∞ theory.

2.5. Termwise convergence. From now on we fix T^* as obtained in the previous section and we consider a time $t \leq T/\alpha$ with $T < T^*$. We shall prove the termwise convergence of each marginal to the solution of the limit hierarchy.

2.5.1. Series truncation, cut-off of high energies and of clustering collision times. The bounds obtained in the previous paragraph imply by the dominated convergence theorem that it is enough to consider finite sums of elementary functions

$$f^{(s,k)}_{N,R,\delta}(t) := \alpha^k \int_{\mathcal{T}_{k,\delta}(t)} \mathbf{T}_s(t-t_1) C_{s,s+1} \mathbf{T}_{s+1}(t_1-t_2) C_{s+1,s+2} \dots \\ \dots \mathbf{T}_{s+k}(t_k) \mathbf{1}_{|V_{s+k}|\leq R} f^{(s+k)}_{N|t=0}\, dT_k\,,$$

$$f^{(s,k)}_{R,\delta}(t) := \alpha^k \int_{\mathcal{T}_{k,\delta}(t)} \mathbf{S}_s(t-t_1) C^0_{s,s+1} \mathbf{S}_{s+1}(t_1-t_2) C^0_{s+1,s+2} \dots \\ \dots \mathbf{S}_{s+k}(t_k) \mathbf{1}_{|V_{s+k}|\leq R} f^{(s+k)}_{|t=0}\, dT_k\,.$$

where R^2 is a cut-off on the high energies and we have defined

$$\mathcal{T}_k(t) := \Big\{ T_k = (t_1,\dots,t_k) \,/\, t_i < t_{i-1} \ \text{ with } \ t_{k+1} = 0 \ \text{ and } \ t_0 = t \Big\}\,,$$

$$\mathcal{T}_{k,\delta}(t) := \Big\{ T_k \in \mathcal{T}_k(t) \,/\, t_i - t_{i+1} \geq \delta \Big\}\,.$$

Indeed defining

$$f^{(s,k)}_N(t) := \alpha^k \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} \mathbf{T}_s(t-t_1) C_{s,s+1} \mathbf{T}_{s+1}(t_1-t_2) C_{s+1,s+2} \dots \\ \dots \mathbf{T}_{s+k}(t_k) f^{(s+k)}_{N|t=0}\, dt_k \dots dt_1$$

$$f^{(s,k)}(t) := \alpha^k \int_0^t \int_0^{t_1} \dots \int_0^{t_{n-1}} \mathbf{S}_s(t-t_1) C^0_{s,s+1} \mathbf{S}_{s+1}(t_1-t_2) C^0_{s+1,s+2} \dots$$

$$\dots \mathbf{S}_{s+k}(t_k) f^{(s+k)}_{|t=0}\, dt_k \dots dt_1\,,$$

one can check that for each given $s \geq 1$ and $t \in [0, T/\alpha]$ there is a constant $C_s > 0$ depending only on β_0, μ_0 and s such that for each $n \geq 1$,

$$\begin{aligned}
&\big\| f^{(s)}_N(t) - \sum_{k=0}^{n} f^{(s,k)}_N(t) \big\|_{L^\infty(\mathcal{D}^\varepsilon_s \times \mathbb{R}^{ds})} + \big\| f^{(s)}(t) - \sum_{k=0}^{n} f^{(s,k)}(t) \big\|_{L^\infty(\mathbb{T}^{ds} \times \mathbb{R}^{ds})} \\
&+ \big\| \sum_{k=0}^{n} (f^{(s,k)}_N - f^{(s,k)}_{N,R,\delta})(t) \big\|_{L^\infty(\mathcal{D}^\varepsilon_s \times \mathbb{R}^{ds})} + \big\| \sum_{k=0}^{n} (f^{(s,k)} - f^{(s,k)}_{R,\delta})(t) \big\|_{L^\infty(\mathbb{T}^{ds} \times \mathbb{R}^{ds})} \\
&\qquad \leq C_s \left(\frac{1}{2}\right)^n + C e^{-C' \beta_0 R^2} + C n^2 \frac{\delta \alpha}{T}
\end{aligned}$$

uniformly in N and $t \leq T/\alpha$, in the Boltzmann-Grad scaling $N\varepsilon^{d-1}\alpha^{-1} = 1$. Theorem 1.5 will therefore follow from the convergence of the elementary functions.

2.5.2. Straightening of trajectories. The main step of the proof now consists in decomposing the previous truncated functions according to the history of collisions: we write

$$f^{(s,k)}_{N,R,\delta}(t) = \sum_{J,S} \Big(\prod_{i=1}^{k} j_i \Big) f^{(s,k)}_{N,R,\delta}(t, J, S) \quad \text{and} \quad f^{(s,k)}_{R,\delta}(t) = \sum_{J,S} \Big(\prod_{i=1}^{k} j_i \Big) f^{(s,k)}_{R,\delta}(t, J, S)$$

with

$$\begin{aligned}
f^{(s,k)}_{N,R,\delta}(t, J, S) := \alpha^k \int_{\mathcal{T}_{k,\delta}(t)} &\mathbf{T}_s(t - t_1) C^{j_1,\sigma_1}_{s,s+1} \mathbf{T}_{s+1}(t_1 - t_2) C^{j_2,\sigma_2}_{s+1,s+2} \\
&\dots \mathbf{T}_{s+k}(t_k - t_{k+1}) \mathbf{1}_{|V_{s+k}| \leq R} f^{(s+k)}_{N|t=0} dT_k\,, \\
f^{(s,k)}_{R,\delta}(t, J, S) := \alpha^k \int_{\mathcal{T}_{k,\delta}(t)} &\mathbf{S}_s(t - t_1) C^{0,j_1,\sigma_1}_{s,s+1} \mathbf{S}_{s+1}(t_1 - t_2) C^{0,j_2,\sigma_2}_{s+1,s+2} \\
&\dots \mathbf{S}_{s+k}(t_k - t_{k+1}) \mathbf{1}_{|V_{s+k}| \leq R} f^{(s+k)}_{|t=0} dT_k\,,
\end{aligned}$$

where $J := (j_1, \dots, j_k) \in \{+, -\}^k$ and the $\pm$ signs were introduced in (2.3) to distinguish incoming from outgoing collisions, while $S := (\sigma_1, \dots, \sigma_k)$ with σ_i in $\{1, \dots, s + i - 1\}$ is the label of the particle colliding with particle $s + i$.

Each one of the functionals $f^{(s,k)}_{N,R,\delta}(t, J, S)$ and $f^{(s,k)}_{R,\delta}(t, J, S)$ can be viewed as the contribution associated with some dynamics, which of course is not the actual dynamics in physical space: the characteristics associated with the operators $\mathbf{T}_{s+i}(t_i - t_{i+1})$ and $\mathbf{S}_{s+i}(t_i - t_{i+1})$ are followed backwards in time between two consecutive times t_i and t_{i+1}, and collision terms (associated with $C^{j_i,\sigma_i}_{s+i,s+i+1}$ and $C^{0,j_i,\sigma_i}_{s+i,s+i+1}$) are seen as source terms, in which, in the words of Lanford [29], "additional particles" are "adjoined" to the marginal. These dynamics are therefore referred to as "pseudo-trajectories".

The end of the proof of Theorem 1.5 consists in straightening out the BBGKY pseudo-trajectories, for them to become asymptotically close to the Boltzmann pseudo-trajectories (which are straight lines between each collision time t_i and t_{i+1}). This is the most technical part of the proof, as between two collision times t_i and t_{i+1}, the BBGKY pseudo-trajectories are not always straight lines since recollisions may occur. These recollisions are eliminated

recursively: when a new particle $s+i$ is adjoined at time t_i, given the other particles (numbered from 1 to $s+i-1$), it is possible to choose the velocity and impact parameter of that new particle $s+i$ in a set of almost full measure as N goes to infinity so that after collision or scattering with particle (this depends on whether the particle is incoming or outgoing), the set of $s+i$ particles will stay at a prescribed distance ε_0 one from another for all times $t \leq t_i - \delta$. The main point here is that this geometric argument needs to be applied only a finite number of times since the series has been truncated. It is also important at this stage that velocities are not too big, and that collision times do not cluster. The previous preparation steps are therefore crucial here.

We shall not present the details of the construction, which is rather long and technical, but to give a flavor of the argument let us state one typical geometric result which plays an important role in the proof. In the following we denote by dist the distance on the torus.

Lemma 2.2. *Let $x_1, x_2 \in \mathbb{T}^d$ be given such that $\mathrm{dist}(x_1, x_2) \geq \varepsilon_0 \gg \varepsilon$, and a velocity v_1 such that $|v_1| \leq R < \infty$. Given $\delta, t > 0$, there is a set $K(x_1 - x_2)$ of small measure:*

$$|K(x_1 - x_2)| \leq CR^d \left(\left(\frac{\varepsilon}{\varepsilon_0} \right)^{d-1} + \left(\frac{\varepsilon_0}{R\delta} \right)^{d-1} + \left(Rt \right)^d \varepsilon_0^{d-1} \right)$$

such that for any velocity $v_2 \notin (v_1 + K(x_1 - x_2))$, with $|v_2| \leq R$, then

(i) *there is no collision over $[0,t]$ by the backward flow: for any $\tau \in [0,t]$, one has $\mathrm{dist}(x_1 - v_1\tau, x_2 - v_2\tau) > \varepsilon$;*

(ii) *the particles are well separated after a time δ: for all times $\tau \in [\delta, t]$, there holds $\mathrm{dist}(x_1 - v_1\tau, x_2 - v_2\tau) > \varepsilon_0$.*

The parameter ε_0 ensures that the pseudo-trajectories are separated and therefore do not recollide. Result (ii) is the main point enabling one to proceed with an inductive proof : with large probability, the pseudo-trajectories in both hierarchies can be coupled and will remain very close to each other up to time 0. At time 0, the cloud of particles will have positions almost identical in both hierarchies up to small shifts of order $n\varepsilon$. As the initial densities $f^{(n)}_{N|t=0}$ and $f^{(n)}_{|t=0}$ are very close in the large N limit, the small shift of the particles can be bounded by using the gradient norm $\|\nabla_x f_0\|_{L^\infty}$.

2.6. Conclusion of the proof. Optimizing the parameters of the estimates obtained in the previous sections

$$n \sim C_1 |\log \varepsilon| \,, \quad R^2 \sim C_2 |\log \varepsilon|$$

for some sufficiently large constants C_1 and C_2, and

$$\delta = \varepsilon^{(d-1)/(d+1)} \,, \quad \varepsilon_0 = \varepsilon^{d/(d+1)}$$

we find that the total error is smaller than

$$\left\| f^{(s)}(t) - f^{(s)}_N(t) \right\|_{L^\infty(K)} \leq C\varepsilon^b, \qquad \text{for any } b < \frac{d-1}{d+1} \cdot$$

This ends the proof of Theorem 1.5.

3. Long-time asymptotics of a tagged particle

In this section, we sketch the proof of Theorem 1.9 on the diffusive behavior of the tagged particle (see [6] for details of the proof). We shall mainly focus on the first part of Theorem 1.9 which states that the density obeys the heat equation after rescaling. The convergence to the Brownian motion is a strengthening of this result which shows that the rescaled increments of the position become independent in the large N limit.

3.1. Main result. As explained in the introduction (Theorem 1.7), the heat equation can be recovered from (LB) in some large time limit. Thus our goal is to prove that for an initial data close to equilibrium (1.15), the time obtained in Lanford's theorem (Theorem 1.5) can be improved up to a time diverging with N and that the solution of the linear Boltzmann equation remains a good approximation of the tagged particle density over such long times. This is the content of the following Theorem from which Theorem 1.9 can be deduced by applying Theorem 1.7.

Theorem 3.1. *Consider N hard spheres on $\mathbb{T}^d \times \mathbb{R}^d$, initially distributed according to* (1.15). *Then the distribution $f_N^{(1)}(t,x,v)$ of the tagged particle is close to the solution $M_\beta(v) \times \varphi_\alpha(t,x,v)$ of the linear Boltzmann equation* (LB) *with initial data ρ^0, in the sense that for all $\alpha > 1$, in the limit $N \to \infty$, $N\varepsilon^{d-1}\alpha^{-1} = 1$, one has*

$$\left\| f_N^{(1)}(t,x,v) - M_\beta(v)\varphi_\alpha(t,x,v) \right\|_{L^\infty([0,\alpha T]\times\mathbb{T}^d\times\mathbb{R}^d)} \leq C \left[\frac{T\alpha^2}{(\log\log N)^{\frac{A-1}{A}}} \right]^{\frac{A^2}{A-1}} \tag{3.1}$$

where $A \geq 2$ can be taken arbitrarily large. The constant C depends on A and on the upper bound C_0 on the initial data ρ^0.

The proof of Theorem 3.1 is the main goal of this section. We shall rely extensively on the arguments used to derive Theorem 1.5 and show that close to equilibrium, they remain valid for macroscopic time scales up to $o\big(\frac{\log\log N}{\alpha^2}\big)$. This is achieved by using L^∞ bounds which provide a uniform control in time of the densities and allow us to truncate large collision trees in the Duhamel series.

3.2. Invariant measure and maximum principle. Let $M_{N,\beta}$ be the invariant Gibbs measure for the hard sphere dynamics

$$M_{N,\beta}(Z_N) := \bar{\mathcal{Z}}_N^{-1} \mathbf{1}_{\mathcal{D}_\varepsilon^N}(X_N) M_\beta^{\otimes N}(V_N)\,, \quad \text{with} \quad \bar{\mathcal{Z}}_N := \int \mathbf{1}_{\mathcal{D}_\varepsilon^N}(X_N)\, dX_N\,.$$

The initial data given by (1.15) satisfies

$$f_N^0(Z_N) \leq C_0 M_{N,\beta}\,.$$

Since $M_{N,\beta}$ is invariant, the maximum principle implies that this bound remains valid at any time $t > 0$ and the marginals are uniformly bounded in time

$$\sup_t f_N^{(s)}(t,Z_s) \leq C_0 M_{N,\beta}^{(s)}(Z_s) \leq C_0 C^s M_\beta^{\otimes s}(V_s)\,,$$

where the last inequality follows from an argument similar to the one leading to (2.7). Thus the weighted norms (2.8) are uniformly bounded in time

$$\forall t > 0, \qquad |f_N^{(s)}(t)|_{\varepsilon,s,\beta} \leq C_0\, C^s. \tag{3.2}$$

These bounds are a key step to control the size of the collision trees and to show that large collision trees have vanishing probability. Indeed compared to Section 2.4, these estimates imply a global control of the solution in the space $\mathbf{X}_{\varepsilon,\beta,\mu}$ with no deterioration on the parameters β and $\mu = \log C$ with time.

3.3. Removing large collision trees. We are going to show that the contribution of large collision trees in the Duhamel series can be neglected. The time interval $[0,t]$ is split into K intervals of time length h, where h is a parameter to be chosen small enough and $K = t/h$ will be large. A collision tree is said to be admissible (see Figure 3) if it has less than $n_k = A^k$ branching points on the time interval $[t-kh, t-(k-1)h]$, where A is the constant in the inequality (3.1), which will be chosen large. The growth of the admissible collision trees is therefore controlled and we are going to show that the other collision trees do not contribute to the Duhamel series.

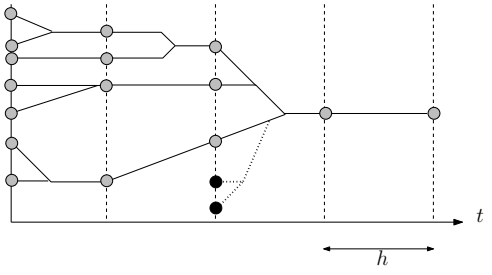


Figure 3. The collision tree depicted in the figure is not admissible for $A = 2$ because there are more than 2^2 collisions during the second time interval. If the black particles were not involved in the collision tree, then the tree would be admissible.

Defining

$$Q_{s,s+n}(t) := \alpha^n \int_0^t \int_0^{t_1} \dots \int_0^{t_{n-1}} \mathbf{S}_s(t-t_1) C_{s,s+1} \mathbf{S}_{s+1}(t_1-t_2) C_{s+1,s+2} \dots \mathbf{S}_{s+n}(t_n)\, dt_n \dots dt_1 \,, \tag{3.3}$$

one can write (2.5) as

$$f_N^{(s)}(t) = \sum_{n=0}^{N-s} Q_{s,s+n}(t) f_N^{(s+n)}(0) \,.$$

In particular, the marginal associated to the tagged particle density $f_N^{(1)}(t)$ can be decomposed as

$$f_N^{(1)}(t) = f_N^{(1,K)}(t) + R_N^K(t) \,, \tag{3.4}$$

where the contribution of the admissible trees is

$$f_N^{(1,K)}(t) := \sum_{m_1=0}^{n_1-1} \dots \sum_{m_K=0}^{n_K-1} Q_{1,M_1}(h) Q_{M_1,M_2}(h) \dots Q_{M_{K-1},M_K}(h)\, f_N^{0(M_K)}$$

and the error term accounts for the contribution of the large trees

$$R_N^K(t) := \sum_{k=1}^{K} \sum_{m_1=0}^{n_1-1} \cdots \sum_{m_{k-1}=0}^{n_{k-1}-1} Q_{1,M_1}(h) \dots Q_{M_{k-2},J_{k-1}}(h) \\ R_{M_{k-1},n_k}(t-kh, t-(k-1)h)\,,$$

with

$$R_{k,n}(t',t) := \int_{t'}^{t} \int_{t'}^{t_1} \cdots \int_{t'}^{t_{n-1}} \mathbf{T}_k(t-t_1) C_{k,k+1} \mathbf{T}_{k+1}(t_1-t_2) C_{k+1,k+2} \cdots \\ \cdots C_{k+n-1,k+n} f_N^{(k+n)}(t_n)\, dt_n \dots dt_1\,.$$

and where we have defined $M_k := 1 + \sum_{i=1}^{k} m_i$. Note that $f_N^{(1,K)}(t)$ is evaluated in terms of the initial data, instead in each term of $R_N^K(t)$ the Duhamel formula is iterated only up to the first time interval $[t-kh, t-(k-1)h]$ where more than $n_k = A^k$ collisions occur.

From the upper bound (2.9) on the collision operator, one can deduce a continuity estimate in terms of the weighted norms (2.8)

$$|Q_{s,s+n}(h) f_{s+n}|_{\varepsilon,s,\frac{\beta}{2}} \leq e^{s-1} \left(C_{d,\beta}\, \alpha\, h\right)^n |f_{s+n}|_{\varepsilon,s+n,\beta}$$

where $C_{d,\beta}$ is a constant. The uniform bound in time on the densities (3.2) enables us to bound from above the collision operators when too many collisions occur on a short time interval h. Choosing $h = \gamma \alpha^{-A/(A-1)}\, t^{-1/(A-1)}$, this leads to an upper bound on the remainder

$$\left\| R_N^K(t) \right\|_{L^\infty(\mathbb{T}^d \times \mathbb{R}^d)} \leq C \gamma^A\,. \tag{3.5}$$

Similar computations lead to a similar decomposition for the Boltzmann hierarchy

$$f^{(1)}(t) = f^{(1,K)}(t) + R^K(t) \quad \text{with} \quad \left\| R^K(t) \right\|_{L^\infty(\mathbb{T}^d \times \mathbb{R}^d)} \leq C \gamma^A. \tag{3.6}$$

Thus the dominant contribution in the decompositions (3.4) and (3.6) is given by the functions $f_N^{(1,K)}(t)$ and $f^{(1,K)}(t)$. To conclude the proof of Theorem 3.1, it remains to show that
$f_N^{(1,K)}(t)$ and $f^{(1,K)}(t)$ are close to each other.

3.4. Conclusion of the proof of Theorem 3.1. Each term of the sum in $f_N^{(1,K)}(t)$ can be shown to converge to the corresponding term in $f^{(1,K)}(t)$ by arguments identical to those developed in Section 2.5 to neglect the influence of the recollisions. Indeed the contribution of a collision tree with s collisions in the BBGKY hierarchy will be close to the corresponding contribution in the Boltzmann hierarchy with an error of order $t^s \varepsilon^b$ (with $b < \frac{d-1}{d+1}$) if no recollision of the pseudo-trajectories occur. This error term is small because the collision trees have been truncated in order to contain less than A^K particles and K can be chosen much smaller than $\log\log(\varepsilon)/\log A$ by tuning $\gamma \simeq \frac{(\alpha^2 T)^{A/(A-1)}}{\log\log N}$. As the remainder $R_N^K(t)$ in (3.5) can be controlled as well in terms of γ, the proof of Theorem 3.1 is complete. The parameter A can be chosen arbitrarily large.

3.5. Convergence to the Brownian motion. We turn now to the second part of Theorem 1.9 and prove the convergence in law of the tagged particle to a Brownian motion.

The first marginal of the Boltzmann hierarchy can be interpreted as the distribution of a single particle $(\bar{x}(t), \bar{v}(t))$ interacting with an ideal gas at density α and temperature $1/\beta$. This particle changes direction at random times of order $1/\alpha$ due to collisions. Rephrased in probalistic terms, the velocity $\{\bar{v}(t)\}_{t\geq 0}$ is a continuous Markov process with generator given by the operator αL associated to the linear Boltzmann equation. When the density α of the background gas increases, the frequency of collisions increases by α. Thus after a time $\alpha\tau$, the particle has encountered $\alpha^2\tau$ random kicks which is the correct rescaling to observe a diffusive behaviour at the macroscopic scale (τ, x) when α diverges. The position of the tagged particle $\bar{x}(\alpha\tau) = \bar{x}(0) + \displaystyle\int_0^{\alpha\tau} \bar{v}(s)\, ds$ is an additive functional of this Markov chain taking values in $\mathbb{T}^d$. We consider the rescaled process $\bar{x}(\alpha\tau)$ taking values in the torus $\mathbb{T}^d$. Since L has a spectral gap, the invariance principle holds for the ideal tracer $\bar{x}(\alpha\tau)$ (see [28] Theorem 2.32 page 74) which converges to a Brownian motion. The Maxwellian distribution M_β is the invariant measure of this process and the diffusion coefficient κ_β (1.13) can be recovered as the variance of the position for any coordinate $k \leq d$

$$\kappa_\beta = \mathbb{E}_{M_\beta}\left[\bar{v}_k\, L^{-1} \bar{v}_k\right] .$$

This implies the convergence of the rescaled finite dimensional marginals towards the ones of the brownian motion B with variance κ_β, i.e. that for any smooth functions $\{\psi_i\}_{i\leq \ell}$ taking values in $\mathbb{T}^d$ and times $\tau_1 < \tau_2 < \cdots < \tau_\ell$

$$\lim_{\alpha\to\infty} \mathbb{E}\Big(\psi_1\big(\bar{x}(\alpha\tau_1)\big) \dots \psi_\ell\big(\bar{x}(\alpha\tau_\ell)\big)\Big) = \mathbb{E}\Big(\psi_1\big(B(\tau_1)\big) \dots \psi_\ell\big(B(\tau_\ell)\big)\Big). \tag{3.7}$$

We have shown that the first particle in the Boltzmann hierarchy behaves as a Markov chain. We turn now to the convergence of the rescaled tagged particle $\Xi(\tau) = x_1(\alpha\tau)$ to a brownian motion when N and $\alpha \ll \sqrt{\log\log N}$ are diverging (with $N\varepsilon^{d-1}\alpha^{-1} = 1$). For this, one needs to check (see [10]) :

- the convergence of the marginals of the tagged particle sampled at different times $\tau_1 < \tau_2 < \cdots < \tau_\ell$

$$\lim_{N\to\infty} \mathbb{E}\Big(\psi_1\big(\Xi(\tau_1)\big) \dots \psi_\ell\big(\Xi(\tau_\ell)\big)\Big) = \mathbb{E}\Big(\psi_1\big(B(\tau_1)\big) \dots \psi_\ell\big(B(\tau_\ell)\big)\Big) . \tag{3.8}$$

- the tightness of the sequence, i.e. that is for any $\tau \in [0, T]$

$$\forall \delta > 0, \qquad \lim_{\eta\to 0} \lim_{N\to\infty} \mathbb{P}\left(\sup_{\tau<s<\tau+\eta} \big|\Xi(s) - \Xi(\tau)\big| \geq \delta\right) = 0 . \tag{3.9}$$

We sketch below the main steps for the convergence of the time marginals (3.8). The tighness follows by similar comparison arguments (see [6]). As for the convergence of the tagged particle density to the heat equation, we proceed by comparison of the microscopic dynamics with the Boltzmann hierarchy and conclude by using the limit (3.7). We fix $\Psi_\ell = \{\psi_1, \dots, \psi_\ell\}$ a collection of continuous functions in $\mathbb{T}^d$. The density at time t of the tagged particle $f^{(1)}_{N,\Psi_\ell}(t)$ weighted by Ψ_ℓ is defined for any test function Φ as

$$\int_{\mathbb{T}^d\times\mathbb{R}^d} dZ_1 f^{(1)}_{N,\Psi_\ell}(t, Z_1)\Phi\big(Z_1(t)\big) = \mathbb{E}\Big(\psi_1\big(x_1(t_1)\big) \dots \psi_\ell\big(x_1(t_\ell)\big)\, \Phi\big(Z_1(t)\big)\Big)$$

$$= \int_{\mathbb{T}^{Nd}\times\mathbb{R}^{Nd}} dZ_N f_N(0,Z_N)\ \psi_1\big(x_1(t_1)\big)\dots\psi_\ell\big(x_1(t_\ell)\big)\,\Phi\big(Z_1(t)\big).$$

The Duhamel formula can be applied to rewrite $f^{(1)}_{N,\Psi_\ell}(t)$ as a series

$$f^{(1)}_{N,\Psi_\ell}(t) = \sum_{m_1+\dots+m_\ell=0}^{N-1} Q_{1,1+m_1}(t-t_\ell)\Big(\psi_\ell Q_{1+m_1,1+m_1+m_2}(t_\ell-t_{\ell-1})\Big(\psi_{\ell-1}\dots$$
$$Q_{1+m_1+\dots+m_{\ell-1},1+m_1+\dots+m_\ell}(t_1)\Big) f_N^{(m_1+\dots+m_\ell+1)}(0), \tag{3.10}$$

where the operator $Q_{n,m}$ was introduced in (3.3). Note that this reformulation of the Duhamel series is close in spirit to the one introduced in [30] to encode the trajectory of the tagged particle. An analogous Duhamel formula holds for the density of the first particle in the Boltzmann hierarchy. Thus a coupling of the trajectories in both hierarchies (similar to the one used in section 3.4) shows that

$$\lim_{N\to\infty} \mathbb{E}\Big(\psi_1\big(\bar{x}(\alpha\tau_1)\big)\dots\psi_\ell\big(\bar{x}(\alpha\tau_\ell)\big)\Big) - \mathbb{E}\Big(\psi_1\big(\Xi(\tau_1)\big)\dots\psi_\ell\big(\Xi(\tau_\ell)\big)\Big) = 0\,.$$

This implies the convergence of the finite dimensional time-marginals (3.8) and ends the proof of Theorem 1.9.

References

[1] R. Alexander, *The Infinite Hard Sphere System*, Ph.D. dissertation, Dept. Mathematics, Univ. California, Berkeley, 1975.

[2] _____, *Time Evolution for Infinitely Many Hard Spheres*, Commun. in Math. Phys. **49** (1976), 217–232.

[3] C. Bardos, R. Santos, and R. Sentis, *Diffusion approximation and computation of the critical size*, Trans. AMS (2) **284** (1984), 617–649.

[4] C. Bardos and S. Ukai, *The classical incompressible Navier-Stokes limit of the Boltzmann equation*, Math. Models and Methods in the Appl. Sci. **1** (1991), 235–257.

[5] H. van Beijeren, O.E. III Lanford, J.L Lebowitz, and H. Spohn, *Equilibrium Time Correlation Functions in the Low Density Limit*, Jour. Stat. Phys. **22** (1980), 237–257.

[6] T. Bodineau, I. Gallagher, and L. Saint-Raymond, *The Brownian motion as the limit of a deterministic system of hard-spheres*, in revision at Inventiones Mathematicae.

[7] N. Bogoliubov, *Problems of Dynamical Theory in Statistical Physics*, in Studies in Statistical Mechanics, J. de Boer and G.E. Uhlenbeck, Ed. Interscience, New York, 1962.

[8] C. Boldrighini, L.A. Bunimovich, and Y. Sinai, *On the Boltzmann equation for the Lorentz gas*, Journal of statistical physics, **32** (3) (1983), 477–501.

[9] M. Born and H. S. Green, *A general kinetic theory of liquids. I. The molecular distribution functions*, Proc. Roy. Soc. London. Ser. A. **188** (1946), 10–18.

[10] P. Billingsley, *Probability and measure*, Wiley Series in Probability and Mathematical Statistics, John Wiley & Sons, 1995.

[11] L. Bunimovich and Ya. Sinai, *Statistical properties of Lorentz gas with periodic configuration of scatterers*, Comm. Math. Phys. **78** (4), (1980/81), 479–497.

[12] S. Chapman and T.G. Cowling, *The mathematical theory of non-uniform gases: An account of the kinetic theory of viscosity, thermal conduction, and diffusion in gases*, Cambridge University Press, New York, 1960.

[13] C. Cercignani, V. I. Gerasimenko, and D. I. Petrina, *Many-Particle Dynamics and Kinetic Equations*, Kluwer Academic Publishers, Netherlands, 1997.

[14] C. Cercignani, R. Illner, and M. Pulvirenti, *The Mathematical Theory of Dilute Gases*, Springer Verlag, New York NY, 1994.

[15] R. DiPerna and P.-L. Lions, *On the Cauchy problem for Boltzmann equations: global existence and weak stability*, Ann. of Math. (2) **130**(2) (1989), 321–366.

[16] D. Enskog. *Kinetische Theorie der Vorgänge in mässig verdünten Gasen.* Uppsala : Almqvist & Wiksell, 1917

[17] L. Erdős, *Lecture notes on quantum Brownian motion*, Quantum Theory from Small to Large Scales: Lecture Notes of the Les Houches Summer School, **95**, Oxford University Press (2010).

[18] L. Erdős, M. Salmhofer, and H. T. Yau, *Quantum diffusion of the random Schrodinger evolution in the scaling limit*, Acta mathematica **200**(2) (2008), 211–278.

[19] R. Esposito, R. Marra, and H.T. Yau, *Navier-Stokes equations for stochastic particle systems on the lattice*, Comm. Math. Phys. **182** (1996), 395–456.

[20] I. Gallagher, L. Saint-Raymond, and B. Texier, *From Newton to Boltzmann : the case of hard-spheres and short-range potentials*, Zürich Lecture Adv. Math., European Mathematical Society, 2013.

[21] G. Gallavotti, *Rigorous theory of the Boltzmann equation in the Lorentz gas*, Università di Roma, Instituto di Fisica, Nota Interna **358** (1972).

[22] F. Golse and L. Saint-Raymond, *The Incompressible Navier-Stokes Limit of the Boltzmann Equation for Hard Cutoff Potentials*, J. Math. Pures Appl. **91** (2009), 508–552.

[23] H. Grad, *On the kinetic theory of rarefied gases*, Comm. Pure Appl. Math. **2** (1949), 331–407.

[24] ———, *Asymptotic theory of the Boltzmann equation II*, Rarefied Gas Dynamics (Proc. of the 3rd Intern. Sympos. Palais de l'UNESCO, Paris, 1962) Vol. I, 26–59.

[25] D. Hilbert, *Sur les problèmes futurs des mathématiques*, in Compte-Rendu du 2ème Congrès International de Mathématiques, Gauthier-Villars, Paris (1902), 58–114.

[26] ———, *Begründung der kinetischen Gastheorie*, Math. Ann. **72** (1912), 562–577.

[27] J. G. Kirkwood, *The Statistical Mechanical Theory of Transport Processes I. General Theory*, Journal of Chemical Physics **14** (1946), 180–202.

[28] T. Komorowski, C. Landim, and S. Olla, *Fluctuations in Markov processes. Time symmetry and martingale approximation*, Grundlehren der Mathematischen Wissenschaften, **345** Springer, Heidelberg, 2012.

[29] O. E. Lanford, *Time evolution of large classical systems*, Lect. Notes in Physics **38**, J. Moser ed., 1–111, Springer Verlag (1975).

[30] J. Lebowitz and H. Spohn, *Steady state self-diffusion at low density*, J. Statist. Phys. **29** (1982), 39–55.

[31] J. Lukkarinen and H. Spohn, *Weakly nonlinear Schrödinger equation with random initial data*, Inventiones mathematicae **183** (1) (2011), 79–188.

[32] J. Marklof, *Kinetic transport in crystals*, Proceedings of the XVI International Congress on Mathematical Physics, Prague 2009, World Scientific 2010, 162–179.

[33] T. Nishida, *Fluid dynamical limit of the nonlinear Boltzmann equation to the level of the compressible Euler equation*, Comm. Math. Phys. **61** (1978), 119–148.

[34] ———, *A note on a theorem by Nirenberg*, Journal of Differential Geometry, **12** (1977), 629–633.

[35] S. Olla, S. Varadhan, and H.-T. Yau, *Hydrodynamical limit for a Hamiltonian system with weak noise*, Commun. Math. Phys. **155** (1993), 523–560.

[36] G. Papanicolaou, *Asymptotic analysis of transport processes*, Bull. AMS **81** (1975), 330–392.

[37] M. Pulvirenti, C. Saffirio, and S. Simonella, *On the validity of the Boltzmann equation for short range potentials*, arXiv:1301.2514.

[38] J. Quastel and H.-T. Yau, *Lattice gases, large deviations, and the incompressible Navier-Stokes equations*, Ann. of Math. **148** (1998), 51–108.

[39] L. Saint-Raymond, *Hydrodynamic limits of the Boltzmann equation*, Lecture Notes in Mathematics, Springer-Verlag **1971**, 2009.

[40] S. Simonella, *Evolution of correlation functions in the hard sphere dynamics*, arXiv:1205.2789.

[41] H. Spohn, *Large scale dynamics of interacting particles*, Springer-Verlag **174**, 1991.

[42] K. Uchiyama, *Derivation of the Boltzmann equation from particle dynamics*, Hiroshima Math. J. **18** (1988), no. 2, 245–297.

[43] S. Ukai, *The Boltzmann-Grad Limit and Cauchy-Kovalevskaya Theorem*, Japan J. Indust. Appl. Math. **18** (2001), 383–392.

[44] C. Villani, *Limites hydrodynamiques de l'équation de Boltzmann*, Sém. Bourbaki 2000/01, Exp 893, Astérisque **282** (2002), 365–405.

[45] J. Yvon, *La théorie statistique des fluides et l'équation d'état*, Actual. Sci. et Indust. **203** (Paris, Hermann), 1935.

CNRS & École Polytechnique, Centre de Mathématiques Appliquées
E-mail: Thierry.Bodineau@polytechnique.edu

Université Paris-Diderot, Institut de Mathématiques de Jussieu-Paris Rive Gauche
E-mail: gallagher@math.univ-paris-diderot.fr

Université Pierre et Marie Curie & École Normale Supérieure, Département de Mathématiques et Applications
E-mail: Laure.Saint-Raymond@ens.fr

The mathematical analysis of black holes in general relativity

Mihalis Dafermos

Abstract. The mathematical analysis of black holes in general relativity has been the focus of considerable activity in the past decade from the perspective of the theory of partial differential equations. Much of this work is motivated by the problem of understanding the two celebrated cosmic censorship conjectures in a neighbourhood of the Schwarzschild and Kerr solutions. Recent progress on the behaviour of linear waves on black hole exteriors as well as on the full non-linear vacuum dynamics in the black hole interior puts us at the threshold of a complete understanding of the stability–and instability–properties of these solutions. This talk will survey some of these developments.

Mathematics Subject Classification (2010). Primary 83C57; Secondary 83C75.

Keywords. Einstein equations, general relativity, black holes, cosmic censorship.

1. Introduction

There is perhaps no other object in all of mathematical physics as fascinating as the black holes of Einstein's general relativity.

The notion as such is simpler than the mystique surrounding it may suggest! Loosely speaking, the black hole region $\mathcal{B}$ of a Lorentzian 4-manifold $(\mathcal{M}, g)$ is the complement of the causal past of a certain distinguished ideal boundary at infinity, denoted $\mathcal{I}^+$ and known as *future null infinity*; in symbols

$$\mathcal{B} = \mathcal{M} \setminus J^-(\mathcal{I}^+). \tag{1.1}$$

In the context of general relativity, where our physical spacetime continuum is modelled by such a manifold $\mathcal{M}$, this ideal boundary at infinity $\mathcal{I}^+$ corresponds to "far-away" observers in the radiation zone of an isolated self-gravitating system such as a collapsing star. Thus, the black hole region $\mathcal{B}$ is the set of those spacetime events which cannot send signals to distant observers like us.

It is remarkable that the simplest non-trivial spacetimes $(\mathcal{M}, g)$ solving the Einstein equations in vacuum

$$\mathrm{Ric}(g) = 0, \tag{1.2}$$

the celebrated *Schwarzschild* and *Kerr* solutions, indeed contain non-empty black hole regions $\mathcal{B} \neq \emptyset$. Moreover, both these spacetimes fail to be future causally geodesically complete, i.e. in physical language, there exist freely falling observers who live for only finite

Proceedings of the International Congress of Mathematicians, Seoul, 2014

proper time. The two properties are closely related in the above examples as all such finitely-living observers must necessarily enter the black hole region $\mathcal{B}$. Far-away observers in these examples, on the other hand, live forever; the asymptotic boundary future null infinity $\mathcal{I}^+$ is itself complete.

In the early years of the subject, the black hole property was widely misunderstood and the incompleteness of the above spacetimes was considered a pathology that would surely go away after perturbation. The latter expectation was shattered by Penrose's celebrated *incompleteness theorem* [68] which implies in particular that the incompleteness of Schwarzschild and Kerr is in fact a stable feature when viewed in the context of dynamics. We have now come to understand the presence of black holes not at all as a pathology but rather as a blessing, shielding the effects of incompleteness from distant observers, allowing in particular for a complete future null infinity $\mathcal{I}^+$. This motivated Penrose to formulate an ambitious conjecture known as *weak cosmic censorship* which states that for generic initial data for the Einstein vacuum equations (1.2), future null infinity $\mathcal{I}^+$ is indeed complete. In the language of partial differential equations, this can be thought of as a form of *global existence* still compatible with Penrose's theorem.

A positive resolution of the above conjecture would be very satisfying but would still not resolve all conceptual issues raised by the Schwarzschild and Kerr solutions. For it is reasonable to expect that our physical theory should explain the fate not just of far-away observers but of *all* observers, including those who choose to enter black hole regions $\mathcal{B}$. In the exact Schwarzschild case, such observers are destroyed by infinite tidal forces, while in the exact Kerr case, they cross a *Cauchy horizon* to live another day in a region of spacetime which is no longer determined by initial data. The former scenario is an omenous prediction indeed–but one we have come to terms with. It is the latter which is in some sense even more troubling, as it represents a failure of the notion of prediction itself. This motivates yet another ambitious conjecture, *strong cosmic censorship*, also originally due to Penrose, which says that for generic initial data for (1.2), the part of spacetime uniquely determined by data is inextendible. In the language of partial differential equations, this conjecture can be thought of as a statement of *global uniqueness*. For this conjecture to be true, the geometry of the interior region of Kerr black holes would in particular have to be unstable.

Despite the ubiquity of black holes in our current astrophysical world-picture, the above conjectures–even when restricted to a neighbourhood of the explicit solutions Schwarzschild and Kerr–are not mathematically understood. More specifically, we can ask the following stability and instability questions concerning the Schwarzschild and Kerr family:

1. *Are the exteriors to the black hole regions $\mathcal{B}$ in Schwarzschild and Kerr stable under the evolution of (1.2) to perturbation of data? In particular, is the completeness of null infinity $\mathcal{I}^+$ a stable property?*
2. *What happens to observers who enter the interior of the black hole region $\mathcal{B}$ of such perturbations of Kerr? Are the smooth Cauchy horizons of Kerr unstable?*

If our optimistic expectations on these questions are in fact not realised by the theory, then this may fundamentally change our understanding of general relativity and perhaps also our belief in it!

The global analysis of solutions to the Einstein vacuum equations (1.2) without symmetry was largely initiated in the monumental proof [23] of the non-linear stability of Minkowski space by Christodoulou and Klainerman in 1993. As with the stability of Minkowski space,

Question 1. would be a statement of global existence and stability, but now concerning a highly non-trivial geometry. Question 2., on the other hand, not only concerns a non-trivial geometry but appears to concern a regime where solutions may become unstable and in fact singular (at least, if strong cosmic censorship is indeed true!); the prospect of proving anything about such a regime seemed until recently quite remote. A number of rapid developments in the last few years, however, concerning linear wave equations on black hole backgrounds as well as the analysis of the fully non-linear Einstein equations in singular–but controlled–regimes have brought a complete resolution of Questions 1. and 2. much closer. The purpose of this talk is to survey some of these developments. In particular, we will describe the following results, which reflect the state of the art concerning our understanding of Questions 1 and 2 above, and had themselves been the subject of a number of open conjectures.

1. *Linear scalar waves on Schwarzschild and Kerr backgrounds remain bounded in the black hole exterior and in fact decay polynomially. Schwarzschild is in fact linearly stable in full linearised gravity.*
2. *For a spherically symmetric toy model, Cauchy horizons are globally stable from the point of view of the metric in L^∞, but unstable at the level of derivatives of the metric, as the Christoffel symbols in any regular frame become singular. For the full vacuum equations (1.2) without symmetry, then, given the stability of the exterior, the above stability statement for the Kerr Cauchy horizon again holds.*

We see in particular that the final part of 2. means that the precise understanding of Questions 1. and 2. is in fact coupled. Note that the result 2. is in fact at odds with the strongest formulations of Question 2 above and this has significant–and slightly troubling–implications as to what versions of strong cosmic censorship are indeed true. This could indicate that some of the conceptual puzzles of general relativity are here to stay!

2. Schwarzschild and Kerr

We begin by reviewing the Schwarzschild and Kerr families.

2.1. The Schwarzschild metric. The *Schwarzschild family* $(\mathcal{M}, g_M)$ represents the simplest non-trivial explicit family of solutions to the Einstein vacuum equations (1.2). These solutions were discovered already in December 1915 [75], the month following Einstein's final formulation of general relativity [43]. The metrics are static and spherically symmetric and can be written in local coordinates as

$$g_M = -(1-2M/r)dt^2 + (1-2M/r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \qquad (2.1)$$

Here, M is a parameter which can be identified with mass. We shall only consider the case $M > 0$. Note that the case $M = 0$ reduces to the flat Minkowski space, which is trivially a solution of (1.2).

In discussing the Schwarzschild solution, we have not yet settled on the ambient manifold $\mathcal{M}$ on which (2.1) should live! Historically, this was indeed only understood later, since the correct differentiable structure of the ambient manifold is not so immediately apparent from

the form (2.1). If we pass, however, to new coordinates (cf. Lemaitre [57]) (t^*, r, θ, ϕ) where

$$t^* = t + 2M \log(r - 2M),$$

we see that the metric expression (2.1) can be rewritten

$$-(1 - 2M/r)(dt^*)^2 + (4M/r)drdt^* + (1 + 2M/r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (2.2)$$

This suggests that we may define our underlying manifold $\widetilde{\mathcal{M}}$ to be precisely

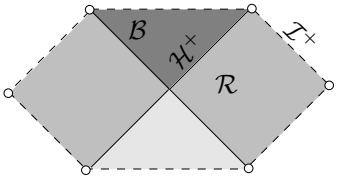
$$\widetilde{\mathcal{M}} = (-\infty, \infty) \times (0, \infty) \times \mathbb{S}^2 \quad (2.3)$$

with coordinates t^*, r, θ, ϕ, on which g_M defined by (2.2) manifestly yields a smooth metric. Let us for now consider $(\widetilde{\mathcal{M}}, g_M)$ as our spacetime.

One easily sees from the form of the metric (2.2) that the region $\mathcal{B} \doteq \{r \leq 2M\}$ has the property that future directed causal curves emanating from $\mathcal{B}$ must stay in $\mathcal{B}$ (i.e. $J^+(\mathcal{B}) = \mathcal{B}$), in particular, they cannot reach large values of r. It turns out that with a suitable definition of the asymptotic boundary future null infinity $\mathcal{I}^+$, $\mathcal{B}$ corresponds also to the black hole region defined in (1.1), and $\mathcal{I}^+$ is moreover complete.[1] The boundary $\mathcal{H}^+ = \{r = 2M\}$ of $\mathcal{B}$ in the spacetime $\mathcal{M}$ is known as the *event horizon*. Note that the static Killing field ∂_t of (2.1) extends to a Killing field ∂_{t*} on $\mathcal{M}$ which is in fact spacelike in the region $\{r < 2M\}$ and null on $\mathcal{H}^+$.

In contrast to the case of Minkowski space $M = 0$ where the above metric (2.2) extends from (2.3) to $\mathbb{R}^{3+1}$ by adding $r = 0$ to the manifold, in the case $M > 0$, the metric becomes singular as $r \to 0$ is approached. In fact, $\{r = 0\}$ can be attached as a spacelike *singular boundary* to which all *future*-incomplete causal geodesics approach. This shows that the manifold $\widetilde{\mathcal{M}}$ is future-inextendible as a suitably regular Lorentzian manifold. It is not, however, *past*-inextendible. It turns out that one can define an even larger ambient manifold $\mathcal{M}$ (by suitably pasting $\widetilde{\mathcal{M}}$ to a copy of itself) so as for (2.2) above to extend to a spherically symmetric solution of (1.2) which is now indeed also past-inextendible. This gives the so-called *maximally extended Schwarzschild solution* $(\mathcal{M}, g)$. See [56, 78]. In what follows, it is this $(\mathcal{M}, g)$ that we shall definitively refer to as the Schwarzschild manifold.

Note that this new manifold $(\mathcal{M}, g)$ does not admit r as a global coordinate, but can be covered by a global system of double null coordinates (U, V) whose range can be normalised to the following shaded bounded subregion $\mathcal{Q}$ of the plane $\mathbb{R}^{1+1}$:



The metric takes the form

$$-\Omega^2(U, V)dUdV + r^2(U, V)(d\theta^2 + \sin^2\theta d\phi^2)$$

[1] This means that if we define a null retarded time coordinate u such that $\partial_u r = -1$ asymptotically at $\mathcal{I}^+$, then $\mathcal{I}^+$ is covered by the u-range $(-\infty, \infty)$.

where Ω and r can be described implicity. The above depiction is known as a *Carter–Penrose diagram* of $(\mathcal{M}, g)$, and gives a concrete realisation of both future null infinity $\mathcal{I}^+$ (as an open constant U-segment of the boundary of $\mathcal{Q}$ in the ambient $\mathbb{R}^{1+1}$) and the singular $\{r = 0\}$ past and future boundaries.

Note that the above manifold is *globally hyperbolic* with a Cauchy hypersurface Σ (possessing two asymptotically flat ends). That is to say, all inextendible causal curves intersect Σ exactly once. When we discuss dynamics in Section 3, this property will allow us to view Schwarzschild $(\mathcal{M}, g)$ as the maximal vacuum Cauchy development of data on Σ.

2.2. The Kerr metrics. The Schwarzschild family sits as the 1-parameter $a = 0$ subfamily of a larger, 2-parameter family $(\mathcal{M}, g_{M,a})$, discovered in 1963 by Kerr [52]. The parameter a can be identified with rotation. The latter metrics are less symmetric when $a \neq 0$–they are only *stationary* and *axisymmetric*–and are given explicitly in local coordinates by the expression

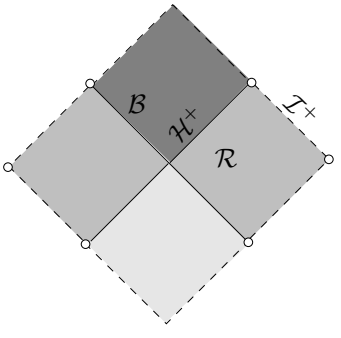
$$g_{M,a} = -\frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\phi\right)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left(a\,dt - (r^2+a^2)d\phi\right)^2 \tag{2.4}$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta, \qquad \Delta = r^2 - 2Mr + a^2.$$

We will only consider the case of parameter values $0 \leq |a| < M$, $M > 0$, where $\Delta = (r - r_+)(r - r_-)$ for $r_+ > r_- > 0$. The case $|a| = M$ is special and is known as the *extremal* case.

Again, by introducing $t^* = t^*(t, r)$ but now also a change $\phi^* = \phi^*(\phi, r)$, the metric can be rewritten in analogy to (2.2) so as to make it regular at $r = r_+$, which will again correspond to the event horizon $\mathcal{H}^+$ of a black hole $\mathcal{B}$. An additional transformation can now make the metric regular at $r = r_-$ and allows a further extension into $r < r_-$. The set $r = r_-$ will correspond to a so-called *Cauchy horizon* $\mathcal{CH}^+$ separating a globally hyperbolic region from part of the spacetime which is no longer determined by Cauchy data. Our convention will be to <u>not</u> include the latter extensions into our ambient manifold $\mathcal{M}$, which will, however, as in Schwarzschild, be "doubled" by appropriately pasting two $r > r_-$ regions. For us, the Kerr spacetime $(\mathcal{M}, g_{M,a})$ will thus again be globally hyperbolic with a two-ended asymptotically flat Cauchy hypersurface Σ as in the Schwarzschild case, and, in the language of Section 3, will again be the maximal vacuum Cauchy development of data on Σ. See



It is, however, precisely the existence of these further extensions to $r < r_-$ which leads to the question of strong cosmic censorship.

The Kerr solutions are truly remarkable objects with a myriad of interesting geometric properties beyond the mere fact of the presence of a black hole region $\mathcal{B}$, for instance, their having a non-trivial ergoregion $\mathcal{E}$ to be discussed in Section 4.2.1. Even the very existence in closed form of the family is remarkable, since simply imposing the symmetries manifest in the above expression (2.4) is by dimensional considerations clearly insufficient to ensure that the Einstein equations (1.2) should admit closed-form solutions. It turns out that the metrics (2.4) enjoy several "hidden" symmetries. For instance, they possess an additional non-trivial Killing *tensor* and they are moreover *algebraically special*. It is in fact through the latter property that they were originally discovered [52].

2.3. Uniqueness. A natural question that arises is whether there are other *stationary* solutions of (1.2) containing black holes $\mathcal{B}$ besides the Kerr family $g_{M,a}$.

If we impose in addition that our solutions be *axisymmetric* then indeed, the Kerr family represents the unique family of black hole solutions (with a connected horizon). See [11, 72] for the original treatments and also [24].

The expectation that the Kerr solutions are unique even without imposing axisymmetry stems from a pretty rigidity argument due to Hawking [47]. Under certain assumptions, including the real analyticity of the metric, he showed that stationary black holes are necessarily also axisymmetric, and thus, the above result applies to infer uniqueness.

The assumption of real analyticity is physically unmotivated, however, and leaves open the possibility that there may yet still be other smooth (but non-analytic) black hole solutions of (1.2). An important partial result has recently been proven in [1], where it is shown (generalising Hawking's rigidity argument using methods of unique continuation) that the Kerr family is indeed unique in the smooth class *provided one restricts to stationary spacetimes suitably near the Kerr family.* In particular, this means that the Kerr family is at the very least isolated in the family of all stationary solutions.

In view of this latter fact, it indeed makes sense to focus on the Kerr family, in particular, to entertain the question of its "asymptotic stability". Before turning to this, however, we must first make some general comments about dynamics for the Einstein equations (1.2).

3. Dynamics of the Cauchy problem

One of the early triumphs of the theory of partial differential equations applied to general relativity was the proof that the Einstein equations (1.2) indeed give rise to an unambiguous notion of *dynamics*. In the language of partial differential equations, this corresponds to the *well-posedness* of the Cauchy problem for (1.2), proven by Choquet-Bruhat [13] and Choquet-Bruhat–Geroch [14].

We will state the foundational well-posedness statement as Theorem 3.1 of Section 3.1 below. We will then proceed in Sections 3.2 and 3.3 to illustrate global aspects of the problem of dynamics with the statement of the stability of Minkowski space and with the formulation of the cosmic censorship conjectures, already mentioned in the introduction. This will prepare us for our study of the dynamics of black holes in Sections 4 and 5.

3.1. Well-posedness. Before formulating the well-posedness theorem, we must first understand what constitutes an initial state. In view of the fact that the Einstein equations (1.2) are second order, one expects to prescribe initially a triple $(\Sigma^3, \bar{g}, K)$, where $(\Sigma^3, \bar{g})$ is a Riemannian 3-manifold and K is an auxiliary symmetric 2-tensor to represent the second fundamental form. We say that a Lorentzian 4-manifold $(\mathcal{M}, g)$ is a *vacuum Cauchy development* of $(\Sigma^3, \bar{g}, K)$ if $(\mathcal{M}, g)$ solves (1.2) and there exists an embedding $i : \Sigma \to \mathcal{M}$ such that $i(\Sigma)$ is a Cauchy hypersurface[2] in $\mathcal{M}$ and $\bar{g}$ and K are indeed the induced metric and second fundamental form of the embedding.

The classical Gauss and Codazzi equations of submanifold geometry immediately imply the following *necessary* conditions on $(\Sigma^3, \bar{g}, K)$ for the existence of such an embedding:

$$\bar{R} + (\mathrm{tr}K)^2 - |K|^2_{\bar{g}} = 0, \qquad \overline{\mathrm{div}}K - d\,\mathrm{tr}K = 0. \tag{3.1}$$

We will thus call a triple $(\Sigma^3, \bar{g}, K)$ satisfying (3.1) a *vacuum initial data set*. In her seminal [13], Choquet-Bruhat proved that for regular $(\Sigma^3, \bar{g}, K)$, the conditions (3.1) are also *sufficient* for the existence of a development and for a local uniqueness statement. In the langauge of partial differential equations, this is the analogue of *local well posedness*.

We are all familiar from the theory of ordinary differential equations that local existence and uniqueness immediately yields the existence of a unique *maximal* solution $x : (-T_-, T_+)$, where $-\infty \leq T_- < T_+ \leq +\infty$. In general relativity, maximalising Choquet-Bruhat's local statement is non-trivial as there is not a common ambient structure on which all solutions are defined so as for them to be readily compared. Such a maximalisation was obtained in

Theorem 3.1 (Choquet-Bruhat–Geroch [14]). *Let $(\Sigma^3, \bar{g}, K)$ be a smooth vacuum initial data set. Then there exists a unique smooth vacuum Cauchy development $(\mathcal{M}, g)$ with the property that if $(\widetilde{M}, \widetilde{g})$ is any other vacuum Cauchy development, then there exists an isometric embedding $i : (\widetilde{\mathcal{M}}, \widetilde{g}) \to (\mathcal{M}, g)$ commuting with the embeddings of Σ.*

The above object $(\mathcal{M}, g)$ is known as the *maximal vacuum Cauchy development.* It is indicative of the trickiness of the maximalisation procedure that the original proof [14] of the above theorem appealed in fact to Zorn's lemma to infer the existence of $(\mathcal{M}, g)$. This made the theorem appear non-constructive, a most unappealing state of affairs in view of its centrality for the theory. A constructive proof has recently been given by Sbierski [73].

For convenience, we have stated Theorem 3.1 in the smooth category, even though it follows from a more primitive result expressed in Sobolev spaces H^s of finite regularity. In the original proofs, this requisite H^s space was high and did not admit a natural geometric interpretation. In a monumental series of papers (see [54]) surveyed in another contribution to these proceedings [79], this regularity has been lowered to $\bar{g} \in H^2$, which can in turn be related to natural geometric assumptions concerning curvature and other quantities.

3.2. Global existence and stability of Minkowski space. With the notion of dynamics well defined, we now turn to the prototype global existence and stability statement, the monumental *stability of Minkowski space* [23].

The result states that small perturbations of trivial initial data 1. lead to geodesically complete maximal vacuum Cauchy developments, with a complete future null infinity $\mathcal{I}^+$

[2] In particular, developments are globally hyperbolic in the sense described at the end of Section 2.1. Global hyperbolicity is essential for the solution to be uniquely determined by data.

and no black holes, 2. remain globally close to Minkowski space and in fact, 3. settle back down asymptotically to Minkowski space:

Theorem 3.2 (Stability of Minkowski space, Christodoulou and Klainerman [23]). *Let $(\Sigma^3, \bar{g}, K)$ be a smooth vacuum initial data set satisfying a global smallness assumption, i.e. suitably close to trivial initial data. Then the maximal vacuum Cauchy development $(\mathcal{M}, g)$ satisfies the following:*

1. *$(\mathcal{M}, g)$ is geodesically complete and moreover, one can attach a boundary $\mathcal{I}^+$ which is itself complete, and $\mathcal{M} = J^-(\mathcal{I}^+)$.*[3]
2. *$(\mathcal{M}, g)$ remains globally close to Minkowski space,*
3. *$(\mathcal{M}, g)$ asymptotically settles down to Minkowski space (at a suitably fast rate).*

In the language of partial differential equationss, the geodesic completeness of statement 1. can be thought of as a geometric formulation of "global existence". Statement 2. then corresponds to "orbital stability" while statement 3. corresponds to "asymptotic stability". Due to the supercriticality of the Einstein equations, the only known mechanism for showing long-time control of a solution is by exploiting its dispersive properties, which here arise due to the radiation of waves to null infinity $\mathcal{I}^+$. As a result, the more primitive statements 1. and 2. can only be obtained in the proof by *using* strong decay rates to flat space, i.e. the full quantitative version of 3. *Thus, the proof of all statements above is strongly coupled.*

The original proof of this theorem has been surveyed in a previous preceedings volume [19] for this conference series. Let us only briefly mention here the central role played by obtaining (in a bootstrap setting) decay of weighted energy quantities associated to the Riemann curvature tensor expressed in a null frame (which satisfies the Bianchi equations) and then coupling these with elliptic and transport estimates for the structure equations satisfied by the connection coefficients, schematically

$$\nabla\!\!\!/\Gamma = \Gamma \cdot \Gamma + \psi, \qquad \nabla\!\!\!/\psi = \not{D}\psi + \Gamma \cdot \psi \tag{3.2}$$

where Γ denotes a generic connection coefficient and ψ denotes a generic curvature component. The problem is especially difficult precisely because the rate of decay of waves to null infinity $\mathcal{I}^+$ is borderline in $3+1$ dimensions. Thus, stability is not true for the generic equation of the degree of nonlinearity of (1.2), but requires identifying special, null-type[4] structure in (3.2). We will return to some of these aspects of the proof when we discuss black holes.

3.3. Penrose's incompleteness theorem and the cosmic censorship conjectures. The explicit examples of Schwarzschild and Kerr indicate that the geodesic completeness of Theorem 3.2 cannot hold for general asymptotically flat data if the global smallness assumption is dropped. In the early years of the subject, one could entertain the hope that this was an artifice of the high degree of symmetry of these special solutions. As mentioned already in the introduction, this was falsified by the following corollary to Penrose's 1965 incompleteness theorem:

[3]Note that the statement $\mathcal{M} = J^-(\mathcal{I}^+)$ represents the fact that these perturbed spacetimes do not contain a non-trivial black hole region $\mathcal{B}$.

[4]In contrast, the classical null condition [53] does not hold when the Einstein equations (1.2) are written in harmonic gauge. See, however, the remarkable proof in [58].

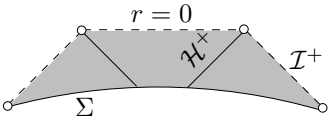
Theorem 3.3 (Corollary of Penrose's incompleteness thoerem [68]). *Let $(\Sigma^3, \bar{g}, K)$ be a smooth vacuum data set sufficiently close to the data corresponding to Schwarzschild or Kerr. Then the maximal vacuum Cauchy development $(\mathcal{M}, g)$ is future causally geodesically incomplete.*

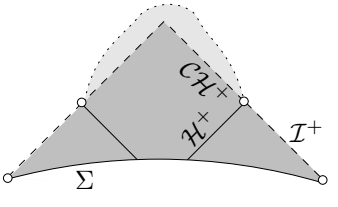
As noted already in the introduction, in the specific examples of Schwarzschild and Kerr, the above incompleteness is "hidden" in black hole regions. That is to say, all finitely-living observers γ must cross $\mathcal{H}^+$ into the region $\mathcal{B}$. In particular, this allows for the asymptotic boundary $\mathcal{I}^+$ to still be complete, cf. the second part of statement 1. of Theorem 3.2. This property is appealing because it means that if one is only interested in far-away observers, one need not further ponder the significance of incompleteness as the theory gives predictions for all time at $\mathcal{I}^+$. This motivates the following conjecture, originally formulated by Penrose, which, if true, would promote this feature to a generic property of solutions to (1.2):

Conjecture 3.4 (Weak cosmic censorship). *For generic asymptotically flat vacuum initial data sets, the maximal vacuum Cauchy devlopment $(\mathcal{M}, g)$ possesses a complete null infinity $\mathcal{I}^+$.*[5]

In the language of partial differential equations, this conjecture can be thought of as the version of *global existence* which is still compatible with Theorem 3.3.

While the above conjecture would indeed explain the possibility of far-away observation for all time, it does not do away with the puzzles opened up by the geodesic incompleteness of Theorem 3.3 from the point of view of fundamental theory. As remarked already, it is reasonable to expect that our theory gives predictions for all observers, not just "far-away" ones. The examples of Schwarzschild and Kerr tell us that the incompleteness of Theorem 3.3 may have very different origin. The Schwarzschild manifold $(\mathcal{M}, g)$ is inextendible in a very strong sense: incomplete geodesics approach what can be thought of as a spacelike singularity corresponding to $r = 0$, and not only do these observers witness infinite curvature but they are torn apart by infinite tidal forces:





Kerr, on the other hand, terminates in what can be viewed as a smooth Cauchy horizon $\mathcal{CH}^+$, across which the solution is *smoothly* extendible to a larger spacetime (the lighter shaded region) which is no longer however uniquely determined from Σ.[6] In the latter case, we see that the maximal Cauchy development is maximal *not because it is inextendible as a smooth*

[5] This particular formulation is due to Christodoulou [18], who in particular, gives a precise general meaning for possessing a complete null infinity. Note also that this conjecture was originally stated without the assumption of generic. The necessity of genericity is to be expected in view of the existence of the spherically symmetric examples [16, 17].

[6] Recall that our conventions on the definition of the ambient Schwarzschild $(\mathcal{M}, g_M)$ and Kerr manifolds $(\mathcal{M}, g_{M,a})$ in Sections 2.1 and 2.2 are precisely so they be the maximal vacuum Cauchy developments of initial data $(\Sigma, \bar{g}, K)$.

solution of (1.2) but because such extensions necessarily fail to be globally hyperbolic and thus cannot be viewed as Cauchy developments.

As explained in the introduction, we have largely come to terms with the former possibility exhibited by Schwarzschild. It gives the theory closure as all observers are accounted for: They either live forever or are destroyed by infinite tidal forces[7]. The implications of the existence of Cauchy horizons, however, as in the Kerr case, would be quite problematic, for it restricts the ability of classical general relativity to predict the fate of macroscopic objects.

The above unattractive feature of Kerr motivated Penrose to formulate his celebrated *strong*[8] *cosmic censorship conjecture*:

Conjecture 3.5 (Strong cosmic censorship). *For generic asymptotically flat vacuum data sets, the maximal vacuum Cauchy development* $(\mathcal{M}, g)$ *is inextendible as a suitably regular Lorentzian manifold.*

The above conjecture can be thought colloquially as saying that *"Generically, the future is determined by initial data"* since the notion of inextendibility captures the idea that there is not a bigger spacetime where the maximal Cauchy development embeds, and which would thus not be uniquely determined by Cauchy data. It can thus be considered, in the language of partial differential equations, to be a statement of *global uniqueness*.

Here the necessity of requiring genericity in the formulation of Conjecture 3.5 is clear from the start. The Kerr solutions do not satisfy the required inextendibility property. Thus, for the above conjecture to be true, this feature of Kerr must be unstable. It is not just wishful thinking that leads to Conjecture 3.5! See Section 5.1.

Finally, let us remark already that the question of how "suitably regular" should be defined in the formulation of Conjecture 3.5 is a subtle one, as will become apparent in view of Section 5.2 below.

4. The stability of the black hole exterior

To make progress on the general understanding of the theory, and in particular, the cosmic censorship conjectures of Section 3.3, we begin by looking at dynamics of (1.2) in a neighbourhood of the Kerr family. With the language of the Cauchy problem developed above, we may now turn to discuss what is one of the central open questions in classical general relativity–the *non-linear stability of the Kerr family* in its exterior region. This represents not only a fundamental test of weak cosmic censorship but a milestone result in itself with important implications for our current working assumption of the ubiquity of objects described by Kerr metrics in our observable universe.

4.1. The conjecture. We begin with a more precise formulation of the conjecture, taken from [29]:

Conjecture 4.1 (Nonlinear stability of the Kerr family). *For all vacuum initial data sets* $(\Sigma, \bar{g}, K)$ *sufficiently "near" data corresponding to a subextremal* ($|a_0| < M_0$) *Kerr metric* g_{a_0,M_0}*, the maximal vacuum Cauchy development spacetime* $(\mathcal{M}, g)$ *satisfies:*

[7] Speculation on what happens to their quantum ashes is beyond the scope of both classical general relativity and this article.

[8] We note that this conjecture is neither stronger nor weaker than Conjecture 3.4. See [18].

1. $(\mathcal{M}, g)$ *possesses a complete null infinity* $\mathcal{I}^+$ *whose past* $J^-(\mathcal{I}^+)$ *is bounded in the future by a smooth affine complete event horizon* $\mathcal{H}^+ \subset \mathcal{M}$,
2. $(\mathcal{M}, g)$ *stays globally close to* g_{a_0,M_0} *in* $J^-(\mathcal{I}^+)$,
3. $(\mathcal{M}, g)$ *asymptotically settles down in* $J^-(\mathcal{I}^+)$ *to a nearby subextremal member of the Kerr family* $g_{a,M}$ *with parameters* $a \approx a_0$ *and* $M \approx M_0$.

We have explicitly excluded the extremal case $|a| = M$ from the conjecture for reasons to be discussed in Section 4.2.5. In particular, the smallness assumption on data will depend on the distance of the initial parameters a_0, M_0 to extremality.

One can compare the above with our formulation of Theorem 3.2. Statement 1. above contains the statement of weak cosmic censorship restricted to a neighbourhood of Schwarzschild. As explained in Section 3.3, in the language of partial differential equations, this is the analogue of "global existence" still compatible with Theorem 3.3. Statement 2. can be thought to represent "orbital stability", whereas statement 3 represents "asymptotic stability". As in our discussion of the proof of the stability of Minkowski space, all these questions are coupled; it is only by identifying and exploiting the dispersive mechanism (i.e. a quantitative version of 3.) that one can show the completeness of null infinity $\mathcal{I}^+$ and orbital stability. In particular, it is essential to identify the final parameters a and M.

Like any non-linear stability result, the first step in attacking the above conjecture is to linearise the equations (1.2) around the Schwarzschild and Kerr solutions. The resulting system of equations is of considerable complexity; we will indeed turn to this in Section 4.3 below. But first, let us discuss what can be thought of a "poor man's" linearisation, namely the study of the *linear scalar wave equation*

$$\Box_g \psi = 0 \tag{4.1}$$

on a fixed Schwarzschild and Kerr background.

4.2. A poor man's stability result: $\Box_g \psi = 0$ on Kerr. The study of (4.1) in the Schwarzschild case goes back to the classic paper of Regge and Wheeler [71] which considered the formal analysis of fixed modes. The first definitive result about actual solutions of (4.1) is due to Kay and Wald [51] and gives that solutions of $\Box_g \psi = 0$ on Schwarzschild arising from regular localised initial data remain uniformly *bounded* in the exterior, up to and including $\mathcal{H}^+$.

The last decade has seen a resurgence in interest in this problem so as to prove not just boundedness but *decay* and to handle not just Schwarzschild but the *general subextremal Kerr* case. Many researchers have contributed to this understanding [2, 5, 7, 32, 33, 36, 44, 81] which progressed from the Schwarzschild case $a = 0$ to the very slowly rotating case $|a| \ll M$ and finally to the general subextremal case $|a| < M$. This programme has culminated in the following result:

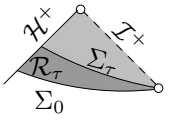
Theorem 4.2 ("Poor man's" linear stability of Kerr [39, 41]). *For Kerr exterior backgrounds in the full subextremal range* $|a| < M$*, general solutions* ψ *of* (4.1) *arising from regular localised data remain bounded and decay at a sufficiently fast polynomial rate through a hyperboloidal foliation of spacetime.*

See also [8, 34, 42, 50] for analysis of the wave equation on (Schwarzschild) Kerr-(anti) de Sitter backgrounds.

A complete survey of the proof of Theorem 4.2 is beyond the scope of this article, but it is worth discussing briefly the salient geometric properties of the Schwarzschild and Kerr families which enter into the analysis.

4.2.1. The conserved energy and superradiance. The existence of conserved energy identities is often crucial for boundedness results. Recall that to every Killing field X^μ, by Noether's theorem, there is a corresponding conserved 1-form associated to solutions ψ of (4.1) formed by contracting X^μ with the energy-momentum tensor $T_{\mu\nu}[\psi] = \partial_\mu\psi\partial_\nu\psi - \frac{1}{2}g_{\mu\nu}\partial^\alpha\psi\partial_\alpha\psi$. If the Killing field is causal, then the flux terms on suitably oriented spacelike or null hypersurfaces are non-negative definite. Let us examine this in the context of our problem.

We first consider the Schwarzschild case $a = 0$. As explained in Section 2.1, the static Killing field ∂_t is then timelike in the black hole exterior, becoming null at the horizon $\mathcal{H}^+$. The associated energy identity applied in a region $\mathcal{R}_\tau$

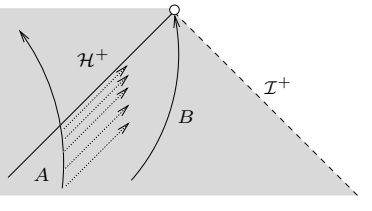


indeed gives nonnegative definite flux terms, and thus yields a useful conservation law for solutions ψ of (4.1)–*but barely!* After obtaining higher order estimates via further commutations of (4.1) by Killing fields and applying the usual Sobolev estimates, this is sufficient to estimate ψ and its derivatives pointwise *away from the horizon*. Since this energy is degenerate where ∂_t becomes null, it is, however, insufficient to obtain uniform pointwise control of the solution and its derivatives *up to and including* $\mathcal{H}^+$. The original boundedness proof of Kay and Wald [51] overcame this problem in a clever manner, but using very fragile structure associated to the exact Schwarzschild metric.

In the Kerr case, for all non-zero values $a \neq 0$, things become much worse. For there is now a region $\mathcal{E}$ in the black hole exterior where the stationary Killing field ∂_t is spacelike! This is known as the *ergoregion*. As a result, the energy flux corresponding to ∂_t is no-longer non-negative definite and thus does not yield even a degenerate global boundedness in the exterior. This is the phenomenon of *superradiance*; there is in particular no *a priori* bound on the flux of radiation to null infinity $\mathcal{I}^+$.

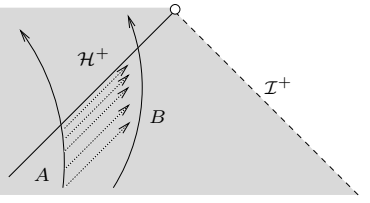
Before understanding how this problem is overcome, we must first discuss two other phenomena, the celebrated *red-shift effect* and the difficulty caused by the presence of *trapped null geodesics*.

4.2.2. The redshift. The *red-shift effect* was first discussed in a paper of Oppenheimer–Snyder [64]. One considers two observers A and B as depicted:



The more adventurous observer A falls in the black hole whereas observer B for all time stays outside. Considering a signal emited by A at a constant frequency according to her watch, in the geometric optics approximation, the frequency of the signal as measured by observer B goes to zero as B's proper time goes to infinity–i.e. it is shifted infinitely *to the red* in the electromagnetic spectrum.

For general sub-extremal black holes, there is a localised version of this effect at the horizon $\mathcal{H}^+$:



If both observers A and B fall into the black hole and are connected by time translation $A = \phi_\tau B$ where ϕ_τ is the Lie flow of the Killing field ∂_t, then the frequency measured by B is shifted to the red by a factor exponential in τ.

It turns out that the above geometric optics argument can be captured by the *coercivity properties* of a physical space energy identity near $\mathcal{H}^+$, corresponding to a well-chosen transversal vector field N to $\mathcal{H}^+$. Such a vector field was introduced in [33] and the construction was generalised in the Epilogue of [38] to arbitrary Killing horizons with positive surface gravity $\kappa > 0$.[9] The good coercivity properties do not hold globally however, and thus to obtain a useful estimate one must combine the energy identity of N with additional information.

In the Schwarzschild case $|a| = 0$, it is precisely the conserved energy estimate discussed in Section 4.2.1 with which one can combine the above red-shift estimate to obtain finally the uniform boundedness of the non-degenerate N-energy. One can moreover further *commute* (4.1) with N preserving the red-shift property at the horizon [37, 38] to again obtain a higher order N-energy estimate, from which then pointwise boundedness follows using standard Sobolev inequalities. This gives a simpler and more robust understanding of Kay and Wald's original [51]. See [38].

In the Kerr case $a \neq 0$, however, in view of the absense of any global a priori energy estimate, it turns out that in order to apply the N identity, one needs some understanding of dispersion. *Thus, the problems of boundedness and decay are coupled.* For the latter, however, it would seem that we have to understand a certain high-frequency obstruction to decay caused by so-called *trapped null geodesics*.

4.2.3. Trapped null geodesics. Again, we begin with the Schwarzschild case. It is well known (cf. [47]) that the hypersurface $r = 3M$ is generated by null geodesics which neither cross the horizon $\mathcal{H}^+$ nor escape to null infinity $\mathcal{I}^+$. They are the precise analogue of *trapped rays* in the classical obstacle problem. In the context of the latter, the presence of a single such ray is sufficient to falsify certain quantitative decay bounds [70]. A similar result holds in the general Lorentzian setting [74]. Weaker decay bounds can still hold, however, if the dynamics of geodesic flow around trapping is "good", that is to say, the trapped null geodesics are themselves dynamically unstable in the context of geodesic flow.

[9] Note that the above positivity property breaks down in the extremal case $|a| = M$ as this is characterized precisely by $\kappa = 0$. See Section 4.2.5 below.

It turns out that Schwarzschild geometry indeed exhibits "good" trapping. The programme of capturing this by *local integrated energy decay estimates* with degeneration was initiated by [5]. See [7, 33, 35]. From these and the red-shift identity of Section 4.2.2, the full decay statement of Theorem 4.2 in the $a = 0$ case can now be inferred directly by a black box method [36]. See also [80].

The Schwarzschild results [7, 33, 35] exploited the fact that not only is the structure of trapping "good" from the point of view of geodesic flow in phase space, but it is localised at the codimensional-1 hypersurface $r = 3M$ of *physical space*. The latter feature is broken in Kerr for all $a \neq 0$. Nonetheless, in the case $|a| \ll M$, analogues of local integrated energy decay could still be shown using either Carter's separability [38, 40], complete integrability of geodesic flow [81], or, commuting the wave equation with the non-trivial Killing tensor [2]. Each of these methods effectively frequency localises the degeneration of trapping and uses the hidden symmetries of Kerr discussed in Section 2.2; implicitly, these proofs all show that when viewed in phase space, the structure of trapping remains "good".[10] The above [2, 38, 40, 81] all use the assumption $|a| \ll M$ in a second essential way, so as to treat superradiance as a small parameter; in particular, this allows one to couple integrated local energy decay with the red-shift identity of Section 4.2.2 and obtain, simultaneously, both boundedness and decay.

Although the problems of boundedness and decay are indeed coupled, a more careful examination shows that one need not understand *trapping* in order to obtain boundedness. Our earlier result [37] had in fact showed that, exploiting the property that superradiance is governed by a small parameter and the ergorergion lies well within the region of coercivitiy properties of the red-shift identity, one could prove boundedness using dispersion *only for the "superradiant part"* of the solution, which is itself *not trapped*. This in fact allowed one to infer boundedness for (4.1) on suitable metrics only assumed C^1 close to Schwarzschild, for which one cannot appeal to structural stability of geodesic flow.

It turns out that it is the above insight which holds the key to the general $|a| < M$ case. Remarkably, one can show that, for the entire subextremal range, not only is trapping always good, but *the superradiant part is never trapped*. The latter is particularly suprising since when viewed in physical space, there do exist trapped null geodesics in the ergorergion for a close to M. The above remarks are sufficient to construct frequency localised vector field multipliers yielding integrated local energy decay in the high frequency regime. See the original treatment in [39].

4.2.4. Finite frequency obstructions. There is one final new difficulty that appears in the general $|a| < M$ case: excluding the possibility of finite frequency exponentially growing superradiant modes or resonances.

The absense of the former was proven in a remarkable paper of Whiting [83]. Whiting's methods were very recently extended to exclude resonances on the axis by Shlapentokh-Rothman in [76]. These proofs depend heavily on the algebraic symmetry properties of the resulting radial o.d.e. associated to Carter's separation of (4.1)–*yet another miracle of the Kerr geometry!* Using a continuity argument in a, it is sufficient in fact to appeal to the result [76] on the real axis. This is the final element of the proof of Theorem 4.2. See [41] for the full details.

[10] Note that the latter fact can also be inferred from structural stability properties of geodesic flow. See [84].

4.2.5. The extremal case and the Aretakis instability. Let us finally note that the precise form (see [41]) of Theorem 4.2 does *not* in fact hold without qualification for the extremal case $|a| = M$. This is related precisely to the degeneration of the red-shift of Section 4.2.2.

Theorem 4.3 (Aretakis [3, 4]). *For extremal Kerr $|a| = M$, for generic solutions of ψ, translation invariant transversal derivatives on the horizon fail to decay, and higher-order such derivatives grow polynomially.*

Decay results for axisymmetric solutions of (4.1) in the case of $|a| = M$ have been obtained in [4], but the non-axisymmetric case is still open and may be subject to additional instabilities. It is on account of Theorem 4.3 that we have excluded $|a| = M$ from Conjecture 4.1. The nonlinear dynamics around extremality promise many interesting features! See [63].

4.3. The full linear stability of Schwarzschild. We have motivated our study of (4.1) as a "poor man's" linearisation of (1.2). Let us turn now to the actual linearisation of (1.2) around black hole backgrounds, that is to say, the true problem of linear stability.

Very recently, with G. Holzegel and I. Rodnianski, we have obtained the full analog of Theorem 4.2 for the linearised Einstein equations around Schwarzschild.

Theorem 4.4 (Full linear stability of Schwarzschild [30]). *Solutions for the linearisation of the Einstein equations around Schwarzschild arising from regular admissible data remain bounded in the exterior and decay (with respect to a hyperboloidal foliation) to a linearised Kerr solution.*

The additional difficulties of the above thorem with respect to the scalar wave equation (4.1) lie in the highly non-trivial structure of the resulting coupled system equations. As in the non-linear stability of Minkowski space, a fruitful way of capturing this structure is with respect to the structure equations and Bianchi equations captured by a null frame. Linearising (3.2), we schematically obtain

$$\nabla\!\!\!/\Gamma^{(1)} = \Gamma^{(1)} \cdot \Gamma^{(0)} + \psi^{(1)}, \qquad \nabla\!\!\!/\psi^{(1)} = \not{D}\psi^{(1)} + \Gamma^{(1)} \cdot \psi^{(0)} + \Gamma^{(0)} \cdot \psi^{(1)}, \tag{4.2}$$

where $\Gamma^{(1)}$, $\psi^{(1)}$ now denote linearised spin coefficients and curvature components, respectively, and $\Gamma^{(0)}$, $\psi^{(0)}$ now denote background terms. Note that in the case of Minkowski space, $\psi^{(0)} = 0$ and thus the equations for $\psi^{(1)}$ decouple from those for $\Gamma^{(1)}$ and admit a coercive energy estimate via contracting the Bel-Robinson tensor with ∂_t [22]. Already in the Schwarzschild case, however, $\psi^{(0)} \neq 0$ and the two sets of equations in (4.2) are coupled. A fundamental difficulty is the absense of an obvious coercive energy identity for the full system (4.2), or even just the Bianchi part. Thus, even obtaining a degenerate boundedness statement, cf. Section 4.2.1, is now non-trivial.

Our approach expresses (4.2) with respect to a suitably normalised null frame associated to a double null foliation. We then introduce a novel quantity, defined explicitly as

$$P = \not{D}_2^\star \not{D}_1^\star \left(-\rho^{(1)}, \sigma^{(1)}\right) + \frac{3}{4}\rho_0 (tr\chi)_0 \left(\hat{\chi}^{(1)} - \underline{\hat{\chi}}^{(1)}\right)$$

together with a dual quantity $\underline{P}$. Here $\rho^{(1)}$, $\sigma^{(1)}$ denote particular linearised components of the Riemann tensor, $\hat{\chi^{(1)}}$ and $\underline{\hat{\chi}}^{(1)}$ denote the linearised shears of the foliation, ρ_0 and $\mathrm{tr}\chi_0$ are Schwarzschild background terms and $\not{D}_2^\star$ and $\not{D}_1^\star$ denote the first order angular differential operators of [23].

The quantity P decouples from (4.2) and satisfies the Regge–Wheeler equation

$$\Omega\nabla\!\!\!/_3(\Omega\nabla\!\!\!/_4(r^5P))-(1-2Mr^{-1})\Delta\!\!\!/(r^5P)+(4r^{-2}-6Mr^{-3})(1-2Mr^{-1})(r^5P)=0 \quad (4.3)$$

Like (4.1), the above equation does indeed admit a conserved coercive energy estimate. The first part of our proof obtains a complete understanding of P, which is a relatively easy generalisation of Theorem 4.2 restricted to $a=0$;

Proposition 4.5. *Solutions P of (4.3) arising from regular localised data satisfy boundedness and integrated local energy decay (non-degenerate at the horizon and with "good weights" at infinity, cf. [36]) and decay polynomially with respect to a hyperboloidal foliation.*

See also [6]. Given Proposition 4.5, one can then exploit a hierarchial structure in (4.2) to estimate, one by one, all other quantitites, schematically denoted $\Gamma^{(1)}$,$\psi^{(1)}$, by integration as transport equations in L^2. From integrated local energy decay and boundedness for P, one obtains integrated local energy decay and boundedness for each quantity, after a suitable linearised Kerr solution is subtracted. It is essential here that one uses the full strength of Proposition 4.5 with respect to the non-degeneration at the horizon and the "good" weights at infinity.

It is interesting to compare our approach to the formal mode analysis of the physics literature (see [12]). There one attempts to recover everything from the linearised curvature components $\alpha^{(1)}$ and $\underline{\alpha}^{(1)}$, which also decouple and satisfy the so-called Bardeen–Press equation[11]. In contrast to (4.3), however, this equation does not admit an obvious coercive conserved energy, but it can nonetheless be shown that it does not admit growing modes. From this one can in principle formally recover control of other quantities for fixed modes [12]. This approach, however, fails to yield an estimate beyond fixed modes, precisely because of the absense of a mode-independent energy estimate for Bardeen–Press. Note that when viewed in frequency space, our P can be related to $\alpha^{(1)}$ by the transformation theory of Chandrasekhar [12].

We reiterate finally that in the above argument, obtaining even boundedness for the full system (4.2) required the dispersive part of Proposition 4.5. Thus we see that, even at the linear level, there does not appear to be a pure "orbital stability" result; just as in the nonlinear theory, boundedness is coupled to showing quantitative decay.

4.4. The road to conjecture 4.1. Before turning in Section 5 to the black hole interior, let us revisit our fully nonlinear problem of Conjecture 4.1.

The issue of using decay rates as in Theorem 4.2 in a nonlinear setting satisfying a null condition has been addressed in a scalar problem by Luk [59]. See also [49].

As we described in Section 4.1, to prove Conjecture 4.1, one must identify (and linearise around) the asymptotic parameters to which the solution will asymptote–and for every open set of initial data, these parameters will generically have $a\neq 0$. It follows that until the analogue of Theorem 4.4 has been obtained for Kerr, at the very least for the very slowly rotating regime $|a|\ll M$, then one expects that there is no open set in the moduli space of initial data which can be handled.

It is worth mentioning, however, that there is a restricted version of Conjecture 4.1 which can in principle be studied using only the Schwarzschild linear stability result. If axisym-

[11] In the Kerr case, this generalises to the Teukolsky equation. See [12].

metry is imposed on the initial data and one moreover imposes that the initial angular momentum vanishes, then, since angular momentum does not radiate to null infinity under the assumption of axisymmetry, one expects that the solution should approach a Schwarzschild black hole and thus should be amenable to study using only Theorem 4.4. This is the content of ongoing work.

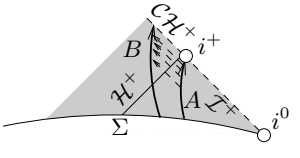
We mention finally that under *spherical symmetry*, one can formulate an analogous problem to that of Conjecture 4.1 concerning the Einstein–scalar field system (see [15]) or the Einstein–Maxwell–scalar field system (to be discussed in the next section).[12] The analogue of Conjecture 4.1 is then proven in [15, 26, 32]. The above problem retains few of the difficulties described in Section 4.2–in particular, it does not exhibit superradiance or trapping. Moreover, on the nonlinear side, it is interesting to note that spherical symmetry breaks the supercriticality of the Einstein equations, so in particular, allows 1., 2. and 3. to be proven separately. Nonetheless, the above models have been especially important as a source for intuition on the stability and instability properties of black hole *interiors*. We turn to this now.

5. The black hole interior and singularities

We now turn to the interior of Kerr black holes and strong cosmic censorship.

5.1. The blue-shift instability. In Section 3.3, we motivated Penrose's strong cosmic censorship by little other than wishful thinking–the possibility of Cauchy horizons is so problematic that we hope that generically they cannot form. There is indeed, however, a heuristic argument that suggests that at least the Kerr Cauchy horizon may be unstable.

The argument, due to Penrose [67], goes as follows. Let A and B be again two observers, where B now enters the black hole whereas A remains for all time outside. If A sends a signal to B, then the frequency measured by B becomes infinitely high as B's proper time approaches his Cauchy horizon-crossing time.



That is to say, the signal is infinitely shifted *to the blue*.

As with the red-shift effect discussed in Section 4.2.2, this effect should be reflected in the behaviour of waves, but now as an instability. This was in fact studied numerically in [77] for the related case of the scalar wave equation (4.1) on Reissner–Nordström background.[13] In view of the role of (4.1) as a "poor-man's linearisation" of (1.2), the above heuristic arguments were the first indication that the smooth Cauchy-horizon behaviour of Kerr could be unstable.[14]

[12] Recall that in view of Birkhoff's theorem [47], the only spherically symmetric vacuum solutions are Schwarzschild.

[13] Reissner–Nordström $(\mathcal{M}, g_{M,Q})$ is a spherically symmetric family of solutions to the Einstein–Maxwell equations and for $Q \neq 0$ has a Cauchy horizon similar to Kerr.

[14] For an another manifestation of the blue-shift instability when solving the Einstein equations *backwards* in the

A general result due to Sbierski [74] shows that the geometric optics argument is sufficient to falsify a quantitative energy boundedness result analogous to the precise statement of Theorem 4.2 in the exterior. Suprisingly, however, it turns out that the blue-shift instability is not strong enough for ψ to blow up in L^∞.

Theorem 5.1 (Franzen [45]). *Solutions ψ of the wave equation (4.1) as in Theorem 4.2 remain pointwise bounded $|\psi| \leq C$ on sub-extremal Kerr for $a \neq 0$ (or Reissner–Nordström $Q \neq 0$) in the black hole interior, up to and including $\mathcal{CH}^+$.*

This result, whose proof uses as an input the result of Theorem 4.2 restricted to $\mathcal{H}^+$, can be thought of as the first indication that rough stability results hold all the way to $\mathcal{CH}^+$. To explore this, however, let us first turn to certain spherically symmetric toy models.

5.2. Spherically symmetric toy-models. With the Schwarzschild case as the only example to go by, Penrose had originally speculated [67] that the blue-shift instability in the fully nonlinear setting would give rise to a spacelike singularity[15].

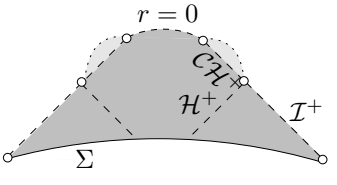
The simplest toy model with a true wave-like degree of freedom where this can be studied is the Einstein–Maxwell[16]–real scalar field system

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu} \doteq 8\pi\left(\frac{1}{4\pi}\left(F_\mu{}^\lambda F_{\lambda\nu} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right) + \partial_\mu\psi\partial_\nu\psi - \frac{1}{2}g_{\mu\nu}\partial^\alpha\phi\partial_\alpha\phi\right) \tag{5.1}$$

$$\nabla^\mu F_{\mu\nu} = 0, \qquad \nabla_{[\lambda}F_{\mu\nu]} = 0, \qquad \Box_g\psi = 0, \tag{5.2}$$

under spherical symmetry. It turns out that for this toy model, Penrose's expectation does not hold as stated: At least a part of the boundary of the maximal development is a null Cauchy horizon through which the metric is at least continuously extendible:

Theorem 5.2 (C^0-stability of a piece of the Cauchy horizon, [25, 27]). *For all two-ended asymptotically flat spherically symmetric initial data for (5.1)–(5.2) with non-vanishing charge, the maximal development can be extended through a non-empty Cauchy horizon $\mathcal{CH}^+$*



as a spacetime with C^0 metric.

The above theorem depends in fact also on joint work with Rodnianski [32] on the exterior region (cf. the end of Section 4.4) which obtains upper polynomial bounds for the decay of ψ on $\mathcal{H}^+$. Heuristic and numerical [10, 46] work suggests a precise asymptotic tail, in particular, polynomial *lower bounds* on $\mathcal{H}^+$. With this as an *assumption*, one can obtain the following

exterior, see [29].

[15] In fact, one still often sees an alternative formulation of Conjecture 3.5 as the statement that "Generically, singularities are spacelike".

[16] The pure scalar field model, whose study was pioneeered by Christodoulou [15], does not admit Cauchy horizons emanating from i^+. The system (5.1)–(5.2) is the simplest generalisation that does, in view of the fact that it admits Reissner–Nordström as an explicit solution.

Theorem 5.3 (Weak null singularities, [27]). *For spherically symmetric initial data as above where a pointwise lower bound on $\partial_v\psi$ is assumed to hold asymptotically along the event horizon $\mathcal{H}^+$ that forms, then the above Cauchy horizon $\mathcal{CH}^+$ is singular: The Hawking mass (thus the curvature) diverges and, moreover, the extension of Theorem 5.2 fails to have locally square integrable Christoffel symbols.*

The above two theorems confirmed a scenario which had been suggested on the basis of previous arguments of Hiscock [48], Israel–Poisson [69] and Ori [65] as well as numerical studies of the above system [9, 10]. In view of the blow up of the Hawking mass, the phenomenon was dubbed *mass inflation*. The type of singular boundary exhibited by the above theorem, where the Christoffel symbols fail to be square integrable but the metric continuously extends, is known as a *weak null singularity*.

The above results apply to general solutions, not just small perturbations of Reissner–Nordström. In the stability context, it turns out that the $r = 0$ piece is absent, and the entire bifurcate Cauchy horizon is globally stable:

Theorem 5.4 (Global stability of the Reissner–Nordström Cauchy horizon [28]). *For small, spherically symmetric perturbations of Reissner–Nordström, the maximal development is extendible beyond a bifurcate null horizon as a manifold with continuous metric. The Carter–Penrose diagramme is as in the Reissner–Nordström case. In particular, there is no spacelike part of the singularity.*

Note that the above is precisely the result that one obtains by naively extrapolating Theorem 5.1 to the fully non-linear theory, identifying ψ with the metric.

Corollary 5.5 (Bifurcate weak null singularities, [28]). *Under the assumptions of Theorem 5.4 and the additional asssumption of Theorem 5.3 on both event horizons, the Cauchy horizons $\mathcal{CH}^+$ represent bifurcate weak null singularities and the extensions fail to have locally square integrable Christoffel symbols.*

The ultimate spherically symmetric toy model is that of the Einstein-Maxwell–*charged* scalar field system, that is when the scalar field is complex-valued and carries charge and is directly coupled with the Maxwell field through this charge, besides the gravitational coupling through the Einstein equations (as in (5.1)). In his Cambridge Ph.D. thesis [55], J. Kommemi has shown an analogue of Theorem 5.2 for this model, *given an a priori decay assumption on the horizon*.

5.3. Beyond toy models: Einstein vacuum equations without symmetry. Whereas the above work [27, 32, 55] more or less definitively resolves the issue of the appearance of weak null singularities in spherically symmetric toy models, one could still hold out hope that the vacuum Einstein equations (1.2) do not allow for the formation of such singularities but favour spacelike singularities as in the Schwarzschild case. In contrast to the spherically symmetric "toy" world, for the Einstein vacuum equations without symmetry there is really no numerical work available on this problem and very little heuristics (see however [66]).

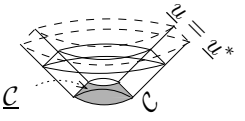
5.3.1. Luk's vacuum weak null singularities. The first order of business is thus to construct examples of local patches of vacuum spacetime with a weak null singular boundary. This has recently been accomplished in a breakthrough paper of J. Luk [60], based in part on his previous work with Rodnianski [61, 62] on impulsive gravitational waves.

Luk's spacetimes have no symmetries and are constructed by solving a characteristic initial value problem with characteristic data of a prescribed singular behaviour. The problem reduces to showing existence in a rectangular domain as well as propagation of the singular behaviour. This is given in:

Theorem 5.6 (Luk [60]). *Consider characteristic initial data for the Einstein vacuum equations on a bifurcate null hypersurface $\mathcal{C} \cup \underline{\mathcal{C}}$ whose spherical sections are parameterised by affine $\underline{u} \in [0, \underline{u}^*))$ and $u \in [0, u^*))$, resepectively, and where the outgoing shear $\hat{\chi}$ (and sufficient angular derivatives) satisfies*

$$|\hat{\chi}| \sim |\log(\underline{u}^* - \underline{u})|^{-p} |\underline{u}^* - \underline{u}|^{-1}. \tag{5.3}$$

Then the maximal development can be covered by a double null foliation terminating in a null boundary $\underline{u} = \underline{u}^*$



through which the metric is continuously extendible. The singular behaviour (5.3) propagates, making this boundary a weak null singularity.

Moreover, in analogy with the Luk–Rodnianski theory of two interacting impulsive gravitational waves [62], Luk obtained

Theorem 5.7 (Luk [60]). *Consider again characteristic data as above but such that both outgoing shears $\hat{\chi}$ and $\underline{\hat{\chi}}$ (and sufficient angular derivatives) satisfy*

$$|\hat{\chi}| \sim |\log(\underline{u}^* - \underline{u})|^{-p} |\underline{u}^* - \underline{u}|^{-1}, \qquad |\underline{\hat{\chi}}| \sim |\log(u^* - u)|^{-p} |u^* - u|^{-1}, \tag{5.4}$$

and moreover, the data satisfies an appropriate smallness condition. Then the maximal development can be covered by a double null foliation which terminates in a bifurcate null hypersurface $\{u^*\} \times [0, \underline{u}^*] \cup [0, u^*] \times \{\underline{u}^*\}$ through which the metric is continuously extendible. Relations (5.4) propagate, making the boundary of spacetime a bifurcate weak null singularity.

Note that in Luk–Rodnianski theory [61, 62], (5.3) is replaced by the assumption that $\hat{\chi}$ is discontinuous but *bounded*. Thus, it was possible in [61, 62] to interpret the Einstein equations beyond these null hypersurfaces, which interact simply passing through each other, leaving in their wake a regular spacetime. Here, however, the boundaries are much more singular ($\hat{\chi}$ is not in any L^p for $p > 1$), and thus, the solution cannot be interpreted beyond them, even as a weak solution of (1.2).[17]

In the short space of this article, it is impossible to give an overview of the proofs of the above theorems. As in several of the results we have discussed, the proof expresses (3.2) with respect to a null frame attached to a double null foliation, and moreover, relies on a renormalisation of this system which removes the most singular components (extending ideas from [61, 62]). This does not completely regularise the system, however, and a fundamental role is played by a hierarchy of largeness/smallness which is preserved in evolution by special null structure of (3.2). These ideas are in turn related to the seminal work of Christodoulou [20] on the dynamic formation of trapped surfaces, surveyed in another article in these proceedings [21], and his *short pulse method*.

[17] In particular, the name "weak null singularity" is in some sense unfortunate!

5.3.2. The global stability of the Kerr Cauchy horizon. Putting together essentially all the ideas form Sections 5.2–5.3.1, we have very recently obtained the following result in upcoming joint work with J. Luk.

Theorem 5.8 (Global stability of the Kerr Cauchy horizon [31]). *Consider characteristic initial data for* (1.2) *on a bifurcate null hypersurface* $\mathcal{H}^+ \cup \mathcal{H}^-$ *where* $\mathcal{H}^\pm$ *have future-affine complete null generators and their induced geometry is globally close to and dynamically approaches that of the event horizon of Kerr with* $0 < |a| < M$ *at a sufficiently fast polynomial rate. Then the maximal development can be extended beyond a bifurcate Cauchy horizon* $\mathcal{CH}^+$ *as a Lorentzian manifold with* C^0 *metric. All finitely-living observers pass into the extension.*

Let us note explicitly that a corollary of the above theorem *together with a successful resolution of Conjecture 4.1* would be the following definitive statement

Corollary 5.9. *If Conjecture is 4.1 is true then the Cauchy horizon of the Kerr solution is globally stable and the* C^0*-inextendibility formulation and the "generically, spacetime singularities are spacelike" formulation of strong cosmic censorship are both false.*

5.3.3. The future for strong cosmic censorship. In view of the toy-model results of Theorem 5.3 and Corollary 5.5, all is not lost for strong cosmic censorship. A version of the inextendibility requirement in the formulation of strong cosmic censorship which is compatible with the result of Theorem 5.3 for the toy problem and may still be true for the vacuum without symmetry is the statement that *"*$(\mathcal{M}, g)$ *be inextendible as a Lorentzian manifold with locally square integrable Christoffel symbols"*. This formulation is due to Christodoulou [20] and would guarantee that there be no extension which can be interpreted as a weak solution of (1.2). It is an interesting open problem to obtain this in a neighbourhood of the Kerr family. This naturally separates into the following two statements:

Conjecture 5.10.

1. *Under a suitable assumption on the data on* $\mathcal{H}^+$ *in Theorem 5.8, then* $\mathcal{CH}^+$ *is a weak null singularity, across which the metric is inextendible as a Lorentzian manifold with locally square integrable Christoffel symbols.*
2. *The above assumption on* $\mathcal{H}^+$ *holds for the data of Conjecture 4.1, provided the latter are generic.*

One can in fact localise the result of Theorem 5.8 to apply to spacetimes with one asymptotically flat end, provided they satisfy the assumption on $\mathcal{H}^+$, and one can infer again a non-empty piece of null singular boundary $\mathcal{CH}^+$. Thus, all black holes which asymptotically settle down in their exterior region to Kerr with $0 < |a| < M$ will have a non-empty C^0-Cauchy horizon, which, assuming a positive resolution to Conjecture **??**, will correspond to a weak null singularity.

Do the above Cauchy horizons/weak null singularities "close up" the whole maximal development as in the above two-ended case? Or will they give way to a spacelike (or even more complicated) singularity? These questions may hold the key to understanding strong cosmic censorship beyond a neighbourhood of the Kerr family.

Acknowledgements. The author is grateful to G. Holzegel, J. Luk, I. Rodnianski and Y. Shlapentokh-Rothman for comments on this manuscript and to D. Christodoulou for many inspiring discussions over the years.

References

[1] S. Alexakis, A. D. Ionescu and S. Klainerman, *Uniqueness of smooth stationary black holes in vacuum: small perturbations of the Kerr spaces*, arXiv:0904.0982.

[2] L. Andersson and P. Blue, *Hidden symmetries and decay for the wave equation on the Kerr spacetime*, arXiv:0908.2265.

[3] S. Aretakis, *Horizon instability of extremal black holes*, arXiv:1206.6598.

[4] ———, *Decay of axisymmetric solutions of the wave equation on extreme Kerr backgrounds* J. Funct. Analysis **263** (2012), 2770–2831.

[5] P. Blue and A. Soffer, *Semilinear wave equations on the Schwarzschild manifold. I. Local decay estimates*, Adv. Differential Equations **8** (2003), no. 5, 595–614.

[6] ———, *The wave equation on the Schwarzschild metric II: Local decay for the spin 2 Regge Wheeler equation* J. Math. Phys. **46** (2005) 012502.

[7] P. Blue and J. Sterbenz, *Uniform decay of local energy and the semi-linear wave equation on Schwarzschild space* Comm. Math. Phys. **268** (2006), 481–504.

[8] J.-F. Bony and D. Häfner, *Decay and non-decay of the local energy for the wave equation in the de Sitter-Schwarzschild metric*, Comm. Math. Phys. **282** (2008), 697–719.

[9] P. Brady and J. D. Smith, *Black hole singularities: a numerical approach* Phys. Rev. Lett. **75** (1995), no. 7, 1256–1259.

[10] L. M. Burko, *Structure of the black hole's Cauchy-horizon singularity* Phys. Rev. Lett. **79** (1997), no. 25, 4958–4961.

[11] B. Carter, *Axisymmetric black hole has only two degrees of freedom* Phys. Rev. Lett. **26** (1971), 331–333.

[12] S. Chandrasekhar, *The mathematical theory of black holes*, Oxford University Press, Oxford 1983.

[13] Y. Choquet-Bruhat, *Théoreme d'existence pour certains systèmes d'équations aux dérivées partielles non linéaires* Acta Math. **88** (1952), 141–225.

[14] Y. Choquet-Bruhat and R. Geroch, *Global aspects of the Cauchy problem in general relativity* Comm. Math. Phys. **14** (1969), 329–335.

[15] D. Christodoulou, *A mathematical theory of gravitational collapse* Comm. Math. Phys. **109** (1987), 613–647.

[16] ———, *Examples of naked singularity formation in the gravitational collapse of a scalar field* Ann. of Math. **140** (1994), no. 3, 607–653.

[17] ———, *The instability of naked singularities in the gravitational collapse of a scalar field* Ann. of Math. **149** (1999), no. 1, 183–217.

[18] ———, *On the global initial value problem and the issue of singularities* Class. Quantum Gravity **16** (1999) A23–A35.

[19] ———, *The stability of Minkowski spacetime*, pp. 1113–1121 in Proceedings of the International Congress of Mathematicians, Kyoto 1990, The Mathematical Society of Japan, Springer-Verlag, 1991.

[20] ———, *The formation of black holes in general relativity* European Mathematical Society (EMS), Zürich, 2009.

[21] D. Christodoulou, these proceedings.

[22] ———, *Asymptotic properties of linear field equations in Minkowski space* Comm. Pure Appl. Math. **43** (1990). 137–199.

[23] D. Christodoulou and S. Klainerman, *The global nonlinear stability of the Minkowski space* Princeton University Press, Princeton, NJ, 1993.

[24] P. Chruściel and J. L. Costa, *On uniqueness of stationary vacuum black holes*, arXiv:0806.0016.

[25] M. Dafermos, *Stability and instability of the Cauchy horizon for the spherically-symmetric Einstein-Maxwell-scalar field equations*, Ann. Math. **158** (2003), 875–928.

[26] ———, *Spherically symmetric spacetimes with a trapped surface* Class. Quantum Grav. **22** (2005), no. 11, 2221–2232.

[27] ———, *The interior of charged black holes and the problem of uniqueness in general relativity* Comm. Pure Appl. Math. **58** (2005), 445–504.

[28] ———, *Black holes without spacelike singularities*, to appear in CMP, arXiv:1201.1797.

[29] M. Dafermos, G. Holzegel and I. Rodnianski, *A scattering theory construction of dynamical black hole spacetimes*, arXiv:1306.5534.

[30] ———, *The full linear stability of Schwarzschild under gravitational perturbations*, preprint.

[31] M. Dafermos and J. Luk, *On the stability of the Kerr Cauchy horizon*, preprint.

[32] M. Dafermos and I. Rodnianski, *A proof of Price's law for the collapse of a self-gravitating scalar field*, Invent. Math. **162** (2005), 381–457.

[33] ———, *The redshift effect and radiation decay on black hole spacetimes* Comm. Pure Appl. Math. **62** (2009), no. 7, 859–919.

[34] ———, *The wave equation on Schwarzschild-de Sitter spacetimes*, arXiv:0709.2766v1 [gr-qc].

[35] ———, *A note on energy currents and decay for the wave equation on a Schwarzschild background*, arXiv:0710.0171.

[36] ———, *A new physical-space approach to decay for the wave equation with applications to black hole spacetimes*, in XVIth International Congress on Mathematical Physics, P. Exner (ed.), World Scientific, London, 2009, pp. 421–433

[37] ———, *A proof of the uniform boundedness of solutions to the wave equation on slowly rotating Kerr backgrounds*, Invent. Math. **185** (2011), 467–559.

[38] ———, *Lectures on black holes and linear waves*, in Evolution equations, Clay Mathematics Proceedings, Vol. 17. Amer. Math. Soc., Providence, RI, 2013, pp. 97–205, arXiv:0811.0354.

[39] ———, *The black hole stability problem for linear scalar perturbations*, in Proceedings of the Twelfth Marcel Grossmann Meeting on General Relativity, T. Damour et al (ed.), World Scientific, Singapore, 2011, pp. 132–189.

[40] ———, *Decay for solutions of the wave equation on Kerr exterior spacetimes I–II: The cases* $|a| \ll M$ *or axisymmetry*, arXiv:1010.5132.

[41] M. Dafermos, I. Rodnianski, and Y. Shlapentokh-Rothman, *Decay for solutions of the wave equation on Kerr exterior spacetimes III: The full sub-extremal case* $|a| < M$, arXiv:1402.7034.

[42] S. Dyatlov, *Quasi-normal modes and exponential energy decay for the for the Kerr–de Sitter black hole* Comm. Math. Physics **306** (2011), 119–163.

[43] A. Einstein, "*Die Feldgleichungen der Gravitation.* Sitzungsber. d. Preuss. Akad. d. Wissenschaften (1915), 844–847.

[44] F. Finster, N. Kamran, J. Smoller, and S. T. Yau, *Decay of solutions of the wave equation in Kerr geometry* Comm. Math. Phys. **264** (2006), 465–503.

[45] A. Franzen, *Boundedness of a massless scalar wave in Reissner–Nordström and Kerr interior backgrounds*, Ph. D. thesis, Universiteit Utrecht, *in preparation*.

[46] C. Gundlach, R. Price and J. Pullin, *Late-time behavior of stellar collapse and explosition. II: Nonlinear evolution* Phys. Rev. D **49** (1994), 890–899.

[47] S. W. Hawking and G. F. R. Ellis, *The large scale structure of space-time* Cambridge University Press, London-New York, 1973.

[48] W. A. Hiscock, *Evolution of the interior of a charged black hole* Phys. Lett. **83A** (1981), 110–112.

[49] G. Holzegel, *Ultimately Schwarzschildean spacetimes and the black hole stability problem*, arXiv:1010.3216.

[50] G. Holzegel and J. Smulevici, *Decay properties of Klein-Gordon fields on Kerr-AdS spacetimes* Comm. Pure Appl. Math. **66** (2013), no. 11, 1751–1802.

[51] B. Kay and R. Wald, *Linear stability of Schwarzschild under perturbations which are nonvanishing on the bifurcation* 2*-sphere* Class. Quant. Grav. **4** (1987), no. 4, 893–898.

[52] R. Kerr, *Gravitational field of a spinning mass as an example of algebraically special metrics* Phys. Rev. Lett. **11** (1963), 237–238.

[53] S. Klainerman, *The null condition and global existence to nonlinear wave equations* Nonlinear systems of partial differential equations in applied mathematics, Part 1, 293–326, Lectures in Appl. Math. **23** Amer. Math. Soc., Providence, RI, 1986.

[54] S. Klainerman, I. Rodnianski and J. Szeftel, *The Bounded L2 Curvature Conjecture*, arXiv:1204.1767.

[55] J. Kommemi, *The global structure of spherically symmetric charged scalar field spacetimes*, Ph. D. thesis, University of Cambridge, 2013.

[56] M. Kruskal, *Maximal extension of Schwarzschild metric* Phys. Rev. **119** (1960), 1743–1745.

[57] G. Lemaitre, *L'Univers en Expansion* Publication du Laboratoire d'Astronomie et de Géodésie de l'Université de Louvain **9** (1932), 171–205.

[58] H. Lindblad and I. Rodnianski, *The global stability of Minkowski space-time in harmonic gauge* Ann. of Math. **171** (2010), 1401–1477.

[59] J. Luk, *The null condition and global existence for nonlinear wave equations on slowly rotating Kerr spacetimes*, JEMS **15** (2013), no. 5, 1629–1700.

[60] ———, *On weak null singularities in general relativity*, arXiv:1311.4970.

[61] J. Luk and I. Rodnianski, *Local propagation of impulsive gravitational waves*, available online at http://www.arxiv.org/abs/arXiv:1209.1130

[62] ———, *Nonlinear interactions of impulsive gravitational waves for the vacuum Einstein equations*, arXiv:1301.1072.

[63] K. Murata, H. S. Reall, N. Tanahashi, *What happens at the horizon(s) of an extreme black hole?*, arXiv:1307.6800.

[64] J. R. Oppenheimer and H. Snyder, *On continued gravitational contraction* Phys. Rev. **56** (1939), 455–459.

[65] A. Ori, *Inner structure of a charged black hole: an exact mass-inflation solution* Phys. Rev. Lett. **67** (1991), 789–792.

[66] ———, *Perturbative approach to the inner structure of a rotating black hole* Gen. Relativity Gravitation **29** (1997), no. 7, 881–929.

[67] R. Penrose, in Battelle Rencontres, edited by C. M. DeWitt and J. A. Wheeler (W.A. Bejamin, New York, 1968) pg 222.

[68] ———, *Gravitational collapse and space-time singularities* Phys. Rev. Lett. **14**, 57–59.

[69] E. Poisson and W. Israel, *Inner-horizon instability and mass inflation in black holes* Phys. Rev. Lett. **63** (1989), no. 16, 1663–1666.

[70] J. Ralston, *Solutions of the wave equation with localized energy* Comm. Pure Appl. Math. **22** (1969), 807–823.

[71] T. Regge and J. Wheeler, *Stability of a Schwarzschild singularity* Phys. Rev. **108** (1957), 1063–1069.

[72] D.C. Robinson, *Uniqueness of the Kerr black hole*, Phys. Rev. Lett. **34** (1975), 905–906.

[73] J. Sbierski, *On the Existence of a Maximal Cauchy Development for the Einstein Equations - a Dezornification*, arXiv:1309.7591v2.

[74] ———, *Characterisation of the Energy of Gaussian Beams on Lorentzian Manifolds - with Applications to Black Hole Spacetimes*, arXiv:1311.2477v1.

[75] K. Schwarzschild, *Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie* Sitzungsber. d. Preuss. Akad. d. Wissenschaften **1** (1916), 189–196.

[76] Y. Shlapentokh-Rothman, *Quantitative Mode Stability for the Wave Equation on the Kerr Spacetime*, arXiv:1302.6902, to appear in Ann. Henri Poincaré.

[77] M. Simpson and R. Penrose, *Internal instability in a Reissner–Nordström black hole* Internat. J. Theoret. Phys. **7** (1973), no. 3, 183–197.

[78] J. L. Synge, *The gravitational field of a particle* Proc. Roy. Irish Acad. **53** (1950), 83–114.

[79] J Szeftel, these proceedings.

[80] D. Tataru, *Local decay of waves on asymptotically flat stationary space-times* Amer. J. Math. **135** (2013), 361–401.

[81] D. Tataru and M. Tohaneanu, *Local energy estimate on Kerr black hole backgrounds*, IMRN (2011), no. 2, 248–292.

[82] A. Vasy, *Microlocal analysis of asymptotically hyperbolic and Kerr-de Sitter spaces, with an appendix by Semyon Dyatlov.* Invent. Math. **194** (2013), 381–513.

[83] B. Whiting, *Mode stability of the Kerr black hole* J. Math. Phys. **30** (1989), 1301.

[84] J. Wunsch and M. Zworski, *Resolvent estimates for normally hpyerbolic trapped sets*, arXiv:1003.4640v7.

DPMMS, University of Cambridge, Wilberforce Road, Cambridge CB3 0WB, UK; Fine Hall, Department of Mathematics, Washington Road, Princeton NJ 08544 USA

E-mail: dafermos@math.princeton.edu

Ancient solutions to geometric flows

Panagiota Daskalopoulos

Abstract. We will discuss ancient and eternal solutions to geometric parabolic equations. These are special solutions that exist for time $-\infty < t \leq T$, with $T \leq +\infty$. They often appear as blow up limits near a singularity. Their classification often results to the better understanding of the singularities of the flow. We will address the classification of ancient solutions to the Ricci flow on surfaces and the Yamabe flow on S^n and point out future open directions. The results in this article are joint work of the author with the collaborators M. del Pino, R. Hamilton and N. Sesum.

Mathematics Subject Classification (2010). Primary 53C44; Secondary 35K55, 35B08.

Keywords. Ancient solutions, geometric evolution equations, Ricci flow, Yamabe flow.

1. Introduction

We will discuss recent progress concerning the classification of *ancient solutions* to geometric evolution equations. More precisely, we will focus on the classification of ancient solutions to the Ricci flow on surfaces and the Yamabe flow on S^n, $n \geq 3$. In addition we will present the construction of new ancient solutions to the Yamabe flow that are of type II.

Ancient solutions often appear as blow up limits at a *finite time singularity* of the flow and their classification provides important information about the behavior of solutions near their singularities. In the Ricci flow and the Mean curvature flow the classification of ancient solutions is also important for performing surgery near a singularity.

Definition 1.1. A solution to a parabolic equation is called ancient if the solution is defined for all time $t \in (-\infty, T)$, for some $T < \infty$. If the solution is defined for all time $t \in (-\infty, +\infty)$ it is called eternal.

Shrinking solitons are often examples of ancient solutions to geometric flows, while *steady solitons* are examples of eternal solutions. However, there often exist other ancient or eternal solutions which are *not solitons* but they can be interpreted as special solutions obtained by the *gluing* of one or more solitons. The natural question is whether these special solutions and the solitons are the only non-trivial ancient or eternal solutions of the flow. In what follows, we will address recent results and open problems that are related to this question.

A well known result by S.T. Yau [57] dated back on 1975 asserts that on a complete non-compact Riemannian manifold M^n of dimension $n \geq 2$ with nonnegative Ricci curvature, any positive harmonic function u must be constant. This is the analogue of Liouville's

Proceedings of the International Congress of Mathematicians, Seoul, 2014

Theorem for harmonic functions on $\mathbb{R}^n$. A natural question is whether the analogue of Yau's theorem holds for *positive ancient or eternal* solutions of the *heat equation*

$$u_t = \Delta u \qquad \text{on } M^n \times (-\infty, T), \qquad T \leq \infty$$

under the same assumptions on M^n. The answer to this question is negative, as it can be easily seen by the eternal solution to the heat equation on $M = \mathbb{R}^1$ given by $u(x,t) = e^{x+t}$. However, P. Souplet and Q. Zhang [55] showed that the analogue of Yau's theorem holds under an extra growth assumption at infinity. Namely, if u is a positive ancient solution to the heat equation on $M \times (-\infty, T)$ such that

$$u(x,t) = e^{o(d(x)+\sqrt{|t|})} \qquad \text{as } d(x) \to \infty$$

then u is a *constant*. Also, if u is any ancient solution to the heat equation such that

$$u(x,t) = o(d(x) + \sqrt{|t|}) \qquad \text{as } d(x) \to \infty$$

then u is a *constant*. Note that this last estimate is also sharp in the spatial direction due to the example $u(x,t) = x$. The result in [55] is based on local Li-Yau type gradient estimates for positive solutions (bounded or not) to the heat equation (c.f. in the works of P. Li and S.T. Yau [42] and R. Hamilton [35] for previous related results).

A simple and well studied model of semi linear diffusion which is often used as a prototype in the analysis of singularities of more complex geometric flows, is the equation

$$u_t = \Delta u + |u|^{p-1}u \tag{1.1}$$

for different exponents $p > 1$. In [46], F. Merle and H. Zaag provided the classification of *positive bounded eternal* solutions to the equation

$$w_\tau = \Delta w - \frac{y}{2} \cdot \nabla w - \frac{w}{p-1} + w^p, \qquad (y,\tau) \in \mathbb{R}^n \times \mathbb{R} \tag{1.2}$$

in the sub-critical range of exponents $1 < p < (n+2)/(n-2)$ following previous classification results by Y. Giga and R. Kohn [27, 28]. Equation (1.2) arises in the singularity analysis of solutions to equation (1.1) at a blow up point (a,T). It also provides a model for the singularity analysis in various geometric flows, including neckpinches in the Mean curvature flow (c.f. in [1]) and the Ricci flow (c.f. in [2, 3]).

If a solution u of (1.1) has a finite time singularity at a point (a,T), then

$$w_a(y,\tau) := (T-t)^{\frac{1}{p-1}}\, u(x,t), \quad y = \frac{x-a}{\sqrt{T-t}},\ \tau = -\log(T-t)$$

satisfies equation (1.2). The study of u near a blow up point (a,T) is equivalent to the study of the long time behavior of the rescaled solution w_a, which after taking the limit over a sequence $\tau_n \to \infty$, converges to an eternal solution of equation (1.2). It follows by the results of Y. Giga and R. Kohn in [27, 28] (see also in [26]) that $\|w_a\|_{L^\infty} \leq C$ and

$$\lim_{\tau\to\infty} w_a(y,\tau) = \lim_{t\to T}(T-t)^{\frac{1}{p-1}}\, u(x,t) = \kappa$$

where $\kappa := (p-1)^{-\frac{1}{p-1}}$ is the steady state of equation (1.2). However, this result is only pointwise in a.

Uniform estimates on the rescaled solution w_a as $\tau \to \infty$ that are independent of a and also in some sense on the initial data were shown in [46]. These estimates are heavily based on the following classification result.

Theorem 1.2 (F. Merle and H. Zaag '98 [46]). *If w is a nonnegative bounded eternal solution of equation* (1.2)*, then w is independent of the spatial variable y and hence*

$$w = 0 \ \textit{or} \ w = \kappa \ \textit{or} \ w(s) = \phi(s - s_0) \ \textit{for some} \ s_0 \in \mathbb{R}$$

with $\phi(s) = \kappa(1 + e^s)^{-\frac{1}{(p-1)}}$.

A previous classification result, which states that if $w_{\pm\infty} = 0$ or $w_{\pm\infty} = \kappa$, then $w \equiv 0$ or $w \equiv \kappa$ was proven by Giga and Kohn in [27]. The result in [27] simply follows by the monotonicity of the Lyapunov functional

$$E(w) = \frac{1}{2}\int |\nabla w|^2 d\mu + \frac{1}{2(p-1)}\int w^2 d\mu - \frac{1}{p+1}\int w^{p+1} d\mu$$

where $d\mu = (4\pi)^{-\frac{n}{2}} e^{-\frac{|y|^2}{4}} dy$ that is shown to be decreasing in time under (1.2).

The significance of Theorem 1.2 is the classification of the eternal solutions to (1.2) that connect the steady states $w_{-\infty} = \kappa$ and $w_{+\infty} = 0$. These solutions are shown to be independent of y, hence given by $\phi(s)$. Its proof is heavily involved and strongly relies on analyzing the behavior of the solution w near $\tau \to -\infty$ in terms of its projections on the positive, zero and negative eigenspaces of the linearized operator $\mathcal{L}v := \Delta v - \frac{1}{2} y \cdot \nabla v + v$ at $w = \kappa$. Notice that in terms of $v := w - \kappa$, equation (1.2) takes the form

$$v_\tau = \mathcal{L}v + f(v) \tag{1.3}$$

with superlinear error $f(v) := (v + \kappa)^p - \kappa^p - p\kappa^{p-1}v$. Equation (1.2), with different nonlinear error functions $f(v)$, often arises in the analysis of singularities in geometric flows, in particular in neck-pinches in the mean curvature flow [1] and the Ricci flow [2, 3].

While the classification of entire solutions to nonlinear elliptic equations often appears in the literature, it is surprising that there are not many results available regarding the classification of ancient or eternal solutions to parabolic PDE.

The remarkable results by R. Hamilton which characterize eternal solutions to the Mean curvature flow [37] and Ricci flow [34] as *translating solitons*, have been widely used in the classification of *slowly forming* singularities. The main assumption in these results, which rely on the strong maximum principle and the Li-Yau differential Harnack inequalities, is that the curvature of the eternal solution assumes a space-time interior maximum. This assumption holds only under very special circumstances and rules out other interesting solutions.

In what follows we will discuss results and open problems concerning the classification of ancient and eternal solutions to the Ricci flow and the Yamabe flow without any other assumption.

In [18], the author jointly with R. Hamilton and N. Sesum, established the classification of ancient solutions $g(\cdot, t)$ to the Ricci flow on a compact surface that exists for $t \in (-\infty, T)$ and becomes spherical at time $t = T$. In addition to their role in the classification of singularities, ancient solutions to the 2-dim Ricci flow appear in *quantum field theory*, as they describe trajectories of the renormalization group equations of certain asymptotically free local quantum field theories in the ultra-violet regime (c.f. in [24] and its references). It was shown in [18] that the metric $g(\cdot, t)$ is either a family of contracting spheres or one of the King-Rosenau solutions [40, 51]. The latter is a *type II ancient solution* which is given in

closed form and asymptotically, as $\tau \to -\infty$, it resembles a surface obtained by the gluing of two cigars (Barenblatt self similar solutions). The King-Rosenau solution and the classification theorem in [18] will be further discussed in section 2.

The work in [18] follows an earlier result by the author with R. Hamilton and N. Sesum in [17], where the classification of ancient, convex and compact solutions to the curve shortening flow was shown.

A well known open question is the classification of ancient compact solutions to the Ricci flow on a compact manifold M^3 of dimension $n = 3$. These solutions typically arise as singularity models [30, 31, 48, 49]. In [48], G. Perelman constructed a class of rotationally symmetric type II ancient solutions to the Ricci flow on S^3. This is the higher dimensional analogue of the King-Rosenau solutions, however in this case are not given in closed form. These solutions are κ-noncollapsed. It is widely conjectured that the contracting spheres and the Perelman solution are the only κ-noncollapsed ancient solutions to the Ricci flow on S^3. The classification of type I ancient solutions to the Ricci flow has been addressed in [9, 47]. However, the answer to the above conjecture remains open.

One may argue that the higher dimensional analogue of the Ricci flow on S^2 is the *Yamabe flow* on S^n. The Yamabe flow was introduced by R. Hamilton in 1989 as an approach to solve the *Yamabe problem* on manifolds of positive conformal Yamabe invariant. Since then, there has been an extensive study of the long time existence and convergence to this flow on a compact manifolds in the articles [6, 7, 13, 33, 54, 58] among others.

In the case where the background manifold is the sphere S^n with the standard spherical metric g_{S^n}, the Yamabe flow evolving a metric $g = v^{\frac{4}{n-2}}(\cdot, t)\, g_{S^n}$ takes the form a *fast diffusion equation* satisfied by the function v. The analogue of the King-Rosenau solution exists in this flow and was found by J.R. King [40] (see also in [8]). However, in contrast with the King-Rosenau solution of the 2-dim Ricci flow and the Perelman solution of the 3-dim Ricci flow, the King solution of the Yamabe flow is a type I ancient solution. Asymptotically, as $t \to -\infty$, it resembles a surface obtained by the gluing of two Barenblatt self-similar solutions.

In analogy with the classification result for the 2-dim Ricci flow in [18], one may ask whether the contracting spheres and the King solutions are the only compact ancient solutions of the Yamabe flow on S^n, $n \geq 3$. This was shown not to be true, by the author jointly with M. del Pino and N. Sesum. Indeed, in [16] new type II ancient compact solutions to the Yamabe flow on S^n were constructed. These solutions are rotationally symmetric and converge, as $t \to -\infty$, to a tower of k spheres, $k \geq 2$. We will refer them as *moving towers of bubbles*. Their curvature operator changes sign. The construction in [16] may be viewed as a parabolic analogue of the gluing of k exact solutions to the rescaled equation (the spheres) with narrow cylindrical necks to obtain a new ancient solution to the Yamabe flow. It will be further discussed in section 3.

2. Ancient solutions to the Ricci flow

In this section we will discuss the classification of ancient solutions to the Ricci flow on compact surfaces and also briefly discuss the higher dimensional case $n \geq 3$. Consider an ancient solution g_{ij} of the *Ricci flow*

$$\frac{\partial g_{ij}}{\partial t} = -2\, R_{ij} \tag{2.1}$$

on a two-dimensional surface M that exists for time $t \in (-\infty, T)$, $T < \infty$.

The Ricci flow was introduced by R. Hamilton in 1981 in his seminal work [30] as an analytical and geometric tool to approach the resolution of the geometrization conjecture of William Thurston. We refer the interested reader to the articles by R. Hamilton [30–32, 36], G. Perelman [48–50] and S. Brendle & R. Schoen [10, 11] for some of the fundamental results on the Ricci flow and its applications.

Definition 2.1. An ancient solution to the Ricci flow (2.1) on a surface M is of type I, if it satisfies

$$\limsup_{t\to-\infty} (|t| \max_M |Rm|(\cdot,t)) < \infty.$$

A solution which is not of type I, will be called of type II. Here *Rm* denotes the Riemannian curvature of the metric.

In two dimensions one has

$$R_{ij} = \frac{1}{2} R\, g_{ij}$$

where R is the scalar curvature of the surface. Moreover, on an ancient non-flat solution one has $R > 0$. It is well known by the works of R. Hamilton [29] and B. Chow [12] that a compact surface evolving by the Ricci flow (2.1) will become spherical at its extinction time T, which means that after a normalization, the normalized flow converges to a spherical metric, that we will refer to as the *limiting sphere*.

One of the remarkable properties of the two-dimensional Ricci flow is its *conformal invariance* (it preserves the conformal class). Since on an ancient non-flat solution the scalar curvature $R > 0$, by the Uniformization theorem one may parametrize the Ricci flow by the limiting sphere at time T, namely express

$$g(\cdot,t) = u(\cdot,t)\, g_{S^2}$$

where g_{S^2} denotes the standard metric on S^2. This metric can be written as

$$g_{S^2} = d\psi^2 + \cos^2\psi\, d\theta^2 \tag{2.2}$$

where ψ, θ denote the global coordinates on S^2. It follows that (2.1) is equivalent to the *logarithmic fast-diffusion* equation for the conformal factor u, namely

$$u_t = \Delta_{S^2} \log u - 2 \qquad \text{on } S^2 \times (-\infty, T) \tag{2.3}$$

where Δ_{S^2} denotes the Laplace Beltrami operator on S^2.

From the PDE point of view, it is natural to consider the *pressure function* $v := u^{-1}$ which evolves by

$$v_t = v\,\Delta_{S^2} v - |\nabla_{S^2} v|^2 + 2v^2, \qquad \text{on } S^2 \times (-\infty, T). \tag{2.4}$$

Typical examples of ancient *type I* solutions to the 2-dim Ricci flow are the *contracting spheres* given by a pressure

$$v_S(\psi,t) = \frac{1}{2(T-t)}.$$

These are examples ancient shrinking Ricci solitons. Explicit examples of ancient *type II* solutions are the *King-Rosenau* solutions [40, 51] which are defined by a pressure

$$v_K(\psi,t) = \mu\big(\coth(2\mu(T-t)) - \tanh(2\mu(T-t))\,\sin^2\psi\big)$$

for a parameter $\mu > 0$. These solutions are often called, in the context of *quantum field theory*, as the *sausage model* [24]. They are particularly interesting because they are not solitons. They are rotationally symmetric solutions and can be visualized (near $t = -\infty$) as two *cigars* (Barenblatt self-similar solutions) *glued* together to form a compact solution.

In [18], the author jointly with R. Hamilton and N. Sesum established the following classification result:

Theorem 2.2. *Let g be a smooth ancient compact solution to the Ricci flow* (2.1) *in dimension $n = 2$. Then, the solution g is either one the contracting spheres or one of the King-Rosenau solutions.*

Discussion on the proof of Theorem 2.2. Because the King-Rosenau solutions *are not solitons* (self-similar) they cannot be captured via previously developed techniques. In addition, the standard *Lyapunov functional* for this equation cannot be used, because when evaluated on the King-Rosenau solutions, it becomes unbounded as $t \to -\infty$. A different suitable Lyapunov functional was introduced in [18] and its monotonicity was used to show that the solution $v(\cdot, t)$ of (2.4) converges, as $t \to -\infty$, in the $C^{1,\alpha}$ norm (for any $\alpha < 1$) to a solution v_∞ of the steady state equation

$$Lv_\infty := v_\infty \Delta_{S^2} v_\infty - |\nabla_{S^2} v_\infty|^2 + 2\, v_\infty^2 = R_\infty\, v_\infty \qquad \text{a.e. on } \ S^2 \tag{2.5}$$

where $R_\infty := \lim_{t\to-\infty} R(\cdot, t)$, with R denoting the scalar curvature of the evolving surface. In particular, this shows that

$$R_\infty = 0 \quad \text{a.e. on } S^2.$$

However, R_∞ is not continuous; actually you expect that R_∞ is discontinuous at exactly two points (which are the tips of the two cigars that are "glued" to make the King-Rosenau solution).

Equation (2.5) fails to be strictly elliptic. Indeed, the limit v_∞ will vanish at points on S^2 that actually coincide with the points of discontinuity of R_∞. This causes the major technical obstacle in the classification of the backward limits v_∞. Using various geometric considerations and a priori estimates on the elliptic equation (2.5), it was shown in [18] that v_∞ vanishes at *exactly two points* and that R_∞ is *discontinuous* at those exactly two points. By choosing a suitable parametrization of the flow (using the conformal invariance of the equation) it follows that

$$v_\infty = C\, \cos^2 \psi, \qquad C \geq 0$$

(where ψ, θ are the global coordinates on S^2). In other words, the backward limit is either *trivial* ($C = 0$) or a *cylinder* ($C > 0$).

When $C = 0$, then it was shown in [18] that the isoperimetric ratio $I(t)$ for the evolving surface satisfies $I(t) \equiv 1$, for all t, forcing the ancient solution to be one of the contracting spheres. For this result a crucial role plays the suitable application of the isoperimetric estimate shown by R. Hamilton in [38].

When $C > 0$, then the ancient solution must be one of the King-Rosenau solutions. This is shown in [18] by expressing the metric g_{ij} in Euclidean coordinates (via the stereographic projection of the background sphere S^2) namely $g_{ij} = \bar{u}\, g_{\mathbb{R}^2} = \bar{v}^{-1}\, g_{\mathbb{R}^2}$, with $g_{\mathbb{R}^2}$ denoting the standard Euclidean metric, and considering the scaling invariant quantity

$$Q(x, y, t) := \bar{v}\, \big[\big(\bar{v}_{xxx} - 3\bar{v}_{xyy}\big)^2 + \big(\bar{v}_{yyy} - 3\bar{v}_{xxy}\big)^2 \big]. \tag{2.6}$$

It was remarked to the authors by S. Brendle that in complex variable notation $z := x + iy$, one has

$$Q = \bar{v}|\nabla_{zzz}\bar{v}|.$$

Also, in [14] an interesting geometric interpretation of the quantity Q was given by B. Chow.

One observes that $Q \equiv 0$ on all three: the King-Rosenau solution, the cigar solution and the cylinder. Actually the classification follows by establishing that

$$Q(\cdot, t) \equiv 0, \qquad \text{for all } t \in (-\infty, T) \tag{2.7}$$

and concluding from this identity that $\bar{v}$ must one of the King-Rosenau solutions. To achieve that $Q \equiv 0$, one shows, via the maximum principle, that $Q_{\max}(t)$ *is decreasing in* t. By also showing that $\lim_{t\to-\infty} Q_{\max}(t) = 0$ (using the classification of the backward limit that was discussed above) one concludes that (2.7) holds. One then concludes that the ancient solution must be one of the King-Rosenau solutions, finishing the proof of the Theorem.

The classification of two-dimensional *complete non-compact eternal solutions* of the Ricci flow was previously given by the author and N. Sesum in [19] (see also in [15, 34]). In fact, in this case one observes that the Ricci flow is equivalent to the fast diffusion equation

$$\frac{\partial u}{\partial t} = \Delta \log u, \qquad \text{on } \mathbb{R}^2 \times \mathbb{R} \tag{2.8}$$

since the evolving metric g may be expressed as $g = u\, g_{\mathbb{R}^2}$, where $g_{\mathbb{R}^2}$ denotes the standard Euclidean metric on $\mathbb{R}^2$. The following classification result was shown by the author and N. Sesum in [19]:

Theorem 2.3. *Let $g = u\, g_{\mathbb{R}^2}$ be a positive smooth eternal solution of the Ricci flow* (2.8) *on $\mathbb{R}^2 \times \mathbb{R}$ which defines a complete metric. Assume that for each $t \in \mathbb{R}$, the metric $g(\cdot, t)$ has finite width and bounded scalar curvature R. Then, u is a translating soliton of the form*

$$U(x, t) = \frac{2}{\beta\,(|x - x_0|^2 + \gamma\, e^{2\beta t})} \tag{2.9}$$

for some $x_0 \in \mathbb{R}^2$ and some constants $\beta > 0$ and $\gamma > 0$.

The assumption on the bound on the width of the metric g in Theorem 2.3 was removed by S.C. Chu in [15]. Theorems 2.2 and 2.3 together with the results in [15, 34] provide the classification of *all ancient or eternal* complete solutions to the 2-dim Ricci flow.

The classification of ancient compact solutions to the Ricci flow on a *three dimensional compact manifold* M^3 remains an open question. Such solutions arise as singularity models [30, 36, 48, 49].

In [48], G. Perelman introduced the notion of a *κ-noncollapsed* solution to the n-dimensional Ricci flow which has played a fundamental role in the classification of singularities. In [49], G. Perelman constructed a class of non-collapsed (as $t \to -\infty$) rotationally symmetric type II ancient solutions to the Ricci flow on S^3. This is the higher dimensional analogue of the King-Rosenau solutions. We will refer to it as the *Perelman solution*. It is conjectured that the the only *κ-noncollapsed* (as $t \to -\infty$) ancient solutions of the Ricci flow on S^3 are the Einstein (contracting spheres) and the Perelman solutions. On the other hand, V. Fateev constructed in [24] an interesting example of a collapsed ancient solution to the Ricci flow in dimension three (c.f. also in [5] for generalizations in higher dimensions).

L. Ni [47] showed that any type I ancient solution to the Ricci flow which is κ-noncollapsed and has positive operator has constant sectional curvature. More generally, S. Brendle, G. Huisken and C. Sinestrari [9] obtained the classification of ancient solutions to the Ricci flow in any dimension, under a suitable uniform curvature pinching condition, which forces the solution to have constant sectional curvature. The example of V. Fateev [24] shows that the pinching condition cannot be removed. The classification of ancient κ-noncollapsed solutions in the general case remains a challenging open question.

3. Ancient solutions to the Yamabe flow on S^n

Let (M, g_0) be a compact manifold without boundary of dimension $n \geq 3$. If $g = v^{\frac{4}{n-2}} g_0$ is a metric conformal to g_0, the scalar curvature R of g is given in terms of the scalar curvature R_0 of g_0 by

$$R = v^{-\frac{n+2}{n-2}} \big(- \bar{c}_n \Delta_{g_0} v + R_0 \, v \big)$$

where Δ_{g_0} denotes the Laplace-Beltrami operator with respect to g_0 and $\bar{c}_n = \frac{4(n-1)}{n-2}$.

In 1989 R. Hamilton introduced the *Yamabe flow*

$$\frac{\partial g}{\partial t} = -R \, g \tag{3.1}$$

as an approach to solve the *Yamabe problem* on manifolds of positive conformal Yamabe invariant, via parabolic PDE methods.

We refer the reader to the seminal work by R. Schoen [52] from 1984 which had resolved the Yamabe problem via elliptic variational methods. Previous important developments on this problem, under a positivity assumption on the scalar curvature, are those of N. Trudinger [56] and T. Aubin [4].

In its variational formulation, the Yamabe flow is the negative L^2-gradient flow of the total scalar curvature, restricted to a given conformal class. It may be interpreted as deforming a Riemannian metric to a conformal metric of constant scalar curvature, when this flow converges.

R. Hamilton [33] showed the existence of the normalized Yamabe flow (which is the re-parametrization of (3.1) to keep the volume fixed) for all time; moreover, in the case of an initial metric with negative scalar curvature, he established the exponential convergence of the flow to a metric of constant scalar curvature.

Since then, there have been a number of important works on the convergence of the Yamabe flow on a compact manifold without boundary to a metric of constant scalar curvature. B. Chow [13] showed the convergence of the flow, under the conditions that the initial metric is locally conformally flat and of positive Ricci curvature. The convergence of the flow for any locally conformally flat initial metric was shown by R. Ye [58]. H. Schwetlick and M. Struwe [54] obtained the convergence of the Yamabe flow on a general compact manifold under a suitable Kazdan-Warner type of condition that rules out the formation of bubbles and it is verified (via the positive mass Theorem) in dimensions $3 \leq n \leq 5$. The convergence result, in its full generality, was established by S. Brendle [6, 7] (up to a technical assumption, in dimensions $n \geq 6$, on the rate of vanishing of Weyl tensor at the points at which it vanishes): starting with any smooth metric on a compact manifold, the normalized Yamabe flow converges to a metric of constant scalar curvature. This provides a parabolic PDE proof of the Yamabe conjecture, as R. Hamilton had originally imagined.

In the special case where the background manifold M_0 is the sphere S^n and g_0 is the standard spherical metric g_{S^n}, the Yamabe flow evolving a metric

$$g = v^{\frac{4}{n-2}}(\cdot,t)\, g_{S^n}$$

takes (after rescaling in time by a constant) the form of the *fast diffusion equation*

$$(v^{\frac{n+2}{n-2}})_t = \Delta_{S^n} v - c_n v, \qquad c_n := \frac{n(n-2)}{4}. \tag{3.2}$$

Starting with any smooth metric g_0 on S^n, it follows by the results in the works [13, 23, 58] that the solution of (3.2) with initial data g_0 will become singular at some finite time $t < T$ and that v becomes spherical at time T, which means that after a normalization, the normalized flow converges to the spherical metric. In addition, v becomes extinct at T.

A metric $g = v^{\frac{4}{n-2}}\, g_{S^n}$ may also be expressed as a metric on $\mathbb{R}^n$ via stereographic projection. It follows that if $g = \bar{v}^{\frac{4}{n-2}}(\cdot,t)\, g_{\mathbb{R}^n}$ (where $g_{\mathbb{R}^n}$ denotes the standard metric on $\mathbb{R}^n$) evolves by the Yamabe flow (3.1), then $\bar{v}$ satisfies (after a rescaling in time) the fast diffusion equation on $\mathbb{R}^n$

$$(\bar{v}^p)_t = \Delta \bar{v}, \qquad p := \frac{n+2}{n-2}. \tag{3.3}$$

Observe that if $g = \bar{v}^{\frac{4}{n-2}}(\cdot,t)\, g_{\mathbb{R}^n}$ represents a smooth solution when lifted on S^n, then $\bar{v}(\cdot,t)$ satisfies the growth condition

$$\bar{v}(y,t) = O(|y|^{-(n-2)}), \qquad \text{as } |y| \to \infty.$$

Definition 3.1. The solution $g = v^{\frac{4}{n-2}}\, g_0$ to (3.1) is called ancient if it exists for all time $t \in (-\infty, T)$, where $T < \infty$. We will say that the ancient solution g is of type I, if it satisfies

$$\limsup_{t \to -\infty} \left(|t| \max_{M_0} |Rm|\,(\cdot,t) \right) < \infty$$

(where *Rm* is the Riemannian curvature of metric g). An ancient solution which is not of type I, will be called of type II.

Explicit examples of ancient solutions to the Yamabe flow on S^n are the *contracting spheres*, given by the conformal factor

$$v_S(p,t) = \left(\frac{4}{n+2}\, c_n\, (T-t) \right)^{\frac{n-2}{4}} \tag{3.4}$$

and the *King solutions* that can be expressed on $\mathbb{R}^n$ in closed from $g = \bar{v}_K(\cdot,t)\, g_{\mathbb{R}^n}$, where $\bar{v}_K$ is the radial function

$$\bar{v}_K(r,t) = \left(\frac{a(t)}{1 + 2b(t)\, r^2 + r^4} \right)^{\frac{n-2}{4}}. \tag{3.5}$$

It follows that the coefficients $a(t)$ and $b(t)$ satisfy a certain system of ODEs. This solution was discovered by J.R. King [40] (c.f. also in [8]). It is *not a soliton* and may be visualized, as $t \to -\infty$, as two Barenblatt self-similar solutions "glued" together to form a compact

solution to the Yamabe flow. It is the analogue of the King-Rosenau solution of the 2-dim Ricci flow which was discussed in the previous section, however the King solution is a type I ancient solution while the King-Rosenau solution is of type II. This reflects to the fact that the cigar self-similar solution of the 2-dim Ricci flow is a type II translating soliton while the Barenblatt self-similar solution of the Yamabe flow is a type I shrinker.

In analogy with the classification result by the author with R. Hamilton and N. Sesum in [18] (Theorem 2.2 in section 2), a natural question to raise is whether the analogous statement holds true for the Yamabe flow, that is, whether the contracting spheres and the King solution are the only compact ancient solutions to the Yamabe flow. This occurs not to be the case as the following discussion shows.

In [16], the author, jointly with M. del Pino and N. Sesum, constructed ancient radially symmetric solutions of the Yamabe flow (3.2) on S^n other than the contracting spheres (3.4) and the King solutions (3.5). The new solutions, as $t \to -\infty$, may be visualized (after re-normalization) as towers of k-spheres joint by short necks. Their curvature operator changes sign and they are *type II ancient* solutions. We will refer to them as *moving towers of bubbles*.

In what follows we will present the ansatz of the construction of the moving towers of bubbles, assuming for simplicity that $k = 2$ (which means that as $t \to -\infty$ the solution will resemble two spheres joint by a short neck). At the end, we will discuss the generalization to k-bubbles, $k > 2$.

Let $g = \bar{v}^{\frac{4}{n-2}}(\cdot, t)\, g_{\mathbb{R}^n}$ be a radially symmetric solution of (3.3) which becomes extinct at time T, namely $\bar{v} = \bar{v}(r, t)$ is a radial function on $\mathbb{R}^n$ that vanishes at T. One may introduce the cylindrical change of variables

$$u(x, \tau) = (T - t)^{-\frac{1}{p-1}} r^{\frac{2}{p-1}}\, \bar{v}(r, t), \quad r = e^x,\ t = T(1 - e^{-\tau})$$

where we recall that $p := (n + 2)/(n - 2)$. In this language equation (3.3) becomes

$$(u^p)_\tau = u_{xx} + \alpha u^p - \beta u, \quad \beta = \frac{(n-2)^2}{4}, \quad \alpha = \frac{p}{p-1} = \frac{n+2}{4}. \tag{3.6}$$

By suitable scaling we can make the two constants α and β in (3.6) equal to 1, so that from now on we will consider the equation

$$(u^p)_\tau = u_{xx} + u^p - u. \tag{3.7}$$

The steady states of equation (3.7), namely the solutions w of the equation

$$w_{xx} + w^p - w = 0, \qquad w(\pm\infty) = 0 \tag{3.8}$$

are given in closed form

$$w(x) = \left(\frac{k_n\, \lambda\, e^{\gamma x}}{1 + \lambda^2\, e^{2\gamma\, x}} \right)^{\frac{n-2}{2}} = (2\, k_n\, \mathrm{sech}(\gamma x + \log \lambda))^{\frac{n-2}{2}} \tag{3.9}$$

with parameter $\lambda > 0$ and constants γ, k_n given by

$$\gamma = \frac{1}{\sqrt{\beta}} = \frac{2}{n-2} \qquad \text{and} \qquad k_n = \left(\frac{4n}{n-2} \right)^{\frac{1}{2}}.$$

It is known that w is the only even, positive solution of (3.8), given in cylindrical coordinates, after stereographic projection, geometrically representing the conformal metric for a sphere. Observe that

$$w(x) = O(e^{-|x|}), \qquad \text{as } |x| \to \infty. \tag{3.10}$$

Since the new solution resembles two bubbles joint by a neck, one chooses the *ansatz* for an ancient solution $u(x,\tau)$ of (3.7) to be of the form

$$u(x,\tau) = (1+\eta(\tau))\, z(x,\tau) + \psi(x,\tau) \tag{3.11}$$

with

$$z(x,\tau) = w(x+\xi(\tau)) + w(x-\xi(\tau)) \tag{3.12}$$

for suitable parameter functions $\eta(\tau), \xi(\tau)$. The perturbation function $\psi(x,\tau)$ will converge to zero, as $\tau \to -\infty$, in a suitable norm. More precisely,

$$\xi(\tau) = \xi_0(\tau) + h(\tau)$$

for a suitable parameter function $h(\tau)$. Both parameter functions $h(\tau)$ and $\eta(\tau)$ decay in $|\tau|$, as $\tau \to -\infty$. It turns out that

$$\xi_0(\tau) := \frac{1}{2}\log(2b\,|\tau|)$$

which is a solution to

$$\dot{\xi} + b\, e^{-2\xi} = 0$$

for a specific constant $b > 0$. This equation is derived as a consequence of adjusting the parameters $h(\tau)$ and $\eta(\tau)$ so that the perturbation function ψ satisfies suitable orthogonality conditions which guarantee the solvability of the linearized equation around each bubble.

The following result was shown in [16].

Theorem 3.2 (The existence of the tower of bubbles [16]). *There exist numbers τ_0 and $b > 0$ and a solution $u(x,\tau)$ to (3.7) defined on $\mathbb{R} \times (-\infty, \tau_0]$, of the form (3.11)-(3.12) with $\xi := \frac{1}{2}\log(2b\,|\tau|) + h(\tau)$, such that the functions $\psi(x,\tau)$, $\eta(\tau)$ and $h(\tau)$ tend to zero in appropriate norms as $\tau \to -\infty$. Moreover, u defines a radially symmetric ancient solution to the Yamabe flow* (3.1) *on S^n which is of type II and its Ricci curvature changes its sign.*

The construction that leads to the proof of Theorem 3.2 can be generalized to give ancient solutions which may be visualized, as $\tau \to -\infty$, as a tower of k spheres joined by short necks. We refer to them as *moving towers of bubbles*. In terms of equation (3.7), for a given $k \geq 2$ we look for a solution of (3.7) of the form

$$u(x,\tau) = \sum_{j=1}^{k} (1+\eta_j(\tau)) w(x-\xi_j(\tau)) + \psi(x,\tau) \tag{3.13}$$

where the functions ξ_j are ordered and symmetrically arranged,

$$\xi_1(\tau) < \xi_2(\tau) < \ldots < \xi_k(\tau), \quad \xi_j(\tau) = -\xi_{k-j+1}(\tau). \tag{3.14}$$

The following was shown in [16]:

Theorem 3.3. *Given $k \geq 2$, there exist numbers τ_0 and $b > 0$ and a solution $u(x,\tau)$ to (3.7) defined on $\mathbb{R} \times (-\infty, \tau_0]$, of the form (3.13)-(3.14), with*

$$\xi_j(\tau) = \xi_{0j}(\tau) + h_j(\tau), \qquad \xi_{0j}(\tau) = (j - \frac{k+1}{2}) \log(b\,|\tau|) + \gamma_j \tag{3.15}$$

for certain explicit constants γ_j, where the functions $\psi(x,\tau)$, $\eta_j(\tau)$ and $h_j(\tau)$ tend to zero in appropriate norms as $\tau \to -\infty$.

The functions ξ_{0j} in the above statement solve the *first order Toda system*

$$b^{-1}\dot{\xi}_j(\tau) \; + \; e^{-(\xi_{j+1}-\xi_j)} - e^{-(\xi_j - \xi_{j-1})} = 0 \quad j = 1, \dots, k, \quad \tau \in (-\infty, -\tau_0]$$

with the conventions

$$\xi_0 \equiv -\infty, \quad \xi_{k+1} \equiv +\infty.$$

Discussion on the proof of Theorem 3.2. We will next indicate the main steps that were used in the proof of Theorem 3.2, referring the interested reader to [16] for the proofs of the results.

(i) One first defines the Banach space in which the error of approximation ψ given in (3.11) belongs and its associated norm. Also, one defines the spaces for the parameter functions $\eta(\tau)$ and $h(\tau)$ and their associated norms.

(ii) Using the ansatz (3.11)-(3.12) for the solution u, one shows that the perturbation term ψ satisfies an equation of the form

$$pz^{p-1}\partial_\tau \psi = \psi_{xx} - \psi + pz^{p-1}\psi + pz^{p-1}E(\psi) \tag{3.16}$$

where $E(\psi)$ denotes the nonlinear error term and z is given by (3.12). It is well known that w and w' are the eigenvectors of the approximating linear operator

$$L_0\psi := -\frac{1}{pw^{p-1}}\big(\psi_{xx} - \psi + pw^{p-1}\psi\big)$$

corresponding to the eigenvalues $\lambda_{-1} < 0$ and $\lambda_0 = 0$ of this operator, respectively. It is also well known that all the other eigenvalues of L_0 are positive.

(iii) The first part of the article [16] concerns with the study of *ancient solutions* to the linear problem

$$pz^{p-1}\partial_\tau \psi = \psi_{xx} - \psi + pz^{p-1}\psi + pz^{p-1}f, \qquad -\infty < \tau < \tau_0 \leq +\infty. \tag{3.17}$$

Assuming certain orthogonality conditions on f with respect to the eigenvectors w and w' of L_0, one establishes the existence of an ancient solution to the linear problem (3.17), satisfying the appropriate energy and L^2 estimates. The latter means that one can bound the weighted L^2-norm of a solution in terms of the weighted L^2-norm of the right hand side f. One also establishes certain weighted $W^{2,\sigma}$ estimates for solutions to (3.17). It follows that the solution ψ belongs to the Banach space which is the intersection of these L^2 and weighted $W^{2,\sigma}$ spaces. Let us denote by T the linear operator between our defined Banach spaces, so that $T(f)$ is the solution to the linear problem (3.17) satisfying the appropriate orthogonality conditions.

(iv) The second part of the article [16] concerns with the study of the nonlinear equation (3.16). One applies the linear theory to the nonlinear equation to establish the existence of a solution ψ to (3.16), by solving the equation $T(E(\psi)) = \psi$. One first shows that this can be achieved, under the assumption that $E(\psi)$ satisfies the right orthogonality conditions with respect to w and w'. The main tool in this proof is the fixed point Theorem and subtle estimates of the error terms in the appropriate norms.

(v) In the final part of the proof in [16] one shows how to adjust the parameters $\eta(\tau)$ and $h(\tau)$ so that the error term $E(\psi)$ in (3.16) indeed satisfies the right orthogonality conditions. One sees that this is equivalent to solving a certain nonlinear system of ODE's for $\eta(\tau)$, $h(\tau)$. The existence of solutions to this system is shown via the fixed point Theorem and subtle a'priori estimates.

Remark 3.1. Theorem 3.2 shows that the classification of ancient solutions to the compact Yamabe flow on S^n poses a rather difficult task. On the other hand, it gives a new way for constructing special solutions. It shows how one may glue two or more ancient solutions of a parabolic equation, in our case of equation (3.7), to construct a new ancient solution of the same equation. This parabolic gluing becomes more and more apparent as $\tau \to -\infty$, since as $\tau \to +\infty$ it is known that the conformal factor approaches the standard sphere.

Remark 3.2. Gluing techniques relying on linearization and perturbation theory have been used in many elliptic settings, such as the gluing of manifolds of constant scalar curvature to produce another manifold of constant scalar curvature (c.f. in [43, 44, 53]) and the gluing of two constant mean curvature surfaces to produce another constant mean curvature surface [39]. In the context of elliptic semilinear equations such methods have been used in [22, 25]. Also, embedded self similar solutions of the mean curvature flow have been constructed in [45] with the use of elliptic gluing techniques. The work in [16] shows that such techniques can be used also in the parabolic setting. One expects that the methods in [16] could be adopted to other geometric flows as well.

References

[1] Angenent, S. B. and Velázquez, J. J. L., *Degenerate neckpinches in mean curvature flow*, J. Reine Angew. Math. **482** (1997), 15–66.

[2] Angenent, S.B., Isenberg, J., and Knopf, D., *Formal matched asymptotics for degenerate Ricci flow neckpinches* Nonlinearity **24** (2011), 2265–2280.

[3] Angenent, S. B., Caputo, M.C., and Knopf, D., *Minimally invasive surgery for Ricci flow singularities*, J. Reine Angew. Math. **672** (2012), 39–87.

[4] Aubin, T., *The scalar curvature*, Differential geometry and relativity, Vol. 3, Reidel, Dordrecht (1976), 5–18, J. Math. Pures Appl. (9) **55** (1976), 269–296.

[5] Bakas, I., Kong, S., and Ni, L., *Ancient solutions of Ricci flow on spheres and generalized hopf fibrations* , J. Reine Angew. Math. **663** (2012), 20–248.

[6] Brendle, S., *Convergence of the Yamabe flow for arbitrary initial energy*, J. Differential Geom. **69** (2005), 217–278.

[7] ———, *Convergence of the Yamabe flow in dimension 6 and higher*, Invent. Math. **170** (2007), 541–576.

[8] ———, *An analogue of the Rosenau solution for Yamabe flow*, private communication.

[9] Brendle, S., Huisken, G. and Sinestrari, C., *Ancient solutions to the Ricci flow with pinched curvature*, Duke Math. J. **158** (2011), 537–551.

[10] Brendle, S. and Schoen, R., *Classification of manifolds with weakly 1/4-pinched curvatures*, Acta Math. **200** (2008), 1–13.

[11] ———, *Manifolds with 1/4-pinched curvature are space forms*, J. Amer. Math. Soc. **22** (2009), 287–307.

[12] Chow, B., *The Ricci flow on the* 2*-sphere*, J.Diff.Geom. **33** (1991), 325–334.

[13] ———, *The Yamabe flow on locally conformally flat manifolds with positive Ricci curvature*, Comm. Pure Appl. Math. **65** (1992), 1003–1014.

[14] ———, On a formula of Daskalopoulos, Hamilton and Sesum, arXiv:1209.1784.

[15] Chu, S.C., *Type II ancient solutions to the Ricci flow on surfaces*, Comm. Anal. Geom. **15** (2007), 195–216.

[16] Daskalopoulos, P., del Pino, M., and Sesum, N. *Type II ancient compact solutions to the Yamabe flow*, arXiv:1306.0859.

[17] Daskalopoulos, P., Hamilton, R., and Sesum, N., *Classification of compact ancient solutions to the curve shortening flow*, J. Differential Geom. **84** (2010), 455–464.

[18] ———, *Classification of ancient compact solutions to the Ricci flow on surfaces*, J. Differential Geom. **91** (2012), 171–214.

[19] Daskalopoulos, P. and Sesum, N., *Eternal solutions to the Ricci flow on* R^2, Int. Math. Res. Not. (2006) Art. ID 83610, 20 pp.

[20] ———, *Type II extinction profile of maximal solutions to the Ricci flow in* R^2, J. Geom. Anal. **20** (2010), 565–591.

[21] ———, *The classification of locally conformally flat Yamabe solitons*, Adv. Math. **240** (2013), 346–369.

[22] del Pino, M., Dolbeault, J., and Musso, M., *"Bubble-tower" radial solutions in the slightly supercritical Brezis-Nirenberg problem*, J. Differential Equations **193** (2003), 280–306.

[23] del Pino, M. and Sáez, M., *On the extinction profile for solutions of* $u_t = \Delta u^{(N-2)/(N+2)}$, Indiana Univ. Math. J. **50** (2001), 611–628.

[24] Fateev, V. A., *The sigma model (dual) representation for a two-parameter family of integrable quantum field theories*, Nuclear Phys. B **473** (1996), 509–538.

[25] Ge, Y., Jing, R. and Pacard, F., *Bubble towers for supercritical semilinear elliptic equations*, J. Funct. Anal. **221** (2005), no. 2, 251–302.

[26] Giga, Y. and Kohn, R. V., *Asymptotically self-similar blow-up of semilinear heat equations*, Comm. Pure Appl. Math. **38** (1985), 297–319.

[27] _____, *Characterizing blowup using similarity variables*, Indiana Univ. Math. J. **36** (1987), 1–40.

[28] _____, *Nondegeneracy of blowup for semilinear heat equations*, Comm. Pure Appl. Math. **42** (1989), 845–884.

[29] Hamilton, R., *The Ricci flow on surfaces*, Mathematics and General Relativity **71** (1988), 237–261.

[30] _____, *Three-manifolds with positive Ricci curvature*, J. Differential Geom. **17** (1982), 255–306.

[31] _____, *Four-manifolds with positive curvature operator*, J. Differential Geom. **24** (1986), 153 –179.

[32] _____, *Four-manifolds with positive isotropic curvature*, Comm. Anal. Geom. **5** (1997), 1–92.

[33] _____, *Lectures on geometric flows* (1989) unpublished.

[34] _____, *Eternal solutions to the Ricci flow*, J. Diff. Geom. **38** (1993), 1–11.

[35] _____, *A matrix harnack estimate for the heat equation*, Comm. Anal. Geom. **1** (1993), 113–126.

[36] _____, *The formation of singularities in the Ricci flow*, Surveys in Differential Geometry **II**, International Press, Cambridge, Mass. (1995), 7–136.

[37] _____, *Harnack estimate for the mean curvature flow*, J. Differential Geom. **41** (1995), 215–226.

[38] _____, *An isoperimetric estimate for the ricci flow on the two-sphere*, Annals of Math. Studies **137**, Princeton Univ. Press (1996).

[39] Kapouleas, N., *Complete constant mean curvature surfaces in Euclidean three-space*, Ann. of Math. **131** (1990), 239-330.

[40] King, J.R., *Exact polynomial solutions to some nonlinear diffusion equations*, Physica. D **64** (1993), 39–65.

[41] Koch, G., Nadirashvili, N., Seregin, G. A., and Sverák, V. *Liouville theorems for the Navier-Stokes equations and applications*, Acta Math. **203** (2009), 83–105.

[42] Li, P. and Yau, S. T., *On the parabolic kernel of the Schrödinger operator*, Acta Math. **156** (1986), 153–201.

[43] Mazzeo, R., Pacard, F., and Pollack, D., *Connected sums of constant mean curvature surfaces in Euclidean 3 space*, J. Reine Angew. Math. **536** (2001), 115–165.

[44] Mazzeo, R., Pollack, D., and Uhlenbeck, K., *Connected sum constructions for constant scalar curvature metrics*, Topol. Methods Nonlinear Anal. **6** (1995), 207–233.

[45] Nguyen, X.H., *Construction of Complete Embedded Self-Similar Surfaces under Mean Curvature Flow*, arXiv:1004.2657.

[46] Merle, F. and Zaag, H. *Optimal estimates for blowup rate and behavior for nonlinear heat equations*, Comm. Pure Appl. Math. **51** (1998), 139–196.

[47] Ni, L., *Closed type I ancient solutions to Ricci flow in Recent Advances in Geometric Analysis*, Adv. Lec. Math. **11** International Press, Somerville, Mass. (2010), 147–150.

[48] Perelman, G., *The entropy formula for the Ricci flow and its geometric applications*, arXiv:math/0211159.

[49] ———, *Ricci flow with surgery on three-manifolds*, arXiv:math/0303109.

[50] ———, *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds*, arXiv:math/0307245.

[51] Rosenau, P., *Fast and superfast diffusion processes*, Phys. Rev. Lett. **74** (1995), 1056–1059.

[52] Schoen, R., *Conformal deformation of a Riemannian metric to constant scalar curvature*, J. Differential Geom. **20** (1984), 479–495.

[53] ———, *The existence of weak solutions with prescribed singular behavior for a conformally invariant scalar equation*, Comm. Pure and Appl. Math. **XLI** (1988), 317-392.

[54] Schwetlick, H. and Struwe, M., *Convergence of the Yamabe flow for "large" energies*, J. Reine Angew. Math. **562** (2003), 59–100.

[55] Souplet, P. and Zhang, Q., *Sharp gradient estimate and Yau's Liouville theorem for the heat equation on noncompact manifolds*, Bull. London Math. Soc. **38** (2006), 1045–1053.

[56] Trudinger, N., *Remarks concerning the conformal deformation of Riemannian structures on compact manifolds*, Ann. Scuola Norm. Sup. Pisa **22** (1968), 265–274.

[57] Yau, S. T., *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. **28** (1975), 201–228.

[58] Ye, R., *Global existence and convergence of Yamabe flow*, J. Differential Geom. **39** (1994), 35–50.

Department of Mathematics, Columbia University, New York, USA
E-mail: pdaskalo@math.columbia.edu

Quantized vortex filaments in complex scalar fields

Robert L. Jerrard

Abstract. We survey a family of problems in which one seeks to prove that, for a complex-valued function solving a semilinear partial differential equation, energy concentrates, in certain scaling limits, around a codimension 2 submanifold solving a geometric problem. The equations in question arise from physical models, and the energy concentration sets are often naturally interpreted as "quantized vortex filaments." One can hope to describe these vortex filaments in a variety of types of PDE, including elliptic (describing an equilibrium of a physical system), parabolic (often describing flow toward an equilibrium) and hyperbolic or dispersive (describing different kinds of oscillations and wave propagation). There are a large number of results about elliptic and parabolic equations, although some significant open problems remain, and less is known about hyperbolic and (especially) dispersive equations.

Mathematics Subject Classification (2010). 35B40, 35B25, 49Q15.

Keywords. Semilinear PDEs, singular limit, geometric measure theory, vortex filament, Ginzburg-Landau.

1. Introduction

In this note we survey a class of problems in which one seeks to establish a relationship between complex-valued functions solving certain semilinear partial differential equations, on the one hand, and codimension 2 submanifolds of $\mathbb{R}^N$ for some $N \geq 3$ that solve certain geometric problems, on the other hand.

The relationship that we have in mind can be informally stated in a number of related ways, including for example:

"most level sets of solutions of the PDE are either nearly trivial or close to a solution of the geometric problem."

or

"energy of solutions of the the PDE concentrates around a submanifold solving the geometric problem."

One can attempt to prove this sort of statement for equations of elliptic, parabolic, hyperbolic or Schrödinger type, and one would expect the associated geometric problem to have the same type as the semilinear PDE. Many results of this sort are known for elliptic and parabolic equations, and much less is known about the hyperbolic and Schrödinger cases.

Proceedings of the International Congress of Mathematicians, Seoul, 2014

We will focus on these sorts of questions for equations related to the (scaled) Ginzburg-Landau[1] functional

$$E_\varepsilon(u) := \tfrac{1}{|\ln \varepsilon|} \int_\Omega \frac{|\nabla u|^2}{2} + \frac{1}{4\varepsilon^2}(|u|^2-1)^2, \qquad u \in H^1_{loc}(\Omega;\mathbb{C}) \tag{1.1}$$

where we take Ω to be an open subset of $\mathbb{R}^N$ for some $N \geq 3$; for evolution equations we normally take $\Omega = \mathbb{R}^N$. In this context, it is often natural to interpret an energy concentration set as a "quantized vortex filament". We are interested in this functional, and in the equations below, when $0 < \varepsilon \ll 1$.

Associated to $E_\varepsilon(\cdot)$ we can consider the equations

$$-\Delta u + \frac{1}{\varepsilon^2}(|u|^2-1)u = 0 \tag{1.2}$$

$$u_t - \Delta u + \frac{1}{\varepsilon^2}(|u|^2-1)u = 0 \tag{1.3}$$

$$u_{tt} - \Delta u + \frac{1}{\varepsilon^2}(|u|^2-1)u = 0 \tag{1.4}$$

$$i|\log\varepsilon| u_t - \Delta u + \frac{1}{\varepsilon^2}(|u|^2-1)u = 0. \tag{1.5}$$

The scaling factor $|\ln\varepsilon|$ in (1.5) will be motivated later. These are, respectively

- the Euler-Lagrange equation for E_ε;
- the L^2 gradient flow for E_ε;
- the Euler-Lagrange equation for the Minkowskian analog of E_ε, in which the term $|\nabla u|^2$ is replaced by $\eta^{\alpha\beta}\,\partial_\alpha u\,\partial_\beta u$, where η denotes the Minkowski metric;
- an infinite-dimensional Hamiltonian system for which the Hamiltonian is exactly the Ginzburg-Landau energy E_e. This equations is known as the Gross-Pitaevskii equation. The reason for the factor $|\log\varepsilon|$ will appear below.

For (1.4) or (1.5) in high dimensions, it is convenient to replace the nonlinearity $(|u|^2-1)u$ by one of the form $f(|u|^2)u$, where $\operatorname{sign}(f(s)) = \operatorname{sign}(s-1)$, and satisfying growth conditions that render the equation globally well-posed in the energy space.

When one considers the geometry of codimension 2 submanifolds, the formal counterpart of the Ginzburg-Landau functional is simply the functional that associates to a submanifold its $N-2$-dimensional Hausdorff measure:

$$\mathcal{A}(\Gamma) := \mathcal{H}^{N-2}(\Gamma), \qquad \Gamma \text{ a codimension 2 submanifold of } \Omega \subset \mathbb{R}^N. \tag{1.6}$$

A precise asymptotic relationship between E_ε and $\mathcal{A}$ (or rather, an extension of $\mathcal{A}$ to a more suitable functional setting) is described in Section 3 below. Associated to this functional, parallel to the above, we can consider the following family of problems for codimension 2 submanifolds of Ω, or of $(0,T)\times\Omega$, in the case of evoution problems:

$$\text{mean curvature } = 0 \tag{1.7}$$

[1]"Ginzburg-Landau" is in some ways a misnomer, but a common and convenient one, and one that we will use frequently.

$$\text{velocity - mean curvature} = 0 \tag{1.8}$$

$$\text{Minkowskian mean curvature } = 0 \tag{1.9}$$

$$\text{velocity - } \mathbb{J}(\text{mean curvature}) = 0. \tag{1.10}$$

These are, at least on a formal level

- the Euler-Lagrange equation for $\mathcal{A}$;
- the L^2 gradient flow for $\mathcal{A}$;
- the Euler-Lagrange equation for the Minkowskian analog of $\mathcal{A}$;
- an infinte-dimensional Hamiltonian system for which the Hamiltonian is exactly the $N-2$ area functional $\mathcal{A}$.

In the Schrödinger-type flow (1.10), at a point p in an oriented codimension 2 submanifold $\Gamma \subset \mathbb{R}^N$, we write $\mathbb{J}$ for the operator $N_p\Gamma \to N_p\Gamma$ corresponding to rotation by 90 degrees in the 2-dimensional normal plane $N_p\Gamma$ to Γ at p, with the sense of the rotation fixed in some way using the orientation of Γ. In the physical case $N = 3$, which is also the only case in which anything nontrivial is known about (1.10), the equation reduces to the *binormal curvature flow* of curves, which can be written parametrically in the form

$$\partial_t \gamma = \partial_s \gamma \times \partial_{ss} \gamma, \qquad\qquad |\partial_s \gamma|^2 = 1$$

for $\gamma : \mathbb{R} \times M^1 \to \mathbb{R}^3$, where M^1 is the circle or the real line. Here $\partial_{ss}\gamma$ is the curvature vector of the curve at the point $\gamma(t, s)$, and $\mathbb{J} = \partial_s \gamma \times$.

As we will recall, results relating the elliptic problems (1.2) and (1.7), and the parabolic problems (1.3) and (1.8), began to appear 20 or more years ago, and for both classes of problems, deep and definitive results about certain questions have been known for more than 10 years. Nonetheless, some open problems remain even in these classical areas. Another issue that has been well-understood for more than 10 years is the convergence, in a suitable sense, of the family of functionals $E_\varepsilon \to \mathcal{A}$, as $\varepsilon \to 0$. Much less is known about the wave and Schrödinger problems, and the few existing results are mostly rather recent.

We close this introduction by mentioning some of the many related problems that we will not discuss in any detail. These include

- questions about relationships between *real*-valued functions solving equations like (1.2) - (1.4), and *hypersurfaces* solving geometric problems like (1.7) - (1.9). (In this setting there is no Schrödinger equation, and no notion of binormal). There is a huge body of literature on these problems, which are better-understood than the questions about $\mathbb{C}$-valued functions and codimension 2 submanifolds on which we focus.
- gauge theoretic versions of the above problems. These are significant in many physical applications. In general, for the family of questions that we consider, there are a number of results about $U(1)$ gauge theories, such as the Abelian Higgs model, whereas much less is known about nonabelian gauge theories.
- parallel relations betwen functions $\Omega \subset \mathbb{R}^2 \to \mathbb{C}$ and codimension 2 submanifolds of Ω, *i.e.* collections of points. In this situation, the functional that governs the geometry of the submanifolds, corresponding to $\mathcal{A}$, simply counts the number of points, and the $\varepsilon \to 0$ limit of solutions of equations such as (1.2) - (1.5) is governed, at least in certain situations, by the "next-order" energy, which involves interactions between points, see [7].

2. Vorticity, energy, and balance laws

2.1. Some physical quantities. The equations that we study arise in quantum physics, as models of wave functions associated to superfluids (1.5) or simplified models (in which the magnetic field is neglected) of superconductors (1.2), (1.3). In these settings, the codimension 2 submanifolds in which we are interested are naturally interpreted as "vortex submanifolds".

Indeed, in models coming from quantum mechanics, if $u \in H^1_{loc}(\Omega;\mathbb{C})$ with Ω an open subset of $\mathbb{R}^N$, we introduce the quantities

$$|u|^2 := \text{ density} \tag{2.1}$$

$$e_\varepsilon(u) := \frac{|\nabla u|^2}{2} + \frac{1}{4\varepsilon^2}(|u|^2-1)^2 = \text{ energy density} \tag{2.2}$$

$$ju := -\frac{i}{2}(\bar u\, du - u\, d\bar u) \;=\; \text{momentum 1-form} \tag{2.3}$$

$$Ju := \frac{1}{2} d\, ju = \text{ vorticity 2-form.} \tag{2.4}$$

When considering the wave equation (1.4) we will also encounter the quantity

$$\ell_\varepsilon(u) := \frac{-|\partial_t u|^2 + |\nabla u|^2}{2} + \frac{1}{4\varepsilon^2}(|u|^2-1)^2, \tag{2.5}$$

which is just the Minkowskian analog of the energy density. The names that we have given in (2.1) - (2.4) are reasonable on physical grounds. The definition of vorticity has the appealing feature that if we write $u = u^1 + iu^2$, then

$$\begin{aligned} Ju = du^1 \wedge du^2 \;&=\; \sum_{i<j} (\partial_i u^1 \partial_j u^2 - \partial_i u^2 \partial_j u^1) dx^i \wedge dx^j \\ &= \text{ the pullback by } u \text{ of the area form on } \mathbb{C}. \end{aligned} \tag{2.6}$$

Thus Ju is a sort of Jacobian determinant, which motivates the notation Ju. This has a couple of useful consequences. First, if $N = 3$ and we identify the vorticity vector field as the one dual to Ju in a rather natural sense, then

$$\text{vorticity vector field } \;=\; \nabla u^1 \times \nabla u^2,$$

and it follows that integral curves of the vorticity vector field are exactly level curves of u, where u is smooth enough and nondegenerate. In $\mathbb{R}^N$, it is similarly true that where u is smooth and nondegenerate, one can associate to the vorticity 2-form a distribution of $N-2$-planes, and that level sets of u are integral manifolds of the vorticity distribution. It also follows from (2.6) that if

$$\varphi \in \mathcal{D}^{N-2}(\mathbb{R}^N) := \{\text{smooth, compactly supported } N-2\text{-forms on } \mathbb{R}^N\ \}$$

then

$$\int_{\mathbb{R}^N} \varphi \wedge Ju \;=\; \int_{z\in\mathbb{C}} \left(\int_{u^{-1}\{z\}} \varphi \right) d\,\text{area} \tag{2.7}$$

if u is smooth enough, in which case $u^{-1}\{z\}$ is a smooth $N-2$-dimensional submanifold with a natural induced orientation, for almost every $z \in \mathbb{C}$. Formulas of the above sort continue to hold for less smooth u, including $u \in H^1(\Omega;\mathbb{C})$, provided the integrals $\int_{u^{-1}\{z\}} \varphi$ on the right-hand side are understood in a suitable (weak) sense.

2.2. How to picture a vortex filament. The simplest solutions of the elliptic Ginzburg-Landau system (1.2) that possess a "vortex filament" have the form

$$u_\varepsilon(x_1,\dots,x_N) = f(\frac{r}{\varepsilon})e^{i\theta}, \quad \text{where } r=\sqrt{x_1^2+x_2^2} \text{ and } e^{i\theta}=\frac{x_1+ix_2}{r} \tag{2.8}$$

for some smooth f, which among other attributes[2] satisfies

$$f(0)=0, \qquad f'>0, \qquad f(s)\to 1 \text{ as } s\to\infty. \tag{2.9}$$

More generally, formal expansions of the equations (1.2)-(1.5) suggest that at least some solutions should, to leading order, resemble a function of the form (2.8), (2.9), in a suitable coordinate system adapted to the local geometry of some vortex submanifold, and possibly perturbed by a multiplicative phase or in other ways.

It is straightforward to compute the limiting behaviour, as $\varepsilon\to 0$, of quantities such as the vorticity and energy density for a family of functions of the form (2.8), (2.9). One finds that, writing $\Gamma := \{x\in\mathbb{R}^N : x_1=x_2=0\}$,

$$\int \phi \frac{e_\varepsilon(u_\varepsilon)}{|\log\varepsilon|}\,dx \;\longrightarrow \pi\int_\Gamma \phi\, d\mathcal{H}^{N-2} \qquad \text{for all } \phi\in C_c(\mathbb{R}^N) \tag{2.10}$$

$$\int \varphi\wedge Ju_\varepsilon \;\longrightarrow \pi\int_\Gamma \varphi \qquad \text{for all } \varphi\in\mathcal{D}^{N-2}(\mathbb{R}^N) \tag{2.11}$$

$$\int S : \frac{\nabla u_\varepsilon\otimes\nabla u_\varepsilon}{|\log\varepsilon|}\,dx \longrightarrow \pi\int_\Gamma S : P^\perp\, d\mathcal{H}^{N-2} \quad \text{for all } S\in C_c(\mathbb{R}^N; M^{N\times N}) \tag{2.12}$$

as $\varepsilon\to 0$. In the final assertion above.

- $\nabla u_\varepsilon\otimes\nabla u_\varepsilon$ is the matrix whose i,j entry is $\frac{1}{2}(u_{x_i}\bar{u}_{x_j}+\bar{u}_{x_i}u_{x_j})$,
- $A:B$ denotes the inner product $A:B := \mathrm{Tr}(A^TB)$ in the space $M^{N\times N}$ of $N\times N$ matrices,
- at $x\in\Gamma$, $P^\perp(x)$ denotes the matrix corresponding to projection onto the 2-dimensional normal space $(T_x\Gamma)^\perp$.

More generally, the same formal computations mentioned above suggest that (2.10) - (2.12) may hold, under good conditions, for sequences of solutions of (1.2) - (1.5) (with some modifications for the wave equation (1.4)), for some Γ that is enough like an $N-2$-dimensional submanifold that one can at least define some version of $T_x\Gamma$ almost everywhere, and hence make sense of $P^\perp$.

2.3. Some balance laws. In every case, some parallels between the equation and the geometric problem are visible on the level of balance laws. These can be seen as providing insight into known connections, and lending support to conjectured connections, between the PDEs and the geometric problems.

On the PDE side:

- a solution $u_\varepsilon\in H^1(\Omega;\mathbb{C})$ of the elliptic equation (1.2) satisfies

$$\int \nabla X : (I-\frac{\nabla u_\varepsilon\otimes\nabla u_\varepsilon}{e_\varepsilon(u_\varepsilon)})\;\frac{e_\varepsilon(u_\varepsilon)}{|\log\varepsilon|}\,dx \;=\; 0 \qquad \text{for all } X\in C_c^\infty(\Omega;\mathbb{R}^N). \tag{2.13}$$

[2]In fact f solves the equation $-f''-\frac{1}{r}f'+\frac{1}{r^2}f+\frac{1}{\varepsilon^2}(f^2-1)f=0$, the details of which will not matter for our discussion.

(This, and the identities below, are written in a way that seeks to emphasize their similarities to the geometric balance laws that follow.)

- a smooth solution $u_\varepsilon : (0,T)\times\mathbb{R}^N \to \mathbb{C}$ of the parabolic equation (1.3) satisfies

$$\frac{d}{dt}\int \phi\, \frac{e_\varepsilon(u_\varepsilon)}{|\log\varepsilon|}dx = -\int \phi\, \frac{|\partial_t u_\varepsilon|^2}{|\log\varepsilon|}\,dx \;-\int \nabla^2\phi : (I - \frac{\nabla u_\varepsilon\otimes\nabla u_\varepsilon}{e_\varepsilon(u_\varepsilon)})\,\frac{e_\varepsilon(u_\varepsilon)}{|\log\varepsilon|}\,dx \tag{2.14}$$

for all $\phi\in C_c^\infty(\mathbb{R}^N)$.

- a smooth solution $u_\varepsilon : (0,T)\times\mathbb{R}^N \to \mathbb{C}$ of the wave equation (1.4) satisfies

$$\int DX : (\eta - \frac{(\eta\, Du_\varepsilon)\otimes(\eta\, Du_\varepsilon)}{\ell_\varepsilon(u_\varepsilon)})\;\frac{\ell_\varepsilon(u_\varepsilon)}{|\log\varepsilon|}\,dx\,dt \;=\; 0 \tag{2.15}$$

for $X\in C_c^\infty(\mathbb{R}^{1+N};\mathbb{R}^{1+N})$, where $Du = (\partial_t u, \nabla u)$ and $\eta = \mathrm{diag}(-1,1,\dots,1)$ represents the Minkowski metric. This is in fact the exact counterpart of (2.13) in Minkowski spacetime.

- Finally, a smooth solution $u_\varepsilon : (0,T)\times\mathbb{R}^N \to \mathbb{C}$ of the Gross-Pitaevskii equation (1.5) satisfies

$$\frac{d}{dt}\int \varphi\wedge Ju_\varepsilon \;=\; \int \nabla(\star d\varphi) : \frac{\nabla u_\varepsilon\otimes\nabla u_\varepsilon}{|\log\varepsilon|}\,dx \qquad \text{for all } \varphi\in\mathcal{D}^{N-2}(\mathbb{R}^N) \tag{2.16}$$

where $\nabla(\star d\varphi)$ denotes the $N\times N$ matrix obtained as the gradient of the vector field $\star d\varphi$ that is dual in a natural way to the $N-1$-form $d\varphi$. In particular, if $N=3$ and we identify the 1-form φ with a vector field, then we may identify $\nabla(\star d\varphi)$ with the 3×3 matrix $\nabla(\nabla\times\varphi)$.

The factor of $|\log\varepsilon|^{-1}$ on the right-hand of (2.16) side is more or less necessary, for our purposes (compare (2.12)), and can be seen as the reason for our choice of scaling for the Gross-Pitaevskii equation (1.5).

On the geometric side, we have a family of balance laws with very similar structure.

- A smooth minimal surface $\Gamma\subset\Omega$ satisfies an identity we can write as

$$\int_\Gamma \nabla X : (I - P^\perp)\, d\mathcal{H}^{N-2} \;=\; 0 \qquad \text{for all } X\in C_c^\infty(\Omega;\mathbb{R}^N). \tag{2.17}$$

- If $(\Gamma_t)_{t\in(0,\infty)}$ is a smooth family of codimension 2 submanifolds evolving by mean curvature (1.8), then

$$\frac{d}{dt}\int_{\Gamma_t}\phi\, d\mathcal{H}^{N-2} = -\int_{\Gamma_t}\Big[\phi|H|^2 + \nabla^2\phi : (I-P^\perp)\Big]d\mathcal{H}^{n-2}dx \tag{2.18}$$

for all $\phi\in C_c^\infty(\mathbb{R}^N)$, where H denotes the mean curvature vector along Γ_t.

- If Γ is a smooth timelike submanifold of $\mathbb{R}^{1+N}$ with vanishing mean curvature, then

$$\int_\Gamma DX : (\eta - P^\perp_{mink})\, d\lambda^{1,N-2} \;=\; 0 \qquad \text{for all } X\in C_c^\infty(\mathbb{R}^{1+N};\mathbb{R}^{1+N}) \tag{2.19}$$

where $\lambda^{1,N-2}$ denotes the Minkoskian area measure on a codimension 2 submanifold, see (6.1) below, and $P^{\perp}_{mink}$ denotes projection with respect to the Minkowski metric onto the (Minkowskian) orthogonal complement of $T_{(t,x)}\Gamma$. Explicitly, the integrand can be written $\partial_\alpha X_\beta(\eta^{\alpha\beta} - n_1^\alpha n_1^\beta - n_2^\alpha n_2^\beta)$, where n_1, n_2 satisfy $\eta_{\alpha\beta} n_i^\alpha n_j^\beta = \delta_{ij}$ and $\eta_{\alpha\beta} n_i^\alpha \tau^\beta = 0$ for all $\tau \in T_{(t.x)}\Gamma$. Identity (2.19) is in fact the exact counterpart of (2.17) in Minkowski spacetime.

- if $(\Gamma_t)_{t\in(0,\infty)}$ is a smooth enough family of codimension 2 oriented submanifolds evolving by binormal mean curvature (1.10), then

$$\frac{d}{dt}\int_{\Gamma_t} \varphi = \int_{\Gamma_t} \nabla(\star d\varphi) : P^{\perp}\, d\mathcal{H}^{N-2} \qquad \text{for all } \varphi \in \mathcal{D}^{N-2}(\mathbb{R}^N). \tag{2.20}$$

2.4. Passage to limits on a formal level. Note that if one has a sequence of solutions $(u_\varepsilon)_{\varepsilon\in(0,1]}$ of the elliptic equation (1.2) for which one is somehow able to verify that (2.10), (2.12) hold for some limiting Γ, then one can directly deduce from (2.13) that the limiting Γ satisfies (2.17).

It is similarly true for the Gross-Pitaevskii equation (1.5) that the identity (2.16) should rather directly converge to the identity (2.20), if one has a sequence of solutions that can *somehow* be shown to satisfy (2.11) and (2.12) at every t, for some limiting family $(\Gamma_t)_{t\in(0,T)}$. Of course, one would expect this to be much harder for an equation of Schrödinger type than for an elliptic equation, and indeed this seems to be the case.

For the parabolic equations, to carry out a parallel (or perhaps slightly weaker) passage to the limit, one needs both to know that (2.10), (2.12) hold, and to relate the L^2 density $|\partial_t u_\varepsilon|^2$ of the velocity field to the squared mean curvature of the limiting object, and for the hyperbolic equation, one needs Lorenz-invariant analogs of (2.10), (2.12). So the parallels between the PDEs and the geometric problems are more subtle for these models, on the level of balance laws, but still plainly visible.

2.5. The necessity of geometric measure theory. In the $\varepsilon \to 0$ limit of equations (1.2) - (1.5), the energy/vorticity concentration sets associated to a sequence of solutions are known or believed not to be smooth embedded submanifolds, but rather to have singular points for suitable data. Thus any attempt to describe the global geometry of these concentration sets must employ a notion of solution of the associated geometric problem that is insensitive to singularities. So one is naturally lead to consider weak solutions of the geometric problems (1.7) - (1.10).

For submanifolds of codimension at least 2, geometric measure theory provides the most natural[3] framework for studying these weak solutions. Indeed, the balance laws (2.17) and (2.18), which appear to have some natural affinity with corresponding PDE identities (2.13) and (2.14), form the basis for the definitions of *stationary varifold* and *Brakke flow* — these are geometric measure theory notions of weak solutions of the minimal surface problem and motion by mean curvature, respectively.

In other words, if one is somehow able to carry out the passage to limits discussed in Section 2.4 above, in the elliptic case, the object one ends up with, satisfying (2.17) or some relaxed version thereof, is *exactly* a stationary varifold. And in the parabolic case, when a

[3] In the case of *scalar* equations and hypersurfaces, notions of weak solution based on the maximum principle are available at least for elliptic and parabolic problems, and provide an alternative framework for the sort of questions we consider here, see for example [27].

parallel passage to limits can be justified, one ends up with an object satisfying (a suitable relaxed version of) the balance law (2.18), and such an object is *exactly* a Brakke flow. As we describe below, this program has been carried out for both elliptic and parabolic equations, prominently in work of Bethuel and collaborators, such as [8, 12]. A key ingredient in these results is provided by measure-theoretic analysis of Ambrosio and Soner [3], which shows roughly speaking that the desired conclusions follow in the parabolic (and hence elliptic) cases if one can prove suitable lower density bounds on the limiting energy measure. This strategy was first implemented in a pioneering paper of Ilmanen [30] concerning the *scalar* Ginzburg-Landau heat flow.

In a similar way, the identities (2.19) and (2.20) can serve as the basis for definitions of weak solutions of the hyperbolic and Schrödonger type geometric evolution problems (1.9) and (1.10). Such weak solutions, motivated in part by the problems that we consider here, have begun only very recently to be developed, see [5] and [35] respectively.

3. Convergence of functionals

The following theorem makes precise a sense (known as *Gamma-convergence*) in which the Ginzburg-Landau functionals E_ε converge to the codimension 2 area functional in the limit as $\varepsilon \to 0$. Some definitions from geometric measure theory, used in the statement of the theorem, are collected in Appendix A. For now we simply note that an "*i.m.* rectifiable boundary" may be thought of as a possibly non-smooth, oriented, homologically trivial "submanifold" of $\mathbb{R}^N$, and the mass $\mathbf{M}(\cdot)$ is the natural generalization to this setting of the area functional $\mathcal{A}(\cdot)$ appearing above. In particular, to every smooth, embedded, oriented submanifold $M \subset \mathbb{R}^N$, one uniquely may associate an *i.m.* rectifiable boundary T_M, and then $\mathbf{M}(T_M) = \mathcal{A}(M)$.

Theorem 3.1. *Let Ω be an open subset of $\mathbb{R}^N$, $N \geq 3$.*

1. Compactness. *If $(u_\varepsilon)_{\varepsilon\in(0,1]}$ is a sequence in $H^1(\Omega;\mathbb{C})$ such that*

$$\limsup_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) < \infty$$

then there exists a subsequence ε_k and an $N-2$-dimensional i.m. *rectifiable boundary Γ in Ω, such that*

$$\|Ju_{\varepsilon_k} - \pi\Gamma\|_{C_0^{0,\alpha}(\Omega)^*} \to 0 \qquad \text{for all } \alpha \in (0,1]. \tag{3.1}$$

2. Lower bound. *If $(u_{\varepsilon_k})_{k\in\mathbb{N}}$ is a sequence in $H^1(\Omega;\mathbb{C})$ such that* (3.1) *holds for some $N-2$-dimensional* i.m. *rectifiable boundary Γ, then*

$$\liminf_{k\to\infty} E_{\varepsilon_k}(u_{\varepsilon_k}) \geq \pi\mathbf{M}(\Gamma). \tag{3.2}$$

3. Upper bound. *If Γ is any $N-2$-dimensional* i.m. *rectifiable boundary in Ω, then there exists a sequence $(u_\varepsilon)_{\varepsilon\in(0,1]}$ in $H^1(\Omega;\mathbb{C})$ such that* (3.1) *holds (without passing to a subsequence) and*

$$\lim_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) = \pi\mathbf{M}(\Gamma). \tag{3.3}$$

Moreover, whenever both (3.1) *and* (3.3) *hold, one has*

$$\frac{1}{\pi|\log\varepsilon_k|} e_{\varepsilon_k}(u_{\varepsilon_k}) \rightharpoonup \mu_\Gamma \qquad \text{weakly as measures, as } \varepsilon_k \to 0, \tag{3.4}$$

where μ_Γ is the mass measure *associated to Γ, see Appendix A.*

The compactness and lower bound assertions were first proved in [37], and the upper bound is due to [1], which in fact proved substantially more general results and also presented new proofs of the other parts of the theorem. A key ingredient in both proofs is provided by "vortex ball" constructions and associated estimates from [31, 52]. Some of the results in these references assume that Ω is bounded, but this restriction is unnecessary and has been dropped in later work.

Remark 3.2. The theorem implies various bounds for the vorticity Ju_ε in terms of the energy $E_\varepsilon(u_\varepsilon)$. For example, arguments in [37] easily imply that there exists some $\alpha, C > 0$ such that for any compact $K \subset \Omega$,

$$\limsup_{\varepsilon\to 0} \|\eta_{\varepsilon^\alpha} * Ju_\varepsilon\|_{L^1(K)} \;\le\; CE_\varepsilon(u_\varepsilon).$$

Here $\eta_{\varepsilon^\alpha}$ is a smoothing kernel supported on a ball of radius ε^α. This should be contrasted with the obvious estimate

$$\|Ju_\varepsilon\|_{L^1(\Omega)} \;\le\; C\int_\Omega |\nabla u_\varepsilon|^2 \;\le\; C|\log\varepsilon| E_\varepsilon(u_\varepsilon),$$

which is sharp in the sense that for any $\varepsilon \in (0,1]$, one can construct u_ε such that $\|Ju_\varepsilon\|_{L^1(\Omega)} \ge c|\log\varepsilon| E_\varepsilon(u_\varepsilon)$.

Thus, the energy (scaled as above) does *not* control the total vorticity $\|Ju_\varepsilon\|_{L^1}$. But by averaging on appropriate small scales, one can exploit cancellations to gain a factor $|\log\varepsilon|^{-1} \ll 1$ (when $\varepsilon \lll 1$), and as a result L^1 norm of the "macroscopic part" of the vorticity $\eta_{\varepsilon^\alpha} * Ju$ is indeed controlled by the energy.

Remark 3.3. The convergence in Theorem 3.1 is weak enough that is does not imply anything about convergence of solutions of any of the evolution equations (1.3)-(1.5) to the geometric evolution problems (1.8)-(1.10), and has only limited implications about convergence of the elliptic problems, as we discuss below. Nonetheless, estimates from Theorem 3.1 and related results provide an important ingredient in many results about the elliptic or parabolic problems, and in all the (much smaller number of) known results about wave and Schrödinger problems.

4. Elliptic problems

One can ask two sorts of questions about the relationship between the elliptic Ginzburg-Landau equation (1.2) and the geometry of minimal surfaces:

- Given a solution of (1.2) with $0 < \varepsilon \ll 1$, or a sequence of such solutions, do the energy and/or vorticity concentrate near a surface of vanishing mean curvature?
- Given a surface of vanishing mean curvature, can one find a solution of (1.2) with $0 < \varepsilon \ll 1$ whose energy and/or vorticity concentrate nearby?

4.1. Associating minimal surfaces to solutions of (1.2)**.** Theorem 3.1 directly implies that for sequences of *energy-minimizing* solutions u_ε of (1.2) with appropriate boundary data, the energy and vorticity concentrate, in the sense of (3.1) and (3.4), around a mass-minimizing current. In particular, statements of this type are proved in [1], when one considers the Dirichlet problem for (1.2) with boundary data in the natural Sobolev space $H^{1/2}(\partial\Omega;\mathbb{C})$.

However, both the earliest and the strongest results linking the elliptic Ginzburg-Landau equation with the minimal surface problem rely not on purely variational techniques, but instead mostly[4] on PDE arguments, including powerful elliptic regularity results tailored to the Ginzburg-Landau setting, initiated and developed by [43, 45, 50] among others. Mature results in this direction, such as [8–10, 15], show for example that for any sequence $(u_\varepsilon)_{\varepsilon\in(0,1]}$ of solutions of (1.2), *not* necessarily energy-minimizing and satisfying *only* uniform bounds on the rescaled energy

$$E_\varepsilon(u_\varepsilon) \le C \tag{4.1}$$

one can extract a subsequence such that (3.1) holds, and moreover $e_\varepsilon(u_\varepsilon) \rightharpoonup \mu_*$ weakly as measures, where μ_* is a Radon measure that is concentrated and bounded from below on the support of Γ, and has vanishing (generalized) mean curvature in what is known as the *varifold* sense. This result states exactly that the measure μ_* satisfies a relaxed[5], measure-theoretic version of the identity (2.17) discussed above.

4.2. Associating solutions of (1.2) **to minimal surfaces.** Surprisingly little is known about the complementary problem. The simplest nontrivial questions of this sort, in the Ginzburg-Landau context, arise when Ω is a smooth, bounded, simply connected, open subset of $\mathbb{R}^3$. Then for $p, q \in \partial\Omega$ we write

$$\ell(p,q) := \text{ the (open) line segment from } p \text{ to } q, \qquad d(p,q) := |p-q|.$$

We say that $\ell(p,q)$ is a *local minimizer* of the "arclength functional" $\mathcal{A}$ if $\ell(p,q) \subset \Omega$ and the pair (p,q) is a local minimizer of $d : \partial\Omega\times\partial\Omega \to \mathbb{R}$. (Thus, we are considering the arclength functional $\mathcal{A}$ with "natural boundary conditions"), We can similarly define a critical point of $\mathcal{A}$, nondegenerate critical point, isolated local minimizer, and so on. One can then ask, given some critical point of $\mathcal{A}$ (possibly satisfying other hypotheses), can one find a nearby solution of the elliptic Ginzburg-Landau system (1.2) (also with natural boundary conditions, which we tacitly assume for the duration of this section)?

A satisfactory answer to this question is known *only* when $\ell(p,q)$ is an *isolated local minimizer* of $\mathcal{A}$. In this case, it was proved in [46] that for any $\theta \in \mathbb{N}$, there exist (locally energy-minimizing) solutions $(u_\varepsilon)_{\varepsilon\in(0,\varepsilon_0]}$ whose vorticity and energy concentrate in the sense of (3.1), (3.4) about the 1-current Γ of constant multiplicity θ associated to the segment $\ell(p,q)$.

The main ingredients in [46] are Theorem 3.1 above and the general Kohn-Sternberg scheme for relating local minimizers and Gamma-convergence [40], and the proof would extend with no difficulties to yield Ginzburg-Landau local minimizers in other, more general situations where one has an *i.m.* rectifiable boundary Γ that is an isolated local minimizer of

[4]Pure PDE techniques are often supplemented by variational estimates. In particular, a common strategy, first developed in [6], studies the 1-form ju, defined in (2.3), by a Hodge decomposition, with control over d^*ju coming directly from the PDE, and control over $d\,ju = 2Ju$ coming from Theorem 3.1 or related estimates, which do not use any equation.

[5]For example, the measure can have non-constant multiplicity on Γ, and tangent planes may be understood in a weak sense.

the mass functional $\mathbf{M}(\,\cdot\,)$ with respect to suitable boundary conditions.

However, as soon as one considers more general critical points[6] only much weaker results are available. In fact, all that is known about this is that if $\ell(p,q)$ is a *nondegenerate* critical point of $\mathcal{A}$, then there exists a sequence $(u_\varepsilon)_{\varepsilon\in(0,\varepsilon_0]}$ of solutions to the Ginzburg-Landau equations (1.2), with natural (Neumann) boundary conditions, such that

$$\frac{1}{\pi}E_\varepsilon(u_\varepsilon) \to d(p,q). \tag{4.2}$$

This is consistent with the energy and vorticity concentrating around $\ell(p,q)$ in the sense of (3.1), (3.4), but is a much weaker statement — one about convergence of critical *values* rather than convergence of critical *points*. These facts are proved in [39], which establishes a general result relating critical points of a limiting functional with those of a sequence of functionals that converges in the sense of Gamma-convergence, then deduces the results described above from Theorem 3.1.

Some open problems include:

- The paper [46] mentioned above shows that it can happen that $\theta > 1$ vortex filaments cluster around a segment $\ell(p,q)$. Can one give a more precise description of the way in which this clustering occurs? An interesting conjecture and suggestive computations are given in [20].
- Can one improve the results of [39] to construct solutions of (1.2) for which vorticity and energy can be proved to concentrate around the segment $\ell(p,q)$?

It is worth noting that questions of this sort are very well-understood when one considers *scalar* semilinear elliptic equations and minimal *hypersurfaces*; see for example [23, 24] and [21, 22, 41, 49] respectively, which establish very strong results by some version of Lyapunov-Schmidt reduction. These techniques seem to be difficult to implement, however, for Ginzburg-Landau equations (for $\mathbb{C}$-valued functions), due in part to poor spectral properties of certain linearized operators and (related) difficulties in controlling the phase of complex-valued functions.

5. Parabolic problems

The balance laws (2.14) and (2.18) mentioned above play an important role in work on the convergence of the parabolic Ginzburg-Landau system (1.3) to the codimension 2 mean curvature flow (1.8).

In particular, the first results in this direction (see [38, 42]) employ an argument first developed by Soner (see the 1995 lectures [56] for the case of a scalar Ginzburg-Landau heat flow) which deduces strong results rather directly from the balance law (2.14). The main point is a computation which shows that if $(\Gamma_t)_{t\in[0,T]}$ is a codimension 2 mean curvature flow, smoothly embedded in $\mathbb{R}^N$ for every $t \in [0,T]$, then there exist constants $C, \delta > 0$, depending on the geometry of $(\Gamma_t)_{t\in[0,T]}$ but independent of $\varepsilon \in (0,1]$, and a smooth

[6]For example, if p, q are two points in $\partial\Omega$ such that $d(p,q) = \operatorname{diam}(\Omega)$ and $\ell(p,q) \subset \Omega$, then $\ell(p,q)$ is a critical point, but not a local minimizer, of the arclength functional.

function η such that

$$\eta(t,x)\begin{cases} = \frac{1}{2}\operatorname{dist}^2(x,\Gamma_t) & \text{if } \operatorname{dist}(x,\Gamma_t) \le \delta, \\ \ge \frac{1}{2}\delta^2 & \text{if not,} \end{cases}$$

and

$$\frac{d}{dt}\int_{\mathbb{R}^N} \eta(t,x)\frac{e_\varepsilon(u_\varepsilon)}{|\log\varepsilon|}dx \;\le\; C\int_{\mathbb{R}^N} \eta(t,x)\frac{e_\varepsilon(u_\varepsilon)}{|\log\varepsilon|}dx \tag{5.1}$$

for $0 \le t < T$, for any solution u_ε of the Ginzburg-Landau heat flow (1.3) on $\mathbb{R}^N$. It then follows by Grönwall's inequality that if the energy $e_\varepsilon(u_\varepsilon)$ is concentrated around Γ_0 at time $t = 0$, in the sense that

$$\int_{\mathbb{R}^N} \eta(0,x)\frac{e_\varepsilon(u_\varepsilon)}{|\log\varepsilon|}\,dx \to 0 \quad \text{as } \varepsilon \to 0$$

then it remains concentrated around Γ_t at time t, at least until the first time at which Γ_t develops singularities or self-intersections.

The proof of (5.1) relies on the fact that when η is taken as a test function in the balance law (2.14), certain remarkable cancellations occur. These are a consequence of algebraic properties of η which encode the fact that it is built around a codimension 2 mean curvature flow (1.8). Related ideas were used earlier by Ilmanen [29] and Ambrosio and Soner [2] in proofs of results that establish compatibility between the Brakke flow and other notions of weak solution of mean curvature flow.

The strongest results on the parabolic equation (1.3) were established in a landmark paper of Bethuel, Orlandi and Smets [12], following important earlier contributions such as [44, 57] and (particularly) the paper [3] of Ambrosio and Soner mentioned above. Bethuel *et al* prove in [12] powerful regularity results which imply that, given any sequence of solutions of the Ginzburg-Landau heat flow in $\mathbb{R}^N$ for any $N \ge 3$, satisfying only uniform energy bounds (4.1) on the initial data, one can find a subsequence for which the energy measures converge to a Brakke flow, globally in t. As mentioned above, a Brakke flow is a measure theoretic weak solution of the mean curvature flow, and is characterized by the fact that it satisfies a relaxed version of the balance law (2.18).

Some significant questions are left open by the work described above. For example,

- Assume that $(u_\varepsilon)_{\varepsilon\in(0,1]}$ is a sequence of solutions of the Ginzburg-Landau heat flow in $\mathbb{R}^3$, and that the energy measures converge globally to a Brakke flow that can be identified, for $0 \le t < T_{sing}$, with a smooth curve that develops a self-intersection at time T_{sing}. The results described above show that the energy concentration set is a weak solution of mean curvature flow for all t, but can one say *which* weak solution emerges (maybe only generically) from a self-intersection?

This seems to be a very difficult question, and should probably first be considered in the simplest possible situation, such as for nearly parallel vortex filaments. A related problem, simpler but still very subtle, involving the collision of point vortices in the $2d$ Ginzburg-Landau heat flow, has been analyzed in depth in works of Bethuel, Orlandi, Smets [11, 13, 14] and Serfaty [54, 55].

Another question is:

- Is it possible to give a purely variational proof of the convergence of the Ginzburg-Landau heat flow to codimension 2 mean curvature flow?

General criteria permitting the "Gamma-convergence of gradient flows" have been developed in an important paper of Sandier and Serfaty [53]. These criteria require variational convergence in a significantly stronger sense than is known from the works summarized in Theorem 3.1 – a sort of "C^1 Gamma-convergence" (see [53]). An improvement of Theorem 3.1 of this sort might also shed some light on questions discussed at the end of Section 4.

6. Hyperbolic problems

We recall some aspects of the geometry of submanifolds of $(1+N)$-dimensional Minkowski space, which we write $\mathbb{R}^{1+N}$. Suppose that $\mathcal{U} = (a,b) \times U$, for $U \subset \mathbb{R}^{N-k}$ open, and that $\Psi : \mathcal{U} \to \mathbb{R}^{1+N}$ is a nondegenerate map of the form

$$(t,y) \in \mathcal{U} \mapsto \Psi(t,y) = (t, \psi(t,y)) \in \mathbb{R}^{1+N}$$

parametrizing a piece of a codimension k submanifold Γ. We will write

$$g_{ab} := \eta_{\alpha\beta} \partial_a \Psi^\alpha \partial_b \Psi^\beta, \qquad a,b = 0, \dots, N-k, \quad \alpha, \beta = 0, \dots, N$$

(here and below, we identify y_0 and t) for the induced metric on Γ, in local coordinates, and

$$g = \det(g_{ab}), \qquad (g^{ab}) = (g_{ab})^{-1}.$$

A submanifold Γ is said to be *timelike* if the induced metric is everywhere Lorentzian (that is, has signature $(-,+,\dots,+)$.) This holds if and only if $g < 0$ everywhere. The Minkowskian area of a timelike submanifold is defined by

$$\mathcal{A}_{mink}\big(\Psi(\mathcal{U})\big) := \int_{\mathcal{U}} \sqrt{-g}\, dy_0 \dots dy_{N-k}$$

for the image of a coordinate patch, and more generally may be defined via a partition of unity. In this setting, we also write $\lambda^{1,N-k}$ to denote the Minkowskian area measure, defined by

$$\int_{\Psi(\mathcal{U})} f \, d\lambda^{1,N-k} := \int_{\mathcal{U}} f(\Psi) \sqrt{-g}\, dy_0 \dots dy_{N-k} \qquad \text{for } f \in C_c(\mathbb{R}^{1+N}). \tag{6.1}$$

Note that all these definitions are independent of the parametrization.

The *Minkowskian mean curvature vector* H_{mink} is the first variation of the Minkowskian area functional. In local coordinates it may be written as

$$H_{mink} = \frac{1}{\sqrt{|g|}} \partial_a \left(\sqrt{|g|} g^{ab} \partial_b \Psi \right).$$

For a timelike submanifold, the equation $H_{mink} = 0$ is a quasilinear hyperbolic equation, in suitable coordinates.

Speculations about relationships between equations like the Ginzburg-Landau wave equation (1.4) and Minkowskian minimal surfaces date back to the early '70s in the physics literature, starting with a seminal paper of Nielsen and Olesen [48], and entered the (applied) mathematics literature through the formal asymptotic analysis of Neu [47]. The first rigorous

results about this problem were given by Bellettini, Novaga and Orlandi [4], who showed that certain measure-theoretic estimates, if they could be proved, would suffice to justify the passage to limits from (2.15) to (2.19), or a relaxed version thereof, suitable for describing singular surfaces of vanishing Minkowskian mean curvature.

The strongest results to date about the dynamics of vortex filaments in the Ginzburg-Landau wave equation are summarized in the following.

Theorem 6.1. *Let $\Gamma \subset (-T,T)\times\mathbb{R}^N$ be a smooth embedded timelike codimension $k=2$ submanifold such that $\Gamma_t := \{x\in\mathbb{R}^N : (t,x)\in\Gamma\}$ is compact for every t and $H_{mink}(\Gamma)=0$.*

Then for every $\varepsilon\in(0,1]$, there exists a solution of the Ginzburg-Landay wave equation (1.4) *such that for any $T_0<T$, there is a constant C, depending on Γ and T_0 but inepedent of ε, for which*

$$\int_{(-T_0,T_0)\times\mathbb{R}^N} \tilde{d}^2\left(\frac{|u_t|^2+|\nabla u|^2}{2}+\frac{(1-|u|^2)^2}{4\varepsilon^2}\right)\,dt\,dx\le C \tag{6.2}$$

where $\tilde{d}(t,x)=\min\{1,\operatorname{dist}((t,x),\Gamma)\}$, and

$$\left|\int S:(\eta-\frac{(\eta\,Du_\varepsilon)\otimes(\eta\,Du_\varepsilon)}{\ell_\varepsilon(u_\varepsilon)})\;\frac{\ell_\varepsilon(u_\varepsilon)}{|\log\varepsilon|}\,dx\,dt - \int_\Gamma S:(\eta-P^\perp_{mink})\,d\lambda^{1,N-2}\right| \le \frac{C}{|\ln\varepsilon|^{1/2}}\|S\|_{W^{1,\infty}} \tag{6.3}$$

for all $S\in C^\infty_c(\mathbb{R}^{1+N};M^{(1+N)\times(1+N)})$.

The significance of (6.3) is that it not only establishes that the wave equation balance law (2.15) converges to its geometric counterpart (2.19) as long as the associated submanifold Γ remains smooth, but it also provides an estimate of the rate of convergence.

Conclusion (6.3) implies in particular that

$$\int_{(-T_0,T_0)\times\mathbb{R}^N}\left(\frac{|u_t|^2+|\nabla u|^2}{2}+\frac{(1-|u|^2)^2}{4\varepsilon^2}\right)\,dt\,dx\ge c|\log\varepsilon|.$$

In light of this, the first conclusion (6.2) shows that the logarithmically diverging part of the energy concentrates near Γ, where $\tilde{d}$ vanishes.

In the definition of the function $\tilde{d}$ appearing in (6.2), we simply understand $\operatorname{dist}((t,x),\Gamma)$ with respect to the Euclean metric on $\mathbb{R}^{1+N}$. One could also use a Minkowsian notion of distance; in some way this would be more natural but it is a little harder to describe and in any case would yield an equivalent estimate.

Theorem 6.1 was proved in [32], which also proved similar results relating the real scalar wave equation and timelike hypersurfaces with vanishing Minkowksian mean curvature. These results were proved under additional topological restrictions on Γ_t; more recent work [28] has shown (among other extensions of results of [32]) how to remove this assumption in the scalar case, and the arguments carry over with no change to the situation described in Theorem 6.1. The proof is carried out by weighted energy estimates in Gaussian normal coordinates adapted to the submanifold Γ, where "normal" is understood with respect to the Minkowski metric. It relies crucially on variational estimates, in the spirit of Theorem 3.1, that establish certain strong stability properties of "vortex filaments". The weighted energy estimates implicitly use a form of the balance law (2.15).

We mention a few of the many related open questions.

- Is it ever possible to describe the dynamics of vortex filaments in solutions of the Ginzburg-Landau wave equation, or related equations[7], globally in time or beyond the onset of singularities for associated geometric flow?

Any progress on this will almost surely require a more precise description of vortex filaments than is given in [32], possibly involving a symplectic orthogonal decomposition or similar ideas. Such an approach is followed in the only result of this type that we know of, due to Cuccagna [16], who proves that for the real scalar Ginzburg-Landau wave equation in 3 dimensions with initial data a very small, smooth perturbation of a flat interface, solutions scatter to a flat interface. In this situation the associated geometric dynamics are very nearly trivial. Techniques of this general character were used in the first work [25] to show that interfaces in the real scalar analog of (1.3) evolve by (codimension 1) mean curvature flow, before the onset of singularities.

- Recently, motivated in part by problems considered here, Bellettini, Novaga and Orlandi [5] introduced the notion of *Lorentzian varifolds*, which in particular yields a definition of weak solutions of the equation "$H_{mink} = 0$" based essentially on the balance law (2.19), and analogous to the classical theory of (Euclidean) varifolds. Basic issues about this, such as weak-strong uniqueness, remain open, and one may also wonder whether there is any prospect for a reasonable (partial) regularity theory.

7. Schrödinger-type problems

Unlike the problems considered above, the validity of the (expected) relationship between the Gross-Pitaevskii equation (1.5) and Schrödinger type geometric flow (1.10) is almost completely open, and may be stated as the following

Conjecture 7.1. *Let $(\Gamma_t)_{t\in[0,T]}$ be a smooth compact embedded binormal curvature flow*[8] *in $\mathbb{R}^3$. Then there exists a sequence $(u_\varepsilon)_{\varepsilon\in(0,1]}$ of solutions of the Gross-Pitaevskii equation* (1.5) *in* 3 *space dimensions such that*

$$\int \varphi \wedge Ju_\varepsilon(t) \to \pi \int_{\Gamma_t} \varphi \qquad \text{for all } \varphi \in \mathcal{D}^1(\mathbb{R}^3). \tag{7.1}$$

and

$$\int \phi \frac{e_\varepsilon(u_\varepsilon(t))}{|\log \varepsilon|} dx \ \to \pi \int_{\Gamma_t} \phi \, d\mathcal{H}^1 \qquad \text{for all } \phi \in C_c(\mathbb{R}^N). \tag{7.2}$$

[7]This question is probably easiest to address not for (1.4), but instead for some equation for which better spectral estimates are available, such as the real scalar counterpart of (1.4) or for a gauge theory such as the Abelian Higgs model, for which some results parallel to those of Theorem 6.1 have been established in [18]. It might also be prudent to consider situations in which the associated geometric evolution has good properties. For example, $1+1$-dimensional timelike surfaces of vanishing mean curvature admit a very explicit description, which for suitable data yields nontrivial but very simple dynamics for large $|t|$.

[8]Thus Γ_t is the oriented image of $\gamma(t,\cdot)$ for some $\gamma : [0,T] \times \mathbb{R} \to \mathbb{R}^3$ satisfying

$$\partial_t \gamma = \partial_s \gamma \times \partial_{ss} \gamma, \qquad |\partial_s \gamma|^2 = 1,$$

with $\gamma(t,s) = \gamma(t, s+\ell)$ for some $\ell > 0$, and injective on $\{t\} \times [0,\ell)$ for all $t \in [0,T]$.

One can more generally pose the conjecture in $\mathbb{R}^N$ for $N \geq 3$, once the equation (1.5) is modified in a way that guarantees global well-posedness in the energy space.

A stronger form of the conjecture posits that for any sequence $(u_\varepsilon)_{\varepsilon\in(0,1]}$ of smooth solutions of (1.5), conclusions (7.1),(7.2) are valid for all $t \in [0,T]$ as long as they are satisfied when $t = 0$.

The main partial results toward the above conjecture (many, but not all, of which are proved in $\mathbb{R}^N$ for general $N \geq 3$) are the following:

- it is known that there are families of translating solutions of (1.5) that converge in the sense of (7.1), (7.2) to certain translating binormal curvature flows, see [9, 17].
- The conjecture is known to hold if (7.1), (7.2) hold when $t = 0$, *and* if Γ_0 is a circle, see [33].
- The conjecture holds for *equivariant*[9] solutions of the Gross-Pitaevskii equation such that (7.1), (7.2) hold at time $t = 0$, and Γ_0 is a finite union of circles. (See [36], which in fact considers a slightly different problem, but the proof applies with very minor modifications to the situation described here.)
- If (7.1), (7.2) hold when $t = 0$, then there exists a family of 1-dimensional *i.m.* rectifiable boundaries $(\Lambda_t)_{t>0}$ such that $\mathbf{M}(\Lambda_t) \leq \mathbf{M}(\Lambda_0)$ and, after passing to a subsequence if necessary,

$$\int \varphi \wedge Ju_\varepsilon(t) \to \pi \int_{\Lambda_t} \varphi \qquad \text{for all } \varphi \in \mathcal{D}^1(\mathbb{R}^3) \tag{7.3}$$

  for all $t \geq 0$, see [33].
- Moreover, *if* $\mathbf{M}(\Lambda_t) = \mathbf{M}(\Lambda_0)$ for all $t \in [0,T]$, then $\Lambda_t = \Gamma_t$ for $t \in [0,T]$. (This follows from results of [33] combined with with the weak-strong uniqueness theorem of [35] described below.)

These results employ a strategy based on attempting a rigorous passage to limits from the balance law (2.16) for the Gross-Pitaevskii equation to the corresponding balance law (2.20) for the binormal curvature flow. To fully implement this strategy, one needs good estimates of the quantity

$$\int_{\mathbb{R}^3} S : \frac{\nabla u \otimes \nabla u}{|\log \varepsilon|} - \pi \int_\Gamma S : P^\perp \, d\mathcal{H}^{N-2}, \qquad S \in C_c^\infty(\mathbb{R}^N; M^{N\times N}) \tag{7.4}$$

compare (2.16) and (2.20). Refined variational estimates from [33], in the spirit of Theorem 3.1, show that the quantity appearing in (7.4) can be controlled by some combination of the vorticity and energy, but these estimates, which are essentially sharp, are too weak to allow a straightforward passage from (2.16) to (2.20). These unfavorable estimates, which are one major obstacle (among several) to the proof of Conjecture 7.1, are related to the same bad spectral properties of certain linearized operators that make Lyapunov-Schmidt reduction difficult for the elliptic Ginzburg-Landau equations.

In the above framework, it is natural to consider a definition of weak solutions of the binormal curvature flow — that is, the $N = 3$ case of (1.10) — based on the balance law (2.20).

[9] That is, solutions of the form $u(t,x,y,z) = f(t,r,z)\frac{x+iy}{r}$, where $r = \sqrt{x^2+y^2}$. Even with this symmetry, vorticity concentration around several circles is much harder to analyze than the case of non-equvariant solutions with vorticity concentrating around a single circle, in which certain key technical obstacles can be circumvented.

One such definition was proposed in [33], and a slightly improved notion of weak solution was recently put forward in [35], which establishes some properties of this class of weak solutions, such as existence of solutions for initial data which is merely a closed rectifiable curve, and the existence of solutions that exhibit change of topology. Such weak solutions need not be unique, but it is shown in [35] that they enjoy a *weak-strong uniqueness* property — a weak solution that agrees with a smooth solution (say, $L^\infty([0,T];W^{3,\infty}(\mathbb{R}/\ell\mathbb{Z};\mathbb{R}^3))$ at time $t = 0$ continues to do so as long as the smooth solution does not develop self-intersections. This is proved by defining what might be called a *relative entropy* of the weak solution with respect to the smooth solution, and showing that its growth can be controlled; the proof thus shows that a weak solution that is close to a smooth solution at $t = 0$ (in a particular sense) remains nearby for some interval of time. This argument has been adapted in [34] to show that if u, v are Schrödinger maps $S^1 \to S^2$ and v is smooth (say, $C([0,T];H^3(\mathbb{R}/\ell\mathbb{Z};\mathbb{R}^3))$ then

$$\inf_{\sigma\in\mathbb{T}^1} \|u - \tau_\sigma v\|_{L^2} \le C(v)\|u^0 - v^0\|_{L^2}.$$

This estimate is interesting partly because the equation supercritical in L^2; the critical space is $H^{1/2}$. Also, u is only required to belong to $L^\infty([0,T];H^{1/2}(\mathbb{R}/\ell\mathbb{Z};\mathbb{R}^3)$ which, in addition to being a critical space, is more or less the weakest space in which the equation can be given a meaning in the sense of distributions.

The most prominent open question in this area, and indeed in the family of problems discussed in this paper, is Conjecture 7.1. But there are also a number of interesting questions related to the binormal mean curvature flow and its weak solutions, among them:

- Results of [35] imply in particular the existence of weak solutions of the binormal curvature flow in 3 dimensions for initial data a regular planar polygon. A very interesting numerical and theoretical study of this problem is carried out in [19], which produces explicit formulas for candidate weak solutions and investigates number-theoretic properties of these curves. Parallel behavior is well-understood for the linear Schrödinger equation on S^1 (see for example [51]), and also for periodic NLS with subcritical nonlinearities (this can be treated as a perturbation of the linear case, see [26]), but remains mysterious in the (supercritical) setting of the binormal curvature flow. A first step could be to see whether the candidate solutions constructed in [19] are in fact weak solutions in the sense of [35].
- As far as we know, nothing is established about well-posedness, or other properties, of the binormal mean curvature flow when $N \ge 4$, although the equation may be seen as the canonical Schrödinger-type counterpart of the (Eulidean and Minkowskian) minimal surface equations and mean curvature flow.

A. Appendix

We recall some terminology from geometric measure theory, used in the statement of Theorem 3.1 and elsewhere.

A *k-current* in an open set $\Omega \subset \mathbb{R}^N$ is a bounded linear functional on the space $\mathcal{D}^k(\Omega)$ of C^∞ compactly supported k-forms in Ω. An oriented k-dimensional embedded submanifold

M in Ω can be associated with the current T_M defined by

$$T_M(\varphi) := \int_M \varphi \qquad \text{for } \varphi \in \mathcal{D}^k(\Omega). \tag{A.1}$$

The space of k-currents also contains some objects that are a little less regular, as well as many objects that are *much* less regular. We will encounter only the former, an example of which is the current of the form

$$\varphi \in \mathcal{D}^k(\Omega) \mapsto \int_{F(A)} \varphi = \int_A F^{\#}\varphi \tag{A.2}$$

where A is a compact subset of $\mathbb{R}^k$ and $F : A \to \mathbb{R}^N$ is only Lipschitz (in general not injective, for example), and $F^{\#}\varphi$ denotes the pullback of φ by F. More generally, an *integer multiplicity*[10] *rectifiable* k-current is one that can be obtained as a limit (with respect to the mass norm, defined below) of currents of the form (A.2). In the case $k = 1$, such currents can always be written as

$$\Gamma(\varphi) = \sum_{i=1}^{\infty} \int_{\gamma_i} \varphi, \qquad \varphi \in \mathcal{D}^1(\Omega) \tag{A.3}$$

where γ_i is the (oriented) image of an injective Lipschitz map $(0,1) \to \mathbb{R}^N$, and the collection $(\gamma_i)_{i=1}^{\infty}$ has finite mass in the sense defined below.

Given a function $u \in H^1(\Omega;\mathbb{C})$, we can associate to the vorticity Ju the $N-2$-current defined by

$$\varphi \in \mathcal{D}^{N-2}(\Omega) \mapsto \int_\Omega \varphi \wedge Ju.$$

This has a geometric interpretation given by (2.7), as a kind of average of currents associated to level sets of u.

The mass in Ω of a k-current T, denoted $\mathbf{M}_\Omega(T)$, is defined by

$$\mathbf{M}_\Omega(T) := \sup\{T(\varphi) : \varphi \in \mathcal{D}^k(\Omega), \max_x |\varphi(x)| \le 1\}$$

where for concreteness we use the Eulidean norm on k-covectors. We will almost always drop the subscript and simply write $\mathbf{M}$. It is a straightforward consequence of the definition that for T_M as in (A.1),

$$\mathbf{M}(T_M) = \mathcal{H}^k(M).$$

If M is merely immersed, then we can still define T_M as in (A.1), and in this case $\mathbf{M}(T_M)$ corresponds to the $\mathcal{H}^k$ measure with a weight that counts the algebraic multiplicity. This is in fact the right notion of "k-dimensional area" in Theorem 3.1. In particular, for an *i.m.* rectifiable 1-dimensional current Γ, the representation on the right-hand side of (A.3) can be chosen with the property that

$$\mathbf{M}(\Gamma) = \sum_i \mathcal{H}^1(\gamma_i).$$

Whenever a current T has locally finite mass, there exists a Radon measure μ_T such that

$$\mathbf{M}_U(T) = \mu_T(U) \quad \text{ for every open } U \subset \Omega.$$

[10] we will generally use the abbreviation *i.m.*

We will refer to this as the *mass measure*.

The boundary of k-current T is a $k-1$-current ∂T defined by

$$\partial T(\varphi) = T(d\varphi).$$

The definition is arranged so that $\partial T_M = T_{\partial M}$ for T_M as in (A.1) — this is just Stokes' Theorem.

A k-current T in Ω is said to be *a boundary* if there exists some $k+1$-current S such that $T = \partial S$ in Ω.

Acknowledgements. The author's research was supported by the National Science and Engineering Research Council of Canada under Operating Grant 261955.

References

[1] Alberti, G., Baldo, S., and Orlandi, G., *Variational convergence for functionals of Ginzburg-Landau type*, Indiana Univ. Math. J. **5**4 (2005), 1411–1472.

[2] Ambrosio, L. and Soner, H.M., *Level set approach to mean curvature flow in arbitrary codimension*, J. Differential Geom. **43** (1996), no. 4, 693–737.

[3] _____, *A measure-theoretic approach to higher codimension mean curvature flows*, Dedicated to Ennio De Giorgi. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **25** (1997), no. 1-2, 27–49 (1998).

[4] Bellettini, G., Novaga, M., and Orlandi, G., *Time-like minimal submanifolds as singular limits of nonlinear wave equations*, Phys. D **239** (2010), no. 6, 335–339.

[5] _____, *Time, Lorentzian varifolds and applications to relativistic strings*, Indiana Univ. Math. J. **61** (2012), no. 6, 2251–2310.

[6] Bethuel, F., Bourgain, J., Brezis, H., and Orlandi, G., *$W^{1,p}$ estimates for solutions to the Ginzburg- Landau equation with boundary data in $H^{1/2}$*, C. R. Acad. Sci. Paris SÂŐr. I Math. **333** (2001), no. 12, 1069–1076.

[7] Bethuel, F., Brezis, H., and Hèlein, F., *Ginzburg-Landau vortices*, Progress in Nonlinear Differential Equations and their Applications, 13. Birkhäuser Boston, Inc., Boston, MA, 1994.

[8] Bethuel, F., Brezis, H., and Orlandi, G., *Asymptotics for the Ginzburg-Landau equation in arbitrary dimensions*, J. Funct. Anal. **186** (2001), no. 2, 432–520.

[9] Bethuel, F., Orlandi, G., and Smets, D., *Vortex rings for the Gross-Pitaevskii equation*, J. Eur. Math. Soc. **6** (2004), no. 1, 17–94.

[10] _____, *Improved estimates for the Ginzburg-Landau equation: the elliptic case*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **4** (2005), no. 2, 319–355.

[11] _____, *Collisions and phase-vortex interactions in dissipative Ginzburg-Landau dynamics*, Duke Math. J. **130** (2005), no. 3, 523–614.

[12] ———, *Convergence of the parabolic Ginzburg-Landau equation to motion by mean curvature*, Ann. of Math. (2) **163** (2006), no. 1, 37–163.

[13] ———, *Quantization and motion law for Ginzburg-Landau vortices*, Arch. Ration. Mech. Anal. **183** (2007), no. 2, 315–370.

[14] ———, *Dynamics of multiple degree Ginzburg-Landau vortices*, Comm. Math. Phys. **272** (2007), no. 1, 229–261.

[15] Bourgain, J., Brezis, H., and Mironescu, P., *$H^{1/2}$ maps with values into the circle: minimal connections, lifting, and the Ginzburg-Landau equation*, Publ. Math. Inst. Hautes Études Sci. **99** (2004), 1–115.

[16] Cuccagna, S., *On asymptotic stability in 3D of kinks for the ϕ^4 model*, Trans. Amer. Math. Soc. **360** (2008), no. 5, 2581–2614.

[17] Chiron, D., *Vortex helices for the Gross-Pitaevskii equation*, J. Math. Pures Appl. (9) **84** (2005), no. 11, 1555–1647.

[18] Czubak, M. and Jerrard, R.L., *Topological defects in the Abelian Higgs model*, preprint, 2013.

[19] de la Hoz, F. and Vega, L., *Vortex Filament Equation for a Regular Polygon*, preprint, 2013.

[20] del Pino, M. and Kowalczyk, M., *Renormalized energy of interacting Ginzburg-Landau vortex filaments*, J. Lond. Math. Soc. (2) **77** (2008), no. 3, 647–665

[21] del Pino, M., Kowalczyk, M., and Wei, J., *On De Giorgi's conjecture in dimension $N \geq 9$*, Ann. of Math. (2) **174** (2011), no. 3, 1485–1569.

[22] ———, *Entire solutions of the Allen-Cahn equation and complete embedded minimal surfaces of finite total curvature in $\mathbb{R}^3$*, J. Differential Geom. **93** (2013), no. 1, 67–131.

[23] del Pino, M., Kowalczyk, M., Pacard, F., and Wei, J., *The Toda system and multiple-end solutions of autonomous planar elliptic problems*, Adv. Math. **224** (2010), no. 4, 1462–1516.

[24] del Pino, M., Kowalczyk, M., Wei, J., and Yang, J., *Interface foliation near minimal submanifolds in Riemannian manifolds with positive Ricci curvature*, Geom. Funct. Anal. **20** (2010), no. 4, 918–957.

[25] de Mottoni, P. and Schatzman, M., *Geometrical evolution of developed interfaces*, Trans. Amer. Math. Soc. **347** (1995), no. 5, 1533–1589.

[26] Erdogan, M.B. and Tzirakis, N., *Talbot effect for the cubic nonlinear Schrödinger equation on the torus*, preprint, 2013.

[27] Evans, L. C., Soner H. M., and Souganidis, P. E., *Phase transitions and generalized motion by mean curvature*, Comm. Pure Appl. Math. **45** (1992), no. 9, 1097–1123.

[28] Galvão-Sousa and B., Jerrard, R.L., *Accelerating fronts in semilinear wave equations*, Rend. del Circ. Mat. di Palermo, to appear.

[29] Ilmanen, T., *Elliptic regularization and partial regularity for motion by mean curvature*, Mem. Amer. Math. Soc. **108** (1994), no. 520.

[30] ———, *Convergence of the Allen-Cahn equation to Brakke's motion by mean curvature*, J. Differential Geom. **38** (1993), no. 2, 417–461

[31] Jerrard, R. L., *Lower bounds for generalized Ginzburg-Landau functionals*, SIAM J. Math. Anal. **30** (1999), no. 4, 721–746.

[32] ———, *Defects in semilinear wave equations and timelike minimal surfaces in Minkowski space*, Anal. PDE **4** (2011), no. 2, 285–340.

[33] ———, *Vortex filament dynamics for Gross-Pitaevsky type equations*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **1** (2002), no. 4, 733–768.

[34] Jerrard, R.L. and Smets, D., *On Schrödinger maps from T^1 to S^2*, Ann. Sci. Norm. Supér **45** (2012), no. 4, 637–680 (2013).

[35] ———, *On the motion of a curve by its binormal curvature*, J. Eur. Math. Soc., to appear.

[36] ———, *Vortex dynamics for the two dimensional non homogeneous Gross-Pitaevskii equation*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), to appear.

[37] Jerrard, R. L. and Soner, H. M., *The Jacobian and the Ginzburg-Landau Energy*, Calc. Var. PDE **14** (2002), 151–191.

[38] ———, *Scaling limits and regularity results for a class of Ginzburg-Landau systems*, Ann. Inst. H. Poincaré Anal. Non Linéaire **16** (1999), no. 4, 423–466.

[39] Jerrard, R.L. and Sternberg, P., *Critical points via Γ-convergence: general theory and applications*, J. Eur. Math. Soc. **11** (2009), no. 4, 705–753.

[40] Kohn, R.V. and Sternberg, P., *Local minimisers and singular perturbations*, Proc. Roy. Soc. Edinburgh Sect. A **111** (1989), no. 1-2, 69–84.

[41] Kowalczyk, M., *On the existence and Morse index of solutions to the Allen-Cahn equation in two dimensions*, Ann. Mat. Pura Appl. (4) **184** (2005), no. 1, 17–52.

[42] Lin, F.-H., *Complex Ginzburg-Landau equations and dynamics of vortices, filaments, and codimension-2 submanifolds*, Comm. Pure Appl. Math. **51** (1998), no. 4, 385–441.

[43] Lin, F.-H. and Rivière, T., *Complex Ginzburg-Landau equations in high dimensions and codimension two area minimizing currents*, J. Eur. Math. Soc. **1** (1999), no. 3, 237–311.

[44] ———, *A quantization property for moving line vortices*, Comm. Pure Appl. Math. **54** (2001), no. 7, 826–850.

[45] ———, *A quantization property for static Ginzburg-Landau vortices*, Comm. Pure Appl. Math. **54** (2001), no. 2, 206–228.

[46] Montero, J.A., Sternberg, P., and Ziemer, W.P., *Local minimizers with vortices in the Ginzburg-Landau system in three dimensions*, Comm. Pure Appl. Math. **57** (2004), no. 1, 99–125.

[47] Neu, J. C., *Kinks and the minimal surface equation in Minkowski space*, Phys. D **43** (1990), no. 2-3, 421–434.

[48] Nielsen, H. B. and Olesen, P., *Vortex-line models for dual strings*, Nuclear Phys., B **61** (1973), 45–61.

[49] Pacard, F. and Ritoré, M., *From constant mean curvature hypersurfaces to the gradient theory of phase transitions*, J. Differential Geom. **64** (2003), no. 3, 359–423.

[50] Rivière, T., *Line vortices in the U(1)-Higgs model*, ESAIM Contrôle Optim. Calc. Var. **1** (1995/96), 77–167.

[51] Rodnianski, I., *Fractal solutions of the Schrödinger equation*, Contemp. Math. **255** (2000), 181–187.

[52] Sandier, E., *Lower bounds for the energy of unit vector fields and applications*, J. Funct. Anal. **152** (1998), no. 2, 379–403.

[53] Sandier, E. and Serfaty, S., *Gamma-convergence of gradient flows with applications to Ginzburg-Landau*, Comm. Pure Appl. Math. **57** (2004), no. 12, 1627–1672.

[54] Serfaty, S., *Vortex collisions and energy-dissipation rates in the Ginzburg-Landau heat flow. I. Study of the perturbed Ginzburg-Landau equation*, J. Eur. Math. Soc. **9** (2007), no. 2, 177–217.

[55] ———, *Vortex collisions and energy-dissipation rates in the Ginzburg-Landau heat flow. II. The dynamics*, J. Eur. Math. Soc. **9** (2007), no. 3, 383–426.

[56] Soner, H.M., *Front propagation, Boundaries, interfaces, and transitions (Banff, AB, 1995)*, 185–206, CRM Proc. Lecture Notes, 13, Amer. Math. Soc., Providence, RI, 1998.

[57] Wang, C.Y., *Limits of solutions to the generalized Ginzburg-Landau functional*, Comm. Partial Differential Equations **27** (2002), no. 5-6, 877–906.

University of Toronto, Department of Mathematics, University of Toronto, Toronto, CA M5S 3G3
E-mail: rjerrard@utoronto.ca

Homogenization theory in nonlinear partial differential equations

Ki-Ahm Lee

Abstract. In this paper, we are going to consider recent works in Homogenization Theory in Nonlinear Partial Differential Equations, which concerns the derivation of a macroscopic homogenized (or effective) equation for the system with oscillating microscopic structures. Mainly we focus on topics related with the homogenization for high oscillation, nonvariational problems, lower dimensional oscillations or equations of nondivergence type.

Mathematics Subject Classification (2010). 35B27, 35K55, 35R35.

Keywords. Homogenization, nonlinear partial differential equations, free boundary problems.

1. Introduction to homogenization

The homogenization theory is concerned with the derivation of homogenized or effective equations satisfied by the limit of solutions of differential equations with rapidly oscillating data. Such problems can be considered to find the macroscopic behavior of a system with heterogeneous oscillations in the microscopic scale. Suppose that there is an alloy consisting of two different matters as white and blue materials at Figure 1.1. At the macroscopic scale (or $\varepsilon \to 0$), the alloys have been discovered to possess stable physical properties such as heat transfer, electric conductivity which differ from those of materials at micro scale.

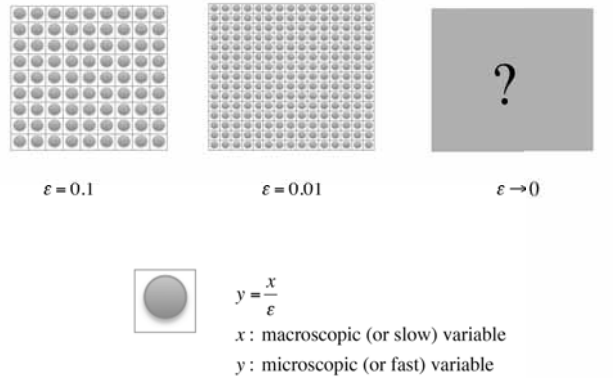


Figure 1.1. Homogenization, [11]

Proceedings of the International Congress of Mathematicians, Seoul, 2014

At mathematical models of microscopically non-homogeneous media, various small scale characteristics are usually described by oscillating functions. Let $\Omega \subset \mathbb{R}^n$ be a smooth domain and assume that $a_{ij}(x,y)$, $f(x,y)$ and $g(x,y)$ are periodic in y-variable i.e. $a_{ij}(x,y+m) = a_{ij}(x,y)$, $f(x,y+m) = f(x,y)$ and $g(x,y+m) = g(x,y)$ for $m \in \mathbb{Z}^n$.

We consider u_ε satisfying the following Dirichlet value problems:

$$\begin{cases} \nabla_i \left(a_{ij} \left(x, \frac{x}{\varepsilon} \right) \nabla_j u_\varepsilon \right) = f \left(x, \frac{x}{\varepsilon} \right) & \text{in } \Omega \\ u_\epsilon = g \left(x, \frac{x}{\varepsilon} \right) & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

in Divergence form or

$$\begin{cases} a_{ij} \left(x, \frac{x}{\varepsilon} \right) D_{ij} u_\varepsilon = f \left(x, \frac{x}{\varepsilon} \right) & \text{in } \Omega \\ u_\epsilon = g \left(x, \frac{x}{\varepsilon} \right) & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

in non-Divergence form

1.1. Example at one dimension ([11]). Let us consider one dimensional simple problem to find the behavior of solutions, u_ε, and the homogenized equation satisfied by the limit, u. For $\Omega = [0,1]$, set $a(y)$ to be 1-periodic i.e. $a(y) = a(y+1)$ and $a_\varepsilon(x) = a\left(\frac{x}{\varepsilon}\right)$ ε-periodic as Figure 1.2.

$$\begin{cases} \frac{d}{dx}\left(a_\varepsilon(x) \frac{d}{dx} u_\varepsilon(x) \right) = 0, \quad 0 < x < 1, \\ u_\varepsilon(x) = 0 \quad \text{at } x = 0, \quad u_\varepsilon(x) = 1 \quad \text{at } x = 1. \end{cases} \tag{1.3}$$

Then we have

$$u_\varepsilon(x) = x + \frac{\int_0^{nx} (\frac{1}{a(y)} - 1) dy}{\int_0^n \frac{dy}{a(y)}} = u(x) + \varepsilon w\left(\frac{x}{\varepsilon}\right),$$

where $u(x) = x$ and w is 1-periodic i.e. $w(y+1) = w(y)$.
$w(y)$ is called as a corrector since $w(y)$ is used to correct the limit function $u(x)$ to be a solution of ε-problem, (1.3). And $u_\varepsilon(x)$ converges to $u(x)$ as Figure 1.2.

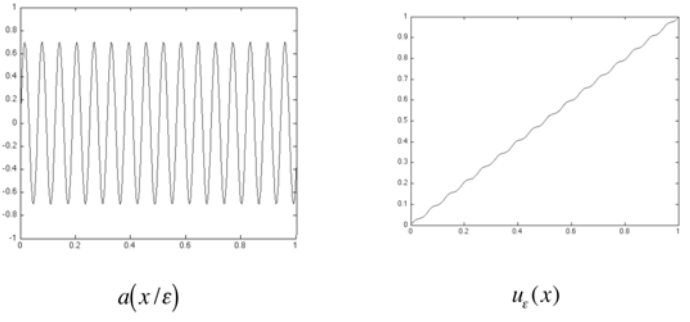


Figure 1.2. 1-dimensional problem

The homogenized equation satisfied by the limit solution u will be

$$\frac{d}{dx}\left(\overline{a} \frac{d}{dx} u(x) \right) = 0, \quad \text{on } 0 < x < 1$$

where $\overline{a} = \frac{1}{\int_0^1 \frac{d\bar{x}}{a(\bar{x})}}$ is an effective diffusion coefficient.

1.2. Outline. Recently there have been important developments in the theory of homogenizations by various authors especially in viscosity method: L. Caffarelli, P. E. Souganidis, H. Ishii, H. Matano, L.C. Evans, I.C. Kim, A. Mellet, S. N. Armstrong, H. Shahgholian and so on. In addition, young researchers have been involved actively: W. Feldman, Sunghoon Kim, Sunghan Kim, B. Orcan, M. Strömqvist, M. Yoo and so on. At Section 2, some known methods are introduced briefly. And then I would like to address some important issues at the theory of homogenization that I have focused on recent years. Section 3 is on the study of the high oscillation where the regularity of oscillating solution lacks especially for fully nonlinear equations. Discrete regularity has been introduced, which will give us the concept of convergence and homogenization. Section 4 is on the homogenization of nonvariational solutions which will have nontrivial effective equations that may not be a simple average. And section 5 focuses on the homogenization theory when the oscillation happens on the lower dimensional surfaces. Number theory and ergodic theory play important roles to find the effect of lower dimensional oscillation. Finally, at section 6, we discuss the concept of compatibility condition for equations of nondivergence type with Neumann boundary value problems for the homogenization theory of soft inclusions.

2. Known methods

In this section, we are going to review briefly some of known methods for the introduction of homogenization theory.

2.1. Multiscale expansion method. Consider u_ε satisfying

$$\left(a_{ij}\left(\frac{x}{\varepsilon}\right)u_{\varepsilon,i}\right)_j = f \quad \text{in } \Omega, \quad u_\varepsilon = 0 \quad \text{on } \partial\Omega. \tag{2.1}$$

Let us consider the following expansion of u_ε with respect to ε:

$$u_\varepsilon(x) = u_0(x, x/\varepsilon) + \varepsilon u_1(x, x/\varepsilon) + \varepsilon^2 u_2(x, x/\varepsilon) + \cdots .$$

where $u_i(x, y)$ is an 1-periodic in y. x is called as a macroscopic(or slow) variable and $y = x/\varepsilon$ as a microscopic(or fast) variable. For heuristic argument, we may assume that $u_i(x, y)$ are smooth. Then $\nabla u_i\left(x, \frac{x}{\varepsilon}\right) = \nabla_x u_i + \frac{1}{\varepsilon}\nabla_y u_i$. Putting u_ε in the equation (2.1),

we have

$$\frac{1}{\varepsilon^2}L_1 u_o + \frac{1}{\varepsilon}(L_1 u_1 + L_2 u_o) + (L_1 u_2 + L_2 u_1 + L_3 u_o) + \varepsilon(\cdots) = f$$

where

$$\begin{cases} (a)\ L_1 v = (a_{ij} v_{y_i})_{y_j}, \\ (b)\ L_2 v = (a_{ij} v_{x_i})_{y_j} - (a_{ij} v_{y_i})_{x_j}, \\ (c)\ L_3 v = (a_{ij} v_{x_i})_{x_j}. \end{cases}$$

(a) For each x, u_0 is a uniformly bounded function in $y \in \mathbb{R}^n$ satisfying $L_1 u_o = 0$. Louville's theorem implies u_o is independent on y, which means $u_0(x, y) = u_0(x)$.

(b) $L_1 u_1 = -L_2 u_o = -(a_{ij})_{y_j} u_{x_i}$ and the linearity of the equation tells us that $u_1(x, y)$ will be of the form

$$u_1(x, y) = w^i(y) u_{x_i} + \tilde{u}_1(x)$$

where

$$L_1 w^i = -(a_{ij}(y))_{y_j} \quad \text{in } Q \quad \text{and } w^i \text{ is 1-periodic} \tag{2.2}$$

for $Q = \{x : 0 < x_i < 1 \quad \text{for } i = 1, \cdots, n\}$

(c) For given u_0 and u_1, u_2 satisfies

$$L_1 u_2 = f - L - 2u_1 - L_3 u_o,$$

which has a 1-periodic solution (in the variable y) only if the integral of the right hand side over Q is zero. The compatibility condition is

$$\int_Q L_2 u_1 + L_3 u_o dy = \int f dy = \langle f \rangle,$$

which gives us the following homogenized equation

$$a^*_{ij} u_{ij} = f \quad \text{in } \Omega \quad \text{and} \quad u = 0 \quad \text{on } \Omega \tag{2.3}$$

where

$$a^*_{ij} = \int_Q \left[a_{ij}(y) - a_{jk}(y) w^i_{y_k}(y) \right] dy.$$

2.1.1. Energe method. The energy method was developed by Tartar to justify the homogenized equation (2.3). The solution of (2.1) is defined the following weak formulation

$$\int_\Omega a_{ij} \left(\frac{x}{\varepsilon} \right) u^\varepsilon_i \phi_j dx = \int_\Omega f(x) \phi(x) dx \tag{2.4}$$

for $\phi(x) \in H^1_o(\Omega)$.

When we try to justify the heuristic computation above, we can be in trouble since the left-hand side of (2.4) involves two weakly converging sequences in L^2. The convergence of LHS is not generally true.

One of key idea is to replace the test function by weakly converging test function (so called *oscillating test function*).

$$\phi_\varepsilon(x) = \phi(x) + \varepsilon \phi_i w^i(x/\varepsilon)$$

where w^i is the solution of (2.2). The convergence of an integral of the product of two weakly converging sequences in (2.4), can be justified by div-curl lemma.

2.2. $\Gamma-$ Convergence. The solution of (2.1) can be achieved by taking the minimum of the following energy

$$\min_{u \in H_o(\Omega)} \left(\frac{1}{2} \int_\Omega a_{ij} \left(\frac{x}{\varepsilon} \right) u_i u_j dx - \int_\Omega f u dx \right).$$

In general, let us consider the convex, periodic and C^1-energy function $W(y, \lambda)$ such that $0 \le c|\lambda|^p \le W(y, \lambda) \le C(1 + |\lambda|^p)$

$$I_\varepsilon = \int_\Omega \left(W \left(\frac{x}{\varepsilon}, \nabla v(x) \right) - f(x) v(x) \right) dx.$$

Its homogenized functional is defined by

$$I(v) = \int_\Omega (W_{hom}(\nabla v(x)) - f(x)v(x))\, dx,$$

where $W_{hom}(\lambda) = \inf_{w \in H^{1,p}_{\#}} \int_Y W(\lambda + \nabla w(y))dy$ and $H^{1,p}_{\#}$ is the collection of periodic $H^{1,p}$-functions. To show the convergence of I_ε to I, De Giorgi introduced the concept of Γ-convergence. The I is a Γ-convergence of I_ε if and only if

(i) all sequence u^ε converging to u satisfy $I(u) \le \liminf_{\varepsilon \to 0} I_\varepsilon(u^\varepsilon)$

(ii) and there exists at least one sequence u^ε converging to u s.t. $I(u) = \lim_{\varepsilon \to 0} I_\varepsilon(u^\varepsilon)$.

2.3. Viscosity method.

2.3.1. Definition of viscosity solution. Roughly, a viscosity solution is a continuous function having a comparison principle when it is compared with another classical solution.

(1) $F(D^2u, x) = f(x)$ is uniformly elliptic if

$$\lambda \|N\| \le F(M + N, x) - F(M, x) \le \Lambda \|N\|,$$

for all symmetric matrix $N \ge 0$ and M.

(2) A continuous solution u is a viscosity subsolution[resp. supersolution] when the following condition holds:
if $\phi \in C^2(\Omega)$ and $u - \phi$ has a local maximum at x_o then

$$F(D^2\phi(x_o), x_o) \ge f(x_o)$$

[resp. if $u - \phi$ has a local minimum at x_o then $F(D^2\phi(x_o), x_o) \le f(x_o)$].

2.3.2. Nondivergence equations. Let us consider the homogenization theory of fully nonlinear equations with oscillating data:

$$F\left(D^2u_\varepsilon, \frac{x}{\varepsilon}\right) = 0 \quad \text{in } \Omega, \quad u_\varepsilon = 0 \quad \text{on } \partial\Omega \tag{2.5}$$

(a) (Convergence) By comparing u_ε with suitable barriers, u_ε is shown to be uniformly bounded, and then Hölder estimate of u_ε, [1], says

$$||u_\varepsilon||_{C^\alpha(B_{1/2})} \le C||u_\varepsilon||_{L^\infty(B_1)}$$

So there is a limit u such that $u_\varepsilon \to u$ as $\varepsilon \to 0$ (up to a subsequence).

(b) (Existence of the corrector) If we attempt to solve $F(M + D^2w, Y) = 0$, then we need a compatibility condition. For instance, if $F = \triangle$, we need $\int_Y tr(M) = 0$. In general, we can show that there is a constant λ such that there is a 1-periodic function $w(y)$ satisfying

$$F(M + D^2w, y) = \lambda,$$

Then we define

$$\lambda = \overline{F}(M).$$

A heuristic reasoning is the following. Set $P(x) = \frac{1}{2}M_{ij}x_ix_j + \xi_i x_i + a$.

$P(x)$ is correctiable.
$\Leftrightarrow P_\varepsilon(x) = P(x) + \varepsilon^2 w\left(\frac{x}{\varepsilon}\right)$ satisfies ε-problem, (2.5).
$\Leftrightarrow F(D^2P_\varepsilon(x), x/\varepsilon) = F(M + D^2w(y), y) = \lambda = \overline{F}(M) = 0.$
$\Leftrightarrow \overline{F}(D^2P) = 0$

Theorem 2.1 ([10]). *The homogenization equations satisfied by u is*

$$\overline{F}(D^2u(x)) = 0 \tag{2.6}$$

Proof. It is enough to check that if P is a quadratic polynomial touching u from below at x_o, $\overline{F}(D^2P) \le 0$. Suppose that $\overline{F}(D^2P) > 0$, then there is $\delta > 0$ s.t. $P = \delta|x - x_o|^2 + Q$ and $\overline{F}(D^2Q) = 0$.

Set $Q_\varepsilon = Q + \varepsilon^2 w\left(\frac{x}{\varepsilon}\right)$. $Q^\varepsilon + \gamma(\varepsilon)$ is a vertical translation of Q_ε to touch u^ε and $Q_\varepsilon + \gamma(\varepsilon) < u^\varepsilon$ on ∂B_δ.

Two different solutions are touching each other, which is a contradiction. □

3. Large oscillation

At Section 2, we considered uniform Hölder estimates for u_ε where the oscillation of u_ε gets smaller as ε goes to zero. In this section we are going to consider obstacle problems in perforated domains where the L^∞-oscillations of u_ε stay large even though ε goes to zero, so called as a large oscillation.

3.1. Perforated domains. First let us introduce perforated domains.

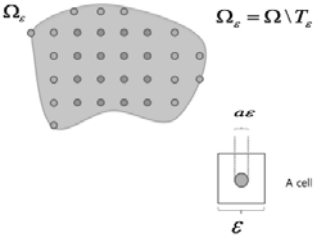


Figure 3.1. Perforated Domain

Let Ω be a bounded connected subset of $\mathbb{R}^n$. For each $\varepsilon > 0$, we cover $\mathbb{R}^n$ by the cubes $P_m^\varepsilon (m \in \mathbb{Z}^n)$ of the size 2ε. Then from each cube, P_m^ε, we remove a ball, $T_m^{a_\varepsilon}$, of the radius a_ε which has the center at the center of the cube, P_m^ε. Then we can produce a domain that is perforated by spherical identical holes (see Figure 3.1). Let

$$\mathbb{R}^n_{a_\varepsilon} = \mathbb{R}^n \setminus \cup_{m\in\varepsilon\mathbb{Z}^n} T_m^{a_\varepsilon}$$

$$\Omega_{a_\varepsilon} = \Omega \cap \mathbb{R}^n_{a_\varepsilon} = \Omega \setminus \cup_{m\in\varepsilon\mathbb{Z}^n} T_m^{a_\varepsilon}.$$

Then we are going to consider highly oscillating obstacles $\varphi_\varepsilon(x)$ which is zero in Ω_{a_ε} and $\varphi(x)$ on each hole $T_m^{a_\varepsilon}$.

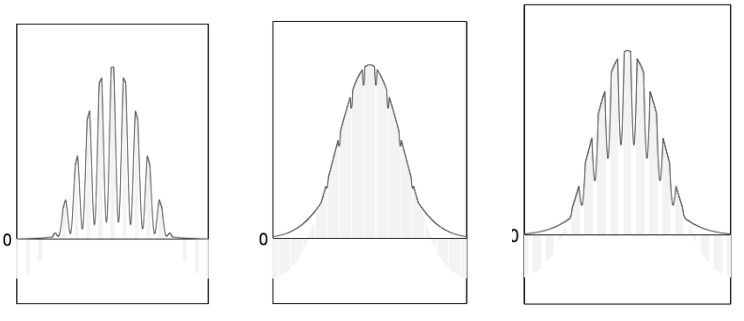
$$\varphi_\varepsilon(x) = \quad \varphi(x)\chi_{(\cup_{m\in\varepsilon\mathbb{Z}^n} T_m^{a_\varepsilon})}$$

Then $\varphi_\varepsilon(x)$ will oscillate more rapidly between 0 and $\varphi(x)$ as ε goes to zero.

3.2. Obstacle problems with highly oscillating obstacles. Now we can consider the standard obstacle problem asking the least viscosity supersolution of Laplace operator above the given oscillating obstacle: find the smallest viscosity super-solution $u_\varepsilon(x)$ s.t.

$$\begin{cases} \triangle u_\varepsilon \le 0 & \text{in } \Omega \\ u_\varepsilon(x) \ge \varphi_\varepsilon(x) & \text{in } \Omega \\ u_\varepsilon(x) = 0 & \text{on } \partial\Omega \end{cases} \tag{3.1}$$

where $\varphi_\varepsilon(x) \le 0$ on $\partial\Omega$ and φ_ε is positive at a point in Ω. The concept of viscosity solution and its regularity can be found at [1].



(a) subcritical (b) supercritical (c) critical

Figure 3.2. This figures show the oscillation of u_ε when the decary rate of a_ε is subcritical (a) , supercritical (b), and critical (c).

We are interested in the limit of the u_ε as ε goes to zero. Then there are three possibles cases. First, if the decay rate a_ε of the radius of column is too high w.r.t. ε, the limit solution will not get the influence from the existence of the obstacle. Hence it will satisfy the Laplace equation without any obstacle, Figure3.2 (a). Second, on the contrary, if the decay rate a_ε is too slow, the limit solution will be influenced fully by the existence of the obstacle and then become a solution of the obstacle problem with the obstacle $\varphi(x)$, Figure3.2-(b). We are interested in the last case when the decay rate a_ε is critical so that the limit solution will have partial influence from the obstacle, Figure3.2-(c). Then we are able to show that the limit function becomes a solution for an operator which has the original operator, that is a Laplace operator, and an additional term that comes from the influence of the oscillating obstacles. Naturally we ask what is the critical rate a_ε^* of the size of the obstacle so that there is non-trivial limit $u(x)$ of $u_\varepsilon(x)$ in the last case, Figure3.2-(c), and what is the homogenized equation satisfied by the limit function u.

Theorem 3.1 ([6]). *Let u_ε be the least viscosity super solution of* (3.1).

(1) *There is a continuous function u such that $u_\varepsilon \xrightarrow{w} u$ in Ω with respect to L^∞-norm. And for any $\delta > 0$, there is a subset $D \subset \Omega$ and ε_o such that , for $0 < \varepsilon < \varepsilon_o$, $u_\varepsilon \to u$ uniformly in D as $\varepsilon \to 0$ and $|\Omega \backslash D| < \delta$.*

(2) *Let $a_\varepsilon^* = \varepsilon^{\alpha_*}$ for $\alpha_* = \frac{n}{n-2}$ in $\mathbb{R}^n (n \geq 3)$ and $a_\varepsilon^* = e^{-\frac{1}{\varepsilon^2}}$ in $\mathbb{R}^2$.*

(a) *For $c_o \alpha_\varepsilon^* \leq \alpha_\varepsilon \leq C_o \alpha_\varepsilon^*$, u is a viscosity solution of*

$$\begin{aligned} \triangle u + k_{B_{r_o}} (\varphi - u)_+ = 0 \quad & \text{in } \Omega \\ u = 0 \quad & \text{on } \partial\Omega. \end{aligned}$$

where $k_{B_{r_o}}$ is the capacity of B_{r_o} if $r_o = \lim_{\varepsilon \to 0} \frac{\alpha_\varepsilon}{\alpha_\varepsilon^*}$ exists.

3.3. Fully nonlinear operator. The viscosity method above can be extended to a fully nonlinear uniformly elliptic operator positively homogeneous of degree one. We define the fully nonlinear equation of nondivergence type as follows:

$$F(D^2 u(x), x) = f(x)$$

where f and u are given in Ω. We also call F positively homogeneous of degree one if $F(tM) = tF(M)$ for $t > 0$. We assume that F is uniformly elliptic and positively homogeneous of degree one.

We are going to consider fully nonlinear version of (3.1): find the least viscosity supersolution u_ε such that

$$\begin{cases} F(D^2 u_\varepsilon(x)) \leq 0 & \text{in } \Omega \\ u_\varepsilon(x) \geq \varphi_\varepsilon(x) & \text{in } \Omega \\ u_\varepsilon(x) = 0 & \text{on } \partial\Omega \end{cases} \tag{3.2}$$

where φ_ε satisfies the same condition in (3.1). At [6], we show that a homogeneous fully nonlinear equation has a homogeneous solution as the Laplace equation has a fundamental solution.

Proposition 3.2 ([6]). There is a solution $V(x)$ for $F(D^2 V) = 0$ in $\mathbb{R}^n \backslash \{0\}$ which is one of the following three types with a positive function $\Phi(\theta)$ where $\theta = \frac{x}{|x|}$.

- **(Type I)** $V(x) = |x|^{-\lambda} \Phi(\theta)$ (Example: Laplacian in $\mathbb{R}^n$ and $n \geq 3$.)
- **(Type II)** $V(x) = \log(|x|) + \Phi(\theta)$. (Example: Laplacian in $\mathbb{R}^2$.)
- **(Type III)** $V(x) = |x|^\lambda \Phi(\theta)$. (Example: Laplacian in $\mathbb{R}$.)

Theorem 3.3 ([6]).

(1) *There is a continuous function u such that $u_\varepsilon \xrightarrow{w} u$ in Ω with respect to L^∞-norm. And for any $\delta > 0$, there is a subset $D \subset \Omega$ such that $u_\varepsilon \to u$ uniformly in D and $|\Omega \backslash D| < \delta$.*

(2) *In (Type III), for any a_ε, the limit u is a least viscosity super solution of*

$$F(D^2 u) \leq 0, \quad u \geq \varphi \quad \text{in } \Omega \quad \text{with } u = 0 \text{ on } \partial\Omega$$

(3) *Set* $a_\varepsilon^* = \varepsilon^{\alpha_*}$ *for* $\alpha_* = \frac{\lambda+2}{\lambda}$ *in (Type I) and* $a_\varepsilon^* = e^{-\frac{1}{\varepsilon}}$ *in (Type II). Then If there are* $0 < c_o < C_o < \infty$ *such that* $c_o a_\varepsilon^* \le a_\varepsilon \le C_o a_\varepsilon^*$, *there is a uniform elliptic operator* $\overline{F}(D^2u, (\varphi - u)_+)$ *such that the limit* u *is a viscosity solution of*

$$\begin{aligned} \overline{F}(D^2u, (\varphi - u)_+) = 0 \quad & \text{in } \Omega \\ u = 0 \quad & \text{on } \partial\Omega. \end{aligned}$$

3.4. Critical rates. To find a critical rate, a_ε^*, we consider a global periodic solution of periodic obstacle problems having constant height called as a corrector, w_ε.

(a) subcritical (b) supercritical (c) critical

Figure 3.3. This figures show the oscillation of w_ε when the decay rate of a_ε is subcritical (a) , supercritical (b), and critical (c)

The existence and behavior of w_ε have been achieved by the construction of super- or sub-solutions using fundamental solution or homogeneous solution $V(x)$. At (a), the obstacle is too thin, the oscillation of w_ε goes infinity while the oscillation of w_ε goes zero at (b) since it is too thick. So at the critical rate, a_ε^* at (c), the oscillation of w_ε will be of order one with which we can correct the limit profile, u, to be almost a solution of ε-problem.

3.5. Uniform discrete estimate. It is clear that the oscillation of u_ε is of order one even though ε goes to zero as we can observe at Figure 3.2. So it may be more appropriate to estimate in terms of integral quantities. For the linear equation, there is a uniform $H^1(\Omega)$-estimate but it is not clear how to estimate u_ε uniformly for nonlinear equation of nondivergence type.

3.5.1. Discrete gradient estimate. Even though u_ε oscillates with order one, the values at discrete points, equivalent in mod one, changes with small oscillations.

Lemma 3.4 (Discrete Gradient Estimate). Let $he \in \mathbb{Z}^n$ with a unit vector e. There is a uniform constant $C > 0$ such that the differential quotient satisfies

$$|\triangle_e^h u_\varepsilon(x)| = \frac{|u_\varepsilon(x + \varepsilon he) - u(x)|}{\varepsilon h} \le C$$

whenever $x, x + \varepsilon he \in \Omega$.

Idea of Proof. We apply maximum principle on $Z = \sup_{\Omega_\varepsilon} |\triangle_e u_\varepsilon|^2$.

(i) Z satisfies a linearized equation having maximum principle in Ω_ε, which implies there is no maximum in $\Omega_\varepsilon \backslash \partial\Omega$.

(ii) The boundary discrete gradient estimate can be achieved after the construction of a super-solution having linear growth from the boundary called as a barrier.

(a) First we consider the convex domain where the plane-like solution $\xi \cdot x + \varepsilon v_\varepsilon$ with oscillations plays as a barrier. And then we go through the homogenization.

This solution h_ε will be bended like a concave function if $f = -M$ for large constant $M > 0$ since $\overline{F}$ is uniformly elliptic.

(b) Now we consider $\tilde{h}_\varepsilon = \max(K - h_\varepsilon, 0)$, which has a flat spot at set close to a small ball and it grows linearly away from the flat spot. If Ω satisfies the uniform exterior sphere condition, we can use $\tilde{h}_\varepsilon$ as a barrier. □

3.5.2. ε-Flatness.

Lemma 3.5 (ε-Flatness). Let u_ε be the viscosity solution of (3.1) or (3.2). There are a uniform constant $C > 0$ and $\alpha \in (0, 1)$ such that, if $B_{\delta_0} \subset \Omega$, then, for any $R > 0$,

$$|u_\varepsilon(x) - u_\varepsilon(y)| \le C\varepsilon$$

for $x, y \in B_{\varepsilon R} \cap \Omega_{\sqrt{a_\varepsilon}} \subset B_{\delta_0}$.

Theorem 3.6 (Global ε-Lipschitz Estimate). *There is uniform constants $C > 0$ such that*

$$|u_\varepsilon(x) - u_\varepsilon(y)| \le C(|x - y| + \varepsilon)$$

for $x, y \in \Omega_\varepsilon$

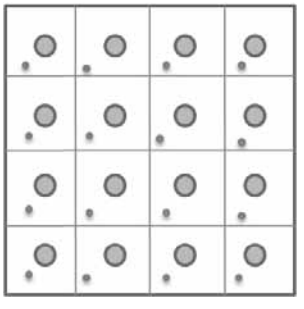


Figure 3.4. Discrete Points

Idea of Proof.

- The values of u_ε on the red dot at Figure 3.5.2 satisfies the discrete gradient estimate, Lemma 3.4.
- u_ε at each cell has ε-Flatness outside of a small neighborhood of holes, saying that the values is almost constant with ε-error in a cell except for very small region.
- For any two points x and y, There will be points x^* and y^* denoted as red dots at Figure 3.5.2 at the same cell respectively. Then the values of u_ε at x and y will be close to u_ε at x^* and y^* with ε-error. □

From the Arzela-Ascoli Theorem, we have a uniform convergence to a limit.

Corollary 3.7. *There is a Lipschitz continuous function u which is a limit of a subsequence of $\{u_\varepsilon\}$.*

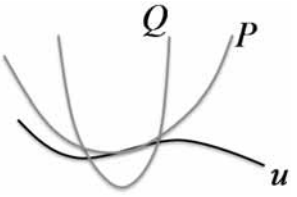


Figure 3.5. Viscosity Method

3.6. Homogenization. We are going to show the limit u is a viscosity sub- and super-solution.

Claim: u is a subsolution. Otherwise there is $P(x) = \frac{1}{2}P_{ij}x_ix_j + \xi_i x_i + c$ touching u at $x_0 = 0$ from above and

$$\overline{F}(P, \xi, 0) < -2\delta_0 < 0.$$

Choose another quadratic polynomial $Q(x)$ as the figure 3.6 and apply correctors to correct $Q(x)$:

$$Q_\varepsilon(x) = Q(x) + \varepsilon^2 w_\varepsilon \left(x, \frac{x}{\varepsilon}\right).$$

Then $Q_\varepsilon(x)$ will be a super-solution of (3.2) and $u_\varepsilon(x) \leq Q_\varepsilon(x)$ in a neighborhood of $x_0 = 0$. From the uniform convergence u_ε to $u(x)$,

$$u(0) \leq Q(0),$$

which is a contradiction.

3.7. Related works. The homogenization theory of fully nonlinear parabolic equations and degenerate equations of Porous Medium type have been studied at [13]. It is very crucial to construct a barrier keeping degeneracy and high oscillations, which have been obtained through the homogenization of nonlinear elliptic eigenvalue problems in perforated domain. And the homogenization of soft inclusions in perforated domain for nonlinear equations of nondiveregnce type has been studied at [18] , where we propose a new compatibility condition for nondivergence-type equations which was unknown. We also consider the homogenizations of thin obstacles at [17] where a lower dimensional obstacle may goes through perforated domains partially, Section 5. So number theory plays very important role.

4. Nonvariational problems

In this section, we are going to consider homogenization of nonvariational problems, whose effective equations could be different from a simple average that we can find frequently at variational problems. Such phenomenon can be addressed at flame propagations.

4.1. Flame propagation. The following equation arises in the modeling of the combustion of a premixed gas (thermodiffusive approximation):

$$\partial_t u = \triangle u - f(x)\beta_\delta(u), \quad x \in \mathbb{R}^n. \tag{4.1}$$

The reaction term is given by $\beta_\delta(s) = \frac{1}{\delta}\beta(\frac{s}{\delta})$, with $0 < \delta < 1$ and with $\beta(s)$ a Lipschitz function satisfying ï£ijï£ij $\beta(s) > 0$ if $x \in (0,1), \beta(s) = 0$ otherwise, $M = \int_0^1 \beta(s)\, ds$.

This model is usually referred to as the ignition temperature model. The function $f(x)$ is positive and is related to the combustion rate; it is independent of the space variable when the media is perfectly homogeneous. In this paper, we assume that heterogeneities arise in the premixed gas over a small scale (of order ε) and in a periodic manner. This amounts to writing $f_\varepsilon(x) = f\left(\frac{x}{\varepsilon}\right)$ with $f(x+k) = f(x)$ for all $k \in \mathbb{Z}^n$. As $\delta \to 0$, we have the a free boundary problem of flame type: for a unit direction e and a periodic function f satisfying $0 < \lambda \le f \le \Lambda < \infty$, u_ε satisfies

$$\begin{cases} u_{\varepsilon,t} = \triangle u_\varepsilon - f\left(\frac{x}{\varepsilon}\right)\beta_\delta(u_\varepsilon) & \text{in } \mathbb{R}^n \\ u_\varepsilon \to 0 \quad \text{as } x \cdot e \to -\infty \\ u_\varepsilon \to 1 \quad \text{on } x \cdot e \to \infty \end{cases} \tag{4.2}$$

4.2. Stationary flame. Let us consider a simple case when f depends only one variable that is $f(x) = f(x_1)$.

Assume u_ε satisfys the following free boundary value problems:

$$\begin{cases} \triangle u_\varepsilon = 0 \quad \text{in } \Omega(u_\varepsilon) = \{u_\varepsilon > 0\} \\ |\nabla u_\varepsilon|^2 = 2f\left(\frac{x_1}{\varepsilon}\right) \quad \text{on } \partial\Omega(u_\varepsilon)\backslash\overline{B_1} \\ u_\varepsilon = 1 \quad \text{on } \partial B_1, \end{cases} \tag{4.3}$$

which can be given by taking $\delta \to 0$ at (4.2). We can find there are multiple solutions. Let v_ε be the minimizer of the corresponding energy

$$E_\varepsilon(w) = \frac{1}{2}\int_{B_R\backslash B_1} \|\nabla w\|^2 ds + \int_{B_R\backslash B_1} f\left(\frac{x_1}{\varepsilon}\right)\chi_{\{w>0\}} dx$$

satisfying the given boundary condition and u_ε is the smallest solution. If u and v are the limits of u_ε and v_ε, then the slops of u and v are $\sqrt{2\langle f\rangle}$ and $\sqrt{2\max f}$ respectively as Figure 4.1-(a) for $n = 1$. This example tells us that *the homogenization of the smallest solution u_ε of nonvariational problem is genuinely different from that of solutions, v_ε, of variational problem.*

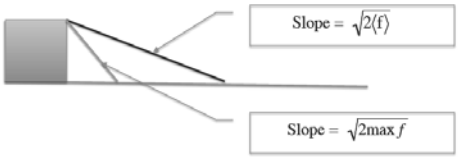
Now let us consider two dimensional problems, $n = 2$. First, *the limit of energy minimizer, v, has circular free boundary*, since the effective free boundary condition is constant, $\sqrt{2\langle f\rangle}$, [4]. Now we cut off the limit of energy minimizer by the planes with the maximum slope by taking the minimum between v and the plane. The smallest solution u lies under the minimum of v and the plane, and u *has a flat spot.*

Theorem 4.1 ([5]). *Let the dimension $n = 2$ and $f(x) = f(x_1)$.*

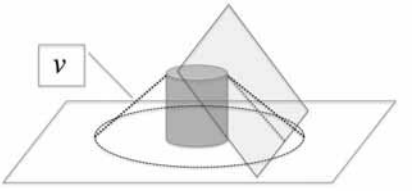
(1) *The free boundary, $\partial\Omega(u) = \partial\{u > 0\}$, is C^1.*

(2) *On the flat spot, $|\nabla u| \in [\sqrt{2\langle f\rangle}, \sqrt{2\max f}]$. $|\nabla u|$ achieves $\sqrt{2\max f}$ at the center of the flat spot and decreases to $\sqrt{2\langle f\rangle}$ as the point goes to the end points of the flat spot.*

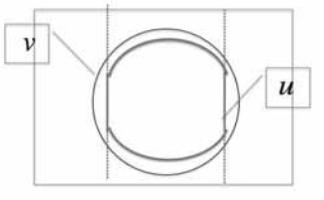
(3) *the similar statements are true for general periodic function f.*



(a) Energy minimizer and the smallest



(b) Cutting with planes



(c) Flat spot on the free boundary

Figure 4.1. Stationary Flames

For a more general periodic function, $f(x+k) = f(x)$ for $k \in \mathbb{Z}^2$ with a line symmetry, the key difference is that a plane is not a stationary flame any more. At [4, 5], for each direction ν, we found plane-like solutions, $w^\nu(x)$, which is trapped between two parallel planes with a uniform distance. Then the free boundary of $w^\nu(x)$ oscillates with a uniform size. If we rescale back $w^\nu(x)$ to ε-problem, the oscillation of free boundary of plane-like solution, $w^\nu_\varepsilon(x)$, goes to zero as $\varepsilon \to 0$ and the slope of the limit, $\alpha(\nu)$, of $w^\nu_\varepsilon(x)$ will be the maximum of possible slope of the limit u as the $\max f$ in one dimension.

4.3. Pulsating flame and homogenization. At [7–9], the homogenization of pulsating wave has been studied when the thickness of the flame is much smaller than the periodicity of reaction materials, $\frac{\delta}{\varepsilon} \to 0$. Pulsating wave is a traveling wave-like oscillating solution trapped by two standard traveling wave with the same constant reaction rate and with the same speed. To track down the property of free boundary, we need to scale the space and time with the same scale and then the heat equation in $\Omega(u_\varepsilon)$ converges to Laplace equation, which means $\overline{u} = \frac{1}{\varepsilon} u_\varepsilon(\varepsilon x, \varepsilon t)$ is close to harmonic function having the same reaction rate, but the boundary is moving with an effective speed. So the effective reaction rate for traveling wave is proved to be the minimum of the slope of plane-like stationary flame for the direction in which the traveling wave moves. In addition the oscillation of free boundary of u_ε will converges to zero uniformly.

5. Lower dimensional oscillation

Homogenization of lower dimensional oscillation has been studied actively recently. The contents in this section can be referred to [3, 16, 17, 19].

5.1. Examples. First, let us consider examples where the oscillation of data occurs on lower dimensional hypersurface.

5.1.1. Dirichlet or Neumann boundary value problems. We are going to consider oscillating boundary data. Set $\Omega \subset \mathbb{R}^n$ to be a smooth domain. Assume that $f(x,y)$ and $g(x,y)$ are periodic in y-variable that is $g(x,y+m) = g(x,y)$ for $m \in \mathbb{Z}^n$. We consider u_ε satisfying

$$\triangle u_\varepsilon = f\left(x, \frac{x}{\varepsilon}\right) \text{ in } \Omega, \quad u_\epsilon = g_\varepsilon(x) \text{ on } \partial\Omega \tag{5.1}$$

for $g_\varepsilon(x) = g\left(x, \frac{x}{\varepsilon}\right)$, or

$$F\left(D^2 u_\epsilon, \frac{x}{\epsilon}\right) = f\left(x, \frac{x}{\varepsilon}\right) \text{ in } \Omega, \quad u_\epsilon = g_\varepsilon(x) \text{ on } \partial\Omega. \tag{5.2}$$

We can also consider Neumann boundary value problem by replacing the Dirichlet boundary data by a Neumann boundary data, $\frac{\partial u_\epsilon}{\partial \nu} = g_\varepsilon(x)$ on $\partial\Omega$.

5.1.2. Thin obstacles. Let $\Gamma = \Gamma_\nu$ be a hyperplane with a surface measure σ, defined by $\Gamma_\nu = \{x \in \mathbb{R}^n : x \cdot \nu = x^0 \cdot \nu\}$ for given $\nu \in S^{n-1}$ and $x^0 \in \mathbb{R}^n$.

The set $\Gamma_\varepsilon = \Gamma \cap T_\varepsilon$ describes the intersection between the hyper-plane and the periodic background. Then, for a given $\psi \in L^\infty(\Omega) \cap H^1(\Omega)$ such that $\psi \le 0$ on $\partial\Omega$, we define the obstacle $\psi_\varepsilon = \psi\chi_{\Gamma_\varepsilon}$ and the admissible set $\mathcal{K}_{\psi_\varepsilon} = \{v \in H_0^1(\Omega) : v \ge \psi_\varepsilon\}$. The inequality in $\mathcal{K}_{\psi_\varepsilon}$ is to be interpreted in the sense of trace, i.e. $\text{Trace}_{\Gamma_\varepsilon}(u_\varepsilon - \psi) \ge 0$ on Γ_ε and $u_\varepsilon \ge 0$ a.e. in $\Omega \setminus \Gamma_\varepsilon$. We consider the following thin obstacle problem, for $f \in L^2(\Omega)$: find $u_\varepsilon \in \mathcal{K}_{\psi_\varepsilon}$ and

$$\int_\Omega \nabla u_\varepsilon \cdot \nabla(v - u_\varepsilon) dx \ge \int_\Omega (v - u_\varepsilon) f dx, \quad \text{for all } v \in \mathcal{K}_{\psi_\varepsilon}. \tag{5.3}$$

The variational inequality (5.3) has a unique solution $u_\varepsilon \in \mathcal{K}_{\psi_\varepsilon}$ which can be obtained as the unique minimizer of the strictly convex and coercive functional

$$J(v) := \int_\Omega \frac{1}{2}|\nabla v|^2 - fv dx, \quad v \in \mathcal{K}_{\psi_\varepsilon}. \tag{5.4}$$

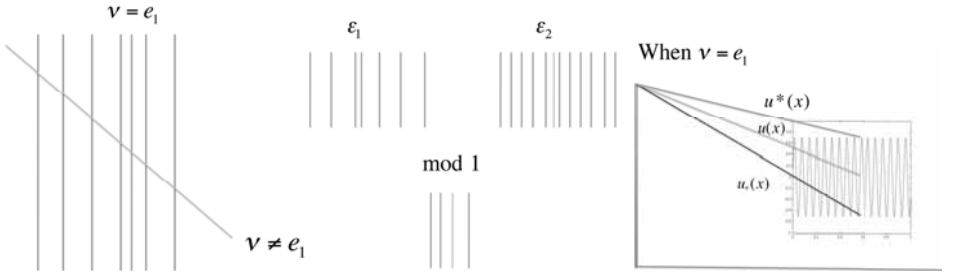
As $\varepsilon \to 0$, we are interested in the asymptotic behavior of u_ε. We want to determine $u = \lim_{\varepsilon\to 0} u_\varepsilon$ in terms of an effective equation that it solves. Main difficulty comes from the fact that the perforated surface Γ_ε does not have a lattice structure in the sense that the perforations are not evenly spaced, and this introduces a substantial difficulty.

5.1.3. Flame propagation. The free boundary of moving flame is also a oscillating hypersurface which will share the same difficulties and flavors as the oscillating boundary data and thin obstacles.

5.2. Observations. Let us consider some examples to address the main issues.

5.2.1. Example 1: One dimensional problem. Let us consider a simple example to find out main issues. Set $\Omega = \{x : 0 < x \cdot \nu < R\}$ and $g(x) = g(x_1)$. Consider u_ε satisfying

$$\triangle u_\varepsilon = 0 \text{ in } \Omega, \quad u_\varepsilon(x) = M \text{ on } x \cdot \nu = 0, \quad u_\varepsilon(x) = g\left(\frac{x}{\varepsilon}\right) \text{ on } x \cdot \nu = R. \tag{5.5}$$



(a) directions (b) mod 1 oscillation (c) $u^*(x), u(x)$, and $u_*(x)$

Figure 5.1. (a)(b): the fixed boundary may oscillate at a configuration of mod one. (c):the boundary data may changes as $\varepsilon \to 0$.

We may have the following three possible limits:

$$u^* = \limsup_{\varepsilon \to 0} u_\varepsilon, \qquad u_* = \liminf_{\varepsilon \to 0} u_\varepsilon,$$

and u which is the limit of a sequence u_{ε_j} of Energy minimizers at ε_j-problems s.t.

$$E(u_{\varepsilon_j}) \to \liminf_{\varepsilon \to 0} E(u_\varepsilon).$$

From the symmetry of the boundary, the solution u_ε is an one dimensional solution, which will be a part of planes.

Homogenized equations for u^* and u_*: Set $\overline{g}^*(e_1, R) = \sup_y g(y)$ and $\overline{g}_*(e_1, R) = \inf_y g(y)$.

And for $\nu \neq e_1$, $\overline{g}^*(\nu, R) = \overline{g}_*(\nu, R) = \langle g \rangle$= the simple average on a unit cell. For $\nu = e_1$, we will have

$$\begin{cases} \triangle u^* = 0 & \text{in } \Omega \\ u^*(x) = M & \text{on } x \cdot \nu = 0 \\ u^*(x) = \overline{g}^*(\nu, R) & \text{on } x \cdot \nu = R, \end{cases} \quad \text{and} \quad \begin{cases} \triangle u_* = 0 & \text{in } \Omega \\ u_*(x) = M & \text{on } x \cdot \nu = 0 \\ u_*(x) = \overline{g}_*(\nu, R) & \text{on } x \cdot \nu = R. \end{cases} \tag{5.6}$$

Homogenized equations for u: u is a one dimensional energy minimizer, which implies u is a part of planes. From a simple computation, we can show that u is a solution of

$$\triangle u = 0 \text{ in } \Omega, \quad u(x) = M \text{ on } x \cdot \nu = 0, \quad u(x) = \overline{g}(\nu, M, R) \text{ on } x \cdot \nu = R, \tag{5.7}$$

where

$$\overline{g}(e_1, M, R) = \begin{cases} \overline{g}^*(e_1, R) & \text{for } M > \sup g \\ \overline{g}_*(e_1, R) & \text{for } M < \inf g \\ M & \text{for } \inf g \leq M \leq \sup g \end{cases} \tag{5.8}$$

and for $\nu \neq e_1$, $\overline{g}(\nu, M, R) = \overline{g}^*(\nu, R) = \overline{g}_*(\nu, R) = \langle g \rangle$.

5.2.2. Main difficulties. We will face the following difficulties on the homogenization with lower dimensional oscillation.

(i) From Figure 5.1-(a) , we can find 2-dimensional straight line with rational direction (mod 1) covers unit square partially. So the average $\overline{g}$ of g_ε along the hyperplanes may be different from the simple average $\langle g \rangle$. And the average $\overline{g}$ may depend on the direction and a point on it.

(ii) Let $\partial\Omega = \{x \cdot \nu = R\}$. The boundary oscillates as $\varepsilon \to 0$. For example $x \cdot \nu = R_2$ (mod ε) oscillates as $\varepsilon \to 0$ as Figure 5.1-(b) .

(iii) There could be multiple possible limits of subsequences of u_ε: the largest limit $u^*(x)$, a limit $u(x)$, and the smallest limit $u_*(x)$, Figure 5.1-(c). Each different limits of u_ε depends on the choice of boundary data and may have different process of homogenization. Any value between maximum of g and minimum of g could be taken by some limit of u_ε.

(iv) There could be no solution in the Neumann Problems due to the compatibility condition.

5.2.3. Example 2: $g(x) = g(x_1)$ in a convex domain. Let Ω be convex, $L = \{x : x_1 = 0, 1 \quad \text{and} \quad -l \le x_2 \le l\} \subset \partial\Omega$, and strictly convex on $\partial\Omega \backslash L$. Let $g(x) = g(x_1)$. **If ν is an irrational direction, we can show**

$$\overline{g}^*(\nu_x, x) = \overline{g}_*(\nu_x, x) = \langle g \rangle,$$

for the ergodic property (or Weyl's Lemma) for the irrational direction, Section 5.4. If u is a limit of a subsequence u_{ε_i}, then u satisfies

$$\triangle u = 0 \text{ in } \Omega, \quad u = \langle g \rangle \text{ on } \partial\Omega \backslash L, \quad \overline{g}_* \le u(x) \le \overline{g}^* \text{ on } L. \tag{5.9}$$

Remark 5.1. For each $\overline{g}_* \le G \le \overline{g}^*$, there is $\{u_{\varepsilon_i}\}$ such that $u_{\varepsilon_i} = G$ on $x \cdot e_1 = R$.

Lemma 5.2. Let $\Omega = \{-R_1 < x_1 < R_2\}$ and $u_\varepsilon = g(x_1/\varepsilon)$. Then there is a smooth periodic function $g(x_1)$ of one variable x_1 with a periodicity one such that u^* nor u_* is not a solution.

Lemma 5.3. There is a a smooth periodic function $g(x_1)$ of one variable x_1 with a periodicity one and a subsequence $\{u_{\varepsilon_i}\}$ converging to u such that u is not continuous on $\partial\Omega$.

Remark 5.4.

(i) From this lemma, we can find the selection of the boundary data in a part of boundary that may depends on the section of boundary data in the other part of boundary. We there is a nonlocal effect in the homogenization, which should be analyzed at future work.

(ii) The regularity of the effective data could be interesting questions. At the homogenization of stationary flame with oscillating reaction rates, [5], we show the effective reaction rate is continuous since that free boundary (or interface between burnt zone and fresh zone) is automatically selected by the equation, not given.

5.3. Uniform distribution (mod 1). This section contains a general discussion of uniform distribution (mod 1), which will be used for the homogenization.

(i) As we can find at Figure 5.1-(a), the hyperplane (mod 1) covers a unit square partially when the normal direction is rational while it covers a unit square fully for the irrational normal direction. Such phenomenon can be expressed as the well-known ergodic theory.

(ii) At application of ergodic theory to homogenization, the region considered is of order one while the periodicity is of order ε. After a scaling changing the periodicity to one, we will consider a part of hyperplane of size $\frac{1}{\varepsilon}$, not of infinite size. So if a hyperplane with irrational direction is very closed to one of rational direction, its behavior is more line a rational direction in a region of oder $\frac{1}{\varepsilon}$. So it is very important convergence rate of quantities.

An introduction to the theory of uniform distribution can be found [14]. And we will summarize some of standard theory.

Thus we are led to study the distribution mod 1 of sequences of this type. We start by considering sequences $k\alpha$ with $k \in \mathbb{Z}$ and $\alpha \in \mathbb{R}$. This can be generalized to the higher dimensional case.

First we define the notion of uniform distribution.

Definition 5.5 (Uniform distribution mod 1).

(i) Let $\{x_j\}_{j=1}^{\infty}$ be given sequence of real numbers. For a positive integer N and a subset E of $[0,1]$, let the counting function $A(E;\{x_j\};N)$ be defined as the numbers of terms x_j, $1 \le j \le N$, for which $x_j \in E$ (mod 1).

(ii) The sequence of real numbers $\{x_j\}$ is said to be uniformly distributed modulo 1 if for every pair a, b of real numbers with $0 \le a < b \le 1$ we have

$$\lim_{N\to\infty} \frac{A([a,b);\{x_j\};N)}{N} = b - a. \tag{5.10}$$

(iii) Let $\{x_j\}_{j=1}^{\infty}$ be a sequence of real numbers. The discrepancy of its N first elements is the number

$$D_N(\{x_j\}_{j=1}^{N}) = \sup_{0\le a<b\le 1} \left| \frac{A([a,b);\{k\alpha\},N)}{N} - (b-a) \right|. \tag{5.11}$$

If $x_j = j\alpha$, we simply write $D_N(\alpha)$.

The uniformly distributed mod 1 of $\{x_j\}$ is equivalent to

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_{[0,1]} f dx. \tag{5.12}$$

for any $f \in C[0,1]$ (upon extending f periodically to $\mathbb{R}$). Then we have well-known the following ergodic theorem.

Lemma 5.6 (Weyl's Lemma). For each an irrational number α, $k\alpha\,(mod\,1)$, for $k \in \mathbb{Z}$, is distributed uniformly on $[0,1]$.

And note that

$$|A([0,N^{-\frac{1}{2}});\{x_j\},N)-N^{-\frac{1}{2}}N|=N\left|\frac{A([0,N^{-\frac{1}{2}});\{x_j\},N)}{N}-N^{-\frac{1}{2}}\right|\le ND_N(\alpha), \tag{5.13}$$

by definition of the discrepancy. The estimate for the discrepancy is the following.

Theorem 5.7 ([12, 14]).

(i) $\frac{ND_N(\alpha)}{\log N\log(\log N)}\to\frac{2}{\pi^2}$ *in measure w.r.t.* α *as* $N\to\infty$. *In particular, this result is true for a.e.* α *in a bounded set.*

(ii) *For a.e.* $\alpha\in\mathbb{R}$ *holds* $D_N(\alpha)=O\left(\frac{\log^{2+\delta}N}{N}\right)$ *for any* $\delta>0$.

5.4. Linear equations with Dirichlet boundary value problem, [16].

Lemma 5.8. Assume that g is continuous. $H(x,\nu)=\{y:\ y\cdot\nu_x\le x\cdot\nu_x\}$. Set $\overline{g}^*(x,\nu)=\limsup_{\varepsilon\to0}\fint_{\partial H(x,\nu_x)}g\left(x,\frac{y}{\varepsilon}\right)d\sigma_y$, and $\overline{g}_*(x,\nu)=\liminf_{\varepsilon\to0}\fint_{\partial H(x,\nu_x)}g\left(x,\frac{y}{\varepsilon}\right)d\sigma_y$. Then if ν is an irrational direction, then $\overline{g}^*(\nu,x)=\overline{g}_*(\nu,x)=\langle g\rangle$ where $\langle g\rangle=\fint_{Q_1}g(y)\,dy$ for the unit cell $Q_1=\{x:\ 0\le x_i\le1\}$.

Theorem 5.9. *For any sequence of* u_ε, *there is a subsequence converging uniformly to* u *in any compact subset* Ω' *of* Ω *and* u *satisfies*

$$\begin{cases}\triangle u=\overline{f}(x)\ \text{in}\ \Omega\\ \overline{g}_*(x,\nu_x)\le\liminf_{x\to x_0}u\le\limsup_{x\to x_0}u\le\overline{g}^*(x,\nu_x)\quad \text{for}\ x_0\in\partial\Omega.\end{cases} \tag{5.14}$$

Definition 5.10. We say a domain D satisfies the Irrational Direction Dense Condition, IDDC, if all but countably many points on ∂D are irrational

Corollary 5.11. *If* Ω *satisfies Irrational Direction Dense Condition , there is a unique continuous functions* u *such that*

(i) u_ε *converges uniformly to* u *in any compact subset* Ω' *of* Ω,

(ii) *and* u *is a viscosity solution of*

$$\begin{cases}\triangle u=\overline{f}(x) & \text{in}\ \Omega\\ u=\langle g\rangle(x) & \text{on}\ \partial\Omega\end{cases} \tag{5.15}$$

Lemma 5.12.

$$\limsup_{\varepsilon\to0}\int_{\partial\Omega}g\left(y,\frac{y}{\varepsilon}\right)f(y)d\sigma_y=\int_{\partial\Omega}\overline{g}^*(y,\nu_y)f(y)d\sigma_y$$

and

$$\liminf_{\varepsilon\to0}\int_{\partial\Omega}g\left(y,\frac{y}{\varepsilon}\right)f(y)d\sigma_y=\int_{\partial\Omega}\overline{g}_*(y,\nu_y)f(y)d\sigma_y.$$

Idea of Theorem 5.9. By applying Lemma on the Green's representation,

$$u_\varepsilon=\int_\Omega G(x,y)f\left(x,\frac{x}{\varepsilon}\right)dx+\int_{\partial\Omega}P(x,y)g\left(x,\frac{x}{\varepsilon}\right)d\sigma_x,$$

we have

$$u^*(x)\le\int_\Omega G(x,y)\langle f\rangle(x)dx+\int_{\partial\Omega}P(x,y)\overline{g}^*(x)d\sigma_x. \qquad \square$$

5.5. Fully nonlinear equations with Dirichlet boundary value problem, [19].

5.5.1. Correctors for Dirichlet problem. For the fully nonlinear case, we will have the following issues:

- There is no Green's representation formula.
- What's the average, $\overline{g}$, of $g(y)$ with respect to each Fully Nonlinear Equation? How can we find $\overline{g}^*(p,\nu)$ and $\overline{g}_*(p,\nu)$ without an integral?

We can resolve the issues through *correctors* which will tell us how the oscillating boundary data will be homogenized out by Fully Nonlinear Equations.

Proposition 5.13 (Correctors). For a $x_0 \in \mathbb{R}^n$, and $\nu \in S^{n-1}$ such that $p \cdot \nu \neq 0$, there are periodic solutions $w_\varepsilon(\cdot\,; x_0, \nu)$, and constants $\overline{g}^*(x_0,\nu)$ and $\overline{g}_*(x_0,\nu)$ such that

$$\begin{cases} F\left(D^2 w_\epsilon, \frac{x}{\varepsilon}\right) = 0 & H(\nu, x_0) = \text{ in } \{x : x\cdot\nu > x_0\cdot\nu\}, \\ w_\epsilon = g\left(p, \frac{x}{\varepsilon}\right) & \partial H(\nu, x_0) \text{ on } \{x : x\cdot\nu = x_0\cdot\nu\}. \end{cases} \tag{5.16}$$

and

$$\limsup_{\varepsilon\to 0} \lim_{t\to\infty} w_\varepsilon(x + t\nu; x_0, \nu) = \overline{g}^*(x_0,\nu),$$
$$\liminf_{\varepsilon\to 0} \lim_{t\to\infty} w_\varepsilon(x + t\nu; x_0, \nu) = \overline{g}_*(x_0,\nu)$$

for all $x \in \Omega'$ where Ω' is a compact subset of $H(\nu, x_0)$.
And if ν is an irrational direction, $\overline{g}^*(p,\nu) = \overline{g}_*(p,\nu)$.

Lemma 5.14 (Flatness away from the boundary). For $H(p,\nu) = \{x : x\cdot\nu < p\cdot\nu\}$ and for $\delta_0 > 0$, there is a $\rho_\varepsilon \to 0$ such that

$$\operatorname{osc}_{H(p,\nu)_{(-\delta_0)}} w_\varepsilon < C\rho_\varepsilon$$

for a uniform constant $C(\lambda, \Lambda, \delta_0) > 0$.

When ν is rational, εM_ν-periodicity of the solution and Hölder regularity of the correctors in the interior give such results. If ν is irrational, there is almost periodicity with errors $\varepsilon \cdot \frac{1}{\varepsilon^{1/4}}$ and Hölder regularity. $\overline{g}^*(p,\nu) \neq \overline{g}_*(p,\nu)$ in the previous Examples.

Lemma 5.15. For any irrational direction ν, there is a unique $w(x;\nu)$ up to discrete translation such that $w_\varepsilon(x; x_0, \nu) = w\left(\frac{x}{\varepsilon}; \nu\right)$. And then $\overline{g}^*(x_0,\nu) = \overline{g}_*(x_0,\nu) = \overline{g}(\nu)$.

Let say there are two correctors, w_1 and w_2, passing through two different points x_1 and x_2 in a unit cell. There is a sequence of discrete transformation τ_k such that $\tau_k(x_2)$ converges to x_1. By the uniqueness of Dirichlet problem, $\tau_k(w_2)$ converges to w_1. So the averages values from w_1 and w_2 will be equal.

Lemma 5.16. We have the following continuity results.

(i) For each fixed $\nu \in S^{n-1}$, $\overline{g}^*(p,\nu)$ and $\overline{g}_*(p,\nu)$ are continuous in p variable.

(ii) And $\overline{g}(x,\nu)$ is continuous on $\partial\Omega \times \{\nu : \nu \text{ is irrational}\}$

(iii) Set $\overline{g}(x) = g(x, \nu_x)$, which is well-defined at irrational points. For each $x_0 \in \partial\Omega$ satisfying $-\nu(x_0) \in \mathcal{D}_\delta$, there exists a neighborhood $B_r(x_0)$ such that if $x_1, x_2 \in B_r(x_0)$ are irrational, then

$$|\overline{g}(x_1) - \overline{g}(x_2)| \le C\|g\|_{\mathcal{C}^2(\square)}\delta \tag{5.17}$$

where the constant C only depends on n, λ and Λ.

Remark 5.17. For each fixed p, $\overline{g}^*(p, \nu)$ and $\overline{g}_*(p, \nu)$ may be discontinuous at the rational direction ν. But it is continuous at the irrational directions due to the uniqueness of correctors up to infinite translation.

5.5.2. Homogenized equations for Dirichlet problem .

Theorem 5.18. *For u_ε be a solution of* (5.19)*, there are continuous functions $\overline{g}^*(\cdot,\cdot)$ and $\overline{g}_*(\cdot,\cdot)$ defined on $\partial\Omega \times S^{n-1}$ such that u, a limit of any subsequence of u_ε, satisfies*

(i)
$$\overline{F}(D^2u) = \overline{f}(x) \text{ in } \Omega, \quad \overline{g}_*(\nu_x, x) \le u \le \overline{g}^*(\nu_x, x) \text{ on } \partial\Omega \tag{5.18}$$

(ii) *and $\overline{g}_*(\nu_x, x) = \overline{g}^*(\nu_x, x) = \overline{g}(\nu_x, x)$ for any irrational direction $\nu_x \in S^{n-1}$. where ν_x is an outward unit normal vector to $\partial\Omega$ at x.*

- $\overline{F}(D^2u) = \overline{f}(x)$ comes from the standard homogenization.
- Any subsequence of u_ε has a convergent subsequence such that $u_\varepsilon(p + \varepsilon x)$ converges to a corrector $w(x)$. So $|w(x) - u_\varepsilon(p+\varepsilon x)| < \frac{C}{R^\gamma}$ on any compact subset in $x \cdot \nu > R$ for a small $\varepsilon > 0$, which implies $\overline{g}_*(\nu_p, p) - \frac{C}{R^\gamma} < u_\varepsilon(p + \varepsilon x) < \overline{g}^*(\nu_p, p) + \frac{C}{R^\gamma}$.

Corollary 5.19. *If Ω satisfies Irrational Direction Dense Condition , there is a unique continuous functions u such that*

(i) *u_ε converges uniformly to u in any compact subset Ω' of Ω,*

(ii) *and u is a viscosity solution of*

$$\overline{F}(D^2u) = \overline{f}(x) \text{ in } \Omega, \quad u = \overline{g}(\nu_x, x) \text{ on } \partial\Omega \tag{5.19}$$

5.6. Neumann problem.

5.6.1. Example. $g(x) = g(x_1)$ in a convex domain

Let Ω be convex, $L = \{x : x_1 = 0, 1 \quad \text{and} \quad -l \le x_2 \le l\} \subset \partial\Omega$, and strictly convex on $\partial\Omega\backslash L$. Let $g(x) = g(x_1, x_2)$.
If $\nu \neq e_1$,

$$\overline{g}^*(\nu_x, x) = \overline{g}_*(\nu_x, x) = \langle g \rangle$$

$$\triangle u = \langle f \rangle \text{ in } \Omega, \quad \frac{\partial u}{\partial \nu} = \langle g \rangle \text{ on } \partial\Omega\backslash L, \quad \frac{\partial u}{\partial \nu}(x) = \overline{g} \text{ on } L \tag{5.20}$$

for $\overline{g}$ satisfying $\langle f \rangle \text{vol}(\Omega) = \langle g \rangle \text{Area}(\partial\Omega\backslash L) + 4\overline{g}\, l$ from the compatibility condition. If $\overline{g} > \overline{g}^*$ or $\overline{g} < \overline{g}_*$, then **there is no solution for small $\varepsilon > 0$.** Hence we can consider the homogenization when a subsequence of u_ε has a limit. The homogenization for Neumann boundary value problem comes from the similar arguments in Dirichlet Boundary value problems, [3].

5.7. Thin obstacles. Let u_ε be a solution thin obstacle problems (5.3), [17] . To describe the effective equation for $u = \lim_{\varepsilon\to 0} u_\varepsilon$, we introduce the averaged capacity, depending on a direction ν. The capacity of compact set, A, is defined as the infimum energy among functions that is one on A and zero at infinity for $n \geq 3$ or on the boundary of B_R.

Figure 5.2 tells us the hyper plane pass through each holes only with lower dimensional pieces, $\widetilde{T} \cap \Gamma_\nu(s)$ for $\Gamma_\nu(s) := \Gamma + s\nu, \quad s \in \mathbb{R}$. The influence of thin obstacles can be measured with the capacity of $\widetilde{T} \cap \Gamma_\nu(s)$ for different s. And then the total influence will be averaged out in s variables due to minimizing the Energy, $J(u)$, at Section 5.1.2.

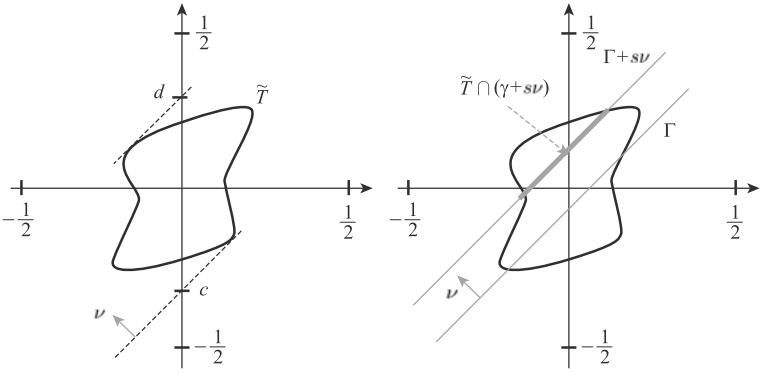


Figure 5.2. The shape of $\widetilde{T}$ and $\widetilde{T} \cap (\Gamma + s\nu)$.

Definition 5.20 (Averaged Capacity). Suppose Γ is a hyper plane in $\mathbb{R}^n$ with normal $\nu \in S^{n-1}$. If $\widetilde{T} \subset \mathbb{R}^n$ and $f(s) = \mathrm{cap}(\widetilde{T} \cap \Gamma_\nu(s))$ is integrable, we set

$$\mathrm{cap}_\nu(\widetilde{T}) := \int_{-\infty}^{\infty} f(s)ds \tag{5.21}$$

and call this quantity the *averaged capacity* of $\widetilde{T}$ with respect to ν. The set $\widetilde{T} \cap \Gamma_\nu(s)$ is illustrated in Figure 5.2.

Theorem 5.21. *Assume $n \geq 3$ and for a given $\nu \in S^{n-1}$ and $x^0 \in \mathbb{R}^n$, let Γ be the hyperplane defined in Section 5.1.2. Let u_ε be the solution to* (5.3) *and set $a_\varepsilon = \varepsilon^{\frac{n}{n-1}}$. Then, for a.e. $\nu \in S^{n-1}$, $u_\varepsilon \rightharpoonup u$ in $H_0^1(\Omega)$ where u is the unique minimizer of*

$$J_\nu(v) := \int_\Omega \frac{1}{2}|\nabla v|^2 - fv dx + \frac{1}{2}\,\mathrm{cap}_\nu(\widetilde{T}) \int_\Gamma ((\psi - v)^+)^2 d\sigma, \quad v \geq 0. \tag{5.22}$$

In particular, u is the solution of

$$-\Delta u = \mathrm{cap}_\nu(\widetilde{T})(\psi - u)^+ d\sigma + f\chi_{\{u>0\}}. \tag{5.23}$$

For the general domain, we will approximate the boundary by a tangent plane locally. Then two errors may occur. The first error is from curvature of the boundary. And the second is the error between average number of holes and the number of holes in a ball with a radius of order $\frac{1}{\varepsilon}$, which can be measured by the *discrepancy* of the sequence $\omega_{r_\varepsilon} = \omega_{r_\varepsilon}(x_i) =$

$\{k' \cdot \alpha(x_i) : k' \in \varepsilon^{-1} Q'_{r_\varepsilon}(x_i) \cap \mathbb{Z}^{n-1}\}$. To apply the results on the hyperplane we require two errors are lower order than the main terms, which is possible for $n \geq 5$ or domains in general dimension with geometric restrictions. More careful analysis is required for the lower dimensional case.

6. Equations of nondivergence type, [18]

6.1. Soft inclusion. In this section, we will consider a generalization of the following soft inclusions where the diffusion coefficients are zero on the holes:

$$\begin{cases} \triangle u_\varepsilon = f(x) & \text{in } \Omega_\varepsilon \\ \dfrac{\partial u_\varepsilon}{\partial \nu} = 0 & \text{on } \partial T_\varepsilon, \quad u_\varepsilon = \varphi(x) \text{ on } \partial\Omega \setminus \partial T_\varepsilon. \end{cases} \tag{6.1}$$

We refer the details for (6.1) to [15]. In this section, we consider the following semi-linear equation of non-divergence type:

$$\begin{cases} L\left(D^2 u_\varepsilon, u_\varepsilon, x, \frac{x}{\varepsilon}\right) = f\left(x, \frac{x}{\varepsilon}\right) & \text{in } \Omega_\varepsilon \\ G\left(Du_\varepsilon(x), \frac{x}{\varepsilon}\right) = 0 & \text{on } \partial T_\varepsilon, \quad u_\varepsilon = \varphi(x) \text{ on } \partial\Omega \setminus \partial T_\varepsilon \end{cases} \tag{6.2}$$

where

$$\begin{aligned} L\left(D^2 u_\varepsilon, u_\varepsilon, x, \frac{x}{\varepsilon}\right) &= A_{ij}\left(\frac{x}{\varepsilon}\right) D_{ij} u_\varepsilon + c\left(u_\varepsilon, x, \frac{x}{\varepsilon}\right), \\ G\left(Du_\varepsilon(x), \frac{x}{\varepsilon}\right) &= b^i\left(\frac{x}{\varepsilon}\right) D_i u_\varepsilon(x), \quad \varphi \in \mathcal{C}^2(\overline{\Omega}). \end{aligned} \tag{6.3}$$

We assume that L is uniformly elliptic, $b^i(y)$ satisfies the uniform oblique condition, coefficients are Hölder continuous, and f and c are uniformly continuous with respect to y variable. The main difficulty for nondivergence type operator is the lack of compatibility condition to have periodic solution because it doesn't have the integration by part.

6.2. Homogenization. We usually expect the existence of a controllable solution when the equation satisfies a compatibility condition. So we find the compatibility condition for equations of nondivergence type with Neumann boundary values from the existence of global periodic solution called as a first corrector. First we show the existence of a compatibility constant for the Neumann Problem.

Theorem 6.1 (Compatibility Condition). *Consider the equation defined as follow:*

$$\begin{cases} A_{ij}(y) D_{ij} v(y) = f(y) & \text{in } \mathbb{R}^n \setminus T^a \\ b^i(y)\left(\xi^i + D_i v(y)\right) + \gamma = g(y) & \text{on } \partial T^a. \end{cases} \tag{6.4}$$

Assume that

$$\|A_{ij}\|_{C^\alpha(\mathbb{R}^n \setminus T^a)} + \|b^i(a\,\cdot)\|_{C^{1,\alpha}(\frac{1}{a}(\mathbb{R}^n \setminus T^a))} \leq \Lambda \tag{6.5}$$

and that $\|f\|_{C^\alpha(\mathbb{R}^n \setminus T^a)} + \|g\|_{C^{1,\alpha}(\mathbb{R}^n \setminus T^a)}$ is bounded. Then, for any given $\xi \in \mathbb{R}^n$, there is a unique constant $\gamma = \gamma(\xi; (A_{ij}), b^i, f, g, a)$ such that Equation(6.4) *admits a bounded solution v.*

Definition 6.2.

(i) We shall call $\gamma(\xi; (A_{ij}), b^i, f, g, a)$ a compatibility constant of Equation (6.4).

(ii) Now suppose that $f = g = 0$. If $\gamma = \gamma((A_{ij}), b^i) = 0$ for all $\xi \in \mathbb{R}^n$ and the radius of holes a, then we say that (A_{ij}) and b^i (or Equation (6.2)) satisfies the compatibility condition.

When we make the asymptotic expansion of u_ε at $x_0 \in \Omega$:

$$u_\varepsilon = u_0 + \varepsilon v\left(\frac{x}{\varepsilon}; \xi\right) + \varepsilon^2 w_\varepsilon\left(\frac{x}{\varepsilon}\right) + o(\varepsilon^2), \tag{6.6}$$

we show v is the first corrector for the compatibility condition and the second corrector has been used to find the homogenized equation, [18].

Theorem 6.3. *Let u_ε be a viscosity solution of* (6.2)*. Suppose that our equation satisfies*

(1) *the equation* (6.2) *satisfies the compatibility condition,*

(2) *u_ε is bounded uniformly on ε, and $u^* = u_*$ on the $\partial\Omega$ where u^* and u_* are limsup and liminf of u_ε as $\varepsilon \to 0$..*

(3) *$0 < a \leq a_0$ for uniform constant a_0*

Then, there exists an uniformly elliptic operator $\overline{L}$. And u_ε, solution of (6.2)*, converges to u, solution of the equation* (6.7)*, uniformly.*

$$\begin{cases} \overline{L}(D^2u, u, x) = 0 & \text{in } \Omega \\ u = \varphi(x) & \text{on } \partial\Omega. \end{cases} \tag{6.7}$$

We can also find a discrete gradient estimate and ε-Lipschitzness as Section 3.5.

Corollary 6.4. *Assume also that Ω satisfies an exterior sphere condition. Then u_ε converges uniformly to u_0 which is the solution of* (6.7)*.*

References

[1] X. Cabre and L. Caffarelli, *Fully Nonlinear Elliptic Equation,* Vol. **43**. American Mathematical Society. Colloquium Publication.(1983).

[2] Crandall, Michael G., Ishii, Hitoshi, and Lions, Pierre-Louis, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. (N.S.) **27** (1992), no. 1, 1–67.

[3] Choi, Sunhi, Kim, Inwon, and Lee, Ki-Ahm, *Homogenization of Neumann boundary data with fully nonlinear operator*, Analysis & PDE, Vol. **6** (2013), No. 4, 951–972.

[4] Caffarelli, L. and Lee, Ki-Ahm, *Homogenization of nonvariational viscosity solutions*, Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. (5) **29** (2005), no. 1, 89–100.

[5] _____, *Homogenization of oscillating free boundaries: the elliptic case*, Comm. Partial Differential Equations **32** (2007), no. 1-3, 149–162.

[6] ———, *Viscosity Method in Homogenization of Highly Oscillating Obstacles*, Indiana Univ. Math. J. **57** (2008), no. 4, 1715–1741.

[7] Caffarelli, L., Lee, Ki-Ahm, and Mellet, A., *Singular limit and homogenization for flame propagation in periodic excitable media*, Archive for Rational Mechanics and Analysis, **172** (2004), no. 2, 153–190.

[8] ———, *Homogenization and flame propagation in periodic excitable media: The asymptotic speed of Propagation*, Comm. Pure Appl. Math. **59** (2006), no. 4, 501–525.

[9] ———, *Flame propagation in one-dimensional stationary ergodic media*, Math. Models Methods Appl. Sci. **17** (2007), no. 1, 155–169.

[10] Evans, Lawrence C., *Periodic homogenisation of certain fully nonlinear partial differential equations*, Proc. Roy. Soc. Edinburgh Sect. A **120** (1992), no. 3-4, 245–265.

[11] Ulrich Hornung, *Homogenization and Porous Media* (Interdisciplinary Applied Mathematics), Springer, 1997 edition.

[12] H. Kesten, *The discrepancy of random sequences* $\{kx\}$, Acta Arith., **10** (1964/1965), 183–213.

[13] Kim, Sunghoon and Lee, Ki-Ahm, *Viscosity method for Homogenization of Parabolic Nonlinear Equations in Perforated Domains*, J. Differential Equations **251** (2011), 2296–2326.

[14] L. Kuipers and H. Niederreiter, *Uniform distribution of sequences,* Wiley-Interscience [John Wiley & Sons], New York, 1974. Pure and Applied Mathematics.

[15] Jikov, V. V., Kozlov, S. M., and Oleĭnik, O. A., *Homogenization of differential operators and integral functionals*, Translated from the Russian by G. A. Yosifian, Springer-Verlag, Berlin, 1994.

[16] Ki-ahm Lee and H. Shahgholian, *Homogenization of the boundary value for the Dirichlet Problem* (preprint).

[17] Lee, Ki-Ahm, Strömqvist, Martin, and Yoo, Minha, *Highly Oscillating Thin Obstacles Advances in Mathematics,* Vol. **237**, nb. 1 (2013), 286–315

[18] Lee, Ki-Ahm and Yoo, Minha, *Viscosity Method for Homogenization of Fully Nonlinear Equations with Neumann data in Perforated Domains*, Arkiv for Rational Mechanics and Analysis, Volume **206**, Number 1 (2012), 297–332

[19] Ki-ahm Lee and Minha Yoo, *Homogenization of fully nonlinear elliptic equations with oscillating Dirichlet boundary data* (preprint).

Department of Mathematical Sciences, Seoul National University , Seoul 151-747, Korea
E-mail: kiahm@snu.ac.kr

Academic wages, singularities, phase transitions and pyramid schemes

Dedicated to the memory of Gary Becker (1930-2014)

Robert J. McCann

Abstract. In this lecture we introduce a mathematical model which couples the education and labor markets, in which steady-steady competitive equilibria turn out to be characterized as the solutions to an infinite-dimensional linear program and its dual. In joint work with Erlinger, Shi, Siow and Wolthoff, we use ideas from optimal transport to analyze this program, and discover the formation of a pyramid-like structure with the potential to produce a phase transition separating singular from non-singular wage gradients. Wages are determined by supply and demand. In a steady-state economy, individuals will choose a profession, such as worker, manager, or teacher, depending on their skills and market conditions. But these skills are determined in part by the education market. Some individuals participate in the education market twice, eventually marketing as teachers the skills they acquired as students. When the heterogeneity amongst student skills is large, so that it can be modeled as a continuum, this feedback mechanism has the potential to produce larger and larger wages for the few most highly skilled individuals at the top of the market. We analyze this phenomena using the aforementioned model. We show that a competitive equilibrium exists, and it displays a phase transition from bounded to unbounded wage gradients, depending on whether or not the impact of each teacher increases or decreases as we pass through successive generations of their students. We specify criteria under which this equilibrium will be unique, and under which the educational matching will be positive assortative. The latter turns out to depend on convexity of the equilibrium wages as a function of ability, suitably parameterized.

Mathematics Subject Classification (2010). Primary 91B66; Secondary 49N15, 35Q91, 91B68, 91B69.

Keywords. steady state, competitive equilibrium, multisector matching, heterogeneous types, education, labor, convex wages, singularity, phase transition, pyramid scheme, optimal transport, infinite-dimensional linear program.

1. Introduction

The last half century has seen much fruitful interaction between economics and mathematics. Still, the relationship between these two subjects is far less developed than the long standing affair between mathematics and physics. This is good news for mathematicians, in the sense that much work remains to be done: economics provides a ready source of interesting mathematical problems, so far only modestly tapped, as well as an area of application where mathematical developments again have a chance to prove transformative, as they did during

Proceedings of the International Congress of Mathematicians, Seoul, 2014

the development of the theories of gravitation and quantum mechanics in the last century, and more recently in areas like statistical physics and string theory.

For mathematicians working in optimal transportation, it has been a source of considerable satisfaction to discover that their work has found diverse applications in economics; see [3, 6, 7, 9] for examples. My work with Figalli and Kim [11] in particular suggests that curvature may have a heretofore unanticipated role to play in economic theory. It is also the case that economists have exploited ideas from optimal transportation in surprising ways [14, 25], and in some cases have been ahead of mathematicians in anticipating significant developments, such as the gradient flow framework discussed in Sonnenschein's myopic price dynamics [28], some twenty-five years before it was linked to the heat equation by Jordan, Kinderlehrer and Otto [19]. (The triangle with physics is complete, in the sense that Cullen and Purser were using Hamiltonian dynamics in the same energy landscape as Sonnenschein to study atmospheric and oceanic flows [8].) All this is perhaps less surprising given the fact that one of the cornerstones of optimal transport — the Kantorovich-Koopmans duality theorem — lead its mathematician and physicist-trained inventors to share the 1975 Nobel Memorial Prize in Economics. A few years before that, future Nobel Laureate Lloyd Shapley had worked out the fundamental results concerning stable matchings in the non-transferable and transferable utility settings with Gale [13] and with Shubik [27], respectively. As Shapley and Shubik discovered, the transferable utility version of this problem can also be reformulated as a discrete optimal transport problem. This discovery was generalized and extended to a continuum setting by Gretzky, Ostroy and Zame [18].

In the present synopsis I will sketch the results of a joint work with three economist colleagues at the University of Toronto, and one former PhD student, which draws inspiration from the foregoing. It concerns a matching model coupling the education and labor sectors. For precise statements and proofs of all the ideas discussed here, see our joint paper [10].

As is often the case in economic models, the problem is to understand how supply equilibrates with demand to determine prices — or in this case wages – in each of these markets. What is supplied and demanded in these markets are skills; we focus primarily on cognitive skills for simplicity. The phenomenon of interest to us is whether, in the limit of a large population displaying a bounded range of skills, competition may lead to wages which display singularities: for example, does the the ratio of the highest to the average salary tend to a finite or an infinite number, as the size of the population tends to infinity? We investigate this question in a competitive equilibrium model, which means individuals choose those teachers, occupations and partners who reward them most generously, under the assumption that such decisions are made at the level of individuals (or groups) too small to affect market conditions such as the wage profile $v(k)$ as a function of skill level $k \in [0, \bar{a}[$. Skills can be continuously distributed in this interval, as a reflection of the large size of the population.

The education market plays a special role in our study. The output produced by this market is the enhancement of salable skills, taking initial student and teacher skills as its input. However, some participants match twice in the education market, first as students, and later as teachers. This creates a feedback mechanism which — depending the effectiveness of the educational technology — has the potential to create a pyramid, in which there is greater and greater demand for the most skilled teachers, who occupy positions closer and closer to the apex. Even if we assume that wages outside the education market are bounded, in a steady-state model the wage which a highly skilled teacher can command depends on the potential future earnings of their students, some of whom will be teachers, who may have many generations of students, whose wages must be determined in part by the internal

dynamics of the education market insofar as these students again will again have the choice to become teachers — and will do so provided they can find other students willing to pay them sufficient tuition. As adults on the labor market, they also have alternatives to teaching: namely they could engage in production by working or managing a team consisting of N' workers and one manager, whose output is a known function $N'b_L$ of the team members' skills.

Our model depends on $c \geq 0$ and positive parameters θ, θ', N, N' and $\bar{a}$ satisfying $\max\{\theta, \theta\} < 1 \leq N$. Here N represents the number of students each teacher can teach, and the extent θ to which she succeeds at transmitting her cognitive skills to her students. Similarly, N' represents the number of workers each manager can manage, and θ' represents the extent to which a manager's skills influence the quality of the work produced by those whom she manages. All skills lie in the interval $A = [0, \bar{a}[$ or its closure $\bar{A}$.

A worker of skill $a \in \bar{A}$ working with a manager of skill $k \in \bar{A}$ produces a continuously differentiable output $b_L((1-\theta')a+\theta' k)$ given by $b_L \in C^1(\bar{A})$. A student of skill a studying with a teacher of skill k becomes an adult of skill $z(a,k) = (1-\theta)a + \theta k$. The acquired skill z may have some external value $cb_E(z)$ to the student — in addition to the wage $v(z)$ it commands on the market. Both b_L and $b_E \in C^1(\bar{A})$ are assumed to be fixed uniformly convex increasing functions hereafter, satisfying

$$0 < \underline{b}_{E/L} = b_{E/L}(0), \tag{1.1}$$

$$0 < \underline{b}'_{E/L} = b'_{E/L}(0), \tag{1.2}$$

$$0 < \underline{b}''_{E/L} = \inf_{a \in A} b''_{E/L}(a), \tag{1.3}$$

where $\underline{b}''_{E/L}$ is defined as the largest constant for which $b_{E/L}(a) - \underline{b}''_{E/L}|a|^2/2$ is convex on $\bar{A}$. A typical example would be $b_E(a) = e^a = b_L(a)$ as in [22].

At each period of our model, students are born into the population with cognitive skills distributed randomly throughout the closure $\bar{A}$ of an interval $A = [0, \bar{a}[\subset \mathbf{R}$ according to some Borel probability distribution $\alpha \geq 0$. They live only for two periods. During the first period, each student $a \in \bar{A}$ seeks to enhance his cognitive skill by studying with whichever teacher $k \in \bar{A}$ provides the best educational value to him. During the second period, the student becomes an adult armed with skill $z = z(a,k) = (1-\theta)a + \theta k$, who then enters the workforce as a worker, manager or teacher earning a wage $v(z)$, whose steady-state value we seek to determine.

Although we cannot find $v(z)$ explicitly, we are able to characterize it as the solution of a variational problem: an infinite-dimensional linear program whose non-standard form complicates its analysis considerably. What has been achieved with Erlinger, Shi, Siow and Wolthoff [10] is an analysis of the existence, uniqueness, and characteristics of the solutions to this linear program under suitable technical hypotheses. These include requiring the initial distribution α of student skills to satisfy a doubling condition at the top skill type $\bar{a}$, meaning there exists $C < \infty$ such that

$$\int_{[\bar{a}-2\Delta a, \bar{a}]} \alpha(da) \leq C \int_{[\bar{a}-\Delta a, \bar{a}]} \alpha(da) \tag{1.4}$$

for all $\Delta a > 0$. Under suitable assumptions, we deduce the wage profile $v(k)$ is strictly convex and increasing, but displays a phase transition from having bounded to unbounded gradients as the product $N\theta$ increases through 1. More precisely, if $N\theta > 1$, so that the

net impact of a teacher increases as one passes from each generation of their students to the next, then $v'(a) \sim |\bar{a} - a|^{-\frac{\log\theta}{\log N}-1}$ as $a \to \bar{a}$ (unless an even stranger pathology occurs — see Theorem 6.1). Since this singularity is integrable, our analysis suggests $\lim_{a\to\bar{a}} v(a)$ is finite.

Along the way, we establish various conclusions about the behavior of the model in different parameter regimes, such as which ranges of skill types will be displayed by workers, managers and teachers, and who will match with whom in each sector (education and labor) of our market. Before describing our model and conclusions more precisely, let us mention some antecedents.

The role of teachers and the individual investment required to develop human capital has been examined in the context of a steady-growth model for specialization by Becker and Murphy, who also recognized the relevance of the long lineages of teachers which may form [2]. The economics of superstars had been analyzed before that by Rosen [26]; convexity of wages play a key role in his study, as they do in ours. The formation of finite-depth pyramids (or management layers) in the context of a labor market model has also been investigated by Garicano [16] and followed up with Rossi-Hansberg [17], though the absence of feedback makes their model quite different from ours. Another explanation for inflated levels of executive compensation has been proposed by Gabaix and Landier [12]. Finally, the possibility of allowing parameters such as θ, θ', N and N' to vary endogenously across the population to model heterogeneity of communication skills was a feature of our original four-author model [22], which we have chosen to suppress in the logarithmically reparameterized five-author sequel [10]. This suppression facilitates a more penetrating analysis of some phenomena of interest in their simplest form.

2. A competitive equilibrium model

Fix θ, θ', N, N' and $\bar{a}$ positive with $\max\{\theta, \theta'\} < 1 \leq N$ and $A = [0, \bar{[}$, a probability measure α on $\bar{A}$, and education and labor production functions $b_{E/L} \in C^1(\bar{A})$ satisfying (1.1)–(1.3). Set $K = [0, \bar{k}[= A$ and $c \geq 0$.

A competitive equilibrium requires the wage profile $v(k)$ as a function of skill level $k \in \bar{A}$ to be related in a certain way to other quantities which reflect the behavior of students and adults in our model. Educational decisions of such agents are captured by a probability measure $d\epsilon(a,k) \geq 0$ on $\bar{A}^2$, which represents the fraction of students with skill a who choose to study with a teacher of skill k, and vice versa. Labor market decisions are recorded by a measure $d\lambda(a,k) \geq 0$ on $\bar{A}^2$, representing the number of workers of skill a who choose to work with a manager of skill k, and vice versa. The net lifetime utility of a student of skill a will be denoted by $u(a)$.

The *support* of any (Borel) measure μ on $\mathbf{R}^m$ refers to the smallest closed set $\operatorname{Spt}\mu \subset \mathbf{R}^m$ carrying the full mass of μ. The *push-forward* of any measure μ on $\mathbf{R}^m$ through a Borel map $f : \mathbf{R}^m \longrightarrow \mathbf{R}^n$ refers to the measure $f_{\#}\mu$ assigning mass $\mu[f^{-1}(N)]$ to each set $N \subset \mathbf{R}^n$. Thus $\epsilon^1 = \pi^1_{\#}\epsilon$ and $\epsilon^2 = \pi^2_{\#}\epsilon$ denote the marginal projections of ϵ through the coordinate maps $\pi^i(x_1, x_2) = x_i$.

We say $\epsilon \geq 0$ and $\lambda \geq 0$ represent a *steady-state* for our model if

$$\epsilon^1 = \alpha \qquad \text{and} \tag{2.1}$$

$$\lambda^1 + \frac{1}{N'}\lambda^2 + \frac{1}{N}\epsilon^2 = z_{\#}\epsilon, \tag{2.2}$$

where $z(a,k) = (1-\theta)a + \theta k$ is the skill attained by a student a through studying with teacher k. Here the first identity requires the initial distribution of student skills to be given by α, while the second requires that the current distribution of (worker + manager + teacher) skills in the population be reproduced at the next generation through education. This is the steady-state constraint. We denote the set of non-negative measures (ϵ, λ) satisfying (2.1)–(2.2) by $R(\alpha)$ — which of course depends also on N, N' and θ. Note that λ will not be a probability measure; rather its mass coincides with the fraction $(1 - \frac{1}{N})/(1 + \frac{1}{N'})$ of the adult population who choose to become workers.

We say a pair of payoffs $u, v : \bar{A} \longrightarrow]0, \infty]$ are *stable* if

$$u(a) + \frac{1}{N}v(k) \geq cb_E(z(a,k)) + v(z(a,k)) \qquad \text{and} \tag{2.3}$$

$$v(a) + \frac{1}{N'}v(k) \geq b_L((1-\theta')a + \theta' k) \qquad \text{on } \bar{A} \times \bar{K}, \text{ and} \tag{2.4}$$

$$\tfrac{N}{N-1}(u(k) - cb_E(k)) \geq v(k) \geq \frac{N'}{N'+1}b_L(k) \qquad > 0. \tag{2.5}$$

The wage constraint (2.4) reflects the stability of matchings in the labor sector. If the reverse inequality held, the output $N'b_L$ produced by N' adults of skill a and one of skill k would be sufficient to allow all $N'+1$ of them to improve their wages by abandoning their occupations to collaborate by forming N' new worker-manager pairs. Similarly, constraint (2.3) is a stability constraint on the education market, which ensures that no N students of ability a plus one teacher of ability k all have the incentive to abandon their institutions to form a school with each other. Together, these two constraints imply (2.5) at any point $a = k$ where v is finite; we have included it only to show the payoffs are positive and that v cannot diverge unless u does.

We must also specify in what class of functions the payoffs u, v must lie. Since we wish to allow for the possibility that the payoffs $u, v : A \longrightarrow [0, \infty[$ become unbounded at the upper endpoint $\bar{a}$ of the half-open interval A, we define the feasible set F_0 to consist of pairs $(u, v) = (u_0 + u_1, v_0 + v_1)$ satisfying (2.3)–(2.5) which differ from bounded continuous functions $u_0, v_0 \in C(\bar{A})$ by non-decreasing lower semicontinuous functions $u_1, v_1 : \bar{A} \longrightarrow [0, \infty]$ which are real-valued on A.

Together, a pair of stable payoffs $(u, v) \in F_0$ and steady-state matchings $(\epsilon, \lambda) \in R(\alpha)$ form a *competitive equilibrium* if they satisfy the budget constraint

$$\text{equality holds } \epsilon\text{-a.e. in (2.3), and } \lambda\text{-a.e. in (2.4).} \tag{2.6}$$

In other words, the productivity of λ-a.e. worker-manager team must be sufficient to the pay the worker's wage plus a fraction $\frac{1}{N'}$ of the manager's; similarly ϵ-a.e. student-teacher pair must generate future earnings v for the student, which together with any non-labor compensation cb_E for skills acquired through education, must be sufficient to leave utility $u(a)$ for the student after a fraction $\frac{1}{N}$ of his teacher's salary has been paid.

3. A variational approach

Since it is not obvious whether such equilibria exist or how to find them, we begin by recharacterizing them variationally. Consider the problem of minimizing the expected net utility

$\alpha(u)$ over the population α of students:

$$LP_* := \inf_{(u,v)\in F_0} \int_{[0,\bar{a}]} u(a)d\alpha(a). \tag{3.1}$$

This is a linear minimization over the convex set of stable payoffs $(u, v) \in F_0$. As an infinite-dimensional linear program whose domain includes pairs of continuous functions on $\bar{A}$ satisfying two stability contraints, this problem has a linear programming dual, which turns out to be a maximization involving pairs of measures on $\bar{A}^2$:

$$LP^* := \max_{(\epsilon,\lambda)\in R(\alpha)} \int_{\bar{A}\times\bar{K}} [cb_\theta(a,k)d\epsilon(a,k) + \tilde{b}_{\theta'}(a,k)d\lambda(a,k)], \tag{3.2}$$

where $b_\theta(a,k) = b_E(z(a,k))$ and $\tilde{b}_{\theta'}(a,k) = b_L((1-\theta')a + \theta'k)$. It can be interpreted as a social planners problem, which is to maximize the production $c\epsilon(b_\theta) + \lambda(\tilde{b}_{\theta'})$ of the two sectors in question (education and labor) over steady-state measures $(\epsilon, \lambda) \in R(\alpha)$.

If F_0 consisted solely of continuous bounded functions $u, v \in C(\bar{A})$, it would be easy to see $LP_* \geq LP^*$ via the argument of the proposition below; equality would then follow from a standard application of the Fenchel-Rockafellar duality theorem. The fact that F_0 includes unbounded functions makes the 'obvious' inequality $LP_* \geq LP^*$ much more subtle to prove. It is for this purpose that we exploit the doubling condition (1.4) on α to establish $LP_* = LP^*$ [10].

We also assert that Shapley and Shubik's insight [27] extends from single-sector, single-stage matching problems to the current multisectorial steady-state setting:

Proposition 3.1 (Optima v. equilibria). *The pair $(u, v) \in F_0$ and $(\epsilon, \lambda) \in R(\alpha)$ constitutes a competive equilibrium* (2.1)–(2.6) *if and only if (u, v) minimizes the primal problem* (3.1) *and (ϵ, λ) maximizes its dual problem* (3.2)*.*

Idea of proof. We sketch a proof here, side-stepping the subtlety mentioned above, by assuming boundedness of u and v to ensure that all integrals in question converge. Integrating the stability constraint (2.3) for the education market against ϵ yields

$$\alpha(u) - c\epsilon(b_\theta) \geq (z_\#\epsilon)(v) - \frac{1}{N'}\epsilon^2(v) \tag{3.3}$$

$$\geq \lambda^1(v) + \frac{1}{N'}\lambda^2(v) \tag{3.4}$$

$$\geq \lambda(\tilde{b}_{\theta'}), \tag{3.5}$$

where $(\epsilon, \lambda) \in R(\alpha)$ has been used to obtain (3.3)–(3.4), and the stability constraint for the labor market (2.4) has been used in (3.5). This shows $LP_* \geq LP^*$. Moreover, the conditions for equality in (3.3) and (3.5) coincide precisely with the budget constraints (2.6). Thus any competitive equilibrium forces (u, v) to minimize the primal linear program, and (ϵ, λ) to maximize its dual. Conversely, since we have independently deduced $LP^* = LP_*$ using the Fenchel-Rockafellar duality theorem, any bounded pair of optimizers $(u, v) \in F_0$ and $(\epsilon, \lambda) \in R(\alpha)$ must saturate the chain of inequalities above, hence satisfy the budget constaint and form a competitive equilibrium. □

Having established the equivalence between equilibrium and optimality, it is natural to want to establish the existence of minimizers for the primal problem and maximizers for the

dual. As is typically the case in Fenchel-Rockafellar duality, existence of optimizers for the dual problem comes for free: it set in the Banach space dual to $(C(\bar{A}^2), \|\cdot\|_\infty)$, which is a space of measures whose unit ball is well-known to be weak-$*$ compact. Since ϵ and λ both belong to this unit ball, it is easy to extract a subsequential limit from a maximizing sequence, and this limit is the maximizer. To show the primal infimum is attained is much more subtle, since the only obvious bound on u (and hence v) is in $L^1(\bar{A}, \alpha)$. To address it, we shall need to learn more about what to expect in terms of the structure of any optimal (u, v).

4. Existence and structure of optimal wages

Given stable $(u, v) \in F_0$, the convex functions $b_E(z)$, $z(a, k) = (1-\theta)a + \theta k$ and $\tilde{b}_{\theta'}(k', k) := b_L((1-\theta')k' + \theta' k)$ can be used to define the wages implicitly available to an individual of cognitive skill k employed as a worker, manager, or teacher, respectively:

$$v_w(k) := \max_{k' \in \bar{A}} \tilde{b}_{\theta'}(k, k') - \tfrac{1}{N'} v(k'), \tag{4.1}$$

$$v_m(k) := N' \max_{k' \in \bar{A}} \tilde{b}_{\theta'}(k', k) - v(k'), \qquad \text{and} \tag{4.2}$$

$$v_t(k) := N \max_{a \in \bar{A}} c b_E(z(a, k)) + v(z(a, k)) - u(a) \qquad \text{where} \tag{4.3}$$

$$\infty - \infty := \infty. \tag{4.4}$$

Notice that v_m and v_w are suprema of convex functions of k; hence inherit uniform convexity directly from (1.3). It is not obvious whether or not v_t is convex — unless v is convex, in which case v_t is convex and inherits uniform convexity from b_E when $c > 0$. Similarly, convexity of

$$\bar{u}(a) := \max_{k \in \bar{A}} c b_E(z(a, k)) + v(z(a, k)) - \tfrac{1}{N} v(k) \tag{4.5}$$

is not obvious, unless v is convex, in which case $\bar{u}$ is a convex function which inherits uniform convexity when $c > 0$. These observations play a crucial role in our proof that $\alpha(u)$ attains its minimum on F_0.

Our strategy is the following: first we minimize (3.1) on the smaller set $F_0 \cap C_0$ consisting of pairs of *convex* non-decreasing functions $(u, v) \in F_0$. For $c > 0$, we then hope to show the minimizer over this restricted set is actually uniformly convex and increasing, in the sense that its first two derivatives are bounded away from zero. In this case the convexity and monotonicity constraints do not bind, so the minimum over the smaller set $F_0 \cap C_0$ also minimizes $\alpha(u)$ over the larger set F_0. (The existence of minimizers in case $c = 0$ can then be handled by taking a limit $c \to 0^+$ and relying on the compactness properties of the set of convex functions. The question of whether or not *uniform* convexity of u and v remains true in this limit requires a more subtle analysis in [10]; its conclusion is appended to Theorem 4.1 below.)

Stability (2.3) implies that the students' net lifetime utility satisfies $u \geq \bar{u}$, which corresponds to the fact that, in a competitive equilibrium, every student chooses to study with the teacher who represents the best educational investment for him. On the other hand, since we seek to minimize the expectation $\alpha(u)$, it costs no generality to assume this bound is

saturated, meaning $u = \bar{u}$. Stability (2.3)–(2.5) also implies $v \geq \bar{v} := \max\{v_w, v_m, v_t\}$, which corresponds to the fact that, in a competitive equilibrium, each adult chooses the most financially rewarding occupation and professional partners for him or herself. Under the plausible hypothesis $v = \bar{v}$, our existence argument would be complete (at least in case $c > 0$). Unfortunately, we can only really expect $v = \bar{v}$ on the set of skills represented in the adult population, which might form a complicated subset of A and vary considerably along a minimizing sequence in F_0. Our strategy for circumventing this difficulty is to perturb both the primal and dual problems artificially, to ensure that adult skills populate the entire range A at some minimal level $\delta > 0$, solve the perturbed problems, and then take a limit $\delta \to 0$. In this way, we arrive at:

Theorem 4.1 (Existence of minimizing wages). *Fix $c \geq 0$ and positive θ, θ', N, N' and $\bar{a} = \bar{k}$ with $\max\{\theta, \theta'\} < 1 \leq N$ and $A = [0, \bar{a}] = K$. Let α be a Borel probability measure on $\bar{A}$ satisfying the doubling condition* (1.4) *at $\bar{a}$, and define $z(a,k) = (1-\theta)a + \theta k$, $b_\theta = b_E \circ z$ and $\tilde{b}_{\theta'}(a,k) = b_L((1-\theta')a + \theta' k)$, where $b_{E/L} \in C^1(\bar{A})$ satisfy* (1.1)–(1.3)*. Then infimum* (3.1) *is attained by functions $(u,v) \in F_0$ satisfying $v = \max\{v_w, v_m, v_t\}$ and $u = \bar{u}$ on $\bar{A} = [0, \bar{a}]$, where the $v_{w/m/t}$ and $\bar{u}$ are defined by* (4.1)–(4.5)*; here $u, v : \bar{A} \longrightarrow]0, \infty]$ are continuous, convex, non-decreasing, and — except perhaps at $\bar{a}$ — real-valued. For $j \in \{1, 2\}$, if $N\theta^j \geq 1$ then $d^j v/dk^j \geq \underline{b}_L^{(j)} \min\{(1-\theta')^j, (\theta')^j N'\}$.*

5. Who matches with whom?

We next try to understand which adults will choose to become workers, managers, or teachers, and with whom they will collaborate. The convex wages we have just shown to exist provide a key tool in this endeavor.

On the one hand, the slopes of v_w and v_m are inherited from $(1-\theta')b_L$ and $\theta' N' b_L$ according to (4.1)–(4.3), so if $(1-\theta')$ is very different from $\theta' N'$, we will have each worker being more skilled than each manager, or vice versa. How different these parameters must be depends on the range of slopes possessed by $b_L \in C^1(\bar{A})$, as reflected in the ratio $\bar{b}'_L / \underline{b}'_L$. Here

$$\bar{b}'_{E/L} = b'_{E/L}(\bar{a}) = \sup_{a \in \bar{A}} b'_{E/L}(a)$$

and $\underline{b}'_L$ is from (1.2). Similarly, v_t inherits its slopes from $(cb_E + v)N\theta$, so taking $N\theta c \underline{b}'_E$ large enough relative to the parameters mentioned above ensures that the cognitive skills of each teacher will exceed those of all managers and workers. However if c is small or vanishes, meaning education has little or no value outside the labor market, things become more subtle, as in our Proposition 5.2 below. See also the numerical simulations of [22].

The other major tool that we have at our disposal is the knowledge that the functions

$$f(a,k) := u(a) + \frac{1}{N} v(k) - c b_E(z(a,k)) - v(z(a,k))$$
$$g(a,k) := v(a) + \frac{1}{N'} v(k) - b_L((1-\theta')a + \theta' k)$$

are non-negative throughout $\bar{A}^2$ by the stability of $(u,v) \in F_0$, yet f vanishes ϵ a.e. and g vanishes λ-a.e. by the budget constraint (2.6). In other words, ϵ is supported on the set where f is minimized, and λ on the set where g is minimized. Thus we expect the derivatives of

f and g to vanish ϵ-a.e. and λ-a.e. respectively, provided these derivatives exist; that is, we expect

$$\frac{u'(a)}{1-\theta} = [cb'_E + v']_{(1-\theta)a+\theta k} = \frac{v'(k)}{N\theta} \qquad \text{to hold } \epsilon\text{-}a.e. \text{ and} \tag{5.1}$$

$$\frac{v'(a)}{1-\theta'} = b'_L((1-\theta')a + \theta' k) = \frac{v'(k)}{N'\theta'} \qquad \text{to hold } \lambda\text{-}a.e. \tag{5.2}$$

Similarly, we expect the Hessians of f and g to be non-negative definite ϵ-a.e. and λ-a.e. respectively, provided these derivatives exist:

$$\frac{u''(a)}{(1-\theta)^2} \geq [cb''_E + v'']_{(1-\theta)a+\theta k} \leq \frac{v''(k)}{N\theta^2} \qquad \text{should hold } \epsilon\text{-}a.e. \text{ and} \tag{5.3}$$

$$\frac{v''(a)}{(1-\theta')^2} \geq b''_L((1-\theta')a + \theta' k) \leq \frac{v''(k)}{N'(\theta')^2} \qquad \text{should hold } \lambda\text{-}a.e., \tag{5.4}$$

plus $\det D^2 f \geq 0$ and $\det D^2 g \geq 0$ should hold ϵ-a.e. and λ-a.e. respectively. In fact, for each $(a,k) \in \operatorname{Spt}\epsilon$ we can show the first equality in (5.1) holds provided $a \in \operatorname{Dom} Du$, while the second equality holds provided $k \in \operatorname{Dom} Dv$; here $\operatorname{Dom} Du$ denotes the subset of $]0,\bar{a}[$ where u is differentiable, and $\operatorname{Dom} D^2 u$ denotes the further subset where the non-decreasing function $u'(a)$ has a derivative in the sense of Lebesgue. (It is straightforward to see for each $(a,k) \in \operatorname{Spt}\lambda$ that the first equality in (5.2) holds provide $a \in \operatorname{Dom} Du$, and the second equality holds provided $k \in \operatorname{Dom} Dv$.)

Assuming α has no atoms, convexity of u ensures $\operatorname{Dom} Du$ constitutes a set of full measure; likewise $\operatorname{Dom} D^2 u$ is a set of full measure when α is absolutely continuous with respect to Lebesgue. If $c > 0$ or convexity of v is strict, (5.1) can be solved to identify the skill

$$k = k_t(a) = \frac{1}{\theta}(cb'_E + v')^{-1}\left(\frac{u'(a)}{1-\theta}\right) - \frac{1-\theta}{\theta}a \tag{5.5}$$

of each teacher who teaches students of skill $a \in \operatorname{Dom} Du$. It is less transparent to see that the convex function v is differentiable at almost every adult skill level, since the distribution of adult skills $z_{\#}\alpha$ is not prescribed, but rather determined by the model. It is therefore useful to know whether or not $z_{\#}\alpha$ can have atoms, assuming α does not. The following lemma rules out atoms in $z_{\#}\alpha$ provided ϵ is positive assortative; it also shows $z_{\#}\alpha$ inherits absolute continuity with respect to Lebesgue from α in that case. *Positive assortativity* simply means $\operatorname{Spt}\epsilon$ is a non-decreasing subset of $\mathbf{R}^2$, so that the skill of each teacher cannot decrease as a function of the skill of the students they teach.

Lemma 5.1 (Endogenous distribution of adult skills). *Fix $\theta \in]0,1[$ and a Borel probability measure $\alpha \geq 0$ on $\bar{A}$ with $A = [0,\bar{a}[$. Set $z(a,k) = (1-\theta)a + \theta k$. If $\epsilon \geq 0$ on $\bar{A}^2$ has $\alpha = \epsilon^1$ as its left marginal, then for each $\bar{a} - \Delta a \in A$ the corresponding distribution $\kappa = z_{\#}\epsilon$ of adult skills satisfies*

$$\int_{[\bar{a}-\Delta a,\bar{a}]} d\kappa(a) \leq \int_{[\bar{a}-\frac{1}{1-\theta}\Delta a,\bar{a}]} d\alpha(a). \tag{5.6}$$

Thus κ has no atom at $\bar{a}$ unless α does.

In addition, if ϵ is positive assortative and α has no atoms, then κ has no atoms and $\epsilon = (id \times k_t)_{\#}\alpha$ for some non-decreasing map $k_t : \bar{A} \longrightarrow \bar{A}$. uniquely determined α-a.e. by κ.

Moreover, if $d\alpha(a) = \alpha^{ac}(a)da$ *is given by a density* $\alpha^{ac} \in L^1(A)$*, then* $d\kappa(a) = \kappa^{ac}(a)da$ *is given by a related density* $\kappa^{ac} \in L^1(A)$ *satisfying*

$$\alpha^{ac}(a) = (1 + \theta(k_t'(a) - 1))\, \kappa^{ac}(z(a, k_t(a))) \tag{5.7}$$

for Lebesgue-a.e. $a \in A$*. In this case* $\|\kappa^{ac}\|_{L^\infty(A)} \le \frac{1}{1-\theta}\|\alpha^{ac}\|_{L^\infty(A)}$.

Our next theorem shows that positive assortativity of ϵ holds as long the equililbrium payoffs $(u, v) \in F_0$ are *strictly* convex. It also explains when and in what sense equilibria will be unique. Before stating it we cite a proposition which details more elaborate consequences of the foregoing analysis concerning who will work, manage and teach. For the phase transition which we plan to describe, it is particularly relevant to have criteria such as (c) below, ensuring that even for $c = 0$, the most skilled individuals will be teachers. It is also essential for the theorem which follows, to know that the skill levels of the academic descendants of almost every given teacher are only finite in number. By contrast, there will certainly be teachers whose academic ancestors populate countably many skill types.

Note that in the following proposition, (c) and (d) together imply (e), meaning at least one of the two inequalities $N\theta \ge 1$ or $c \ge 0$ is strict. Also note $N'\theta' \ge \bar{b}_L'/\underline{b}_L'$ and $N\theta \ge \bar{b}_L'/\underline{b}_L'$ are sufficient for (b) and (c), respectively.

Proposition 5.2 (Specialization by type; the educational pyramid). *Fix* $A = [0, \bar{a}[$ *with* $\bar{a} > 0$*, and* $c \ge 0$*. Extend convex, nondecreasing* $u, v : A \longrightarrow \mathbf{R}$ *lower semicontinuously to* $\bar{A}$ *and suppose* $v = \max\{v_w, v_m, v_t\}$*, where* $v_{w/m/t}$ *are from* (4.1)–(4.4)*.*

If (a) $N\theta c\underline{b}_E' \ge \bar{b}_L' \max\{N'\theta', 1 - \theta'\}$ *then all teacher types lie weakly above all of the manager and worker types.*

If (b) $N'\theta' > (1 - \theta') \sup_{a\in A} b_L'(1 - \theta')a + \theta'\bar{a})/b_L'(\theta' a)$ *then all of the worker types lie weakly below all of the manager types.*

If (c) $N\theta \ge \sup_{0\le z\le k} b_L'((1 - \theta')z + \theta'\bar{a})/(b_L'(\theta' z) + \frac{c}{N'\theta'} b_E'(z))$ *and (b) holds, and* $f(a, k) := u(a) + \frac{1}{N}v(k) - cb_E(z(a, k)) - v(z(a, k))$ *vanishes at some* $(a, k) \in A \times A$ *where* $v(z(a, k)) = v_m(z(a, k))$*, then* $v > v_m$ *on* $]k, \bar{a}]$*. In other words, no manager (or worker) can have a type higher than a teacher of managers.*

If (d) $N\theta \ge 1$*, then any student of type* $a \in A$ *will be weakly less skilled than his teacher, and strictly less skilled if (e) either* $c > 0$ *or* $N\theta > 1$ *in addition.*

If (f) either $c > 0$ *or* $v'(0) > 0$*, then (d)–(e) imply all academic descendants of a teacher with skill* $k \in A$ *will display one of at most finitely many* $d = d(k)$ *distinct skill types, unless differentiability of* v *fails at* k*. However,* $d(k)$ *may diverge as* $k \to \bar{a}$*, in which case* $v'(k) \to +\infty$ *at a rate we can estimate.*

We are finally in a position to state our positive assortativity and uniqueness results.

Theorem 5.3 (Positive assortative and unique optimizers). *Adopting the hypotheses and notation of Theorem 4.1, if* $(\epsilon, \lambda) \in R(\alpha)$ *maximize the dual problem* (3.2)*, then the labor matching* λ *is positive assortative. Moreover, there exist a pair of maximizers* (ϵ, λ) *for which the educational matching* ϵ *is also positive assortative.*

If there exist minimizing payoffs $(u, v) \in F_0$ *for the primal problem* (3.1) *which are strictly convex and increasing, (as for example if either* $c > 0$ *or* $N\theta^2 \ge 1$*), then any maximizing* ϵ *and* λ *are positive assortative. If, in addition,* α *is free from atoms then the maximizing* ϵ *and* λ *are unique. If, in addition, hypotheses (d)-(f) from Proposition 5.2 hold, then* u' *and* v' *exist and are uniquely determined* α*-a.e. and* $(z_\#\epsilon)$*-a.e. respectively.*

If, in addition, α dominates some absolutely continuous measure whose support fills $\bar{A}$, and $(u_0, v_0) \in F_0$ is any other minimizer with $v_0 : A \longrightarrow \mathbf{R}$ locally Lipschitz then $u_0 = u$ holds α-a.e., meaning u_0 is unique.

Regarding the marginals of λ as fixed, positive assortativity follows from the fact that λ is chosen to maximize a surplus $\tilde{b}_{\theta'}(a, k) = b_L((1-\theta)a + \theta k)$ whose cross-partial derivatives are positive (owing to the uniform convexity of b_L). Such results have played a celebrated role in the economics literature since the work of Mirrlees on taxation [24], Becker on marriage [1], and Spence on educational signaling in the labor market [29]; in the mathematical literature they date to Lorentz' earlier work on rearrangement inequalities [20]. The positive assortativity of ϵ cannot be derived in quite the same way, since it is $z_{\#}\epsilon - \epsilon^2/N$ rather than ϵ^2 which is fixed in the maximization (3.2). However, it is strongly suggested by (5.3) and (5.5), and can be rigorously derived from cross-partial derivatives of the expressions appearing in the suprema (4.3)–(4.5), whose positivity relies on the uniform convexity of the endogenous wage profile $v(k)$ established in Theorem 4.1. In view of (5.3) and (5.4), one can also view this convexity as propagating from the wage of each adult to the wage of their teacher; it takes finitely many steps to reach any teacher's skill level in the educational pyramid, by Proposition 5.2(f).

Uniqueness of ϵ and λ follow from positive assortativity once their marginal distributions are known. My favorite proof of this fact appears in [21]. The distribution of student skills $\epsilon^1 = \alpha$ is specified a priori, and the distribution of teacher skills ϵ^2 can be worked out from the equilibrium payoffs (u, v) using the student-teacher skill correspondence $k = k_t(a)$ given by (5.5). This follows a strategy which has become standard in optimal transportation since the work of Brenier [4], Caffarelli [5], Gangbo and myself [15]. To specify the marginals of λ uniquely requires sorting out who will be a worker and who will be a manager, allowing for the possibility that their skill distributions λ^1 and λ^2 may overlap. A precedent for deriving uniqueness in such settings appears in work with Trokhimtchouk [23].

6. Transition to unbounded wage gradients

Finally, we are in a position to address our motivating question, which is the possibility of singularities in the wage profile at the apex $\bar{a}$ of the skills pyramid $A = [0, \bar{a}[$. By analyzing the recursion (5.1) relating the wage of each teacher to the future earnings of their students, we are able to prove the following singularity alternative:

Theorem 6.1 (Wage behavior and density of top-skilled adults). *Adopting the hypotheses and notation of Theorem 4.1, let α be given by a Borel probability density $\alpha^{ac} \in L^\infty(A)$ which is continuous and positive at the upper endpoint of $A = [0, \bar{a}[$. Suppose $(\epsilon, \lambda) \in R(\alpha)$ and convex $(u, v) \in F_0$ optimize the primal and dual problems* (3.1)–(3.2)*, and (i) $\bar{a} \in (\mathrm{Spt}\,\epsilon^2) \setminus \mathrm{Spt}(\lambda^1 + \lambda^2)$, meaning all adults with sufficiently high skills become teachers; (ii) the educational matching ϵ is positive assortative, meaning a non-decreasing correspondence $k = k_t(a)$ relates the ability of α-a.e. student a to that of his teacher; (iii) k_t is differentiable at $\bar{a}$, and (iv) v is differentiable on some interval $]\bar{a} - \delta, \bar{a}[$. Then for $N\theta \neq 1$,*

$$v'(a) = \frac{const}{|\bar{a} - a|^{\frac{\log N\theta}{\log N}}} - \frac{\bar{cb}'_E}{1 - \frac{1}{N\theta}} + o(1) \tag{6.1}$$

as $a \to \bar{a}$.

In Proposition 5.2 and Theorem 5.3 we have already seen that $N'\theta'$, $N\theta$ and $N\theta^2$ large enough guarantee (i) and (ii). We do not know conditions which guarantee (iii)-(iv). However (iii) follows from (5.5) if $b_E, u, v \in C^2$ near $\bar{a}$, so our theorem guarantees that some nearby singularity is produced when (i)–(ii) hold. The difference quotients whose limit defines $k_t'(\bar{a})$ are bounded under the conditions of Proposition 5.2(d), in which case we judge failure of (iii) to be less likely that the smooth gradient blow-up predicted by (6.1). We judge an accumulation (iv) of non-differentiabilities $A \setminus \mathrm{Dom}\, Du$ at $\bar{a}$ to be even more unlikely. On the other hand, the leading order behavior of (6.1) changes, depending on whether the influence $(N\theta)^d$ of a given teacher grows or decays as one moves through successive generations $d \geq 1$ of their academic descendants. This strongly suggests a sharp transition from bounded to unbounded wage gradients, at the critical value $N\theta = 1$ where this influence remains constant from generation to generation. Note however that the singularity (6.1) in the gradient is integrable, so that even when it is present the wages $v(k)$ tend to a finite limit as $k \to \bar{k}$. This answers the question raised at the outset: at least in the context of the present model, the maximum wage tends to a finite multiple of the average wage in the large population limit; its sensitivity to skill level, however, can be bounded or unbounded, depending on the effectiveness of education.

Acknowledgements. The author thanks Gary Becker and Yann Brenier for key references, plus Rosemonde Lareau-Dussault and his collaborators Alice Erlinger, Xianwen Shi, Aloysius Siow and Ronald Wolthoff for many stimulating interactions. He acknowledges partial support of his research by Natural Sciences and Engineering Research Council of Canada Grant 217006-08.

References

[1] G.S. Becker, *A theory of marriage. Part I*, J. Political Econom. **81** (1973), 813–846.

[2] G.S. Becker and K.M. Murphy, *The division of labor, coordination costs, and knowledge*, Quart. J. Econom. **107** (1992), 1137–1160.

[3] M. Beiglböck, P. Henry-Labordère, and F. Penkner, *Model-independent bounds for option prices—a mass transport approach*, Finance Stoch. **17** (2013), 477–501.

[4] Y. Brenier, *Décomposition polaire et réarrangement monotone des champs de vecteurs*, C.R. Acad. Sci. Paris Sér. I Math. **305** (1987), 805–808.

[5] L. Caffarelli, *Allocation maps with general cost functions*, In P. Marcellini et al, editors, *Partial Differential Equations and Applications*, Lecture Notes in Pure and Appl. Math. number 177, pages 29–35. Dekker, New York, 1996.

[6] G. Carlier, *Duality and existence for a class of mass transportation problems and economic applications*, Adv. Math. Econom. **5** (2003), 1–21.

[7] P.-A. Chiappori, R.J. McCann, and L. Nesheim, *Hedonic price equilibria, stable matching and optimal transport: equivalence, topology and uniqueness*, Econom. Theory **42** (2010), 317–354.

[8] M.J.P Cullen and R.J. Purser, *An extended Lagrangian model of semi-geostrophic frontogenesis*, J. Atmos. Sci., **41** (1984), 1477–1497.

[9] I. Ekeland, *Existence, uniqueness and efficiency of equilibrium in hedonic markets with multidimensional types*, Econom. Theory **42** (2010), 275–315.

[10] A. Erlinger, R.J. McCann, A. Siow, X. Shi, and R. Wolthoff, *Academic wages and pyramid schemes: a mathematical model*. In preparation.

[11] A. Figalli, Y.-H. Kim, and R.J. McCann, *When is multidimensional screening a convex program?*, J. Econom Theory **146** (2011), 454–478.

[12] X. Gabaix and A. Landier, *Why has CEO compensation increased so much?*, Quart. J. Econom. **123** (2008), 49–100.

[13] D. Gale and L.S. Shapley, *College admissions and the stability of marriage*, Amer. Math. Monthly **69** (1962), 9–15.

[14] A. Galichon and B. Salanié, *Cupid's invisible hand: social surplus and identification in matching models*, Working paper, 2011.

[15] W. Gangbo and R.J. McCann, *The geometry of optimal transportation*, Acta Math. **177** (1996), 113–161.

[16] L. Garicano, *Hierarchies and the organization of knowledge in production*, J. Political Econom. **108** (2000), 874–904.

[17] L. Garicano and E. Rossi-Hansberg, *Organization and inequality in a knowledge economy*, Quarterly J. Econom. **121** (2006), 1383–1435.

[18] N.E. Gretsky, J.M. Ostroy, and W.R. Zame, *The nonatomic assignment model*, Econom. Theory **2** (1992), 103–127.

[19] R. Jordan, D. Kinderlehrer and F. Otto, *The variational formulation of the Fokker-Planck equation*, SIAM J. Math. Anal. **29** (1998), 1–17.

[20] G.G. Lorentz, *An inequality for rearrangements]*, Amer. Math. Monthly **60** (1953), 176–179.

[21] R.J. McCann, *Exact solutions to the transportation problem on the line*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci., **455** (1999), 1341–1380.

[22] R.J. McCann, X. Shi, A. Siow, and R. Wolthoff, *The organization of the labor market with communication and cognitive skills*, Preprint at http://www.math.toronto.edu/mccann/publications.

[23] R.J. McCann and M. Trokhimtchouk, *Optimal partition of a large labor force into working pairs*, Econom. Theory **42** (2010), 375–395.

[24] J.A. Mirrlees, *An exploration in the theory of optimum income taxation*, Rev. Econom. Stud. **38** (1971), 175–208.

[25] J.-C. Rochet and P. Choné, *Ironing, sweeping and multidimensional screening*, Econometrica **66** (1998), 783–826.

[26] S. Rosen, *The economics of superstars*, Amer. Econom. Rev. **71** (1982), 845–858.

[27] L.S. Shapley and M. Shubik, *The assignment game I: The core*, Internat. J. Game Theory **1** (1972), 111–130.

[28] H. Sonnenschein, *Price dynamics based on the adjustment of firms*, Amer. Econom. Rev. **72** (1982), 1088–1096.

[29] M. Spence, *Job market signaling*, Quarterly J. Econom. **87** (1973), 355–374.

Department of Mathematics, University of Toronto, Toronto ON Canada M5S 2E4
E-mail: mccann@math.toronto.edu

On singularity formation in Hamiltonian evolution equations

Pierre Raphaël

Abstract. Hamiltonian evolution equations arise in the description of nonlinear phenomenons in various instances from nonlinear optics to astrophysics or fluid mechanics, but the description of most even simplified models still remains a mathematical challenge. Substantial progress have been made since the 1980's for the qualitative description of solutions through the importation and mixing of various ideas from dynamical systems, functional analysis, harmonic analysis and the calculus of variations. I will report in this survey on recent progress on the study of one specific scenario: singularity formation, that is the ability for non linear waves to concentrate their energy while propagating in some nonlinear medium. A new methodology has emerged in the last two decades on canonical models like the non linear Schrödinger or wave equations both for the construction and the classification of singular regimes, with applications also to parabolic models. A special class of solutions plays a distinguished role in the structure of the corresponding blow up bubbles: the solitary wave.

Mathematics Subject Classification (2010). Primary 35Q41, 35Q51, 35Q53, 35Q55; Secondary 37K40.

Keywords. Non linear Hamiltonian equations, nonlinear Schrödinger equations, singularity formation, solitons.

1. Introduction

The propagation of waves in a nonlinear medium has been the subject of an intense investigation for the physics of the 20th century with the discovery of many deeply non linear phenomenons. Let us for example consider the propagation of light in a non linear crystal. The laser beam is submitted to two main effects: dispersion, that is the tendency of light to spread in space while propagating, and focusing induced by the optical properties of the crystal. The system of PDE's underlying the description of the laser beam should involve the full non linear Maxwell's equations which is a complicated nonlinear vectorial system, [70]. In suitable regimes and neglecting some effects in a first approximation, one can derive in a nowadays canonical way the "envelope" equation which in the most simplified models typically takes the form of a non linear Schrödinger equation:

$$(NLS)\quad \begin{cases} i\partial_t u + \Delta u + u|u|^2 = 0 \\ u_{|t=0} = u_0 \end{cases}, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^d, \ \ u(t,x) \in \mathbb{C}. \tag{1.1}$$

In the physical model, the time variable is in fact a space variable corresponding to the direction of propagation of the nonlinear wave packet. The concentration of the laser beam at

Proceedings of the International Congress of Mathematicians, Seoul, 2014

a point in space which is typically observed if the crystal is focusing enough then corresponds to the concentration of the wave packet solution to (1.1) at a finite time $0 < T < +\infty$. In other words, for a strong enough nonlinearity, focusing may overcome dispersion and a singularity forms by concentration of the energy of the wave locally in space even though the data was initially smooth and well localized.

The question then becomes: can we see on the infinite dimensional Hamiltonian system (NLS) that such a singularity will form, and if yes, what should it look like? Beautifully enough, the proof that something like singularity formation will hold for the system (1.1) is almost as old as its derivation in the 1950's thanks to a spectacular algebraic identity[1] which *prevents the flow from being global*, but this says very little on the structure of the possible singularity formation. Moreover, in many canonical situations, this miraculous algebra is lost. For example the question of singularity formation for the Navier Stokes equation of incompressible fluids raised by Leray [33] in 1934 is open and at the heart of one of the Millenium Prizes of the Clay Institute.

The end of the 20th century has seen the development of numerical methods for the study of simplified canonical models like (NLS) with the first qualitative predictions of singular regimes, and the emergence of rigorous mathematical tools for the qualitative understanding of linear and nonlinear PDE's. This movement has accelerated tremendously in the past fifteen years under the influence of tools and intuition coming from various areas of mathematics and in particular variational methods, dynamical systems, functional analysis and harmonic analysis. The qualitative description of nonlinear PDE's has entered a new age with the quantitative descriptions of fully nonlinear regimes in wave dynamics.

I will report in these notes on the activity related to the construction and possibly classification of blow up bubbles for canonical models like semilinear (NLS) models. An essential role is played in the analysis by a fundamental nonlinear object discovered more than hundred years ago in fluid mechanics [30]: the solitary wave.

1.1. The critical space. Let us consider the (NLS) equation with power nonlinearity:

$$(NLS)\quad \left\{ \begin{array}{l} i\partial_t u + \Delta u + u|u|^{p-1} = 0 \\ u_{|t=0} = u_0 \end{array} \right. ,\ \ (t,x) \in \mathbb{R}\times\mathbb{R}^d,\ \ u(t,x) \in \mathbb{C}. \tag{1.2}$$

This is an infinite dimensional Hamiltonian system with Hamiltonian (or energy):

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u(t,x)|^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^d} |u(t,x)^{p+1} dx = E(u_0) \tag{1.3}$$

where the $-$ sign in front of the potential energy reflects the focusing nature of the problem. There also holds the conservation of mass:

$$\int_{\mathbb{R}^d} |u(t,x)|^2 dx = \int_{\mathbb{R}^d} |u_0(x)|^2 dx. \tag{1.4}$$

We recall the definition of the Sobolev norm:

$$\|u\|_{H^s}^2 = \|u\|_{\dot{H}^s}^2 + \|u\|_{L^2}^2,\ \ \|u\|_{\dot{H}^s} = \||D|^s u\|_{L^2},\ \ s \geq 0$$

[1] the virial identity, see (1.8).

where we defined the Fourier multiplier $\widehat{|D|^s u} = |\xi|^s \hat{u}$. The critical scaling exponent is an important phenomenological number related to the invariance by scaling of (1.2) and is computed as follows: if $u(t,x)$ is a solution, then for all $\lambda > 0$, so is

$$u_\lambda(t,x) = \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x), \tag{1.5}$$

and then the critical Sobolev space $\dot{H}^{s_c}$ is the one which norm (in space) does not see the scaling symmetry:

$$\|u_\lambda(t,\cdot)\|_{\dot{H}^{s_c}} = \|u(\lambda^2 t,\cdot)\|_{\dot{H}^{s_c}} \text{ i.e. } s_c = \frac{d}{2} - \frac{2}{p-1}.$$

Since the conservation laws of energy (1.3) and mass (1.4) are the *only* a priori information on the flow and live at the level of Sobolev regularity respectively $\dot{H}^1$ and $L^2 = \dot{H}^0$, H^1 is the natural *energy space* in which to look for solutions. The special cases $s_c \in \{0,1\}$ are respectively the mass critical and energy critical cases and correspond to a specific algebra where the scaling symmetry meets a conservation law. The fact that the phenomenological number s_c is essential for the qualitative description of the solutions is a consequence of the analysis.

1.2. Global existence and blow up: the virial law. Local solvability in time of an infinite dimensional system like (1.2) is now well understood [16] using a reformulation à la Cauchy-Lipschitz in a suitable functional space thanks to dispersive estimates à la Strichartz [69]: for $s_c < 1$, given $u_0 \in H^1$, there exists a unique $u \in \mathcal{C}([0,T), H^1)$ solution to (1.2) for some maximal time of existence $0 < T = T(u_0) \leq +\infty$, and:

$$T < +\infty \text{ implies } \lim_{t \uparrow T} \|u(t,\cdot)\|_{H^1} = +\infty. \tag{1.6}$$

Now whether solutions are global or not is in general a very difficult problem. For the specific case of (NLS), the existence of finite time blow up solutions can be obtained from a spectacular algebra: the virial law.

Theorem 1.1 (Global well posedness and finite time blow up, [16, 17]). *If $s_c < 0$, then all H^1 solutions are global ($T = +\infty$). If $s_c \geq 0$, let $u_0 \in \Sigma = H^1 \cap \{xu \in L^2\}$, then*

$$E(u_0) < 0 \text{ implies } T < +\infty. \tag{1.7}$$

Indeed, for $s_c < 0$, a simple interpolation estimate ensures that the non positive potential term in the total energy (1.3) is in fact controlled by the kinetic energy, and hence the a priori control of mass and energy implies the control of the full H^1 norm of the solution $\|u(t,\cdot)\| \lesssim \|u_0(\cdot)\|_{H^1}$ which together with the blow up criterion (1.6) implies $T = +\infty$. For $s_c \geq 0$, this bound becomes critical and indeed singularity formation may occur due to the following algebra, [17]: for $s_c \geq 0$,

$$\frac{d^2}{dt^2} \int_{\mathbb{R}^d} |x|^2 |u(t,x)|^2 dx \leq 16 E_0 \tag{1.8}$$

and hence for $E_0 < 0$, the positive variance $\int_{\mathbb{R}^d} |x|^2 |u(t,x)|^2 dx$ lies below an inverted parabola which touches zero in finite time, hence preventing the flow from being global.

This argument for showing singularity formation is spectacular by its simplicity and its strength as it not only shows singularity formation, but also the fact that the phenomenon occurs for an open set of data $E_0 < 0$ and hence displays some stability. Such kind of identities are in fact fairly common, and for example allowed Sideris to prove singularity formation for 3d compressible gas dynamics [68]. However this argument breaks down in many situations (like incompressible fluid mechanics) and anyway says very little on the structure of the singularity.

1.3. The solitary wave. Our approach for the study of singularity formation is to focus on the flow near a special object which is of particular relevance for nonlinear dynamics: the solitary wave. A soliton or solitary wave is a special gobal in time solution of the wave mechanics for which dispersion and focusing balance exactly. It was discovered in the setting of fluid mechanics by Korteweg-de Vries [30] at the end of the 19th century in the form of a traveling wave which propagates without deformation at the surface of water on a very long distance. But this kind of object is in fact present in a very wide area of physics in the form of stationary, periodic or traveling wave solutions. In the setting of (NLS) equation for $s_c < 1$, the solitary wave corresponds to a time periodic solution:

$$u(t,x) = Q(x)e^{it} \text{ ie } \Delta Q - Q + Q|Q|^{p-1} = 0. \tag{1.9}$$

The classification of all solutions to the non linear elliptic problem (1.9) is open, but the classification of the *ground state* solution is an important breakthrough of the 1980's:

Theorem 1.2 (Classification of the ground state solitary wave, [15], [31]). *Let $s_c < 1$, then there exists a unique up to translation solution to*

$$\Delta Q - Q + Q|Q|^{p-1} = 0, \;\; Q > 0, \;\; Q \in H^1(\mathbb{R}^d). \tag{1.10}$$

Moreover, the solution has up to translation radial symmetry and decays exponentially fast as $|x| \to +\infty$.

In some very special instances (the integrable case $p = 3$, $d = 1$ for (NLS)) in the subcritical regime $s_c < 0$, one can show that the solitary wave is in some sense the universal attractor of *all* solutions which asymptotically decouple into a sum of solitary waves traveling at different speeds and a remaining dispersion. In the blow up setting $s_c \geq 0$, we will show similarly that the solitary wave plays a distinguished role in the analysis and occurs as the universal *blow up profile* for a large class of blow up bubbles, or equivalently the universal attractor of the flow *after renormalization*.

2. The mass critical problem

The mass critical case $s_c = 0$ ie $p = 1 + \frac{4}{d}$ corresponds to the smallest nonlinearity for which blow up is possible, and is directly related in dimension $d = 2$ to the self focusing of a laser beam in nonlinear optics.

2.1. Small data scattering. In the pioneering work [76], Weinstein obtained the celebrated variational characterization of the ground state solution Q to (1.10).

Theorem 2.1 (Varational characterization of the ground state, [76]). *For all $u_0 \in H^1$, there holds the lower bound on the energy* (1.3)*:*

$$E(u) \geq \frac{1}{2}\|\nabla u\|_{L^2}^2 \left[1 - \left(\frac{\|u\|_{L^2}}{\|Q\|_{L^2}}\right)^{\frac{4}{d}}\right]. \tag{2.1}$$

Moreover, $E(u) = 0$ and $\|u\|_{L^2} = \|Q\|_{L^2}$ imply $u \equiv Q$ up to symmetries.

Equivalently, the ground state Q attains the best constant in the Gagliardo-Nirenberg interpolation estimate

$$\int_{\mathbb{R}^d} |u|^{2+\frac{4}{d}} \leq C^* \left(\int_{\mathbb{R}^d} |\nabla u|^2\right)\left(\int_{\mathbb{R}^d} |u|^2\right)^{\frac{2}{d}}$$

which compares the size of the kinetic and potential energy in (1.3). This variational characterization of the ground state is the first step towards the *dynamical* characterization of the solitary wave:

Theorem 2.2 (Scattering below the ground state). *Let $u_0 \in H^1$ with $\|u_0\|_{L^2} < \|Q\|_{L^2}$, then the corresponding solution to* (1.2) *is global and scatters as $t \to \pm\infty$.*

Hence any initial data smaller than the ground state (in the L^2 topology) is global in time and asymptotically attracted by the linear dynamics in $\pm\infty$. There is a long history on this problem. The fact that the solution is global in time follows directly from the lower bound (2.1) and the conservation of mass (1.4) which for $\|u_0\|_{L^2} < \|Q\|_{L^2}$imply an upper bound $\|u(t,\cdot)\|_{H^1} \lesssim \|u_0\|_{H^1}$, and the conclusion follows from the blow up criterion (1.6). The fact that the solution is attracted by the linear flow is a much more subtle issue. For (1.2), it can be easily proved for $u_0 \in \Sigma$ using an additional structure of the equation: the pseudo conformal symmetry, see [6]. However the sharp statement (2.2) in the sole energy space H^1 relies on the dynamical classification of the ground state solitary wave as the "minimal non scattering solution" as first obtained in the breakthrough paper by Kenig and Merle [23] in the energy critical case. This is a long and difficult way to go, and we refer to [10, 26] for an overview of this recently solved problem.

2.2. Minimal blow up elements. The global existence criterion of Theorem 2.2 is sharp. Indeed, there exists a minimal mass blow up solution. This is a consequence of the pseudo conformal symmetry: if $u(t,x)$ solves the mass critical (NLS), then so does

$$v(t,x) = \frac{1}{|t|} u\left(-\frac{1}{t}, \frac{x}{t}\right) e^{i\frac{|x|^2}{4t}}. \tag{2.2}$$

Applying this to the non dispersive soliton solution $u(t,x) = Q(x)e^{it}$ yields the *explicit blow up solution*:

$$S(t,x) = \frac{1}{|t|}\left(Qe^{it\frac{|y|^2}{4}}\right)\left(\frac{x}{|t|}\right) e^{-\frac{i}{t}}, \quad t \in \mathbb{R}^*. \tag{2.3}$$

This solution corresponds to the smooth and well localized Cauchy data

$$S(-1) = Q(x)e^{-i\frac{|x|^2}{4}+i}$$

which has minimal mass $\|S(t,\cdot)\|_{L^2} = \|Q\|_{L^2}$ and concentrates at the origin in time and space

$$|S(t,\cdot)|^2 \rightharpoonup \|Q\|_{L^2}^2 \delta_{x=0}$$

at the rate of concentration

$$\|\nabla S(t)\|_{L^2} \sim \frac{1}{|t|}. \tag{2.4}$$

This is an explicit example of "blow up bubble". In 1992, Merle obtained the first *dynamical classification* of the solitary wave:

Theorem 2.3 (Classification of the minimal blow up element, [43]). *Let $u_0 \in H^1$ with $\|u_0\|_{L^2} = \|Q\|_{L^2}$. Assume that $T < +\infty$. Then $u \equiv S$ up to the symmetries of the flow.*

In other words, there is a complete rigidity of the minimal blow up element. Conceptually, this proof is the first step towards the classification of "minimal compact elements of the flow" at the heart of the breakthrough "Kenig-Merle route map" for the proof of global existence [23, 24], see [66] for a further discussion of the connections between these results. Merle's proof however had a fundamental weakness: both the existence and uniqueness proofs were based on the explicit use of the pseudo-conformal symmetry (2.2), and it has been long believed that indeed the existence of this object was deeply tied to the pseudo conformal symmetry. In a joint work with J. Szeftel, we revisited in [65] the existence and uniqueness of the minimal element for an inhomogeneous problem

$$i\partial_t u + \Delta u + k(x)u|u|^2 = 0, \;\; x \in \mathbb{R}^2 \tag{2.5}$$

for a suitable class of inhomogeneity k which breaks the symmetry group and the pseudo conformal symmetry.

Theorem 2.4 (Dynamical classification of the minimal blow up element, [65]). *Let $x_0 \in \mathbb{R}^2$ with*

$$k(x_0) = 1 \;\; \textit{and} \;\; \nabla^2 k(x_0) < 0.$$

Then there exists an energy threshold $E^* > 0$ such that for all higher energies $E_0 > E^*$, there exists a unique up to phase shift H^1 minimal mass blow up solution to (2.5) *which blows up at time $T = 0$, at the point x_0 and with energy level E_0.*

Theorem 2.4 relies on a dynamical proof which involves a complete understanding of the minimal blow up bubble, and in particular the computations of the *large* deviations induced by the inhomogeneity $k(x)$. This analysis has been recently extended to (NLS) on manifolds [4] and other classes of dispersive PDE's [27, 37]. We also refer to [11] for connected results in the energy critical case. The outcome is that the existence of minimal blow up bubbles is the generic situation and is disconnected to the existence of the pseudo conformal symmetry.

2.3. Log-log blow up. The minimal mass blow up regime is obviously unstable by perturbation of the data, and there has been in the 1980's an intense search for the structure of the *stable* singularity formation. Numerics and very clever formal arguments have led to several conjectures among which Landman, Papanicolaou, Sulem, Sulem [32] "log-log" law as the generic blow up speed for solutions to (1.2):

$$\|\nabla u(t)\|_{L^2} \sim \left(\frac{\log|\log(T-t)|}{T-t}\right)^{\frac{1}{2}} \;\; \text{as} \;\; t \to T.$$

The double log correction to self similarity has long remained a mysterious issue from the analytic point of view. Galina Perelman gave in [57] the first rigorous construction of such a blow up bubble in dimension $d = 1$. In a series of papers joint with F. Merle, we revisited the log-log analysis in dimensions $d \leq 5$ and obtained classification results of the flow near the solitary wave. The first outcome of this analysis is the universality of the blow up bubble in the critical L^2 space and the quantization of the mass focused at the singularity.

Theorem 2.5 (Universality of the blow up bubble, [47, 49]). *Let $d \leq 5$. Let $\alpha^* > 0$ denote a small enough universal constant. Let $u_0 \in H^1$ with*

$$\|u_0\|_{L^2} < \|Q\|_{L^2} + \alpha^*, \tag{2.6}$$

and assume that the corresponding solution to (1.2) *blows up in finite time $T < +\infty$. Then there exist parameters $(\lambda(t), x(t), \gamma(t)) \in \mathcal{C}^1([0,T), \mathbb{R}_+^* \times \mathbb{R}^d \times \mathbb{R})$ with*

$$\lim_{t\to T} \lambda(t) = 0, \quad \lim_{t\to T} x(t) = x(T) \in \mathbb{R}^d, \quad \lim_{t\to T} \gamma(t) = +\infty$$

and an asymptotic profile $u^* \in L^2$ such that

$$u(t) - \frac{1}{\lambda(t)^{\frac{d}{2}}} Q\left(\frac{x - x(t)}{\lambda(t)}\right) e^{i\gamma(t)} \to u^* \ \text{ in } \ L^2 \ \text{ as } \ t \to T. \tag{2.7}$$

In other words, the singular part of *any* blow up bubble with mass slightly above the minimal mass is completely classified and corresponds to a trajectory $(x(t), \lambda(t), \gamma(t))$ drawn on the manifold of solitary waves generated by the large symmetry group of the equation. Blow up occurs at a point, and the mass which is concentrated at the singularity is the *universal quantum* $\|Q\|_{L^2}^2$ independent of the data. We then obtained the existence and stability of the log log blow up regime.

Theorem 2.6 (Hamiltonian condition for log-log blow up [46, 48]). *Let $d \leq 5$. Let $\alpha^* > 0$ denote a small enough universal constant. Let $u_0 \in H^1$ with*

$$\|u_0\|_{L^2} < \|Q\|_{L^2} + \alpha^* \ \text{ and } \ E(u_0) < 0, \tag{2.8}$$

then the corresponding solution to (1.2) *blows up in finite time $T < +\infty$. Moreover, the concentration rate $\lambda(t)$ in* (2.7) *is given by*

$$\lambda(t) = \left[\sqrt{2\pi} + o(1)\right] \left(\frac{T-t}{\log|\log(T-t)|}\right)^{\frac{1}{2}} \ \text{ as } \ t \to T. \tag{2.9}$$

This result is a considerable improvement on the virial theorem (1.7) since we show that the Hamiltonian condition $E_0 < 0$ not only implies blow up, it forces the solution to dynamically enter the open set of log log blow up and the associated universal regime. It is easy to show using the pseudo-conformal symmetry (2.2) that there exist blow up solutions in the log log regime with $E_0 > 0$. The general statement is then that the log-log set is open:

Theorem 2.7 (Stability of log-log blow up [59]). *Let $d \leq 5$. Let $\alpha^* > 0$ denote a small enough universal constant. The set of initial data with small super critical mass* (2.6) *for which the solution to (NLS) blows up in finite time in the log-log regime* (2.9) *is open in the energy space H^1.*

The proof of these results relies on the introduction of a modulated decomposition of the flow

$$u(t,x) = \frac{1}{\lambda(t)^{\frac{d}{2}}}(Q+\varepsilon)\left(t, \frac{x-x(t)}{\lambda(t)}\right)e^{i\gamma(t)}, \ \ \|\varepsilon(t,\cdot)\|_{H^1} \ll 1.$$

This decomposition holds near the singularity as a consequence of the variational characterization of the ground state and the conservation laws of mass and energy. Hence in any blow up regime, the ground state solitary wave Q is a good approximation of the blow up profile, and this is the starting point for a perturbative analysis. The sharp description of the blow up bubble now relies on the extraction of the finite dimensional and possibly universal dynamic for the evolution of the geometrical parameters $(\lambda(t), x(t), \gamma(t))$ which is coupled to the infinite dimensional dispersive dynamic driving the small excess of mass $\varepsilon(t)$. Classification results like the openness of the loglog set deeply rely on new *monotonicity formula* which existence was unexpected in the dispersive setting.

2.4. Threshold dynamics. Let us insist again that the minimal mass blow up solution $S(t,\cdot)$ given by (2.3) does not blow up in the log-log regime. This solution is unstable, but Bourgain and Wang [5] could stabilize this blow up dynamic on a finite codimensional center stable manifold and show that given a limiting profile $u^* \in H^1$ flat enough at the origin, one can build a solution to (1.2) which blows up at $t=0$ at $x=0$ and satisfies:

$$u(t) - S(t) \to u^* \ \text{ in } \ H^1 \ \text{ as } \ t \uparrow 0, \tag{2.10}$$

see also [28]. We proved in [55] that these solutions are threshold dynamics between the open set of finite time log log blow up, and the open set of solutions which scatter to the right.

Theorem 2.8 (Strong instability of Bourgain Wang solutions, [55]). *Let $d = 1, 2$. Then any H^1 neighborhood of the data of the Bourgain Wang solution* (2.10) *contains data which lead respectively to finite time blow up in the log log regime, or global existence and scattering at $+\infty$.*

2.5. Classification of the flow near the ground state. Theorem 2.8 is a first step towards the complete description of the flow near the ground state for the mass critical (NLS) which however is still not complete. There may in particular exist other blow up regimes near Q in the energy topology. An important step however has been achieved on another very much related problem: the mass critical (gKdV) equation

$$(gKdV) \ \begin{cases} \partial_t u + \partial_x(\partial_x^2 u + u^5) = 0 \\ u_{|t=0} = u_0 \end{cases}, \ \ (t,x) \in \mathbb{R}\times\mathbb{R}. \tag{2.11}$$

This one dimensional model shares a lot of the mass critical (NLS) structure: the same conservation laws of mass and energy, the same mass critical scaling, and the same solitary wave which for (gKdV) takes the form of a traveling wave

$$u(t,x) = Q(x-t).$$

However, both the pseudo conformal symmetry (2.3) and the virial identity (1.8) are lost, and hence the sole existence of blow up solutions has been a long standing open problem. In their series of seminal works [34, 35, 42], Martel and Merle have obtained the first existence

result of blow up dynamics together with a preliminary description of the blow up bubble. We revisited the analysis in [36–38] and obtained the first complete classification of the flow near the ground state in the mass critical setting. Let the set of initial data

$$\mathcal{A} = \left\{ u_0 = Q + \varepsilon_0 \text{ with } \|\varepsilon_0\|_{H^1} < \alpha_0 \text{ and } \int_{y>0} y^{10}\varepsilon_0^2 < 1 \right\}, \tag{2.12}$$

and consider the L^2 tube around the family of solitary waves

$$\mathcal{T}_{\alpha^*} = \left\{ u \in H^1 \text{ with } \inf_{\lambda_0>0,\, x_0\in\mathbb{R}} \left\| u - \frac{1}{\lambda_0^{\frac{1}{2}}} Q\left(\frac{\cdot - x_0}{\lambda_0}\right) \right\|_{L^2} < \alpha^* \right\}.$$

Then the dynamics for data in $\mathcal{A}$ is classified as follows:

Theorem 2.9 (Rigidity of the flow in $\mathcal{A}$, [36]). *Let $0 < \alpha_0 \ll \alpha^* \ll 1$ and $u_0 \in \mathcal{A}$. Let $u \in \mathcal{C}([0,T), H^1)$ be the corresponding solution to* (2.11)*. Then one of the following three scenarios occurs:*
(Blow up)*: the solution blows up in finite time $0 < T < +\infty$ in the universal regime*

$$\|u(t)\|_{H^1} = \frac{c_{u_0} + o(1)}{T-t} \quad \text{as } t \to T, \ \ c_{u_0} > 0. \tag{2.13}$$

(Soliton)*: the solution is global and converges asymptotically to a solitary wave.*
*(*Exit)*: the solution leaves the tube $\mathcal{T}_{\alpha^*}$ at some time $0 < t_u^* < +\infty$.*
Moreover, the scenarios (Blow up) *and* (Exit) *are stable by small perturbation of the data in $\mathcal{A}$.*

Equivalently, the codimension one center stable manifold of global solutions asymptotically attracted by the solitary wave separates the open set of finite time blow up with universal blow up speed, and the open set of solutions which escape the nonlinear dynamics. The proof is delicate but most likely canonical and relies on the derivation of new Lyapounov functionals based on mixed Energy/Morawetz estimates in the continuation of [34, 45, 61]. It remains to understand the long time dynamics in the (Exit) regime. This question is in fact related to the behavior of the minimal blow up element. We first claim that this object exists and is unique:

Theorem 2.10 (Existence and uniqueness of the minimal mass blow up element, [37]). *There exists a unique up to symmetries H^1 minimal mass blow up solution $\|S(t,\cdot)\|_{L^2} = \|Q\|_{L^2}$. This solution blows up at $T = 0$ and is globally defined and H^1 bounded for $t > 0$.*

The existence and uniqueness of the minimal element lies in the continuation of Theorem 2.4. The behavior of $S(t,\cdot)$ as $t \to +\infty$ is at this point an open problem. For (NLS), the explicit global in time formula (2.3) implies that the minimal element scatters as $t \to +\infty$. We conjecture that the same holds true for (gKdV) which would complete the classification of the flow in $\mathcal{A}$:

Theorem 2.11 ((Exit) is scattering, [25, 37]). *Assume that $S(t)$ scatters as $t \to +\infty$. Then any solution in the* (Exit) *scenario is global for positive time and scatters as $t \to +\infty$.*

The topology (2.12) of $\mathcal{A}$ is stronger than the energy topology due to the presence of L^2 weights. We showed in [38] that this additional assumption is essential in the sense that there exist blow up solutions with slower decay to the right and blow up speed

$$\|\nabla u(t)\|_{L^2} \sim \frac{1}{(T-t)^\nu} \text{ for arbitrary } \nu > 2, \tag{2.14}$$

and even infinite time blow up solutions with exponential growth. This shows that classification theorems close to the ground state heavily rely on a suitable choice of the topology, and that arbitrarily slow blow up regimes can be reached through a suitable preparation of the data, see [29] for related results in the energy critical case.

3. The energy subcritical problem $0 < s_c < 1$

The singularity formation is poorly understood in the mass super critical zone, in particular because the blow up profile is no longer given by the solitary wave. The beautiful program by Nakanishi and Schlag [56] gives a very good description of the flow near the ground state but without the description of the associated instability by blow up. In a series of joint works with Merle and Szeftel, we have obtained qualitative results on mass super critical blow up bubbles with the discovery of a new type of blow up scenario.

3.1. Blow up of the critical norm. The critical $\dot{H}^{s_c}$ norm plays a distinguished role in the analysis since it does not see the scaling symmetry (1.5) which is at the heart of the concentration mechanism. In the mass critical case, this norm remains bounded since it is conserved by the flow. We proved in [50] that the situation for $0 < s_c < 1$ is quite different:

Theorem 3.1 (Blow up of the critical norm, [50]). *Let $0 < s_c < 1$, $p < 5$ and $d \geq 2$. There exists a universal constant $\gamma(d,p) > 0$ such that the following holds true. Let $u_0 \in H^1$ with radial symmetry and assume that the corresponding solution to* (1.2) *blows up in finite time $T < +\infty$. Then there holds the lower bound for t close enough to T:*

$$\|u(t)\|_{\dot{H}^{s_c}} \geq |\log(T-t)|^{\gamma(d,p)}.$$

Related results were proved for the Navier Stokes equation [13]. The logarithmic lower bound is sharp in some regimes, [53], but there also exist regimes where the critical norm blows up polynomially, [54]. The proof relies on the quantification of a Liouville type theorem, see [22] for further extensions to the wave equation.

3.2. On bounds on the blow up rate. We now address the question of upper and lower bounds on blow up rate for general solutions. First, a direct consequence of the blow up criterion (1.6) and the scaling symmetry (1.5) is the universal *scaling lower bound*:

Proposition 3.2 ([6]). *Let $u_0 \in H^1$ and assume that the corresponding solution to* (1.2) *blows up in finite time $T < +\infty$, then for t close enough to T:*

$$\|\nabla u(t)\|_{L^2} \gtrsim \frac{1}{(T-t)^{\frac{1-s_c}{2}}}. \tag{3.1}$$

A simple observation by Merle is that for $0 < s_c < 1$ and $u_0 \in \Sigma$, the brute force time integration of the virial law (1.8) not only implies finite time blow up for $E_0 < 0$, it also immediately yields an upper bound on the blow up rate:

$$\int_0^T (T-t)\|\nabla u(t)\|_{L^2}^2 dt < +\infty. \tag{3.2}$$

Interestingly enough, this bound fails for $s_c = 0$ due to (2.4), and in fact *there exists no known upper bound on blow up rate in the mass critical case*. The example of the (gKdV) problem with the speeds (2.14) indicates that there may simply not be any polynomial upper bound. For $s_c > 0$, the estimate (3.2) can be improved in the radial class:

Theorem 3.3 (Sharp upper bound on blow up rate for radial data, [54]). *Let $d \geq 2$, $0 < s_c < 1$, $p < 5$. Let the interpolation number:*

$$\alpha = \frac{5-p}{(p-1)(d-1)}, \quad 0 < \alpha < 1. \tag{3.3}$$

Let $u_0 \in H^1$ with radial symmetry and assume that the corresponding solution to (1.2) *blows up in finite time $T < +\infty$. Then there holds the space time upper bound:*

$$\int_t^T (T-\tau)\|\nabla u(\tau)\|_{L^2}^2 d\tau \leq C(u_0)(T-t)^{\frac{2\alpha}{1+\alpha}}. \tag{3.4}$$

We will see below that this upper bound is in fact *sharp*.

3.3. Stable self similar blow up. The same formal arguments and numerical experiments which led to the log-log law conjecture for the mass critical problem yield in the super critical range the self similar law

$$\|\nabla u(t)\|_{L^2} \sim \frac{1}{(T-t)^{\frac{1-s_c}{2}}} \tag{3.5}$$

as the generic stable blow up speed in the super critical case, [70], hence saturating the lower bound (3.1). A difficulty towards a rigorous proof is the construction of suitable blow up profiles supporting the law (3.5). This leads for (NLS) to a complicated non variational system of coupled elliptic PDE's. It is in particular believed that the ground state solitary wave is no longer the correct blow up profile. In [53], we could bypass this difficulty for $0 < s_c \ll 1$ by bifurcating in some sense from the log-log analysis [46], [48] of the mass critical case to show the existence and stability of a self similar blow up regime.

Theorem 3.4 (Existence and stability of self similar solutions, [53]). *Let $1 \leq d \leq 5$ and $0 < s_c \ll 1$. Then there exists an H^1 open set of initial data such that the corresponding solution to* (1.2) *blows up in finite time $T < +\infty$ with the self similar speed* (3.5)*.*

This is a first intrusion into the mass super critical regime, the main open problem being the physical case $d = p = 3$ which for the moment is out of reach.

3.4. Ring solutions. A new class of blow up solutions has been discovered in [60] in the super critical regime $p = 3, d = 2, s_c = \frac{1}{2}$: these are *standing ring* blow up solutions which are radially symmetric solutions which concentrate on a fixed sphere (and not a point). Let us say that even in the parabolic setting, there are few known examples of non trivial geometries for the blow up set in space, and typically only point or sphere singularities are known.

Theorem 3.5 (Standing ring blow up for $p = 5$, $d \geq 2$, [60, 64]). *Let $d \geq 2$ and $p = 5$. Let Q be the one dimensional ground state solution to (1.9). Then there exists an open subset of smooth radially symmetric data such that the corresponding solution $u(t)$ to (1.2) blows up in finite time $0 < T < +\infty$ according to the following dynamics: there exist $\lambda(t) > 0$, $r(t) > 0$, $\gamma(t) \in \mathbb{R}$ and $u^* \in L^2$ such that*

$$u(t,r) - \frac{1}{\lambda(t)^{\frac{1}{2}}} Q\left(\frac{r - r(t)}{\lambda(t)}\right) e^{i\gamma(t)} \to u^*(r) \ \text{ in } \ L^2 \ \text{ as } \ t \to T. \tag{3.6}$$

Here the radius of the singular circle converges $\lim_{t\uparrow T} r(t) = r(T) > 0$, and the scaling parameter $\lambda(t)$ satisfies the log-log law (2.9).

These standing ring solutions see locally around the blow up sphere the log-log regime of the one dimensional mass critical quintic $p = 5$ (NLS). The robustness of the tools developed for the study of the log-log regime are here essential to control the full flow. The underlying mechanism of reduction of dimension can be generalized to cylindrical symmetry [21, 77].

In [14], Fibich, Gavish and Wang conjectured on the basis of formal expansions and numerical experiments the existence of *collapsing ring* solutions which concentrate on a ring which radius vanishes at blow up time. We give a rigorous proof in [54] and beautifully enough, these solutions saturate the general upper bound rate (3.4) which is therefore sharp in the radial class.

Theorem 3.6 (Collapsing ring solutions, [54]). *Let $d \geq 2$, $0 < s_c < 1$ and $p < 5$. Let $0 < \alpha < 1$ be given by* (3.3) *and let the Galilean shift $\beta_\infty = \sqrt{\frac{5-p}{p+3}}$. Let Q be the one dimensional mass subcritical ground state. Then there exists a solution $u \in \mathcal{C}([-1,0), H^1)$ of* (1.2) *with radial symmetry which blows up at time $T = 0$ according to the following dynamics: there exist geometrical parameters $(r(t), \lambda(t), \gamma(t)) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \times \mathbb{R}$ such that:*

$$u(t,r) - \frac{1}{\lambda^{\frac{2}{p-1}}(t)} \left(Q e^{-i\beta_\infty y}\right)\left(\frac{r - r(t)}{\lambda(t)}\right) e^{i\gamma(t)} \to 0 \ \text{ in } \ L^2(\mathbb{R}^d) \tag{3.7}$$

with the sharp asymptotic laws:

$$r(t) \sim |t|^{\frac{\alpha}{1+\alpha}}, \ \ \lambda(t) \sim |t|^{\frac{1}{1+\alpha}}, \ \ \gamma(t) \sim |t|^{-\frac{1-\alpha}{1+\alpha}} \ \text{ as } \ t \uparrow 0. \tag{3.8}$$

Moreover, the blow up speed admits the equivalent:

$$\|\nabla u(t)\|_{L^2} \sim \frac{1}{(T-t)^{\frac{1}{1+\alpha}}} \ \text{ as } \ t \uparrow 0 \tag{3.9}$$

which saturates the upper bound (3.4).

These solutions see after renormalization a L^2 *subcritical* NLS type motion with Galilean drift. Moreover, even though the problem is mass super critical, the mass conservation law still plays a central role in the analysis and the concentration occurs at the L^2 level of regularity:

$$|u(t)|^2 \rightharpoonup c\|Q\|_{L^2}^2 \delta_{x=0} \ \text{ as } \ t \uparrow 0, \ \ c > 0. \tag{3.10}$$

The ring solutions of Theorem 3.6 are *minimal blow up bubbles* again and their construction lies within the framework of construction of minimal elements introduced in [65]. These solutions are believed to be stable by radial perturbation and strongly unstable by non radial perturbation, [14], but both these statements are open.

4. The energy critical problem $s_c = 1$

Energy critical models naturally arise in a number of models (general relativity, crystal physics, particle physics and ferromagnetism) with a strong geometric structure, [75]. The corresponding wave maps, Schrödinger maps and harmonic heat flow have attracted a considerable attention for the past twenty years in connection with conjectures concerning both the existence of blow up bubbles and their classification. We will show an unexpected conceptual connection with the mass critical case $s_c = 0$.

4.1. Geometric wave and parabolic equations. A harmonic map between say $\mathbb{R}^d$ and an embedded Riemanian manifold (M, g) is a critical point of the Dirichlet energy

$$\int_{\mathbb{R}^d} |\nabla u(x)|^2 \, dx$$

for maps $u : \mathbb{R}^d \to M$, and hence a solution to

$$\mathbb{P}_{T_u M}(\Delta u) = 0$$

where $\mathbb{P}_{T_u M}$ is the projection onto the tangent space to M at u. The associated gradient flow is the parabolic heat flow evolution equation

$$\partial_t u = \mathbb{P}_{T_u M} \Delta u, \;\; (t, x) \in \mathbb{R} \times \mathbb{R}^d, \;\; u(t, x) \in M,$$

and the dispersive Hamiltonian version is the Schrödinger map problem

$$J \partial_t u = \mathbb{P}_{T_u M} \Delta u, \;\; (t, x) \in \mathbb{R} \times \mathbb{R}^d, \;\; u(t, x) \in M$$

where J acts like a $\frac{\pi}{2}$ rotation matrix in $T_u M$. In the special case $d = 2$ with the $\mathbb{S}^2$ sphere target, these equations become the energy critical harmonic heat flow to the 2-sphere

$$\text{(HHF)} \quad \begin{cases} \partial_t v = \Delta v + |\nabla v|^2 v \\ v_{|t=0} = v_0 \end{cases} \quad (t, x) \in \mathbb{R} \times \mathbb{R}^2, \;\; v(t, x) \in \mathbb{S}^2 \tag{4.1}$$

and the energy critical Schrödinger map problem

$$\text{(Smap)} \quad \begin{cases} v \wedge \partial_t v = \Delta v + |\nabla v|^2 v, \\ v_{|t=0} = v_0 \end{cases} \quad (t, x) \in \mathbb{R} \times \mathbb{R}^2, \;\; v(t, x) \in \mathbb{S}^2 \tag{4.2}$$

which are the building blocks of the Landau-Lifschitz model of ferromagnetism. The wave analogue is the energy critical wave map problem

$$\text{(WM)} \quad \begin{cases} \partial_{tt} v - \Delta v = (|\nabla v|^2 - |\partial_t v|^2) v, \\ v_{|t=0} = v_0, \;\; \partial_t v_{|t=0} = v_1 \end{cases} \quad (t, x) \in \mathbb{R} \times \mathbb{R}^2, \;\; v(t, x) \in \mathbb{S}^2. \tag{4.3}$$

All these problems are *energy critical* since the scaling symmetry $u(\lambda^i t, \lambda x)$ with $i \in \{1, 2\}$ leaves the Dirichlet energy invariant. Hence the Cauchy theory is critical in the energy space $\dot{H}^1$ and the a priori control of the Dirichlet energy is no longer enough to ensure the global existence of the flow: the bounded kinetic energy may *concentrate*.

4.2. Global existence and harmonic maps. In the parabolic case thanks to dissipation, the pioneering works by Struwe [71] ensures that if a singularity formation occurs, then the solution must bubble off a non trivial harmonic map on a sequence in time:

$$u(t_n\lambda(t_n)x) \to Q \quad \text{as} \quad t_n \to T$$

locally in space, where Q is a non trivial harmonic map. If the target manifold is such that it does not admit any non trivial harmonic map, as is typically the case for a negatively curved target, it follows that the flow is global. This strategy of proof of global existence was propagated to the dispersive wave map case in the presence of symmetries in [8], and finally in the general case by Sterbenz, Tataru in [73, 74] in the continuity of the Kenig-Merle minimal element approach [24].

This series of outstanding results yields global existence in the absence of non trivial harmonic maps, but this is typically not the case for the sphere target $\mathbb{S}^2$ where non trivial harmonic maps exist. The simplest way to produce such maps is to look for so called corotational maps

$$u(r,\theta) = \left| \begin{array}{l} \sin\phi(r)\cos(k\theta) \\ \sin\phi(r)\sin(k\theta) \\ \cos\phi(r) \end{array} \right.$$

which map the origin to the north pole and infinity to the south pole

$$\phi(0) = 0, \quad \lim_{r\to+\infty} \phi(r) = \pi.$$

The homotopy number $k \in \mathbb{N}^*$ measures the non trivial topological degree of the map. In a given homotopy class of corotational functions, the infimum of the Dirichlet energy is attained on the ground state harmonic map

$$Q_k(r) = 2\tan^{-1}(r^k).$$

The existence of blow up solutions in the vicinity of the Q_k harmonic map and the description of the associated singularity formation have been the subject of intense investigations for the past thirty years, with a series of breakthrough results in the past ten years. The difficulty is that there is little a priori information on the flow, and hence the need for new tools to compute explicitly the solution. We collect in the following the main results for each model.

4.3. Blow up for the energy critical wave map. Singularity formation for the wave map problem (4.3) has been mostly studied in the setting of corotational symmetry which is preserved by the flow: the solution is of the form

$$u(t,r,\theta) = \left| \begin{array}{l} \sin\phi(t,r)\cos(k\theta) \\ \sin\phi(t,r)\sin(k\theta) \\ \cos\phi(t,r) \end{array} \right. \tag{4.4}$$

where $\phi(t,r)$ satisfies the semilinear energy critical wave equation

$$\partial_{tt}\phi - \partial_r^2\phi - \frac{\partial_r\phi}{r} + \frac{k^2\sin(2\phi)}{r^2} = 0. \tag{4.5}$$

The first result of singularity formation for the $k = 1$ case is due to Krieger, Schlag, Tataru [29] who construct a *continuum* of blow up bubbles.

Theorem 4.1 (Continuums of blow up bubble for $k = 1$, [29]). *Let $k = 1$, there exist corotational initial data arbitrarily close to the ground state harmonic map Q_1 such that the corresponding solution to* (4.5) *concentrates in finite time*

$$u(t,r) - Q\left(\frac{r}{\lambda(t)}\right) \to u^* \ \text{ in } \ \dot{H}^1 \ \text{ as } \ t \to T$$

at the speed

$$\lambda(t) = (T-t)^{\nu}, \ \ \nu > 1. \tag{4.6}$$

The Krieger-Schlag-Tataru solutions are conjectured to correspond to non generic blow up bubbles. The first stable blow up regime was constructed by Rodnianski, Sterbenz [67] for large homotopy number $k \geq 4$ with a partial description of the associated blow up bubble. We revisited in [61] the blow up analysis and obtained for all homotopy number $k \geq 1$ the sharp description of the stable blow up scenario.

Theorem 4.2 (Stable blow up for all $k \geq 1$, [61]). *Let $k \geq 1$. There exists a set of $\mathcal{C}^\infty$ initial data arbitrarily close to Q_k in the energy topology such that the corresponding solution to* (4.5) *blows up in finite time*

$$u(t,r) - Q\left(\frac{r}{\lambda(t)}\right) \to u^* \ \text{ in } \ \dot{H}^1 \ \text{ as } \ t \to T$$

at the quantized rates:

$$\lambda(t) = \begin{cases} c_k(1+o(1))\frac{T-t}{|\log(T-t)|^{\frac{1}{2k-2}}} \text{for } \ k \geq 2, \\ (T-t)e^{-\sqrt{|\log(T-t)|}+O(1)} \text{for } \ k = 1 \end{cases} \quad \text{as } \ t \to T. \tag{4.7}$$

Moreover, the associated singularity formation is stable by smooth corotational perturbation of the data.

Theorem 4.2 builds a deep unexpected connection between the energy critical wave problem map and the (NLS) mass critical problem. The sharp logarithmic corrections in the quantized laws (4.7) are directly connected to the log-log correction to self similarity (2.9). A new systematic approach is implemented for the computation of the blow up speed which completely avoids previous approaches based on "matched asymptotics", [2], and here the far out polynomial decay of the ground state ∇Q_k -which is exponential for $s_c < 1$- is a new essential feature of the analysis. Finally, a robust mixed Energy/Morawetz method for the control of the flow is implemented which draws a route map for the construction of blow up bubbles.

4.4. Blow up for the Schrödinger map. The question of the singularity formation for the Schrödinger map has been a long standing open problem. There are two additional difficulties in this case. First of all, the "phase" invariance of the (NLS) equation adds a degree of freedom by rotation and the cororational structure (4.4) is no longer preserved by the flow. The smallest symmetry group corresponds to equivariant flows

$$u(t,x) = e^{k\theta R} \left| \begin{array}{l} u_1(t,r) \\ u_2(t,r) \\ u_3(t,r), \end{array} \right. \quad R = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which still generate a system for the radially symmetric unit vector (u_1, u_2, u_3), and not a scalar equation like (4.5). Moreover, the equation displays a *quasilinear* structure through the non linear $u \wedge \partial_t u$ term which complicates the derivation of energy estimates. A pioneering spectacular result in the field is obtained by Guan, Gustaffson, Nakanishi, Tsai [18, 19] and is a negative answer to singularity formation for large homotopy number.

Theorem 4.3 (Global existence for the Schrödinger map $k \geq 3$, [18, 19]). *Let $k \geq 3$. Let u_0 be a smooth equivariant initial data close enough to Q_k in the energy critical topology. Then the corresponding solution to* (4.2) *is global and asymptotically attracted up to symmetries by another ground state harmonic map.*

In other words, the harmonic map is orbitally asymptotically stable in the energy critical topology. We prove in [51] that the situation is very different for $k = 1$ and obtain the first result of singularity formation for this problem.

Theorem 4.4 (Blow up for the Schrödinger map, [51]). *Let $k = 1$. There exists a codimension one manifold of $\mathcal{C}^\infty$ initial data arbitrarily close to Q_1 in the energy critical topology such that the corresponding solution to* (4.2) *blows up in finite time*

$$u(t,x) - e^{\Theta(t)R} Q_1\left(\frac{x}{\lambda(t)}\right) \to u^* \ \text{ in } \ \dot{H}^1 \ \text{ as } \ t \to T \tag{4.8}$$

at the quantized rate

$$\lambda(t) = \frac{T-t}{|\log(T-t)|^2}\left(c_{u_0} + o_{t\to T}(1)\right), \ \ c_{u_0} > 0, \tag{4.9}$$

and for an asymptotically converging rotation:

$$\Theta(t) = \Theta(u_0) + o_{t\to T}(1). \tag{4.10}$$

The proof exhibits the dynamical system driving the concentration $\lambda(t)$ and the phase rotation $\Theta(t)$, and here a new unexpected instability occurs: generically, an initial data which starts concentrating will start rotating, and the energy will be transferred from concentration to rotation hence acting against blow up. However this asymptotic rotation around the e_z axis can happen towards the left or the right, and these two dynamics are separated by a threshold regime where the solution exactly concentrates as it stops rotating: this is the solution of Theorem 4.4. This instability by rotation will also occur for the wave map and the heat flow for generic equivariant perturbations. This shows that for the full system, blow up, at least near the ground state harmonic map, is never stable, a typical vectorial effect.

4.5. Blow up for the harmonic heat flow. The parabolic heat flow in corotational symmetry (4.4) reduces to the semi linear radially symmetric heat equation:

$$\partial_t \phi - \partial_r^2 \phi - \frac{\partial_r \phi}{r} + \frac{k^2 \sin(2\phi)}{r^2} = 0. \tag{4.11}$$

Again, all solutions are global in time for $k \geq 2$ with asymptotic stability of harmonic map for $k \geq 3$, [18, 19, 72], and the existence of blow up solutions for $k = 1$ was known using maximum principle techniques [7]. However, the sharp description of the blow up bubble was still mostly open, [1]. We construct in [62, 63] the *quantized family* of blow up bubbles as conjectured in [3].

Theorem 4.5 (Quantized blow up bubbles for the harmonic heat flow, [62, 63]). *Let $k = 1$. Then for all $L \in \mathbb{N}^*$, there exists a codimension $L-1$ manifold of corotational initial data $u_0(r)$ arbitrarily close to Q_1 in the energy critical topology such that the corresponding solution to* (4.11) *blows up in finite time $T > 0$ by bubbling off a harmonic map:*

$$u(t,r) - Q_1\left(\frac{r}{\lambda(t)}\right) \to u^* \text{ in } \dot{H}^1 \text{ as } t \to T \tag{4.12}$$

at the excited rate:

$$\lambda(t) = (c_{u_0}) + o_{t\to T}(1))\frac{(T-t)^L}{|\log(T-t)|^{\frac{2L}{2L-1}}}, \ c_{u_0} > 0. \tag{4.13}$$

In particular, the fundamental regime $L = 1$ is stable by smooth corotational perturbation of the data, and higher modes posses $L-1$ direction of instabilities. The computation of the full quantized family of blow up rates is a success of the "tail computation" initiated in [61] in the presence of polynomially decaying solitons and relies on the derivation of the full dynamical system driving the evolution of each mode. In principle, since the proof is very robust and relies solely on energy type estimates, the analysis can be propagated to the wave and Schrödinger map to produce the full family of quantized blow up bubbles. This shades a new light on the difference between the Krieger, Schlag, Tataru solutions with the continuum (4.6) and the quantized rates (4.7), (4.9), (4.13): the data leading to (4.6) have limited regularity and propagate a singularity on the light cone, see also [58], while the speeds (4.7), (4.9), (4.13) can be achieved for arbitrarily smooth data, which is of course essential to address the parabolic problem.

We conjecture that (4.13) describes the full family of corotational blow up solutions near the ground state Q_1 *in a suitable smooth topology*, and similarily for the wave and Schrödinger map.

5. On the energy super critical problem $s_c > 1$

The energy super critical problem $s_c > 1$ for (NLS) is the frontier of current research. Despite some important efforts, even the global existence of the flow in the defocusing case is open. This zone is of particular interest since most problems from fluid mechanics live there. Moreover any reasonable smooth nonlinearity in large dimension typically generates an energy super critical nonlinearity. But there are so far very few qualitative descriptions of truly nonlinear regimes in this range of exponents.

5.1. Energy super critical radial heat equation. The only problem which is truly better understood is the radially symmetric super critical heat equation

$$(NLH) \quad \begin{cases} \partial_t u = \Delta u + u|u|^{p-1}, \\ u_{|t=0} = u_0 \end{cases} \qquad s_c > 1$$

for which the maximum principle is available and yields for example for free global existence in the defocusing case. There is an immense literature on the study of singularity formation for the heat equation, but most works deal with the energy subcritical case $s_c < 1$. In the

energy super critical case, two types of blow solutions are known. First "type I" blow up solutions which correspond to the ODE blow up

$$\|u(t,\cdot)\|_{L^\infty} \sim \frac{c_u}{(T-t)^{\frac{1}{p-1}}}$$

in accordance with the exact blow up solution to $\dot{u} = u|u|^{p-1}$. Now for $d \geq 11$ and

$$p > p_{JL}(d) = 1 + \frac{4}{d-4-2\sqrt{d-1}}$$

the so called Joseph-Lundgren exponent, there is a change of nature in the problem and another kind of "type II" blow up solutions is predicted in the seminal work by Herrero-Velasquez [20] for which

$$\lim_{t\uparrow T} (T-t)^{\frac{1}{p-1}} \|u(t,\cdot)\|_{L^\infty} = +\infty. \tag{5.1}$$

The situation is clarified in the breakthrough works by Matano and Merle [39], [40]:

Theorem 5.1 (Universality of type I blow up, [39, 40]). *Let $s_c > 1$. For $p < p_{JL}$, all radially symmetric finite time time blow up solutions to (NLH) are of type I. For $p > p_{JL}$, there exist type II blow up solutions which are threshold dynamics.*

Using the Matano-Merle machinery, Mizoguchi [41] could revisit the Herrero-Velasquez construction and completely classify the structure of type II blow up bubbles.

Theorem 5.2 (Classification of type II blow up, [41]). *Let $d \geq 11$ and $p > p(d) > p_{JL}$. Then there exists a discrete sequence $(\mu_j)_{j\geq 1}$ with*

$$\mu_j > 0, \quad \lim_{j\to+\infty} \mu_j = +\infty \tag{5.2}$$

such that for a large class of radially symmetric initial data, if the solution blows up in finite time with type II (5.1)*, then*

$$\|u(t,\cdot)\|_{L^\infty} \sim \frac{c_u}{(T-t)^{\frac{1}{p-1}(1+\mu_j)}} \text{ for some } j \geq 1, \ c_u > 0.$$

This sequence of results is extraordinary by its strength and generality, but the techniques are absolutely based on Lyapounov functionals attached to the maximum principle which would completely break down for the non radial case or any dispersive model.

5.2. Type II blow up for the energy super critical (NLS) equation. In the recent work [52], we completely revisit the construction of type II blow up solutions of [20, 41] and show that the strategy developed for the study of the energy critical problems admits an unexpected extension to the energy super critical setting. The solitary wave is here the stationary solution $u(t,x) = Q(|x|)$ where Q is the unique radially symmetric solution to

$$\begin{cases} Q'' + (d-1)\frac{Q'}{r} + Q^p = 0 \\ Q'(0) = 0, \ \ Q(0) = 1 \end{cases},$$

Theorem 5.3 (Type II blow up for the energy super critical NLS, [52]). *Let $d \geq 11$, and let $(\mu_j)_{j\geq 1}$ be the sequence* (5.2)*. Pick a generic odd integer $p > p_{JL}$ and an integer $j \geq 1$. Then there exists a set of $\mathcal{C}_c^\infty$ radially symmetric initial data such that the corresponding solution to (NLS) blows up by concentrating a universal bubble*

$$u(t,r) = \frac{1}{\lambda(t)^{\frac{2}{p-1}}}(Q+\varepsilon)\left(\frac{r}{\lambda(t)}\right)e^{i\gamma(t)}$$

at the quantized rate

$$\lambda(t) = (c_{u_0} + o_{t\to T}(1))(T-t)^{\frac{1+\mu_j}{2}}, \ \ c_{u_0} > 0 \tag{5.3}$$

and for an asymptotically frozen phase $\lim_{t\to T}\gamma(t) = \gamma(T) \in \mathbb{R}$. The blow up is type II:

$$\limsup_{t\uparrow T}\|u(t)\|_{\dot{H}^s} \begin{cases} < +\infty \ \textit{for all} \ \ s < s_c, \\ = +\infty \ \textit{for} \ \ s \geq s_c. \end{cases} \tag{5.4}$$

The proof of Theorem 5.3 pushes the energy critical technology to its limit in order to deal to deal with the very different behavior of norms (5.4) above and below scaling, but still relies solely on robust mixed energy/Morawetz estimates. The derivation of the quantized family of excited blow up regimes initiated in [63] is an essential step. The analogous result for the wave equation is obtained in the companion paper [9]. This works open a breach into the super critical world which is the frontier of current research.

5.3. Perspectives. The understanding of the formation of blow up bubbles has considerably evolved in the last fifteen years and there is now a fairly robust method for the construction of such bubbles at least for relatively simple canonical models like (NLS). A first important direction of research is the derivation of classification results i.e. the understanding of all possible blow up bubbles. The most advanced result in this direction is the classification of bounded solutions for the energy critical wave equation by Duyckaerts, Kenig and Merle [12, 44] which together with the complete understanding of the quantized blow up bubbles as in Theorems 4.5 and 5.3 clearly opens up new perspectives to completely classify all blow up dynamics. A second axis of research is related to the question of continuation after blow up. (NLS) equations are typically derived under assumptions which are violated when the solution becomes large due to singularity formation, and new effects should be taken into account which arrest blow up. But the understanding of the corresponding regimes requires first a good understanding of the structure of the forming blow up bubble which is the new regime to be perturbed. Here solutions become typically highly oscillatory and sometimes turbulent, and there are many fascinating conjectures on the structure of the associated non linear flows, [70]. A third axis of research is the investigation of more complicated and more realistic systems, in particular in connections with plasmas physics (Zakharov's equations, Maxwell's equations, ...) and of course fluid mechanics. Here the universality and the robustness of our approach which applies to all regimes (critical and super critical) and relies solely on energy estimates gives us hope for a better understanding of this new generation of models which until very recently were essentially out of reach.

References

[1] Angenent, S.B., Hulshof, J., and Matano, H., *The radius of vanishing bubbles in equivariant harmonic map flow from D^2 to $\mathbb{S}^2$*, SIAM J. Math. Anal. **41**, no 3, 1121–1137.

[2] Bizoń, P., Ovchinnikov, Yu. N., and Sigal, I. M., *Collapse of an instanton*, Nonlinearity **17** (2004), no. 4, 1179–1191.

[3] Van den Berg, G.J.B., Hulshof, J., and King, J., *Formal asymptotics of bubbling in the harmonic map heat flow*, SIAM J. Appl. Math. **63**, no 5, 1682–1717.

[4] Boulenger, T., *Ph.D Thesis*, Paris XI (2012).

[5] Bourgain, J. and Wang, W., *Construction of blowup solutions for the nonlinear Schrödinger equation with critical nonlinearity*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **25** (1997), no. 1-2, 197–215.

[6] Cazenave, Th., *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics **10** (2003).

[7] Chang, K-C., Ding, W.Y., and Ye, R., *Finite-time blow-up of the heat flow of harmonic maps from surfaces*, J. Differential Geom. **36** (1992), 507–515.

[8] Christodoulou, D. and Tahvildar-Zadeh, A. S., *On the regularity of spherically symmetric wave maps*, Comm. Pure Appl. Math. **46** (1993), no. 7, 1041–1091.

[9] Collot, C., *Type II blow up for the energy super critical wave equation*, preprint 2014.

[10] Dodson, B., *Global well-posedness and scattering for the defocusing, L^2-critical nonlinear Schrd̈inger equation when $d \geq 3$*, J. Amer. Math. Soc. **25** (2012), no. 2, 429–463.

[11] Duyckaerts, T. and Merle, F., *Dynamic of threshold solutions for energy-critical NLS*, Geom. Funct. Anal. **18** (2009), no. 6, 1787–1840.

[12] Duyckaerts, T., Kenig, C.E., and Merle, F., *Profiles of bounded radial solutions of the focusing, energy-critical wave equation*, Geom. Funct. Anal. **22** (2012), no. 3, 639–698.

[13] Escauriaza, L., Seregin, G.A., and Svverak, V., *$L_{3,\infty}$-solutions of Navier-Stokes equations and backward uniqueness*, Uspekhi Mat. Nauk **58** (2003), no. 2(350), 3–44; translation in Russian Math. Surveys **58** (2003), no. 2, 211–250.

[14] Fibich, G., Gavish, N., and Wang, X-P., *Singular ring solutions of critical and supercritical nonlinear Schrödinger equations*, Phys. D **231** (2007), no. 1, 55–86.

[15] Gidas, B., Ni, W.M., and Nirenberg, L., *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), 209–243.

[16] Ginibre, J. and Velo, G., *On a class of nonlinear Schrödinger equations. I. The Cauchy problem, general case*, J. Funct. Anal. **32** (1979), no. 1, 1–32.

[17] Glassey, R. T., *On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equations*, J. Math. Phys. **18** (1977), no. 9, 1794–1797.

[18] Guan, M., Gustafson, S., and Tsai, T-P, *Global existence and blow-up for harmonic map heat flow*, J. Differential Equations **246** (2009), no. 1, 1–20.

[19] Gustafson, S., Nakanishi, K., and Tsai, T-P.; *Asymptotic stability, concentration and oscillations in harmonic map heat flow, Landau Lifschitz and Schrödinger maps on* $\mathbb{R}^2$, Comm. Math. Phys. **300** (2010), no 1, 205–242.

[20] Herrero, M.A. and Velázquez, J.J.L., *Explosion de solutions des equations paraboliques semilinéaires supercritiques*, C. R. Acad. Sci. Paris **319** (1994), 141–145.

[21] Holmer, J. and Roudenko, S., *A class of solutions to the 3D cubic nonlinear Schrödinger equation that blows up on a circle*, Appl. Math. Res. Express. (2011), no. 1, 23–94.

[22] Killip, R., Stovall, B., and Visan, M., *Blowup behaviour for the nonlinear Klein-Gordon equation*, Math. Ann. **358** (2014), no. 1-2, 289–350.

[23] Kenig C.E. and Merle, F., *Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case*, Invent. Math. **166** (2006), 645–675.

[24] Kenig, C.E. and Merle, F., *Global well-posedness, scattering and blow-up for the energy-critical focusing non-linear wave equation*, Acta Math. **201** (2008), no. 2, 147–212.

[25] C.E. Kenig, G. Ponce, and L. Vega, *On the concentration of blow up solutions for the generalized KdV equation critical in* L^2, Contemp. Math., **263**, Amer. Math. Soc., Providence, RI, 2000.

[26] Killip, R. and Visan, M., *The focusing energy-critical nonlinear Schrödinger equation in dimensions five and higher*, Amer. J. Math. **132** (2010), no. 2, 361–424.

[27] Krieger, J., Lenzmann, E., and Raphaël, P., *Nondispersive solutions to the* L^2*-critical Half-Wave Equation*, Arch. Ration. Mech. Anal. **209** (2013), no. 1, 61–129.

[28] Krieger, J. and Schlag, W., *Non-generic blow-up solutions for the critical focusing NLS in 1-D*, J. Eur. Math. Soc. **11** (2009), no. 1, 1–125.

[29] Krieger, J., Schlag, W., and Tataru, D., *Renormalization and blow up for charge one equivariant critical wave maps*, Invent. Math. **171** (2008), no. 3, 543–615.

[30] Korteweg, D.J. and de Vries, G., *On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves*, Philos. Mag. **539** (1895), 422–443.

[31] Kwong, M. K., *Uniqueness of positive solutions of* $\Delta u - u + u^p = 0$ *in* R^n, Arch. Rational Mech. Anal. **105** (1989), no. 3, 243–266.

[32] Landman, M. J., Papanicolaou, G. C., Sulem, C., and Sulem, P.-L., *Rate of blowup for solutions of the nonlinear Schrödinger equation at critical dimension*, Phys. Rev. A (3) **38** (1988), no. 8, 3837–3843.

[33] Leray, J., *Sur le mouvement d'un liquide visqueux emplissant l'espace*, Acta. Math. **63** (1934), 193–248.

[34] Martel, Y. and Merle, F., *Stability of blow-up profile and lower bounds for blow-up rate for the critical generalized KdV equation*, Ann. of Math. (2) **155** (2002), 235–280.

[35] _____, *Blow up in finite time and dynamics of blow up solutions for the L^2-critical generalized KdV equation*, J. Amer. Math. Soc. **15** (2002), 617–664.

[36] Martel, Y., Merle, F., and Raphaël, P., *Blow up for the critical gKdV equation I: dynamics near the solitary wave*, to appear in Acta. Math.

[37] _____, *Blow up for the critical gKdV equation II: minimal mass blow up*, peprint 2013, submitted.

[38] _____, *Blow up for the critical gKdV equation III: exotic regimes*, to appear in Ann. Scuola. Norm.

[39] Matano, H. and Merle, F., *Classification of type I and type II behaviors for a supercritical nonlinear heat equation*, J. Funct. Anal. **256** (2009), no. 4, 992–1064.

[40] _____, *On nonexistence of type II blowup for a supercritical nonlinear heat equation*, Comm. Pure Appl. Math. **57** (2004), no. 11, 1494–1541.

[41] Mizoguchi, N., *Rate of type II blowup for a semilinear heat equation*, Math. Ann. **339** (2007), no. 4, 839–877.

[42] Merle, F., *Existence of blow-up solutions in the energy space for the critical generalized KdV equation*, J. Amer. Math. Soc. **14** (2001), 555–578.

[43] _____, *On uniqueness and continuation properties after blow-up time of self-similar solutions of nonlinear Schrödinger equation with critical exponent and critical mass*, Comm. Pure Appl. Math. **45** (1992), 203–254.

[44] _____, *Asymptotics for critical Nonlinear Dispersive Equations*, ICM Proceddings Seoul 2014.

[45] Merle, F. and Raphaël, P., *Blow up dynamic and upper bound on the blow up rate for critical nonlinear Schrödinger equation*, Ann. Math. **161** (2005), no. 1, 157–222.

[46] _____, *Sharp upper bound on the blow-up rate for the critical nonlinear Schrödinger equation*, Geom. Funct. Anal. **13** (2003), no. 3, 591–642.

[47] _____, *On universality of blow up profile for L^2 critical nonlinear Schrödinger equation*, Invent. Math. **156** (2004), 565–672.

[48] _____, *Sharp lower bound on the blow up rate for critical nonlinear Schrödinger equation*, J. Amer. Math. Soc. **19** (2006), no. 1, 37–90.

[49] _____, *Profiles and quantization of the blow up mass for critical nonlinear Schrödinger equation*, Comm. Math. Phys. **253** (2005), no. 3, 675–704.

[50] _____, *Lower bound on the blow up rate of the critical norm for some radial L^2 super critical NLS equations*, Amer. J. Math. **130** (2008), no. 4, 945–978.

[51] Merle, F., Raphaël, P., and Rodnianski, I., *Blow up dynamics for smooth equivariant solutions to the energy critical Schrödinger map*, Invent. Math. **193** (2013), no. 2, 249–365.

[52] ———, *Type II blow up for the energy supercritical NLS*, preprint 2014.

[53] Merle,F., Raphaël, P., and Szeftel, J., *Stable self similar blow up dynamics for slightly L^2 supercritical NLS equations*, Geom. Funct. Anal. **20** (2010), no. 4, 1028–1071.

[54] ———, *Collapsing ring blow up solutions to the L^2 super critical NLS*, to appear in Duke Math Jour.

[55] ———, *Instability of Bourgain Wang solutions for the L^2 critical NLS*, Amer. J. Math. **135** (2013), no. 4, 967–1017.

[56] Nakanishi, K. and Schlag, W., *Invariant manifolds and dispersive Hamiltonian evolution equations*, Zurich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich, 2011.

[57] Perelman, Ga., *On the blow-up phenomenon for the critical nonlinear Schrödinger equation in 1D*, Ann. Henri Poincare **2** (2001), 605–673.

[58] Perelman, G., *Blow up dynamics for equivariant critical Schrödinger maps*, arXiv: 1212.6709.

[59] Raphaël, P., *Stability of the log-log bound for blow up solutions to the critical non linear Schrödinger equation*, Math. Ann. **331** (2005), no. 3, 577–609.

[60] ———, *Existence and stability of a solution blowing up on a sphere for a L^2 supercritical nonlinear Schrödinger equation*, Duke Math. J. **134** (2006), no. 2, 199–258.

[61] Raphaël, P. and Rodnianski, I., *Stable blow up dynamics for critical corotational wave maps and the equivariant Yang Mills problems*, Publ. Math. Inst. Hautes Etudes Sci. **115** (2012), 1–122.

[62] Raphaël, P. and Schweyer, R., *Stable blow up dynamics for the 1-corotational energy critical harmonic heat flow*, Comm. Pure Appl. Math. **66** (2013), no. 3, 414–480.

[63] ———, *Quantized slow blow up dynamics for the corotational energy critical harmonic heat flow*, to appear in Anal PDE.

[64] Raphaël, P. and Szeftel, J., *Standing ring blow up solutions to the N-dimensional quintic nonlinear Schrödinger equation*, Comm. Math. Phys. **290** (2009), no. 3, 973–996.

[65] ———, *Existence and uniqueness of minimal blow up solutions to an inhomogeneous mass critical NLS equation*, J. Amer. Math. Soc. **24** (2011), no. 2, 471–546.

[66] Raphaël, P., *Concentration compacité à la Kenig-Merle*, séminaire Bourbaki (2011).

[67] Rodnianski, I. and Sterbenz, J., *On the formation of singularities in the critical $O(3)$ σ-model*, Ann. of Math. (2) **172** (2010), no. 1, 187–242.

[68] Sideris, T.C., *Formation of singularities in three-dimensional compressible fluids*, Comm. Math. Phys. **101** (1985), no. 4, 475–485.

[69] Strichartz, R., *Restriction Fourier transform of quadratic surfaces and decay of solutions of the wave equations*, Duke Math. Jour. **44** (1977), 705–714.

[70] Sulem, C. and Sulem, P.L., *The nonlinear Schrödinger equation. Self-focusing and wave collapse*, Applied Mathematical Sciences **139**, Springer-Verlag, New York, 1999.

[71] Struwe, M., *On the evolution of harmonic mappings of Riemannian surfaces*, Comment. Math. Helv. **60** (1985), no. 4, 558–581.

[72] ———, *The evolution of harmonic maps*, Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990), 1197–1203, Math. Soc. Japan, Tokyo, 1991.

[73] Sterbenz, J. and Tataru, D., *Regularity of wave-maps in dimension* $2+1$, Comm. Math. Phys. **298** (2010), no. 1, 231–264.

[74] ———, *Energy dispersed large data wave maps in* $2+1$ *dimensions*, Comm. Math. Phys. **298** (2010), no. 1, 139–230.

[75] Yang, Y., *Solitons in field theory and nonlinear analysis*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2001.

[76] Weinstein, M.I., *Nonlinear Schrödinger equations and sharp interpolation estimates*, Comm. Math. Phys. **87** (1983), 567–576.

[77] Zwiers, I., *Standing ring blowup solutions for cubic nonlinear Schrödinger equations*, Anal. PDE **4** (2011), no. 5, 677–727.

Laboratoire J.A Dieudonné, Université de Nice-Sophia Antipolis, 06000 Nice, France
Institut Universitaire de France, Université de Nice Sophia Antipolis, 06000 Nice, France
E-mail: praphael@unice.fr

Regularity estimates for parabolic integro-differential equations and applications

Luis Silvestre

Abstract. We review some regularity results for integro-differential equations, focusing on Hölder estimates for equations with rough kernels and their applications. We show that if we take advantage of the integral form of the equation, we can obtain simpler proofs than for second order equations. For the equations considered here, the Harnack inequality may not hold.

Mathematics Subject Classification (2010). Primary 45E99; Secondary 35K99.

Keywords. Nonlocal equations, regularity.

1. Introduction

We consider parabolic integro-differential equations of the following general form

$$u_t + b(t,x)\cdot\nabla u + \int_{\mathbb{R}^d} (u(x+y) - u(x) - y\cdot\nabla u(x)\,\chi_{B_1}(y))K(t,x,y)\,\mathrm{d}y = f. \quad (1.1)$$

The equations corresponds to generators of discontinuous Levy processes with drift, but without a diffusion part. The purpose of this article is to discuss the regularization effect of the integral part of the equation under general conditions on b and K. We present a Hölder regularity result for the solution u to the equation (1.1) for f bounded and under some conditions on b and K which are discussed in section 3. It is important that there is no smoothness assumption on the kernel values $K(t,x,y)$ with respect to the variables x and t.

Integro-differential equations are a natural, fractional order, generalization of classical parabolic equations. Indeed, second order parabolic equations arise as asymptotic limits of integral equations. In this respect, we can classify regularity estimates for integral equations in two types. A regularity estimate that is uniform in the order of the equation (provided only that it is bounded away from zero) can be passed to the limit to local equations and is thus a generalization of regularity estimates for second order parabolic equations (for example the results in [9, 10, 26] and [34]). The term *robust* is sometimes used for this type of estimates. In other types of regularity results, the estimates depend on the order of the equation and blow up as it approaches two. These estimates are the ones which take advantage of the non local integral structure of the equation. The loss of the estimates in the classical limit is certainly an undesirable quality. However, it is interesting to understand how the non local structure of the equation can be used to our advantage. In fact, this second type of estimates usually have simpler proofs even than their classical local counterparts.

Proceedings of the International Congress of Mathematicians, Seoul, 2014

In this article, we discuss several results in the area. We also show a proof of Hölder estimates for solutions to a rather general form of the equation (1.1). The proof we show gives an estimate which is **not** robust. Our methods here depend strongly on the integral structure of the equation. In exchange, we can analyze a large class of kernels K and still have a rather short and clean proof. The Hölder estimates are based on a version of the weak Harnack inequality (Theorem 5.1) which has interest in itself and is not true for second order parabolic equations in non divergence form.

The Hölder estimates are closely related with the Harnack inequality. For the equations we consider in this paper we prove the Hölder estimates but we show that the Harnack inequality does not hold.

Note that these estimates are a fractional order version of the classical theory for second order parabolic equations due to Krylov and Safonov [32]. In comparison, the proofs for second order equations are much more complicated than then proofs of the non-robust estimates for integral equations. Also, the Harnack inequality certainly holds in the second order case.

We review related results and a brief history of the subject in section 2. In the last section we give a quick summary of some of the main applications of the regularity estimates.

2. A review on regularity results

2.1. Classical results for second order equations. Integro-differential equations are a natural extension of second order equations of elliptic and parabolic type. There are two types of regularity results for second order equations, those which apply to equations in divergence form and those for equations in non divergence form.

The first result for elliptic and parabolic equations with rough coefficients and without any smallness condition is the classical result of De Giorgi, Nash and Moser which was obtained in the late 1950's. This result provides a Hölder estimate for equations of the form

$$u_t - \partial_i a_{ij}(x,t)\partial_j u \partial_j u = 0.$$

where the coefficients a_{ij} are assumed to satisfy the point-wise bounds

$$\lambda I \leq \{a_{ij}\} \leq \Lambda I. \tag{2.1}$$

No assumption is made in terms of the regularity of a_{ij} with respect to either x or t. The Harnack inequality also holds for this type of equations.

This classical result plays a crucial role in the regularity theory of solutions to equations in divergence form. It was the key to solve Hilbert's 19th problem. It is essential for all its applications that no regularity assumption on the coefficients is necessary.

The corresponding result for equations in non divergence form was obtained in 1979 by Krylov and Safonov [32, 33]. It applies to equations of the form

$$u_t - a_{ij}(x,t)\partial_{ij} u = 0,$$

with identical assumptions on the coefficients a_{ij}. Before the result of Krylov and Safonov, the only regularity results available for equations in non divergence form applied to either continuous coefficients a_{ij} or coefficients with small oscillation (i.e. $|a_{ij} - \delta_{ij}| < \varepsilon$). Note

that any of these extra assumptions would allow us to approximate the equation locally with an equation with constant coefficients. The result of Krylov and Safonov is more delicate because it deals with a different scale invariant class of equations.

The Hölder estimate and Harnack inequality by Krylov and Safonov have multiple applications. It is a central result in the study of regularity of solutions to fully non linear elliptic equations. These are equations of the form $F(D^2u) = 0$ where F is an arbitrary nonlinear function which satisfies $\lambda I \leq \partial F/\partial X_{ij} \leq \Lambda I$. The canonical examples of equations of this form come from the study of stochastic games and are the Bellman equation

$$u_t - \sup_r a_{ij}^r \partial_{ij} u = 0,$$

or the Isaacs equation

$$u_t - \inf_s \sup_r a_{ij}^{rs} \partial_{ij} u = 0.$$

Note that even tough the matrices a_{ij}^r or a_{ij}^{rs} in these equations may be independent of x and t, a different one is chosen at every point (x,t) and we have no a priori estimate on the optimal choice $a_{ij}(x,t)$ other than a quantitative point estimate like (2.1). That's why a result like the Hölder estimates by Krylov and Safonov guarantees some initial regularity for solutions to this kind of problems. More importantly, the derivative of the solution also satisfies an equation with (a priori) rough coefficients, which we can use to deduce that the solutions are $C^{1,\alpha}$ both in space and time. This is the best regularity currently known for the Isaacs equation and in fact it is known to be optimal in high dimensions [37, 38]. The solutions to the Isaacs equation in 2D are always $C^{2,\alpha}$ in space. It is still an outstanding open problem whether singular solutions exists in dimensions 3 or 4. For the Bellman equation, the solutions are always $C^{2,\alpha}$ in space, and therefore classical, due to the celebrated theorem proved independently by Evans [22] and Krylov [31].

2.2. Integro-differential equations. Just like in the second order case, the first regularity results that appear for integro-differential equations were for equations with a variational structure. They are a fractional order version of the classical results by De Giorgi Nash and Moser for second order equations in divergence form. See [1, 5, 8, 23, 26, 30] among others. In this article we will concentrate in results of non variational form. There is a good survey on Hölder estimates for divergence form integro-differential equations in [28].

The first Hölder estimate, together with a Harnack inequality, was obtained by Bass and Levin [4] for elliptic integro-differential equations of the form

$$\int_{\mathbb{R}^d} \left(u(x+y) - u(x) - y \cdot \nabla u(x) \chi_{B_1}(y)\right) K(x,y) \, \mathrm{d}y = 0,$$

assuming that $K(x,y) = K(x,-y)$ and

$$\frac{\lambda}{|y|^{d+\alpha}} \leq K(x,y) \leq \frac{\Lambda}{|y|^{d+\alpha}}, \tag{2.2}$$

for positive constants λ and Λ uniform in x and y.

Note that there is no regularity assumption of K with respect to x. The assumption (2.2) is a uniform ellipticity condition of order $\alpha \in (0,2)$ comparable to (2.1) for the second order case.

The result in [4] is obtained using probabilistic techniques and the estimates are not robust since the constants in the Hölder estimate are not uniformly bounded as $\alpha \to 2$.

Just as in the case of Krylov-Safonov theorem for second order equations, this result applies to a scale invariant class of equations. The assumption (2.1) which gives us a pointwise bound for the coefficients a_{ij} in the second order case can be reproduced in the integro-differential case by the assumption (2.2), but there are also many other alternatives. For each point x, the possible kernels are non negative functions (in terms of y) from the full space $\mathbb{R}^d$ into $\mathbb{R}$. This is naturally a much richer class than the coefficients of a second order equation, which is just a symmetric matrix in $\mathbb{R}^{d\times d}$.

The original result of Bass and Levin was extended to more general classes of equations in [2, 3, 41, 45]. The last one was the first one to use purely analytic methods instead of probabilistic techniques.

The first robust Hölder estimate and Harnack inequality appear in [9] for symmetric kernels satisfying the same condition (2.2). In this paper, the Hölder estimates were used to derive a $C^{1,\alpha}$ estimate for the non local Isaacs equation from stochastic games driven by Levy processes,

$$\inf_r \sup_s \int_{\mathbb{R}^d} (u(x+y) - u(x) - y\cdot\nabla u(x)\chi_{B_1}(y))\, K^{rs}(y)\, \mathrm{d}y = 0.$$

This result was extended to some Isaacs equations with variable coefficients in [11]. Moreover, the integro-differential Bellman equation

$$\inf_r \int_{\mathbb{R}^d} (u(x+y) - u(x) - y\cdot\nabla u(x)\chi_{B_1}(y))\, K^{r}(y)\, \mathrm{d}y = 0,$$

has classical solutions in the class $C^{\alpha+\varepsilon}$ if all kernels K satisfy (2.2) plus some smoothness condition with respect to y [10].

There is a number of variations of the results mentioned above, including, among others, generalizations to parabolic equations [35], to non symmetric kernels [13] and to kernels with logarithmic singularities [27].

As mentioned above, the condition (2.2) is a version of uniform ellipticity of fractional order. However, given that the possible choices of kernel measures are so rich, there are several possible alternative definitions of uniform ellipticity. It is not clear at the moment what the optimal condition should be, and there is probably never going to be one.

In [6] and [39], the lower bound condition in (2.2) was relaxed. The authors observed that it is enough to let it hold only in a cone of directions, which can change from point to point, but must have a uniform width. This is a remarkable improvement which shows that the previous condition (2.2) was far from optimal. It turns out that for this kind of integro-differential equations the Hölder estimates hold but the Harnack inequality does not. This is also quite remarkable, since the two properties are closely related to each other, and the Hölder estimates are often proved as a consequence of the Harnack inequality. Note that actually the Harnack inequality is claimed to be true in [6], but there are some issues in the proof and a counterexample was given in [39] using ideas from [7].

In this article we prove a version of the Hölder estimate for parabolic integro-differential equations. We keep the lower bound of (2.2) but we relax the upper bound. For this type of equations we show that the Harnack inequality does not hold either. We give a rather simple proof at the expense of making the estimate not robust (that is, the constants blow up as $\alpha \to 2$).

In a work in progress with Russell Schwab, we are working on a robust estimate for parabolic equations whose kernels satisfy the same upper bound condition as in this paper, but whose lower bound only holds in a cone of directions.

Note that a robust estimate like the one in [9] in particular implies the Krylov Safonov theorem about the regularity of uniformly elliptic second order equations. We can recognize the main ideas of the proof of Krylov and Safonov in the proof of the corresponding estimates for integral equations in [9]. In particular, there is some replacement for the Alexandrov-Bakelman-Pucci estimate. However, a perfect nonlocal analog for the ABP estimate is unknown. More precisely, the following is an open problem. Assume

$$\int_{\mathbb{R}^d} (u(x+y) - u(x) - y \cdot \nabla u(x) \chi_{B_1}(y))\, K(x,y)\, \mathrm{d}y = -\chi_A \qquad \text{in } B_1,$$
$$u(x) \leq 0 \qquad \text{for all } x \notin B_1$$

Is there an estimate of the form

$$\max_{B_1} u \leq \mu(|A|),$$

for any function μ so that $\mu(m) \to 0$ as $m \to 0$?

Here χ_A denotes the characteristic function of the set A.

In spite of its apparent simplicity, the estimate above has only been proved for a very special class of kernels [24].

3. A class of kernels

In this section, we describe a special set of assumptions that we make on the kernels for the result that we prove in this article.

The integro-differential equations that we study have an associated order $\alpha \in (0,2)$. The assumptions on the kernel K depend on this value. Typically, we will look at kernels K such that $K(t,x,y) \approx |y|^{-d-\alpha}$, but our assumptions are more general than that. We assume the following two inequalities.

$$K(t,x,y) \geq \frac{\lambda}{|y|^{d+\alpha}}, \tag{3.1}$$

$$\int_{\partial B_r} K(t,x,y)\, \mathrm{d}S(y) \leq \Lambda r^{-1-\alpha} \text{ for all } r > 0. \tag{3.2}$$

The estimates (3.1) and (3.2) also make sense for some singular jump measures instead of the absolutely continuous measure $K(t,x,y)\ \mathrm{d}y$. We stick to the absolutely continuous form only for the sake of clarity.

Note that (3.2) is more general than the usual assumption $K(t,x,y) \leq \Lambda |y|^{-d-\alpha}$. In particular (3.2) allows us to consider kernels containing singular measures. An extra factor $(2-\alpha)$ would be needed in the right hand side of both (3.1) and (3.2) in order to pass to the limit as $\alpha \to 2$ and recover uniformly parabolic equations in non divergence form. Since the estimates in this article are not uniform in α, it makes no difference to have this factor or not for the purposes of our results.

We do not assume that K is symmetric, (i.e. $K(t,x,y) \neq K(t,x,-y)$. Note that the purpose of the gradient term $y \cdot \nabla u(x)\chi_{B_1}(x)$ in the integrand in (1.1) is for the integral to be well defined around the origin. When K is symmetric, this term can be safely ignored by computing the integral in the principal value sense. For non symmetric kernels K, this term is necessary for the integral to make sense if $\alpha \geq 1$. The choice of the radius of the ball B_1 is arbitrary. If we replace B_1 with B_R for any other value of R, the difference of the integral operators would be absorbed by the gradient term. This ambiguity in the structure of the integral operator is inconvenient for the proofs in this article because it depends on scale. In our proofs we often rescale solutions of the equation and we need our assumptions to be invariant by this scaling. In order to avoid this ambiguity, we modify the structure of the equation depending on whether $\alpha < 1$ or $\alpha > 1$. We use the following notation

$$\delta_y u(x) := \begin{cases} u(x+y) - u(x) & \text{if } \alpha < (0,1), \\ u(x+y) - u(x) - y \cdot \nabla u(x)\chi_{B_1}(y) & \text{if } \alpha = 1, \\ u(x+y) - u(x) - y \cdot \nabla u(x) & \text{if } \alpha \in (1,2), \end{cases} \tag{3.3}$$

In any case, we study an equation of the form

$$u_t + b(t,x) \cdot \nabla u + \int_{\mathbb{R}^d} \delta_y u(x) K(t,x,y) \, \mathrm{d}y = f \tag{3.4}$$

The case $\alpha = 1$ is special. In this case we need to assume an additional symmetry assumption in the kernel K. Assuming $K(t,x,y) = K(t,x,-y)$ would be enough, but we make a mildly more general assumption

$$\int_{\partial B_r} yK(t,x,y) \, \mathrm{d}S(y) = 0 \text{ for every } r > 0. \tag{3.5}$$

Some symmetry assumption like (3.5) is required to make the proof using the current methods. See [14] for a more precise description of assumptions that work in the elliptic case with $\alpha = 1$.

These alternative structures for the non symmetric integral operators depending on the order α were stated in the work of Hongjie Dong and Doyoon Kim for elliptic nonlocal equations [19] and [20].

The drift term does not contribute to the regularization of the solution. We need to be able to control it with the integral part. We will assume that $b(t,x)$ is bounded if $\alpha \geq 1$ and that $b(t,\cdot)$ is uniformly bounded in $C^{1-\alpha}$ if $\alpha < 1$. For the right hand side f, we always assume it is a bounded function.

Note that the class of equations of the form (1.1) is no different from (3.4). Indeed, if $\alpha < 1$ then $yK(y)$ is integrable at the origin. Thus, the integral of $y \cdot \nabla u(x)\chi_{B_1}(y)$ is of the form $\tilde{b} \cdot \nabla u(x)$ and therefore it is absorbed into the first order term of (3.4). In the case $\alpha > 1$, $yK(y)$ is integrable at infinity. Thus, the integral of $y \cdot \nabla u(x)\chi_{\mathbb{R}^d \setminus B_1}(y)$ can also be absorbed in the first order term.

4. Extremal operators and viscosity solutions

Note that the equation (1.1) does not have a variational form and therefore we cannot define a weak solution in the sense of distributions. In order to make sense of the equation in the

viscosity sense, we define the extremal Pucci-like operators which represent the maximum and minimum possible value of the integral term in (3.4).

Definition 4.1. Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be bounded in $\mathbb{R}^d$ and $C^{1,1}$ at x. We define $M_\alpha^\pm \varphi$ in the following way.

$$M_\alpha^+ \varphi(x) = \sup \left\{ \int_{\mathbb{R}^d} \delta_y \varphi(x) K(y)\, \mathrm{d}y : \text{for all } K(y) \text{ satisfying (3.2) and (3.1), and also (3.5) if } \alpha = 1 \right\}$$

The extremal operator M_α^- is defined similarly exchanging the $\sup$ with an $\inf$.

We will omit the subscript α whenever this value is clear to avoid notation clutter.

The operators M_α^+ and M_α^- have an explicit formula which is given in the next lemma.

Lemma 4.2. *Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be bounded in $\mathbb{R}^d$ and $C^{1,1}$ at x. The operators M_α^+ and M_α^- have the following formula*

$$M_\alpha^+ \varphi(x) = \int_{\mathbb{R}^d} \delta_y \varphi(x) \frac{\lambda}{|y|^{d+\alpha}}\, \mathrm{d}y + \omega_d(\Lambda - \lambda) \int_0^\infty \left(\sup_{y \in \partial B_r} \delta_y \varphi(x)^+ \right) r^{-1-\alpha}\, \mathrm{d}r,$$

$$M_\alpha^- \varphi(x) = \int_{\mathbb{R}^d} \delta_y \varphi(x) \frac{\lambda}{|y|^{d+\alpha}}\, \mathrm{d}y + \omega_d(\lambda - \Lambda) \int_0^\infty \left(\sup_{y \in \partial B_r} \delta_y \varphi(x)^- \right) r^{-1-\alpha}\, \mathrm{d}r,$$

where ω_d represents the surface area of ∂B_1 in $\mathbb{R}^d$.

The proof of the lemma above is a straight forward interpretation of Definition 4.1

The following is a rather simple proposition stating essentially that $M_\alpha^\pm \varphi$ is well defined for $\varphi \in C^2$ and has some basic stability properties.

Proposition 4.3. *Assume φ is a continuous bounded function in $\mathbb{R}^d$ so that $\varphi \in C^2(B_2)$. Then $M_\alpha^+ \varphi$ and $M_\alpha^- \varphi$ are continuous in B_1. Moreover, if φ_k is a uniformly bounded sequence of such functions such that $\varphi_k \to \varphi$ locally uniformly in $\mathbb{R}^d$ and $\varphi_k \to \varphi$ in $C^2(B_2)$, then $M_\alpha^\pm \varphi_k \to M_\alpha^\pm \varphi$ uniformly in B_1.*

Proof. Let $K(y)$ be one kernel satisfying (3.1) and (3.2). Let Lu be the linear (translation invariant) operator

$$Lu(x) := \int_{\mathbb{R}^d} \delta_y u(x) K(y)\, \mathrm{d}y. \tag{4.1}$$

If $x \in B_1$, then we can estimate $\delta_y u(x)$ with respect to the norms of u. If $x \in B_1$ and $x + y \in B_{3/2}$, we have

$$|\delta_y u(x)| \le \begin{cases} |y| |\nabla u|_{L^\infty(B_2)} & \text{if } \alpha > 1, \\ |y|^2 |D^2 u|_{L^\infty(B_2)} \chi_{|y|<1} & \text{if } \alpha = 1 \text{ and } |y| < 1, \\ 2|u|_{L^\infty}(B_{3/2}) \chi_{|y|<1} & \text{if } \alpha = 1 \text{ and } |y| \ge 1, \\ |y|^2 |D^2 u|_{L^\infty(B_2)} & \text{if } \alpha > 1. \end{cases}$$

Also, if $x \in B_1$ and $x + y \notin B_{3/2}$,

$$|\delta_y u(x)| \leq \begin{cases} |u(x+y)| + |u|_{L^\infty(B_1)} & \text{if } \alpha > 1, \\ |u(x+y)| + |u|_{L^\infty(B_1)} + |y||\nabla u|_{L^\infty(B_1)}\chi_{|y|<1} & \text{if } \alpha = 1 \text{ and } |y| < 1, \\ |u(x+y)| + |u|_{L^\infty(B_1)} & \text{if } \alpha = 1 \text{ and } |y| \geq 1, \\ |u(x+y)| + |u|_{L^\infty(B_1)} + |y||\nabla u|_{L^\infty(B_1)} & \text{if } \alpha > 1. \end{cases}$$

All these inequalities, combined with the assumption (3.2) for $K(y)$ tell us that the expression in (4.1) is integrable and

$$|Lu(x)| \leq C\left(\|u\|_{C^2(B_2)} + \||x|^\beta u(x)\|_{L^\infty(\mathbb{R}^d)}\right). \tag{4.2}$$

Here β is any non negative number so that $\beta < \alpha$. The constant C depends on λ, Λ, d, α and β, but not on the particular choice of the kernel $K(y)$. In particular it also holds for the suppremum and infimum value of $Lu(x)$ for all admissible choices of K, and that is $M_\alpha^+ u(x)$ and $M_\alpha^- u(x)$.

From (4.2), we see that if φ_n is a uniformly bounded sequence so that $\varphi_n \to \varphi$ in $C^2(B_2)$ and locally uniformly in $\mathbb{R}^d$, then $L\varphi_n \to L\varphi$ uniformly in B_1 and at a rate independent of K. In particular $M_\alpha^\pm \varphi_n$ converges to $M_\alpha^\pm \varphi$ uniformly in B_1, which proves the second part of the proposition.

Assume now that $u \in C^3(B_2) \cap C^1(\mathbb{R}^d)$, since $\partial_i[Lu] = L[\partial_i u]$, we deduce from (4.2) that in this case Lu is Lipschitz continuous in B_1.

If φ is any bounded continuous function in $\mathbb{R}^d$ which is C^2 in B_2, we can approximate it with a bounded sequence $\varphi_n \in C^3(B_2) \cap C^1(\mathbb{R}^d)$, which converges to φ in $C^2(B_2)$ and locally uniformly in $\mathbb{R}^d$. Therefore, $L\varphi_n \to L\varphi$ uniformly in B_1, and therefore $L\varphi$ is continuous in B_1. □

Note that the fact that u solves an equation of the form (3.4), for some kernel K satisfying our assumptions, is equivalent to the fact that u satisfies the following two inequalities.

$$u_t + b \cdot \nabla u - M_\alpha^+ u \leq 0, \tag{4.3}$$

$$u_t + b \cdot \nabla u - M_\alpha^- u \geq 0. \tag{4.4}$$

Even though we defined M_α^+ and M_α^- using kernels which do not depend on t and x, ultimately the choice of kernel in $M_\alpha^\pm u(t,x)$ is different at every point. Thus, the equations (4.3) and (4.4) are equivalent to (3.4) without any regularity assumption of $K(t,x,y)$ with respect to t and x.

The advantage of the inequalities (4.3) and (4.4) with respect to the equation (3.4) is that the former can be defined in the viscosity sense.

Definition 4.4. Assume b and f are continuous in some domain $\Omega \subset \mathbb{R} \times \mathbb{R}^d$. We say that (4.3) (resp. (4.4)) holds in Ω if every time there is a function $\varphi : (t_0 - \varepsilon, t_0] \times B_\varepsilon(x_0)$ such that φ is left differentiable in time and point-wise second differentiable in space at (t_0, x_0) and

$$\begin{aligned} &\varphi(t_0, x_0) = u(t_0, x_0), \\ &\qquad \varphi \geq u \text{ in } (t_0 - \varepsilon, t_0] \times B_\varepsilon(x_0) \ \ (\text{resp. } \leq), \end{aligned}$$

Then, if we construct the function

$$v(t,x) = \begin{cases} \varphi(t,x) & \text{if } (t,x) \in (t-\varepsilon, t_0] \times B_\varepsilon(x_0), \\ u(t,x) & \text{otherwise,} \end{cases}$$

we get

$$v_t(t_0,x_0) + b(t_0,x_0)\cdot\nabla v(t_0,x_0) - M_\alpha^+ v(t_0,x_0) \le f(t_0,x_0).$$

(resp. M_α^- and $\ge$)

Note that M^+ and M^- satisfy the relation $M^+\varphi = -M^-(-\varphi)$. Moreover, u satisfies $u_t + b\cdot\nabla u - M^- u \ge f$ in the viscosity sense, if and only if $-u$ satisfies $(-u)_t + b\cdot\nabla(-u) - M^+(-u) \le -f$.

One of the keys in the study of regularity properties of second order equations in non divergence form lies in the difficulty to obtain estimates in integral form. This is achieved estimating the measure of some contact sets or through the Alexandrov-Bakelman-Pucci inequality. The following lemma is a simple property of viscosity solutions of the integral equations we consider in this paper. At the same time, it is a crucial ingredient in our regularity theory since it provides a simple integral quantity associated with every point which can be touched by one side with a smooth function.

Lemma 4.5. *Assume* (3.1) *holds in the viscosity sense. Let φ be a test function as in the Definition 4.4 such that $u(t_0, x_0+y) \ge \varphi(t_0, x_0+y)$ for all $y \in \mathbb{R}^d$. Then, we have*

$$\begin{aligned}\varphi_t(t_0,x_0) + b(t_0,x_0)\cdot\nabla\varphi(t_0,x_0) - M^-\varphi(t_0,x_0) \\ - \int_{\mathbb{R}^d} (u(t_0,x_0+y) - \varphi(t_0,x_0+y)) \frac{\lambda}{|y|^{d+\alpha}}\,\mathrm{d}y \ge f(t_0,x_0).\end{aligned}$$

Proof. Let v be the function constructed out of u, φ and some $\varepsilon > 0$ in Definition 4.4. Note that if $|y| > \varepsilon$ we have $\delta_y v(t_0,x_0) = (u(t_0,x_0+y) - \varphi(t_0,x_0+y)) + \delta_y\varphi(t_0,x_0)$. Moreover, $v_t(t_0,x_0) = \varphi_t(t_0,x_0)$ and $\nabla v(t_0,x_0) = \nabla\varphi(t_0,x_0)$. Therefore

$$\begin{aligned} f(t_0,x_0) &\le v_t(t_0,x_0) + b(t_0,x_0)\cdot\nabla v(t_0,x_0) - M_\alpha^+ v(t_0,x_0), \\ &= \varphi_t(t_0,x_0) + b(t_0,x_0)\cdot\nabla\varphi(t_0,x_0) - M^-\varphi(t_0,x_0) \\ &\qquad - \int_{\mathbb{R}^d\setminus B_\varepsilon} (u(t_0,x_0+y) - \varphi(t_0,x_0+y)) \frac{\lambda}{|y|^{d+\alpha}}\,\mathrm{d}y \end{aligned}$$

We finish the proof taking $\varepsilon \to 0$ and applying the monotone convergence theorem. □

Remark 4.6. Applying the previous result to $-u$, we can obtain a corresponding result for viscosity sub-solutions of (4.3).

5. The weak Harnack inequality

We use the following notation for parabolic cylinders

$$\begin{aligned} Q_r &:= (-r^\alpha, 0] \times B_r, \\ Q_r(t_0,x_0) &:= (t_0,x_0) + Q_r = (t_0 - r^\alpha, t_0] \times B_r(x_0). \end{aligned}$$

The following result is the weak Harnack inequality and is the key ingredient in the proof of the Hölder estimates. Its proof is inspired by a similar (but somewhat weaker) result for a particular case of (1.1) which appeared in [43]. Some of the ideas can be traced back to [41] for the elliptic case.

Theorem 5.1. *Let u be a function which satisfies the following inequality in the viscosity sense.*

$$u_t + \Lambda|\nabla u| - M^- u \geq -a \text{ in } Q_1,$$

Assume that $u \geq 0$ in Q_1, then

$$\min_{[-1/2,0]\times B_{1/2}} u \geq c \left(\int_{(-1,-1/2]\times\mathbb{R}^d} \frac{u(t,x)}{(1+|x|)^{d+\alpha}} \, \mathrm{d}x \, \mathrm{d}t \right) - a,$$

where c is a positive constant depending on λ, Λ, α and the dimension d.

Proof. Multiplying u by a scalar, we can assume that

$$\int_{(-1,-1/2]\times\mathbb{R}^d} \frac{u(t,x)}{(1+|x|)^{d+\alpha}} \, \mathrm{d}x \, \mathrm{d}t = 1. \tag{5.1}$$

We should then prove that there is a small constant c so that

$$\min_{[-1/2,0]\times B_{1/2}} u \geq c - a,$$

for any value of a. Of course this inequality is non trivial when $a < c$ only.

Let θ be the constant, depending on dimension and α only, so that

$$\int_{(-1,-1/2]\times\mathbb{R}^d} \frac{\theta}{(1+|x|)^{d+\alpha}} \, \mathrm{d}x \, \mathrm{d}t = 1/2.$$

From (5.1), we deduce that

$$\int_{(-1,-1/2]\times\mathbb{R}^d} \frac{(u(t,x)-\theta)^+}{(1+|x|)^{d+\alpha}} \, \mathrm{d}x \, \mathrm{d}t \geq 1/2.$$

Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be a smooth, non negative function supported in $B_{3/4}$ so that $\varphi \equiv 1$ in $B_{1/2}$. We will construct a bound from below of the form $m(t)\varphi(x) - (1+t)a$. The function m is the solution of the following ODE, for some positive constants c_0 and C_1.

$$m(-1) = 0, \tag{5.2}$$

$$m'(t) = c_0 \left(\int_{\mathbb{R}^d} \frac{(u(t,x)-\theta)^+}{(1+|x|)^{d+\alpha}} \, \mathrm{d}x \right) - C_1 m(t). \tag{5.3}$$

The ODE above has the explicit solution

$$\begin{aligned} m(t) &= c_0 \int_{-1}^{t} \int_{\mathbb{R}^d} \frac{(u(t,x)-\theta)^+}{(1+|x|)^{d+\alpha}} e^{C_1(s-t)} \, \mathrm{d}x \, \mathrm{d}s \\ &\geq \frac{c_0}{e^{C_1}} \iint_{[-1,-1/2]\times\mathbb{R}^d} \frac{(u(t,x)-\theta)^+}{(1+|x|)^{d+\alpha}} \, \mathrm{d}x \, \mathrm{d}t. \end{aligned}$$

Therefore, if we proved that $u(t,x) \geq m(t)\varphi(x) - (1+t)a$, we would finish the proof with $c = c_0 e^{-C_1}/2$. Let us assume the contrary and let $\varepsilon > 0$ be an arbitrarily small number. Since $m(-1) = 0$ and φ is supported in $B_{3/4}$, there exists a first crossing point (t_0, x_0) so that

$$\begin{aligned} u(t_0,x_0) &= m(t_0)\varphi(x_0) - (1+t_0)a - \varepsilon, \\ u(t,x) &\geq m(t)\varphi(x) - (1+t)a - \varepsilon \text{ for every } t < t_0 \text{ and } x \in \mathbb{R}^d. \end{aligned}$$

We observe that we can use $m(t)\varphi(x) - (1+t)a - \varepsilon$ as a test function for Definition 4.4 and from Lemma 4.5 we have

$$\begin{aligned} m'(t_0)\varphi(x_0) - a + \Lambda m(t_0)|\nabla\varphi(x_0)| - m(t_0)M^-\varphi(t_0,x_0) \\ - \int_{\mathbb{R}^d} (u(t_0,x_0+y) - \varphi(t_0,x_0+y)) \frac{\lambda}{|y|^{d+\alpha}}\,\mathrm{d}y \geq -a. \end{aligned}$$

The a terms cancel out and we get

$$\begin{aligned} m'(t_0)\varphi(x_0) + \Lambda m(t_0)|\nabla\varphi(x_0)| - m(t_0)M^-\varphi(t_0,x_0) \\ - \int_{\mathbb{R}^d} (u(t_0,x_0+y) - \varphi(t_0,x_0+y)) \frac{\lambda}{|y|^{d+\alpha}}\,\mathrm{d}y \geq 0. \end{aligned} \tag{5.4}$$

The last integral term is crucial to obtain the contradiction.

The first condition that we need in the choice of c_0 is that it must be chosen small enough so that $m(t) < \theta$ for all $t \in [-1,0]$. This is guaranteed simply by the condition $c_0 \leq \theta$. Under this condition, we obtain the following

$$\begin{aligned} &\int_{\mathbb{R}^d} (u(t_0,x_0+y) - \varphi(t_0,x_0+y)) \frac{\lambda}{|y|^{d+\alpha}}\,\mathrm{d}y \\ &\geq \int_{\mathbb{R}^d} (u(t,y) - \theta)^+ \left(\min_{x_0 \in B_{1/2}} \frac{\lambda}{|y-x_0|^{d+\alpha}} \right) \mathrm{d}y \\ &\geq c_0 \left(\int_{\mathbb{R}^d} \frac{(u(t,x)-\theta)^+}{(1+|x|)^{d+\alpha}}\,\mathrm{d}x \right) \end{aligned}$$

The last inequality holds provided that $c_0 \leq \lambda(5/4)^{d+\alpha}$.

Substituting back into (5.4), we have

$$m'(t_0)\varphi(x_0) + \Lambda m(t_0)|\nabla\varphi(x_0)| - m(t_0)M^-\varphi(x_0) - c_0 \left(\int_{\mathbb{R}^d} \frac{(u(t,x)-\theta)^+}{(1+|x|)^{d+\alpha}}\,\mathrm{d}x \right) \geq 0.$$

Recalling the ODE (5.3) and that $\varphi \leq 1$, it follows that

$$-C_1 m(t_0)\varphi(x_0) + \Lambda m(t_0)|\nabla\varphi(x_0)| - m(t_0)M^-\varphi(x_0) \geq 0.$$

Since $u(t_0,x_0) = m(t_0)\varphi(x_0) - a(t+1) - \varepsilon$, then certainly $m(t_0) > 0$ and we can factor it out from the previous inequality.

$$-C_1\varphi(x_0) + \Lambda|\nabla\varphi(x_0)| - M^-\varphi(x_0) \geq 0.$$

We are left with choosing C_1 large enough in order to contradict this last inequality. Note that we can certainly do so in the set $\{x : \varphi(x) > \rho\}$ for any fixed $\rho > 0$. However, we must address the fact that $\varphi(x_0)$ might be arbitrarily small.

The key to solve this extra difficulty is to observe that $M^-\varphi(x) > 0$ wherever $\varphi(x) = 0$. Indeed $M^-f(x) > 0$ if f achieves its global minimum at x. Moreover, also $\nabla\varphi = 0$ wherever $\varphi = 0$. Since φ is C^2, then $M^-\varphi$ is continuous. Let $-\delta := \min\{M^-\varphi(x) : \varphi(x) = 0\}$. Then there is a $\rho > 0$ so that if $\varphi(x) < \rho$ then $|\nabla\varphi(x)| < \delta/(2\Lambda)$ and $M^-\varphi(x) < -\delta/2$. Therefore, every time $\varphi(x) < \rho$ we have

$$-C_1\varphi(x_0) + \Lambda|\nabla\varphi(x_0)| - M^-\varphi(x_0) \leq -C_1\varphi(x_0) < 0.$$

For the points x where $\varphi(x) \geq \rho$, we choose C_1 large enough so as to obtain a contradiction. This finishes the proof. □

Note that the lower bound provided in Theorem 5.1 involves a weighted integral of u in the full space $\mathbb{R}^d$. This is obviously something that would not be expected for a local equation. The following is an immediate corollary of Theorem 5.1 in which we simply replace the integral in the right hand side by an integral in a subdomain. In this way, we obtain the weaker version of the result which is more similar to the weak Harnack inequality for local equations.

Corollary 5.2. *Let u be a function which satisfies the following inequality in the viscosity sense.*

$$u_t + \Lambda|\nabla u| - M^-u \geq -a \text{ in } Q_1,$$

Assume that $u \geq 0$ in Q_1, then

$$\min_{[-1/2,0]\times B_{1/2}} u \geq c\left(\int_{(-1,-1/2]\times B_{1/2}} u(t,x)\,\mathrm{d}x\,\mathrm{d}t\right) - a,$$

where c is a positive constant depending on λ, Λ, α and the dimension d.

Even the result of Corollary 5.2 is not true for second order equations. Instead, the integral on the right hand side must be replaced with the L^ε norm of u in $[-1,-1/2]\times B_{1/2}$ (See theorem 4.15 in [25]). It is relatively simple to construct stationary examples of the form $u(t,x) = |x|^{-p}$ for large p, to check that indeed a small power ε is required in the second order case if Λ/λ is large.

6. Hölder estimates

We first state the Hölder estimates in the case $\alpha \geq 1$. This case is relatively easier than $\alpha < 1$ because the diffusion is of higher order than the drift. The proof is a rather standard iterative improvement of oscillation. If the drift term vanishes, the same proof works for all $\alpha \in (0,2)$.

Theorem 6.1. *Assume $\alpha \geq 1$. Let $u : [-1,0]\times\mathbb{R}^d \to \mathbb{R}$ be a continuous bounded function which satisfies the following two inequalities in the viscosity sense*

$$u_t + \Lambda|\nabla u| - M^-u \geq -A \text{ in } Q_1,$$
$$u_t - \Lambda|\nabla u| - M^+u \leq A \text{ in } Q_1.$$

Then, $u \in C^\gamma(Q_{1/2})$ and there is an estimate

$$\|u\|_{C^\gamma(Q_{1/2})} \le C\left(\|u\|_{L^\infty([-1,0]\times\mathbb{R}^d} + A\right).$$

Here the constant C depends on λ, Λ, α, and the dimension d.

Proof. Replacing u with $u/(\|u\|_{L^\infty} + A/\varepsilon_0)$, we can assume $\|u\|_{L^\infty} \le 1/2$ and $A \le \varepsilon_0$. We must now find a universal upper bound for the Hölder norm of u in $Q_{1/2}$. We will prove that there for all $r > 0$,

$$\operatorname*{osc}_{Q_r} u \le r^\gamma. \tag{6.1}$$

This shows that u is C^γ in space and $C^{\gamma/\alpha}$ in time at the point $(0,0)$. The regularity is extended to the cylinder $Q_{1/2}$ by a standard scaling and translation argument. Therefore, we only need to show (6.1).

Note that since we have $\|u\|_{L^\infty} \le 1/2$, then (6.1) holds for all $r \ge 1$. We will show it holds for all $r > 0$ by induction in r. The inductive step we need to prove is that if (6.1) holds for all $r > r_0$, with $r_0 \le 1$, then it also holds for all $r > r_0/2$.

So, assume (6.1) holds for all $r > r_0$. Let us consider the rescaled function

$$u_{r_0}(t,x) := (2R)^{-\gamma} u\left((2R)^\alpha t, 2Rx\right).$$

This is the scaling for which the values of u_{r_0} in $Q_{1/2}$ correspond to the values of u in Q_{r_0}. The function u_{r_0} satisfies the equations

$$u_t + {r_0}^{\alpha-1}\Lambda|\nabla u| - M^- u \ge -{r_0}^\alpha \varepsilon_0 \text{ in } Q_{1/r_0}, \tag{6.2}$$

$$u_t - {r_0}^{\alpha-1}\Lambda|\nabla u| - M^+ u \le {r_0}^\alpha \varepsilon_0 \text{ in } Q_{1/r_0}. \tag{6.3}$$

Since $r_0 < 1$ and $\alpha \ge 1$, both powers of r_0 in the formula above are less or equal to one. In particular

$$u_t + \Lambda|\nabla u| - M^- u \ge -\varepsilon_0 \text{ in } Q_{1/r_0},$$
$$u_t - \Lambda|\nabla u| - M^+ u \le \varepsilon_0 \text{ in } Q_{1/r_0}.$$

Moreover, from the inductive hypothesis (6.1) for $r \ge r_0$, we have that

$$\operatorname*{osc}_{Q_r} u_{r_0} \le r^\gamma \text{ for all } r \ge 1/2. \tag{6.4}$$

Let $m := \min_{Q_1} u_{r_0}$ and $M = \max_{Q_1} u_{r_0}$. From (6.4) we know that $M - m \le 1$. Therefore, for every point (t,x) in Q_1 we have at least one of the inequalities $u_{r_0}(t,x) \le m + 1/2$ or $u(t,x) \ge M - 1/2$. Thus, one of the following two statements will hold

$$\left|\{u_{r_0} < m + 1/2\} \cap [-1,-1/2] \times B_{1/2}\right| > \frac{1}{4}|B_{1/2}| \text{ or} \tag{6.5}$$

$$\left|\{u_{r_0} > M - 1/2\} \cap [-1,-1/2] \times B_{1/2}\right| > \frac{1}{4}|B_{1/2}| \tag{6.6}$$

Without loss of generality, we assume (6.6) (otherwise, we will proceed with the rest of the proof with $-u_{r_0}$ instead of u_{r_0}).

Note that from (6.4), we deduce that $u_{r_0}(t,x) > M - 1$ in Q_1 and $u_{r_0}(t,x) > M - |x|^\gamma$ for all $x \notin B_1$ and $t \in [-1,0]$.

Let $v(t,x)$ be the non negative function

$$v(t,x) = (u_{r_0}(t,x) - M + 2^\gamma)^+ .$$

Note that $v \geq u_{r_0}$ and for $t \in [-1,0]$ we have $v(t,x) - u(t,x) \leq (|x|^\gamma - 2^\gamma)^+$. In particular, $u(t,x) = v(t,x)$ if $t \in [-1,0]$ and $x \in B_2$.

Let $U(x) = (|x|^\gamma - 2^\gamma)^+$, so that $0 \leq v(t,x) - u(t,x) \leq U(x)$. The function v satisfies the following equation (in the viscosity sense)

$$v_t + \Lambda|\nabla v| - M^- v \geq -\varepsilon_0 - M^- U \text{ in } Q_1.$$

Taking γ small, we can make $M^- U$ arbitrarily small in B_1. Therefore, for small enough γ,

$$v_t + \Lambda|\nabla v| - M^- v \geq -2\varepsilon_0 \text{ in } Q_1.$$

We now apply Corollary 5.2 to v and obtain a lower bound in $Q_{1/2}$,

$$\min_{Q_{1/2}} v \geq c \int_{[-1,-1/2]\times B_{1/2}} v \, \mathrm{d}x \, \mathrm{d}t - 2\varepsilon_0.$$

We now apply (6.6) and obtain

$$\min_{Q_{1/2}} v \geq c(2^\gamma - 1/2) - 2\varepsilon_0 > \delta.$$

This lower bound δ does not depend on γ or ε_0 provided that ε_0 is sufficiently small.

Bringing this information back into u_{r_0}, this means that $u_{r_0} \geq M - 2^\gamma + \delta$ in $Q_{1/2}$. Here we also choose γ sufficiently small so that $2^\gamma - \delta < 1 - \delta/2$. Therefore we have that $\operatorname{osc}_{Q_{1/2}} u_{r_0} \leq 1 - \delta/2$. This also means that

$$\operatorname*{osc}_{Q_{r_0/2}} u \leq (1 - \delta/2) {r_0}^\gamma.$$

In particular $\operatorname{osc}_{Q_r} u \leq (1 - \delta/2) {r_0}^\gamma$ for all $r < r_0$. Choosing γ sufficiently small one last time so that $2^{-\gamma} > 1 - \delta/2$, we proved that $\operatorname{osc}_{Q_r} u \leq r^\gamma$ for all $r \in [r_0/2, r_0]$.

This finishes the proof of (6.1) by induction in r. □

We now state and prove the corresponding theorem for $\alpha < 1$. In this case the interaction between the diffusion and drift is more subtle and we must make a change of variables partially following the flow in order to obtain the necessary cancellation to prove the theorem. This idea originated in [44].

Theorem 6.2. *Assume $\alpha < 1$. Let $u : [-1,0] \times \mathbb{R}^d \to \mathbb{R}$ be a continuous bounded function which satisfies the following two inequalities in the viscosity sense*

$$u_t + b(t,x) \cdot \nabla u - M^- u \geq -A \text{ in } Q_1,$$
$$u_t + b(t,x) \cdot \nabla u - M^+ u \leq A \text{ in } Q_1.$$

Assume that b is a continuous vector field and $\|b(t,\cdot)\|_{C^{1-\alpha}(B_1)}$ is bounded uniformly in t. Then, $u \in C^\gamma(Q_{1/2})$ and there is an estimate

$$\|u\|_{C^\gamma(Q_{1/2})} \leq C \left(\|u\|_{L^\infty([-1,0]\times\mathbb{R}^d} + A \right).$$

Here the constant C depends on λ, Λ, α, and the dimension d.

Proof. The general strategy of the proof is similar to the proof of Theorem 6.1. The problem in the iterative argument is that in the rescaled equations (6.2) and (6.3), the factor ${r_0}^{\alpha-1}$ is large for small values of r_0. It is crucial that this factor remains bounded for the induction argument to succeed.

The solution is to change the shape of the parabolic cylinders we use, so that they follow the flow. Let us defined the following modified parabolic cylinders.

$$\tilde{Q}_r := \{(t,x) : |x - X(t)| < r \wedge t \in (-r^\alpha, 0]\}.$$

Here $X(t)$ is one solution to the backward ODE

$$\begin{aligned} X(0) &= 0, \\ X'(t) &= b(t, X(t)), \quad \text{for } t < 0. \end{aligned}$$

The corresponding scaled function $\tilde{u}_{r_0}$ is now

$$\tilde{u}_{r_0} = {r_0}^{-\gamma} u({r_0}^\alpha t, r_0(X(t) + x)).$$

This function solves the equations

$$\begin{aligned} u_t + {r_0}^{\alpha-1} \left(b(t,x) - b(t,X(t))\right) |\nabla u| - M^- u \geq -{r_0}^\alpha \varepsilon_0 \text{ in } Q_{1/r_0}, \\ u_t - {r_0}^{\alpha-1} \left(b(t,x) - b(t,X(t))\right) |\nabla u| - M^+ u \leq {r_0}^\alpha \varepsilon_0 \text{ in } Q_{1/r_0}. \end{aligned}$$

The Hölder continuity assumption on b assures that ${r_0}^{\alpha-1} |b(t,x) - b(t,X(t))| \leq \Lambda$ for some constant $\Lambda > 0$. This allows us to continue with the rest of the proof as in Theorem 6.1. □

7. Failure of the Harnack inequality

The Harnack inequality is a property of non negative solutions to some elliptic and parabolic equations. For parabolic equations, it says that there is some universal constant C so that if u is a solution of the equation in Q_1 which is non negative in $[-1,0] \times \mathbb{R}^d$, then

$$\sup_{[-3/4,-1/2]\times B_{1/2}} u \leq C \inf_{[-1/4,0]\times B_{1/2}} u.$$

The Harnack inequality holds for some integral equations, for example see [4], [9] and [1]. Interestingly enough, in the situation of [1], the Harnack inequality holds, but the solution of the equation is not necessarily continuous.

It turns out, however, that the Harnack inequality fails for the type of equations we consider in this paper. In this section, we construct a counterexample.

Let e_1 be a the unit vector $(1, 0, \dots)$ in $\mathbb{R}^d$. Consider the following integral operator

$$Lu(x) = \int_{\mathbb{R}^d} \frac{\delta_y u(x)}{|y|^{d+\alpha}} \, \mathrm{d}y + \int_{\mathbb{R}} \frac{\delta_{y_1 e_1} u(x)}{|y_1|^{1+\alpha}} \, \mathrm{d}y.$$

This operator is in fact the same as

$$Lu = -c_1(-\Delta)^{\alpha/2} u - c_2(-\partial_{x_1 x_1})^{\alpha/2} u.$$

for some positive constants c_1 and c_2. We look at the solution to the problem

$$\begin{aligned} u_t - Lu &= f(x) \qquad \text{in } (0,\infty)\times\mathbb{R}^d, \\ u(-1,x) &= 0 \qquad \text{for } x\in\mathbb{R}^d. \end{aligned} \tag{7.1}$$

where $f=\chi_{Q_\varepsilon}$ is the indicator function of the set

$$K_\varepsilon = \{x\in\mathbb{R}^d : |x_1-4|<\varepsilon \text{ and } |x'|<\varepsilon\}.$$

We used the notation $x=(x_1,x')$. The operator L we are considering here is an integral operator with respect to a singular measure (singular along the axis $y'=0$). That is the only reason why it does not have the form (1.1). Indeed, in (1.1) we implicitly assumed that for every (t,x) the integral equation has a absolutely continuous measure with density $K(t,x,\cdot)$. That is a choice for convenience of notation only. Indeed, the non negative solution u to the equation (7.1) satisfies the two inequalities in the viscosity sense

$$\begin{aligned} u_t - M^- u &\geq 0 \qquad \text{in } Q_1, \\ u_t - M^+ u &\leq 0 \qquad \text{in } Q_1. \end{aligned}$$

So, it is a valid candidate for a Harnack inequality. However, we will prove that

$$\lim_{\varepsilon\to 0} \frac{u(0,(0,x'))}{u(-1/2,0)} = 0, \tag{7.2}$$

provided that $x'\neq 0$. This contradicts the parabolic Harnack inequality.

The intuition behind this counterexample is similar to the constructions given in [7] and [39] (for very different types of kernels). The equation we consider is the generator of a Levy process with frequent purely horizontal jumps. For $\varepsilon \ll 1$, a process starting at the origin would have a much higher chance to exit the domain at a point with $|x_1|<\varepsilon$ than a process that starts outside of a band $|x_1|>\delta$.

In order to verify (7.2), we will compute the solution to (7.1) almost explicitly. In order to write a formula for u we will use the heat kernel associated with L and Duhamel formula.

The heat kernel associated with this equation is explicit in Fourier side:

$$\hat{H}(t,\xi) = c\exp(-t|\xi|^\alpha - t|\xi_1|^\alpha).$$

In real variables, $H(t,x)$ is not explicit. But we know it must be the convolution of the heat kernel of $(-\Delta)^{\alpha/2}$ and the heat kernel of $(-\partial_{x_1x_1})$. That is

$$H(t,x) = \Phi_d(t,x) * \Phi_1(t,x),$$

where

$$\Phi_d(t,x) = t^{-d/\alpha}\Phi_d(1,x/t^{1/\alpha}) =: t^{-d/\alpha}\varphi_d\left(\frac{x}{t^{1/\alpha}}\right).$$

The exact formula for the d-dimensional fractional hear kernel φ_d is not known. However, we know that

$$\varphi_d(x) \approx (1+|x|)^{-d-\alpha}.$$

Here we use the symbol $\approx$ to say that the ratio between the left hand side and right hand side is bounded below and above by positive constants.

The heat kernel $\Phi_1(t,x)$ is simply the one dimensional heat kernel of the fractional Laplacian in the variable x_1. Thus, it is a singular measure supported on the line $x'=0$ with a density of the form

$$t^{-1-\alpha}\varphi_1\left(\frac{x_1}{t^{1/\alpha}}\right).$$

As before, we have $\varphi_1(x)\approx(1+|x|)^{-1-\alpha}$.

Using Duhamel formula, and following routine arithmetic manipulations, we arrive at a formula for $u(T,x)$.

$$u(T,x)=\int_0^{T+1}\int_{K_\varepsilon}\int_{\mathbb{R}} t^{-\frac{d+1}{\alpha}}\varphi_d\left(\frac{x-z-y_1e_1}{t^{1/\alpha}}\right)\varphi_1\left(\frac{y_1}{t^{1/\alpha}}\right)\,\mathrm{d}y_1\,\mathrm{d}z\,\mathrm{d}t.$$

We will first estimate $u(-1/2,0)$ in terms of ε. That is, we set $T=-1/2$ and $x=0$.

Assume $\varepsilon\ll 1$. We obtain a lower bound by restricting the domain of integration to a smaller domain.

$$u(-1/2,0)\geq\int_{K_\varepsilon}\int_0^{|z'|^\alpha}\int_{-4-\varepsilon}^{-4+\varepsilon} t^{-\frac{d+1}{\alpha}}\varphi_d\left(\frac{-z-y_1e_1}{t^{1/\alpha}}\right)\varphi_1\left(\frac{y_1}{t^{1/\alpha}}\right)\,\mathrm{d}y_1\,\mathrm{d}t\,\mathrm{d}z,$$

In this whole domain of integration we have

$$\varphi_d\left(\frac{-z-y_1e_1}{t^{1/\alpha}}\right)\geq c\left(1+\frac{\varepsilon}{t^{1/\alpha}}\right)^{-d-\alpha}\geq c\varepsilon^{-d-\alpha}t^{d/\alpha+1},$$
$$\varphi_1\left(\frac{y_1}{t^{1/\alpha}}\right)\geq c\left(1+\frac{4}{t^{1/\alpha}}\right)^{-1-\alpha}\geq ct^{1+1/\alpha}$$

Therefore

$$\begin{aligned}u(-1/2,0)&\geq c\varepsilon^{-d-\alpha+1}\int_{K_\varepsilon}\int_0^{|z'|^\alpha}t^2\,\mathrm{d}t\,\mathrm{d}z,\\&=c\varepsilon^{2\alpha+1}\end{aligned}$$

Now we estimate $u(0,(0,x'))$ from above in terms of ε. That is, we set $T=-1/2$ and $x=(0,x')$ for some non zero $x'\in\mathbb{R}^{d-1}$ with $|x'|<1/2$. Using that $\varphi_d(x)\leq C|d|^{-d-\alpha}$ and $\varphi_1(x)\leq C|x|^{-1-\alpha}$, we get

$$u(0,(0,x'))\leq C\int_0^{T+1}\int_{K_\varepsilon}\int_{\mathbb{R}} t^{1-\frac{1}{\alpha}}\left(|x'-z'|+|x_1-z_1-y_1|\right)^{-d-\alpha}\varphi_1\left(\frac{y_1}{t^{1/\alpha}}\right)\mathrm{d}y_1\,\mathrm{d}z\,\mathrm{d}t.$$

Since $x'\neq 0$, then $|x'-z'|>|x'|/2$ provided that $\varepsilon<|x'|/2$. That is, $|x'-z'|$ is of order one as $\varepsilon\to 0$. Therefore

$$\begin{aligned}u(0,(0,x'))&\leq C\int_0^{T+1}\int_{K_\varepsilon}\int_{\mathbb{R}} t^{1-\frac{1}{\alpha}}\varphi_1\left(\frac{y_1}{t^{1/\alpha}}\right)\,\mathrm{d}y_1\,\mathrm{d}z\,\mathrm{d}t,\\&=C\int_0^{T+1}\int_{K_\varepsilon} t\,\mathrm{d}z\,\mathrm{d}t=C\varepsilon^d.\end{aligned}$$

So, if $d>1+2\alpha$, we obtain that $u(0,(0,x'))\ll u(-1/2,0)$ as $\varepsilon\to 0$. Therefore the Harnack inequality does not hold.

Remark 7.1. It is not clear whether the condition $d > 1 + 2\alpha$ is a limitation of this construction or the standard Harnack inequality actually holds for $d \leq 1 + 2\alpha$. We leave it as an open question.

8. Applications

In this last section we give a brief summary of applications to the estimates of Theorems 6.1 and (6.2).

8.1. Fully nonlinear parabolic equations. One of the canonical applications of the Hölder estimates presented in this paper is the $C^{1,\gamma}$ regularity for solutions to the parabolic Isaacs equation.

$$u_t - \inf_a \sup_b \int_{\mathbb{R}^d} \delta_y u(x) K_{ab}(y)\, \mathrm{d}y = 0.$$

The Isaacs equation models the value function for the optimal strategy in a zero-sum stochastic game. For the purpose of this article, we consider games driven by Levy processes without diffusion. We point out that discontinuous Levy processes have a number of applications in finance [46] and physics [36].

It is easy to check that if all kernels K_{ab} satisfy the assumptions (3.2) and (3.1), then the incremental quotients $v_h(x) = (u(x+h) - u(x))/|h|$ satisfy the assumptions of Theorem (6.1) or Theorem (6.2). This quickly leads to a $C^{1,\gamma}$ regularity result for u at least if the equation holds in the whole space $(0,\infty) \times \mathbb{R}^d$. If the equation holds in a bounded domain, there are some extra difficulties. The method is explained in [40]. The result there applies to a less general class of kernels but it is robust, since it is based on the Hölder estimates from [34].

8.2. Active scalar equations. There exist several active scalar equations of the form

$$\theta_t + B(\theta) \cdot \theta + (-\Delta)^{\alpha/s}\theta = 0, \tag{8.1}$$

that have attracted attention in recent years. Here $B(\theta)$ is a vector field which depends on the solution θ of the equation. This dependence makes the equation non linear. Some examples of B which are of interest are the following.

- **Conservation laws with fractional diffusion.** $B(\theta) = F_i'(\theta)$. See for example [21].
- **Surface quasi-geostrophic equation.** $B(\theta) = R^\perp\theta$, where R stands for the Riesz tranform. See for example [12, 16, 29] among many others.
- **Modified surface quasi-geostrophic equation.** $B(\theta) = R^\perp(-\Delta)^{1-\alpha}\theta$, where R stands for the Riesz tranform. See for example [15].
- **Incompressible flow in porous media.** $B(\theta) = (0, -\theta) - \nabla p$, so that $\operatorname{div} B = 0$. See for example [17].

The solution to any of these equations is a priori bounded in L^∞ from the maximum principle. The key step in order to prove that they posses a classical global solution is to be able to obtain a regularity estimate for the solution which goes beyond L^∞. Once a Hölder

estimate is established, it can be bootstrapped into higher regularity using the result from [42] in any of the models above.

Theorem 6.1 gives us a Hölder estimate for conservation laws with critical fractional diffusion $\alpha = 1$. It also gives us a Hölder estimate for the modified surface quasi-geostrophic equation if $\alpha \in (0, 1)$. Therefore, the classical well-posedness of both models follows.

The study of either the surface quasi-geostrophic model or the fluid in porous media with critical diffusion $\alpha = 1$ does not follow immediately from Theorem 6.1. This is because Theorem 6.1 requires the vector field to be bounded and in these cases $B(\theta)$ is a priori only controlled in $L^\infty((0,\infty), BMO)$. A version of Theorem 6.1 for vector fields $b \in L^\infty(BMO)$ was given in [12] provided that $\operatorname{div} b = 0$, but the result are of very different nature. Indeed, the result in [12] is based on the variational structure of the equation and uses De Giorgi's technique.

8.3. The space homogeneous Boltzmann equation. The Boltzmann equation models the evolution of dilute gasses. In the space homogeneous case, the equation takes the form

$$f_t = Q(f, f),$$

where

$$Q(g, f) = \int_{\mathbb{R}^d} \int_{S_1} (g(v'_*)f(v') - g(v_*)f(v))\ \mathrm{d}\sigma\ \mathrm{d}v_*$$

and we write

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \qquad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma.$$

For every fixed g, the operator $f \mapsto Q(g, f)$ is an integro-differential operator which has the form

$$Q(g, f) = R_g f(v) + \int_{\mathbb{R}^d} \delta_{v'} f(v) K_g(v, v' - v)\ \mathrm{d}v'.$$

The function R_g and the kernel K_g can be computed in terms of g, although the formula is quite involved.

Under some conditions, the kernel K_f satisfies the assumptions (3.2) and (3.1) and consequently Theorems 6.1 and 6.2 may be used to prove a local Hölder continuity result for the Boltzmann equation.

In order to apply the result of Theorems 6.1 and 6.2 we would need to consider a collision kernel without Grad's angular cutoff condition. Moreover, we should know a priori that f is bounded below in order to guarantee that K_f satisfies (3.1). This last assumption in particular is quite undesirable. In a work in progress of Russell Schwab and the author, we are developing a more general Hölder estimate which, for some collision kernels, would only depend on observable quantities associated with f (mass, energy and entropy).

Note that the regularity of the solutions to the inhomogeneous Boltzmann equation is rather well understood by completely different methods [18].

References

[1] Martin T. Barlow, Richard F. Bass, Zhen-Qing Chen, and Moritz Kassmann, *Non-local Dirichlet forms and symmetric jump processes*, Trans. Amer. Math. Soc. **361**(4) (2009), 1963–1999.

[2] Richard F Bass and Moritz Kassmann, *Harnack inequalities for non-local operators of variable order*, Transactions of the American Mathematical Society (2005), 837–850.

[3] _____, *Hölder continuity of harmonic functions with respect to operators of variable order*, Communications in Partial Differential Equations, **30**(8) (2005), 1249–1259.

[4] Richard F Bass and David A Levin, *Harnack inequalities for jump processes*, Potential Analysis, **17**(4) (2002), 375–388.

[5] _____, *Transition probabilities for symmetric jump processes*, Trans. Amer. Math. Soc. **354**(7) (2002), 2933–2953 (electronic).

[6] C. Bjorland, L. Caffarelli, and A. Figalli, *Non-local gradient dependent operators*, Adv. Math. **230**(4-6) (2012), 1859–1894.

[7] Krzysztof Bogdan and Paweł Sztonyk, *Harnack's inequality for stable Lévy processes*, Potential Anal. **22**(2) (2005), 133–150.

[8] Luis Caffarelli, Chi Hin Chan, and Alexis Vasseur, *Regularity theory for parabolic nonlinear integral operators*, J. Amer. Math. Soc. **24**(3) (2011), 849–869.

[9] Luis Caffarelli and Luis Silvestre, *Regularity theory for fully nonlinear integro-differential equations*, Communications on Pure and Applied Mathematics **62**(5) (2009), 597–638.

[10] _____, *The evans-krylov theorem for nonlocal fully nonlinear equations*, Annals of mathematics **174**(2) (2011), 1163–1187.

[11] _____, *Regularity results for nonlocal equations by approximation*, Archive for rational mechanics and analysis **200**(1) (2011), 59–88.

[12] Luis Caffarelli and Alexis Vasseur, *Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation*, Ann. of Math. **171**(3) (2010), 1903–1930.

[13] Héctor Chang Lara and Gonzalo Dávila, *Regularity for solutions of nonlocal, nonsymmetric equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire **29**(6) (2012), 833–859.

[14] Hector A. Chang Lara, *Regularity for fully non linear equations with non local drift*, preprint arXiv:1210.4242, 2013.

[15] Peter Constantin, Gautam Iyer, and Jiahong Wu, *Global regularity for a modified critical dissipative quasi-geostrophic equation*, preprint arXiv:0803.1318, 2008.

[16] Peter Constantin and Vlad Vicol, *Nonlinear maximum principles for dissipative linear nonlocal operators and applications*, Geometric And Functional Analysis **22**(5) (2012), 1289–1321.

[17] Diego Córdoba, Francisco Gancedo, and Rafael Orive, *Analytical behavior of two-dimensional incompressible flow in porous media*, Journal of mathematical physics **48**(6) (2007), 065206.

[18] Laurent Desvillettes and Bernt Wennberg, *Smoothness of the solution of the spatially homogeneous boltzmann equation without cutoff*, Comm. Partial Differential Equations **29**(1-2) (2004), 133–155.

[19] Hongjie Dong and Doyoon Kim, *Schauder estimates for a class of non-local elliptic equations*, preprint arXiv:1103.5069, 2011.

[20] _____, *On L_p-estimates for a class of non-local elliptic equations*, Journal of Functional Analysis **262**(3) (2012), 1166–1199.

[21] Jérôme Droniou and Cyril Imbert, *Fractal first-order partial differential equations*, Archive for Rational Mechanics and Analysis **182**(2) (2006), 299–331.

[22] Lawrence C. Evans, *Classical solutions of fully nonlinear, convex, second-order elliptic equations*, Comm. Pure Appl. Math. **35**(3) (1982), 333–363.

[23] Matthieu Felsinger and Moritz Kassmann, *Local Regularity for Parabolic Nonlocal Operators*, Comm. Partial Differential Equations **38**(9) (2013), 1539–1573.

[24] Nestor Guillen and Russell W Schwab, *Aleksandrov–bakelman–pucci type estimates for integro-differential equations*, Archive for Rational Mechanics and Analysis **206**(1) (2012), 111–157.

[25] Cyril Imbert and Luis Silvestre, *An introduction to fully nonlinear parabolic equations*, An Introduction to the Kähler-Ricci Flow, Springer, 2013, pp. 7–88.

[26] Moritz Kassmann, *A priori estimates for integro-differential operators with measurable kernels*, Calculus of Variations and Partial Differential Equations **34**(1) (2009), 1–21.

[27] Moritz Kassmann and Ante Mimica, *Intrinsic scaling properties for nonlocal operators*, preprint arXiv:1310.5371, 2013.

[28] Moritz Kassmann and Russell Schwab, *Regularity results for nonlocal parabolic equations*, preprint arXiv:1305.5418, 2013.

[29] A. Kiselev, F. Nazarov, and A. Volberg, *Global well-posedness for the critical 2D dissipative quasi-geostrophic equation*, Invent. Math. **167**(3) (2007), 445–453.

[30] Takashi Komatsu, *Uniform estimates for fundamental solutions associated with non-local Dirichlet forms*, Osaka J. Math. **32**(4) (1995), 833–860.

[31] N. V. Krylov, *Boundedly inhomogeneous elliptic and parabolic equations*, Izv. Akad. Nauk SSSR Ser. Mat. **46**(3) (1982) 487–523, 670.

[32] Nikolai Vladimirovich Krylov and Mikhail V Safonov, *A certain property of solutions of parabolic equations with measurable coefficients*, Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya **44**(1) (1980) 161–175.

[33] NV Krylov and MV Safonov, *Estimate of the possibility of diffusion process hitting into a set of positive measure*, Doklady Akademii Nauk SSSR **245**(1) (1979), 18–20.

[34] Héctor Chang Lara and Gonzalo Dávila, *Regularity for solutions of non local parabolic equations*, Calculus of Variations and Partial Differential Equations **49**(1-2) (2014), 139–172.

[35] _____, *Regularity for solutions of non local parabolic equations*, Calc. Var. Partial Differential Equations **49**(1-2) (2014), 139–172.

[36] Ralf Metzler and Joseph Klafter, *The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics*, Journal of Physics A: Mathematical and General **37**(31) (2004), R161.

[37] Nikolai Nadirashvili and Serge Vlăduţ, *Nonclassical solutions of fully nonlinear elliptic equations*, Geometric and Functional Analysis **17**(4) (2007), 1283–1296.

[38] ———, *Singular solutions of hessian elliptic equations in five dimensions*, Journal de Mathématiques Pures et Appliquées **100**(6) (2013), 769–784.

[39] Marcus Rang, Moritz Kassmann, and Russell W Schwab, *Hölder regularity for integro-differential equations with nonlinear directional dependence*, preprint arXiv:1306.0082, 2013.

[40] Joaquim Serra, *Regularity for fully nonlinear nonlocal parabolic equations with rough kernels*, preprint arXiv:1401.4521, 2014.

[41] Luis Silvestre, *Holder estimates for solutions of integro-differential equations like the fractional laplace*, Indiana University mathematics journal **55**(3) (2006), 1155–1174.

[42] ———, *On the differentiability of the solution to an equation with drift and fractional diffusion*, preprint arXiv:1012.2401, 2010.

[43] ———, *On the differentiability of the solution to the hamilton–jacobi equation with critical fractional diffusion*, Advances in mathematics **226**(2) (2011), 2020–2039.

[44] ———, *Hölder estimates for advection fractional-diffusion equations*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **11**(4) (2012), 843–855.

[45] Renming Song and Zoran Vondraček, *Harnack inequality for some classes of Markov processes*, Math. Z. **246**(1-2) (2004), 177–202.

[46] Peter Tankov, *Financial modelling with jump processes*, CRC Press, 2004.

Dept. of Mathematics, University of Chicago, 5734 S. University Avenue, Room 208C, Chicago, Illinois 60637

E-mail: luis@math.uchicago.edu

The resolution of the bounded L^2 curvature conjecture in general relativity

Sergiu Klainerman, Igor Rodnianski, and Jérémie Szeftel

Abstract. This paper reports on the recent proof of the bounded L^2 curvature conjecture. More precisely we show that the time of existence of a classical solution to the Einstein-vacuum equations depends only on the L^2-norm of the curvature and a lower bound of the volume radius of the corresponding initial data set.

Mathematics Subject Classification (2010). Primary 83C05; Secondary 35Q75, 58J45.

Keywords. Einstein equations, Cauchy problem, rough solutions, null structure, bilinear estimates.

1. Introduction

This paper reports on the recent proof of the bounded L^2 curvature conjecture. More precisely we show that the time of existence of a classical solution to the Einstein-vacuum equations depends only on the L^2-norm of the curvature and a lower bound of the volume radius of the corresponding initial data set.

The entire proof of the conjecture is contained in the sequence of papers [20, 33–37].

1.1. Initial value problem for the Einstein vacuum equations. We consider the Einstein vacuum equations (EVE),

$$\mathbf{Ric}_{\alpha\beta} = 0 \tag{1.1}$$

where $\mathbf{Ric}_{\alpha\beta}$ denotes the Ricci curvature tensor of a four dimensional Lorentzian space time $(\mathcal{M}, \mathbf{g})$. (1.1) corresponds to an evolution problem. An initial data set consists of a three dimensional manifold Σ_0 together with a Riemannian metric g and a symmetric 2-tensor k on Σ_0. For a given initial data set (Σ_0, g, k), the Cauchy problem consists in finding a metric $\mathbf{g}$ satisfying (1.1) and an embedding of Σ_0 in $\mathcal{M}$ such that the metric induced by $\mathbf{g}$ on Σ_0 coincides with g and the 2-tensor k is the second fundamental form of the embedding.

Remark 1.1. Since physically one should not be able to distinguished between different coordinates systems, a solution of the Cauchy problem can be unique only modulo a diffeomorphism.

The equations (1.1) are overdetermined and the initial data set (Σ_0, g, k) has to satisfy the following compatibility conditions known as the constraint equations

$$\begin{cases} \nabla^j k_{ij} - \nabla_i \mathrm{tr} k = 0, \\ R_{scal} - |k|^2 + (\mathrm{tr} k)^2 = 0, \end{cases} \tag{1.2}$$

Proceedings of the International Congress of Mathematicians, Seoul, 2014

where the covariant derivative ∇ is defined with respect to the metric g, R_{scal} is the scalar curvature of g, and tr$k$ is the trace of k with respect to the metric g.

In this paper we restrict ourselves to asymptotically flat initial data sets with one end. More precisely, (Σ_0, g, k) is such that Σ_0 minus a compact set is diffeomorphic to $\mathbb{R}^3$ minus a ball, with $g_{ij} - \delta_{ij}$ and k_{ij} satisfying a suitable rate of fall off at infinity in this coordinates system.

The first local existence and uniqueness result for (EVE) was established by Y.C. Bruhat, see [3], with the help of wave coordinates which allowed her to cast the Einstein vacuum equations in the form of a system of nonlinear wave equations to which one can apply[1] the standard theory of nonlinear hyperbolic systems. The optimal, classical[2] result states the following.

Theorem 1.2 (Classical local existence [7, 8]). *Let (Σ_0, g, k) be an initial data set for the Einstein vacuum equations* (1.1). *Assume that Σ_0 can be covered by a locally finite system of coordinate charts, related to each other by C^1 diffeomorphisms, such that $(g, k) \in H^s_{loc}(\Sigma_0) \times H^{s-1}_{loc}(\Sigma_0)$ with $s > \frac{5}{2}$. Then there exists a unique[3] (up to an isometry) globally hyperbolic development $(\mathcal{M}, \mathbf{g})$, verifying* (1.1)*, for which Σ_0 is a Cauchy hypersurface[4] .*

Our goal will be to lower the assumptions of the previous theorem on the regularity of the initial data set. To motivate our result, let us first emphasize in the next section why pushing for rough solutions is a main theme in nonlinear evolution PDEs.

1.2. The quest for rough solutions in nonlinear evolution PDEs.

1.2.1. First examples. To illustrate the role played by rough solutions in nonlinear evolution PDEs, let us consider a nonlinear evolution equation possessing a conserved quantity which is positive definite and in particular controls a norm in a certain functional space. We refer to the conserved quantity as the energy, its associated functional space as the energy space, and its associated norm as the energy norm. One can then classify such evolution equations into three categories[5]

1. One can prove a local existence result with a time of existence which depends only on the energy norm of the initial data. This case is referred as *energy subcritical*.
2. One can prove a local existence result for initial data in the energy space, but with a time of existence which does not only depend on the energy norm of the initial data (i.e. there is no uniform lower bound on the size of the time interval of existence for initial data with a given energy norm). This case is referred as *energy critical*.
3. One can not prove a local existence result for initial data in the energy space. This case is referred as *energy supercritical*.

[1] The original proof in [3] relied on representation formulas, following an approach pioneered by Sobolev, see [29].

[2] Based only on energy estimates and classical Sobolev inequalities.

[3] The original proof in [7, 8] actually requires one more derivative for the uniqueness. The fact that uniqueness holds at the same level of regularity than the existence has been obtained in [25].

[4] That is any past directed, in-extendable causal curve in $\mathcal{M}$ intersects Σ_0.

[5] One usually defines the criticality relative to the behavior of the energy under some notion of scaling. Here, we instead classify the equation with respect to an ability to prove local existence results. These two classifications agree in most cases and our choice only aims at simplifying the exposition.

As the energy supercritical case as it is still vastly open, we will focus on the two other cases. In the energy subcritical case, one can pile up time intervals of existence provided by the local existence result which are all of the same size since they only depend on the energy norm of the data which is conserved. One infers global existence for any initial data. A nice illustration of this procedure is provided by [11] in the case of the classical Yang-Mills equations in dimension 1+3, where the energy space is the Sobolev space H^1.

In the energy critical case, the conjecture is that global existence holds for any data below the energy of the first nontrivial stationary solution or solitary wave. This conjecture has been proved in a large number of cases over the last thirty years. A spectacular achievement of this method is the recent proof of the conjecture in the case of the 2+1 wave map problem in [21, 31, 32, 38].

A key step in the large data results mentioned above, both for the energy subcritical and energy critical cases, is a local well-posedness result at the level of the energy space, which is typically a low regularity well-posedness result. In the next section, we discuss another example of a nonlinear evolution partial differential equation for which making sense of rough solutions plays a fundamental role.

1.2.2. The proof of the weak cosmic censorship in spherical symmetry. In this section, we briefly discuss the influential proof by Christodoulou [6] of the weak cosmic censorship conjecture for the Einstein equations coupled to a scalar field in spherical symmetry[6] .

Let us first recall the weak cosmic censorship conjecture of Penrose. The starting point of this conjecture is the existence of space times containing singularities, the most famous example being the Schwarzschild space-time which is spherically symmetric and contains a singularity at $r = 0$. Now, the existence of space-times containing a singularity could be considered as an undesirable feature from the point of view of physics. To come to term with such space-times, Penrose formulated the celebrated weak cosmic censorship conjecture[7] .

Conjecture 1.3 (Weak cosmic censorship). *For generic asymptotically flat initial data set, singularities are hidden by a black hole.*

In view of this conjecture, singularities are acceptable as they not visible by an observer at infinity. At the moment, it is still an open problem in general, but the conjecture has been proved in the case of spherical symmetry in [6] for the Einstein equations coupled to a scalar field. This seminal work relies on the rough well posedness result of [5]. This well posedness result - which involves regularity assumptions at the level of a weighted bounded variation (BV) norm - allows in particular to make sense of solutions with a jump along the backward null cone from the singularity. This jump turns out to be essential in generating arbitrarily small perturbations of a given solution containing a singularity which are still strong enough to cover the singularity with a black hole[8] .

The result in [6] provides thus yet another example of the importance of making sense of rough solutions for nonlinear evolution PDEs. It motivates our main result, which concerns well-posedness of rough solutions for the Einstein equations in the absence of symmetry.

[6] Due to Birkhoff's theorem, the Einstein vacuum equations are non dynamical in spherical symmetry. To obtain a dynamical problem and yet retain the advantage of working in spherical symmetry, one adds a matter field to the right-hand side of the Einstein equations, a scalar field being the simplest possibility. For the sake of simplicity, we do not explicitly write down the equations in this case.

[7] He also introduced the strong cosmic censorship conjecture which despite its name is independent of the weak cosmic censorship conjecture.

[8] In the proof, the singularity is actually covered by a trapped region which, as it turns out, is enough.

Now, BV norms are only adapted to hyperbolic problems in 1+1 dimension (and hence to spherical symmetry). This will force us to abandon BV norms and to instead measure the regularity of our solution using L^2 based norms which are the only norms which are propagated by the evolution in higher dimensions.

1.3. The resolution of the bounded L^2 curvature conjecture.

1.3.1. The bounded L^2 curvature conjecture. In this section, we consider the problem of going beyond the classical local existence result stated in Theorem 1.2. To make the discussion more tangible it is worthwhile to recall the form of the Einstein vacuum equations in the wave gauge. Assuming given coordinates x^α, verifying[9]

$$\Box_{\mathbf{g}} x^\alpha = 0, \ \alpha = 0, \dots, 3, \tag{1.3}$$

the metric coefficients $\mathbf{g}_{\alpha\beta} = \mathbf{g}(\partial_\alpha, \partial_\beta)$, with respect to these coordinates, satisfy the system of quasilinear wave equations[10] ,

$$\Box_{\mathbf{g}} \mathbf{g}_{\alpha\beta} = F_{\alpha\beta}(\mathbf{g}, \partial\mathbf{g}) \tag{1.4}$$

where $F_{\alpha\beta}$ are quadratic functions of $\partial\mathbf{g}$, i.e. the first order derivatives of $\mathbf{g}$ with respect to the coordinates x^α. In the harmonic coordinates, the wave operator on the curved background $\mathbf{g}$ is given by $\Box_{\mathbf{g}} = \mathbf{g}^{\mu\nu}\partial_\mu\partial_\nu$. Equation (1.4) is obtained by expressing the Ricci tensor $\mathbf{Ric}_{\alpha\beta}$ in terms of the components of $\mathbf{g}$ and its first and second order derivatives. Conversely, to verify that the solution of (1.4) yields a solution to the Einstein vacuum equations (1.1), one has additionally to show that the coordinates conditions (1.3) propagate. This holds for solutions of (1.4), as was observed by Choquet-Bruhat, provided these coordinates conditions are satisfied initially on Σ_0 and (Σ_0, g, k) satisfies the constraint equations (1.2).

In a first approximation we may compare (1.4) with the semilinear wave equation,

$$\Box\phi = F(\phi, \partial\phi) \tag{1.5}$$

with F quadratic in $\partial\phi$. Using standard energy estimates - i.e. differentiating (1.5) $s-1$ times, multiplying it with $\partial_t\partial^{s-1}\phi$, integrating by parts and using Gronwall's lemma - one obtains the following control for the Sobolev norm H^s of ϕ

$$\|\phi(t)\|_s \lesssim \|\phi(0)\|_s \exp\left(C_s \int_0^t \|\partial\phi(\tau)\|_{L^\infty} d\tau\right). \tag{1.6}$$

The classical exponent $s > 3/2 + 1$ arises simply from the Sobolev embedding of H^r, $r > 3/2$ into L^∞.

To go beyond the classical exponent, see [26], one has to replace Sobolev inequalities with Strichartz estimates of, roughly, the following type,

$$\left(\int_0^t \|\partial\phi(\tau)\|_{L^\infty}^2 d\tau\right)^{1/2} \lesssim C\left(\|\partial\phi(0)\|_{H^{1+\epsilon}} + \int_0^t \|\Box\phi(\tau)\|_{H^{1+\epsilon}}\right)$$

[9] $\Box_{\mathbf{g}}$ is the covariant wave operator $\mathbf{g}^{\alpha\beta}\mathbf{D}_\alpha\mathbf{D}_\beta$.

[10] Nonlinear wave equations are either semilinear or quasilinear according to whether the higher order terms - here the terms containing second order derivatives - are linear or nonlinear.

where $\epsilon > 0$ can be chosen arbitrarily small. This leads to a gain of $1/2$ derivatives, i.e. we can prove well-posedness for equations of type (1.5) for any exponent $s > 2$.

The same type of improvement in the case of quasilinear wave equations requires a highly non-trivial extension of such estimates for wave operators with non-smooth coefficients. The first improved regularity results for quasilinear wave equations of the type,

$$g^{\mu\nu}(\phi)\partial_\mu\partial_\nu\phi = F(\phi, \partial\phi) \tag{1.7}$$

with $g^{\mu\nu}(\phi)$ a non-linear perturbation of the Minkowski metric $m^{\mu\nu}$, are due to [1, 2, 13, 40, 41]. The best known results for equations of type (1.4) were obtained in [14] and [28]. According to them one can lower the Sobolev exponent $s > 5/2$ in Theorem 1.2 to $s > 2$. It turns out, see [22], that these results are sharp in the general class of quasilinear wave equations of type (1.4). However, the Einstein equations enjoy a special structure, and it was conjectured in [12] that one can obtain a well-posedness result at the level of $s = 2$[11] .

Conjecture 1.4 (Bounded L^2 curvature conjecture). *The Einstein- vacuum equations admit local Cauchy developments for initial data sets (Σ_0, g, k) with locally finite L^2 curvature and locally finite L^2 norm of the first covariant derivatives of k*[12] .

1.3.2. The bounded L^2 curvature theorem. In this section, we state our main result which gives a positive answer to the above conjecture. We assume the space-time $(\mathcal{M}, \mathbf{g})$ to be foliated by the level surfaces Σ_t of a time function t. Let T denote the unit normal to Σ_t, and let k the the second fundamental form of Σ_t, i.e. $k_{ab} = -\mathbf{g}(\mathbf{D}_a T, e_b)$, where $e_a, a = 1, 2, 3$ denotes an arbitrary frame on Σ_t and $\mathbf{D}_a T = \mathbf{D}_{e_a} T$. We assume that the Σ_t foliation is maximal, i.e. we have

$$g^{ab}k_{ab} = 0 \tag{1.8}$$

where g is the induced metric on Σ_t.

We also recall below the definition of the volume radius on a general Riemannian manifold M.

Definition 1.5. Let $B_r(p)$ denote the geodesic ball of center p and radius r. The volume radius $r_{vol}(p, r)$ at a point $p \in M$ and scales $\leq r$ is defined by

$$r_{vol}(p, r) = \inf_{r' \leq r} \frac{|B_{r'}(p)|}{r^3},$$

with $|B_r|$ the volume of B_r relative to the metric on M. The volume radius $r_{vol}(M, r)$ of M on scales $\leq r$ is the infimum of $r_{vol}(p, r)$ over all points $p \in M$.

Our main result is the following.

Theorem 1.6 (Main theorem). *Let $(\mathcal{M}, \mathbf{g})$ an asymptotically flat solution to the Einstein vacuum equations* (1.1) *together with a maximal foliation by space-like hypersurfaces Σ_t*

[11] The curvature tensor of g and the first order derivatives of the second fundamental form k are both at the level of two derivatives of $\mathbf{g}$. Thus, Conjecture 1.4 is at the level of two derivatives of $\mathbf{g}$ in L^2 which indeed corresponds to the case $s = 2$.

[12] As we shall see, from the precise theorem stated below, other weaker conditions, such as a lower bound on the volume radius, are needed.

defined as level hypersurfaces of a time function t. Assume that the initial slice (Σ_0, g, k) is such that the Ricci curvature $Ric \in L^2(\Sigma_0)$, $\nabla k \in L^2(\Sigma_0)$, and Σ_0 has a strictly positive volume radius on scales ≤ 1, i.e. $r_{vol}(\Sigma_0, 1) > 0$.

1. L^2 ***regularity.*** *There exists a time*

$$T = T(\|Ric\|_{L^2(\Sigma_0)}, \|\nabla k\|_{L^2(\Sigma_0)}, r_{vol}(\Sigma_0, 1)) > 0$$

and a constant

$$C = C(\|Ric\|_{L^2(\Sigma_0)}, \|\nabla k\|_{L^2(\Sigma_0)}, r_{vol}(\Sigma_0, 1)) > 0$$

such that the following control holds on $0 \leq t \leq T$:

$$\|\mathbf{R}\|_{L^\infty_{[0,T]}L^2(\Sigma_t)} \leq C,\ \|\nabla k\|_{L^\infty_{[0,T]}L^2(\Sigma_t)} \leq C \text{ and } \inf_{0\leq t\leq T} r_{vol}(\Sigma_t, 1) \geq \frac{1}{C}.$$

2. ***Higher regularity.*** *Within the same time interval as in part (1) we also have the higher derivative estimates*[13] *,*

$$\sum_{|\alpha|\leq m} \|\mathbf{D}^{(\alpha)}\mathbf{R}\|_{L^\infty_{[0,T]}L^2(\Sigma_t)} \leq C_m \sum_{|i|\leq m} \Big[\|\nabla^{(i)} Ric\|_{L^2(\Sigma_0)} + \|\nabla^{(i)}\nabla k\|_{L^2(\Sigma_0)}\Big], \tag{1.9}$$

where C_m depends only on the previous C and m.

Let us comment on Theorem 1.6.

1. As mentioned in the previous section, the well posedness result of [28] in H^s with $s > 2$ is sharp for general quasilinear wave equations of type (1.4). To do better, one needs to take into account the so called *null structure*, i.e. the special nonlinear structure of the Einstein equations. In particular, Theorem 1.6 is the first well posedness result in which the full structure of the quasilinear hyperbolic system, not just its principal part, plays a crucial role.
2. The assumptions of Theorem 1.6 concern the L^2 norm of the curvature tensor of g and of the first covariant derivatives of the second fundamental form k which are all invariant in the sense that these objects can be defined without reference to any coordinates system[14] . This allows, when working in the framework of the solutions constructed in Theorem 1.6, to retain an essential property of the Einstein equations, namely the freedom to pick a coordinates system (see Remark 1.1).
3. The part of Theorem 1.6 dealing with the propagation of higher order regularity provides a continuation argument for the Einstein equations; that is the space-time constructed by evolution from smooth data can be smoothly continued, together with a time foliation, as long as the curvature of the foliation and the first covariant derivatives of its second fundamental form remain L^2- bounded along the leaves of the foliation. In fact, Theorem 1.6 implies the break-down criterion previously obtained in [19] and improved in [24, 42]. Furthermore, this break-down criterion involves only invariant assumptions, and hence provides information on true singularities (as opposed to coordinates singularities).

[13] Assuming that the initial has more regularity so that the right-hand side of (1.9) makes sense.

[14] Note that this is not the case for instance of the result in [14] where one has to choose a fixed coordinates system with respect to which the metric coefficients are in H^s for $s > 2$.

4. One may wonder whether the solutions constructed in Theorem 1.6 are as rough as possible. To discuss this issue, observe that the light cones of a Lorentzian space-time $(\mathcal{M}, \mathbf{g})$ can be obtained as the level hypersurfaces of a solution u to the Eikonal equation
$$\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0.$$
Now, a byproduct of the proof of Theorem 1.6 is the fact that L^2 bounds on the curvature is the minimum requirement to control solutions u to the Eikonal equation (see Remark 4.1), and hence to make sense of light cones. As far as light cones are fundamental objects in Lorentzian space-times, it is reasonable to expect their control to be necessary to make sense of solutions to the Einstein equations. For this reason, we conjecture that Theorem 1.6 is optimal.

The entire proof of Theorem 1.6 is contained in the sequence of papers [20, 33–37]. In the rest of this paper, we discuss the general strategy of the proof as well as the main steps.

2. Sketch of the proof of the main theorem

2.1. Strategy of the proof. As mentioned earlier, the well posedness result of [28] in H^s for $s > 2$ is sharp for general quasilinear wave equations of type (1.4). To do better one needs to take into account the special structure of the Einstein equations and rely on a class of estimates which go beyond Strichartz estimates, namely the so called bilinear estimates. In the case of semilinear wave equations, such as Wave Maps, Maxwell-Klein-Gordon and Yang-Mills, the first results which make use of bilinear estimates go back to [9–11]. In the particular case of the Yang-Mills equation the main observation was that, after the choice of a special gauge (Coulomb gauge), the most dangerous nonlinear terms exhibit a special, null structure so that the system reduces to the following schematic form
$$\Box\phi = Q_{ij}(\phi, \nabla^{-1}\phi) + \nabla^{-1}(Q_{ij}(\phi, \phi)) + l.o.t, \tag{2.1}$$
where ϕ is vector valued[15] and Q_{ij} is the null form given by
$$Q_{ij}(\phi, \psi) = \partial_i\phi\partial_j\psi - \partial_i\psi\partial_j\phi, \ \ i, j = 1, 2, 3, \tag{2.2}$$
for which one can apply the bilinear estimates derived in [9]. With the help of these estimates one was able to derive a well posedness result, in the flat $1+3$ dimensional Minkowski space, for the exponent $s = 1$[16] .

To carry out a similar program in the case of the Einstein equations one would need, at the very least to

1. Exhibit the null structure, i.e. provide a coordinate condition, relative to which the Einstein vacuum equations verify an appropriate version of the null condition.
2. Exploit the null structure, i.e. prove bilinear estimates for the null quadratic terms appearing in the previous step.

[15] Note that (2.1) is a system. In particular, the schematic notation $Q_{ij}(\phi, \phi)$ should be understood as being a linear combination of terms of the type $Q_{ij}(\phi^m, \phi^l)$ where ϕ^l and ϕ^m denote components of the vector valued function ϕ.

[16] This corresponds precisely to the $s = 2$ exponent in the case of the Einstein-vacuum equations.

Concerning the coordinate condition, let us first mention that it is a priori not at all clear what it should be, or even if there is one for that matter.

Remark 2.1. The only known structural condition related to the classical null condition, called the *weak null condition* [23], tied to wave coordinates, fails the test. Indeed, the following simple system in Minkowski space

$$\Box\phi = 0, \qquad \Box\psi = \phi \cdot \Delta\phi$$

verifies the weak null condition and yet, according to [22], it is ill posed for $s = 2$. Coordinate conditions, such as spatial harmonic[17] , also do not seem to work.

We rely instead on a Coulomb type condition, for orthonormal frames, adapted to a maximal foliation. Such a gauge condition appears naturally if we adopt a Yang-Mills description of the Einstein field equations using Cartan's formalism of moving frames, see [4]. It is important to note nevertheless that it is not at all a priori clear that such a choice would do the job. Indeed, the null form nature of the Yang-Mills equations in the Coulomb gauge is only revealed once we commute the resulting equations with the projection operator $\mathcal{P}$ on the divergence free vectorfields. Such an operation is natural in that case, since $\mathcal{P}$ commutes with the flat d'Alembertian. In the case of the Einstein equations, however, the corresponding commutator term $[\Box_{\mathbf{g}}, \mathcal{P}]$ generates[18] a whole host of new terms and it is quite a miracle that they can all be treated by an extended version of bilinear estimates.

Concerning bilinear estimates, let us mention that these types of estimates where only available for the wave operator on the Minkowski space-time. This forces us to find an appropriate geometric framework to extend these estimates to the wave operator on a curved space-time. To this end, we need to rely on a plane wave representation - a parametrix - for solutions of the wave equation on a curved background. Moreover, this parametrix, unlike in the flat case, is only an approximate solution of the wave equation. In other words, when applying the wave operator to the parametrix, we obtain an error term which needs to be controlled.

Furthermore, there is another ingredient needed to establish bilinear estimates on a curved space-time. Numerous bilinear estimates need to be derived, and it turns out that the proof of several of these estimates reduces to sharp $L^4(\mathcal{M})$ Strichartz estimates for a localized version of the parametrix.

Finally, the above discussion leads to the following four steps which constitute the basic strategy of our main theorem

A. Exhibit the null structure by recasting the Einstein vacuum equations as a quasilinear Yang-Mills theory[19] .

B. Prove appropriate bilinear estimates for solutions ϕ to the scalar wave equation on a curved space-time $\Box_{\mathbf{g}}\phi = 0$.

[17] Maximal foliation together with spatial harmonic coordinates on the leaves of the foliation would be the coordinate condition closest in spirit to the Coulomb gauge.

[18] Note also that additional error terms are generated by projecting the equations on the components of the frame.

[19] The classical Yang-Mills equations are semilinear, i.e. they are defined on a given (Lorentzian) background. Here, we recast the Einstein vacuum equations as a Yang-Mills theory on the background $(\mathcal{M}, \mathbf{g})$ solution to (1.1). As the background is not given but instead the unknown of the problem itself, we obtain a quasilinear analog of the Yang-Mills equations.

C. Construct an effective progressive wave representation Φ_F (parametrix) for solutions to the scalar linear wave equation $\Box_{\mathbf{g}}\phi = F$, derive appropriate bounds for both the parametrix and the corresponding error term $E = F - \Box_{\mathbf{g}}\Phi_F$ and use them to derive the desired bilinear estimates.

D. Prove sharp $L^4(\mathcal{M})$ Strichartz estimates for a localized version of the parametrix of step C.

While Step A is purely algebraic, Steps B, C and D all require to establish estimates. The main difficulty is to implement these steps using only hypothetical L^2 bounds for the space-time curvature tensor, consistent with the statement of our main theorem. To achieve this, we crucially need to exploit the null structure of the equations at every stage in the proof.

In the rest of the paper, we comment on each of these steps. We start with Step A in the next section. We then show how to conclude the proof of the main theorem when assuming Steps B, C and D. Finally, we discuss Steps B, C and D.

2.2. The Yang-Mills formalism (Step A). We cast the Einstein-vacuum equations in a Yang-Mills form which corresponds to step A in the strategy outlined above. This relies on the Cartan formalism of moving frames. The idea is to give up on a choice of coordinates and instead express the Einstein vacuum equations in terms of the connection 1-forms associated to moving orthonormal frames, i.e. vectorfields e_α, which verify,

$$\mathbf{g}(e_\alpha, e_\beta) = \mathbf{m}_{\alpha\beta} = \text{diag}(-1, 1, 1, 1).$$

The connection 1-forms (they are to be interpreted as 1-forms with respect to the external index μ with values in the Lie algebra of $so(3,1)$), defined by the formulas,

$$(\mathbf{A}_\mu)_{\alpha\beta} = \mathbf{g}(\mathbf{D}_\mu e_\beta, e_\alpha) \tag{2.3}$$

verify the equations,

$$\mathbf{D}^\mu \mathbf{F}_{\mu\nu} + [\mathbf{A}^\mu, \mathbf{F}_{\mu\nu}] = 0 \tag{2.4}$$

where, denoting $(\mathbf{F}_{\mu\nu})_{\alpha\beta} := \mathbf{R}_{\alpha\beta\mu\nu}$,

$$(\mathbf{F}_{\mu\nu})_{\alpha\beta} = \big(\mathbf{D}_\mu \mathbf{A}_\nu - \mathbf{D}_\nu \mathbf{A}_\mu - [\mathbf{A}_\mu, \mathbf{A}_\nu]\big)_{\alpha\beta}. \tag{2.5}$$

In other words we can interpret the curvature tensor as the curvature of the $so(3,1)$-valued connection 1-form $\mathbf{A}$. Note also that the covariant derivatives are taken only with respect to the *external indices* μ, ν and do not affect the *internal indices* α, β. We can rewrite (2.4) in the form,

$$\Box_{\mathbf{g}} \mathbf{A}_\nu - \mathbf{D}_\nu(\mathbf{D}^\mu \mathbf{A}_\mu) = \mathbf{J}_\nu(\mathbf{A}, \mathbf{DA}) \tag{2.6}$$

where,

$$\mathbf{J}_\nu = \mathbf{D}^\mu([\mathbf{A}_\mu, \mathbf{A}_\nu]) - [\mathbf{A}_\mu, \mathbf{F}_{\mu\nu}].$$

Observe that the equations (2.4)-(2.5) look just like the Yang-Mills equations on a fixed Lorentzian manifold $(\mathcal{M}, \mathbf{g})$ except, of course, that in our case $\mathbf{A}$ and $\mathbf{g}$ are not independent but rather connected by (2.3), reflecting the quasilinear structure of the Einstein equations. Just as in the case of [9], which establishes the well-posedness of the Yang-Mills equation in Minkowski space in the energy norm (i.e. $s = 1$), we rely in an essential manner on a Coulomb type gauge condition. More precisely, we take e_0 to be the future unit normal to

the Σ_t foliation and choose e_1, e_2, e_3 an orthonormal basis to Σ_t, in such a way that we have, essentially $div\, A = \nabla^i A_i = 0$, where A is the spatial component of $\mathbf{A}$. It turns out that A_0 satisfies an elliptic equation while each component $A_i = \mathbf{g}(\mathbf{A}, e_i)$, $i = 1, 2, 3$ verifies an equation of the form,

$$\Box_{\mathbf{g}} A_i = -\partial_i(\partial_0 A_0) + A^j \partial_j A_i + A^j \partial_i A_j + \text{ l.o.t.} \tag{2.7}$$

with l.o.t. denoting nonlinear terms which can be treated by more elementary techniques (including non sharp Strichartz estimates).

2.3. The proof of the bounded L^2 curvature theorem. In this section, assuming that step B holds - which corresponds to having appropriate bilinear estimates at our disposal - we conclude the proof Theorem 1.6. Now, to be in position to use these bilinear estimates, we first need to reduce the problem to a wave equation. In view of (2.7), we thus need to eliminate $\partial_i(\partial_0 A_0)$. To this end, we need to project (2.7) onto divergence free vectorfields with the help of a non-local operator which we denote by $\mathcal{P}$. In the case of the flat Yang-Mills equations, treated in [9], this leads to an equation of the form,

$$\Box A_i = \mathcal{P}(A^j \partial_j A_i) + \mathcal{P}(A^j \partial_i A_j) + \text{ l.o.t.}$$

where both terms on the right exhibit the null structure[20] . In our case however, the operator $\mathcal{P}$ does not commute with $\Box_{\mathbf{g}}$. It turns out, fortunately, that the terms generated by commutation can still be estimated by an extended class of bilinear estimates which includes contractions with the curvature tensor. Thus, we obtain in the end schematically for A_i

$$\Box_{\mathbf{g}} A_i = \text{null forms} + \text{ l.o.t.}, \tag{2.8}$$

where up to (cubic) lower order terms, the quadratic terms exhibit the null structure.

We are now in position to conclude the proof of our main theorem. Recall that the A_i are connection coefficients, and hence the curvature is a the level of one derivative of the A_i (see (2.5)). In particular, controlling the curvature tensor in L^2 corresponds to the control of first order derivatives of A_i in L^2. In other words, we need to run the energy estimate for the wave equation (2.8). In the case of the standard wave equation on the Minkowski space-time, the energy estimate is based on the usual timelike Killing vectorfield ∂_t. In our case, the corresponding vectorfield $e_0 = T$ (the future unit normal to Σ_t) is not Killing. This leads to another class of trilinear error terms. That is to say, to control the energy estimates for the wave equation (2.8) we need trilinear estimates, while to control the null forms in the right-hand side we need bilinear estimates. Assuming these bilinear and trilinear estimates hold, we finally control first order derivatives of A_i in L^2 and hence the curvature tensor in L^2. This concludes the proof of Theorem 1.6.

The rest of the paper is organized as follows. In section 3, we discuss the derivation of the bilinear estimates[21] which corresponds to Step B. This derivation relies on Step C and Step D which we discuss respectively in section 4 and 5.

[20] This corresponds to (2.1) where the null structure manifests itself in the presence of the null form Q_{ij} in the right-hand side.

[21] As we have seen, trilinear estimates have to be derived as well, but we skip this part for the sake of simplicity.

3. Bilinear estimates (Step B)

3.1. The plane wave representation on a curved space-time. In order to establish bilinear estimates on a curved space-time, we need to rely on a plane wave representation formula[22] for solutions of scalar wave equations,

$$\Box_{\mathbf{g}}\phi = 0.$$

To build such a plane wave representation, consider a plane wave

$$e^{i\lambda\ {}^{\omega}u(t,x)},\ \lambda \in [0,+\infty),\ \omega \in \mathbb{S}^2,$$

with λ and ω parameters corresponding to Fourier variables in $\mathbb{R}^3$ in spherical coordinates. We compute

$$\Box_{\mathbf{g}}(e^{i\lambda\ {}^{\omega}u}) = (-\lambda^2\mathbf{g}^{\alpha\beta}\partial_\alpha({}^{\omega}u)\partial_\beta({}^{\omega}u) + i\lambda\Box_{\mathbf{g}}({}^{\omega}u))e^{i\lambda\ {}^{\omega}u}.$$

The first term turns out to be the most dangerous one[23] . This motivates to choose ${}^{\omega}u$ solution to the Eikonal equation

$$\mathbf{g}^{\alpha\beta}\partial_\alpha({}^{\omega}u)\partial_\beta({}^{\omega}u) = 0,$$

in which case we obtain

$$\Box_{\mathbf{g}}(e^{i\lambda\ {}^{\omega}u}) = i\lambda\Box_{\mathbf{g}}({}^{\omega}u)e^{i\lambda\ {}^{\omega}u}.$$

This yields in general an approximate solution to $\Box_{\mathbf{g}}(\phi) = 0$. We then superpose these plane waves to obtain a full plane wave representation.

In the particular case of the standard wave equation on the Minkowski space-time, we recover the well-known plane wave representation which is an exact solution[24] . We have

$$\phi = \sum_{\pm}\int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\lambda\ {}^{\omega}u_{\pm}(t,x)} f_{\pm}(\lambda\omega)\lambda^2 d\lambda d\omega, \tag{3.1}$$

where ${}^{\omega}u_{\pm}(t,x) = \pm t + x\cdot\omega$ and $f_{\pm}$ can be explicitly computed in terms of the Fourier transform of the initial data set $(\phi(0,.),\partial_t\phi(0,.))$ of ϕ. In the general case, we superpose the basic plane waves as in the right-hand side of (3.1), and choose ${}^{\omega}u_{\pm}$ solution of the Eikonal equation with the following asymptotic behavior on Σ_0

$${}^{\omega}u_{\pm}(0,x) \sim x\cdot\omega \text{ when } |x| \to +\infty.$$

This asymptotic behavior is necessary to be able to generate any initial data of the wave equation

In view of the previous paragraph, we consider the following representation formula[25]

$$\phi_f(t,x) = \int_{\mathbb{S}^2}\int_0^{\infty} e^{i\lambda\ {}^{\omega}u(t,x)}\, f(\lambda\omega)\lambda^2 d\lambda d\omega \tag{3.2}$$

[22] We follow the proof of the bilinear estimates outlined in [15] which differs substantially from that of [9] and is reminiscent of the null frame space strategy used by Tataru in his fundamental paper [39].

[23] λ should be understood as a Fourier variable corresponding to a derivative in physical space. The λ^2 term hence costs 2 derivatives while the wave equation only recovers one. Thus, this term is problematic as it induces a derivative loss.

[24] This is special to the flat case. In the general case, we only obtain an approximate solution.

[25] (3.2) actually corresponds to the representation formula for a half-wave. The full representation formula corresponds to the sum of two half-waves as in (3.1). Since the bilinear estimates are identical for both half waves, we only consider one of them for simplicity.

where f represents schematically the initial data[26], and where ${}^{\omega}u$ is a solution of the eikonal equation[27],

$$\mathbf{g}^{\alpha\beta}\partial_\alpha({}^{\omega}u)\,\partial_\beta({}^{\omega}u)=0, \tag{3.3}$$

with the following asymptotic behavior on Σ_0

$${}^{\omega}u(0,x)\sim x\cdot\omega \text{ when } |x|\to+\infty.$$

Remark 3.1. (3.3) is a nonlinear transport equation. Hence, ${}^{\omega}u$ needs to be prescribed not only at infinity on Σ_0 as explained above, but everywhere on Σ_0. This choice of ${}^{\omega}u$ on Σ_0 turns out to be delicate and is discussed in section 4 (see Step C1 and related subsequent comments).

Remark 3.2. Note that (3.2) is a parametrix for a scalar wave equation. The lack of a good parametrix for a tensorial wave equation forces us to develop a strategy based on writing the main equation in components relative to a frame, i.e. instead of dealing with the tensorial wave equation (2.6) directly, we consider the system of scalar wave equations (2.7). Unlike the flat case, this "scalarization" procedure produces several terms which are potentially dangerous, and it is fortunate, as in yet another manifestation of a hidden null structure of the Einstein equations, that they can still be controlled by the use of an extended[28] class of bilinear estimates.

3.2. Bilinear estimates on a curved space-time. The bilinear estimates all involve after some reductions the null form Q_{ij} introduced in (2.2). Let us briefly explain how the structure of Q_{ij} is exploited to derive these estimates. For simplicity, we focus on two specific bilinear estimates[29].

The first example of a bilinear estimate on a curved space-time aims at controlling the $L^2(\mathcal{M})$ norm of the null form $Q_{ij}(\phi_f,\psi)$, where ϕ_f is given by (3.2). We compute

$$\begin{aligned}Q_{ij}(\phi_f,\psi)&=Q_{ij}\left(\int_{\mathbb{S}^2}\int_0^{+\infty}e^{i\lambda\,{}^{\omega}u(t,x)}f(\lambda\omega)\lambda^2d\lambda d\omega,\psi\right)\\&=\int_{\mathbb{S}^2}\int_0^{+\infty}Q_{ij}(e^{i\lambda\,{}^{\omega}u(t,x)},\psi)f(\lambda\omega)\lambda^2d\lambda d\omega\\&=i\int_{\mathbb{S}^2}\int_0^{+\infty}e^{i\lambda\,{}^{\omega}u(t,x)}Q_{ij}({}^{\omega}u,\psi)f(\lambda\omega)\lambda^3d\lambda d\omega.\end{aligned}$$

Now we have

$$Q_{ij}({}^{\omega}u,\psi)=\partial_i({}^{\omega}u)\partial_j\psi-\partial_j({}^{\omega}u)\partial_i\psi. \tag{3.4}$$

The fundamental observation which ultimately allows us to derive a bilinear estimate in this case is the fact that the structure of Q_{ij} is such that

$$Q_{ij}({}^{\omega}u,\psi) \text{ is tangent to the level hypersurfaces of } {}^{\omega}u, \tag{3.5}$$

[26] Here f is in fact at the level of the Fourier transform of the initial data and the norm $\|\lambda f\|_{L^2(\mathbb{R}^3)}$ corresponds, roughly, to the H^1 norm of the data .

[27] As we have seen above, we have ${}^{\omega}u(t,x)=\pm t+x\cdot\omega$ in the flat Minkowski space.

[28] involving contractions between the Riemann curvature tensor and derivatives of solutions of scalar wave equations.

[29] The two examples of bilinear estimates discussed here have both an analog in the semilinear case. Indeed, they correspond to estimating the $L^2_{t,x}$ norm of each of the term in the right-hand side of (2.1).

as can be seen from (3.4).

The second example of a bilinear estimate on a curved space-time aims at controlling the following expression

$$\|\nabla^{-1}(Q_{ij}(\phi_{f_1},\phi_{f_2}))\|_{L^2(\mathcal{M})}.$$

First, we decompose ϕ_{f_1} and ϕ_{f_2} in dyadic frequencies according to

$$\phi_f = \sum \phi_{f,p},\ \phi_{f,p} = \int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\lambda\ {}^{\omega}u(t,x)}\psi(2^{-p}\lambda)f(\lambda\omega)\lambda^2 d\lambda d\omega$$

where $\lambda\sim 2^p$ on the support of $\psi(2^{-p}\lambda)$. We infer

$$\|\nabla^{-1}(Q_{ij}(\phi_{f_1},\phi_{f_2}))\|_{L^2(\mathcal{M})} \lesssim \sum_{p\geq q}\|\nabla^{-1}(Q_{ij}(\phi_{f_1,p},\phi_{f_2,q}))\|_{L^2(\mathcal{M})}.$$

It is at this stage that we use the null structure of Q_{ij} by noticing that

$$Q_{ij}(\phi,\psi) = \partial_i(\phi\partial_j\psi) - \partial_j(\phi\partial_i\psi) = \partial_j(\psi\partial_i\phi) - \partial_i(\psi\partial_j\phi) \tag{3.6}$$

so that we may choose which derivative we factorize. We choose to factorize the derivative corresponding to the highest frequency which yields

$$\begin{aligned}\|\nabla^{-1}(Q_{ij}(\phi_{f_1},\phi_{f_2}))\|_{L^2(\mathcal{M})} &\lesssim \sum_{p\geq q}\|\nabla^{-1}\partial(\phi_{f_1,p}\partial\phi_{f_2,q})\|_{L^2(\mathcal{M})}\\ &\lesssim \sum_{p\geq q}\|\phi_{f_1,p}\|_{L^4(\mathcal{M})}\|\partial\phi_{f_2,q}\|_{L^4(\mathcal{M})}.\end{aligned}$$

The last ingredient is the sharp $L^4(\mathcal{M})$ Strichartz of Step D (see section 5) which finally yields

$$\begin{aligned}\|\nabla^{-1}(Q_{ij}(\phi_{f_1},\phi_{f_2}))\|_{L^2(\mathcal{M})} &\lesssim \sum_{p\geq q}2^{-\frac{|p-q|}{2}}\|\lambda f_{1,p}\|_{L^2(\mathbb{R}^3)}\|\lambda f_{2,q}\|_{L^2(\mathbb{R}^3)}\\ &\lesssim \|\lambda f_1\|_{L^2(\mathbb{R}^3)}\|\lambda f_2\|_{L^2(\mathbb{R}^3)}\end{aligned}$$

and concludes the proof of the second example of bilinear estimate.

Remark 3.3. The null structure of Q_{ij} is exploited differently in the two examples of bilinear estimates presented above as can be seen by comparing (3.5) and (3.6).

4. Control of the parametrix (step C)

To prove the bilinear and trilinear estimates of Step B, we need in particular to control the parametrix given by (3.2). To this end, it turns out that it suffices to control the parametrix at initial time (i.e. restricted to the initial slice Σ_0)

$$\phi_f(0,x) = \int_{\mathbb{S}^2}\int_0^{\infty} e^{i\lambda\ {}^{\omega}u(0,x)}\ f(\lambda\omega)\lambda^2 d\lambda d\omega \tag{4.1}$$

as well as the error term[30] corresponding to (3.2)

$$Ef(t,x) = \Box_{\mathbf{g}}\phi_f(t,x) = i\int_{\mathbb{S}^2}\int_0^\infty e^{i\lambda\, {}^\omega u(t,x)}\,\Box_{\mathbf{g}}({}^\omega u) f(\lambda\omega)\lambda^3 d\lambda d\omega. \tag{4.2}$$

This requires the following four sub steps

- **C1** *Make an appropriate choice for the equation satisfied by* ${}^\omega u(0,x)$ *on* Σ_0*, and control the geometry of the foliation of* Σ_0 *by the level surfaces of* ${}^\omega u(0,x)$.
- **C2** *Prove that the parametrix at* $t=0$ *given by* (4.1)[31] *is bounded in* $\mathcal{L}(L^2(\mathbb{R}^3), L^2(\Sigma_0))$ *using the estimates for* ${}^\omega u(0,x)$ *obtained in* **C1**.
- **C3** *Control the geometry of the foliation of* $\mathcal{M}$ *given by the level hypersurfaces of* ${}^\omega u$.
- **C4** *Prove that the error term* (4.2) *satisfies the estimate* $\|Ef\|_{L^2(\mathcal{M})} \lesssim \|\lambda f\|_{L^2(\mathbb{R}^3)}$ *using the estimates for* ${}^\omega u$ *and* $\Box_{\mathbf{g}}({}^\omega u)$ *proved in* **C3**.

To achieve Step C3 and Step C4, we need, at the very least, to control $\Box_{\mathbf{g}}({}^\omega u)$ in L^∞. This issue was first addressed in the sequence of papers [16–18] where an L^∞ bound for $\Box_{\mathbf{g}}({}^\omega u)$ was established, depending only on the L^2 norm of the curvature flux along null hypersurfaces. The proof required an interplay between both geometric and analytic techniques and had all the appearances of being sharp, i.e. we don't expect an L^∞ bound for $\Box_{\mathbf{g}}({}^\omega u)$ which requires bounds on less than two derivatives in L^2 for the metric[32] .

Remark 4.1. It turns out, as a byproduct of the proof of Step C3, that the radius of injectivity of the level hypersurfaces of ${}^\omega u$ is controlled by the L^∞ norm of $\Box_{\mathbf{g}}({}^\omega u)$. Furthermore, this control appears to be sharp. In other words, we expect to loose control over the radius of injectivity in the absence of this bound. Hence, in view of the discussion above, L^2 bounds on the curvature tensor appear to be minimal for the control of the Eikonal equation.

To obtain the L^2 bound for the Fourier integral operator E defined in (4.2), we need, of course, to go beyond uniform estimates for $\Box_{\mathbf{g}}({}^\omega u)$. The classical L^2 bounds for Fourier integral operators of the form (4.2) are not at all economical in terms of the number of integration by parts which are needed. In our case the total number of such integration by parts is limited by the regularity properties of the function $\Box_{\mathbf{g}}({}^\omega u)$. To get an L^2 bound for the parametrix at initial time (4.1) and the error term (4.2) within such restrictive regularity properties we need, in particular:

- In Step C1 and Step C3, a precise control of derivatives of ${}^\omega u$ and $\Box_{\mathbf{g}}({}^\omega u)$ with respect to both ω as well as with respect to various directional derivatives[33] . To get optimal control we need, in particular, a very careful construction of the initial condition for ${}^\omega u$ on Σ_0 and then sharp space-time estimates of Ricci coefficients, and their derivatives, associated to the foliation induced by ${}^\omega u$.

[30] Note that ϕ_f is an exact solution of $\Box_{\mathbf{g}}\phi = 0$ only if $\Box_{\mathbf{g}}({}^\omega u) = 0$. Hence, ϕ_f is an exact solution only in flat space.

[31] (4.1) only corresponds to the value at $t=0$ of a half wave parametrix. The full parametrix at initial data is the sum of two half waves as in (3.1). Step C2 actually corresponds to proving that the parametrix at $t=0$ generates any initial data to the wave equation $\Box_{\mathbf{g}}\phi = 0$ with a suitable control of the corresponding $f_\pm$. We have chosen to provide a more restricted statement of Step C2 to simplify the exposition.

[32] classically, this requires, at the very least, the control of $\mathbf{R}$ in L^∞.

[33] Taking into account the different behavior in tangential and transversal directions with respect to the level surfaces of ${}^\omega u$.

- In Step C2 and Step C4, a careful decompositions of the Fourier integral operators (4.1) and (4.2) in both λ and ω, similar to the first and second dyadic decomposition in harmonic analysis, see [30], as well as a third decomposition, which in the case of (4.2) is done with respect to the space-time variables relying on the geometric Littlewood-Paley theory developed in [18].

Below, we make further comments on Steps C1-C4:

1. *The choice of ${}^{\omega}u(0,x)$ on Σ_0 in Step C1.* Let us note that the typical choice ${}^{\omega}u(0,x) = x \cdot \omega$ in a given coordinate system would not work for us, since we don't have enough control on the regularity of a given coordinate system within our framework. Instead, we need to find a geometric definition of ${}^{\omega}u(0,x)$. A natural choice would be that $u = {}^{\omega}u$ verifies
$$\Box_{\mathbf{g}} u = 0 \text{ on } \Sigma_0$$
which by a simple computation turns out to be the following simple variant of the minimal surface equation[34]
$$div\left(\frac{\nabla u}{|\nabla u|}\right) = k\left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right) \text{ on } \Sigma_0.$$
Unfortunately, this choice does not allow us to have enough control of the derivatives of u in the normal direction to the level surfaces of u. This forces us to look for an alternate equation for u:
$$div\left(\frac{\nabla u}{|\nabla u|}\right) = 1 - \frac{1}{|\nabla u|} + k\left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right) \text{ on } \Sigma_0.$$
This equation turns out to be parabolic in the normal direction to the level surfaces of u, and allows us to obtain the desired regularity in Step C1. A closer inspection reveals its relation to the mean curvature flow on Σ_0.
2. *How to achieve Step C3.* The regularity obtained in Step C1, together with null transport equations tied to the eikonal equation, elliptic systems of Hodge type, the geometric Littlewood-Paley theory of [18], sharp trace theorems, and an extensive use of the structure of the Einstein equations, allows us to propagate the regularity on Σ_0 to the space-time, thus achieving Step C3.
3. *The regularity with respect to ω in Steps C1 and C3.* The regularity with respect to x for u is clearly limited as a consequence of the fact that we only assume L^2 bounds on $\mathbf{R}$. On the other hand, $\mathbf{R}$ is independent of the parameter ω, and one might infer that u is smooth with respect to ω. Surprisingly, this is not at all the case. Indeed, the regularity in x obtained for u in Steps C1 and C3 is better in directions tangent to the level hypersurfaces of u. Now, the ω derivatives of the tangential directions have non zero normal components. Thus, when differentiating the structure equations with respect to ω, tangential derivatives to the level surfaces of u are transformed to non tangential derivatives which in turn severely limits the regularity in ω obtained in Steps C1 and C3.

[34] In the time symmetric case $k = 0$, this is exactly the minimal surface equation.

4. *How to achieve Steps C2 and C4.* The classical arguments for proving L^2 bounds for Fourier operators are based either on a TT^* argument, or a T^*T argument, which requires several integration by parts either with respect to x for T^*T, of with respect to (λ, ω) for TT^*. Both methods would fail by far within the regularity for u obtained in Step C1 and Step C3. This forces us to design a method which allows to take advantage both of the regularity in x and ω. This is achieved using in particular the following ingredients

   - Geometric integrations by parts taking full advantage of the better regularity properties in directions tangent to the level hypersurfaces of u.
   - The standard first and second dyadic decomposition in frequency space, with respect to both size and angle (see [30]), an additional decomposition in physical space relying on the geometric Littlewood-Paley projections of [18] for Step C4, as well as another decomposition involving frequency and angle for Step C2.

   Even with these precautions, at several places in the proof, one encounters log-divergences which have to be tackled by ad-hoc techniques, taking full advantage of the null structure of the Einstein equations.

5. Sharp $L^4(\mathcal{M})$ Strichartz estimates (Step D)

Recall that the parametrix constructed in Step C is also used to prove sharp $L^4(\mathcal{M})$ Strichartz estimates. Indeed the proof of several bilinear estimates of Step B reduces to the proof of sharp $L^4(\mathcal{M})$ Strichartz estimates for the parametrix (3.2) with λ localized in a dyadic shell (see section 3.2).

More precisely, let $j \geq 0$, and let ψ a smooth function on $\mathbb{R}^3$ supported in

$$\frac{1}{2} \leq |\xi| \leq 2.$$

Let $\phi_{f,j}$ the parametrix (3.2) with a additional frequency localization $\lambda \sim 2^j$

$$\phi_{f,j}(t,x) = \int_{\mathbb{S}^2} \int_0^\infty e^{i\lambda\, {}^\omega u(t,x)} \psi(2^{-j}\lambda) f(\lambda\omega) \lambda^2 d\lambda d\omega. \tag{5.1}$$

We will need the sharp[35] $L^4(\mathcal{M})$ Strichartz estimate

$$\|\phi_{f,j}\|_{L^4(\mathcal{M})} \lesssim 2^{\frac{j}{2}} \|\psi(2^{-j}\lambda) f\|_{L^2(\mathbb{R}^3)}. \tag{5.2}$$

The standard procedure for proving[36] (5.2) is based on a TT^* argument which reduces it to an L^∞ estimate for an oscillatory integral with a phase involving ${}^\omega u$. This is then achieved by the method of stationary phase which requires quite a few integrations by parts. In fact the standard argument would require, at the very least[37] , that the phase function $u = {}^\omega u$ verifies,

$$\partial_{t,x} u \in L^\infty,\ \partial_{t,x} \partial_\omega^2 u \in L^\infty. \tag{5.3}$$

[35] Note in particular that the corresponding estimate in the flat case is sharp.

[36] Note that the procedure we describe would prove not only (5.2) but the full range of mixed Strichartz estimates.

[37] The regularity (5.3) is necessary to make sense of the change of variables involved in the stationary phase method.

This level of regularity is, unfortunately, incompatible with the regularity properties of solutions to our eikonal equation (3.3). In fact, based on the estimates for ${}^{\omega}u$ derived in step C3, we are only allowed to assume

$$\partial_{t,x} u \in L^\infty,\ \partial_{t,x}\partial_\omega u \in L^\infty. \tag{5.4}$$

We are thus forced to follow an alternative approach[38] to the stationary phase method inspired by [27, 28] .

References

[1] H. Bahouri and J.-Y. Chemin, *Équations d'ondes quasilinéaires et estimation de Strichartz*, Amer. J. Math., **121** (1999), 1337–1777.

[2] ———, *Équations d'ondes quasilinéaires et effet dispersif*, IMRN, **21** (1999), 1141–1178.

[3] Y. C. Bruhat, *Theoreme d'Existence pour certains systemes d'equations aux derivees partielles nonlineaires.*, Acta Math. **88** (1952), 141–225.

[4] E. Cartan, *Sur une généralisation de la notion de courbure de Riemann et les espaces torsion*, C. R. Acad. Sci. (Paris), **174** (1922), 593–595.

[5] D. Christodoulou, *Bounded variation solutions of the spherically symmetric einstein-scalar field equations*, Comm. Pure and Appl. Math, **46** (1993), 1131–1220.

[6] ———, *The instability of naked singularities in the gravitational collapse of a scalar field*, Ann. of Math., **149** (1999), 183–217.

[7] A. Fischer and J. Marsden, *The Einstein evolution equations as a first-order quasi-linear symmetric hyperbolic system. I*, Comm. Math. Phys. **28** (1972), 1–38.

[8] T. J. R. Hughes, T. Kato, and J. E. Marsden, *Well-posed quasi-linear second-order hyperbolic systems with applications to nonlinear elastodynamics and general relativity*, Arch. Rational Mech. Anal. **63** (1977), 273–394.

[9] S. Klainerman and M. Machedon, *Space-time estimates for null forms and the local existence theorem*, Communications on Pure and Applied Mathematics, **46** (1993), 1221–1268.

[10] ———, *Finite energy solutions of the Maxwell-Klein-Gordon equations*, Duke Math. J. **74** (1994), 19–44.

[11] ———, *Finite Energy Solutions for the Yang-Mills Equations in* $\mathbb{R}^{1+3}$, Annals of Math. **142** (1995), 39–119.

[12] S. Klainerman, *PDE as a unified subject*, Proceeding of Visions in Mathematics, GAFA 2000(Tel Aviv 1999). Geom Funct. Anal. 2000, Special Volume , Part 1, 279–315.

[38] We refer to the approach based on the overlap estimates for wave packets derived in [27] and [28] in the context of Strichartz estimates respectively for $C^{1,1}$ and $H^{2+\epsilon}$ metrics. Note however that our approach does not require a wave packet decomposition.

[13] S. Klainerman, I. Rodnianski, *Improved local well-posedness for quasi-linear wave equations in dimension three*, Duke Math. J. **117**(1) (2003), 1–124.

[14] ———, *Rough solutions to the Einstein vacuum equations*, Annals of Math. **161** (2005), 1143–1193.

[15] ———, *Bilinear estimates on curved space-times*, J. Hyperbolic Differ. Equ. **2**(2) (2005), 279–291.

[16] ———, *Casual geometry of Einstein vacuum space-times with finite curvature flux*, Inventiones **159** (2005), 437–529.

[17] ———, *Sharp trace theorems on null hypersurfaces*, GAFA **16**(1) (2006), 164–229.

[18] ———, *A geometric version of Littlewood-Paley theory*, GAFA **16**(1) (2006), 126–163.

[19] ———, *On a break-down criterion in General Relativity*, J. Amer. Math. Soc. **23** (2010), 345–382.

[20] S. Klainerman, I. Rodnianski, and J. Szeftel, *The Bounded L^2 Curvature Conjecture*, arXiv:1204.1767, 91 p, 2012.

[21] J. Krieger and W. Schlag, *Concentration compactness for critical wave maps*, Monographs of the European Mathematical Society, 2012.

[22] H. Lindblad, *Counterexamples to local existence for quasilinear wave equations*, Amer. J. Math. **118**(1) (1996), 1–16.

[23] H. Lindblad and I. Rodnianski, *The weak null condition for the Einstein vacuum equations*, C. R. Acad. Sci. **336** (2003), 901–906.

[24] D. Parlongue, *An integral breakdown criterion for Einstein vacuum equations in the case of asymptotically flat spacetimes*, arXiv:1004.4309, 88 p, 2010.

[25] F. Planchon and I. Rodnianski, *Uniqueness in general relativity*, preprint.

[26] G. Ponce and T. Sideris, *Local regularity of nonlinear wave equations in three space dimensions*, Comm. PDE **17** (1993), 169–177.

[27] H. F. Smith, *A parametrix construction for wave equations with $C^{1,1}$ coefficients*, Ann. Inst. Fourier (Grenoble) **48** (1998), 797–835.

[28] H.F. Smith and D. Tataru, *Sharp local well-posedness results for the nonlinear wave equation*, Ann. of Math. **162** (2005), 291–366.

[29] S. Sobolev, *Methodes nouvelle a resoudre le probleme de Cauchy pour les equations lineaires hyperboliques normales*, Matematicheskii Sbornik, **1**(43) (1936), 31–79.

[30] E. Stein, *Harmonic Analysis*, Princeton University Press, 1993.

[31] J. Sterbenz and D. Tataru, *Regularity of Wave-Maps in dimension $2+1$*, Comm. Math. Phys. **298**(1) (2010), 231–264.

[32] ———, *Energy dispersed large data wave maps in* $2+1$ *dimensions*, Comm. Math. Phys. **298**(1) (2010), 139–230.

[33] J. Szeftel, *Parametrix for wave equations on a rough background I: Regularity of the phase at initial time*, arXiv:1204.1768, 145 p, 2012.

[34] ———, *Parametrix for wave equations on a rough background II: Construction of the parametrix and control at initial time*, arXiv:1204.1769, 84 p, 2012.

[35] ———, *Parametrix for wave equations on a rough background III: Space-time regularity of the phase*, arXiv:1204.1770, 276 p, 2012.

[36] ———, *Parametrix for wave equations on a rough background IV: Control of the error term*, arXiv:1204.1771, 284 p, 2012.

[37] ———, *Sharp Strichartz estimates for the wave equation on a rough background*, arXiv:1301.0112, 30 p, 2013.

[38] T. Tao, *Global regularity of wave maps I–VII*, preprints.

[39] D. Tataru, *Local and global results for Wave Maps I*, Comm. PDE **23** (1998), 1781–1793.

[40] ———, *Strichartz estimates for operators with non smooth coefficients and the nonlinear wave equation*, Amer. J. Math. **122** (2000), 349–376.

[41] ———, *Strichartz estimates for second order hyperbolic operators with non smooth coefficients*, J.A.M.S. **15**(2) (2002), 419–442.

[42] Q. Wang, *Improved breakdown criterion for Einstein vacuum equation in CMC gauge*, Comm. Pure Appl. Math. **65**(1) (2012), 21–76.

Department of Mathematics, Princeton University, Princeton NJ 08544
E-mail: seri@math.princeton.edu

Department of Mathematics, Princeton University, Princeton NJ 08544 & Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139
E-mail: irod@math.princeton.edu & irod@math.mit.edu

Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, 75005 Paris
E-mail: jeremie.szeftel@upmc.fr

Some recent advances in microlocal analysis

András Vasy

Abstract. In this talk we describe some recent developments in microlocal analysis that have led to advances in understanding problems such as wave propagation, the Laplacian on asymptotically hyperbolic spaces and the meromorphic continuation of the dynamical zeta function for Anosov flows.

Mathematics Subject Classification (2010). Primary 35S05; Secondary 35A21, 35P25, 35L05, 58J47, 58J50.

Keywords. pseudodifferential operators, real principal type, radial,points, normally hyperbolic trapping, semiclassical analysis, asymptotically hyperbolic spaces, Kerr-de Sitter space, Melrose's, b-analysis, non-linear equations.

1. Introduction

In this talk we describe some recent developments in microlocal analysis that have led to advances in understanding problems such as wave propagation, the Laplacian on asymptotically hyperbolic spaces and the meromorphic continuation of the dynamical zeta function for Anosov flows. We state some of these results as theorems directly, giving details in the body of the notes.

The first theorem concerns asymptotically hyperbolic spaces, which are n-dimensional manifolds with boundary X_0, with a preferred boundary defining function x, with a complete Riemannian metric g_0 on the interior of X_0 such that $\hat{g}_0 = x^2 g_0$ is Riemannian on X_0 (i.e. up to the boundary) and $|dx|_{\hat{g}_0} = 1$ at ∂X_0. For such metrics the Laplacian is essentially self-adjoint on $\mathcal{C}_c^\infty(X_0^\circ)$, and is positive, and thus the modified resolvent

$$R(\sigma) = (\Delta_{g_0} - (n-1)^2/4 - \sigma^2)^{-1}$$

exists, as a bounded operator on $L^2(dg_0)$ for $\operatorname{Im}\sigma > 0$, $\sigma \notin \imath(0, (n-1)/2]$.

Theorem 1.1 ([73, 74]). *Let (X_0, g_0) be an* even *asymptotically hyperbolic space (in the conformally compact sense) of dimension n. Then the (modified) resolvent of the Laplacian on functions, $R(\sigma) = (\Delta_{g_0} - (n-1)^2/4 - \sigma^2)^{-1}$, continues meromorphically from $\operatorname{Im}\sigma > (n-1)/2$ to $\mathbb{C}$ with finite rank Laurent coefficients at the poles (called* resonances*), and if the geodesic flow on (X, g) is non-trapping, i.e. all geodesics escape to infinity, then in strips $\operatorname{Im}\sigma > s$, $R(\sigma)$ satisfies non-trapping estimates $\|R(\sigma)\|_{\mathcal{L}(\mathcal{Y},\mathcal{X})} \leq C|\sigma|^{-1}$, $\operatorname{Re}\sigma > C_1$, for suitable Hilbert spaces $\mathcal{X}, \mathcal{Y}$.*

Analogous results hold on differential k-forms, with $(n-1)^2/4$ replaced by $(n-2k\pm 1)^2/4$, with the sign $+$ corresponding to closed, and $-$ corresponding to coclosed forms.

Proceedings of the International Congress of Mathematicians, Seoul, 2014

The meromorphic extension was proved by Mazzeo and Melrose [48] and Guillarmou [29], using the 0-pseudodifferential algebra of Mazzeo and Melrose. In this algebra the operator is elliptic in the principal symbol sense, but one also needs to invert the *normal operator*. The latter is sufficiently difficult that (suboptimal, but sufficient for some applications) estimates as $\operatorname{Re}\sigma \to \infty$ in strips were only obtained recently by Melrose, Sá Barreto and Vasy [50] by a semiclassical construction in special cases. Recently Vasy [74, 75] gave a new proof, and proved the non-trapping estimates as well, using a new method, extending a renormalized and conjugated version of the spectral family $\Delta_g - (n-1)^2/4 - \sigma^2$ across ∂X to a new operator P_σ which can be thought of as being defined on a manifold without boundary X, so only 'standard' microlocal analysis is needed. The extended operator P_σ is no longer elliptic, but the additional phenomena are well-understood from the point of view of microlocal analysis: real principal type propagation, radial points and complex absorption. This method also allows for a generalization to differential forms; these were previously studied in the context of Hodge theory by Mazzeo [47]. Also, as a byproduct, it gives a new approach for analyzing the wave equation on asymptotically de Sitter spaces, on which wave propagation was described earlier, without the evenness condition, by Vasy [80] and Baskin [6].

In addition to providing a new proof of the meromorphic continuation of the resolvent, as well as the large σ estimates, this approach also allows for microlocalization of the estimates which is crucial in many applications, such as in the gluing work of Datchev and Vasy [16]. A nice application of this theory, in combination with the exotic pseudodifferential calculus/second microlocal machinery developed by Sjöstrand and Zworski [63], is the work of Datchev and Dyatlov [15], which gave a proof of fractal upper bounds, in terms of the upper Minkowski dimension of the trapped set, for the resonance counting function on even asymptotically hyperbolic spaces with hyperbolic geodesic flow. This in particular applies for convex cocompact quotients of hyperbolic space and gives analogous upper bounds for the counting function of zeros of the Selberg zeta function then. (These quotients have long been studied; see e.g. [58, 67].)

We also point out that the Euclidean analogue of the theorem has a long history (with stronger restrictions at infinity needed). An effective meromorphic continuation was obtained by complex scaling methods due to Aguilar, Balslev, Combes and Simon, and other authors, including the microlocal perspective of Helffer and Sjöstrand; see [62] and the references therein.

The second theorem concerns wave propagation on Kerr-de Sitter spaces. This is particularly interesting since the asymptotic behavior of waves involves resonances, which are poles of a family $\sigma \mapsto P_\sigma^{-1}$, where P_σ is very similar to the P_σ in the asymptotically hyperbolic case; it is an operator on a manifold without boundary. Concretely, Kerr-de Sitter space has a bordification, or partial compactification, $\overline{M}$, with a boundary defining function τ and P_σ is then an operator on ∂M. The extra complication is that this operator is trapping, but the trapping is of a relatively weak type, called normally hyperbolic trapping, which has been analyzed by Wunsch and Zworski [83] and by Dyatlov [23] recently.

Theorem 1.2 (See [41, 75], and see [22] for exact Kerr-de Sitter space). *Let $(\overline{M}, g)$ be a Kerr-de Sitter type space with normally hyperbolic trapping. Then there is $\kappa > 0$ such that solutions of $(\Box_g - \lambda)u = 0$ have an asymptotic expansion $u \sim \sum_j \sum_{k \leq k_j} \tau^{\imath\sigma_j} (\log|\tau|)^k a_{jk}$ $+\tilde{u}$, where $\tilde{u} \in \tau^\kappa H_{\mathrm{b}}^s(\overline{M})$; here σ_j are* resonances *of the associated* normal operator. *For $\lambda = 0$ on Kerr-de Sitter space, the unique σ_j with $\operatorname{Im}\sigma_j \geq 0$ is 0, and the corresponding term is a constant, i.e. waves decay to constants.*

Further, this result is stable under b-perturbations of the metric, with the b-structure understood in the sense of Melrose [54].

In spatially compact parts of Kerr-de Sitter space, $\tau = e^{-t}$ for the usual time function t, i.e. this decay is exponential.

In fact, in a slightly different way, the wave equation for Minkowski-type metrics, more specifically Lorentzian scattering metrics, can also be handled by similar techniques, see [5, 41, 75], for both Cauchy problems and for the Feynman propagator. In fact, Klein-Gordon type equations, even in ultrahyperbolic settings, are also amenable to this type of analysis – in this case in Melrose's scattering framework [49].

A different direction of extending these results is to non-linear equations. In the semilinear setting this was discussed by [41], and then extended to the quasilinear case by Hintz [38]. We briefly discuss this direction at the end of these notes.

The third theorem concerns the dynamical zeta function. It was a conjecture of Smale's, proved by Giulietti, Liverani and Pollicott [26] recently by dynamical systems techniques, but shortly afterwards Dyatlov and Zworski [20] gave a new short proof using microlocal analysis, relying on ideas of Faure and Sjöstrand [24], which are analogous to the setup involved in proving the above theorems, as well as Guillemin's approach to trace formulae [33].

Theorem 1.3 ([20]). *Let X be a compact manifold and $\phi_t : X \to X$ a $\mathcal{C}^\infty$ Anosov flow with orientable stable and unstable bundles. Let $\{\gamma^\sharp\}$ denote the set of primitive orbits of ϕ_t, and $T^\sharp_\gamma$ their periods. Then the Ruelle zeta function $\zeta_R(\lambda) = \prod_{\gamma^\sharp}(1 - e^{i\lambda T^\sharp_\gamma})$, which converges for $\operatorname{Im}\lambda \gg 0$, extends meromorphically to $\mathbb{C}$.*

While of course this lecture cannot cover all of microlocal analysis, at this point we need to point out a particularly glaring omission in the author's opinion: analysis on singular spaces. This has been an extremely active area of study, both in elliptic and non-elliptic settings. The former include for instance N-body scattering, see [72] and references therein for this very extensive topic connected to wave propagation on manifolds with corners, the low energy description of the resolvent of the Laplacian on asymptotically Euclidean spaces [30, 31, 60], scattering theory on geometrically finite hyperbolic manifolds [10, 32, 35], index and K-theory on manifolds with corners and stratified spaces [1, 55], and a general symbolic pseudodifferential theory [2]. The latter include for instance wave propagation on cones, edges, manifolds with corners, see e.g. [7, 46, 51, 52, 78, 79] and references therein.

With these illustrations of the uses of microlocal analysis, we now explain the new developments which facilitated these advances.

2. Pseudodifferential operators

Microlocal, or phase space, analysis in its simplest form concerns itself with the study of functions or distributions on manifolds by means with which one can localize not only in the base manifold, but also conically in the fibers of its cotangent bundle. This corresponds to a description of not only where a distribution lies in, say, a Sobolev space locally, but in which (co)direction this happens. In the most basic setting of $\mathbb{R}^n$, it is closely related to the Fourier transform: one localizes in the base space $\mathbb{R}^n_z$, as well as in conic (i.e. dilation invariant) subsets of $\mathbb{R}^n_\zeta$. For instance, one says that a distribution u on $\mathbb{R}^n$ is *in the Sobolev space H^s*

microlocally near $(z_0,\zeta_0)\in\mathbb{R}^n\times(\mathbb{R}^n\setminus\{0\})=T^*\mathbb{R}^n\setminus o$ if there exists $\phi\in\mathcal{C}^\infty_c(\mathbb{R}^n)$ identically 1 near z_0, and $\psi\in\mathcal{C}^\infty(\mathbb{R}^n)$, homogeneous of degree 0 for $|\zeta|>1$, such that $\psi(t\zeta_0)=1$ for all $t\gg 1$ sufficiently large, and

$$\mathcal{F}^{-1}\psi\mathcal{F}\phi u\in H^s,$$

or equivalently

$$\mathcal{F}^{-1}\langle\zeta\rangle^s\psi\mathcal{F}\phi u\in L^2,$$

where $\mathcal{F}$ is the Fourier transform on $\mathbb{R}^n$, and $\langle\zeta\rangle=(1+|\zeta|^2)^{1/2}$. The operator $A=\mathcal{F}^{-1}\langle\zeta\rangle^s\psi\mathcal{F}\phi$ is a prime example of a *pseudodifferential operator*, a class of operators which includes differential operators. In general, on $\mathbb{R}^n$, a pseudodifferential operator has the form

$$Au(z)=(2\pi)^{-n}\int e^{\imath(z-z')\cdot\zeta}a(z,z',\zeta)\,u(z')\,d\zeta\,dz',$$

with the integral understood as an *oscillatory integral*, where a satisfies appropriate estimates. (For the example above, one can take $a(z,z',\zeta)=\langle\zeta\rangle^s\psi(\zeta)\phi(z')$ independent of z.) A typical class of estimates is

$$|D_z^\alpha D_{z'}^\beta D_\zeta^\gamma a(z,z',\zeta)|\leq C_{\alpha\beta\gamma}\langle\zeta\rangle^{m-|\gamma|};\tag{2.1}$$

this gives Hörmander's symbol class S^m_∞, and the corresponding operators $A\in\Psi^m_\infty$. Another typical class is $S^{m,\ell}$ given by the estimates

$$|D_z^\alpha D_{z'}^\beta D_\zeta^\gamma a(z,z',\zeta)|\leq C_{\alpha\beta\gamma}\langle z\rangle^{\ell_1-|\alpha|}\langle z'\rangle^{\ell_2-|\beta|}\langle\zeta\rangle^{m-|\gamma|};$$

this gives rise to the operators $A\in\Psi^{m,\ell}$ with $\ell=\ell_1+\ell_2$. Both of these classes form an algebra (with composition giving the ring multiplication) over $\mathbb{C}$. An important property is that composition is symbolic, i.e. one can compute AB modulo *residual* operators, i.e. modulo $\Psi^{-\infty}_\infty$, resp. $\Psi^{-\infty,-\infty}$. Note that $\Psi^{m,0}\subset\Psi^m_\infty$, and elements of Ψ^m_∞ are bounded maps between weighted Sobolev spaces $H^{s,r}\to H^{s-m,r}$, where $H^{s,r}=\langle z\rangle^{-r}H^s$, while those of $\Psi^{m,\ell}$ are bounded $H^{s,r}\to H^{s-m,r-\ell}$. This explains the different properties of these algebras: the residual terms are non-compact on, say, L^2, for the first algebra, but they are compact for the second.

Another class of operators that plays a role below is the *semiclassical family*, $A=(A_h)_{h\in(0,1)}$ of operators:

$$A_hu(z)=(2\pi h)^{-n}\int e^{\imath(z-z')\cdot\zeta/h}a(z,z',\zeta,h)\,u(z')\;d\zeta\,dz',$$

where a satisfies estimates in one of the above cases, uniformly in $h\in[0,1)$; indeed, we typically consider smooth h dependence with h-uniform bounds for all derivatives. This gives rise to the operator algebras $\Psi^m_{\infty,\hbar}$ and $\Psi^{m,\ell}_\hbar$. Here the residual terms (modulo which one can do calculations) are in $h^\infty\Psi^m_{\infty,\hbar}$, resp. $h^\infty\Psi^{m,\ell}_\hbar$, i.e. one can do computations modulo rapidly vanishing, as $h\to 0$, errors, not merely compact (in the case of $\Psi^{m,\ell}_\hbar$) error terms.

Pseudodifferential operators can be transferred to compact manifolds without boundary via local coordinate charts, requiring that the Schwartz kernel is $\mathcal{C}^\infty$ away from the diagonal; one checks easily that for the class stated above on $\mathbb{R}^n$, the Schwartz kernel is indeed

$\mathcal{C}^\infty$ away from the diagonal $z = z'$ on $\mathbb{R}^n \times \mathbb{R}^n$. It is worthwhile noticing that differential operators are of the form stated above with polynomial behavior in ζ; the subclass of pseudodifferential operators for which a as above has an asymptotic expansion in terms of homogeneous functions of degree $m - j$, $j \in \mathbb{N}$, is called *classical* or *one-step polyhomogeneous*. In this generality of manifolds, pseudodifferential operators have a well-defined *principal symbol* $[a]$ in $S^m_\infty/S^{m-1}_\infty$; in the case of classical operators, these can be regarded as homogeneous degree m functions on $T^*M \setminus o$.

Since T^*M is not compact even if M is, and as the most interesting objects are homogeneous, it is useful to work instead on the compact (if M is such) space $(T^*M \setminus o)/\mathbb{R}^+ = S^*M$. However, it is even better to compactify the fibers of T^*M as balls to obtain $\overline{T}^*M$; this glues S^*M to T^*M at fiber infinity using 'reciprocal polar coordinates', so $\partial T^*M = S^*M$. The principal symbol a of $A \in \Psi^0_{\mathrm{cl}}$, which can be regarded as a homogeneous degree zero function, is then equivalently a function on ∂T^*M. If $A \in \Psi^m_{\mathrm{cl}}$, then this principal symbol is homogeneous degree m, or a section of a line bundle over ∂T^*M, but it is also well-defined as a function up to a positive multiplier, so e.g. its zero set, which is also called the *characteristic set* $\mathrm{Char}(A)$ of A, is well-defined as a subset of ∂T^*M. The complement of the characteristic set is the *elliptic set* $\mathrm{Ell}(A)$; an operator is *elliptic* if it has empty characteristic set, i.e if it is elliptic at every point. It also makes sense to talk of the operator *wave front set* $\mathrm{WF}'(A)$; a point in ∂T^*M is in the complement of $\mathrm{WF}'(A)$ if it has a neighborhood on which A is given (say, in local coordinates) by an order $-\infty$, i.e. 'really trivial', symbol, as opposed to one of order $m-1$, which is what would be guaranteed by the vanishing of the principal symbol on such a neighborhood. One thinks of operators A which have wave front set in some open set $U \subset \partial T^*M$ as a microlocalizer to U (analogue of the multiplication operator by a compactly supported function on an open set O in M localizing to O); one can also construct microlocal partitions of unity, etc.

In the semiclassical version of this analysis, in addition to the behavior of a at ∂T^*M for all h, one is also interested in the behavior of a *on all of* $\overline{T}^*M$ at $h = 0$. This is geometrically encoded by working with the space $[0,1)_h \times \overline{T}^*M$, then the boundary hypersurfaces are $\{0\}_h \times \overline{T}^*M$ and $[0,1)_h \times \partial\overline{T}^*M$. We call the restriction of a to the former the *semiclassical principal symbol*. If a is a classical, h-dependent, order 0 family, then the *joint principal symbol* can be thought of as a function on $\partial([0,1)_h \times \overline{T}^*M)$, consisting of the standard and the semiclassical principal symbol as its two constituents. Correspondingly, the elliptic set, characteristic set and wave front sets are all subsets of $\partial([0,1)_h \times \overline{T}^*M)$.

Much as we compactified the fibers of the cotangent bundle above, we can also compactify the base space $\mathbb{R}^n$ for $\Psi^{m,\ell}$, we again do it as a ball. Then neighborhoods of points at $\partial\overline{\mathbb{R}^n}$ correspond to (cut off to the exterior of a compact subset of $\mathbb{R}^n$) *conic* regions in $\mathbb{R}^n$. Much as transferring the definition of pseudodifferential operators to manifolds is possible via local coordinate charts, we can transfer $\Psi^{m,\ell}$ to *manifolds with boundary* M, requiring smooth and rapidly (order ∞) decaying Schwartz kernel away from the diagonal in $M \times M$. The resulting algebra of operators is Melrose's *scattering algebra* $\Psi^{m,\ell}_{\mathrm{sc}}(M)$, see [49]. The natural place of microlocalization is a replacement for the standard cotangent bundle, called the *scattering cotangent bundle* ${}^{\mathrm{sc}}T^*M$, which is naturally identified with T^*M over M°. Since we transferred our pseudodifferential operators from $\mathbb{R}^n$, the smooth sections of this bundle are locally (near ∂M) of the form $\sum \zeta_j(z)\, dz_j$, where $\zeta_j \in \mathcal{C}^\infty(\overline{\mathbb{R}^n})$. It is more convenient to express this in polar coordinates on $\mathbb{R}^n$ and then transfer to M: one then sees that these forms are $\tau \frac{dx}{x^2} + \sum \eta_j \frac{dy_j}{dx}$, with x a boundary defining function, y_j local coordinates on ∂M. Dually, the vector fields in $\Psi_{\mathrm{sc}}(M)$ are the scattering vector fields, of the

form $V = a(x^2\partial_x) + \sum b_j(x\partial_{y_j})$, with a, b_j smooth. More invariantly, they are of the form $\rho\mathcal{V}_{\mathrm{b}}(M)$, ρ now a global boundary defining function, where $\mathcal{V}_{\mathrm{b}}(M)$ is the Lie algebra of Melrose's b-vector fields, namely vector fields tangent to ∂M.

In fact, it is useful to compactify the fibers of ${}^{\mathrm{sc}}T^*M$ to obtain the space ${}^{\mathrm{sc}}\overline{T}^*M$. Now the *joint principal symbol* is an object (section of a line bundle, or simply a function for order $(0,0)$) on $\partial{}^{\mathrm{sc}}\overline{T}^*M$, and the elliptic, characteristic and wave front sets are subsets of $\partial{}^{\mathrm{sc}}\overline{T}^*M$. There is even a semiclassical version of this, in which case these objects 'live on' $\partial([0,1)_h \times {}^{\mathrm{sc}}\overline{T}^*M)$, which has three boundary hypersurfaces: 'fiber infinity', 'base infinity' and $h = 0$.

Here we shall mostly consider applications of microlocal analysis to partial differential equations. However, this is by no means the only application. For instance, the field of inverse problems, such as determining a Riemannian metric on a manifold with boundary M from its distance function restricted to the boundary, called the *boundary rigidity problem*, has been using microlocal analysis, in particular the theory of Fourier integral operators, extensively. Such problems have linearizations related to the geodesic X-ray transform (on tensors). By introducing an artificial boundary, given by a level set of a convex function, and using Melrose's scattering pseudodifferential algebra let Stefanov, Uhlmann and Vasy [66, 70] show that the local (in M) boundary rigidity problem is well-posed in a conformal class under suitable conditions. Of course, there were many predecessors of this work in inverse problems using microlocal analysis, see for instance works of Greenleaf, Stefanov and Uhlmann [27, 28, 64, 65].

Returning to manifolds without boundary M, T^*M being symplectic, there is a vector field H_a corresponding to the principal symbol a of A, which is homogeneous of degree $m-1$. For $m = 1$, H_a can be regarded as a vector field on ∂T^*M; for other a, we can multiply a by a positive homogeneous degree $1-m$ function b, then H_{ba} is well-defined, and as $H_{ba} = bH_a + aH_b$, so in $\mathrm{Char}(A)$, H_a is well-defined up to a positive multiple, in particular the integral curves of H_a are well-defined. A bit more precisely, not only is H_a well-defined as a vector field on ∂T^*M, but it is well defined as a vector field in $\mathcal{V}_{\mathrm{b}}(\overline{T}^*M)$ modulo $\rho_{\mathrm{fiber}}\mathcal{V}_{\mathrm{b}}(\overline{T}^*M)$, where on a manifold X, $\mathcal{V}_{\mathrm{b}}(X)$ is the Lie algebra of vector fields tangential to ∂X. Integral curves of H_a inside the characteristic set are called (null)-bicharacteristics.

A useful extension of the setting discussed so far is to allow the operators to have *variable order*. For instance, in the case of $\Psi^m(M)$, one can allow m to be a $\mathcal{C}^\infty$ function on $S^*M = \partial\overline{T}^*M$. For M compact, this is a subset $\Psi^{m_0}(M)$, $m_0 > \sup m$, but it allows for much finer control of regularity. Here one needs to allow symbols which gain less than a full order upon differentiation, so e.g. in the setting of (2.1) one would have, with $\delta \in (0, 1/2)$,

$$|D_z^\alpha D_{z'}^\beta D_\zeta^\gamma a(z, z', \zeta)| \leq C_{\alpha\beta\gamma}\langle\zeta\rangle^{m-|\gamma|+\delta|(\alpha,\beta,\gamma)|};$$

these are symbols in $S^m_{\infty,1-\delta,\delta}$ in the standard notation, and one can take $\delta > 0$ arbitrarily small. One can still talk about ellipticity (including microlocally) by requiring the invertibility (by symbols of order $-m$) of the principal symbol modulo symbols of order -1. In particular, the Sobolev space $H^m(M)$ is defined, with $\tilde{m} = \inf m$, by

$$H^m(M) = \{u \in H^{\tilde{m}}(M) : \ Au \in L^2\},$$

where A is any elliptic element of $\Psi^m(M)$. Such results appeared in the work of Unterberger [71] (with order depending on the underlying space M) and Duistermaat [18]; a more

complete discussion as needed for our purposes is given in [5, Appendix A]. In the semiclassical version the order m can be a function on $\partial([0,1)_h \times \overline{T}^*M)$. If the only place where it needs to vary is at $h = 0$, it can be thought of as a function, constant outside a compact set, on T^*M. In this case one is considering a fixed Sobolev space, but with an h-dependent norm. Such microlocal norms play a key role in the work of Faure and Sjöstrand [24]; see the Anosov flow discussion below for an application. Similarly, in the scattering setting, one can have an order that is a function on $\partial^{\mathrm{sc}}\overline{T}^*M$.

3. Elliptic and non-elliptic Fredholm theory

3.1. Elliptic theory. The basic results in microlocal analysis concern the structures we have already introduced. In order to explain their significance from the perspective of solving PDE, we remark that the Fredholm property of a (pseudo)differential operator P acting between two Hilbert spaces $\mathcal{X}$ and $\mathcal{Y}$, i.e. the range being closed, the nullspace finite dimensional, the range finite codimensional, is equivalent to estimates

$$\|u\|_{\mathcal{X}} \leq C(\|Pu\|_{\mathcal{Y}} + \|u\|_{\tilde{\mathcal{X}}}), \qquad \|v\|_{\mathcal{Y}^*} \leq C(\|P^*v\|_{\mathcal{X}^*} + \|v\|_{\mathcal{Z}}),$$

where the adjoints are taken relative to a fixed space, such as L^2, and $\tilde{\mathcal{X}}, \mathcal{Z}$ are Hilbert spaces, with compact inclusion maps $\mathcal{X} \to \tilde{\mathcal{X}}$, $\mathcal{Y}^* \to \mathcal{Z}$. The simplest example of such an estimate is an elliptic estimate, in which case one can take all the spaces to be standard Sobolev spaces. For instance if $P \in \Psi^m(M)$ is elliptic and M is compact then $\mathcal{X} = H^s$, $\mathcal{Y} = H^{s-m}$, $\mathcal{X}^* = H^{-s}$, $\mathcal{Y}^* = H^{-s+m}$, $\tilde{\mathcal{X}} = H^{\tilde{N}}$, $\mathcal{Z} = H^{N^*}$ work for any $\tilde{N}, N^*$, which are taken to satisfy $\tilde{N} < s$, $N^* < -s + m$ for the application (the compact inclusion map). Thus, elliptic operators are invertible as maps between these Sobolev spaces, up to a finite dimensional obstruction. The analytic version of the Fredholm theory can also be used to show that for $P \in \Psi^m(M)$ elliptic with $m > 0$, the resolvent family $\mathbb{C} \ni \lambda \mapsto (P - \lambda I)^{-1}$ is meromorphic with finite rank smoothing ($\Psi^{-\infty}(M)$) Laurent coefficients if $P - \lambda_0 I$ is invertible for a single value, λ_0, of λ. All of these have natural extension to operators acting between sections of vector bundles E, F; then the principal symbol $a(z,\zeta)$ has values in bundle homomorphisms $\mathrm{Hom}(E_z, F_z)$, and ellipticity is the invertibility of these homomorphisms between finite dimensional vector spaces.

The elliptic estimates can be proved by constructing an approximate inverse, or parametrix, for P, which can be done by inverting the principal symbol (i.e. taking its reciprocal in the scalar setting). They can also be microlocalized, namely for any $P \in \Psi^m(M)$ (not necessarily elliptic) one also has estimates of the form

$$\|Q_1 u\|_{H^s} \leq C(\|Q_2 Pu\|_{H^{s-m}} + \|u\|_{H^{\tilde{N}}}),$$

$Q_1, Q_2 \in \Psi^0(M)$ with $\mathrm{WF}'(Q_1) \subset \mathrm{Ell}(Q_2) \cap \mathrm{Ell}(P)$, i.e. on the elliptic set of P, if one microlocalizes (via Q_1 and Q_2), the analogue of the global estimate holds. Note that as the principal symbol of P^* is the complex conjugate of that of P, P^* is elliptic wherever P is. It is also useful to remark here that the elliptic estimates are valid on variable order Sobolev spaces as well.

3.2. Real principal type propagation. While there are many interesting elliptic operators, such as the Laplacian on functions or differential forms, or the Dirac operator, there are

even more non-elliptic problems of interest, and it turns out that with a bit of effort they fit into similar Fredholm frameworks. If $P \in \Psi^m(M)$ is non-elliptic, it has a non-empty characteristic set. The non-degenerate version of non-ellipticity is if the principal symbol p vanishes simply there; then the characteristic set is smooth. There is a difference between the real and complex valued cases; we are here most interested in the real valued one, then $\mathrm{Char}(P)$ has codimension one. The symplectic structure turns a non-degenerate dp into a non-vanishing Hamilton vector H_p; however H_p may be *radial*, i.e. tangent to the $\mathbb{R}^+$-orbits. Hörmander's propagation of singularities theorem [42], see also [19], is that one can propagate estimates along the bicharacteristic flow in $\mathrm{Char}(P)$; such a statement is meaningful where H_p is not radial. That is, one has estimates

$$\|Q_1 u\|_{H^s} \leq C(\|Q_2 u\|_{H^s} + \|Q_3 P u\|_{H^{s-m+1}} + \|u\|_{H^{\tilde{N}}}), \tag{3.1}$$

where $Q_1, Q_2, Q_3 \in \Psi^0(M)$, provided the elliptic set of Q_3 contains $\mathrm{WF}'(Q_1)$, and the bicharacteristics through all points in $\mathrm{WF}'(Q_1) \cap \mathrm{Char}(P)$ reach the elliptic set of Q_2 while remaining in $\mathrm{Ell}(Q_3)$. The more usual phrasing of this theorem is that if u is in H^s microlocally at a point in $\mathrm{Char}(P)$, then $u \in H^s$ on the maximal bicharacteristic segment through this point, with 'maximal' being with respect to being contained in the complement of $\mathrm{WF}^{s-m+1}(Pu)$, i.e. in the set where Pu is microlocally H^{s-m+1}. This theorem is proved by *positive commutator estimates*, computing

$$\langle \imath P u, A u\rangle - \langle \imath A u, P u\rangle = \langle (\imath[A, P] + \imath(P - P^*)A)u, u\rangle, \tag{3.2}$$

$A = A^* \in \Psi^{m'}(M)$, and the principal symbol of $\imath[A, P] + \imath(P - P^*)A$ in $\Psi^{m+m'-1}(M)$ is $-H_p a - 2\tilde{p}a$ if $\tilde{p}$ is the principal symbol of $\frac{1}{2\imath}(P - P^*) \in \Psi^{m-1}(M)$. One arranges that

$$-H_p a - 2\tilde{p}a = b^2 + e, \tag{3.3}$$

where e has support in the region where the a priori assumptions are imposed (such as $\mathrm{WF}'(Q_2)$ above). Taking B, E with principal symbols b, e, one has $\imath[A, P] + \imath(P - P^*)A = B^*B + E + F$, with F lower order, so substituting into (3.2), one controls Bu in terms of Eu as well as A^*Pu, proving a theorem after a regularization argument.

This theorem, with the proof, is also valid on variable order Sobolev spaces, but only in one direction of flow. Thus, if s is monotone along the Hamilton flow, say s is increasing, then one can propagate H^s estimates in the backward direction, while if s is decreasing, one can propagate H^s estimates in the forward direction. In terms of the sketched proof, the reason for the restriction on the direction is that the Hamilton derivative hitting the weight (giving the order) provides a logarithmically larger term than the other ones, which thus must have a correct sign for the argument to go through; see [5, 71].

We also remark that Hörmander's theorem, with the positive commutator proof, extends easily to systems whose principal symbol is real scalar (a multiple of the identity operator on the vector bundle), and also extends to more general *real principal type systems*, as shown by Dencker [17].

3.3. Complex absorption. Hörmander's theorem, as well as its generalizations, had a key role in understanding propagation phenomena, such as waves. In all these cases estimates propagate, i.e. if one a priori knows that u is well-behaved somewhere (in this case on the wave front set of Q_2) then one can conclude that u is well-behaved somewhere else. From the perspective of Fredholm problems, the problem with this is that the a priori hypothesis

need not ever be fulfilled. One way of dealing with this is called *complex absorption*, see [56]. This means that one considers an artificial operator $Q \in \Psi^m(M)$ with non-negative principal symbol, and replaces P by $P - \imath Q$. Then there is still an analogue of Hörmander's theorem, but one can only propagate estimates in the *forward* direction along H_p. Notice that in the elliptic set of Q one has elliptic estimates even in $\mathrm{Char}(P)$, so the point is that one can propagate estimates from and to this elliptic set, in the forward direction, along the H_p flow. Replacing $P - \imath Q$ by $P + \imath Q$, but Q still having non-negative principal symbol, the estimates can be propagated in the *backward* direction. In particular, this works for $(P - \imath Q)^* = P^* + \imath Q^*$, so *estimates for the adjoint can be propagated in the opposite direction as estimates for the operator*. As an example, if all bicharacteristics of P (in $\mathrm{Char}(P)$) reach the elliptic set of Q in both the forward and the backward direction, which we may call the simplest *non-trapping* scenario, one can piece together elliptic and propagation estimates to conclude that

$$\|u\|_{H^s} \leq C(\|(P - \imath Q)u\|_{H^{s-m+1}} + \|u\|_{H^{\tilde{N}}}),$$

and

$$\|v\|_{H^{s'}} \leq C(\|(P^* + \imath Q^*)v\|_{H^{s'-m+1}} + \|v\|_{H^{N'}}), \qquad s' = -s + m - 1.$$

This corresponds to Fredholm estimates, though one has to be a bit careful as $P - \imath Q$ does not map H^s to H^{s-m+1}. So one lets

$$\mathcal{X} = \{u \in H^s : \ (P - \imath Q)u \in H^{s-m+1}\},$$

which is a Hilbert space in the natural norm; this is the simplest example of a coisotropic space, see [51, Appendix A]. One also lets $\mathcal{Y} = H^{s-m+1}$. Then $P - \imath Q : \mathcal{X} \to \mathcal{Y}$ is Fredholm, and indeed, if P depends on a parameter $\sigma \in \mathbb{C}$, with the principal symbol of P independent of σ, then this is an analytic Fredholm family, with a meromorphic inverse if it is invertible at a single point.

3.4. Radial points. While complex absorption is artificial, though very useful in eliminating dynamics in certain regions of ∂T^*M by making the operator elliptic there, it illustrates an important point: in order to have Fredholm problems, we need the bicharacteristics to reach regions in which we have good a priori control, such as $\mathrm{Ell}(Q)$ above. The most natural setting is that of *radial points*, which were already mentioned earlier as the points at which Hörmander's propagation theorem provides no extra information. Unlike in the settings considered above, in which the Sobolev order s was arbitrary, here there are restrictions on it because the positivity, corresponding to b^2 in (3.3), can only be given by the *weight*, $\rho_{\mathrm{fiber}}^{-m'}$. Thus, it is useful to think of H_p, or rather $\rho_{\mathrm{fiber}}^{m-1} H_p$, as an element of $\mathcal{V}_{\mathrm{b}}(\overline{T}^* M)$ modulo $\rho_{\mathrm{fiber}} \mathcal{V}_{\mathrm{b}}(\overline{T}^* M)$, since this encodes $H_p \rho_{\mathrm{fiber}}$ modulo one order additional vanishing. The results in this setting depend on the sign of the Hamilton derivative of the weight relative to the sign of the Hamilton derivative of the microlocalizer: if they have the same sign, one need not make an assumption like $Q_2 u \in H^s$ in the propagation estimates (3.1) to get a conclusion at the radial set (set of radial points), but if they have the opposite sign, one does need to do this. Since the sign of $H_p \rho_{\mathrm{fiber}}^{-m'}$ depends on the sign of m', this means that the kind of results one gets in the high regularity (which needs m' bigger than a threshold) versus the low regularity (which needs m' smaller than a threshold) are different. Finally, we need to keep in mind the appearance of $\tilde{p}$ in (3.3), which shifts this threshold value from being $m' = 0$. Thus, the estimate in this setting has two parts. We first make the non-degeneracy

assumption that $\rho_{\mathrm{fiber}}^{m-2} H_p \rho_{\mathrm{fiber}} = \mp\beta_0 > 0$ on the radial set $L \subset \mathrm{Char}(P) \subset \partial T^*M$, while $H_p\tilde{p} = \pm\tilde{\beta}\beta_0\tilde{\rho}^{-m+1}$, where we assume for simplicity that $\tilde{\beta}$ is constant on L, which is the case in many applications. Further, assume that L acts as a sink $(-)$ or source $(+)$ in $\mathrm{Char}P \subset \partial\overline{T}^*M$, in a non-degenerate sense; this basically means that there is a quadratic defining function ρ_1 of L in $\mathrm{Char}(P)$ such that $\rho_{\mathrm{fiber}}^{m-1} H_p\rho_1$ is a positive definite. Then

(i) If $s \geq s_0 > (m-1)/2 - \tilde{\beta}$, then for $u \in H^{s_0}$ one has estimates

$$\|Q_1 u\|_{H^s} \leq C(\|Q_3 P u\|_{H^{s-m+1}} + \|u\|_{H^{\tilde{N}}}),$$

Q_j elliptic at L, $j = 1, 3$, with $\mathrm{WF}'(Q_1) \subset \mathrm{Ell}(Q_3)$ and such that all bicharacteristics from points in $\mathrm{WF}'(Q_1)$ tend to L in either the forward $(-)$ or the backward $(+)$ direction, while remaining in $\mathrm{Ell}(Q_3)$.

(ii) If $s < (m-1)/2 - \tilde{\beta}$, then one has estimates

$$\|Q_1 u\|_{H^s} \leq C(\|Q_2 u\|_{H^s} + \|Q_3 P u\|_{H^{s-m+1}} + \|u\|_{H^{\tilde{N}}}),$$

Q_j elliptic at L, $j = 1, 2, 3$, with $\mathrm{WF}'(Q_1) \subset \mathrm{Ell}(Q_3)$ and such that all bicharacteristics from points in $\mathrm{WF}'(Q_1) \setminus L$ tend to $\mathrm{Ell}(Q_2)$ (which is now typically disjoint from L) in either the forward $(+)$ or the backward $(-)$ direction, while remaining in $\mathrm{Ell}(Q_3)$.

Note that if P is replaced by P^*, then $P - P^*$ is replaced by its negative, so $\tilde{\beta}$ defined for P^* is the negative of that defined for P, which means that for P^* (keeping $\tilde{\beta}$ as defined for P) one has

(i) If $s' \geq s'_0 > (m-1)/2 + \tilde{\beta}$, then for $u \in H^{s'_0}$ one has estimates

$$\|Q_1 u\|_{H^{s'}} \leq C(\|Q_3 P^* u\|_{H^{s'-m+1}} + \|u\|_{H^{N'}}),$$

Q_j elliptic at L, $j = 1, 3$, with $\mathrm{WF}'(Q_1) \subset \mathrm{Ell}(Q_3)$ and such that all bicharacteristics of p from points in $\mathrm{WF}'(Q_1)$ tend to L in either the forward $(-)$ or the backward $(+)$ direction, while remaining in $\mathrm{Ell}(Q_3)$.

(ii) If $s' < (m-1)/2 + \tilde{\beta}$, then one has estimates

$$\|Q_1 u\|_{H^{s'}} \leq C(\|Q_2 u\|_{H^{s'}} + \|Q_3 P^* u\|_{H^{s'-m+1}} + \|u\|_{H^{N'}}),$$

Q_j elliptic at L, $j = 1, 2, 3$, with $\mathrm{WF}'(Q_1) \subset \mathrm{Ell}(Q_3)$ and such that all bicharacteristics from points in $\mathrm{WF}'(Q_1) \setminus L$ tend to $\mathrm{Ell}(Q_2)$ in either the forward $(+)$ or the backward $(-)$ direction, while remaining in $\mathrm{Ell}(Q_3)$.

Substituting in $s' = -s + m - 1$, one sees that the condition in (ii) for P^* is equivalent to that in condition (i) for P, and similarly with (i) and (ii) interchanged. This means that if one has non-trapping in the sense that both in the forward and in the backward direction the bicharacteristics escape to radial sets, one has Fredholm estimates, provided one can arrange that the Sobolev spaces are such that one can propagate estimates away from a radial set (case (i)) for P from one of the ends of the bicharacteristics, and for P^* from the other end (as this implies case (ii) for P). Often the numerology in (i) and (ii) is such that the Sobolev spaces H^s must have *variable order*. One can also combine such a Fredholm setup with complex absorption; in this case one can often work with constant order Sobolev spaces.

Notice that radial points are in some sense the best thing that can happen to a non-elliptic problem with real principal symbol: if one has a chaotic Hamilton flow, there is no reason to think that one can propagate regularity from *anywhere*; radial points provide just such a location. This being said, note that the requirements above were weaker than L being radial: roughly speaking, there can still be non-trivial Hamilton flow in L, and we care about the Hamilton dynamics normally to L. This turns out to be important in Kerr-de Sitter black holes, where the conormal bundle of the event horizon at infinity has this kind of structure. (The non-rotating black holes of de Sitter-Schwarzschild spaces give rise to actual radial sets.) Also notice that the estimates we stated were global in a radial set (component) L; one can in fact microlocalize if the set is actually radial, as shown by Haber and Vasy [36], but this becomes impossible when there is a non-trivial Hamilton flow within L, as in Kerr-de Sitter space. We finally remark that the radial point estimates also hold for systems provided the principal symbol is real scalar (a multiple of the identity operator on the fibers of the vector bundle).

3.5. Normally hyperbolic trapping. This lack of ability to microlocalize within a set Γ invariant under the Hamilton flow occurs also in a more degenerate setting, that of *normally hyperbolic trapped sets*. After much earlier work of Gérard and Sjöstrand [25] in the analytic setting, this was analyzed by Wunsch and Zworski [83], to an extent which suffices for the problems we consider here, with refinements by Hintz and Vasy [39], and in more detail by Nonnenmacher and Zworski [57] and by Dyatlov [21, 23]. (The latter is sufficiently precise to locate a sequence of resonances corresponding to Γ, while [57] allows for rather irregular normal dynamics (stable and unstable distributions)!) For us these enter in either the semiclassical, or in the b-settings, with Kerr-de Sitter spaces containing perhaps the prime examples. In the normally hyperbolic setting one drops the non-degeneracy of $\rho_{\mathrm{fiber}}^{m-2} H_p \rho_{\mathrm{fiber}} = \mp\beta_0 > 0$ of the radial setting; in fact, one has a defining function of the boundary hypersurface at which one is doing analysis (so h in the semiclassical setting) which, at L, has vanishing H_p-derivative. (This is automatic in the semiclassical setting!) The subprincipal symbol (in the form of $\tilde{p}$), which shifts the threshold in the radial setting via $\tilde{\beta}$, can still give positivity, but it must have a definite sign to do so. However, one can extend a bit beyond this strict threshold (which cannot be moved by changing the weight, such as h, since H_p annihilates the latter), at the cost of losing powers of the weight relative to the real principal type and radial point settings, provided that the Hamilton dynamics *normally to* Γ is well-behaved. Here, for brevity, we do not discuss details, but the key feature is that, within the characteristic set, there are transversally intersecting smooth codimension 1 manifolds $\Gamma_\pm$ with intersection Γ, with Γ_- and Γ_+ the local stable, resp. unstable, manifolds along the flow. Then one can arrange defining functions $\phi_\pm$ for these such that $H_p\phi_\pm = \mp c_\pm \phi_\pm$, with $c_\pm > 0$ and $H_{\phi_+}\phi_- = \{\phi_+, \phi_-\} > 0$; the latter positivity plays an important role in the control of the subprincipal term in [39, 83]. The functions $c_\pm$ can be chosen in a manner related to the normal hyperbolicity of the flow, namely bounded from below and above, up to an ϵ loss, by the normal minimal and maximal expansion rates. They dictate the size of the 'gap', i.e. the upper bound for the wrong-sign subprincipal symbol, to be (up to an ϵ loss) half of the minimum expansion rate; see [21, 23, 57].

3.6. Semiclassical and scattering settings. These results have natural extensions to the other algebras considered above: $\Psi_\hbar(M)$ and $\Psi_{\mathrm{sc}}(M)$ (as well as its semiclassical version). A straightforward application of the results thus far then is the limiting absorption principle

for *scattering metrics*, introduced by Melrose, modelled on the large ends of cones, including non-trapping estimates if the geodesic flow is *non-trapping*, i.e. all geodesics escape to infinity. (Indeed, this setting is where Melrose started studying Lagrangian sets of radial points, see [49]; the earlier work of Guillemin and Schaeffer was on isolated radial points [34], see also [37] and [36].) Concretely, a scattering metric is a Riemannian metric g on M° of the form

$$g = x^{-4}\,dx^2 + x^{-2}h$$

near ∂M, where h is a smooth symmetric 2-cotensor which restricts to a Riemannian metric on ∂M. This generalizes the Euclidean metric, where one would take M to be the radial compactification of $\mathbb{R}^n$, so $\partial M = \mathbb{S}^{n-1}$ with the round metric h. One then has the following result of Melrose, with the semiclassical version due to Vasy and Zworski:

Theorem 3.1 ([49, 81]). *The Laplacian Δ_g of a scattering metric has spectrum $[0,\infty)$, and for $\lambda > 0$ the limiting resolvents $R(\lambda^2 \pm \imath 0)$ exist as bounded operators $H^{s,r} \to H^{s+2,r-1}$, provided the weight r satisfies $r > 1/2$ at the incoming radial set, $r < 1/2$ on the outgoing radial set.*

Further, if the manifold is non-trapping then one has non-trapping resolvent estimates

$$\|R(\lambda^2 \pm \imath 0)\|_{\mathcal{L}(H^{s,r}_{|\lambda|^{-1}}, H^{s+2,r-1}_{|\lambda|^{-1}})} \leq C|\lambda|^{-1},\ \lambda \gg 1.$$

Here we do not provide further detail, but in fact this scattering framework also works directly for Klein-Gordon equations on *non-trapping Lorentzian scattering metrics* in the sense of Baskin, Vasy and Wunsch [5]; see also [41, Section 5]. Both of these discuss the actual wave equation, which requires b-methods described at the end of these notes, but in fact the Klein-Gordon version is much easier (as far as Fredholm analysis is concerned) as it can be done in the very amenable scattering setting, see [77]. For $\Box_g - \lambda$, $\lambda > 0$, and g of signature $(1, n-1)$, the characteristic set has two components, and within each there are two components of the radial set. One can thus choose the direction of propagation in either component separately. Choosing forward propagation in the base 'time' variable, this is the forward propagator; reversing it one gets the adjoint, the backward propagator. These correspond to the Cauchy problem. However, choosing forward propagation relative to the Hamilton flow, which means propagation in the opposite directions in the two components of the characteristic set relative to the base 'time' variable, gives a Feynman propagator; similarly choosing the backward one relative to the Hamilton flow gives another Feynman propagator. Indeed, even ultrahyperbolic equations are perfectly well-behaved: e.g. if g is a non-degenerate translation invariant metric, then for the corresponding d'Alembertian $\Box_g$, $\Box_g - \lambda$, $\lambda \in \mathbb{R} \setminus \{0\}$, fits into this framework. Here, in general, the radial set has two components, and the Feynman propagator is the only reasonable option – this corresponds to the Cauchy problem being ill-behaved.

4. Applications

4.1. Anosov flows. One of the simplest kinds of differential operator is a vector field. Following earlier work of Faure and Sjöstrand [24], Dyatlov and Zworski [20] adapted a PDE point of view to analyze $\mathcal{C}^\infty$ Anosov flows $\varphi_t : X \to X$ on a compact manifold X, $\phi_t = \exp(tV)$, from the perspective of the generator V. Here the Anosov property means

that the tangent space TX has a continuous (in x) decomposition into a stable subspace $E_s(x)$, an unstable subspace $E_u(x)$, and the neutral direction of $E_0(x) = \mathrm{Span}(V(x))$. Then the differential operator one studies is $P = \frac{1}{\imath}\mathcal{L}_V$ on differential forms, which has scalar principal symbol given by that of V. The key ingredient to the meromorphic continuation of the dynamical zeta function, which can be expressed as a (regularized) trace, is the analysis of $(P-\lambda)^{-1}$ on appropriate function spaces. But with $E_s^*(x)$ and $E_u^*(x)$ the dual bundles, they are sources/sinks, in the generalized sense described above, for the Hamilton flow (which is just the flow of V lifted to $T^*X \setminus o$ from the homogeneous perspective), and the microlocal analysis we discussed yields the desired analytic Fredholm statement for the family $\lambda \mapsto P - \lambda$. A wave front set analysis then allows Dyatlov and Zworski to complete the proof of Theorem 1.3.

4.2. Asymptotically hyperbolic and de Sitter spaces. As a more involved application, the results discussed so far by themselves suffice to show the meromorphic extension of the resolvent of an asymptotically hyperbolic Laplacian together with high energy estimates using $\Psi_\hbar(M)$. We start by recalling the definition of manifolds with *even* conformally compact metrics. These are Riemannian metrics g_0 on the interior of an n-dimensional compact manifold with boundary X_0 such that near the boundary Y, with a product decomposition $[0,\epsilon)_x \times Y$ of a neighborhood U of Y and a boundary defining function x, they are of the form

$$g_0 = \frac{dx^2 + h}{x^2}$$

where h is a family of metrics on $Y = \partial X_0$ depending on x in an even manner, i.e. all odd derivatives of h with respect to x vanish at Y. (There is a much more natural way to phrase the evenness condition due to Guillarmou [29].) Then the dual metric is

$$G_0 = x^2(\partial_x^2 + H),$$

with H the dual metric family of h (depending on x as a parameter), and so

$$\Delta_{g_0} = (xD_x)^2 + \imath(n-1+x^2\gamma)(xD_x) + x^2\Delta_h,$$

with γ even, and Δ_h the x-dependent family of Laplacians of h on Y. We then consider the spectral family

$$\Delta_{g_0} - \frac{(n-1)^2}{4} - \sigma^2$$

of the Laplacian. In addition to working with finite σ, or σ in a compact set, we also want to consider $\sigma \to \infty$, mostly in strips, with $|\operatorname{Im}\sigma|$ bounded. In that case we should consider σ as a 'large parameter' in the sense of [61]. Such a setting can be converted into a semiclassical one by writing $\sigma = h^{-1}z$, where $h \sim |\sigma|^{-1}$ (one can even take $h = |\sigma|^{-1}$, but this is often not convenient); then the spectral family becomes $h^{-2}(h^2\Delta_{g_0} - h^2\frac{(n-1)^2}{4} - z^2)$.

We show now that if we change the smooth structure on X_0 by declaring that only even functions of x are smooth, i.e. introducing $\mu = x^2$ as the boundary defining function, then after a suitable conjugation and division by a vanishing factor the resulting operator smoothly and non-degenerately continues across the boundary, i.e. continues to $X_{-\delta_0} = (-\delta_0, 0)_\mu \times Y \sqcup X_{0,\mathrm{even}}$, where $X_{0,\mathrm{even}}$ is the manifold X_0 with the new smooth structure. At the level of the principal symbol, i.e. the dual metric, the conjugation is irrelevant, so we can easily see what happens: changing to coordinates (μ, y), $\mu = x^2$, as $x\partial_x = 2\mu\partial_\mu$,

$$G_0 = 4\mu^2\partial_\mu^2 + \mu H = \mu(4\mu\partial_\mu^2 + H),$$

so after dividing by μ, we obtain $\mu^{-1}G_0 = 4\mu\partial_\mu^2 + H$. This is a quadratic form that is positive definite for $\mu > 0$, is Lorentzian for $\mu < 0$, and has a transition at $\mu = 0$ that as we shall see involves radial points. In fact, a similar argument would show that in $\mu < 0$, this dual metric is obtained by similar manipulations performed on the negative of a signature $(1, n-1)$ even *asymptotically de Sitter metric*, i.e. one of the form $\tilde{x}^{-2}(d\tilde{x}^2 - h)$, with $\tilde{x}$ the boundary defining function, and h positive definite at $\tilde{x} = 0$. Then $\mu = -\tilde{x}^2$ gives this form of the metric. Notice that $-\tilde{x}^2$ and x^2 are formally the 'same', i.e. $\tilde{x}$ is formally like $\imath x$, which means that this extension across the boundary is a mathematically precise general realization of a 'Wick rotation'. Correspondingly, in addition to providing a new method of analysis for asymptotically hyperbolic spaces, extension across the boundary also provides a new approach to asymptotically de Sitter analysis, providing an alternative to [6, 80].

To see that the full spectral family of the Laplacian is well behaved, first, changing to coordinates (μ, y), $\mu = x^2$, we obtain

$$\Delta_{g_0} = 4(\mu D_\mu)^2 + 2\imath(n - 1 + \mu\gamma)(\mu D_\mu) + \mu\Delta_h.$$

Now we conjugate by $\mu^{-\imath\sigma/2+(n+1)/4}$, and multiply by $\mu^{-1/2}$ from both the left and right

$$\begin{aligned}\mu^{-1/2}\mu^{\imath\sigma/2-(n+1)/4}(\Delta_{g_0} - \frac{(n-1)^2}{4} - \sigma^2)\mu^{-\imath\sigma/2+(n+1)/4}\mu^{-1/2}\\ = 4\mu D_\mu^2 - 4\sigma D_\mu + \Delta_h - 4\imath D_\mu + 2\imath\gamma(\mu D_\mu - \sigma/2 - \imath(n-1)/4).\end{aligned}$$

This operator is in $\mathrm{Diff}^2(X_{0,\mathrm{even}})$, and now it continues smoothly across the boundary, by extending h and γ in an arbitrary smooth manner; it is now of the form

$$P_\sigma = 4(1 + a_1)\mu D_\mu^2 - 4(1 + a_2)\sigma D_\mu - a_3\sigma^2 + \Delta_h - 4\imath D_\mu + b_1\mu D_\mu + b_2\sigma + c_1$$

with a_j smooth, real, vanishing at $\mu = 0$, b_j and c_1 smooth. This form suffices for analyzing the problem for σ in a compact set, or indeed for σ going to infinity in a strip near the reals. (In [74] a further modification is made to obtain semiclassical ellipticity when σ leaves this strip in an appropriate manner.)

Writing covectors as $\xi\, d\mu + \eta\, dy$, the principal symbol of $P_\sigma \in \mathrm{Diff}^2(X_{-\delta_0})$, including in the high energy sense ($\sigma \to \infty$), is

$$p_{\mathrm{full}} = 4(1 + a_1)\mu\xi^2 - 4(1 + a_2)\sigma\xi - a_3\sigma^2 + |\eta|^2_{\mu,y},$$

and is real for σ real. Correspondingly, the standard principal symbol is

$$p = \sigma_2(P_\sigma) = 4(1 + a_1)\mu\xi^2 + |\eta|^2_{\mu,y},$$

which is real, independent of σ, and elliptic for $\mu > 0$.

Let

$$N^*S \setminus o = \Lambda_+ \cup \Lambda_-, \qquad \Lambda_\pm = N^*S \cap \{\pm\xi > 0\}, \qquad S = \{\mu = 0\};$$

thus $S \subset X_{-\delta_0}$ can be identified with $Y = \partial X_0 (= \partial X_{0,\mathrm{even}})$. Note that $p = 0$ at $\Lambda_\pm$ and H_p is radial there since

$$N^*S = \{(\mu, y, \xi, \eta) :\ \mu = 0,\ \eta = 0\},\ \text{so}\ H_p|_{N^*S} = -4\xi^2\partial_\xi.$$

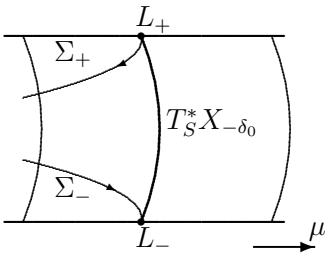


Figure 4.1. The cotangent bundle of $X_{-\delta_0}$ near $S = \{\mu = 0\}$ in a fiber-radially compactified view. The boundary of the fiber compactification is the cosphere bundle $S^*X_{-\delta_0}$; it is the surface of the cylinder shown. $\Sigma_\pm$ are the components of the (classical) characteristic set containing $L_\pm$. They lie in $\mu \leq 0$, only meeting $S^*_S X_{-\delta_0}$ at $L_\pm$. Semiclassically, i.e. in the interior of $\overline{T}^* X_{-\delta_0}$, for $z = h^{-1}\sigma > 0$, only the component of the semiclassical characteristic set containing L_+ can enter $\mu > 0$. This is reversed for $z < 0$.

This corresponds to $dp = 4\xi^2\, d\mu$ at N^*S, so the characteristic set $\Sigma = \{p = 0\}$ is smooth at N^*S.

Let $L_\pm$ be the image of $\Lambda_\pm$ in $S^*X_{-\delta_0}$. Then L_- is a sink and L_+ is a source in the sense that all bicharacteristics nearby converge to $L_\pm$ as the parameter goes to $\mp\infty$. Further, one computes that $\tilde{\beta}|_{L_\pm} = \operatorname{Im}\sigma$. In the other direction, all bicharacteristics reach $\mu = -\epsilon_0$, $\epsilon_0 > 0$ small, so adding complex absorption there assures that we have a Fredholm problem if we make the choice of *propagating all estimates away from L_+ and L_- for P_σ, and towards L_+ and L_- in P^*_σ*. To be precise, we take two copies of $X_{-\delta_0}$, smoothly glued at $\mu = -\epsilon_0$, where complex absorption is introduced, to obtain a compact manifold without boundary X; alternatively, one can work with a single copy, and replace the complex absorption by a boundary working with spaces of extendible distributions for P_σ, and supported distributions for P^*_σ, see [41, Section 2]. This requires that the order of the Sobolev space (for P_σ) be sufficiently high, namely the more negative $\operatorname{Im}\sigma$ becomes, the more positive the Sobolev order must be. Indeed, if $\tilde{f} \in \mathcal{C}^\infty(X)$, then $(P_\sigma - \imath Q_\sigma)^{-1}\tilde{f} \in \mathcal{C}^\infty(X)$ as well (away from poles of this operator). If the geodesic flow is non-trapping then in fact we have semiclassical propagation/radial point estimates, which in turn imply the non-trapping statement of Theorem 1.1.

While this explains why $(P_\sigma - \imath Q_\sigma)^{-1}$ is a well-behaved operator, it may not be obvious how this helps with understanding the resolvent of the Laplacian, $R(\sigma)$. However, this is not hard to see. To make the extension from $X_{0,\mathrm{even}}$ to X more systematic, let $E_s : H^s(X_{0,\mathrm{even}}) \to H^s(X)$ be a continuous extension operator, $R_s : H^s(X) \to H^s(X_{0,\mathrm{even}})$ the restriction map. Then in $\operatorname{Im}\sigma > 0$, when σ is not a pole of either $R(\sigma)$ or $(P_\sigma - \imath Q_\sigma)^{-1}$, we have for $f \in \dot{\mathcal{C}}^\infty(X_0)$,

$$R(\sigma)f = x^{(n+1)/2-i\sigma}x^{-1}R_s(P_\sigma - \imath Q_\sigma)^{-1}E_{s-1}x^{-(n+1)/2+i\sigma}x^{-1}f, \tag{4.1}$$

since a simple computation shows that the right hand side is an element of $L^2(X_0, dg)$ (indeed, it is of the form $x^{(n-1)/2-\imath\sigma}\mathcal{C}^\infty(X_{0,\mathrm{even}})$, since after the application of $(P_\sigma - \imath Q_\sigma)^{-1}$ in the formula, the result is in $\mathcal{C}^\infty(X)$) with $\Delta_{g_0} - (n-1)^2/4 - \sigma^2$ applied to it yielding f, so by the self-adjointness of Δ_{g_0}, it is indeed $R(\sigma)f$. Notice that this uses very strongly that Q_σ has Schwartz kernel supported away from $X_{0,\mathrm{even}} \times X_{0,\mathrm{even}}$ (i.e. more than just $\mathrm{WF}'(Q) \cap S^*X_{0,\mathrm{even}} = \emptyset$).

In fact, in this unified treatment of asymptotically hyperbolic and de Sitter spaces one can even arrange a set up which does not need complex absorption at all, and does not need an artificially added boundary. To do so, given an asymptotically hyperbolic space (X_0, g_0), one can construct a compact manifold X without boundary containing two (disjoint) copies of X_0, connected by an asymptotically de Sitter space; one may call the two copies the 'future' and 'past' copies. (Vice versa, given an asymptotically de Sitter space, one can cap it off by two asymptotically hyperbolic spaces; one may need to take two copies of the de Sitter space, however, for topological reasons.) This is motivated by the structure of the boundary of radially compactified Minkowski space, which has two copies of hyperbolic space in the interior of the future and past light cones at infinity, and a copy of de Sitter space outside these light cones. However, the construction can be made in full generality, see [76, Section 3]. In this case one propagates estimates from the conormal bundle of the boundary of one of the copies of $X_{0,\mathrm{even}}$ (say, the past one) to the other one; for the adjoint the estimates propagate in the opposite direction. Since the threshold regularity for the radial points is the same for both the future and the past copies, this requires variable order Sobolev space; in this case one can actually arrange that the order varies only in the interior of the asymptotically de Sitter space, and depends only on the base, X (not on the location within the fiber of S^*X).

4.3. Kerr-de Sitter spaces. Next we turn to Kerr-de Sitter spaces, which are 4-dimensional Lorentzian space-times. Here an appropriate bordification of the space-time is

$$M_\delta = X_\delta \times [0,\infty)_\tau, \qquad X_\delta = (r_- - \delta, r_+ + \delta)_r \times \mathbb{S}^2,$$

where $r_\pm$ are specified later, and where $\tau = e^{-t}$ in $\tau > 0$ for a more conventional 'time' variable t (that is essentially equivalent to the usual time far from $r_\pm$ in (r_-, r_+)). On this the signature $(1,3)$ dual metric G has the form for appropriate choice of a function $c = c_\pm$,

$$\begin{aligned} G = -\rho^{-2}\Big(&\mu\big(\partial_r \pm c\tau\partial_\tau\big)^2 \pm 2(1+\gamma)(r^2+a^2)\big(\partial_r \pm c\tau\partial_\tau\big)\tau\partial_\tau \\ &\mp 2(1+\gamma)a\big(\partial_r \pm c\tau\partial_\tau\big)\partial_\phi + \kappa\partial_\theta^2 + \frac{(1+\gamma)^2}{\kappa\sin^2\theta}(-a\sin^2\theta\tau\partial_\tau + \partial_\phi)^2\Big), \end{aligned}$$

where r_s, Λ, a constants, $r_s, \Lambda \geq 0$, $\kappa = 1 + \gamma\cos^2\theta$, $\gamma = \frac{\Lambda a^2}{3}$,

$$\rho^2 = r^2 + a^2\cos^2\theta, \qquad \mu = (r^2+a^2)(1 - \frac{\Lambda r^2}{3}) - r_s r,$$

and $\mu(r) = 0$ has two positive roots $r = r_\pm$, $r_+ > r_-$, with $\mp\frac{\partial\mu}{\partial r}|_{r=r_\pm} > 0$; r_+ is the de Sitter end (cosmological horizon), r_- is the Kerr end (event horizon). Physically, r_s is twice the black hole mass, Λ is the cosmological constant, a is the angular momentum of the black hole. Thus, de Sitter space is the case $a = 0$, $r_s = 0$, and Λ can be normalized to be 3; in this case $r_- = 0$ can be removed, and the space-time becomes $M_\delta = X_\delta \times [0,\infty)_\tau$, $X_\delta = \mathbb{B}_{r_+ + \delta}$, with $\mathbb{B}_{r_+ + \delta}$ the ball of radius $r_+ + \delta$. If $a = 0$ still, but $r_s > 0$, then one obtains non-rotating de Sitter-Schwarzschild black holes.

Mellin transforming $\Box_g$ in τ (i.e. Fourier transforming in e^{-t}), with dual parameter σ, one obtains a family of operators P_σ, whose principal symbol in the large parameter sense is given by this dual metric function. One can now check that P_σ *almost* has the same structure as the conjugated extended asymptotically hyperbolic case. Most importantly, if $a = 0$,

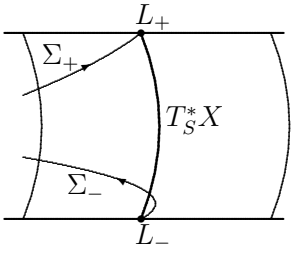


Figure 4.2. The fiber-radially compactified cotangent bundle near the event horizon $S = \{\mu = 0\}$. $\Sigma_\pm$ are the components of the (classical) characteristic set containing $L_\pm$. The characteristic set crosses the event horizon on both components; here the part near L_+ is hidden from view. The projection of this region to the base space is the ergoregion. Semiclassically, i.e. the interior of $\overline{T}^* X_\delta$, for $z = h^{-1}\sigma > 0$, only $\Sigma_{\hbar,+}$ can enter $\mu > a^2$.

$N^*\{r = r_\pm\}$ in X_δ consists of radial points which are sources or sinks; if $a \neq 0$, then instead $N^*\{r = r_\pm\}$ are still 'normally source/sink bundles', as required for our generalized radial points results, but there is non-trivial dynamics within $N^*\{r = r_\pm\}$ corresponding to the black hole rotation (closed orbits, along which ϕ varies and θ is fixed). Further, if one adds complex absorption where μ is small and negative, i.e. just outside $[r_-, r_+] \times \mathbb{S}^2$, then all classical bicharacteristics reach $N^*\{r = r_\pm\}$ or the elliptic set of the complex absorption in both the forward and backward direction, i.e. the problem is *classically non-trapping*, and thus one can make it Fredholm by capping off the space X_δ (e.g. by working with two copies of the space, glued via complex absorption, so the problem is elliptic). Again a useful, but less microlocal, alternative is to add a boundary at $\mu = -\delta_0$, and set up appropriate Fredholm Cauchy-type problems. Although it does not play a role in our analysis, one interesting feature of Kerr-de Sitter wave operators with $a \neq 0$ is that the projection of the characteristic set of P_σ to the base space enters (r_-, r_+); this is called the *ergoregion* – the operator is thus *not* elliptic everywhere between the event horizons. This was considered a major difficulty for the analysis, and was first overcome by Dyatlov [22] by a separation of variables argument; the microlocal analysis described here achieves a similar result in a systematic manner.

However, the operator is semiclassically trapping due to the *photon sphere* in the de Sitter-Schwarzschild case, and its no longer spherically symmetric replacement in general. This trapped set is, however, *normally hyperbolic*. The works of Wunsch and Zworski [83], Hintz and Vasy [39] and Dyatlov [21, 23] give microlocal control at this trapped set, which, combined with gluing constructions of Datchev and Vasy [16], suffices to prove Theorem 1.2.

While Kerr-de Sitter space had not been intensively studied, though there have been works on de Sitter-Schwarzschild space ($a = 0$) [4, 9, 59] and further references in [75], we mention that Kerr space-time has been the subject of intensive research. For instance, polynomial decay on Kerr space was shown recently by Tataru and Tohaneanu [68, 69] and Dafermos, Rodnianski and Shlapentokh [12–14], while electromagnetic waves were studied by Andersson and Blue [3], after pioneering work of Kay and Wald in [43] and [82] in the Schwarzschild setting. While some of these papers employ microlocal methods at the trapped set, they are mostly based on physical space where the phenomena are less clear than in phase space (unstable tools, such as separation of variables, are often used in phase space though). Kerr space is less amenable to immediate microlocal analysis to attack the decay

of solutions of the wave equation due to the singular/degenerate behavior at zero frequency; in some sense it combines the scattering and b-analysis.

4.4. Melrose's b-analysis. While here we used the dilation invariance to reduce to a problem on X_δ, this is easily eliminated. The framework then is Melrose's b-pseudodifferential operator algebra $\Psi_{\mathrm{b}}(M)$, introduced in [53] to study hyperbolic boundary value problems; see [54] for a general setup. On a general manifold M, this microlocalizes the earlier mentioned Lie algebra $\mathcal{V}_{\mathrm{b}}(M)$ of vector fields tangent to the boundary, which are locally of the form $a(x\partial_x) + \sum b_j\partial_{y_j}$. These are the smooth sections of a vector bundle; the dual bundle ${}^{\mathrm{b}}T^*M$ has a local basis $\frac{dx}{x}, dy_j$ over $\mathcal{C}^\infty(M)$. Now from the homogeneous perspective the (standard) principal symbol is a homogeneous function on ${}^{\mathrm{b}}T^*M\setminus o$; for 0th order operators, it can be considered as a function on ${}^{\mathrm{b}}S^*M = ({}^{\mathrm{b}}T^*M \setminus o)/\mathbb{R}^+$.

Much like for $\Psi_{\mathrm{sc}}(M)$, the standard principal symbol does not capture operators modulo (relatively) compact ones. However, unlike the scattering case, there is no other function making up for this deficit (in the case of the scattering algebra, the symbol on ${}^{\mathrm{sc}}T^*_{\partial M}M$), rather it is an operator, called the *normal operator*. This is obtained by 'freezing coefficients' at ∂M to obtain a dilation invariant operator. Together the principal symbol and the Mellin transformed (as it is dilation invariant!) normal operator $\hat{L}(\sigma)$ do allow for a development of Fredholm theory. However, this is a bit more intricate: one has to work with b-Sobolev spaces $H^{s,r}_{\mathrm{b}}(M)$ which have constant weights r, which on the Mellin transform side corresponds to working on the line $\operatorname{Im}\sigma = -r$, but s variable for many non-elliptic problems of interest (though Kerr-de Sitter allows for constant s). Now, at the principal symbol level there are analogues of all of the microlocal ingredients described above; indeed, one also has to allow $L_\pm$ to have a 'normally saddle' structure for Kerr-de Sitter type settings, see [41]. This allows one to conclude that $\hat{L}(\sigma)$ is a Fredholm family on induced spaces. However, in order to have a Fredholm problem on M, one needs that $\hat{L}(\sigma)$ is *invertible* for $\operatorname{Im}\sigma = -r$, so non-symbolic, or 'quantum' objects determine Fredholm properties of L. On the other hand, under this assumption, one indeed has a Fredholm problem, which is perturbation stable in the appropriate sense. This gives the stability of the Kerr-de Sitter problem. In fact, the earlier mentioned Lorentzian scattering metrics, studied by Baskin, Vasy and Wunsch [5], fit into the same general framework.

4.5. Non-linear equations. The final topic we discuss is non-linear PDE. Small data semilinear problems in either non-trapping or, with lower order semilinear terms, normally hyperbolic Kerr-de Sitter type settings can be easily solved by the contraction mapping principle as long as one can work with Sobolev spaces with non-growing weights (i.e. one can choose such a weight r with no resonances σ with $\operatorname{Im}\sigma = -r$), or one has special properties of the resonances for Sobolev spaces $H^{s,0}_{\mathrm{b}}(M)$. In this case, for instance polynomial semilinear terms (for second order equations, to be definite) map $H^{s,r}_{\mathrm{b}}$ to $H^{s-1,r}_{\mathrm{b}}$ for $s > n/2+1$, and thus the Fredholm structure we discussed provides for a Picard iteration for small data; see [41]. The same setting for Lorentzian scattering metrics, generalizing results of Klainerman [44, 45] and Christodoulou [11] is more delicate, both because unlike the saddle points of Kerr-de Sitter space, the radial source/sinks in Minkowski space limit regularity when one is propagating estimates towards them, and also because the reduction to a b-problem involves weights, so there is a more complicated numerology, and one must rely on additional microlocal regularity relative to a pseudodifferential module (which generalizes Klainerman's vector fields), see [41].

While the linear setting of asymptotically Minkowski spaces had well-behaved global dynamics and thus no artificial tools such as complex absorption was needed (we only needed variable order Sobolev spaces), in the Kerr-de Sitter, and indeed localized de Sitter, type settings one needs to 'cut off' the problem. Such a cut off is possible due to the hyperbolic nature of the equations. From the perspective of microlocal analysis it is most conveniently done via complex absorption as discussed above, but this may not provide complete control: one only gets the *exact* solution operator one wants (with supports) if each bicharacteristic is controlled at least at one end by other means, such as radial points or a boundary. Thus, in the non-dilation invariant de Sitter and Kerr-de Sitter type setting it is convenient to consider domains Ω in the manifold M whose (artificial) boundary hypersurfaces (other than those of M, that is) are space-like. (Note that such a Cauchy hypersurface is just as artificial as complex absorption, is less well-behaved microlocally, but has the advantage of giving the supports one wants for time-oriented problems by standard local energy estimates!) As all the complicated phenomena, such as radial points or trapping, or indeed even variable orders of Sobolev spaces, are located away from these artificial boundaries, including these artificial boundaries (as done in [41]) in the framework does not pose significant complications. One obtains, for instance, the small data (here f) well-posedness, with vanishing Cauchy data at the appropriate boundary hypersurface, of Klein-Gordon equations

$$(\Box_g - m^2)u = f + q(u, {}^{\mathrm{b}}du),$$

where q is a polynomial with second order vanishing at $(0,0)$ (so quadratic terms are allowed) if $m > 0$ and the metric g is non-trapping, such as perturbations of asymptotically de Sitter type spaces. Here ${}^{\mathrm{b}}du$ denotes derivatives relative to the b-structure, i.e. the derivatives are given by b-vector fields. If the metric has normally hyperbolic trapping such as Kerr-de Sitter metrics, the losses in derivatives provided by the normally hyperbolic estimates only allow for general non-linearities independent of ${}^{\mathrm{b}}du$ for the contraction mapping argument to go through, though non-linearities depending on derivatives with a particular structure are allowed as well since the loss of derivatives is only microlocally at the trapped set. If $m = 0$ the issue is the 0-resonance, which has resonant state 1, and thus non-linearities q which *only contain derivatives*, and thus annihilate the resonant state, are allowed in the non-trapping asymptotically de Sitter type settings. In either case, one also obtains an expansion at infinity which is generated by the resonances of the Mellin transformed normal operator of the linear problem.

Quasilinear problems require more work. Hintz [38] has developed a framework for b-pseudodifferential operators with Sobolev coefficients, modeled on the Sobolev pseudodifferential operators of Beals and Reed [8]. This framework is sufficient in non-trapping settings, such as perturbations of de Sitter space, to achieve this. More recently, Hintz and Vasy [40] extended this analysis even to normally hyperbolic problems. In this case the contraction mapping is replaced by a use of the Nash-Moser iteration due to the losses in derivatives; the conclusion is a small data global well-posedness and decay result for quasilinear wave equations on Kerr-de Sitter space: for the small mass Klein-Gordon equation without further restrictions (since there is no 0-resonance), while for the actual wave equation for non-linearities containing derivatives (due to the 0-resonance).

Acknowledgements. The author gratefully acknowledges partial support from the NSF under grant number DMS-1068742.

References

[1] P. Albin, E. Leichtnam, R. Mazzeo, and P. Piazza, *The signature package on Witt spaces*, Ann. Sci. Éc. Norm. Supér. (4) **45**(2) (2012), 241–310.

[2] B. Ammann, R. Lauter, and V. Nistor, *Pseudodifferential operators on manifolds with a Lie structure at infinity*, Ann. of Math. (2) **165**(3) (2007), 717–747.

[3] L. Andersson and P. Blue, *Uniform energy bound and asymptotics for the Maxwell field on a slowly rotating Kerr black hole exterior*, Preprint, arXiv:1310.2664, 2013.

[4] A. Bachelot, *Scattering of electromagnetic field by de Sitter-Schwarzschild black hole*, In Nonlinear hyperbolic equations and field theory (Lake Como, 1990), Pitman Res. Notes Math. Ser., Longman Sci. Tech., Harlow **253** (1992), 23–35.

[5] D. Baskin, A. Vasy, and J. Wunsch, *Asymptotics of radiation fields in asymptotically Minkowski space*, Preprint, arxiv:1212.5141, 2012.

[6] D. Baskin, *A parametrix for the fundamental solution of the Klein-Gordon equation on asymptotically de Sitter spaces*, J. Funct. Anal. **259**(7) (2010), 1673–1719.

[7] D. Baskin and J. Wunsch, *Resolvent estimates and local decay of waves on conic manifolds*, J. Differential Geom. **95**(2) (2013), 183–214.

[8] M. Beals and M. Reed, *Microlocal regularity theorems for nonsmooth pseudodifferential operators and applications to nonlinear problems*, Trans. Amer. Math. Soc. **285**(1) (1984), 159–184.

[9] J.-F. Bony and D. Häfner, *Decay and non-decay of the local energy for the wave equation on the de Sitter-Schwarzschild metric*, Comm. Math. Phys. **282**(3) (2008), 697–719.

[10] D. Borthwick, C. Judge, and P. A. Perry, *Selberg's zeta function and the spectral geometry of geometrically finite hyperbolic surfaces*, Comment. Math. Helv. **80**(3) (2005), 483–515.

[11] D. Christodoulou, *Global solutions of nonlinear hyperbolic equations for small initial data*, Comm. Pure Appl. Math. **39**(2) (1986), 267–282.

[12] M. Dafermos and I. Rodnianski, *Decay of solutions of the wave equation on Kerr exterior space-times I-II: The cases of $|a| \ll m$ or axisymmetry*, Preprint, arXiv: 1010.5132, 2010.

[13] ———, *The black hole stability problem for linear scalar perturbations*, In T. Damour et al, editor, Proceedings of the Twelfth Marcel Grossmann Meeting on General Relativity, pp. 132–189. World Scientific, Singapore, 2011. arXiv:1010.5137.

[14] M. Dafermos, I. Rodnianski, and Y. Shlapentokh-Rothman, *Decay for solutions of the wave equation on Kerr exterior spacetimes III: The full subextremal case $|a| < m$*, Preprint, arXiv:1402.7034, 2014.

[15] K. Datchev and S. Dyatlov, *Fractal Weyl laws for asymptotically hyperbolic manifolds*, Geom. Funct. Anal. **23**(4) (2013), 1145–1206.

[16] K. Datchev and A. Vasy, *Gluing semiclassical resolvent estimates via propagation of singularities*, Int. Math. Res. Not. IMRN (23) (2012), 5409–5443.

[17] N. Dencker, *On the propagation of polarization sets for systems of real principal type*, J. Funct. Anal. **46**(3) (1982), 351–372.

[18] J. J. Duistermaat, *On Carleman estimates for pseudo-differential operators*, Invent. Math. **17** (1972), 31–43.

[19] J. J. Duistermaat and L. Hörmander, *Fourier integral operators, II*, Acta Mathematica **128** (1972), 183–269.

[20] S. Dyatlov and M. Zworski, *Dynamical zeta functions for Anosov flows via microlocal analysis*, Preprint, arXiv:1306.4203, 2013.

[21] S. Dyatlov, *Spectral gaps for normally hyperbolic trapping*, Preprint, arXiv:1403.6401, 2014.

[22] ———, *Quasi-normal modes and exponential energy decay for the Kerr-de Sitter black hole*, Comm. Math. Phys. **306**(1) (2011), 119–163.

[23] ———, *Resonance projectors and asymptotics for r-normally hyperbolic trapped sets*, Preprint, arXiv:1301.5633, 2013.

[24] F. Faure and J. Sjöstrand, *Upper bound on the density of Ruelle resonances for Anosov flows*, Comm. Math. Phys. **308**(2) (2011), 325–364.

[25] ———, *Resonances en limite semiclassique et exposants de Lyapunov*, Comm. Math. Phys. **116**(2) (1988), 193–213.

[26] P. Giulietti, C. Liverani, and M. Pollicott, *Anosov flows and dynamical zeta functions*, Ann. of Math. (2) **178**(2) (2013), 687–773.

[27] A. Greenleaf and G. Uhlmann, *Nonlocal inversion formulas for the X-ray transform*, Duke Math. J. **58**(1) (1989), 205–240.

[28] ———, *Estimates for singular Radon transforms and pseudodifferential operators with singular symbols*, J. Funct. Anal. **89**(1) (1990), 202–232.

[29] C. Guillarmou, *Meromorphic properties of the resolvent on asymptotically hyperbolic manifolds*, Duke Math. J. **129**(1) (2005), 1–37.

[30] C. Guillarmou and A. Hassell, *Resolvent at low energy and Riesz transform for Schrödinger operators on asymptotically conic manifolds. I*, Math. Ann. **341**(4) (2008), 859–896.

[31] ———, *Resolvent at low energy and Riesz transform for Schrödinger operators on asymptotically conic manifolds. II*, Ann. Inst. Fourier (Grenoble) **59**(4) (2009), 1553–1610.

[32] C. Guillarmou and R. Mazzeo, *Resolvent of the Laplacian on geometrically finite hyperbolic manifolds*, Invent. Math. **187**(1) (2012), 99–144.

[33] V. Guillemin, *Lectures on spectral theory of elliptic operators*, Duke Math. J. **44**(3) (1977), 485–517.

[34] V. Guillemin and D. Schaeffer, *On a certain class of Fuchsian partial differential equations*, Duke Math. J. **44**(1) (1977), 157–199.

[35] L. Guillopé and M. Zworski, *Scattering asymptotics for Riemann surfaces*, Ann. of Math. (2) **145**(3) (1997), 597–660.

[36] N. Haber and A. Vasy, *Propagation of singularities around a Lagrangian submanifold of radial points*, Bulletin de la SMF, To appear.

[37] A. Hassell, R. B. Melrose, and A. Vasy, *Microlocal propagation near radial points and scattering for symbolic potentials of order zero*, Analysis and PDE **1** (2008), 127–196.

[38] P. Hintz, *Global well-posedness of quasilinear wave equations on asymptotically de Sitter spaces*, Preprint, arXiv:1311.6859, 2013.

[39] P. Hintz and A. Vasy, *Non-trapping estimates near normally hyperbolic trapping*, Preprint, arXiv:1306.4705, 2013.

[40] ———, *Global analysis of quasilinear wave equations on asymptotically Kerr-de Sitter spaces*, Preprint, arXiv:1404.1348, 2014.

[41] ———, *Semilinear wave equations on asymptotically de Sitter, Kerr-de Sitter and Minkowski spacetimes*, Preprint, arXiv:1306.4705, 2013.

[42] L. Hörmander, *On the existence and the regularity of solutions of linear pseudo-differential equations*, Enseignement Math. (2) **17** (1971), 99–163.

[43] B. S. Kay and R. M. Wald, *Linear stability of Schwarzschild under perturbations which are nonvanishing on the bifurcation* 2*-sphere*, Classical Quantum Gravity **4**(4) (1987), 893–898.

[44] S. Klainerman, *The null condition and global existence to nonlinear wave equations*, In Nonlinear systems of partial differential equations in applied mathematics, Part 1 (Santa Fe, N.M., 1984), volume 23 of Lectures in Appl. Math., pp. 293–326. Amer. Math. Soc., Providence, RI, 1986.

[45] ———, *Uniform decay estimates and the Lorentz invariance of the classical wave equation*, Comm. Pure Appl. Math. **38**(3) (1985), 321–332.

[46] G. Lebeau, *Propagation des ondes dans les variétés à coins*, Ann. Sci. École Norm. Sup. (4) **30**(4) (1997), 429–497.

[47] R. Mazzeo, *The Hodge cohomology of a conformally compact metric*, J. Diff. Geom. **28** (1988), 309–339.

[48] R. R. Mazzeo and R. B. Melrose, *Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature*, J. Funct. Anal. **75**(2) (1987), 260–310.

[49] R. B. Melrose, *Spectral and scattering theory for the Laplacian on asymptotically Euclidian spaces*, Marcel Dekker, 1994.

[50] R. B. Melrose, A. Sá Barreto, and A. Vasy, *Analytic continuation and semiclassical resolvent estimates on asymptotically hyperbolic spaces*, Comm. in PDEs **39**(3) (2014), 452–511.

[51] R. B. Melrose, A. Vasy, and J. Wunsch, *Diffraction of singularities for the wave equation on manifolds with corners*, Astérisque **351** (2013), vi+136.

[52] R. Melrose and J. Wunsch, *Propagation of singularities for the wave equation on conic manifolds*, Invent. Math. **156**(2) (2004), 235–299.

[53] R. B. Melrose, *Transformation of boundary problems*, Acta Math. **147**(3-4) (1981), 149–236.

[54] _____, *The Atiyah-Patodi-Singer index theorem*, volume 4 of Research Notes in Mathematics, A K Peters Ltd., Wellesley, MA, 1993.

[55] R. B. Melrose and P. Piazza, *Analytic K-theory on manifolds with corners*, Adv. Math. **92**(1) (1992), 1–26.

[56] S. Nonnenmacher and M. Zworski, *Quantum decay rates in chaotic scattering*, Acta Math. **203**(2) (2009), 149–233.

[57] _____, *Decay of correlations for normally hyperbolic trapping*, Preprint, arXiv: 1302.4483, 2013.

[58] S. J. Patterson, *The limit set of a Fuchsian group*, Acta Math. **136**(3-4) (1976), 241–273.

[59] A. Sá Barreto and M. Zworski, *Distribution of resonances for spherical black holes*, Math. Res. Lett. **4**(1) (1997), 103–121.

[60] D. Sher, *The heat kernel on an asymptotically conic manifold*, Anal. PDE **6**(7) (2013), 1755–1791.

[61] M. A. Shubin, *Pseudodifferential operators and spectral theory*, Springer-Verlag, 1987.

[62] J. Sjöstrand, *A trace formula and review of some estimates for resonances*, In Microlocal analysis and spectral theory (Lucca, 1996), v. **490** of NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., pp. 377–437. Kluwer Acad. Publ., Dordrecht, 1997.

[63] J. Sjöstrand and M. Zworski, *Fractal upper bounds on the density of semiclassical resonances*, Duke Math. J. **137**(3) (2007), 381–459.

[64] P. Stefanov and G. Uhlmann, *Stability estimates for the X-ray transform of tensor fields and boundary rigidity*, Duke Math. J. **123**(3) (2004), 445–467.

[65] _____, *Boundary rigidity and stability for generic simple metrics*, J. Amer. Math. Soc. **18**(4) (2005), 975–1003.

[66] P. Stefanov, G. Uhlmann, and A. Vasy. *Boundary rigidity with partial data*, Preprint, arxiv:1306.2995, 2013.

[67] D. Sullivan. *The density at infinity of a discrete group of hyperbolic motions*, Inst. Hautes Études Sci. Publ. Math., (50):171–202, 1979.

[68] D. Tataru. *Local decay of waves on asymptotically flat stationary space-times*, Amer. J. Math., 135(2):361–401, 2013.

[69] D. Tataru and M. Tohaneanu. *A local energy estimate on Kerr black hole backgrounds*, Int. Math. Res. Not. IMRN, (2):248–292, 2011.

[70] G. Uhlmann and A. Vasy. *The inverse problem for the local geodesic ray transform*, Preprint, arxiv.1210.2084, 2012.

[71] A. Unterberger. *Résolution d'équations aux dérivées partielles dans des espaces de distributions d'ordre de régularité variable*, Ann. Inst. Fourier (Grenoble), 21(2):85–128, 1971.

[72] A. Vasy, *Propagation of singularities in many-body scattering*, Ann. Sci. École Norm. Sup. (4), 34:313–402, 2001.

[73] ———, *Analytic continuation and high energy estimates for the resolvent of the Laplacian on forms on asymptotically hyperbolic spaces*, Preprint, arxiv:1206.5454, 2012.

[74] ———, *Microlocal analysis of asymptotically hyperbolic spaces and high energy resolvent estimates*, volume 60 of MSRI Publications. Cambridge University Press, 2012.

[75] ———, Microlocal analysis of asymptotically hyperbolic and Kerr-de Sitter spaces. *Inventiones Math.*, 194:381–513, 2013. With an appendix by S. Dyatlov.

[76] ———, Resolvents, Poisson operators and scattering matrices on asymptotically hyperbolic and de Sitter spaces. *Preprint, arxiv:1303.5653*, 2013.

[77] ———, *Propagation phenomena*, Lecture Notes, Stanford University, http://math.stanford.edu/~andras/256B.html, 2014.

[78] ———, *Propagation of singularities for the wave equation on manifolds with corners*, Ann. of Math. (2), 168(3):749–812, 2008.

[79] ———, *Diffraction at corners for the wave equation on differential forms*, Comm. Partial Differential Equations, 35(7):1236–1275, 2010.

[80] ———, *The wave equation on asymptotically de Sitter-like spaces*, Adv. Math., 223(1):49–97, 2010.

[81] A. Vasy and M. Zworski, *Semiclassical estimates in asymptotically Euclidean scattering*, Commun. Math. Phys., 212:205–217, 2000.

[82] R. M. Wald. *Note on the stability of the Schwarzschild metric*, J. Math. Phys., 20(6):1056–1058, 1979.

[83] J. Wunsch and M. Zworski. *Resolvent estimates for normally hyperbolic trapped sets*, Ann. Henri Poincaré, 12(7):1349–1385, 2011.

Department of Mathematics, Stanford University, CA 94305-2125, USA
E-mail: andras@math.stanford.edu

Geometric approaches to semilinear elliptic equations

Juncheng Wei

Abstract. A fundamental problem in nonlinear PDE is the classification and construction of entire solutions of nonlinear elliptic equations in $\mathbb{R}^N$ such as $\Delta u + f(u) = 0 \quad \text{in } \mathbb{R}^N$. This is the context of various classical results in the literature like the Gidas-Ni-Nirenberg theorems on radial symmetry, Liouville type theorems, or the achievements around De Giorgi's conjecture. In this paper we review some recent mathematical results on applying geometric approaches towards geometrization of solutions of entire solutions. We focus particularly on two prototype nonlinear elliptic equations: Allen-Cahn equation and nonlinear Schrödinger equation. For Allen-Cahn, we describe the De Giorgi conjecture and the connections with minimal surfaces as well as Toda systems. For nonlinear Schrödinger equation we are interested in new entire solutions with either finite energy or multiple ends. We discuss its surprising connection with the theory of Constant Mean Curvature (CMC) surfaces and Toda system. Applications to gauged Ginzburg-Landau equation as well as Chern-Simons-Higgs will be given.

Mathematics Subject Classification (2010). Primary 35J61; Secondary 35B06, 35B25.

Keywords. De Giorgi's Conjecture, minimal surfaces, Allen-Cahn equation, Nonlinear Schrödinger equation, CMC surfaces, gluing techniques.

1. Part I: Allen-Cahn equation and minimal surfaces

In this section, we survey the studies on entire solutions of Allen-Cahn equation and review its connection with minimal surface theory and Toda system.

1.1. Background.

We are interested the following Allen-Cahn equation

$$\Delta u + u - u^3 = 0 \quad \text{in } \mathbb{R}^N . \tag{1.1}$$

This is a (simplified) model of great importance in physics. It was introduced by Cahn-Hilliard [17] in the description of two-phase separation in fluids and by Allen-Cahn [6] in the mathematical formulation of ordering in binary alloys [6]. Allen-Cahn equation is a prototype for the modeling of interface phenomena in a variety of contexts, for example in pattern formation of mathematical biology such as Gierer-Meinhardt system or diblock copolymer system [66]. It can also be considered as the scalar version of Ginzburg-Landau equation.

The theory of $\Gamma-$convergence reviews a deep connection between Allen-Cahn equation

Proceedings of the International Congress of Mathematicians, Seoul, 2014

and the minimal surface. To see this, we consider the singularly perturbed version of (1.1),

$$\varepsilon^2 \Delta v + v - v^3 = 0 \quad \text{in } \mathbb{R}^N. \tag{1.2}$$

On every bounded domain $\Omega \subset \mathbb{R}^N$, (1.2) is the Euler-Lagrange equation for the action functional

$$J_\varepsilon(v) = \int_\Omega \frac{\varepsilon}{2} |\nabla v|^2 + \frac{1}{4\varepsilon}(1 - v^2)^2.$$

It is easy to see that that the constants $v = \pm 1$ minimize J_ε. They are idealized as two *stable phases* of a material in Ω. It is of interest to analyze configurations in which the two phases coexist. These states are represented by stationary points of J_ε, or solutions v_ε of Equation (1.2), that take values close to $+1$ in a subregion of Ω of and -1 in its complement. The theory of Γ-convergence developed in the 70s and 80s, showed an intricate connection between this problem and the theory of minimal surfaces, see Modica, Mortola, Kohn, Sternberg, [44, 60, 61, 73]. In fact, it is known that for a family u_ε of local minimizers of u_ε with uniformly bounded energy must converge, up to subsequences, in L^1-sense to a function of the form $\chi_E - \chi_{E^c}$ where χ denotes characteristic function, and ∂E has minimal perimeter. Thus the interface between the stable *phases* $u = 1$ and $u = -1$, represented by the sets $[u_\varepsilon = 0]$ approach a minimal hypersurface, see Caffarelli and Córdoba [14, 15], Hutchinson and Tonegawa [39], Röger and Tonegawa [67] for stronger convergence and uniform regularity results.

1.2. From Bernstein's to De Giorgi's conjecture. In 1979 Ennio De Giorgi [19] formulated the following celebrated conjecture concerning entire solutions of equation (1.1).

Conjecture 1.1 (De Giorgi's Conjecture). *Let u be a bounded solution of equation* (1.1) *such that $\partial_{x_N} u > 0$. Then the level sets $[u = \lambda]$ are all hyperplanes,* **at least** *for dimension $N \leq 8$.*

Equivalently, u must depend only on one Euclidean variable so that it must have the form $u(x) = w((x - p) \cdot \nu)$ for some $p \in \mathbb{R}^N$ and some ν with $|\nu| = 1$ and $\nu_N > 0$, where w is the one-dimensional profile

$$w'' + w - w^3 = 0, \quad w(0) = 0, \quad w(\pm\infty) = \pm 1,$$

which is given explicitly by $w(\zeta) := \tanh(\zeta/\sqrt{2})$.

The condition $\partial_{x_N} u > 0$ implies that the level sets of u are all graphs of functions of the first $N - 1$ variables. As we have discussed in Section 1.1, level sets of non-constant solutions are closely connected to minimal hypersurfaces. De Giorgi's conjecture is in fact a parallel to the following classical

Conjecture 1.2 (Bernstein's conjecture). *A minimal hypersurface in $\mathbb{R}^N$, which is also the graph of a smooth entire function of $N - 1$ variables, must be a hyperplane.*

In other words, if Γ is an *entire minimal graph*, namely

$$\Gamma = \{(x', x_N) \mid x' \in \mathbb{R}^{N-1},\ x_N = F(x')\} \tag{1.3}$$

where F solves the minimal surface equation

$$H_\Gamma \equiv \nabla \cdot \left(\frac{\nabla F}{\sqrt{1 + |\nabla F|^2}} \right) = 0 \quad \text{in } \mathbb{R}^{N-1}, \tag{1.4}$$

then Γ must be a hyperplane, hence F must be a linear affine function.

Bernstein's conjecture is known to be true up to dimension $N = 8$, see Simons [72] and references therein, while it is *false* for $N \geq 9$, as proven by Bombieri, De Giorgi and Giusti [12], who found a nontrivial solution to Equation (1.4).

De Giorgi's conjecture has been established for $N = 2$ by Ghoussoub and Gui [30] and for $N = 3$ by Ambrosio and Cabré [8]. Savin [69] proved its validity for $4 \leq N \leq 8$ under the additional assumption

$$\lim_{x_N \to \pm\infty} u(x', x_N) = \pm 1 \quad \text{for all} \quad x' \in \mathbb{R}^{N-1}. \tag{1.5}$$

A counterexample to De Giorgi's conjecture in dimension $N \geq 9$ was believed to exist for a long time. Partial progress in this direction was made by Jerison and Monneau [41] and by Cabré and Terra [13]. The following result shows that De Giorgi's caveat was justified since the conjecture fails for $N \geq 9$.

Theorem 1.3 (del Pino-Kowalczyk-Wei [24]). *Let $N \geq 9$. Then there is an entire minimal graph Γ which is not a hyperplane, such that all $\varepsilon > 0$ sufficiently small there exists a bounded solution $u_\varepsilon(x)$ of equation* (1.1) *whose zero-level set lies close to the dilated $\varepsilon^{-1}\Gamma$. Besides, $\partial_{x_N} u_\varepsilon > 0$ and u_ε satisfies condition* (1.5).

1.3. Outline of the proof: gluing methods. The proof of Theorem 1.3 makes use of the newly developed *infinite dimensional gluing method* which we shall describe briefly.

Let $N = 9$ and assume that Γ is a smooth minimal hypersurface and let ν designate a choice of its unit normal. Points δ-close to Γ can be uniquely represented as

$$x = y + z\nu(y), \quad y \in \Gamma,\ |z| < \delta. \tag{1.6}$$

A well known formula for the Laplacian in these coordinates reads as follows:

$$\Delta_x = \partial_{zz} + \Delta_{\Gamma^z} - H_{\Gamma_z}\partial_z \tag{1.7}$$

where $\Gamma^z := \{y + z\nu(y) \,/\, y \in \Gamma\}$ and Δ_{Γ^z} is the Laplace-Beltrami operator on Γ^z acting on functions of the variable y, and H_{Γ^z} is its mean curvature. Let $k_1, \ldots, k_N$ denote the principal curvatures of Γ. Then we readily have

$$H_{\Gamma^z} = \sum_{i=1}^{N} \frac{k_i}{1 - zk_i}. \tag{1.8}$$

It is reasonable to assume that the solution is a smooth function of the variables (y, ζ), where $\zeta = \varepsilon^{-1} z$, and the equation for $v_\varepsilon(y, \zeta)$ reads

$$\varepsilon^2 \Delta_{\Gamma^{\varepsilon\zeta}} v_\varepsilon - \varepsilon H_{\Gamma^{\varepsilon\zeta}}(y)\, \partial_\zeta v_\varepsilon \ + \ \partial^2_\zeta v_\varepsilon + v_\varepsilon - v_\varepsilon^3 \ = 0, \quad y \in \Gamma, \quad |\zeta| < \delta\varepsilon^{-1}. \tag{1.9}$$

Using expression (1.8) and the fact that Γ is a minimal surface, we expand

$$H_{\Gamma^{\varepsilon\zeta}}(y) = \varepsilon^2 \zeta\, |A_\Gamma(y)|^2 + \varepsilon^3 \zeta^2 \sum_{i=1}^{8} k_i^3 + \cdots, \text{ where } |A_\Gamma|^2 = \sum_{i=1}^{8} k_i^2.$$

For a small $\varepsilon > 0$ we look for a solution u_ε of the form (near Γ_ε),

$$u_\varepsilon(x) = w(\zeta - \varepsilon h(\varepsilon y)) + \phi(\zeta - \varepsilon h(\varepsilon y), y), \quad x = y + \zeta\nu(\varepsilon y), \tag{1.10}$$

where $y \in \Gamma_\varepsilon$, ν is a unit normal to Γ with $\nu_N > 0$, h is a function defined on Γ, which is left as a parameter to be adjusted. Setting $r(y', y_9) = |y'|$, we assume a priori in h that

$$\|(1+r^2)D_\Gamma h\|_{L^\infty(\Gamma)} + \|(1+r)h\|_{L^\infty(\Gamma)} \ \le \ M \tag{1.11}$$

for some large, fixed number M, also with a uniform control on $(1+r^3)D^2_\Gamma h$.

Letting $f(u) = u - u^3$ and using Expression (1.7) for the Laplacian, the equation becomes

$$\begin{aligned} S(u_\varepsilon) := \Delta u_\varepsilon + f(u_\varepsilon) \ &= \Delta_{\Gamma^\zeta_\varepsilon} u_\varepsilon - \varepsilon H_{\Gamma^\zeta_\varepsilon}(\varepsilon y)\, \partial_\zeta u_\varepsilon \\ &\quad + \partial^2_\zeta u_\varepsilon + f(u_\varepsilon) \ = 0, \quad y \in \Gamma_\varepsilon, \ |\zeta| < \delta/\varepsilon. \end{aligned} \tag{1.12}$$

Letting $t = \zeta - \varepsilon h(\varepsilon y)$, we look for u_ε of the form

$$u_\varepsilon(t, y) = w(t) + \phi(t, y)$$

for a small function ϕ. The equation in terms of ϕ becomes

$$\partial^2_t \phi + \Delta_{\Gamma_\varepsilon}\phi + B\phi + f'(w(t))\phi + N(\phi) + E \ = \ 0 \tag{1.13}$$

where B is a small linear second order differential operator, and

$$E = S(w(t)), \quad N(\phi) = f(w+\phi) - f(w) - f'(w)\phi \approx f''(w)\phi^2.$$

While the expression (1.13) makes sense only for $|t| < \delta\varepsilon^{-1}$, it turns out that the equation in the entire space can be reduced to one similar to (1.13) in entire $\mathbb{R} \times \Gamma_\varepsilon$, where E and the undefined coefficients in B are just cut-off far away, while the operator N is slightly modified by the addition of a small nonlinear, nonlocal operator of ϕ. Rather than solving this problem directly we carry out an infinite dimensional form of Lyapunov-Schmidt reduction, which consists two major steps:

Step 1 : Fixing h, we consider a projected problem,

$$\begin{aligned} \partial^2_t \phi + \Delta_{\Gamma_\varepsilon}\phi + B\phi + f'(w(t))\phi + N(\phi) + E \ &= \ c(y)w'(t) \ \text{ in } \mathbb{R} \times \Gamma_\varepsilon, \\ \int_{\mathbb{R}} \phi(t, y) w'(t)\, dt \ &= \ 0 \ \text{ for all } \ y \in \Gamma_\varepsilon. \end{aligned} \tag{1.14}$$

The error of approximation E has roughly speaking a bound $O(\varepsilon^2 r(\varepsilon y)^{-2} e^{-\sigma|t|})$. Thanks to the orthogonal condition and some energy estimates, one can find a solution $\phi = \Phi(h)$ to problem (1.14) with the same bound.

Step 2 : Find h such that the Lagrange multiplier function $c(y) \equiv 0$. Thus the problem is reduced to finding h such that

$$c(y) \int_{\mathbb{R}} {w'}^2 = \int_{\mathbb{R}} (E + B\Phi(h) + N(\Phi(h)))\, w' dt \equiv 0.$$

A lengthy computation yields that this problem is equivalent to

$$\mathcal{J}_\Gamma(h) := \Delta_\Gamma h + |A_\Gamma|^2 h \ = \ c_0 \sum_{i=1}^{8} k_i^3 + \mathcal{N}(h) \quad \text{in } \Gamma, \tag{1.15}$$

where $\mathcal{N}(h)$ is a small operator. A central point is to show that the unperturbed equation (1.15) has a solution $h = O(r^{-1})$, which justifies a posteriori the assumption (1.11) made originally on h. By an explicit positive barrier the operator $\mathcal{J}_\Gamma$ satisfies maximum principle and existence thus follows. The full nonlinear equation is then solved with the aid of contraction mapping principle.

The program towards the counterexample in [8] and [41] mimics the classical program that lead to the proof of Bernstein's conjecture: the existence of the counterexample is reduced to establishing the minimizing character of a *saddle solution* in $\mathbb{R}^8$ that vanishes on Simon's cone. Our approach of direct construction is actually applicable to build solutions, which may be in principle unstable, associated to general minimal surfaces, as we illustrate in the next sections. We should mention that method of infinite dimensional reduction for the Allen Cahn equation in compact settings has precedents with similar flavor in [23, 48, 64]. Using variational approach, local minimizers were built in [44].

1.4. Beyond De Giorgi conjecture: Stable solutions. The complete resolution of De Giorgi's conjecture is not the end of the story but rather the starting point. The assumption of monotonicity in one direction for the solution u in De Giorgi's conjecture implies a form of stability, locally minimizing character for u when compactly supported perturbations are considered in the energy. Indeed, the linearized operator $L = \Delta + (1 - 3u^2)$, satisfies maximum principle since $L(Z) = 0$ for $Z = \partial_{x_N} u > 0$. This implies stability of u, in the sense that its associated quadratic form, namely the second variation of the corresponding energy,

$$\mathcal{Q}(\psi, \psi) := \int_{\mathbb{R}^3} |\nabla \psi|^2 + (3u^2 - 1)\, \psi^2 \tag{1.16}$$

satisfies $\mathcal{Q}(\psi, \psi) > 0$ for all $\psi \neq 0$ smooth and compactly supported.

Stability of u is sufficient for De Giorgi's statement to hold in dimension $N = 2$ ([8, 30]) while it remains an open problem for $3 \leq N \leq 8$. In fact, the monotonicity assumption, together with (1.5), actually implies that u is a *global minimizer*, in the following sense, for any bounded domain Ω

$$\int_\Omega (\frac{1}{2}|\nabla u|^2 + \frac{1}{4}(1 - u^2)^2) \geq \int_\Omega (\frac{1}{2}|\nabla v|^2 + \frac{1}{4}(1 - v^2)^2) \tag{1.17}$$

for any function $v \in H^1(\Omega)$ with $v = u$ on $\partial\Omega$. This fact was observed by Alberti-Ambrosio-Cabre [5] and used by Savin [69].

Naturally, one would ask the following generalized De Giorgi Conjecture.

Conjecture 1.4 (Generalized De Giorgi's conjecture). *Let u be a bounded and* **stable** *solution of equation* (1.1). *Then the level sets $[u = \lambda]$ are all hyperplanes,* **at least** *for dimension $N \leq 7$*

The dimension 7 is again motivated by the study of minimal surface. The generalized De Giorgi's conjecture is in fact a parallel to the following classical statement.

Theorem 1.5 (Generalized Bernstein theorem). *A stable minimal hypersurface must be a hyperplane.*

The stability conjecture for minimal surfaces is known to be true in dimension $N = 3$ by Fischer-Colbrie and Schoen [29], it is *false* for $N \geq 8$, as proven by Bombieri, De Giorgi and Giusti [12], who proved that there is a foliation of Simons's cone in dimension eight

or higher. Yau [77] asked whether one can prove that a complete minimal hypersurface in $\mathbb{R}^{n+1}$ $(N \leq 7)$ is a hyperplane. Although much hard work on this problem has been done, it remains still open in dimensions $3 \leq N \leq 7$.

Using the foliation of the Simon's cone, the following theorem shows that the generalized De Giorgi Conjecture is not true in dimension 8 (and hence higher).

Theorem 1.6 (Pacard-Wei [65]). *Let $N = 8$. Then there exists a stable and bounded solution to (1.1) whose level sets approach one of the foliations of the Simons cone.*

1.5. Finite Morse index solutions in $\mathbb{R}^2$. After the stable solutions, it is natural to consider solutions which are *not too unstable*. These are solutions with finite Morse index. The *Morse index* $m(u)$ is defined as the maximal dimension of a vector space E of compactly supported functions such that

$$\mathcal{Q}(\psi, \psi) < 0 \quad \text{for all} \quad \psi \in E \setminus \{0\}.$$

In view of the discussion so far, it seems natural to associate complete, embedded minimal surfaces Γ with finite Morse index, and solutions of (1.1). In $\mathbb{R}^2$ the only minimal surfaces are *lines*. Therefore a class of solutions to (1.1) with a *finite number of transition lines*, likely to have finite Morse index. This kind of solutions has been recently built in [26]. The location and shape of these lines is governed by the *Toda system*, a classical integrable model for particles moving on a line with exponential forces between any two closest neighbors:

$$\frac{\sqrt{2}}{24} f_j'' = e^{-\sqrt{2}(f_j - f_{j-1})} - e^{-\sqrt{2}(f_{j+1} - f_j)}, \quad j = 1, \dots k. \tag{1.18}$$

For definiteness we take $f_0 \equiv -\infty,\ f_{k+1} \equiv +\infty$. It is known that for any given solution there exist numbers $a_j^{\pm}, b_j^{\pm}$ such that

$$f_j(z) = a_j^{\pm}|z| + b_j^{\pm} + O(e^{-|z|}) \quad \text{as } z \to \pm\infty, \tag{1.19}$$

where $a_j^{\pm} < a_{j+1}^{\pm}$, $j = 1, \dots, k-1$ (long-time scattering).

The role of this system in the construction of solutions with multiple transition lines in the Allen-Cahn equation in bounded domains was discovered in [23]. In entire space the following result holds.

Theorem 1.7 (del Pino-Kowalczyk-Pacard-Wei [26]). *Given a solution f of* (1.18) *if we scale*

$$f_{\varepsilon,j}(z) := \sqrt{2}\,(j - \frac{k+1}{2}) \log \frac{1}{\varepsilon} + f_j(\varepsilon z), \tag{1.20}$$

then for all small ε there is a solution u_ε to (1.1) with k transitions layers $\Gamma_{\varepsilon,j}$ near the lines $x_2 = f_{\varepsilon,j}(x_1)$. More precisely $\Gamma_{\varepsilon,j}$ are graphs of functions:

$$x_1 = f_{\varepsilon,j}(x_2) + h_{\varepsilon,j}(\varepsilon x_2),$$

where $h_{\varepsilon,j}(z) = O(\varepsilon^\alpha)(|z| + 1)$, with some $\alpha > 0$. In addition

$$u_\varepsilon(x_1, x_2) = \sum_{j=1}^{k} (-1)^{j-1} w(x_1 - f_{\varepsilon,j}(x_2) - h_{\varepsilon,j}(\varepsilon x_2)\,) - \frac{1}{2}(1 + (-1)^k) + O(\varepsilon^\alpha). \tag{1.21}$$

In particular, if $k = 2$ and f solves the ODE

$$\frac{\sqrt{2}}{24} f''(z) = e^{-2\sqrt{2}f(z)}, \quad f'(0) = 0,$$

and $f_\varepsilon(z) := \sqrt{2}\log\frac{1}{\varepsilon} + f(\varepsilon z)$, then there exists a solution u_ε to (1.1) in $\mathbb{R}^2$ with

$$u_\varepsilon(x_1, x_2) = w(x_1 + f_\varepsilon(x_2)\,) + w(x_1 - f_\varepsilon(x_2)) \; - 1 \; + O(\varepsilon^\alpha). \tag{1.22}$$

In general in the case of even solutions to the Toda system the deficiency functions $h_{\varepsilon,j}(z)$ decay exponentially as $|z| \to \infty$, c.f. [26].

Remark 1.8. The solutions (1.21) show a major difference between the theory of minimal surfaces and the Allen-Cahn equation, as it is the fact that two separate interfaces *interact*, leading to a major deformation in their asymptotic shapes. We believe that these examples should be prototypical of bounded finite Morse index solutions of (1.1). A finite Morse index solution u should be stable outside a bounded set. If we follow a component of its nodal set along a unbounded sequence, translation and a standard compactness argument leads in the limit to a stable solution. Hence from the result in [20] its profile must be one-dimensional and hence its nodal set is a straight line. This makes it plausible that asymptotically the nodal set of u consists of a *finite, even number of straight lines, the ends*. If this is the case, those lines are not distributed in arbitrarily: Gui [36] proved that if $e_1,\dots e_{2k}$ are unit vectors in the direction of the ends of the nodal set of a solution of (1.1) in $\mathbb{R}^2$, then the balancing formula $\sum_{j=1}^{2k} e_j = 0$ holds.

Another (possibly finite Morse index) solution is known, [22]. This is the so-called saddle solution. It is built by positive barriers with zero boundary data in a quadrant, and then extended by odd reflections to the rest of the plane, so that its nodal set is an infinite cross, hence having 4 straight ends. A similar construction with $2k$ ends has been done in [7].

1.6. Classification of four-ended solutions in $\mathbb{R}^2$. Recall that a 4-ended solution of (1.1) is a solution whose nodal set is, outside a large ball, the union of four curves, each of which is asymptotic to a half affine line and furthermore, along each end, the solution is asymptotic to a heteroclinic solution having this affine line as a nodal set (a slightly different but equivalent definition was given by Gui [37]). The key fact about 4-ended solutions is that, after a rigid motion, they are even and, in the first quadrant

$$Q := \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\};$$

they are monotone functions of the x and the y variables (see [37] for details). We denote the set of four-ended solutions as $\mathcal{M}_4$.

More precisely, if $v \in \mathcal{M}_4$ is a 4-ended solution, there exists a rigid motion g of $\mathbb{R}^2$ such that the function $\bar{v} := v \circ g$ satisfies

$$\bar{v}(x, y) = \bar{v}(-x; y) = \bar{v}(x, -y) = -\bar{v}(-x, -y)$$

in $\mathbb{R}^2$ and

$$\partial_x v < 0 \;\text{ and }\; \partial_y v > 0 \text{ in } Q.$$

The subfamily of $\mathcal{M}_4$ which consists of functions satisfying the above symmetries will be denoted by M_4^{even}. Because of even symmetry and the monotonicity property, the nodal

set of a solution $v \in \mathcal{M}_4^{even}$, restricted to Q, consists of a single curve, which is asymptotic to the half of an affine line Λ. We denote by $\theta \in (0, \frac{\pi}{2})$, the angle between the x-axis and Λ and we define the angle map by

$$\mathcal{A} : \mathcal{M}_4^{even} \to (-\frac{\pi}{4}, \frac{\pi}{4}),\ v \to \frac{\pi}{4} - \theta \tag{1.23}$$

For the saddle solution constructed in [22], we have $\mathcal{A}(v) = 0$, while, for the solutions with almost parallel ends which were constructed in [26] we have $\mathcal{A}(v) \sim \pm\frac{\pi}{4}$. For four-ended solution it is proven that

Theorem 1.9 (Kowalczyk-Liu-Pacard [49, 50]). *$\mathcal{M}_4^{even}$ is diffeomorphic to $\mathbb{R}$ and also that the angle map is surjective. In particular, for each $\alpha \in (-\frac{\pi}{4}, \frac{\pi}{4})$ there exists (at least) one element $v \in \mathcal{M}_4^{even}$ such that $\mathcal{A}(v) = \alpha$.*

Remark 1.10. It is very likely that $\mathcal{A}$ is in fact one-to-one but this is still an open problem.

1.7. Finite Morse index solutions in $\mathbb{R}^3$. Thanks to Weierstrass representation, in $\mathbb{R}^3$ there are plenty of complete, embedded minimal surfaces with finite Morse index. The *Morse index* of the minimal surface Γ, $i(\Gamma)$, has a similar definition relative to the quadratic form for its Jacobi operator $\mathcal{J}_\Gamma := \Delta_\Gamma + |A_\Gamma|^2$.

For more than a century, only two examples of such surfaces were known: the plane and the catenoid. The first nontrivial example was found in 1981 by C. Costa [16]. Later this was generalized to Costa-Hoffman-Meeks surface with arbitrary genus ℓ [38].

In [25] we have established an almost one-to-one correspondence between minimal surfaces in $\mathbb{R}^3$ and the finite Morse index solutions to (1.1):

Theorem 1.11 (del Pino-Kowalcyzk-Wei [25]). *Let $N = 3$ and Γ be a minimal surface embedded, complete with finite total curvature and non-parallel ends, which is in addition nondegenerate. Then for all sufficiently small $\varepsilon > 0$ there exists a solution u_ε of Problem* (1.1) *with the property that $u_\varepsilon(0) \sim \varepsilon^{-1}\Gamma$. Moreover, we have*

$$m(u_\varepsilon) \ = \ i(\Gamma).$$

Besides, the solution is non-degenerate, in the sense that any bounded solution of

$$\Delta\phi \ + \ (1 - 3u_\varepsilon^2)\,\phi \ = \ 0 \quad \text{in } \mathbb{R}^3$$

must be a linear combination of the functions $Z_i,\ i = 1, 2, 3, 4$ defined as

$$Z_i = \partial_i u_\varepsilon, \quad i = 1, 2, 3, \quad Z_4 = -x_2\partial_1 u_\varepsilon + x_1\partial_2 u_\varepsilon.$$

It is well-known that if Γ is a catenoid then $i(\Gamma) = 1$. Moreover, in the Costa-Hoffmann-Meeks surface it is known that $i(\Gamma) = 2\ell + 3$ where ℓ is the genus of Γ. See [62].

Some further comments are in order. In analogy with De Giorgi's conjecture, it seems plausible that qualitative properties of embedded minimal surfaces with finite Morse index should hold for the level sets of finite Morse index solutions of Equation (1.1), provided that these sets are embedded manifolds outside a compact set. As a sample, one may ask if the following two statements are valid:

- *The level sets of any finite Morse index solution u of* (1.1) *in $\mathbb{R}^3$, such that $\nabla u \neq 0$ outside a compact set should have a finite, even number of catenoidal or planar ends with a common axis.*

The above fact does hold for minimal surfaces with finite total curvature and embedded ends as established by Ossermann and Schoen. On the other hand, the above statement should not hold true if the condition $\nabla u \neq 0$ outside a large ball is violated. For instance, let us consider the octant $\{x_1, x_2, x_3 \geq 0\}$. Problem (1.1) in the octant with zero boundary data can be solved by a super-subsolution scheme (similar to that in [22]) yielding a positive solution. Extending by successive odd reflections to the remaining octants, one generates an entire solution (likely to have finite Morse index), whose zero level set does not have the characteristics above: the condition $\nabla u \neq 0$ far away corresponds to *embeddedness of the ends of the level sets.*

An analog of De Giorgi's conjecture for the solutions that follow in complexity the stable ones, namely those with Morse index one, may be the following:

- *A bounded solution u of* (1.1) *in $\mathbb{R}^3$ with $m(u) = 1$, and $\nabla u \neq 0$ outside a bounded set, must be axially symmetric, namely radially symmetric in two variables.*

The solution we found, with transition on a dilated catenoid has this property. This statement would be in correspondence with results by Schoen [70]: if $i(\Gamma) = 1$ and Γ has embedded ends, then it must be a catenoid.

1.8. Mutiple-ended solutions in $\mathbb{R}^3$ and the role of Toda system. The results in [25] provide a connection between a large class of minimal surfaces in $\mathbb{R}^3$ and families of solutions to the Allen Cahn equation, where even Morse index is transmitted. Next we want to show that **more richness** is present in solutions to Allen Cahn with transition layers. A major difference between Allen Cahn and the minimal surface problem, is that two disjoint surfaces **do not interact** in the latter problem, while they do as components of the zero set of solutions to the Allen Cahh equations. These nodal sets are actually solving a form of nonlocal minimal surface problem, which is interesting in its own sake, not just regarding Allen-Cahn as a sort of regularization of the minimal surface problem.

As remarked in [25], the Morse index is a natural quantity to consider in the classification of entire solutions to (1.1). It is natural step beyond De Giorgi's conjecture, to understand "mountain pass solutions" namely those with $m(u) = 1$. In Theorem 1.11, the first example of Morse index one solution is the *catenoidal* axially symmetric solution in [25]. Our next result shows that the structure of Morse index one solutions of (1.1) is more complicated than dilations of a catenoid: there exists an axially symmetric solution of Morse index one whose zero set is *disconnected*.

Theorem 1.12 (Agudelo-del Pino-Wei [1]). *For all sufficiently small $\varepsilon > 0$ there exists an smooth axially symmetric bounded solution $u_\varepsilon(r, x_3)$ to equation* (1.1) *for $N = 3$, with Morse index $m(u_\varepsilon) = 1$ and*

$$u_\varepsilon(r, x_3) = w\,(\,x_3 + \mathrm{q}_\varepsilon(r)\,) - w\,(\,x_3 - \mathrm{q}_\varepsilon(r)\,) - 1 + O(\varepsilon), \quad \textit{uniformly as } \varepsilon \to 0 \tag{1.24}$$

where

$$\mathrm{q}_\varepsilon(r) = \frac{\sqrt{2}}{2}(1 + o(\varepsilon)) \log\left(1 + \varepsilon^2\, r^2\right) + b_0 + \frac{\sqrt{2}}{2} \log \frac{1}{\varepsilon} \tag{1.25}$$

uniformly, as $\varepsilon \to 0$. Here b_0 is an explicit constant.

The solution of the above theorem is in addition even in the x_3-coordinate. The zero level set of u_ε of this result is the union of the graph of a positive radially symmetric function

which asymptotically behaves logarithmically, and its reflection through the plane $x_3 = 0$. We can actually think of this solution as having a parallel with minimal surfaces: If we take two planes $x_3 = \pm A$, their union is a (disconnected) minimal surface. For no solution of the Allen Cahn equation the zero set can be close to this two-plane object. Instead, the Allen Cahn equation produces (for A large) a nonlocal interaction between the corresponding components of the nodal set, which can be quantified, making them diverge logarithmically.

The law governing the interaction of the two components, assumed to be graphs, $x_3 = \pm \mathrm{q}_\varepsilon(r)$, is a perturbation of the Liouville equation

$$\Delta\, \mathrm{q}_\varepsilon - a_0 e^{-2\sqrt{2}\,\mathrm{q}_\varepsilon} = 0, \quad \text{in } \mathbb{R}^2 \tag{1.26}$$

whose radial solutions of (1.26) are given by the one-parameter family of functions

$$\mathrm{q}_\varepsilon(r) = \frac{1}{2\sqrt{2}} \log\left(\frac{\sqrt{2}\, a_0}{4} \left(1 + (\varepsilon r)^2\right)^2 \right) + \frac{\sqrt{2}}{2} \log\left(\frac{1}{\varepsilon}\right).$$

Until now, two families of Morse index 1 axially symmetric solutions have become known: That with a connected, catenoidal zero set constructed in [25], and the two-component constructed in Theorem 1.12. We believe these solutions correspond to limiting situations of a single one-parameter family of solutions, in a similar sense to how four-ended solutions in $\mathbb{R}^2$ are deformed. We will partially solve this question in the next section.

1.9. Connectedness of Two-ended Solutions in $\mathbb{R}^3$. In this section, we show that the solutions constructed in Theorems 1.11 and 1.12 are *connected*. To this end, we consider solutions of (1.1) satisfying in addition

$$\begin{cases} u_{zz} + u_{rr} + r^{-1} u_r + u - u^3 = 0, r \in [0, +\infty), z \in R, \\ u(r, z) = u(r, -z),\ u_r(0, z) = 0, \end{cases} \tag{1.27}$$

and

$$u_z > 0 \text{ for } z > 0; u_r < 0, \text{for } r > 0. \tag{1.28}$$

The following result confirms the belief that the two branches of Morse index one solutions are actually connected in $\mathbb{R}^3$:

Theorem 1.13 (Gui-Liu-Wei [45]). *For each $\kappa \in \left(\sqrt{2}, +\infty\right)$, there exists a solution u_κ to (1.27) and (1.28) which has the following asympotic behavior in the region where $z > 0$:*

$$u_\kappa(r, z) = w(z - \kappa \ln r + c_\kappa) + o(1),\ \textit{as } r \to +\infty, \tag{1.29}$$

where c_κ is a constant depending on κ.

It is natural to conjecture that for each $\kappa \in \left(\sqrt{2}, +\infty\right)$, there should be a unique solution having the asymptotic behavior (1.29). We also conjecture that for $\kappa \in (0, \sqrt{2}]$, there will not be such solutions.

We sketch the main ideas of the proof. First, we show that the set of solutions indeed has a structure of real analytic variety of formal dimension 1. Then we show compactness of solutions with certain natural constraints and we also show uniqueness of solutions on the boundary of the moduli space. Combining these results, we conclude the proof of Theorem 1.13 by applying a structure theorem for real analytic varieties, due to Dancer and Toland.

1.10. Infinite Morse index solutions in $\mathbb{R}^2$ and the Toda soliton train. After the study of finite Morse index solutions we now move on to the research on solutions with infinitely many ends and infinite Morse index.

In $\mathbb{R}^1$ the Allen-Cahn equation reduces to a second order ODE. It is easy to see that except the heteroclinc solution, this ODE also admits periodic solutions with large periods. Clearly, they could be lifted to $\mathbb{R}^2$, yielding periodic solutions depending only on one space variable. Since they are periodic, their Morse index is infinite but because they depend on just one variable they are at the same time one-dimensional. This is the simplest solutions with infinitely many ends. Naturally, one would like to find non-trivial periodic solutions. In this respect, a bifurcation analysis has been carried out in [71] and some nontrivial doubly periodic solutions were found.

As in [26], there is a deep connection between solutions with infinitely many ends and the infinite Toda system. More precisely we are able to construct new entire solutions of Allen-Cahn in $\mathbb{R}^2$ which are singly periodic and whose nodal set is determined by the solutions of the infinite *Toda lattice*:

$$c_* q_i'' = e^{\sqrt{2}(q_{i-1}-q_i)} - e^{\sqrt{2}(q_i-q_{i+1})}, \quad i \in \mathbb{Z}. \tag{1.30}$$

Due to the fact that the number of particles, whose positions are given by the functions $q_i, i \in \mathbb{Z}$, is infinite this system is more complicated than the classical Toda system, however it is still integrable. For us a special solution called *one-soliton* will be of crucial importance and we will describe it now. Given a parameter $k > 0$, let us denote

$$c = c(k) := \frac{\sinh k}{k} > 1.$$

The function

$$S_c(t) := \ln \frac{\cosh k\left(t - \frac{1}{2}\right)}{\cosh k\left(t + \frac{1}{2}\right)},$$

is odd and satisfies, for t large,

$$|S_c(t) + k| \leq Ce^{-2kt}.$$

Direct computation shows that

$$s_i(t; c, \lambda) := S_c(i - c\lambda t) - 2i \ln \lambda, \qquad i \in \mathbb{Z}, \qquad \lambda > 0,$$

solve the Toda lattice equation:

$$s_i'' = e^{s_{i-1}-s_i} - e^{s_i - s_{i+1}}, \quad i \in \mathbb{Z}.$$

This one soliton solution and its properties were studied in a series of papers by Friesecke and Pego [31] and Mizumachi and Pego [32].

Rescaling the Toda lattice equation we see that:

$$q_i = q_i(t; c, \varepsilon) := \frac{1}{\sqrt{2}} s_i\left(\sqrt{\sqrt{2}c_*^{-1}}t; c, \varepsilon\right), \quad i \in \mathbb{Z}, \tag{1.31}$$

is a family of one-soliton solutions of (1.30). The distance $q_i - q_{i-1}$ between two adjacent "particles" is approximately $\sqrt{2}\,|\ln \varepsilon|$, which means they are far away from each other if the parameter ε is small.

Now we can state the following existence result

Theorem 1.14 (Kowalczyk-Liu-Wei [46]). *For each sufficiently small $\varepsilon > 0$ and for each $c > 0$, there exists a vector* e *such that the Allen-Cahn equation (1.1) has a non-trivial singly periodic solution $u_{c,\varepsilon}$ with minimal period* $2\mathsf{e}$. *The nodal set of $u_{c,\varepsilon}$ is close to the family of curves $\{(t, q_i(t; c, \varepsilon))\}_{i\in\mathbb{Z}}$, where $\{q_i(t; c, \varepsilon)\}_{i\in\mathbb{Z}}$ is the one-soliton solution of the Toda lattice given by (1.31). Moreover, it holds*

$$u_{c,\varepsilon}(z) = -u_{c,\varepsilon}(-z),\ u_{c,\varepsilon}(z) = -u_{c,\varepsilon}(z + \mathsf{e}) \tag{1.32}$$

where $\mathsf{e} \in \mathbb{R}^2$ depends on c and ε.

1.11. Interface solutions in higher dimensional catenoid. All the developments mentioned above show the strong connection between the Allen-Cahn equation and the Minimal surfaces theory, but this connection has been only partly explored in lower dimensions, in particular when providing more examples of solutions to the Allen-Cahn Equation in higher dimensions. On the other hand, unlike dimensions $\mathbb{R}^2$ and $\mathbb{R}^3$ in which a large amount of examples of minimal surfaces exist and have been analyzed, in higher dimensions $\mathbb{R}^{N+1}$ there are very few examples of minimal surfaces.

In this section we explore the connection between higher dimensional catenoid and the equation (1.1) for $N \geq 3$. Thus we let M be the $N-$dimensional catenoid, which is described by the graph of the axially symmetric functions $\pm F$ where $F = F(|y|)$ is the unique increasing axially symmetric solution to the minimal graph equation

$$\nabla \cdot \left(\frac{\nabla F}{\sqrt{1 + |\nabla F|^2}} \right) = 0, \quad |y| > 1, \quad y \in \mathbb{R}^N, \quad F(1) = 0, \quad \partial_r F(1) = +\infty. \tag{1.33}$$

The catenoid M has asymptotically parallel flat ends:

$$F(r) = F(r) = \int_1^\infty \frac{1}{\sqrt{s^{2(N-1)} - 1}}\, ds - \frac{r^{2-N}}{N-2} + \mathcal{O}_{L^\infty(\mathbb{R}^N)}\left(r^{4-3N}\right), \quad \text{as } r \to \infty. \tag{1.34}$$

At a first glance, one may think that the proper choice for the approximate nodal set of the solutions predicted in our theorem would be a large dilated version of M, $M_\epsilon = \epsilon^{-1} M$, with $\epsilon > 0$ small. As pointed out in [25], this is not an appropriate global choice. Instead, the parallel ends of M_ϵ must be perturbed in order to obtain a profile for the nodal set that will lead to good sizes in the error and the rule governing this perturbation is the *Liouville Equation*

$$\epsilon \Delta F_\epsilon - a_0\, e^{\frac{-2\sqrt{2}\, F_\epsilon}{\epsilon}} = 0, \quad \text{for } |y| > R_\epsilon. \tag{1.35}$$

We match the functions F and F_ϵ in a C^1 way by considering the additional initial conditions for F_ϵ at some suitable radius $r = R_\epsilon$.

Theorem 1.15 (Agudelo-del Pino-Wei [2]). *For every $N \geq 3$ and any sufficiently small $\epsilon > 0$ there exist a solution u_ϵ to equation* (1.1) *in having the asymptotics*

$$u_\epsilon(x) = w(z)(1 + o(1)), \quad \text{as } \epsilon \to 0$$

where z is the normal direction to the largely dilated surface $\epsilon^{-1}\Sigma$ described above. Furthermore for $\epsilon > 0$ small and for dimensions $3 \leq N \leq 9$, $m(u_\epsilon) = +\infty$.

The above statement shows a dramatic difference between the Allen-Cahn equation and the Theory of Minimal surfaces. In fact it is known ([76]) that for dimensions $3 \leq N \leq 9$, the catenoid M has Morse index 1. In contrast our result states that with the same restriction in the dimension, these solutions are highly unstable.

2. Part II: Nonlinear Schrodinger equation and CMC surfaces

In the second part of this paper, we survey recent results on the study and construction of entire solutions to the nonlinear Schrodinger equation.

2.1. Background. We consider another classical semilinear elliptic problem

$$\Delta u - u + u^p = 0, \quad u > 0, \quad \text{in } \mathbb{R}^N, \text{ where } p > 1. \tag{2.1}$$

Equation (2.1) can be considered as the standing-wave problem for the standard nonlinear Schrödinger equation

$$i\psi_t = \Delta_y \psi + |\psi|^{p-1}\psi,$$

typically $p = 3$, corresponding to that of solutions of the form $\psi(y,t) = u(y)e^{-it}$. It also arises in nonlinear models in Turing's theory biological theory of pattern formation such as Gierer-Meinhardt system [34].

The solutions of (2.1) which decay to zero at infinity are well understood. Problem (2.1) has a radially symmetric solution $w_N(y)$ which approaches 0 at infinity provided that

$$1 < p < \frac{N+2}{N-2} \text{ if } N \geq 3; +\infty \text{ if } N = 1, 2,$$

see [10]. This solution is unique [51], and actually any positive solution to (2.1) which decyas at infinity must be radially symmetric around some point ([33]). This solution will be called *Type I Solution*.

Problem (2.1) and its variations have been broadly treated in the PDE literature in the last two decades. These variations are mostly of one of the two types: (2.1) is changed to a non-autonomous problem with a potential depending on the space variable; or (2.1) is considered in a bounded domain under suitable boundary conditions. Typically, in both versions a small parameter is introduced rendering (2.1) a singular perturbation problem. We refer the reader to the works [35, 42, 52, 55, 56] and references therein. Many constructions in the literature refer to "multi-bump solutions", built by perturbation of a sum of copies of the basic radial bump w_N suitably scaled, with centers adjusted in equilibrium under appropriate constraints on the potential or the geometry of the underlying domain.

2.2. Type II Solutions: Dancer's solution. Much less is known about solutions to this equation in entire space which do not vanish at infinity. For simplicity, we restrict ourselves to the case $N = 2$. A canonical example is thus built from the one-dimensional bump w_1, which we denote in the sequel just by w, namely the unique solution of the ODE

$$w'' - w + w^p = 0,\ w > 0, \quad \text{in } \mathbb{R},\ w'(0) = 0, \quad w(x) \to 0 \quad \text{as } |x| \to +\infty, \tag{2.2}$$

corresponding in phase plane to a homoclinic orbit for the equilibrium 0. Using this function we can define a family of solutions u of equation (2.1) by setting $u(x,z) := w(x-a)$, $a \in \mathbb{R}$. By analogy with the above terminology, we may call these solutions "single bump-lines". A natural question is whether a solution that satisfies $\lim_{|x|\to+\infty} u(x,z) = 0$ and which is in addition even in x must equal $w(x)$. It turns out that this is not true. A second class of solutions which are even both in z and x was discovered by Dancer in [20] via local bifurcation arguments. They constitute a one-parameter family of solutions which are periodic in the z variable and originate from $w(x)$:

$$w_\delta(x,z) = w(x) + \delta w^{\frac{p+1}{2}} \cos(\sqrt{\lambda_1} z) + O(\delta^2) e^{-|x|}.$$

We refer to the functions w_δ in what follows as *Dancer solutions* or *Type II Solution*.

2.3. Type III solutions: Multi-bump line solutions. Using Dancer's solutions, in [27], del Pino, Kowalczyk, Pacard and Wei constructed a new type of solutions of (2.1) in $\mathbb{R}^2$ that have multiple ends in the form of multiple bump-lines. What they actually constructed is a solution $u(x,z)$ which is close, up to lower order terms, to a multi bump-line of the form

$$w_*(x,z)=\sum_{j=0}^{k} w_{\delta_j}(x-f_j(z),z), \tag{2.3}$$

for suitable small numbers δ_j and even functions

$$f_1(z)\ll f_2(z)\ll\cdots\ll f_k(z),$$

which have uniformly small derivatives. The functions f_j cannot be arbitrary and they turn out to satisfy (asymptotically) a second order system of differential equations, the *Toda system* (1.18), as in the Allen-Cahn equation.

The main result in [27] is:

Theorem 2.1 (del Pino-Kowalczyk-Pacard-Wei [27]). *Assume that $N=2$ and $p\geq 2$. Given $k\geq 2$ and rescaled solutions f_ϵ (1.20) of Toda system (1.18) , for any sufficiently small number $\epsilon>0$, there exists a solution u_ϵ of equation (2.1) that has the form*

$$u_\epsilon(x,z)=\sum_{j=1}^{k} w_{\delta_j}(x-f_{\epsilon_j}(z),z)\,(1+o(1)).$$

2.4. Type IV Solution: Malchiodi's $Y-$shaped solutions. On the other hand, in [54], Malchiodi constructed another new kind of solutions with three rays of bumps. More precisely, the solutions constructed in [54] have the form

$$u(x,z)\approx\sum_{j=1}^{3}\sum_{i=1}^{+\infty} w_2((x,z)-iL\vec{l_j}) \tag{2.4}$$

where $\vec{l}_j, j=1,2,3$ are three unit vectors satisfying some balancing conditions ($\mathbf{Y}$-shaped solutions). Here w_2 is the unique ground state solution (Type I Solutions) in $\mathbb{R}^2$.

2.5. Solution type V: Front-Spike solutions. In [68], Santra and I constructed solutions with the coexistence of *both fronts* and *bumps.* More precisely we look for positive solutions of the form

$$u_\sharp(x,z)=w(x-f(z))+\sum_{i=1}^{\infty} w_2((x,z)-\xi_i\vec{e}_1) \tag{2.5}$$

for suitable large $L>0$ and ξ_i's are such that $\xi_1-f(0)=L$ and $\xi_j=jL+\mathcal{O}(1)$ for all $j\geq 1$; w is the unique one-dimensional solution and w_2 is two-dimensional ground state and $\vec{e}_1=(1,0)$.

Because of the new interaction between the fronts and bumps, we are led to considering the following second order ODE:

$$f''(z)=\Psi_L(f,z) \text{ in } \mathbb{R},\ \ f(0)=0,\ \ f'(0)=0, \tag{2.6}$$

where $\Psi_L(f,z)$ is a function measuring the interactions between bumps and fronts. Asymptotically $\Psi_L(f,z) \sim ((f-L)^2+z^2)^{-\frac{1}{2}} e^{-\sqrt{(f-L)^2+z^2}}$. Let $\alpha = \int_0^{+\infty} \Psi(\sqrt{L^2+z^2})dz$.

We prove the following:

Theorem 2.2 (Santra-Wei [68]). *Let $N = 2$. For $p > 2$ and sufficiently large $L > 0$, (2.1) admits a one parameter family of positive solution satisfying*

$$\begin{cases} u_L(x,z) = u_L(x,-z) & \text{for all } (x,z) \in \mathbb{R}^2 \\ u_L(x,z) = \left(w_\delta(x - f(z) - h_L(z), z) + \sum_{i=1}^{\infty} w_2((x,z) - \xi_i \vec{e}_1) \right)(1 + o_L(1)) \end{cases} \tag{2.7}$$

where $\delta = \delta_L$ is a small constant, ω_δ is the Dancer's solution, f is the unique solution of (2.6), $\xi_j \sim jL$ and $o_L(1) \to 0$ as $L \to +\infty$, and the function $\|h_L\|_{C^{2,\mu}_\theta(\mathbb{R})\oplus\mathcal{E}} \le C\alpha^{1+\gamma}$ for some constant $\theta > 0, \gamma > 0$. Moreover, the solution has three ends.

2.6. Solution type VI: Finite-Energy Sign-Changing solutions. Obviously (2.1) is equivariant with respect to the action of the group of isometries of $\mathbb{R}^N$, it is henceforth natural to ask whether or not all finite-energy solutions of (2.1) are radially symmetric. In that regard, the classical result of Gidas, Ni and Nirenberg [33] asserts that all *positive* solutions of (2.1) are indeed radially symmetric. Therefore, nonradial finite energy solutions, if they exist, are necessarily sign-changing solutions. Berestycki and Lions [10] and Struwe [74] have obtained the existence of infinitely many *radially symmetric* sign-changing solutions to (2.1) in the subcritical case.

The existence of *nonradial* sign-changing solutions to (2.1) was first proved by Bartsch and Willem [9] in dimension $N = 4$ and $N \ge 6$. The key idea is to look for solutions invariant under the action of $O(2) \times O(N-2) \subset O(N)$ to recover some compactness property. Later on, this result was generalized by Lorca and Ubilla [53] to handle the $N = 5$ dimensional case. The proofs of both results rely on variational methods and the oddness of the nonlinearity. The question of the existence of nonradial solutions remained open in dimensions $N = 2, 3$.

In [63], Musso, Pacard and Wei constructed unbounded sequences of solutions of (2.1) in any dimensions $N \ge 2$. The solutions they obtained are nonradial, have finite energy and are invariant under the action of $D_k \times O(N-2)$, for some given $k \ge 7$, where $D_k \subset O(2)$ is the dihedral group of rotations and reflections leaving a regular polygon with k sides invariant. Moreover, these solutions are not invariant under the action of $O(2) \times O(N-2)$ and hence they are different from the solutions constructed in [9] and [53].

The energy functional associated to (2.1) is given by

$$\mathcal{E}(u) := \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)\, dx - \frac{1}{p+1} \int_{\mathbb{R}^N} |u|^{p+1}\, dx. \tag{2.8}$$

Denote

$$\mathcal{E} =:= \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla w_N|^2 + w_N^2)\, dx - \frac{1}{p+1} \int_{\mathbb{R}^N} |w_N|^{p+1}\, dx. \tag{2.9}$$

The main result of [63] can be summarized as follows

Theorem 2.3 (Musso-Pacard-Wei [63]). *Assume that $k \geq 7$ is a fixed integer. Then, there exist two sequences of integers, $(m_i)_{i\geq 0}$ and $(n_i)_{i\geq 0}$, tending to $+\infty$, and $(u_i)_{i\geq 0}$, a sequence of nonradial sign-changing solutions of (2.1), whose energy $\mathcal{E}(u_i)$ is equal to*

$$\mathcal{E}(u_i) = k\left((m_i + 2n_i)\mathcal{E}\right) + o(1).$$

Moreover, the solutions u_i are invariant under the action of $D_k \times O(N-2)$ but are not invariant under the action of $O(2) \times O(N-2)$.

2.7. Finite energy solutions without any symmetry. In view of Theorem 2.3, a natural question is the following :

Do all finite-energy solutions of (2.1) have a nontrivial group of symmetry?

Surprisingly, the answer to this question is negative. In fact, we prove the :

Theorem 2.4 (Ao-Musso-Pacard-Wei [3]). *There exist infinitely many solutions of (2.1) which have finite energy but whose maximal group of symmetry reduces to the identity.*

A more precise statement of the above theorem involves the notion of *flexible, balanced and closable planar network*, which we shall describe next.

A finite planar network $\mathcal{N} := (V, E)$ in $\mathbb{R}^2$ is given by its set of vertices $V \subset \mathbf{C}$ and its set of edges E joining the vertices. If $[p, q] \in E$, then the points $p, q \in V$ are called the *end points* of the edge $[p, q]$. Naturally, we identify $[p, q]$ and $[q, p]$. The number of vertices of a given network $\mathcal{N}$ will be denoted by n and its number of edges will be denoted by m. For each $p \in V$, we denote by $V_p \subset V$ the set of vertices $q \in V$ such that $[p, q] \in E$, namely $V_p := \{q \in V \,:\, [p, q] \in E\}$.

On a connected and embedded network, we define two functions: the first is the length function and the second is the force. The *length* of a given network $\mathcal{N}$ is defined to be as the collection of the lengths of the edges of the network, namely

$$\mathbf{L}_{\mathcal{N}} := \left(|p - q|\right)_{[p,q]\in E}.$$

If $\mathcal{N} = (V, E)$ is a network and if $a : E \to \mathbf{R} - \{0\}$ is a function, we will say that $(\mathcal{N}, a)$ is a *weighted network*. The image of $[p, q] \in E$ by a will be denoted by $a_{[p,q]}$. Then for all $p \in V$, we define the *force of the weighted network* $(\mathcal{N}, a)$ *at the vertex* p by

$$\mathbf{F}_{(\mathcal{N},a)} := \left(\mathbf{F}_{(\mathcal{N},a)}(p)\right)_{p\in V}, \quad \mathbf{F}_{(\mathcal{N},a)}(p) := \sum_{q\in V_p} a_{[p,q]}\, \frac{q-p}{|q-p|}$$

which is the collection of all forces at the different vertices of the weighted network $(\mathcal{N}, a)$.

Definition 2.5. A weighted network $(\mathcal{N}, a)$ is said to be *balanced* if $\mathbf{F}_{(\mathcal{N},a)} = 0$. Otherwise, we say that the weighted network $(\mathcal{N}, a)$ is *unbalanced*.

For a balanced weighted network, we want to perturb it so that the perturbed network is still balanced. For this, we need to consider the linear map

$$\begin{array}{rcccl} \Lambda & : & \mathbf{C}^n \times \mathbf{R}^m & \to & \mathbf{C}^n \times \mathbf{R}^m \\ & & (\dot\Phi, \dot a) & \mapsto & \left(\mathrm{D}\mathbf{F}_{(\mathrm{Id},a)}(\dot\Phi, \dot a)\,,\, \mathrm{D}\mathbf{L}_{\mathrm{Id}}(\dot\Phi)\right). \end{array} \tag{2.10}$$

If the weighted network $(\mathcal{N}, a)$ is unbalanced we have proved that Λ has kernel of dimension at least 2 and cokernel of dimension at least 2, while, if the network $(\mathcal{N}, a)$ is balanced, then Λ has kernel of dimension at least 4 and cokernel of dimension at least 3.

Let us now focus on *balanced weighted networks* for which we introduce the notion of flexibility.

Definition 2.6. A balanced weighted network $(\mathcal{N}, a)$ is said to be flexible if the mapping Λ has rank $2n + m - 4$.

The last definition is a little bit intricate:

Definition 2.7. A flexible, balanced network $(\mathcal{N}, a)$ is said to be *closable* if

$$\mathring{\Lambda}(\dot{\Phi}, \dot{a}, \dot{s}) := \left(\mathrm{D}\mathbf{F}_{(\mathrm{Id},a)}(\dot{\Phi}, \dot{a}) \, ; \mathrm{D}\mathbf{L}_{\mathrm{Id}}(\dot{\Phi}) + \dot{s}\, \mathbf{T} \right),$$

has rank $2n + m - 3$.

With these definitions at hand, we now have the following theorem

Theorem 2.8 (Ao-Musso-Pacard-Wei [3]). *Given any balanced, flexible, and closable network, there exists a sequence of finite energy solutions to (2.1) with large number of positive or negative bumps on the dilated nextwrok.*

2.8. Applications: Chern-Simons-Higgs equations and magnetic Ginzburg-Landau. The results developed in the study of nonlinear Schrödinger equation can have important applications in equations in mathematical physics. First one is the magnetic Chern-Simons-Higgs (CSH) equations:

$$-\Delta_A \psi + \lambda^2 (1 - |\psi|^2)(1 - 3|\psi|^2)\psi - \frac{\mu^2}{4} \frac{|\nabla \times A|^2}{|\psi|^4} \psi = 0 \tag{2.11}$$

$$\frac{\mu^2}{4} \nabla \times \left(\frac{\nabla \times A}{|\psi|^2} \right) + \mathrm{Im}(\bar{\psi} \nabla_A \psi) = 0 \tag{2.12}$$

for $\lambda, \mu > 0$ constants, where $\psi : \mathbf{R}^2 \to \mathbf{C}$ and $A : \mathbf{R}^2 \to \mathbf{R}^2$. $\nabla_A = \nabla - iA$ is the covariant gradient, and $\Delta_A = \nabla_A \cdot \nabla_A$. For a vector field A, $\nabla \times A$ is the scalar $\partial_1 A_2 - \partial_2 A_1$ and for scalar ξ, $\nabla \times \xi$ is the vector $(-\partial_2 \xi, \partial_1 \xi)$. The CSH equations arise from the problems in condensed matter physics such as high-temperature superconductivity and quantum and fractional Hall effect ([40]).

Equations (2.11) and (2.12) are Euler-Lagrange equations for the CSH energy functional

$$E(\psi, A) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla_A \psi|^2 + \frac{\mu^2}{4} \frac{(\nabla \times A)^2}{|\psi|^2} + \lambda^2 (1 - |\psi|^2)^2 |\psi|^2, \tag{2.13}$$

In addition to being translational and rotationally invariant, Equations (2.11) and (2.12) have translational and gauge symmetries: solutions are mapped to solutions under the transformations

$$\psi(x) \mapsto \psi(x - z), \qquad A(x) \mapsto A(x - z)$$

for any $z \in \mathbb{R}^2$, and

$$\psi \mapsto e^{i\gamma} \psi, \qquad A \mapsto A + \nabla \gamma$$

for twice differentiable $\gamma : \mathbb{R}^2 \to \mathbb{R}$.

When $\lambda = \frac{1}{\mu}$, minimizers of the CSH energy will satisfy a set of Bogomol'nyi-type self-dual equations ([40]). Using the new variable $u = \ln |\phi|^2$, the self-dual equation was reduced to a single scalar equation

$$\Delta u = \frac{4}{\mu^2} e^u (e^u - 1) + 4\pi \sum_{j=1}^{N} \delta_{p_j}. \tag{2.14}$$

For $\lambda > \frac{1}{\mu}$, by [18], we know that there exists $n-$vortices solutions to the CSH-equation:

$$\psi^{(n)}(x) = f_n(r) e^{in\theta} \qquad and \qquad A^{(n)}(x) = a_n(r) \nabla(n\theta). \tag{2.15}$$

However less is known about other solutions for the non self-dual case $\lambda \neq \frac{1}{\mu}$.

Another physical model is the magnetic Ginzburg Landau equations. The standard macroscopic (or mean field) theory of superconductivity is due to Ginzburg and Landau. Stationary states of superconductors occupying (for simplicity) the plane $\mathbb{R}^2$, are described by pairs (ψ, A), where $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ is the *order parameter* and $A : \mathbb{R}^2 \to \mathbb{R}^2$ is the *magnetic potential*. These states satisfy the system of equations

$$-\Delta_A \psi + \lambda(|\psi|^2 - 1)\psi = 0 \tag{2.16}$$

$$\nabla \times \nabla \times A + \mathrm{Im}(\bar{\psi} \nabla_A \psi) = 0 \tag{2.17}$$

called the *Ginzburg-Landau* (GL) equations. Here $\lambda > 0$ is a constant depending on the material in question: when $\lambda < 1/2$ or $> 1/2$, the material is of type I or II superconductor, respectively. The corresponding energy functional is

$$\mathcal{E}_{gl}(\psi, A) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla_A \psi|^2 + |\nabla \times A|^2 + \frac{\lambda}{2}(|\psi|^2 - 1)^2. \tag{2.18}$$

The Ginzburg-Landau equations on the plane model superconductors which are spatially homogeneous in one direction, when neglecting boundary effects. They also describe equilibrium states of the $U(1)$ Higgs model of particle physics [40]. Equations (2.16) and (2.17) also have translational and gauge symmetries.

First we prove the same result as in [63] for the magnetic Ginzburg-Landau:

Theorem 2.9 (Ting-Wei [75]). *There exists $\varepsilon_0 > 0$ small, and for fixed $\lambda > \frac{1}{2}$ and an integer $k \geq 7$. There exists a sequence, $(u_i)_{i\geq 0} := (\psi_i, A_i)_{i\geq 0}$, of non-radial degree-changing solutions to (2.16) and (2.17) containing km_i vortices, $m_i \to \infty$, invariant under rotations by $\frac{2\pi}{k}$ (but not by rotations in $O(2)$ in general) and reflections in the $x_2 = 0$ line. Each u_i has finite-energy and the energy of u_i approaches $+\infty$ as $i \to +\infty$.*

The same result holds for CSH:

Theorem 2.10 (Ao-Pacard-Ting-Wei [4]). *There exists $\varepsilon_0 > 0$ small, and for fixed $\lambda > \frac{1}{\mu}$ with $\lambda - \frac{1}{\mu} < \varepsilon_0$ and an integer $k \geq 7$. There exists a sequence, $(u_i)_{i\geq 0} := (\psi_i, A_i)_{i\geq 0}$, of non-radial degree-changing solutions to (2.11) and (2.12) containing km_i vortices, $m_i \to \infty$, invariant under rotations by $\frac{2\pi}{k}$ (but not by rotations in $O(2)$ in general) and reflections in the $x_2 = 0$ line.*

For general networks, we can also construct solutions to (2.11)-(2.12) or (2.16)-(2.17) without any symmetry, as in Theorem 2.8.

Theorem 2.11 (Ao-Pacard-Ting-Wei [4]). *There exists $\varepsilon_0 > 0$ small, and for fixed $\lambda > \frac{1}{\mu}$ with $\lambda - \frac{1}{\mu} < \varepsilon_0$, there exists infinitely many solutions of (2.11) and (2.12) which have finite energy but without any symmetry. The same result holds for (2.16)-(2.17) for $\lambda > \frac{1}{2}$.*

2.9. Geometric analogues with CMC theory. One of the striking features of all the *six types of solutions*, which are purely PDE results, is that their counterparts and origins can be found in geometric framework. Indeed, there are many examples where correspondence between solutions of (2.1) and those of some geometric problem can be drawn. To illustrate this, we will concentrate on what is perhaps the most spectacular one: the analogy between the theory of complete constant mean curvature surfaces in Euclidean 3-space and the study of entire solutions of (2.1). In the following we will draw parallels between these geometric objects and families of solutions of (2.1).

The two well-known CMC surfaces in $\mathbb{R}^3$ are spheres and cylinders. In the mid 19th century Delaunay discovered a new family of singly periodic embedded constant mean curvature surfaces of revolution D_τ, parametrized by a parameter $\tau \in (0, 1]$, which interpolate between the cylinder $D_1 = S^1(1) \times \mathbb{R}$ and the singular surface $D_0 := \lim_{\tau \to 0} D_\tau$, which is the union of an infinitely many spheres of radius $1/2$ centered at each of the points $(0, 0, n)$ as $n \in \mathbb{Z}$. When $\tau = 1$, the Delaunay surface is nothing but a right circular cylinder $D_1 = S^1(1) \times \mathbb{R}$, with the unit circle as the cross section. This cylinder is clearly invariant under the continuous group of vertical translations, in the same way that the single bump-line solution of (2.1) is invariant under a one parameter group of translations. It is then natural to agree on the correspondence between the cylinder D_1 and the single bump-line solution $w(x)$. On the other hand the unique radially symmetric, decaying solution of (2.1) corresponds to the sphere. The full correspondence of Delaunay solution should correspond to the family of mountain-pass solutions of (2.1) built as follows (see [11]): Let $S_R = \mathbb{R} \times (0, R)$ and consider a least energy (mountain pass) solution in $H^1(S_R)$ for the the energy

$$\frac{1}{2} \int_{S_R} |\nabla u|^2 + \frac{1}{2} \int_{S_R} u^2 - \frac{1}{p+1} \int_{S_R} u^{p+1},$$

for large $R > 0$, which we may assume to be even in x and with maximum located at the origin. For R very large, this solution, which satisfies zero Neumann boundary conditions, resembles half of the unique radial, decaying solution w_2 of (2.1). Extension by successive even reflections in z variable yields a solution to (2.1) which resembles a periodic array of radially symmetric solutions of (2.1), with a very large period, along the z-axis. While this is not known, these solutions may be understood as a limit of the branch solutions constructed by Dancer.

A CMC surface $\mathcal{S}$ of finite topology is *Alexandrov-embedded*; if $\mathcal{S}$ is properly immersed, and if each end of $\mathcal{S}$ is embedded; there exist a compact manifold M with boundary of dimension three and a proper immersion $F : M \setminus \{q_1, q_2, \cdots, q_m\} \to \mathbb{R}^3$ such that $F|_{\partial M \setminus \{q_1, q_2, \cdots, q_m\}}$ parametrizes M. Moreover, the mean curvature normal of $\mathcal{S}$ points into M. Then we define *tridulonoid* as an Alexandrov embedded CMC surface having zero genus and three ends. Triunduloids are a basic building block for Alexandrov embedded CMC surface with any number of ends. Mazzeo-Pacard [58] established existence of tridunduloid with small necksize or high symmetry.

With these analogies in mind, we can now *translate* the other four types of solutions above into the constant mean curvature surface framework.

The result of Theorem 2.1 corresponds to the connected sum of finitely many copies of the cylinder $S^1(1) \times \mathbb{R}$ which have a common plane of symmetry. The connected sum construction is performed by inserting small catenoidal necks between two consecutive cylinders and this can be done in such a way that the ends of the resulting surface are coplanar. Such a result, in the context of constant mean curvature surfaces, follows at once from [59]. It is observed that, once the connected sum is performed the ends of the cylinder have to be slightly bent and moreover, the ends cannot be kept asymptotic to the ends of right cylinders but have to be asymptotic to Delaunay ends with parameters close to 1, in agreement with the result of Theorem 2.1. However there is a major difference. The Toda system which governs the level sets has found no analogy in the constant mean curvature surfaces. This is mainly due to the strong interactions in the elliptic equations.

Another (older) construction of complete noncompact constant mean curvature surfaces was performed by N. Kapouleas [43] (see also [58]) starting with finitely many halves of Delaunay surfaces with parameter τ close to 0 which are connected to a central sphere. The corresponding solutions of (2.1) are the Type IV solutions constructed by A. Malchiodi in [54].

Type IV and Type V solutions belong to the same *tridunduloid* type of solution for (2.1) in $\mathbb{R}^2$ i.e. a solution having three ends ([57]-[59]). Type VI solutions (Theorems 2.3 and 2.8) corresponds to construction for immersed constant mean curvature surfaces in Euclidean 3-space given in [43].

Acknowledgments. The research of the author is partially supported by NSERC of Canada. I would like to take this opportunity to thank my collaborators Professors Manuel del Pino, M. Kowalczyk, M. Musso, Frank Pacard for sharing their ideas, insights, and for their patience and help.

References

[1] O. Agudelo, M. del Pino, and J, Wei, *Multiple catenoidal end solutions to the Allen-Cahn Equation in* $\mathbb{R}^3$. J. Math. Pures Appl. to appear.

[2] _____, *Higher dimensional catenoid, Liouville equation and Allen-Cahn equation*, preprint 2014.

[3] WW. Ao, M. Musso, F. Pacard, and J. Wei, *Solutions without any symmetry for semi-linear elliptic problems*, Preprint 2012.

[4] W.W. Ao, F. Pacard, F. Ting, and J. Wei, *Multi-vortex non-radial solutions to the magnetic Chern-Simons-Higgs equations and Ginzburg-Landau equations*, Preprint 2014.

[5] G. Alberti, L. Ambrosio, and X. Cabré, *On a long-standing conjecture of E. De Giorgi: symmetry in 3D for general nonlinearities and a local minimality property*, Acta Appl. Math. **65** (2001), no. 1-3, 9–33.

[6] S.M. Allen and J.W. Cahn, *A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening*, Acta Metall. **27** (1979), 1084–1095.

[7] F. Alessio, A. Calamai, and P. Montecchiari, *Saddle-type solutions for a class of semilinear elliptic equations*, Adv. Diff. Eqns. **12** (2007), 361–380.

[8] L. Ambrosio and X. Cabré, *Entire solutions of semilinear elliptic equations in* $\mathbb{R}^3$ *and a conjecture of De Giorgi*, Journal Amer. Math. Soc. **13** (2000), 725–739.

[9] T. Bartsch and M. Willem, *Infinitely many nonradial solutions of a Euclidean scalar field equation*, J. Funct. Anal. **117** (1993), 447–460.

[10] H. Berestycki and P.L. Lions, *Nonlinear scalar field equations, I, Existence of a ground state*, Arch. Rat. Mech. Anal. **82** (1981), 313–347.

[11] H. Berestycki and J. Wei, *On least energy solutions to a semilinear elliptic equation in a strip*, DCDS-A **28** (2010), no.3, 1083–1099.

[12] E. Bombieri, E. De Giorgi, and E. Giusti, *Minimal cones and the Bernstein problem*, Invent. Math. **7** (1969) 243–268.

[13] X. Cabré and J. Terra, *Saddle-shaped solutions of bistable diffusion equations in all of* $\mathbb{R}^{2m}$, J. Eur. Math. Soc. **11**, Issue 4 (2009), 819–943.

[14] L. Caffarelli and A. Córdoba, *Uniform convergence of a singular perturbation problem*, Comm. Pure Appl. Math. **XLVII** (1995), 1–12.

[15] _____, *Phase transitions: uniform regularity of the intermediate layers*, J. Reine Angew. Math. **593** (2006), 209–235.

[16] C.J. Costa, *Example of a complete minimal immersions in* $\mathbb{R}^3$ *of genus one and three embedded ends*, Bol. Soc. Bras. Mat. **15**(1-2) (1984), 47–54.

[17] J. W. Cahn and J. E. Hilliard, *Free energy of a nonuniform system. I. Interfacial energy*, J. Chem. Phys. **28** (1958), 258.

[18] Chen, R.M. and Spirn, D., *Symmetric Chern-Simons-Higgs Vortices*, Comm. Math. Phys. **285** (2009), 1005–1031.

[19] E. De Giorgi, *Convergence problems for functionals and operators,* Proc. Int. Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978), 131–188, Pitagora, Bologna (1979).

[20] E.N. Dancer, *New solutions of equations on* $\mathbf{R}^n$, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **30**(4) (2002), no. 3-4, 535–563.

[21] C. Delaunay, *Sur la surface de revolution dont la courbure moyenne est constante*, Jour. de Mathématiques **6** (1841), 309–320.

[22] H. Dang, P. C. Fife, and L. A. Peletier. *Saddle solutions of the bistable diffusion equation*, Z. Angew. Math. Phys. **43**(6) (1992), 984–998.

[23] M. del Pino, M. Kowalczyk, and J. Wei, *The Toda system and clustering interfaces in the Allen-Cahn equation*, Arch. Rational Mech. Anal. **190** (2008), no. 1, 141–187.

[24] _____, *On De Giorgi's Conjecture in Dimensions* $N \geq 9$, Annals of Mathematics **174** (2011), no. 3, 1485–1569.

[25] ———, *Entire Solutions of the Allen-Cahn equation and Complete Embedded Minimal Surfaces of Finite Total Curvature in* $\mathbb{R}^3$, J. Diff. Geom. **83** (2013), no. 1, 67–131.

[26] M. del Pino, M. Kowalczyk, F. Pacard, and J. Wei, *Multiple end solution to the Allen-Cahn equation in* $\mathbb{R}^2$, J. Func. Anal. **258** (2010), no. 2, 458–503.

[27] ———, *The Toda system and multiple-end solutions of autonomous planar elliptic problem*, Advances in Mathematics **224** (2010), no. 4, 1462–1516.

[28] M. del Pino, M. Kowalczyk, and F. Pacard, *Moduli space theory for the Allen-Cahn equation in the plane*, Trans. Amer. Math. Soc. **365**(2) (2010), 721–766.

[29] D. Fischer-Colbrie and R. Schoen, *The structure of complete stable minimal surfaces in* 3*-manifolds of nonnegative scalar curvature*, Comm. Pure Appl. Math. **33** (1980), no. 2, 199–211.

[30] N. Ghoussoub and C. Gui, *On a conjecture of De Giorgi and some related problems*, Math. Ann. **311** (1998), 481–491.

[31] G. Friesecke and R. L. Pego, *Solitary waves on Fermi-Pasta-Ulam lattices. IV. Proof of stability at low energy*, Nonlinearity, **17**(1) (2004), 229–251.

[32] T. Mizumachi and R. L. Pego, *Asymptotic stability of Toda lattice solitons*, Nonlinearity **21**(9) (2008), 2099–2111.

[33] B. Gidas, W. M. Ni and L. Nirenberg, *Symmetry of positive solutions of nonlinear elliptic equations in* R^N, Adv. Math. Suppl. Stud. 7A (1981), 369–402.

[34] A. Gierer and H. Meinhardt, *A theory of biological pattern formation*, Kybernetik (Berlin) **12** (1972), 30–39.

[35] C. Gui and J. Wei, *Multiple interior peak solutions for some singular perturbation problems*, J. Differential Equations **158** (1999), 1–27.

[36] C. Gui, *Hamiltonian identities for elliptic partial differential equations*, J. Funct. Anal. **254** (2008), no. 4, 904–933.

[37] ———, *Symmetry of some entire solutions to the Allen-Cahn equation in two dimensions*, J. Differential Equations **252**(11) (2012), 5853–5874.

[38] D. Hoffman and W.H. Meeks III, *The asymptotic behavior of properly embedded minimal surfaces of finite topology*, J. Am. Math. Soc. **4**(2) (1989), 667–681.

[39] J.E. Hutchinson and Y. Tonegawa, *Convergence of phase interfaces in the van der Waals-Cahn-Hilliard theory*, Cal. Var. PDE **10**(1) (2000), 49–84.

[40] Jaffe, J., Taubes, C., *Vortices and Monopoles*, Bikhäuser, 1980.

[41] D. Jerison and R. Monneau, *Towards a counter-example to a conjecture of De Giorgi in high dimensions*, Ann. Mat. Pura Appl. **183** (2004), 439–467.

[42] Kang, X. and Wei, J., *On interacting bumps of semi-classical states of nonlinear Schrödinger equations*, Adv. Diff. Eqn. **5**(7-9) (2000), 899–928.

[43] N. Kapouleas, *Compact constant mean curvature surfaces in Euclidean three-space*, J. Differential Geom. **33** (1991), no. 3, 683–715.

[44] R. V. Kohn and P. Sternberg, *Local minimizers and singular perturbations*, Proc. Roy. Soc. Edinburgh Sect. A **11** (1989), 69–84.

[45] C. Gui, Y. Liu, and J. Wei, *Two-ended solutions to the Allen-Cahn equation in dimension three*, preprint 2014.

[46] M. Kowalczyk, Y. Liu, and J. Wei, *Singly periodic solutions of the Allen-Cahn equation and the Toda lattice*, preprint 2014.

[47] M. Kowalczyk, Y. Liu, F. Pacard, and J. Wei, *End-toend construction for the Allen-Cahn equation in the plane*, Cal.Var. PDE to appear.

[48] M. Kowalczyk, *On the existence and Morse index of solutions to the Allen-Cahn equation in two dimensions*, Ann. Mat. Pura Appl. **184**(4) (2005), 17–52.

[49] M. Kowalczyk, Y. Liu, and F. Pacard, *The space of four ended solutions to the Allen-Cahn equation on the plane*, Ann. Inst. H. Poincare Anal. Non Lineeaire **29**(5) (2012), 761–781.

[50] ———, *The classification of four ended solutions to the Allen-Cahn equation on the plane*, Analysis and PDE **6** (2013), no.7, 1675–1718.

[51] M. K. Kwong, *Uniquness of positive solutions of* $\Delta u - u + u^p = 0$ *in* $\mathbb{R}^N$, Arch. Rational Mech. Anal. **105** (1991), 243–266.

[52] F.H. Lin, W.-M. Ni, and J. Wei, *On the number of interior peak solutions for a singularly perturbed Neumann problem*, Comm. Pure Appl. Math. **60** (2007), no. 2, 252–281.

[53] S. Lorca and P. Ubilla, *Symmetric and nonsymmetric solutions for an elliptic equation on* $\mathbb{R}^N$, Nonlinear Anal. **58** (2004), 961–968.

[54] A. Malchiodi, *Some new entire solutions of semilinear elliptic equations on* R^n, Adv. Math. **221** (2009), no. 6, 1843–1909.

[55] A. Malchiodi and M. Montenegro, *Boundary concentration phenomena for a singularly perturbed elliptic problem*, Comm. Pure Appl. Math. **55** (2002), no. 12, 1507–1568.

[56] ———, *Multidimensional boundary layers for a singularly perturbed Neumann problem*, Duke Math. J. **124** (2004), no. 1, 105–143.

[57] R. Mazzeo and F. Pacard, *Constant scalar curvature metrics with isolated singularities*, Duke Math. J. **99** (1999), no. 3, 353–418.

[58] ———, *Constant mean curvature surfaces with Delaunay ends*, Comm. Anal. Geom. **9** (2001), no. 1, 169–237.

[59] R. Mazzeo, F. Pacard, and D. Pollack, *Connected sums of constant mean curvature surfaces in Euclidean 3 space*, J. Reine Angew. Math. **536** (2001), 115–165.

[60] L. Modica, *The gradient theory of phase transitions and the minimal interface criterion*, Arch. Rational Mech. Anal. **98** (1987), no. 2, 123–142.

[61] L. Modica and S. Mortola, *Un esempio di* Γ*-convergenza*, Boll. Unione Mat. Ital. Sez. B **14** (1977), 285–299.

[62] F. Morabito, *Index and nullity of the Gauss map of the Costa-Hoffman-Meeks surfaces*, Indiana Univ. Math. J. **58** (2009), no. 2, 677–707.

[63] M. Musso, F. Pacard and J. Wei, *Finite-energy sign-changing solutions with dihedral symmetry for the stationary nonlinear Schrodinger equation*, J. Eur. Math. Soc. **14** (2012), no.6, 1923–1953.

[64] F. Pacard and M. Ritoré, *From the constant mean curvature hypersurfaces to the gradient theory of phase transitions*, J. Differential Geom. **64** (2003), no. 3, 356–423.

[65] F. Pacard and J. Wei, *Stable Solutions of Allen-Cahn Equation over Simon's cone*, J. Funct. Anal. **264** (2013), no. 5, 1131–1167.

[66] X. Ren and J. Wei, *Oval shaped droplet solutions in the saturation process of some pattern formation problems*, SIAM J. Appl. Math. **70** (2009), no. 4, 1120–1138.

[67] M. Röger and Y. Tonegawa, *Convergence of phase-field approximations to the Gibbs-Thomson law*, Cal. Var. PDE **32** (2008), 111–136.

[68] S. Santra and J. Wei, *New entire positive solution for the nonlinear Schrödinger equation: Coexistence of fronts and bumps*, American J. Math. **135** (2013), no. 2, 443–491.

[69] O. Savin, *Regularity of flat level sets in phase transitions*, Ann. of Math. **169**(2) (2009), no. 1, 41–78.

[70] R. Schoen, *Uniqueness, symmetry, and embeddedness of minimal surfaces*, J. Differential Geom. **18** (1983), 791–809.

[71] J. Shi, *Saddle solutions of the balanced bistabel diffusion equation*, Comm. Pure Appl. Math. **55** (2002), no. 7, 815–830.

[72] J. Simons, *Minimal varieties in riemannian manifolds*, Ann. of Math. **88**(2) (1968), 62–105.

[73] P. Sternberg, *The effect of a singular perturbation on nonconvex variational problems*, Arch. Rational Mech. Anal. **101** (1988), no. 3, 209–260.

[74] M. Struwe, *Multiple solutions of differential equations without the Palais-Smale condition*, Math. Ann. **261** (1982), 399–412.

[75] Ting, F. and Wei, J., *Multi-vortex Non-radial Solutions to the Magnetic Ginzburg-Landau Equations*, Comm. Math. Physics **317** (2013), Issue 1, 69–97.

[76] Tam, L-F. and Zhou, D., *Stability properties for the higher dimensional catenoid in* R^{n+1}, Proc. Amer. Math. Soc. **137** (2009), no. 10, 3451–3461.

[77] ST Yau and R. Schoen, *Differential Geometry*, Science Press, Beijing, 1991.

Department of Mathematics, University of British Columbia, Vancouver BC V6T1Z2 Canada;
Department of Mathematics, Chinese University of Hong Kong, Shatin, NT, Hong Kong
E-mail: jcwei@math.ubc.ca

Duality in Boltzmann equation and its applications

Shih-Hsien Yu

Abstract. In this paper we will survey a quantitative and qualitative development on the Boltzmann equation. This development reveals the dual natures of the Boltzmann equation: The particle-like nature and the fluid-like nature. This dual nature property gives rise to the precise construction of the Green's function for Boltzmann equation around a global Maxwellian state. With the precise structure of the Green's function, one can implement the Green's function to study various problems such as invariant manifolds for the steady Boltzmann flows, time asymptotic nonlinear stability of Boltzmann shock layers and Boltzmann boundary layers, Riemann Problem, and bifurcation problem of boundary layer problem, etc.

Mathematics Subject Classification (2010). Primary 35Q20; Secondary 35A08, 35B40, 35C07, 35L65, 35L67.

Keywords. Boltzmann equation, Green's function, duality.

1. Introduction

The Boltzmann equation for the hard sphere collision model is a basic equation for the thermal non-equilibrium gases or rarefied gas flows:

$$\partial_t \mathsf{f} + \xi \cdot \nabla_{\vec{\boldsymbol{x}}} \mathsf{f} = Q(\mathsf{f})/\kappa,\ \mathsf{f}(\vec{\boldsymbol{x}}, t, \xi) \in \mathbb{R},\ \vec{\boldsymbol{x}}, \xi \in \mathbb{R}^3,\ \kappa > 0. \tag{1.1}$$

Here, $\mathsf{f}(\vec{\boldsymbol{x}}, \xi, t)$ stands for the gas particle velocity density function with velocity $\xi \in \mathbb{R}^3$ at $(\vec{\boldsymbol{x}}, t) \in \mathbb{R}^3 \times \mathbb{R}$; and Q is a bilinear integral operator on the velocity density function $\mathsf{f}(x, t, \xi)$, which represents the mechanism for particle collision. One can regard the collision operator as an equilibrating mechanism. The constant $\kappa > 0$ is the Knudsen's number, which represents the mean free path of the gas flow.

The main purpose to investigate the Boltzmann equation qualitatively and quantitatively is to clarify the following two subjects:

i) Theoretical background of gas-dynamic equation (Euler, Navier-Stokes, or other equations). There is a Ghost effect showing the incompleteness of the classical gas dynamic system (Euler and Navier-stokes), [29].

ii) The behavior of gas where the Knudsen number is not small.

The Boltzmann equation is a particularly interesting equation in terms of its physics nature by varying the size of κ or the sizes of the space-time scales. When $\kappa \gg 1$ or in a small space-time scales, the solution behavior resembles to free particle motions. When

Proceedings of the International Congress of Mathematicians, Seoul, 2014

$\kappa \ll 1$ and space-time scales are large, the balance of the transport nature $\partial_t + \xi \cdot \nabla_x$ and the equilibrating mechanics by Q results in a conventional compressible fluid structure, which is close to the compressible Euler equation for ideal gases by the Hilbert expansion.

With the presence of a physical boundary, the gas flows behave very differently from the conventional fluid mechanics such as the thermal transpiration flows, edge flows, condensation-evaporation problems, etc. mentioned in the monograph by Sone, [29]. Grad, [8–10], also recognized an atypical nature when the presence of boundary. He proposed to have complete studies with the presences of singular layers regarding to boundary, initial data, and shock wave which are the key elements for a deep understanding of the Boltzmann equation.

Since the collision operator Q is a nonlinear integral operator, it attracts attentions of researchers to develop theories on Q such as the exponentially fast convergence to an equilibrium state for a space homogeneous problem, [2, 3]. However, those beautiful results on space homogeneous problems did not provide so much informations to study the space inhomogeneous problems. The first global result on nonlinear theorem with the presence of $\xi \cdot \nabla_{\vec{x}}$ by [33] was due to a better understanding of the spectral property of the linearized Boltzmann equation $(\partial_t + \xi \cdot \nabla_x - \mathsf{L})\mathsf{g} = 0$ in [6], where L is a linear collision operator around a global Maxwellian state. The analysis on the spectrum of $-\xi \cdot \nabla_{\vec{x}} + \mathsf{L}$ is the first analytic establishment on the balance of $\xi \cdot \nabla_{\vec{x}}$ and L.

The mathematical developments on the Boltzmann equation thrilled since late '70 by various groups by different approaches and interests. Mathematically and physically, the collective behavior among $\xi \cdot \nabla_{\vec{x}}$, Q, and a physical boundary is even more interesting and complex. However, one still expects further substantial progress in this regard to achieve the understanding so that this subject is possible. On the other hand from '60 Sone [22–24, 24, 26–28, 30–32] has obtained very interesting theories regarding to boundary phenomena related to the Boltzmann equation and kinetic equations.

In year 2002 a completely different approach in the mathematical analysis for the Boltzmann equation was introduced by Liu and Yu to serve as a primary tool to undertake the analysis for the singular layers arouse from the shock layer, boundary layer, and initial layer as well as to give some partial results on Sone's discoveries. This is an approach based on the dual physical natures "wavelike-particlelike" of the Boltzmann equation. This article is aimed to review this development and its applications towards the problems by Sone and Grad.

2. Some background and motivation for Boltzmann equation and conservation laws

In [6], one considers the spectrum problem

$$(-i\xi \cdot \eta + \mathsf{L})\psi(\eta) = \sigma(\eta)\psi(\eta) \tag{2.1}$$

for the linear Boltzmann equation

$$\mathsf{f}_t + \xi \cdot \nabla_{\vec{x}}\mathsf{f} - \mathsf{L}\mathsf{f} = 0 \tag{2.2}$$

around a global Maxwellian state $\mathsf{M} = \mathsf{M}_{[1,0,\theta]}$ in the Fourier variable $\eta \in \mathbb{R}^3$, where $\mathsf{M}_{[\rho,u,\theta]}(\xi) = \rho\frac{e^{-\frac{|\xi-u|^2}{4\theta}}}{(4\pi\theta)^{3/2}}$. It is asserted that there exist $\kappa_0 > 0$ and $\kappa_1 > 0$ such that for

$|\eta| < \kappa_0$ there are five branches $\sigma_j(|\eta|) \subset \{z \in \mathbb{C} | Re(z) < 0\}$ tangential to the imaginary axis with the asymptotic for $|\eta| \ll 1$

$$\begin{cases} \sigma_1(\eta), \sigma_2(\eta) = \pm ic|\eta| - A_1|\eta|^2 + O(|\eta|^3), \\ \sigma_j(\eta) = -A_j|\eta|^2 + O(1)|\eta|^3 \text{ for } j = 3, 4, 5, \end{cases} \tag{2.3}$$

with $A_j > 0$, where $c = \sqrt{5\theta/3}$ is the speed of sound wave at rest; and there is a spectral gap:

$$\sigma(\eta) \notin \{Re(z) > -\kappa_1\} \text{ for } |\eta| > \kappa_0. \tag{2.4}$$

One can view the spectrum $\sigma(\eta)$ as a balance of the space transport mechanism $\xi \cdot \nabla_{\vec{x}}$ in the Fourier variable η and the linear collision operator L. By this spectrum property in [33], one applied a resolvent approach and a bootstrap approach to yield nonlinear stability of a global Maxwellian state M.

In [11, 21], one expanded the eigenfunction $\psi(\eta)$ in terms of the collision invariants of L so that the relationship between the Boltzmann equation and the hydrodynamic equations is clearer. The expansion of the eigenfunctions gave hints to the introduction of macro-micro decomposition in [14]:

$$\mathsf{f} = \mathsf{P}_0\mathsf{f} + \mathsf{P}_1\mathsf{f} \equiv \mathsf{f}_0 + \mathsf{f}_1, \tag{2.5}$$

where P_0 is a linear combination of finite number of collision invariants related to a local Maxwellian; and one can identify f_0 as a vector in $\mathbb{R}^3$ for a planar wave problem. With this decomposition, one can rewrite the time asymptotic stability for a planar wave perturbation j, $\partial_t \mathsf{j} + \xi^1 \partial_x \mathsf{j} = \frac{\delta Q}{\delta \varphi}\mathsf{j} + Q(\mathsf{j})$, of a Boltzmann shock profile φ coupled with a 3×3 viscous system through the microscopic component j_1 of j:

$$\partial_t F + A(x)F_x = B(x)F_{xx} + O(1)J(\partial_t \mathsf{j}_1),\ F \in \mathbb{R}^3. \tag{2.6}$$

Here, the Boltzmann shock profile φ of (1.1) is a travelling wave solution $\mathsf{f}(x,t) = \varphi(x - st)$ connecting two Maxwellians $\mathsf{M}_{[\rho_\pm, u_\pm, \theta_\pm]}$ given by a hyperbolic shock wave $((\rho_-, u_-, \theta_-), (\rho_+, u_+, \theta_+))$ together with the speed s given the Rankine-Hugoniot condition.

Then, by assuming that the difference of the end states of the shock wave is sufficient small and the total macroscopic component of perturbation is zero, one shows that the Boltzmann shock profile is stable by implementing the energy method for conservation laws by [7]. The consequence of the stability is that the Boltzmann shock profile $\varphi(x, \xi)$, obtained by [1], is a positive-valued function in (x, ξ).

With the micro-macro decomposition, one can implement this energy method to work out the problem about the existence of Knudsen layers (boundary layers) with condition, $|Mach\ Number| \neq 0, 1$, [34]. The energy method was also applied to derive a macroscopic H-theorem, [19], to show the time asymptotic convergent to a hyperbolic rarefaction wave, [20], and to show nonlinear stability of the boundary layer with Mach number less -1, [35]. When $Mach\ Number > -1$, the energy method can not be applied due to the fact that the solution of initial boundary value problem contains singularity at boundary so that the energy method could not be applied. It led to search for a new approach which does not require regularity property of the solution. The right candidate for such a tool is the Green's function since the Boltzmann equation is a semilinear equation.

3. Particlelike-Wavelike duality

One starts to consider problems in planar wave solutions to establish the understanding on the natures of the Boltzmann equation, i.e. $x, \eta \in \mathbb{R}$, and $\xi \in \mathbb{R}^3$.

We start to review the work given in [15]. It begins from the consideration of the Green's function for (2.2). The Green's function can be represented as the inverse transform of the semigroup:

$$\mathbb{G}(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\eta x + (-\xi^1\eta + \mathsf{L})t} d\eta. \tag{3.1}$$

This is an L^2_ξ-operator-valued function in (x,t), where L^2_ξ is the standard Hilbert space, $L^2(\mathbb{R}^3)$. The spectral information $\sigma(\eta)$ of (2.1) given in (2.3) poses a difficulty to obtain the Green's function for any (x,t) since there is no spectral information $\sigma(\eta)$ for all $|\eta| \geq \kappa_0$. In order to cope with the insufficient spectral information due to (2.4), one introduces a long wave-short wave decomposition of the Green's function

$$\begin{aligned} &\mathbb{G}(x,t) = \mathbb{G}_L(x,t) + \mathbb{G}_S(x,t), \\ &\begin{cases} \mathbb{G}_L(x,t) \equiv \dfrac{1}{2\pi}\displaystyle\int_{|\eta|<\varepsilon_0} e^{i\eta x + (-i\eta x + \mathsf{L})t} d\eta, \text{ for a fixed } \varepsilon_0 \in (0,\kappa_0), \\ \mathbb{G}_S(x,t) = \mathbb{G}(x,t) - \mathbb{G}_L(x,t). \end{cases} \end{aligned} \tag{3.2}$$

Here, $\mathbb{G}_L(x,t)$ is a long wave component of the Green's function. The spectrum information (2.3) is the core to build the long wave component for both the Boltzmann equation and linearized compressible Navier-Stokes equations. By complex analysis one can conclude the long wave component $\mathbb{G}_L(x,t)$ satisfies for $t \geq 1$ and $|x| < 2ct$ there exists $C_0 > 0$ such that

$$\|\mathbb{G}_L(x,t)\|_{L^2_\xi} \leq O(1)\left(\frac{e^{-\frac{(x+ct)^2}{C_0 t}} + e^{-\frac{x^2}{C_0 t}} + e^{-\frac{(x-ct)^2}{C_0 t}}}{\sqrt{t+1}}\right); \tag{3.3}$$

$$\|\partial_x^k \mathbb{G}_L\|_{L^2_x(L^2_\xi)} \leq O(1) \text{ for } k = 0, 1, 2, \cdots, \tag{3.4}$$

and one also has that

$$\|\mathbb{G}_S(x,t)\|_{L^2_x(L^2_\xi)} \leq O(1)e^{-t/C_0}, \tag{3.5}$$

where c is the sound speed at rest.

Though $\|\mathbb{G}_S\|_{L^2_x(L^2_\xi)}$ decays exponentially fast, it still does not assert that the $\|\mathbb{G}_S\|_{L^\infty_x(L^2_\xi)}$ decays sufficient fast for the purpose to study the full nonlinear problem with presence of boundaries or shock layers. To resolve the problem for obtaining the estimate for $\|\mathbb{G}_S\|_{L^\infty_x(L^2_\xi)}$, one needs to reconsider the problem (2.2) in the space-time domain instead of the transform domain, and one needs to spell out the linear collision operator L in details in order to catch the physics nature of the Boltzmann equation:

$$\begin{aligned} &\mathsf{L}\mathsf{g}(\xi) = -\nu(\xi)\mathsf{g}(\xi) + \mathsf{K}\mathsf{g}(\xi), \\ &\begin{cases} \nu(\xi) \geq \nu_0(1 + |\xi|), \\ \mathsf{K}\mathsf{g}(\xi) \equiv \displaystyle\int_{\mathbb{R}} K(\xi,\xi_*)\mathsf{g}(\xi_*)d\xi_*, \\ K(\xi,\xi_*) \in C^\infty \text{ for } |\xi - \xi_*| > 0. \end{cases} \end{aligned} \tag{3.6}$$

After spelling L one rearranges (2.2) in the form of particle propagation (ODE along particle path):

$$\begin{cases}(\partial_t+\xi^1\partial_x+\nu)\mathsf{f}=\mathsf{K}\mathsf{f},\\ \mathsf{f}(x,t,\xi)=\delta(x)\delta^3(\xi-\xi_*).\end{cases}\tag{3.7}$$

Then, one can perform the standard Picard's iteration in ODE for finite number of iterations with some cut-off in $K(\xi,\xi_*)$ in the first iteration to yield the following particlelike decomposition:

$$\begin{cases}\mathsf{f}=\mathbb{P}+\mathsf{R},\\ \mathbb{P}\equiv\sum_{k=0}^{2l}\mathsf{f}_k.\end{cases}\tag{3.8}$$

Here, $\mathsf{R}(x,t)$ is the remainder term of the Picard iteration. The functions f_k and $\mathsf{R}(x,t)$ satisfy the property:

$$\begin{aligned}\mathsf{f}_0(x,t)&=e^{-\nu(\xi)t}\delta(x-\xi^1t)\delta^3(\xi-\xi_*),\\ \|\mathsf{f}_k(x,t)\|_{L^2_\xi}&\le O(1)e^{-(|x|+t)/C_0}\text{ for }k=3,\cdots,2l+1,\\ \partial^k_\xi\mathsf{f}_2(x,t,\xi)&<\infty\text{ for }k=0,\cdots,2l,\\ &\begin{cases}(\partial_t+\xi^1\partial_x-\mathsf{L})\mathsf{R}=\mathsf{K}\mathsf{f}_{2l+1},\\ \mathsf{R}|_{t=0}\equiv 0.\end{cases}\end{aligned}\tag{3.9}$$

From the properties (3.9) and (2.3), one can only have property about the remainder $\mathsf{R}(x,t)$ there exists $C_0>0$

$$\|\mathsf{R}(\cdot,t)\|_{L^2_x(L^2_\xi)}\le C_0\text{ for }t>0.\tag{3.10}$$

Here, neither the two decompositions (3.2) nor (3.8) give the global structure of $\|\mathbb{G}(x,t)\|_{L^2_\xi}$ for all (x,t).
Denote

$$\mathsf{M}_l\equiv\underbrace{e^{(-\xi^1\partial_x-\nu(\xi))t}\mathsf{K}\underset{(x,t)}{*}e^{(-\xi^1\partial_x-\nu(\xi))t}\mathsf{K}\underset{(x,t)}{*}\cdots\underset{(x,t)}{*}e^{(-\xi^1\partial_x-\nu(\xi))t}\mathsf{K}}_{2l\ times}\underset{(x,t)}{*}e^{(-\xi^1\partial_x-\nu(\xi))t}.\tag{3.11}$$

Lemma 3.1 (Mixture Lemma [15]). *For each given $l\ge 0$ there exists $O_l>0$ such that*

$$\|\partial^l_x\mathsf{M}_l\mathsf{g}\|_{L^2_x(L^2_\xi)}\le O_l\left(\|\mathsf{g}\|_{L^2_x(L^2_\xi)}+\|\partial^l_\xi\mathsf{g}\|_{L^2_x(L^2_\xi)}\right)\text{ for }t\ge 0.\tag{3.12}$$

Here, $e^{(-\xi^1\partial_x-\nu(\xi))t}$ is a transport mechanism in the space-time domain and K is a mechanism to mix the velocity density distribution ξ at (x,t). This lemma asserts the conversion from the microscopic regularity ∂_ξ to the macroscopic regularity ∂_x with every two mixture of $e^{(-\xi^1\partial_x-\nu(\xi))t}\mathsf{K}\underset{(x,t)}{*}e^{(-\xi^1\partial_x-\nu(\xi))t}\mathsf{K}$. This lemma is about the conversion on the regularity through space convection and microscopic velocity.

3.1. Dual structures. Here, (3.3), (3.4), (3.5), (3.9), (3.10), and (3.12) are facts of simple mathematical analysis except (3.3) required some detailed complex analysis. By each own mathematical approach along, there is no much room to obtain the structure $\|\mathbb{G}(x,t)\|_{L^2_\xi}$. It is strikingly interesting that all those simple estimates binding together will generate the dual natures of the Boltzmann equation. By equating the two decompositions (3.2) and (3.8) together,

$$\begin{cases} \mathbb{P} - \mathbb{G}_S = \mathbb{G}_L - \mathsf{R}, \\ \|\partial_x^l(\mathbb{G}_L - \mathsf{R})\|_{L^2_x(L^2_\xi)} = O_l \text{ for } l \geq 2, \\ \|\mathbb{P} - \mathbb{G}_S\|_{L^2_x(L^2_\xi)} \leq O(1)e^{-t/C_0}. \end{cases} \tag{3.13}$$

The above and Poincare's inequality yield that

$$\|\mathsf{R} - \mathbb{G}_L\|_{L^\infty_x(L^2_\xi)} = \|\mathbb{P} - \mathbb{G}_S\|_{L^\infty_x(L^2_\xi)} \leq O(1)e^{-t/C_1} \text{ for some } C_1 > 0. \tag{3.14}$$

It concludes that the remainder term R and the long wave component $\mathbb{G}_L$ are exponentially close; and the compressible viscous fluid wave structure presented in R and the shortwave component $\mathbb{G}_S(x,t)$ are as follows.

$$\begin{aligned} \|\mathsf{R}(x,t)\|_{L^2_\xi} &\leq O(1)\left(\frac{e^{-\frac{(x+ct)^2}{C_0(t+1)}} + e^{-\frac{x^2}{C_0(t+1)}} + e^{-\frac{(x-ct)^2}{C_0(t+1)}}}{\sqrt{t+1}}\right); \\ \|\mathbb{P} - \mathbb{G}_S(x,t)\|_{L^2_\xi} &\leq O(1)e^{-t/C_1}. \end{aligned} \tag{3.15}$$

In particular, one can have a time lapse property for the remainder term R:

$$\|\mathsf{R}(x,t)\|_{L^2_\xi} \leq O(1)\int_0^t e^{-\tau/C_1}d\tau\left(\frac{e^{-\frac{(x+ct)^2}{C_0(t+1)}} + e^{-\frac{x^2}{C_0(t+1)}} + e^{-\frac{(x-ct)^2}{C_0(t+1)}}}{\sqrt{t+1}}\right). \tag{3.16}$$

This, (3.8), and (3.9) together conclude the particlelike-wavelike structure, $\mathbb{P}(x,t)$-$\mathsf{R}(x,t)$, of the linear Boltzmann equation (2.2).

Remark 3.2. The two decompositions $\mathbb{P}+\mathsf{R}$ and $\mathbb{G}_S+\mathbb{G}_L$ are complimentary to each other in their focusing on particle nature and wave nature. The first one loses its focus when the time frame becomes large. The second one becomes rough when wave length becomes small, since the spectral analysis does not register the physics nature in the space-time variable. The mixture lemma plays a simple role to merge these two complimentary decompositions together by (3.13) to result in the precise structure $\mathsf{R}(x,t)$ in (3.16) and $\mathbb{G}_S(x,t)$ in (3.14).

3.2. Diagonal and off-diagonal hydrodynamic structure. With respect to the macro-micro decomposition $(\mathsf{P}_0, \mathsf{P}_1)$, the representation $\mathsf{P}_0\xi^1\mathsf{P}_0$ of the macroscopic transport ξ^1 is identical to the convection matrix of a linearized Euler equation. The convection matrix can be diagonalised in terms of the Riemann invariants E_j, $j = 1, 2, 3$,

$$\begin{aligned} &\mathsf{P}_0\xi^1\mathsf{P}_0\mathsf{E}_j = \lambda_j\mathsf{E}_j, \\ &(\mathsf{E}_j, \mathsf{E}_k) = \delta^j_k, \\ &\{\lambda_1, \lambda_2, \lambda_3\} = \{-c, 0, c\}, \end{aligned}$$

so that each Riemann invariant E_j propagates along a particular direction $dx/dt = \lambda_j$, where c is the speed of sound wave. Those Riemann invariants E_j and the Green's function $\mathbb{G}(x,t)$ satisfy for $t \geq 1$

$$(\mathsf{E}_l, \mathbb{G}_L(x,t)\mathsf{E}_k) \leq O(1)\frac{e^{-\frac{(x-\lambda_j t)^2}{C_1 t}}}{t^{(3-\delta_j^k-\delta_j^l)/2}} \text{ for } |x-\lambda_j t| < \frac{c}{2}t. \tag{3.17}$$

Remark 3.3. With the time decaying rates, this property (3.17) gives a time-asymptotically diagonalization property. It is the key analytic ingredient to assure the Green's function $\mathbb{G}(x,t)$ to become a powerful instrument to apply to various different problems.

4. Application of the Green's function

After establishing the structure of the Green's functions for planar wave solutions, one had applied those structures to various nonlinear problems. We will outline the applications of the Green's function in this section.

4.1. Pointwise convergence to global Maxwellian state. In [15], one considers a small perturbation of the Boltzmann equation around a global Maxwellian in a 1-D space domain

$$\begin{cases} \mathsf{f}_t + \xi^1 \partial_x \mathsf{f} = \mathsf{L}\mathsf{f} + \mathsf{M}^{-1/2} Q(\mathsf{M}^{1/2}\mathsf{f}), \\ \|\mathsf{f}(x,0)\|_{L^\infty_{\xi,\beta}} \leq O(1)\varepsilon\, e^{-|x|},\ \beta \geq 5/2 \end{cases} \tag{4.1}$$

where $\|\mathsf{g}\|_{L^\infty_{\xi,\beta}}$ is defined by $\|(1+|\xi|)^\beta \mathsf{g}\|_{L^\infty_\xi}$. The Green's function and the lemmas in [13] for nonlinear waves coupling give the structures of the perturbations as follows.

$$\|\mathsf{f}(x,t)\|_{L^\infty_\beta} \leq O(1)\varepsilon \left(\sum_{j=1}^{3} \frac{e^{-\frac{(x-\lambda_j t)^2}{C_0(1+t)}}}{\sqrt{1+t}} + \psi_j(x,t) + e^{-(|x|+t)/C_0} \right), \tag{4.2}$$

where $\psi_j(x,t) = 1/\sqrt{(x-\lambda_j t)^2 + t}$, which is the dissipation wave given in [13].

4.2. Time asymptotic stability of an initial boundary value problem. In [17], one considers a global Maxwellian $\mathsf{M}_{[1,u,\theta]}$ with Mach number $\equiv \ u/\sqrt{5\theta/3} \notin \{-1,0,1\}$ in a half space domain with an imposed homogeneous boundary condition. One begins with the linear Milne's problem:

$$\begin{cases} \mathsf{g}_t + \xi^1 \partial_x \mathsf{g} = \mathsf{L}\mathsf{g}, \\ \mathsf{g}(0,t)|_{\xi^1>0} = 0, \\ \|\mathsf{g}(x,0)\|_{L^\infty_{\xi,3}} \leq e^{-|x|}. \end{cases} \tag{4.3}$$

The Green's function $\mathbb{G}(x,t)$ for (2.2) plays a role to reduce the linear initial boundary problem into a pure boundary value problem by subtracting $\mathsf{h}(x,t) \equiv \int_0^\infty \mathbb{G}(x-y,t)\mathsf{g}(y,0)dy$ from $\mathsf{g}(x,t)$ to result in the boundary value problem:

$$\begin{cases} \partial_t \mathsf{j} + \xi^1 \partial_x \mathsf{j} - \mathsf{L}\mathsf{j} = 0, \\ \mathsf{j}(0,t)|_{\xi^1>0} = -\mathsf{h}(0,t)|_{\xi^1>0}, \\ \mathsf{j}(x,0) \equiv 0, \end{cases} \tag{4.4}$$

where the function h satisfies $\|\mathsf{h}(0,t)\|_{L^\infty_{\xi,3}} \le O(1)\sum_{j=1}^3 \frac{e^{-\lambda_j^2 t}}{\sqrt{t+1}}$ due to the pointwise structure of $\mathbb{G}(x,t)$ and where $\{\lambda_1,\lambda_2,\lambda_3\} \equiv \{u-\sqrt{5\theta/3}, u, u+\sqrt{5\theta/3}\}$. For the problem (4.4) together with a boundary condition $\mathsf{h}(0,t)|_{\xi^1>0}$ with a pointwise structure, a upwind damping mechanism $\gamma\mathsf{B}_+$ was applied to introduce an auxiliary problem

$$\begin{cases} \partial_t \mathsf{j}_a + \xi^1 \partial_x \mathsf{j}_a - \mathsf{L}\mathsf{j}_a = -\gamma \mathsf{B}_+ \mathsf{j}_a, \\ \mathsf{j}_a(0,t)|_{\xi^1>0} = -\mathsf{h}(0,t)|_{\xi^1>0}, \\ \mathsf{j}_a(x,0) \equiv 0. \end{cases} \tag{4.5}$$

This problem can be solved globally by the energy method with an exponentially growing weighted function in x and t, where $0 < \gamma \ll 1$ and the damping mechanism B_+ was introduced in [34] for the construction of a boundary layer. Then, one uses $\mathsf{j}_a(0,t)$ as an approximation to the full boundary data $\mathsf{j}(0,t)$.

The diagonal-off diagonal structure (3.17) and Duhamel's principle are used to justify that the approximated full boundary function $\mathsf{j}_a(0,t)$ is a good approximation to $\mathsf{j}(0,t)$ so that one can form a geometric series $\sum_{k=1}^\infty \mathsf{j}_{a,k}(0,t)$ to represent the full boundary data $\mathsf{j}(0,t)$ and each term satisfies

$$\|\mathsf{j}_{a,k}(0,t)\|_{L^\infty_{\xi,3}} \le O(1)\gamma^{-1/4+k} \sum_{k=0}^\infty \sum_{j=1}^3 \frac{e^{-\lambda_j^2 t/C_0}}{\sqrt{t+1}}. \tag{4.6}$$

This yields the full boundary data $\mathsf{j}(0,t)$. With this data, $\mathbb{G}(x,t)$, and the first Green's identity together, one obtained the pointwise structure of the solution $\mathsf{j}(x,t)$ for all $(x,t) \in \mathbb{R}_+ \times \mathbb{R}_+$. With the precise structure of the linear problem (4.4), the nonlinear time-asymptotic stability follows.

Following the analysis for the nonlinear time asymptotic stability problem for a Maxwellian in half space domain, in [4] one continued to study the time asymptotic pointwise structure for a nonlinear problem around a Knudsen layer. The time asymptotically nonlinear stability problem for a Knudsen layer for the cases $Mach\ number \notin \{-1,0,1\}$ were concluded, and the motivation to introduce the Green's function to study the Knudsen layer was justified in this work.

4.3. Bifurcation of boundary layers. In [18], one started to analyze the Knudsen layer when the Mach number close to 0 and ± 1. The Knudsen layers constructed in [34] are under a condition that the Mach number at the far field does include ± 1 and 0. Indeed, when the Mach numbers are around 0 or ± 1, the physical behaviours of the solutions are rather singular as pointed out by Sone's works listed the reference. The Knudsen layer problem with Mach number near $\{\pm 1, 0\}$ is a bifurcation problem,

$$\begin{cases} -\xi^1 \partial_x \mathsf{F} - Q(\mathsf{F}) = 0 \text{ for } x \in \mathbb{R}_+, \\ \lim\limits_{x\to\infty} \mathsf{F}(x) = \mathsf{M}_{[\rho,u,\theta]}, \\ \mathsf{F}(0,t)|_{\xi^1>0} : \text{ posed}, \end{cases} \tag{4.7}$$

with respect to parameters given by the macroscopic variables of the Maxwellian $\mathsf{M}_{[\rho,u,\theta]}$ at the far field. This is a singular problem due to two facts that the system (4.7) is an infinite dimensional dynamical system and it also possesses a transonic behavior with Mach

number close to ± 1 and a condensation-evaporation nature with Mach number is close to 0. This problem was not ready during the work in [34]. At that time the analytic tools (energy estimates) available were too primitive and too rough to realize the rich natures of the problem. The pointwise structure of the Green's function in (3.17) and the particlelike structure $\mathbb{P}$ given in (3.8) play an essential role to perform a finite dimensional reduction for the dynamical system (4.7). To devise a finite dimensional reduction, one will need to construct invariant manifolds for the system (4.7). One establishes the invariant manifolds from building concrete projection operators $\mathbb{S}_x$, $\mathbb{U}_x$, and $\mathbb{C}_0$ on $L^\infty_{\xi,3}$ for a linear system,

$$\xi^1 \partial_x \mathsf{f} - \mathsf{L}\mathsf{f} = 0, \tag{4.8}$$

i.e. for any $\mathsf{b} \in L^\infty_{\xi,3}$ the functions $\mathbb{S}_x\mathsf{b}$ and $\mathbb{U}_x\mathsf{b}$ give the solutions of (4.8) so that

$$\lim_{x\to\infty} \mathbb{S}_x\mathsf{b} = 0, \tag{4.9}$$

$$\lim_{x\to-\infty} \mathbb{U}_x\mathsf{b} = 0, \tag{4.10}$$

$$\mathbb{C}_0\mathsf{b} \in Range(\mathsf{P}_0), \tag{4.11}$$

$$\mathsf{b} = \lim_{x\to 0+} \mathbb{S}_x\mathsf{b} + \lim_{x\to 0-} \mathbb{U}_x\mathsf{b} + \mathbb{C}_0\mathsf{b}. \tag{4.12}$$

With the pointwise structure (3.17), one can show that the functions $\mathbb{S}_x\mathsf{b}$ and $\mathbb{U}_x$ are

$$\begin{cases} \mathbb{S}_x\mathsf{b} \equiv \displaystyle\int_0^\infty \mathbb{G}(x,s)\xi^1(1-\tilde{\mathsf{B}}_+)\mathsf{b}ds \text{ for } x>0, \\ \mathbb{S}_{0+}\mathsf{b} \equiv \lim_{x\to 0+} \mathbb{S}_x\mathsf{b}, \\ \mathbb{U}_x\mathsf{b} \equiv -\displaystyle\int_0^\infty \mathbb{G}(x,s)\xi^1(1-\tilde{\mathsf{B}}_-)\mathsf{b}ds \text{ for } x<0, \\ \mathbb{U}_{0-}\mathsf{b} \equiv \lim_{x\to 0-} \mathbb{U}_x\mathsf{b}, \\ \mathbb{C}_0\mathsf{b} = \tilde{\mathsf{P}}_0\mathsf{b}, \end{cases} \tag{4.13}$$

$$\begin{cases} \tilde{\mathsf{P}}_0 \equiv \displaystyle\sum_{\mathsf{k}=1}^{3} \tilde{\mathsf{B}}_\mathsf{k}, \\ \tilde{\mathsf{B}}_k\mathsf{g} \equiv \dfrac{(\mathsf{E}_k, \xi^1\mathsf{g})\mathsf{E}_k}{\lambda_k}, \\ \tilde{\mathsf{B}}_\pm \equiv \sum_{\pm\lambda_k>0} \tilde{\mathsf{B}}_k, \end{cases}$$

where $\tilde{\mathsf{P}}_0$, $\tilde{\mathsf{B}}_+$, and $\tilde{\mathsf{B}}_-$ are the Euler flux projection, the upwind Euler flux projection, and downwind Euler flux projection.

The properties (4.9) and (4.10) are due to (3.17). The identity (4.12) is due to the δ-functions in $\mathbb{P}$ (the particlelike wave) to yield a version of Gauss lemma given in Lemma 3 in [18]. Then, one has obtained the projection operators $\mathbb{S}_{0+}$, $\mathbb{U}_{0-}$, and $\mathbb{C}_0$ to the linear stable manifold, linear unstable manifold, and the linear center manifold; and one also has an exponentially decaying structures in $\mathbb{S}_x$ and $\mathbb{U}_x$ of the linear stable flows and linear unstable flows. Thus, with the exponentially decaying structures one can apply the standard construction to obtain the local stable, local center-stable manifold for (4.7).

When the Mach number is close to 0, and ± 1, one needs to compare the structures of the linear stable and linear unstable manifold. When the Mach number is -1, there is a

1-dimensional degeneracy to the center manifold either from the linear stable manifold or linear unstable manifold. One can calculate this degenerated vector and use it to modify the upwind damping $\tilde{\mathsf{B}}_+$ and the projection operator $\mathbb{S}_x$ into

$$\begin{cases} \mathsf{B}_3^{\sharp,\epsilon}\mathsf{g} & \equiv \dfrac{(\xi^1\mathsf{E}_3^\epsilon,\mathsf{g})}{(\xi^1\mathsf{E}_3^\epsilon,\ell_3^\epsilon)}\ell_3^\epsilon, \\ \mathbb{S}_x^{\sharp,\epsilon}\mathsf{g} & \equiv \displaystyle\int_0^\infty \mathbb{G}^\epsilon(x,\tau)[\xi^1(1-\mathsf{B}_3^{\sharp,\epsilon})\mathsf{g}]d\tau \end{cases} \tag{4.14}$$

so that one can verify the continuity of the microscopic component,

$$\mathsf{P}_1\int_0^\infty \mathbb{G}^\epsilon(x,\tau)[\xi^1(1-\mathsf{B}_3^{\sharp,\epsilon})\mathsf{g}]d\tau, \tag{4.15}$$

where ϵ is the difference of the Mach number and -1. Then, by energy estimates one can have the uniformly exponentially decaying structure in x when $\epsilon > 0$ and together with an algebraic condition (148) in [18] on the macroscopic and microscopic component to yield the uniformly exponentially decaying upper bound $e^{-\alpha x}$ for $x > 0$ of $\|\mathbb{S}_x^{\sharp,\epsilon}\|_{L^2_\xi}$ and with the uniform structure in $\epsilon > 0$. By taking the limits $\epsilon \to 0+$, it follows

$$\begin{cases} \mathsf{b} = \mathring{\mathbb{S}}_{0+}\mathsf{b} \oplus \mathring{\mathbb{C}}_0\mathsf{b} \oplus \mathring{\mathbb{U}}_{0-}\mathsf{b}, \\ dim(Range(\mathring{\mathbb{C}}_0)) = 4, \end{cases} \tag{4.16}$$

where $\mathring{\mathbb{S}}_{0+}$, $\mathring{\mathbb{U}}_{0-}$, and $\mathring{\mathbb{C}}_0$ are linear stable manifold and linear unstable manifold, and the linear center manifold. With the uniformly exponential decaying upper bound of $\|\mathring{\mathbb{S}}_x\|_{L^2_\xi}$ for $x > 0$, one can construct the local centre-unstable manifold. By taking the limit of $\epsilon \to 0-$, then one can construct the local unstable and center-stable manifolds; and the dimension of the nonlinear center manifold is 4. Since all Maxwellian states M are equilibrium states of the dynamical system, they are all in the center manifold. Due to the fact that the collision operator is orthogonal to the collision invariant, the macroscopic flux $\vec{\boldsymbol{q}} = \mathsf{P}_0\xi^1\mathsf{M}$ is an invariant 3-vector of the dynamical system. This gives a three constraints to the 4-dimensional center manifold and yields a 1-dimensional invariant manifold in the center manifold with two fixed points corresponding to the Maxwellians $(\mathsf{M}_-^{\vec{\boldsymbol{q}}}, \mathsf{M}_+^{\vec{\boldsymbol{q}}})$, which are related to the end states of a shock wave. Then, by using the coordinate of the linear center manifold and linear stable manifold one can obtain a two scale dynamical system in the center-stable manifold with two co-dimension 2 invariant manifolds at the equilibrium states $\mathsf{M}_-^{\vec{\boldsymbol{q}}}$ and $\mathsf{M}_+^{\vec{\boldsymbol{q}}}$. The flows on the two co-dimension 2 will converge to the equilibrium state with an uniform exponential rate. Otherwise, it behaviours like a Burgers' equation (compressible fluid like). We illustrate the phase diagram of the center-stable manifold of the dynamical system given by (4.7) around a Maxwellian M_0 state with Mach number$= -1$.

The dynamical system on the 1-D invariant curve (center manifold) is a Burgers type ODE (First order ODE). This flow concludes a connecting orbit for the two states $(\mathsf{M}_-^{\vec{\boldsymbol{q}}}, \mathsf{M}_+^{\vec{\boldsymbol{q}}})$. This proves the existence of Boltzmann profile as well as the monotone property of the profile. This monotone property is a problem raised in [14]. Here, the two co-dimension 2 invariant submanifolds of the center-stable manifold define two scalar functions K_- and K_+ on the center-stable manifold so that the function K_- gives the bifurcation of the dynamical system; and the function K_+ defines the hydrodynamics flows patterns, either a slowly expanded pattern for flows in the region $K_+ < 0$ or an exponentially fast compressive wave pattern in the region $\{K_+ > 0\} \cup \{K_- < 0\}$. With these two functions, one

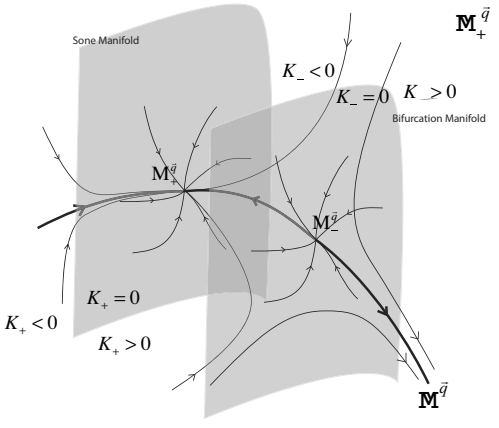


Figure 4.1. Two-scale dynamics on the center-stable manifold $\mathbb{M}_+^{\vec{q}}$ which is the center-stable manifold with macroscopic flux $\vec{\boldsymbol{q}} \equiv \mathsf{P}_0\xi^1\mathsf{M}_-$.

can return to the bifurcation of the Milne's problem (4.7). By Lemma 20 in [18], there is a local 1-1 continuous map $\iota_{\vec{\boldsymbol{q}}}$ from the center-stable manifold with given macroscopic flux $\vec{\boldsymbol{q}}$ to the space $L^\infty_{\xi,3,+}$, which is the space for the imposed boundary data. Thus, the sign of the function $K_-(\iota_{\vec{\boldsymbol{q}}}(\mathsf{b}))$ gives the bifurcation of the Milne's problem around the Mach number =-1. When Mach number is around 0, the result in [18] gives the Sone's bifurcation from condensation to evaporation.

4.4. Linear and nonlinear wave scattering around a Boltzmann shock layer. In [36], one considers the Boltzmann equation around a Boltzmann shock profile, $\varphi(x-st)$:

$$\begin{cases} (\partial_t - s\partial_x\mathsf{F}) + \xi^1\partial_x\mathsf{F} - \mathsf{L}_\varphi\mathsf{F} = Q(\mathsf{F}), \\ \mathsf{F}(x,0) = \mathsf{F}_0(x), \text{ (posed initial data,)} \end{cases} \tag{4.17}$$

where L_φ is a linear collision operator around the shock profile φ. Suppose that the Boltzmann shock profile φ is for a weak 3-shock wave $(\vec{\boldsymbol{u}}_-, \vec{\boldsymbol{u}}_+)$ for a compressible Euler equation as a system of hyperbolic conservation laws:

$$\vec{\boldsymbol{u}}_t + \vec{\mathsf{F}}(\vec{\boldsymbol{u}})_x = 0, \ \vec{\boldsymbol{u}} \in \mathbb{R}^3.$$

One wants to remove the zero total macroscopic mass condition in [14],

$$\int_{\mathbb{R}} \mathsf{P}_0\mathsf{F}_0(x,0)dx = 0 \tag{4.18}$$

for the purpose to investigate the hydrodynamic limits problem for the Boltzmann equation, [9, 10].

The main point is on obtaining the optimal linear wave propagation around the Boltzmann shock layer and to use it to establish the nonlinear wave coupling. The central idea is due to viscous conservation laws. The approach to obtain the linear wave scattering around a shock profile is called the T-C scheme (transverse-compressible scheme). This scheme is

closely related to the Lax's entropy condition for a p-th shock wave and the diffuse waves introduced in [12] to determine the viscous shock profile phase shift. In [36] one uses the Green's functions at two far fields to construct an approximated solution $\mathsf{A}_0(x,t)$ and a local wave front $l_0(t)\varphi'(x)$ to approximate the solution of the linearized problem

$$(\partial_t + (\xi^1 - s)\partial_x - \mathsf{L}_\varphi)\mathsf{f} = 0 \tag{4.19}$$

to yield that $\mathscr{E}_0$, the truncation error for (4.19),

$$\mathscr{E}_0 \equiv (\partial_t + (\xi^1 - s)\partial_x - \mathsf{L}_\varphi)(\mathsf{A}_0(x,t) + l_0(t)\varphi'(x)) \tag{4.20}$$

satisfies that following property:

$$\begin{cases} \displaystyle\int_{\mathbb{R}} (\mathsf{D}_i, \mathscr{E}_0(x,t))dx = 0,\ i = 1,2, \\ \|\mathsf{P}_0\mathscr{E}_0(x,t)\|_{L^\infty_{\xi,3}} \le O(1)\frac{\epsilon^2}{t}e^{-(\epsilon|x|+\epsilon^2 t)/C_0} \text{ for } t \ge \epsilon^{-2}, \\ \|\mathsf{P}_1\mathscr{E}_0(x,t)\|_{L^\infty_{\xi,3}} \le O(1)\frac{\epsilon}{\sqrt{t}}e^{-(\epsilon|x|+\epsilon^2 t)/C_0} \text{ for } t \ge \epsilon^{-2}, \end{cases} \tag{4.21}$$

where $\epsilon \equiv \|\vec{\boldsymbol{u}}_- - \vec{\boldsymbol{u}}_+\|$ and $\{\mathsf{D}_1, \mathsf{D}_2, \mathsf{M}_- - \mathsf{M}_+\}$ are the macroscopic dual vectors of $\{r_1(\vec{\boldsymbol{u}}_-), r_2(\vec{\boldsymbol{u}}_-), \vec{\boldsymbol{u}}_- - \vec{\boldsymbol{u}}_+\}$, and $r_j(\vec{\boldsymbol{u}}_-)$ are the j-th left eigenvectors of $\vec{\mathsf{F}}'(\vec{\boldsymbol{u}}_-)$. The approximated solution $\mathsf{A}_0 + l_0\varphi'$ for (4.19) with the property (4.21) is the T part of the T-C scheme.

Next, one needs to have an exponentially sharp estimate of the output $w(x,t)$ due to the truncation error $\mathscr{E}(x,t)$:

$$\begin{cases} (\partial_t + (\xi^1 - s)\partial_x - \mathsf{L}_\varphi)w = -\mathscr{E}_0, \\ \displaystyle\int_{\mathbb{R}} \mathsf{P}_0 w(x,0)dx = 0. \end{cases} \tag{4.22}$$

This is a system of equations and there is no spectrum gap property to assure an exponential decaying structure though $w(0,t)$ will exponentially converge in time. For the purpose to assert an exponential estimate, one introduced a damping to the system (4.22):

$$\begin{cases} (\partial_t + (\xi^1 - s)\partial_x - \mathsf{L}_\varphi)W_0 = -\mathscr{E}_0 - \gamma\displaystyle\sum_{j=1}(\mathsf{D}_j, W_0)\mathsf{D}_j, \\ \displaystyle\int_{\mathbb{R}} \mathsf{P}_0 W_0(x,0)dx = 0, \end{cases} \tag{4.23}$$

with a small $\gamma > 0$. This system possesses conservation laws:

$$\int_{\mathbb{R}} \mathsf{P}_0 W_0(x,t)dx = 0, \tag{4.24}$$

so that with $\gamma > 0$, (4.24), and energy estimates one shows that this system will decay in time exponentially.

Since the truncation error $\mathsf{P}_0\mathscr{E}_0$ does not possess any transient components, the damping $-\gamma\sum_{j=1}(\mathsf{D}_j, W_0)\mathsf{D}_j$ is essentially virtual. Hence, the solution $W_0(x,t)$ gives an exponentially sharp approximation to the solution $w_0(x,t)$ around $x = 0$. The construction of the

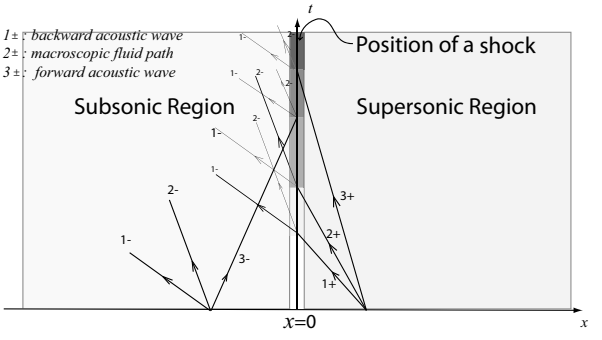
approximated solution $W_0(x,t)$ is called the C-part of the T-C scheme. This part creates another truncation error $-\gamma\sum_{l=1}^{2}(\mathsf{D}_l, W_0)\mathsf{D}_l$. Then, this leads to consider the problem

$$\begin{cases} (\partial_t + (\xi^1 - s)\partial_x - \mathsf{L}_\varphi)\mathsf{f}_1 = \gamma\sum_{l=1}^{2}(\mathsf{D}_l, W_0)\mathsf{D}_l, \\ \mathsf{f}_1(x,0) = 0. \end{cases} \tag{4.25}$$

One repeats the same procedure to give the T-C iteration:
To find A_i and $l_i(t)$ satisfying

$$\begin{cases} \mathscr{E}_i(x,t) \equiv (\partial_t + (\xi^1 - s)\partial_x - \mathsf{L}_\varphi)(\mathsf{A}_i + l_i(t)\varphi') - \gamma\sum_{l=1}^{2}(\mathsf{D}_l, W_{i-1})\mathsf{D}_l, \\ (\partial_t + (\xi^1 - s)\partial_x - \mathsf{L}_\varphi)W_i = -\mathscr{E}_i - \gamma\sum_{j=1}(\mathsf{D}_j, W_i)\mathsf{D}_j, \\ W_i(x,0) = 0, \end{cases} \tag{4.26}$$

and the property (4.21) for $\mathscr{E}_0$ still holds for $\mathscr{E}_i$. Finally, one obtained sharp linear wave scattering structure around the shock profile. The linear wave scattering structure is used to show the pointwise structure of solution of (4.17) as illustrated:



This T-C scheme also works for viscous conservation law. Especially, the sharp pointwise structure gives advantages in the study of the case with presence of boundary in [5].

4.5. Riemann problem for shock wave data. In [37], one considers the initial value problem (4.17) with a shock wave initial data $\mathsf{F}_0(x)$:

$$\mathsf{F}_0(x) = \begin{cases} \mathsf{M}_{\vec{\boldsymbol{u}}_-} \text{ for } x < 0, \\ \mathsf{M}_{\vec{\boldsymbol{u}}_+} \text{ for } x > 0. \end{cases} \tag{4.27}$$

Here, $(\vec{\boldsymbol{u}}_-, \vec{\boldsymbol{u}}_+)$ is a shock wave and $\mathsf{M}_{\vec{\boldsymbol{u}}_\pm}$ are Maxwellians related to the states $\vec{\boldsymbol{u}}_\pm$; and $\|\vec{\boldsymbol{u}}_- - \vec{\boldsymbol{u}}_+\| = \varepsilon \ll 1$.
This problem is a multi-time scale problem. There are five time scales illustrated by the table:

Primary wave	Valid time domain	Scale	Slip
$f(x/t)$	$0 < t < 1$	Hyperbolic scale	Initial Layer
$O(1)$	$t \sim 1$	$O(1)$ scale	Overlapping layer (a)
$\sum_{j=1}^{3} f_j(\frac{x-\lambda_j t}{\sqrt{t}})$	$1 \le t < \varepsilon^{-2}$	Parabolic scale	Overlapping layer (b)
$v(x,t)$	$\varepsilon^{-2} \le t \le \lvert\log\varepsilon\rvert\varepsilon^{-2}$	Nonlinearity formation scale	Overlapping layer (c)
$\mathsf{P}_0\varphi(x)$	$t < \lvert\log\varepsilon\rvert\varepsilon^{-2}$	Time-asymptotic stability scale	Shock layer

In the time scale $0 < t < 1$, the particlelike structure $\mathbb{P}$ of the Green's function and the shock wave initial data force the solution $\mathsf{F}(x,t)$ to behave close to the hyperbolic scale function $f(x/t)$. In the time scale $t \sim 1$, one breaks the collision operator into gain and loss to yield the $O(1)$ structure. When $t \in (1, \varepsilon^{-2})$, one can linearize the problem at the Maxwellian $\mathsf{M}_{\vec{u}_-}$ or $\mathsf{M}_{\vec{u}_+}$, then by the structure (3.17) one concludes that the structures resemble to the convection heat equation with speeds λ_j. When $t \in (\varepsilon^{-2}, \varepsilon^{-2}\log\varepsilon)$, one restricts the macroscopic state on the line segment connecting $\mathsf{M}_{\vec{u}_-}$ and $\mathsf{M}_{\vec{u}_+}$ to form an approximated solution. This restriction carries the spirit of the Chapman-Enskog expansion. One can derive a nonlinear scalar equation close to the viscous Burgers equation. One can use the Hopf-Cole transform effectively to realize the formation of the nonlinear layer. When $t \sim \varepsilon^{-2}\log\varepsilon$, one can use the formed profile by the Burgers-like equation and compare it with the Boltzmann shock profile so that one applies the stability of a shock profile in [37] to yield the global structure of the Riemann problem with a shock wave initial data.

4.6. Future developments. The works done in [4, 15, 17, 18, 36, 37] are for planar wave motions of the Boltzmann equation. When the perturbations are multi-D, the mathematical analysis of the related problems are completely open. Indeed, there are many open problems in physics mentioned in the classical book [29].

About the Boltzmann equation in multi-D, the work in [16] gave the Green's function in 3-D space domain; and gave a wave structure related to Huygen's principle for the 3-D d'Alembert wave equation. In this aspect, it is interesting to consider the shock profile stability under a 3-D perturbations and in particular the multi-D hyperbolic scale waves interact with the viscous shock front. It is also interesting to consider the Riemann problem without assuming the shock wave data. The thermal transpiration flow derived in [29] is an interesting physical phenomenon to distinguish the difference between Boltzmann equation and conventional fluid mechanics. To investigate the geometric effects due to a physical boundary and to relate it with the geometric theory of diffraction would be very interesting as well.

It is also very interesting to complete the Grad's and Sone's program to study the interactions of the singular layers (shock layer, initial layer, and boundary layer) for 1-D problem.

Acknowledgements. The author is in great debt to Professors Sone and Tai-Ping Liu for their deep physics and mathematical insights to the Boltzmann equations during the development on this direction of research; and the author is grateful to Institute of Mathematics, Academia Sinica to host constant visits to Taipei for conducting the research of this subject since 2000. The writing of the paper is supported by the NUS Provost Chair Professorship.

References

[1] Caflisch, R. E. and Nicolaenko, B., *Shock profile solutions of the Boltzmann equation*, Comm. Math. Phys. **89** (1982), 161–194.

[2] Carleman, T., *Sur La Théorie de l'Équation Intégrodifférentielle de Boltzmann*, Acta Mathematica **60** (1933), 91–142.

[3] _____, *Problémes mathématiques dans la théorie cinétique des gaz.* (French) Publ. Sci. Inst. Mittag-Leffler. 2 Almqvist & Wiksells Boktryckeri Ab, Uppsala 1957.

[4] Deng, Shijin, Wang, Weike, and Yu, Shih-Hsien, *Pointwise convergence to Knudsen layers of the Boltzmann equation*, Comm. Math. Phys. **281** (2008), no. 2, 287–347.

[5] _____, *Viscous conservation laws with boundary*, SIAM J. Math. Anal. **44** (2012), no. 4, 2695–2755.

[6] Ellis, R. and Pinsky, M., *The first and second fluid approximations to the linearized Boltzmann equation*, J. Math. Pures Appl. **54** (1975), no. 9, 125–156.

[7] Goodman, J., *Nonlinear asymptotic stability of viscous shock profiles for conservation laws*, Arch. Rational Mech. Anal., **95** (1986), no. 4, 325–344.

[8] Grad, H., *Asymptotic theory of the Boltzmann equation*, Physics of Fluids Vol. **6** (1963), No. 2, 147–181.

[9] _____, *Asymptotic equivalence of the Navier-Stokes and nonlinear Boltzmann equations,* Proc. Symp. Appl. Math. **17**(ed. R.Finn), 154–183, AMS, Providence, 1965.

[10] Grad, H., Proc. Symp. in Appl. Math. **17** (1965), 154–183, *Rarified Gas Dynamics*, I, 26–59 (1963).

[11] S. Kawashima, A. Matsumura, and & T. Nishida, *On the fluid-dynamical approximation to the Boltzmann equation at the level of the Navier-Stokes equation*, Comm. Math. Phys. **70** (1979), no. 2, 97–124.

[12] Liu, Tai-Ping, *Shock waves and diffusion waves*, Nonlinear systems of partial differential equations in applied mathematics, Part 1 (Santa Fe, N.M., 1984), 283–291, Lectures in Appl. Math., 23, Amer. Math. Soc., Providence, RI, 1986.

[13] _____, *Pointwise convergence to shock waves for viscous conservation laws*, Comm. Pure Appl. Math. **50** (1997), no. 11, 1113–1182.

[14] Liu, T.-P. and Yu, S.-H., *Boltzmann Equation: Micro-Macro Decompositions and Positivity of Shock Profiles*, Comm. Math. Phys. **246** (2004), 133–179.

[15] _____, *The Green's function and large-time behavior of solutions for one-dimensional Boltzmann equation,* Comm. Pure. Appl. Math., **57** (2004), no.12, 1543–1608.

[16] _____, *Green's function of Boltzmann equation, 3-D waves,* Bull. Inst. Math. Acad. Sin. (N.S.) **1** (2006), no. 1, 1–78.

[17] _____, *Initial-boundary value problem for one-dimensional wave solutions of the Boltzmann equation,* Comm. Pure Appl. Math., **60** (2007), no. 3, 295–356.

[18] _____, *Invariant Manifolds for steady Boltzmann flows and its applications,* Arch. Rational Mech. Anal., **209** (2013), no. 3, 869–997.

[19] Liu, Tai-Ping, Yang, Tong, and Yu, Shih-Hsien, *Energy method for Boltzmann equation*, Phys. D **188** (2004), no. 3-4, 178–192.

[20] Liu, Tai-Ping, Yang, Tong, Yu, Shih-Hsien, and Zhao, Hui-Jiang, *Nonlinear stability of rarefaction waves for the Boltzmann equation*, Arch. Ration. Mech. Anal. **181** (2006), no. 2, 333–371.

[21] T. Nishida, *Takaaki Fluid Fluid dynamical limit of the nonlinear Boltzmann equation to the level of the compressible Euler equation*, Comm. Math. Phys. **61** (1978), no. 2, 119–148.

[22] Sone, Y., *Kinetic theory analysis of linearized Rayleigh problem*, J. Phys. Soc. Jpn **19** (1964), 1463–1473.

[23] _____, *Effect of sudden change of wall temperature in a rarefied gas*, J. Phys. Soc. Jpn **20** (1965), 222–229.

[24] _____, *Thermal creep in rarefied gas*, J. Phys. Soc. Jpn **21** (1966), 1836–1837.

[25] _____, *Asymptotic theory of flow of rarefied gas over a smooth boundary I*, in: L. Trilling and H. Y. Wachman, eds., Rarefied Gas Dynamics (Academic Press, New York), Vol. **I** (1969), 243–253.

[26] _____, *Asymptotic theory of flow of rarefied gas over a smooth boundary II*, in: D. Dini, ed., Rarefied Gas Dynamics (Editrice Tecnico Scientifica, Pisa), Vol. **II** (1971), 737–749.

[27] _____, *Kinetic theory of evaporation and condensation Linear and nonlinear problems*, J. Phys. Soc. Jpn **45** (1978), 315–320.

[28] _____, *Kinetic theoretical studies of the half-space problem of evaporation and condensation*, Transp. Theory Stat. Phys. **29** (2000), 227–260.

[29] _____, *Molecular Gas Dynamics: Theory, Techniques, and Applications,* Birkhäuser, 2007.

[30] Sone, Y. and Onishi, Y., *Kinetic theory of evaporation and condensation: Hydrodynamic equation and slip boundary condition*, J. Phys. Soc. Jpn **44** (1978), 1981–1994.

[31] Sone, Y., Golse, F., Ohwada, T., and Doi, T., *Analytical study of transonic flows of a gas condensing onto its plane condensed phase on the basis of kinetic theory*, Eur. J. Mech. B/Fluids **17** (1998), 277–306.

[32] Sone, Y., Aoki, K., and Yamashita, K., *A study of unsteady strong condensation on a plane condensed phase with special interest in formation of steady profile*, in: V. Bofli and C. Cercignani, eds., Rarefied Gas Dynamics (Teubner, Stuttgart), Vol. **II** (1986), 323–333.

[33] Ukai, S., *On the existence of global solutions of mixed problem for non-linear Boltzmann equation*, Proc. Japan Acad. **50** (1974), 179–184.

[34] Ukai, S., Yang, T., and Yu, S.-H., *Nonlinear Boundary Layers of the Boltzmann Equation: I. Existence*, Commun. Math. Phys. **236** (2003), 373–393.

[35] ———, *Nonlinear stability of boundary layers of the Boltzmann equation. I. The case* $\mathscr{M}^\infty < -1$, Comm. Math. Phys. **244** (2004), no. 1, 99–109.

[36] Yu, Shih-Hsien, *Nonlinear wave propagations over a Boltzmann shock profile*, J. Amer. Math. Soc. **23** (2010), no. 4, 1041–1118.

[37] ———, *Initial and Shock Layers for Boltzmann Equation*, Arch. Ration. Mech. Anal. **211** (2014), no. 1, 1–60.

Department of Mathematics, National University of Singapore
E-mail: matysh@nus.edu.sg

11. Mathematical Physics

Three lives of the Gelfand-Zeitlin integrable system

Anton Alekseev

Abstract. Gelfand-Zeitlin integrable systems were discovered by Guillemin and Sternberg in 1983, and they represent a standard reference point in the vast world of complete integrability. One of their characteristic features is that action variables satisfy the interlacing inequalities which govern eigenvalues of Hermitian matrices and their principal submatrices. In the paper, we explain that besides the standard Linear Algebra interpretation there are two other, seemingly unrelated situations where interlacing inequalities and Gelfand-Zeitlin systems naturally arise. The first one is combinatorics of planar networks with Boltzmann weights on their edges. Surprizingly, it turns out that maximal weights of multi-paths in planar networks verify exactly the same inequalities as eigenvalues of Hermitian matrices. The second one is tropicalization of Poisson structures. We show that tropicalization of the canonical Poisson structure on the dual Poisson-Lie group $U^*(n)$ produces an integrable system isomorphic to the Gelfand-Zeitlin system. The link between the three topics comes from the synthesis of ideas of tropicalization, Total Positivity and Poisson-Lie groups. As an application, we sketch a new symplectic proof of Horn inequalities for the spectrum of the sum of two Hermitian matrices with given eigenvalues.

Mathematics Subject Classification (2010). Primary 53D17; Secondary 15B57.

Keywords. Gelfand-Zeitlin integrable systems, Poisson structures, planar networks.

1. Introduction

The Gelfand-Zeitlin completely integrable system is a structure at the crossroads between several branches of Mathematics. It originates in Classical Mechanics and Poisson Geometry. The space of Hermitian $n \times n$ matrices $\mathcal{H}_n$ carries a canonical linear Poisson bracket (the Kirillov-Kostant-Souriau bracket). The eigenvalues of a matrix are Casimir functions with respect to this bracket, and symplectic leaves consist of matrices with fixed eigenvalues. The eigenvalues of the principal submatrices ($k \times k$ submatrices sitting in the upper left corner, with $k = 1, \dots, n-1$) are not Casimir functions, but they are in involution with each other. This defines a completely integrable systems (in the sense of Liouville) on each conjugacy class in $\mathcal{H}_n$.

The construction described above was inspired by the work of Gelfand and Zeitlin [16] on the family of bases in irreducible representations of the group $U(n)$. Gelfand-Zeitlin integrable systems for group $U(n)$ and $SO(n)$ were introduced and studied by Guillemin and Sternberg [14]. In particular, they considered applications to Bohr-Sommerfeld quantization of coadjoint orbits. A Gelfand-Zeitlin system provides a set of action-angle variables on the orbit. This observation was applied in [4] to the path integral approach to quantization of coadjoint orbits, and then further applied to the theory of matrix integrals in [31]. The

Proceedings of the International Congress of Mathematicians, Seoul, 2014

Gelfand-Zeitlin systems have also been studied in the framework of complex Poisson Geometry [26], and in the theory of Geometric Quantization on spaces with singularities. They represent a standard reference point in the vast world of integrable systems.

The action variables of the Gelfand-Zeitlin system satisfy an interesting set of *interlacing inequalities*. This is a rather elementary fact from Linear Algebra (and one of the very few facts that we shall actually prove in the text). In the case of $n = 2$, the interlacing inequalities reduce to the inequality $|z| \leq r = \sqrt{x^2 + y^2 + z^2}$ for the norm and one of the components of a 3-dimensional vector $(x, y, z) \in \mathbb{R}^3$. Inequalities become more involved for higher n. For each conjugacy class in $\mathcal{H}_n$, the action variables of the Gelfand-Zeitlin system span an interesting *Gelfand-Zeitlin polytope*.

In this paper, we present two new situations naturally governed by the interlacing inequalities. At first site, they are completely unrelated to the origins of the Gelfand-Zeitlin systems in Linear Algebra and Classical Mechanics on $\mathcal{H}_n$. The first occurrence of interlacing inequalities is in the theory of *planar networks* (planar graphs of special type) equipped with Boltzmann weights on the edges. We shall be interested in maximal weights of paths and multi-paths in the network and in some of its (naturally defined) subnetworks. Surprizingly, the piece-wise linear functions defined by the maximal weights satisfy exactly the same inequalities as eigenvalues of principal submatrices of a Hermitian matrix [7]!

The second scenario that we study is somewhat more involved. In the theory of Poisson-Lie groups (the semi-classical counterpart of the theory of Quantum Groups) one has a natural notion of duality. For G a compact simple Lie group, the dual Poisson-Lie group is a very interesting object. Amont other things, they admit a global linearization given by the (typically transcendental) Ginzburg-Weinstein isomorphism [15]. We study the Poisson-Lie group $U^*(n)$ in the coordinate system given by certain solid minors $\Delta_i^{(k)}$ (and inspired by the Total Positivity and Cluster Algebra theories). We define the *tropicalization* of the Poisson structure by making a change of variables $\Delta_i^{(k)} = \exp(\tau\delta_i^{(k)})$ with $\tau \to +\infty$. Tropicalization produces the following combinatorial data: a polyhedral cone and a completely integrable system with action variables taking values in this cone. By now the reader is expecting the answer: the cone in question is defined by the interlacing inequalities and the integrable system is exactly the Gelfand-Zeitlin integrable system!

While the reasons for these miraculous coincidences are not entirely clear, one possible explanation is as follows: Flaschka-Ratiu [12] discovered an analogue of the Gelfand-Zeitlin integrable system on the dual Poisson-Lie group $U^*(n)$. The Ginzburg-Weinstein isomorphism, and its more precise versions given by the Linearization Lemma [2, 5] establish a link to Hermitian matrices. The group $U^*(n)$ is a subgroup of the group of upper triangular matrices which admit a well-behaved tropicalization naturally leading to the world of planar networks. The work of assembling this puzzle is still in progress. For this reason, we put more emphasis on different manifestations of the Gelfand-Zeitlin system and interlacing inequalities rather then on drawing a unified picture.

As applications of our technique, we compute the asymptotic behavior of the Ginzburg-Weinstein map (for elements of $U^*(n)$ close to the boundary of the corresponding symmetric space). Again, this asymptotics is governed by the set of Gelfand-Zeitlin action variables. Another application is a new symplectic proof [8] of the Horn inequalities which describe the spectrum of a sum of two Hermitian matrices with given eigenvalues. This problem was solved by Klyachko [23] and Knutson-Tao [24] using the techniques of Geometric Invariant Theory. Several other proofs exist in literature, *e.g.* the one by Kapovich-Leeb-Millson [19] using triangle inequalities in symmetric spaces and building. Our proof is based on the

tropicalzation technique and on the control of the Liouville volumes.

Throughout the text, we consistently use the case of $n = 2$ (that is, of 2×2 matrices) as an example. We decided that the advantage of making the text more understandable outweighs the disadvantage of hiding the real difficulties which typically occur for higher n.

We conclude with a number of open questions. Two of them are constructing the Gelfand-Zeitlin integrable system for the group $Sp(2n)$ and finding the planar network (or similar) interpretation for the problem of a spectrum of the product of two unitary matrices with given eigenvalues.

2. Interlacing inequalities obtained in three ways

The interlacing inequalities are inequalities between eigenvalues of Hermitian matrices and their principal submatrices. They play a crucial role in the description of the Gelfand-Zeitlin integrable system. It turns out that exactly the same inequalities come up in combinatorics of planar networks and in Poisson-Geometry of dual Poisson-Lie groups. In this Section, we describe three ways of obtaining the interlacing inequalities.

2.1. Interlacing inequalities and Gelfand-Zeitlin integrable systems. In this Section, we explain the origin of interlacing inequalities in Linear Algebra and recall the Guillemin-Sternberg construction [14] of the Gelfand-Zeitlin integrable system.

2.1.1. Hermitian matrices and interlacing inequalities. Let $\mathcal{H}_n$ be the set of $n \times n$ Hermitian matrices,

$$\mathcal{H}_n = \{a \in \mathrm{Mat}_n(\mathbb{C});\ a^* = a\}.$$

Denote by $\lambda : \mathcal{H}_n \to \mathbb{R}^n$ the map assigning to a the n-tuple of its ordered eigenvalues

$$\lambda : a \mapsto (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n).$$

The map λ is continuous over $\mathcal{H}_n$. Hermitian matrices with n distinct eigenvalues form an open dense subset of $\mathcal{H}_n$. Over this subset, the map λ is real analytic.

For $a \in \mathcal{H}_n$, let $a^{(k)}$ be the $k \times k$ principal submatrix of a sitting at the intersection of the first k rows and the first k columns of a. In particular, $a^{(n)} = a$ and $a^{(1)} = a_{1,1}$. Matrices $a^{(k)}$ are Hermitian, and one can introduce the corresponding eigenvalue maps $\lambda^{(k)} : a^{(k)} \mapsto \mathbb{R}^k$. The maps $\lambda^{(k)}$ for $k = 1, \ldots, n$ give rise the *generalized eigenvalue map* (also called the Gelfand-Zeitlin map) $\Lambda : \mathcal{H}_n \to \mathbb{R}^N$ for $N = n(n+1)/2$. It is convenient to place the eigenvalues $\lambda_i^{(k)}$ in the vertices of a triangular tableau of size n (see Figure 2.1).

Proposition 2.1. *The generalized eigenvalues of a Hermitian matrix verify the interlacing inequalities*

$$\lambda_i^{(k)} \geq \lambda_i^{(k-1)} \geq \lambda_i^{(k)}. \tag{2.1}$$

Proof. The first inequality $\lambda_i^{(k)} \geq \lambda_i^{(k-1)}$ follows from the minimax principle for eigenvalues of a Hermitian matrix:

$$\lambda_i(a) = \max_U \min_{x \in U, ||x||=1} (x, a(x)),$$

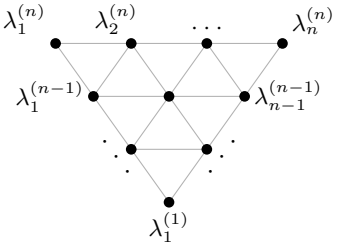


Figure 2.1. Triangular tableau of size n

where $U \subset \mathbb{C}^n$ is a subspace of dimension i, $||\cdot||$ is the standard norm and $(\cdot,\cdot)$ is the standard Hermitian product on $\mathbb{C}^n$. Indeed, the maximum is taken over the subspaces $U \subset \mathbb{C}^k$ for $\lambda_i^{(k)}$ and over the smaller set of subspaces $U \subset \mathbb{C}^{k-1}$ for $\lambda_i^{(k-1)}$.

The second inequality $\lambda_i^{(k-1)} \geq \lambda_i^{(k)}$ is obtained by observing that $\lambda_i(-a) = -\lambda_{n-i+1}(a)$ and by applying it to the first inequality. □

Example 2.2. Let $n = 2$ and make the simplifying assumption that matrices $a \in \mathcal{H}_2$ are traceless. The space of traceless 2×2 Hermitian matrices is isomorphic to $\mathbb{R}^3$. The top eigenvalue of such a matrix $\lambda_1^{(2)} = -\lambda_2^{(2)}$ is then interpreted as the norm r of a vector $a \in \mathbb{R}_3$, and the eigenvalue $\lambda_1^{(1)}$ is its Cartesian projection to the z-axis. In this case, the inequalities take the form $r \geq z \geq -r$.

The interlacing inequalities define a polyhedral Gelfand-Zeitlin cone $\mathcal{C}_{GZ}$ in $\mathbb{R}^N$. All triangular tableaux verifying the interlacing inequalities come up as generalized eigenvalues of Hermitian matrices:

Proposition 2.3. *The image of the generalized eigenvalue map Λ is the Gelfand-Zeitlin cone $\mathcal{C}_{GZ}$.*

Sometimes it is more convenient to use a different parametrization of the Gelfand-Zeitlin cone. Denote $l_0^{(k)} = 0$ for all $k = 0, 1, \ldots, n$ and put

$$l_i^{(k)} = \lambda_1^{(k)} + \cdots + \lambda_i^{(k)}.$$

Proposition 2.4. *The interlacing inequalities are equivalent to the following system of inequalities for the variables $l_i^{(k)}$:*

$$\begin{aligned} l_i^{(k)} + l_{i-1}^{(k-1)} &\geq l_{i-1}^{(k)} + l_i^{(k)}, \\ l_i^{(k)} + l_i^{(k-1)} &\geq l_{i+1}^{(k)} + l_{i-1}^{(k)}. \end{aligned} \tag{2.2}$$

One can place the variables $l_i^{(k)}$ in the nodes of a triangular tableau of size $n+1$. Then, the inequalities (2.2) admit the following visualization (see Figure 2.2). Two adjacent small equilateral triangles covering the tableau form a small rhombus. Such rhombi have three possible orientations, NE (North-East), NW (North-West) and S (South). Inequalities (2.2) are in one-to-one correspondence with rhombi of NE and NW orientations.

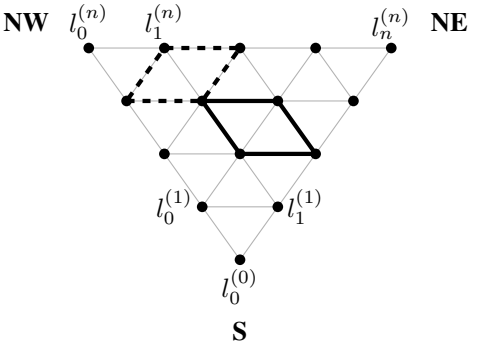


Figure 2.2. Triangular tableau with NE and NW oriented rhombi

For each rhombus with numbers a, b, c, d decorating the corners, the inequality

$$b + d \geq a + c$$

says that the sum of numbers at the end-points of the short diagonal is greater or equal to the sum of numbers at the end-points of the long diagonal (see Figure 2.3).

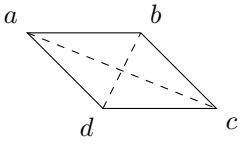


Figure 2.3. $b + d \geq a + c$

We shall denote by $L : \mathcal{H}_n \to \mathbb{R}^N$ the map with components $l_i^{(k)}$. By abuse of notations, we will denote the polyhedral cone defined by inequalities (2.2) (with $l_0^{(k)} = 0$) by $\mathcal{C}_{GZ}$.

Denote by $\mathcal{H}_n^{\mathrm{reg}}$ the subset of $\mathcal{H}_n$ where all interlacing inequalities are strict. On this open dense subset, the generalized eigenvalues $\lambda_i^{(k)}$ and the variables $l_i^{(k)}$ are real analytic functions of $a \in \mathcal{H}_n$.

2.1.2. The Gelfand-Zeitlin integrable system. The Gelfand-Zeitlin map admits a natural interpretation in terms of Poisson and Symplectic Geometry. Recall that the space $\mathcal{H}_n$ can be naturally identified with the dual space to the Lie algebra $\mathfrak{u}(n)$. The pairing is given by

$$\langle a, \xi \rangle = \mathrm{Im}\,\mathrm{Tr}(a\xi),$$

where $a \in \mathcal{H}_n$ is a Hermitian matrix and $\xi \in \mathfrak{u}(n)$ is an anti-Hermitian matrix. Hence, the space $\mathcal{H}_n \cong \mathfrak{u}^*(n)$ carries a canonical linear Kirillov-Kostant-Souriau (KKS) Poisson structure π_{KKS}. On linear functions $f_\xi(a) = \langle a, \xi \rangle$, it is given by

$$\{f_\xi, f_\eta\}_{\mathrm{KKS}}(a) = \langle a, [\xi, \eta]_{\mathfrak{u}(n)} \rangle = f_{[\xi,\eta]_{\mathfrak{u}(n)}}.$$

The symplectic leaves of this bracket are formed by matrices with fixed eigenvalues. For $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$, we denote the corresponding symplectic leaf by $\mathcal{O}_\lambda$.

The following theorem is due to Guillemin-Sternberg [14]:

Theorem 2.5. *Over $\mathcal{H}_n^{\rm reg}$, the map $L : (\mathcal{H}_n, \pi_{\rm KKS}) \to (\mathbb{R}^n, \pi = 0)$ is a Poisson map. On each symplectic leaf, it defines a completely integrable system.*

In other words, components of the map L Poisson commute with each other:

$$\{l_i^{(k)}, l_j^{(m)}\}_{\rm KKS} = 0.$$

Moreover, for each $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$, they define a Poisson commutative subalgebra of functions on $\mathcal{O}_\lambda$ spanned by $m = \dim \mathcal{O}_\lambda/2$ independent functions.

On the subset $\mathcal{H}_n^{\rm reg}$, one can define the differential forms $d\lambda_i^{(k)}$ and Hamiltonian vector fields $v_i^{(k)} = \pi_{\rm KKS}^\sharp(d\lambda_i^{(k)}) = \pi_{\rm KKS}(d\lambda_i^{(k)}, \cdot)$. These vector fields integrate to circle actions [14]:

Theorem 2.6. *On $\mathcal{H}_n^{\rm reg}$, the Hamiltonian vector fields $v_i^{(n)}$ vanish, and the vector fields $v_i^{(k)}$ with $k \neq n$ integrate to commuting circle actions with periods equal to 2π.*

Hence, one obtains a densely defined action of the real torus of dimension $n(n-1)/2$. This torus action is known as the Thimm action, and it admits the following explicit description. Let $a \in \mathcal{H}_n$, and choose a $k \times k$ unitary matrix u which diagonalizes $a^{(k)}$. Denote by

$$U = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$$

the block-diagonal matrix with u sitting in the upper left corner. Then, we have

$$UaU^{-1} = \begin{pmatrix} \lambda_1^{(k)} & \dots & 0 & a_{1,k+1} & \dots & a_{1,n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \lambda_k^{(k)} & a_{k,k+1} & \dots & a_{k,n} \\ a_{k+1,1} & \dots & a_{k+1,k} & a_{k+1,k+1} & \dots & a_{k+1,n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n,1} & \dots & a_{n,k} & a_{n,k+1} & \dots & a_{n,n} \end{pmatrix}$$

The Thimm action of the circles corresponding to $v_i^{(k)}$ with $i = 1, \dots, k$ is given by conjugation of the matrix UaU^{-1} by the diagonal matrices of the form

$$\operatorname{diag}\left(e^{i\theta_1}, \dots, e^{i\theta_k}, 1, \dots, 1\right).$$

One can show that over $\mathcal{H}_n^{\rm reg}$ the matrix elements $a_{k+1,i}$ with $i = 1, \dots, k$ are non vanishing. One can then choose a section for the Thimm action by requiring $a_{k+1,i}$ to be real positive. It gives rise to a system of action-angle variables on $\mathcal{H}_n^{\rm reg}$ with action variables $\lambda_i^{(k)}$ and angle variables $\varphi_i^{(k)} = {\rm Arg}(a_{k+1,i})$. The symplectic forms on the leaves are then given by formula

$$\omega_\lambda = \sum_{k=1}^{n-1} \sum_{i=1}^{k} d\lambda_i^{(k)} \wedge d\varphi_i^{(k)},$$

and the corresponding Liouville volume form is of the form

$$\mathcal{L}_\lambda = \prod_{k,i} d\lambda_i^{(k)} \wedge d\varphi_i^{(k)}.$$

Returning to interlacing inequalities, for $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$ we denote by Δ_λ the intersection of the Gelfand-Zeitlin cone $\mathcal{C}_{GZ}$ with the affine subspace defined by equations $\lambda_i^{(n)} = \lambda_i$. The map L restricts to $\mathcal{O}_\lambda$ and maps it to Δ_λ with generic fibers the orbits of the Thimm action. The push-forward of the Liouville volume form under the map L is the Lebesgue measure on the polytope Δ_λ:

$$L_* \mathcal{L}_\lambda = \chi_{\Delta_\lambda} d\lambda,$$

where χ_{Δ_λ} is the characteristic function of Δ_λ.

Example 2.7. In the case of $n = 2$, symplectic leaves $\mathcal{O}_\lambda$ are either points (if $\lambda_1 = \lambda_2$) of 2-spheres (if $\lambda_1 > \lambda_2$). Gelfand-Zeitlin polytopes are segments $\lambda_1^{(1)} \in \Delta_\lambda = [\lambda_2, \lambda_1]$ and the induced measure is given by

$$L_* \mathcal{L}_\lambda = \chi_{[\lambda_2, \lambda_1]} \, d\lambda_1^{(1)}. \tag{2.3}$$

2.2. Planar networks and inequalities. In this Section, we explain how interlacing inequalities come up in the combinatorics of planar networks with Boltzmann weights on the edges.

2.2.1. Planar networks. Planar networks is a special type of planar graphs which originate in the theory of Total Positivity.

Definition 2.8. A planar network of type n is a finite planar oriented graph Γ drawn on the Cartesian plane with a coordinate system x, y such that

- Γ is contained between two vertical straight lines L and R.
- Edges of Γ are segments of straight lines with positive projection on the x-axis.
- Γ has exactly n sources on L and exactly n sinks on R.

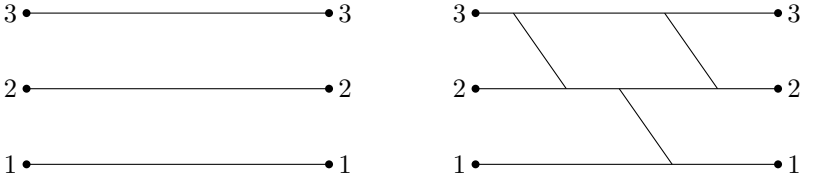


Figure 2.4. Two examples of planar networks, Γ_{hor} and Γ_0

The definition implies that planar networks do not have oriented cycles. We will label sources on L and sinks on R with numbers $\{1, \ldots, n\}$ from bottom to top. We will denote by $V\Gamma$ the set of vertices of Γ and by $E\Gamma$ the set of edges. A *path* in Γ is an oriented path starting on L and ending on R. We denote the set of paths of Γ by $P\Gamma$. An i-path in Γ is a collection of i paths which do not touch each other (that is, have no common vertices and no common edges). The set of i-paths will be denoted by $P_i\Gamma$. We shall denote by $\Gamma^{(k)}$ with $1 \leq k \leq n$ the planar network of type k obtained from Γ by deleting the sources and sinks with labels $k+1, \ldots, n$ and by deleting the edges having them as one of the end points.

Two examples of planar networks are shown on Figure 2.4. The first one is denoted by $\Gamma_{\rm hor}$, and it has exactly n edges connecting sources and sinks with the same label. The second one is denoted Γ_0, and it has a more complicated structure with k slanted edges at the kth floor of the network. Figure 2.4 shows the two examples for $n = 3$, but such networks can be defined for any n.

Denote by $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ the real line with the point $-\infty$ added. The set $\mathbb{T}$ has a natural structure of a semi-ring with operations

$$a +_{\mathbb{T}} b = \max(a, b) \quad , \quad a \cdot_{\mathbb{T}} b = a + b$$

and with $-\infty$ being the neutral element for addition. This semi-ring is called *tropical* and it plays an important role in Tropical Geometry.

We will consider planar networks equipped with weightings on their edges with values in $\mathbb{T}$. Thus, a weighting on Γ is a map $w : E\Gamma \to \mathbb{T}$. We denote the set of all weightings by $W(\Gamma, \mathbb{T})$. We define the following set of functions on $W(\Gamma, \mathbb{T})$:

$$l_i\Gamma(w) = \max_{\gamma \in P_i\Gamma} w(\gamma), \tag{2.4}$$

where

$$w(\gamma) = \sum_{e \in \gamma} w(e).$$

The right hand side of equation (2.4) is well defined if the set of i-paths $P_i\Gamma$ is non empty. In case of $P_i\Gamma = \emptyset$, we set $l_i\Gamma(w) = -\infty$ for all $w \in W(\Gamma, \mathbb{T})$.

We put $l_0\Gamma(w) = 0$ for all $w \in W(\Gamma, \mathbb{T})$ and denote by $l\Gamma : W(\Gamma, \mathbb{T}) \to \mathbb{T}^{n+1}$ the map with components $l_i\Gamma$ for $i = 0, \ldots, n$. It is also convenient to introduce a map $L\Gamma$ with components $l_i^{(k)}(w) = l_i\Gamma^{(k)}(w)$ for $0 \leq i \leq k \leq n$.

Proposition 2.9. *Let Γ be a planar network of type n, and $\Gamma' \subset \Gamma$ a subnetwork which is also to type n. Then,* $\mathrm{im}\,(l\Gamma') \subset \mathrm{im}\,(l\Gamma)$.

Proof. Put the weights of all the edges in $\Gamma \backslash \Gamma'$ equal to $-\infty$. On remaining weights, the map $l\Gamma$ restricts to the map $l\Gamma'$. □

It is interesting that the functions $l_i\Gamma$ are completely expressed in terms of tropical operations: addition and maximum. The following result establishes a link between combinatorics of weighted planar networks and interlacing inequalities [7]:

Theorem 2.10. *Let Γ be a planar network of type n. Then, for all weightings $w \in W(\Gamma, \mathbb{T})$ we have*

$$\begin{array}{lcl} l_i\Gamma^{(k)}(w) + l_{i-1}\Gamma^{(k-1)}(w) & \geq & l_{i-1}\Gamma^{(k)}(w) + l_i\Gamma^{(k-1)}, \\ l_i\Gamma^{(k)}(w) + l_i\Gamma^{(k-1)}(w) & \geq & l_{i+1}\Gamma^{(k)}(w) + l_{i-1}\Gamma^{(k)}(w). \end{array} \tag{2.5}$$

Theorem 2.10 states that the image of $L\Gamma$ is contained in the closure of the cone $\mathcal{C}_{GZ}$ in the natural topology of $\mathbb{T}^N$.

The proof of Theorem 2.10 is based on the following observation. Consider, for instance, the first inequality in (2.5). It can be rewritten in the form

$$\max_{\alpha_1 \in P_i\Gamma^{(k)}, \beta_1 \in P_{i-1}^{(k-1)}} \sum_{e \in \alpha_1 \cup \beta_1} w(e) \geq \max_{\alpha_2 \in P_{i-1}\Gamma^{(k)}, \beta_2 \in P_i^{(k-1)}} \sum_{e \in \alpha_2 \cup \beta_2} w(e).$$

In the sums, we count the edges twice if they belong both to α_1 and β_1 (or to α_2 and β_2). The inequality would follow if for every pair $\alpha_2 \in P_{i-1}\Gamma^{(k)}, \beta_2 \in P_i\Gamma^{(k-1)}$ there is a pair $\alpha_1 \in P_i\Gamma^{(k)}, \beta_1 \in P_{i-1}^{(k-1)}$ with the same set of edges (including mltiplicities). And this turns out to be the case. Figure 2.5 illustrates this principle for $i = 2$. The second inequality in (2.5) can be treated in a similar way.

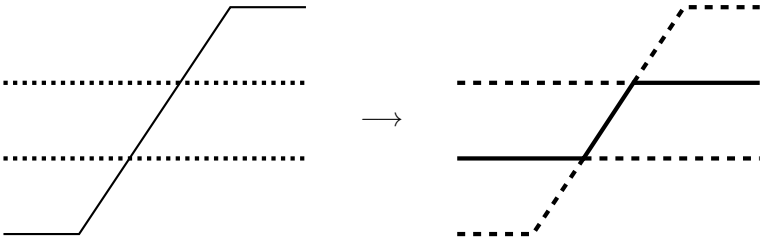


Figure 2.5. A 2-path in $\Gamma^{(k-1)}$ and a 1-path in $\Gamma^{(k)}$ redrawn as a 2-path in $\Gamma^{(k)}$ and a 1-path in $\Gamma^{(k-1)}$

Motivated by Theorem 2.10, we call the map $L\Gamma$ the *tropical Gelfand-Zeitlin map*. One may wonder whether there exist planar networks for which the image of the tropical Gelfand-Zeitlin map coincides with the closure $\overline{\mathcal{C}}_{GZ}$ of the Gelfand-Zeitlin cone. The answer to this question is given by the following theorem [7]:

Theorem 2.11.

$$\mathrm{im}(L\Gamma_0) = \overline{\mathcal{C}}_{GZ}.$$

Let $W^{\mathrm{res}}\Gamma_0 \subset W\Gamma_0$ be the subset of weights w which vanish on all horizontal edges of Γ_0 with the exception of the ones which connect to the sinks. Then, the number of nonvanishing weights is equal to $n(n+1)/2$ and coincides with the number of nonvanishing entries of the triangular tableau. Components $l_i^{(k)}$ of the map $L\Gamma_0$ are piece-wise linear functions on $W^{\mathrm{res}}\Gamma_0 \subset W\Gamma_0$. They define a splitting of $\mathbb{R}^N$ (or $\mathbb{T}^N$) into a union of linearity chambers. On each linearity chamber, the map $L\Gamma_0$ restricts to a linear transformation. Moreover, on each linearity chamber there are unique multi-paths $\gamma_i^{(k)} \subset \Gamma_0^{(k)}$ such that $l_i\Gamma_0^{(k)}(w) = w(\gamma_i^{(k)})$.

The following unexpected result [7] gives the description of the paths $\gamma_i^{(k)}$:

Theorem 2.12. *There is a unique linearity chamber $\mathcal{C}_0 \subset \mathbb{T}^N$ on which the map $L\Gamma_0$ is of full rank (equal to N). On $\mathcal{C}_0$, the map $L\Gamma_0$ restricts to a bijection between $\mathcal{C}_0$ and $\mathcal{C}_{GZ}$. The corresponding multi-paths $\gamma_i^{(k)}$ are shown on Figure 2.6.*

Example 2.13. Consider the case of $n = 2$. Denote the weights on the horizontal edges by a_1 and a_2 (corresponding to the label of the sink) and the weight on the slanted edge by b.

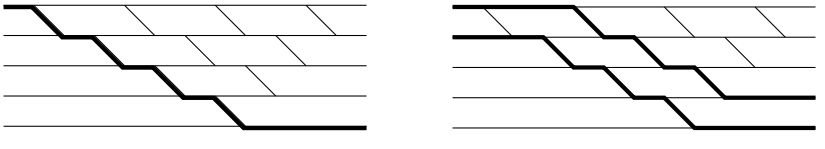


Figure 2.6. The paths $\gamma_1^{(5)}$ and $\gamma_2^{(5)}$

Then, we easily obtain

$$l_1^{(2)} = \max(a_1, a_2, b + a_1), \quad l_2^{(2)} = a_1 + a_2, \quad l_1^{(1)} = a_1.$$

There are several linearity chambers depending on which of the terms wins in the expression for $l_1^{(2)}$. But the resulting linear transformation in non degenerate only if the winning term is $b + a_1$ since the expressions for $l_2^{(2)}$ and $l_1^{(1)}$ do not involve the weight b.

2.3. Inequalities from tropicalization of Poisson brackets. In this Section, we show that the interlacing inequalities come up in the study of tropicalization of Poisson brackets on the dual Poisson-Lie group $U^*(n)$.

2.3.1. Log-canonical Poisson brackets and tropicalization. Let U be a chart in a real Poisson manifold M with coordinates $\{x_1, \dots, x_k\}$ taking values in $\mathbb{R}_+$. The Poisson bracket on M is called *log-canonical* with respect to these coordinates if

$$\{x_i, x_j\} = \pi_{i,j} x_i x_j$$

for some constants $\pi_{i,j} = -\pi_{j,i}$. Another way to present the same concept is as follows: let $K = \mathbb{R}_+^k$ be a group under the component-wise multiplication. Then, log-canonical Poisson brackets on K are exactly those which are invariant under translations. One more approach to log-canonical brackets is as follows: make the change of variables $x_i = \exp(\xi_i)$ and consider K as a graded manifold with $\deg \xi_i = 1$. In these coordinates, log-canonical Poisson brackets become constant

$$\{\xi_i, \xi_j\} = \pi_{i,j}, \tag{2.6}$$

of degree (-2).

Consider a Poisson bracket of more general type,

$$\{x_i, x_j\} = \pi_{i,j} x_i x_j + p_{i,j}(x),$$

where $p_{i,j}$ are Laurent polynomials in variables x. That is, $p_{i,j} = \sum_I c_I x^I$ with $x^I = x_1^{i_1} \dots x_k^{i_k}$. Let $V = \mathbb{R}^k$ and for every pair (i, j) define a polyhedral cone

$$\mathcal{C}_{i,j} = \{\xi \in V;\ \xi_i + \xi_j \geq \sum_{s=1}^k i_s \xi_s \ \forall I \text{ such that } c_I \neq 0\}.$$

The *tropicalization* of the Poisson structure is defined as the closed polyhedral cone $\mathcal{C}(\pi, x) = \cap_{i,j} \mathcal{C}_{i,j}$ equipped with the constant Poisson structure (2.6) which we denote by π_∞. The following statement is an elementary observation:

Proposition 2.14. *Let $\tau \in \mathbb{R}_+, \pi_\tau = \tau^2 \pi$ and introduce coordinate functions $\tau \xi_i = \log(x_i)$. Then, for $\xi \in \mathcal{C}(\pi, \xi)$ we have*

$$\pi_\tau \to_{\tau \to +\infty} \pi_\infty.$$

A similar formalism can be developed for coordinate charts with both real and complex valued coordinate functions $\{x_1, \dots, x_k, z_1, \dots, z_l\}$, where $x_1, \dots, x_k$ take values in $\mathbb{R}_+$ and $z_1, \dots, z_l$ take values in $\mathbb{C}^*$. Again, let $K = \mathbb{R}_+^k \times (\mathbb{C}^*)^l$ be a real Lie group with

point-wise multiplication. A Poisson bivector on K is called log-canonical if it is translation-invariant. Make a change of variables $x_i = \exp(\xi_i), z_a = \exp(\zeta_a + i\phi_a)$ and view K as a graded manifold with coordinates ξ_i, ζ_a of degree $+1$ and angles ϕ_a of degree 0. We shall be especially interested in log-canonical Poisson brackets of degree (-1). The only non vanishing components of such a Poisson bracket are given by

$$\{\xi_i, \phi_a\} = \pi_{i,a} \quad , \quad \{\zeta_a, \phi_b\} = \pi_{a,b}. \tag{2.7}$$

This structure gives rise to a completely integrable system with ξ's and ζ's as action variables and ϕ's as angle variables. Indeed, the condition for the Poisson bracket to be log-canonical and to have degree (-1) implies that $k + l$ functions $\xi_i, i = 1, \ldots, k$ and $\zeta_a, a = 1, \ldots, l$ are in involution.

Again, one can consider Poisson brackets of the form $\pi = \pi^{\log} + \pi'$, where $\pi^{\log}$ is a log-canonical Poisson bracket of degree (-1), and π' is expressed in terms of Laurent polynomials. A procedure similar to the one described above (see [3] for details) gives rise to a tropicalization of π given by a convex polyhedral cone $\mathcal{C}(\pi, x, z) \subset V = \mathbb{R}^{k+l}$ and a constant Poisson bracket (2.7) on $\mathcal{C}(\pi, x, z) \times T^l$. Here T^l is a real torus of dimension l and we again denote the Poisson bracket (2.7) by π_∞. As before, we have the following statement about the limit of scaled Poisson structures:

Proposition 2.15. *Let $\tau \in \mathbb{R}_+, \pi_\tau = \tau\pi$ and introduce coordinate functions*

$$\tau\xi_i = \log(x_i) \quad , \quad \tau\zeta_a + i\phi_a = \log(z_a).$$

Then, for $(\xi, \zeta) \in \mathcal{C}(\pi, x, z)$ we have

$$\pi_\tau \to_{\tau\to+\infty} \pi_\tau.$$

Example 2.16. Let $k = 1, l = 1$ and consider the Poisson bracket of the form

$$\{x, z\} = ixz, \qquad \{x, \bar{z}\} = -ix\bar{z}, \qquad \{z, \bar{z}\} = i(x^2 - x^{-2}).$$

The corresponding cone is given by inequalities $-\zeta \le \xi \le \zeta$, and the Poisson structure π_∞ has the only nonvanishing component $\{\xi, \phi\}_\infty = 1$.

2.3.2. Tropicalization of the canonical Poisson bracket on $U^*(n)$. Recall that compact Lie groups carry canonical multiplicative Poisson structures defined by the classical r-matrix

$$r = \frac{1}{2}\sum_i h_i \otimes h_i + \sum_{\alpha\in\Delta} e_\alpha \otimes e_{-\alpha}.$$

Here $\{h_i\}$ is an orthonormal basis in the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, and Δ_+ is the set of positive roots. By Drinfeld's theory [10, 30], Poisson-Lie groups come in dual pairs. At the level of Lie bialgebras, the duality corresponds to an exchange of the Lie bracket with Lie co-bracket. We shall be particularly interested in the dual Poisson-Lie group $U^*(n)$ which has been studied by Lu and Weinstein [27]. As a Lie group, it is isomorphic to the set of invertible upper triangular matrices with positive reals on the diagonal. The map $A \mapsto AA^*$ establishes an isomorphism between $U^*(n)$ and the set of positive definite Hermitian matrices $\mathcal{H}_n^+$. The action of $U(n)$ by conjugations on $\mathcal{H}_n^+$ induces an action on $U^*(n)$ called the *dressing action*. The Poisson structure on $U^*(n)$ is uniquely defined by the following equation,

$$\pi\left(\operatorname{Im}\operatorname{Tr}(dAA^{-1}\xi), \cdot\right) = \xi_{U^*(n)}.$$

Here dAA^{-1} is the right-invariant Maurer-Cartan form on $U^*(n)$, $\xi \in \mathfrak{u}(n) \cong i\mathcal{H}_n$ is an element of the Lie algebra and $\xi_{U^*(n)}$ is the fundamental vector field defined by the dressing action (recall that the bilinear form $\mathrm{Im}\,\mathrm{Tr}(a\xi)$ defines a duality between $\mathfrak{u}(n)$ and $\mathcal{H}_n$).

The following coordinate system on the group $U^*(n)$ is inspired by the Total Positivity and Cluster Algebra theory [13]. For $1 \leq k \leq n$ and $1 \leq i \leq k$, let $\Delta_i^{(k)} : U^*(n) \to \mathbb{C}$ be the solid minor defined by the rows with numbers $n-k+1, \dots, n-k+i$ and by the last i columns of A (see Figure 2.7). The minors $\Delta_k^{(k)}$ take values in $\mathbb{R}$, and the minors $\Delta_i^{(k)}, i = 1, \dots, k-1$ take values in $\mathbb{C}$. The coordinate chart defined by the minors $\{\Delta_i^{(k)}\}$ is open and dense in $U^*(n)$.

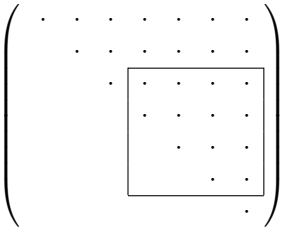


Figure 2.7. A minor $\Delta_i^{(k)}$

The following statement is a reformulation of a theorem by Kogan-Zelevinsky [25]:

Proposition 2.17. *Under the canonical Poisson structure on $U^*(n)$, the Poisson brackets of minors $\Delta_i^{(k)}$ are of the form*

$$\{\Delta_i^{(k)}, \Delta_j^{(l)}\} = \frac{i}{2}\mathrm{sign}(k-l)(C-R)\Delta_i^{(k)}\Delta_j^{(l)},$$

where C is the number of the common rows and R is the number of common columns of $\Delta_i^{(k)}$ and $\Delta_j^{(l)}$.

Proposition 2.17 implies that Poisson brackets of complex conjugate minors are also of log-canonical form:

$$\{\overline{\Delta}_i^{(k)}, \overline{\Delta}_j^{(l)}\} = -\frac{i}{2}\mathrm{sign}(k-l)(C-R)\overline{\Delta}_i^{(k)}\overline{\Delta}_j^{(l)}.$$

However, the mixed Poisson brackets $\{\Delta_i^{(k)}, \overline{\Delta}_j^{(l)}\}$ are *not log-canonical*. It turns out that they are given by (rather complicated) Laurent polynomials in $\Delta_i^{(k)}$'s. Hence, we can address the question of tropicalization of this Poisson structure. We shall denote the tropical versions of the coordinates $\Delta_i^{(k)}$ by $\delta_i^{(k)}$ so as $\Delta_i^{(k)} = \exp(\tau\delta_i^{(k)})$.

Theorem 2.18. *The tropicalization of the canonical Poisson bracket on the Poisson-Lie group $U^*(n)$ in coordinates $\Delta_i^{(k)}$ is given by a pair $(\mathcal{C}(\pi_{U^*(n)}, \Delta), \pi_\infty)$, where $\mathcal{C}(\pi_{U^*(n)}, \Delta)$ is a convex polyhedral cone isomorphic to the Gelfand-Zeitlin cone $\mathcal{C}_{GZ}$ with isomorphism given by the identification $\delta_i^{(k)} = l_i^{(k)}$. The Poisson structure π_∞ on $\mathcal{C}(\pi_{U^*(n)}, \Delta) \times T^{n(n-1)/2}$ is isomorphic to the one of the Gelfand-Zeitlin integrable system. In particular, $\delta_i^{(n)}, i = 1, \dots, n$ are Casimir functions under π_∞.*

Example 2.19. For once, we choose $n = 3$ rather than $n = 2$ as an example. Consider the Poisson bracket $\{\Delta_1^{(3)}, \bar{\Delta}_1^{(3)}\}$ written in the coordinates $\{\Delta_i^{(k)}\}$:

$$\{\Delta_1^{(3)}, \bar{\Delta}_1^{(3)}\} = i\left(\frac{\Delta_3^{(3)}}{\Delta_2^{(2)}}\right)^2 - i\left(\Delta_1^{(1)}\right)^2 - i\Delta_1^{(2)}\bar{\Delta}_1^{(2)} + \\ + i\left(\frac{\Delta_2^{(3)}}{\Delta_1^{(2)}}\right)\left(\frac{\bar{\Delta}_2^{(3)}}{\bar{\Delta}_1^{(2)}}\right) + i\left(\frac{\Delta_1^{(3)}}{\Delta_1^{(2)}}\right)\left(\frac{\Delta_2^{(2)}}{\Delta_1^{(1)}}\right)\left(\frac{\bar{\Delta}_2^{(3)}}{\bar{\Delta}_1^{(2)}}\right) + \\ + i\left(\frac{\bar{\Delta}_1^{(3)}}{\bar{\Delta}_1^{(2)}}\right)\left(\frac{\Delta_2^{(2)}}{\Delta_1^{(1)}}\right)\left(\frac{\Delta_2^{(3)}}{\Delta_1^{(2)}}\right) + i\left(\frac{\Delta_1^{(3)}}{\Delta_1^{(2)}}\right)\left(\frac{\Delta_2^{(2)}}{\Delta_1^{(1)}}\right)^2\left(\frac{\bar{\Delta}_1^{(3)}}{\bar{\Delta}_1^{(2)}}\right).$$

This expression alone gives rise to 6 inequalities:

$$\begin{array}{lll} \delta_1^{(3)} \geq \delta_1^{(2)}, & \delta_1^{(3)} + \delta_1^{(2)} \geq \delta_2^{(3)}, & \delta_1^{(1)} + \delta_1^{(2)} \geq \delta_2^{(2)} \\ \delta_1^{(3)} \geq \delta_1^{(1)}, & \delta_1^{(3)} + \delta_2^{(2)} \geq \delta_3^{(3)}, & \delta_1^{(1)} + 2\delta_1^{(2)} + \delta_1^{(3)} \geq \delta_2^{(2)} + \delta_2^{(3)}. \end{array}$$

The inequalities in the first row are part of the interlacing inequalities for $n = 3$. This *is not the case* for inequalities in the low row, but it turns out that they follow from interlacing inequalities.

3. Applications and open problems

In this Section, we present two applications of planar network and tropicalization techniques. The first one is the asymptotic behavior of the mysterious Ginzburg-Weinstein map in the theory of Poisson-Lie groups. The second one is a new symplectic proof of Horn inequalities for eigenvalues of a sum of two Hermitian matrices. We also give a brief description of several open problems.

3.1. The Ginzburg-Weinstein isomorphism and tropicalization. The Ginzburg-Weinstein map is a mysterious Poisson isomorphism between the dual to the Lie algebra $\mathfrak{g}^*$ and the dual Poisson-Lie group G^* which is valid in the case of G compact. While explicit formulas for this map are only available for $n = 2$, we shall see that for $G = U(n)$ the idea of scaling (with the parameter τ) allows to control the asymptotics of the Ginzburg-Weinstein map for large τ and to express it in terms of the Gelfand-Zeitlin generalized eigenvalue map.

Let G be a compact connected Poisson-Lie group and G^* be the dual Poisson-Lie group. The following fundamental result is due to Ginzburg and Weinstein [15]:

Theorem 3.1. *The dual Poisson-Lie group G^* is isomorphic to the dual of the Lie algebra $\mathfrak{g}^*$ as a Poisson space.*

The Ginzburg-Weinstein isomorphism is a rare example of a global solution of the linearization problem in Poisson Geometry: the Poisson bivector on G^* vanishes at the group unit, and induces a canonically defined linear bivector on $T_eG^* \cong \mathfrak{g}^*$. This linear bivector coincides with the KKS Poisson structure on $\mathfrak{g}^*$.

The original proof [15] of the Ginzburg-Weinstein isomorphism is the existence proof using cohomology arguments. There are several other proofs in the literature which make

use of the Moser Lemma in Symplectic Geometry [2], the dynamical r-matrices [11] and the Stokes data of flat connections with irregular singularities [9]. In the case of $\mathfrak{g} = \mathrm{su}(2)$, there is an explicit albeit cumbersome formula for the Ginzburg-Weinstein map:

$$\gamma : \begin{pmatrix} z & \rho e^{i\phi} \\ \rho e^{-i\phi} & -z \end{pmatrix} \mapsto \begin{pmatrix} e^{z/2} & e^{i\phi} f(r,\rho) \\ 0 & e^{-z/2} \end{pmatrix},$$

where $r = \sqrt{r^2+\rho^2}$ and $f(r,\rho) = \left(e^{r/2} + e^{-r/2} - e^{\rho/2} - e^{-\rho/2}\right)^{1/2}$.

Let us scale the Lie bracket on the Lie algebra $\mathfrak{g}$ by the factor $\tau \in \mathbb{R}_+$ to obtain a new bracket $[\cdot,\cdot]_\tau = \tau[\cdot,\cdot]$ (it is isomorphic to the original bracket for all $\tau \neq 0$). This corresponds to scaling of the Poisson bivector on $G^*, \pi_\tau = \tau\pi_{G^*}$ and to scaling of the Ginzburg-Weinstein map $\gamma_\tau(x) = \gamma(\tau x)$. For large values of τ, the expression for $\gamma_\tau(x)$ simplifies to give

$$\gamma_\tau \begin{pmatrix} z & \rho e^{i\phi} \\ \rho e^{-i\phi} & -z \end{pmatrix} \sim \begin{pmatrix} e^{\tau z/2} & e^{i\phi+\tau r/2} \\ 0 & e^{-\tau z/2} \end{pmatrix}.$$

This phenomenon generalizes to the case of $G = U(n)$. Let $\gamma : \mathrm{u}^*(n) \to U^*(n)$ be the Ginzburg-Weinstein map solving the Flaschka-Ratiu conjecture (intertwining the Gelfand-Zeitlin integrable systems on $\mathrm{u}^*(n)$ and $U^*(n)$, see [12] and [5]). Then, we have the following statement:

Theorem 3.2.

$$\lim_{\tau\to\infty} \frac{1}{\tau} \log |\Delta_i^k(\gamma_\tau(x))| = l_i^k(x).$$

That is, while the Ginzburg-Weinstein map is a rather complicated transcendental map, its asymptotic for large τ is described by the familiar Gelfand-Zeitlin maps $l_i^{(k)}$. Note that for $n \geq 3$ the asymptotic behavior of other naturally defined functions (*e.g.* matrix elements) is significantly more involved.

3.2. A symplectic proof of Horn inequalities. Our second application combines the techniques of the planar network theory and tropicalization with the classical techniques of Symplectic Geometry .

3.2.1. Sums of Hermitian matrices and Horn inequalities. The problem of computing the spectrum of a sum of two Hermitian matrices with given eigenvalues is a classical problem of Linear Algebra. One can formulate it in terms of the set

$$\mathcal{C}_H = \{(r,s,t) \in \mathbb{R}^{3(n+1)};\ \exists\ A, B \in \mathcal{H}_n \text{ such that } l(A) = r, l(B) = s, l(A+B) = t\}.$$

The set $\mathcal{C}_H$ turns out to be a polyhedral cone which belongs to the plane described by the trivial conditions $r_0, s_0, t_0 = 0$ (which result from the definition of the map $l : \mathcal{H}_n \to \mathbb{R}^{n+1}$) and by the equation $t_n = r_n + s_n$ which represents linearity of trace, $\mathrm{Tr}(A+B) = \mathrm{Tr}(A) + \mathrm{Tr}(B)$. Some of the inequalities describing $\mathcal{C}_H$ are rather simple such as

$$r_1 + s_1 \geq t_1$$

expressing the fact that the top eigenvalue of the sum of two Hermitian matrices is bounded by the sum of their top eigenvalues. The complete set of inequalities was conjectured by

Horn [18]. This conjecture was proved by Klachko [21, 22] and by Knutson and Tao [24]. The following elegant description of the set $\mathcal{C}_H$ is due to Knutson-Tao.

Let $l_i^{(k)}$ with $0 \leq i \leq k \leq n$ be entries of a triangular tableau. Consider a polyhedral cone $\mathcal{K}$ contained in the hyperplane $l_0^{(n)} = 0$ and defined by inequalities

$$\begin{array}{lcl} l_i^{(k)} + l_{i-1}^{(k-1)} & \geq & l_{i-1}^{(k)} + l_i^{(k)}, \\ l_i^{(k)} + l_i^{(k-1)} & \geq & l_{i+1}^{(k)} + l_{i-1}^{(k)}, \\ l_i^{(k)} + l_{i-1}^{(k)} & \geq & l_i^{(k+1)} + l_{i-1}^{(k-1)}. \end{array} \tag{3.1}$$

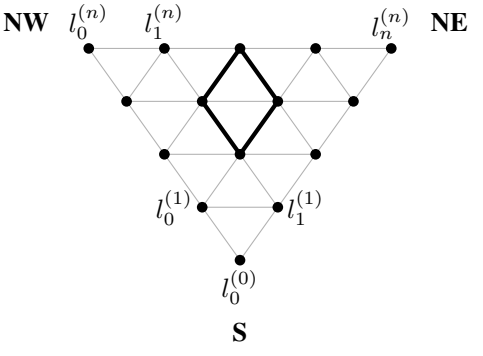


Figure 3.1. Triangular tableau with a South oriented rhombus

Note that the first two inequalities coincide with the interlacing inequalities (2.2). Recall that they correspond to elementary rhombi with NE and NW orientations. The third inequality admits a similar interpretation in term of rhombi oriented to the South (see Figure 3.1).

Define the projection p from triangular tableaux to $\mathbb{R}^{3(n+1)}$ given by

$$r_i = l_i^{(n)}, s_i = l_{n-i}^{(n-i)} - l_n^{(n)}, t_i = l_0^{(n-i)}.$$

These are the labels on the edges of the tableau. They automatically verify the conditions $r_0 = s_0 = t_0 = 0$ and $r_n + s_n = t_n$. Define the Knutson-Tao cone $\mathcal{C}_{KT} = p(\mathcal{K})$ as the image of $\mathcal{K}$ under p. The following theorem is due to Knutson and Tao [24]:

Theorem 3.3.

$$\mathcal{C}_H = \mathcal{C}_{KT}.$$

One interesting question which arises in this context is the interpretation of the labels $l_i^{(k)}$ assigned to the middle nodes of the tableau. We shall return to this issue later in this Section.

Example 3.4. Consider the case of $n = 2$ and assume that the matrices in question are traceless. This implies that $r_2 = s_2 = t_2 = 0$. Then, A and B can be interpreted as vectors in $\mathbb{R}^3$ of lengths r and s. Their sum is the vector $A + B$ of length t. The three lengths satisfy the triangle inequalities

$$r + s \geq t, \quad s + t \geq r, \quad t + r \geq s$$

which can be summarized as

$$|r-s| \leq t \leq r+s.$$

In the case of $n=3$, the tableau has an internal node with label $l_1^{(2)}$ that we denote by x. Two examples of Knutson-Tao inequalities are

$$\begin{array}{lll} l_1^{(3)} + l_0^{(2)} \geq l_1^{(2)} & \Rightarrow & r_1 + t_1 \geq x, \\ l_1^{(2)} + l_2^{(3)} \geq l_1^{(3)} + l_2^{(2)} & \Rightarrow & x + r_2 \geq r_1 + r_3 + s_1. \end{array}$$

Together, they imply one of the Horn inequalities,

$$r_2 + t_1 \geq r_3 + s_1.$$

3.2.2. Planar networks and Horn inequalities. In this Section, we explain the planar network approach to Knutson-Tao and Horn inequalities. The idea is similar to the one which works for interlacing inequalities.

Let Γ_1 and Γ_2 be two planar networks of type n. Denote by $\Gamma_1 * \Gamma_2$ a new planar network obtained by identifying sinks of Γ_1 with sources of Γ_2 (see Figure 3.2). We denote by $l(\Gamma_1, \Gamma_2) : W(\Gamma_1 * \Gamma_2) \to \mathbb{T}^{3(n+1)}$ the map

$$l(\Gamma_1, \Gamma_2) : (w_1, w_2) \to (l\Gamma_1(w_1), l\Gamma_2(w_2), l(\Gamma_1 * \Gamma_2)(w_1 * w_2)).$$

Here w_1 is a weighing on Γ_1, w_2 is a weighting on Γ_2 and $w_1 * w_2$ is the induced weighting on $\Gamma_1 * \Gamma_2$.

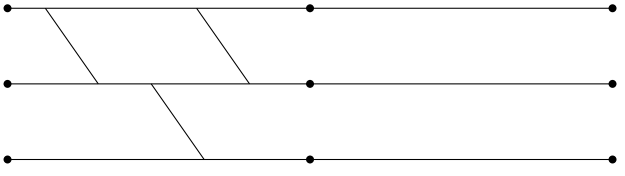


Figure 3.2. The planar network $\Gamma_0 * \Gamma_{\rm hor}$

Recall two examples of planar networks, $\Gamma_{\rm hor}$ and Γ_0, shown on Figure 2.4. The combination of the following two results gives a new proof of the Knutson-Tao Theorem:

Theorem 3.5 ([7]). *For any Γ_1, Γ_2 planar networks of type n, we have*

$$\text{im } l(\Gamma_1, \Gamma_2) \subset \mathcal{C}_{KT}.$$

Moreover, for $\Gamma_1 = \Gamma_0$ and $\Gamma_2 = \Gamma_{\rm hor}$ we have

$$\text{im } l(\Gamma_0, \Gamma_{\rm hor}) = \mathcal{C}_{KT}.$$

Theorem 3.6 ([8]).

$$\text{im } l(\Gamma_0, \Gamma_0) = \mathcal{C}_H.$$

Since the network $\Gamma_{\rm hor} \subset \Gamma_0$ is a subnetwork of Γ_0, we have

$$\mathcal{C}_{KT} = l(\Gamma_0, \Gamma_{\rm hor}) \subset l(\Gamma_0, \Gamma_0) = \mathcal{C}_H.$$

At the same type, Theorem 3.5 implies

$$\mathcal{C}_H = l(\Gamma_0, \Gamma_0) \subset \mathcal{C}_{KT}.$$

Hence $\mathcal{C}_H = \mathcal{C}_{KT}$, as required.

3.2.3. Ideas of proofs. Theorem 3.5 is a statement about combinatorics of paths and weights on planar networks. An important step in the proof is to find the meaning of intermediate weights $l_i^{(k)}$ in the triangular tableau. Recall that $l_i^{(n)} = r_i$ is the maximal weight of an i-path in Γ_1, $l_{n-i}^{(n-i)} - l_n^{(n)} = s_i$ is the maximal weight of an i-path in Γ_2, and $l_0^{(n-i)} = t_i$ is the maximal weight of an i-path in $\Gamma_1 * \Gamma_2$. For $l_i^{(k)}$ with arbitrary $i \leq k$ we assign the following expression:

$$l_i^{(k)} = \max_{\alpha,\beta} \sum_{e \in \alpha \cup \beta} w(e),$$

where α is an $(n-k+i)$-path in Γ_1 and β is an $(n-k)$-path in Γ_2 with the condition that sources of β form a subset of sinks of α. One can also interpret the union $\alpha \cup \beta$ as an $(n-k)$-path in $\Gamma_1 * \Gamma_2$ and an i-path in Γ_1 which do not touch each other.

As in the case of interlacing inequalities, the proof of Knutson-Tao inequalities in the framework of planar networks is achieved by showing that the maximum is taken over a bigger set on one side of the inequality. This can be illustrated by the following example.

Example 3.7. Consider the inequality

$$l_1^{(n-1)} + l_0^{(n)} \geq l_1^{(n)} + l_0^{(n-2)}.$$

The right hand side corresponds to a union of the maximal 1-path in Γ_1 and a maximal 2-path in $\Gamma_1 * \Gamma_2$. Such a configuration can always be reinterpreted as a 2-path in Γ_1 with a 1-path in Γ_2 attached to one of its sinks and a 1-path in $\Gamma_1 * \Gamma_2$, see Figure 3.3. This proves the inequality in question and gives an insight in how the proof works in the general case.

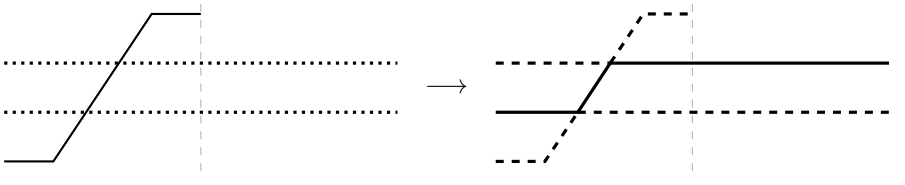


Figure 3.3. A 1-path in Γ_1 and a 2-path in $\Gamma_1 * \Gamma_2$ redrawn as a (2,1)-path and a 1-path in $\Gamma_1 * \Gamma_2$

Theorem 3.6 is of a very different nature. First, recall that for a matrix $A \in \mathrm{Mat}_n(\mathbb{C})$ its singular values are defined as eigenvalues of the positive definite matrix AA^*. Similar to the case of Hermitian matrices, it is convenient to define the maps $l_0^\nabla(A) = 0, l_1^\nabla(A) = \log(\lambda_1(AA^*))$ and in general

$$l_i^\nabla(A) = \log(\lambda_1(AA^*)) + \cdots + \log(\lambda_i(AA^*)).$$

In analogy to the Horn problem, one can introduce the set of solutions of the multiplicative singular value problem:

$$\mathcal{C}_\nabla = \left\{ (r,s,t) \in \mathbb{R}^{3(n+1)} \;\middle|\; \begin{array}{c} \exists\ A, B \in U^*(n) \text{ such that} \\ l^\nabla(A) = r,\ l^\nabla(B) = s,\ l^\nabla(AB) = t \end{array} \right\}.$$

The following result is due to Klyachko [23] (for a symplectic version, see [6]):

Theorem 3.8.

$$\mathcal{C}_\nabla = \mathcal{C}_H.$$

We give a sketch of a proof of Theorem 3.6 in the case of $n = 2$ with the additional simplifying assumption that the Hermitian matrices in question are traceless. We shall first establish an inclusion $\mathrm{im}\, l(\Gamma_0 * \Gamma_0) \subset \mathcal{C}_H$.

Consider a one parameter family of elements of $U^*(2)$ of the form

$$u_\tau = \begin{pmatrix} e^{\tau z/2} & e^{\tau r/2} \\ 0 & e^{-\tau z/2} \end{pmatrix}.$$

Define

$$r(\tau) = \frac{1}{\tau} \log\left(\lambda_1(u_\tau u_\tau^*)\right).$$

It is easy to check that $r(\tau)$ admits a limit when τ tends to infinity and

$$r_\infty = \lim_{\tau\to\infty} r(\tau) = \max(r, |z|).$$

Now consider the product of two matrices of this type

$$\begin{pmatrix} e^{\tau z/2} & e^{\tau r/2} \\ 0 & e^{-\tau z/2} \end{pmatrix} \begin{pmatrix} e^{\tau w/2} & e^{\tau s/2} \\ 0 & e^{-\tau w/2} \end{pmatrix} = \begin{pmatrix} e^{\tau(z+w)/2} & e^{\tau(r-w)/2} + e^{\tau(z+s)/2} \\ 0 & e^{-\tau(z+w)/2} \end{pmatrix}.$$

By the previous calculation, its top singular $\Lambda(\tau)$ behaves in the following way for τ large:

$$t_\infty = \lim_{\tau\to\infty} \frac{1}{\tau} \log(\Lambda(\tau)) = \max(r - w, z + s, |z + w|). \tag{3.2}$$

Note that this is exactly the maximal weight of a 1-path in the planar network $\Gamma_0 * \Gamma_0$ shown on Figure 3.2.3 while r_∞ and s_∞ are weights of the maximal 1-paths in the two copies of Γ_0. Hence, $(r_\infty, s_\infty, t_\infty) \in \mathrm{im}\ (l(\Gamma_0 * \Gamma_0))$. Since the weights on the edges are arbitrary, we obtain the whole image.

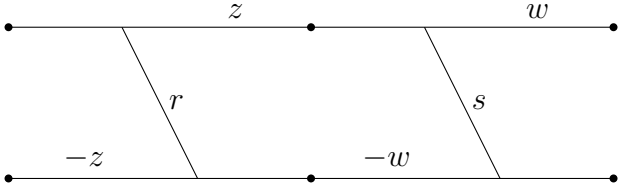


Figure 3.4.

By Klyachko's Theorem 3.8, the triple $(\tau r(\tau), \tau s(\tau), \tau t(\tau)) \in \mathcal{C}_H$ is contained in the Horn cone for all values of τ. Making a rescaling by τ, we see that $(r(\tau), s(\tau), t(\tau)) \in \mathcal{C}_H$. This triple admits a limit $(r_\infty, s_\infty, t_\infty)$ when τ tends to infinity. Since $\mathcal{C}_H$ is a closed cone, we conclude that $(r_\infty, s_\infty, t_\infty) \in \mathcal{C}_H$ and $\mathrm{im}\ (l(\Gamma_0 * \Gamma_0)) \subset \mathcal{C}_H$, as required.

To show the inclusion $\mathcal{C}_H \subset \mathrm{im}(l\Gamma_0 * \Gamma_0)$, we consider products of all possible elements of $U^*(2)$:

$$\begin{pmatrix} e^{\tau z/2} & e^{\tau r/2 + i\phi} \\ 0 & e^{-\tau z/2} \end{pmatrix} \begin{pmatrix} e^{\tau w/2} & e^{\tau s/2 + i\psi} \\ 0 & e^{-\tau w/2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{\tau(z+w)/2} & e^{\tau(r-w)/2+i\phi} + e^{\tau(z+s)/2+i\psi} \\ 0 & e^{-\tau(z+w)/2} \end{pmatrix}.$$

Now the formula (3.2) for t_∞ is only valid away from the locus $r - w = z + s$ where the cancelation between the two exponents is possible. Choose ϵ sufficiently small and consider the set

$$X_\tau = \{(z, w) \in [-r, r] \times [-s, s];\ |\tau(r - w - z - s)| \leq \epsilon\}.$$

Recall that the push-forward of the Liouville measure (2.3) is the Lebesque measure on the rectangle $[-r, r] \times [-s, s]$. The volume of X_τ decreases linearly with τ.

Assume that $\mathrm{im}\ (l(\Gamma_0 * \Gamma_0))$ is strictly smaller than $\mathcal{C}_H$. Then, one can find r and s such that $\mathrm{im}\,(t_\infty)$ is strictly smaller than the segment $[|r - s|, r + s]$. By the previous considerations, the induced measure on the complement of $\mathrm{im}\,(t_\infty)$ must vanish (since we can always increase τ to make the measure of the set X_τ sufficiently small). But this contradicts the Duistermaat-Heckman Theorem stating that the image of the Liouville measure under the moment map is piece-wise polynomial with respect to the Lebesgue measure and non vanishing on the interior of the image.

3.3. Open problems. In this Section, we briefly describe several open problems.

3.3.1. Inequalities and the cluster structure on $U^*(n)$. In Section 2.3.2, we explained that the tropicalization of the canonical Poisson bracket on the dual Poisson-Lie group $U^*(n)$ is described by the Gelfand-Zeitlin integrable system. This result makes use of the coordinate chart $\{\Delta_i^{(k)}\}$ formed by special solid minors on $U^*(n)$. In general, tropicalization may depend on the choice of coordinates. However, there is a set of distinguished coordinate charts defining a cluster structure on $U^*(n)$ (and, in general, on the set of invertible upper triangular matrices). Some of these charts are labeled by the reduced expressions for the longest word in the Weyl group $W = S_n$. The coordinates $\{\Delta_i^{(k)}\}$ correspond to the reduced expression

$$w_0 = s_{n-1}s_{n-2}\dots s_1 s_{n-1}\dots s_2\dots s_{n-1}s_{n-2}s_{n-1}. \tag{3.3}$$

It is natural to expect that tropicalization making use of other cluster charts gives rise to cones isomorphic to the Gelfand-Zeitlin cone, and to integrable systems isomorphic to the Gelfand-Zeitlin integrable system.

Example 3.9. A simple example where this question can be settled in the positive is the comparison of the reduced expression (3.3) with the expression

$$s_1 s_2 \dots s_{n-1} s_1 \dots s_{n-2} \dots s_1 s_2 s_1.$$

The reduced expression above corresponds to the Gelfand-Zeitlin integrable system with the upper left corners replaced by the down right corners as principle submatrices.

3.3.2. The Gelfand-Zeitlin integrable system for $Sp(2n)$. Guillemin and Sternberg described Gelfand-Zeitlin integrable system for the groups $G = U(n)$ and $G = SO(n)$. Since then, several attempts were made to fill the gap for the remaining classical series $Sp(2n)$. Some partial successes were achieved using the inspiration from Representation Theory [28] and the techniques from the theory of complete integrability [17]. Symplectic Gelfand-Zeitlin tableaux were studied in [20] and [29]. However, to the best of our knowledge there

is no construction producing action-angle variables and the cone in the same way as it works for $U(n)$ and $SO(n)$.

Our technique gives a new approach to this challenging problem. Indeed, the cluster coordinates on the Borel subgroup of $Sp(2n,\mathbb{C})$ are again defined by the reduced expressions in the longest word of the corresponding Weyl group. These coordinate charts admit a tropical limit described by planar networks (with extra symmetries). An important step towards solving the problem would be an appropriate analogues of Theorem 3.2 and of Theorem 2.18.

3.3.3. Integrable systems on symplectic multiplicity spaces. Symplectic multiplicity spaces are defined as Hamiltonian reductions of products of coadjoint orbits. Let $G=U(n)$ and consider a triple $\mathcal{O}_\lambda,\mathcal{O}_\mu,\mathcal{O}_\nu$ of sympelctic leaves in $\mathfrak{u}^*(n)\cong\mathcal{H}_n$. The multiplicity space

$$M_{\lambda,\mu,\nu}=(\mathcal{O}_\lambda\times\mathcal{O}_\mu\times\mathcal{O}_\nu)//G$$

is a sympletic space which might have orbifold or more serious singularities. While the symplectic leaves $\mathcal{O}_\lambda$ are equipped with Gelfand-Zeitlin completely integrable systems, the question of constructing action-angle variables on $M_{\lambda,\mu,\nu}$ remains open. In the tropical limit, a possible choice for action variables is given by functions $l_i^{(k)}$ corresponding to intermediate nodes of the triangular tableau. They are defined as maximal weights of multi-paths in a pair of planar networks. It would be very interesting to find a counterpart of this construction in terms of Hermitian or upper triangular matrices. Note that the tropical construction automatically produces a set of functions in involution under the Poisson structure π_∞. Preserving this property would be crucial when one passes from planar networks to the theory of matrices.

3.3.4. The multiplicative problem for the group $SU(n)$. Recall that conjugacy classes of unitary matrices are parametrized by the Weyl alcove Δ_W. Denote by $\sigma:SU(n)\to\Delta_W$ the natural projection.

Example 3.10. For $n=2$, we have $\Delta_W=[0,\pi]$ where a matrix $u\in SU(2)$ is conjugate to a unique diagonal matrix of the form $\mathrm{diag}(e^{i\theta},e^{-i\theta})$, where $\theta=\sigma(u)$.

Define the set

$$S_{SU(n)}=\{(r,s,t)\in\Delta_W^{\times 3};\ \exists\ u,v\in SU(n)\text{ such that }\sigma(u)=r,\sigma(v)=s,\sigma(uv)=t\}.$$

This set is a multiplicative counterpart of the Horn cone. It admits a description in terms of quantum cohomology [1], and it turns out that $S_{SU(n)}$ is polytope contained in $\Delta_W^{\times 3}$. For instance, in the case of $n=2$ the corresponding inequalities take the form

$$|r-s|\le t\le\max(r+s,2\pi-r-s).$$

For r,s much smaller than 2π they reproduce the triangle inequalities of the Horn problem but they deviate from this pattern for bigger values of r,s.

It would be very interesting to find a combinatorial model in terms of planar networks (or similar combinatorial machinery) giving rise to the defining inequalities of the polytope $S_{SU(n)}$.

Acknowledgements. My work was supported in part by the Swiss National Science Foundation through the grants 200021–152812 and 200020–141329 and by the European Re-

search Council through the grant MODFLAT.

I am indebted to L. Faddev and S. Shatashvili who initiated me to Gelfand-Zeitlin systems. I am grateful to I. Davydenkova, E. Meinrenken, M. Podkopaeva, A. Szenes, C. Woodward for a number of fruitful collaborations. I greatly appreciated the discussions with V. Fock, S. Fomin, V. Guillemin, A. Knutson, J.-H. Lu, S. Sternberg, A. Weinstein, J. Weitsmann, A. Zelevinsky who generously shared with me their knowledge. Special thanks to M. Podkopaeva who helped to bring the text in the present form.

References

[1] Agnihotri, S. and Woodward, C., *Eigenvalues of products of unitary matrices and quantum Schubert calculus*, Math. Res. Lett. **5** (1998), no. 6, 817–836.

[2] Alekseev, A., *On Poisson actions of compact Lie groups on symplectic manifolds*, J. Differential Geom. **45** (1997), no. 2, 241–256.

[3] Alekseev, A. and Davydenkova, I., *Inequalities from Poisson brackets*, in the thesis of I. Davydenkova, University of Geneva (2014), available at http://archive-ouverte.unige.ch/unige:34471.

[4] Alekseev, A., Faddeev, L., and Shatashvili, S., *Quantization of symplectic orbits of compact Lie groups by means of the functional integral*, J. Geom. Phys. **5** (1988), no. 3, 391–406.

[5] Alekseev, A. and Meinrenken, E., *Ginzburg-Weinstein via Gelfand-Zeitlin*, J. Differential Geom. **76** (2007), no. 1, 1–34.

[6] Alekseev, A., Meinrenken, E., and Woodward, C., *Linearization of Poisson actions and singular values of matrix products*, Ann. Inst. Fourier (Grenoble) **51** (2001), no. 6, 1691–1717.

[7] Alekseev, A., Podkopaeva, M., and Szenes, A., *The Horn problem and planar networks*, preprint arXiv:1207.0640.

[8] ———, *A symplectic proof of Horn inequalities*, in preparation.

[9] Boalch, P., *Stokes matrices, Poisson Lie groups and Frobenius manifolds*, Invent. Math. **146** (2001), no. 3, 479–506.

[10] Drinfeld, V. G., *Quantum groups*, Proceedings of the International Congress of Mathematicians, Vol. **1, 2** (Berkeley, Calif., 1986), 798–820, Amer. Math. Soc., Providence, RI, 1987.

[11] Enriquez, B., Etingof, and P., Marshall, I., *Comparison of Poisson structures and Poisson-Lie dynamical r-matrices*, Int. Math. Res. Not. (2005), no. 36, 2183–2198.

[12] Flaschka, H. and Ratiu T., *A convexity theorem for Poisson actions of compact Lie groups*, Ann. Sci. École Nor. Sup. (4) **29** (1996), no. 6, 787–809.

[13] Fomin, S. and Zelevinsky, A., *Double Bruhat cells and total positivity*, J. Amer. Math. Soc. **12** (1999), no. 2, 335–380.

[14] Guillemin, V. and Sternberg S., *The Gel'fand-Cetlin system and quantization of complex flag manifolds*, J. Funct. Anal. **52** (1983), no. 1, 106–128.

[15] Ginzburg, V.L. and Weinstein, A., *Lie-Poisson structure on some Poisson Lie groups*, J. Amer. Math. Soc. **5** (1992), no. 2, 445–453.

[16] Gel'fand, I.M. and Cetlin, M.L., *Finite-dimensional representations of the group of unimodular matrices*, Doklady Akad. Nauk SSSR (N.S.) **71** (1950), 825–828.

[17] Harada, M., *The symplectic geometry of the Gel?fand-Cetlin-Molev basis for representations of* $Sp(2n, \mathbb{C})$, J. Symplectic Geom. **4** (2006), no. 1, 1–41.

[18] Horn, A., *Eigenvalues of sums of Hermitian matrices*, Pacific J. Math. **12** (1962), 225–241.

[19] Kapovich, M., Leeb, B., and Millson, J., *The generalized triangle inequalities in symmetric spaces and buildings with applications to algebra*, Mem. Amer. Math. Soc. **192** (2008), no. 896.

[20] Kirillov, A.A., *A remark on the Gel'fand-Tsetlin pattern for symplectic groups*, J. Geom. Phys. **5** (1988) no. 3, 473–482.

[21] Klyachko, A., *Stable bumdles, representation theory and Hermitian operators*, Selecta Math. (N.S.) **4** (1998), 419–445.

[22] ———, *Vector bundles, linear representations, and spectral problems*, Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), 599–613, Higher Ed. Press, Beijing, 2002.

[23] ———, *Random walks on symmetric spaces and inequalities for matrix spectra. Special Issue: Workshop on Geometric and Combinatorial Methods in the Hermitian Sum Spectral Problem (Coimbra, 1999)*, Linear Algebra Appl. **319** (2000), no. 1-3, 37–59.

[24] Knutson, A. and Tao, T., *The honeycomb model of* $GL_n(\mathbb{C})$ *tensor products. I*, Proof of the saturation conjecture. J. Amer. Math. Soc. **12** (1999), no. 4, 1055–1090.

[25] Kogan, M. and Zelevinsky, A., *On symplectic leaves and integrable systems in standard complex semisimple Poisson-Lie groups*, Int. Math. Res. Not. 2002, no. 32, 1685–1702.

[26] Kostant, B. and Wallach, N., *Gelfand-Zeitlin theory from the perspective of classical mechanics. I*, Studies in Lie theory, 319–364, Progr. Math. **243**, Birkhäuser Boston, MA, 2006.

[27] Lu, J.-H. and Weinstein, A., *Groupoïdes symplectiques doubles des groupes de Lie-Poisson*, C. R. Acad. Sci. Paris Sér. I Math. **309** (1989), no. 18, 951–954.

[28] Molev, A., *A basis for representations of symplectic Lie algebras*, Comm. Math. Phys. **201** (1999) no.3, 591–618.

[29] Proctor, R., *Odd symplectic groups*, Invent. Math. **92** (1988) no. 2, 307–332.

[30] Semenov-Tian-Shansky, M., *Dressing Transformations and Poisson Group Actions*, Publ. RIMS Kyoto Univ. **21** (1985), 1237–1260.

[31] Shatashvili, S., *Correlation functions in the Itzykson-Zuber model*, Commun. Math. Phys. **154** (1993), 421–432.

Section de mathématiques, Université de Genève, 2-4 rue du Lièvre, c.p. 64, 1211 Genève 4, Suisse
E-mail: Anton.Alekseev@unige.ch

Macdonald processes, quantum integrable systems and the Kardar-Parisi-Zhang universality class

Ivan Corwin

Abstract. Integrable probability has emerged as an active area of research at the interface of probability/mathematical physics/statistical mechanics on the one hand, and representation theory/integrable systems on the other. Informally, integrable probabilistic systems have two properties: (1) It is possible to write down concise and exact formulas for expectations of a variety of interesting observables (or functions) of the system. (2) Asymptotics of the system and associated exact formulas provide access to exact descriptions of the properties and statistics of large universality classes and universal scaling limits for disordered systems. We focus here on examples of integrable probabilistic systems related to the Kardar-Parisi-Zhang (KPZ) universality class and explain how their integrability stems from connections with symmetric function theory and quantum integrable systems.

Mathematics Subject Classification (2010). 82C22, 82B23, 60H15, 33D52.

Keywords. KPZ, symmetric polynomials, quantum integrable systems, Bethe ansatz, TASEP.

1. Integrable probabilistic systems in the KPZ class

A primary aim of statistical mechanics and probability theory is to describe aggregate behavior of disordered microscopic systems driven by noise. Many systems include self-averaging mechanisms which result in the appearance of deterministic (law of large number) behavior on macroscopic scales. A central problem is to characterize the behavior of such systems between microscopic disorder and macroscopic order. On critical mesoscopic scales, large classes of systems seem to share universal fluctuation behaviors. This belief in "universality classes" is bolstered by (non-rigorous) physical arguments, extensive numerics, some experimental results and, recently, a growing body of mathematical proof coming from the field of integrable probability. "Integrable" or "exactly solvable" models play a key role in probing the nature and extent of universality classes. Due to enhanced algebraic structure they are often amenable to detailed analysis, thus providing the most complete access to various phenomena such as phase transition, scaling exponents, and fluctuation statistics.

The success of integrable probability in describing universal behaviors is quite strking for the non-equilibrium statistical mechanics problem of describing random interface growth. In this section we provide examples of integrable probabilistic systems whose analysis deepens our understanding of random $(1+1)$-dimensional random interface growth and the Kardar-Parisi-Zhang (KPZ) universality class. Through these examples we also demonstrate connections to interacting particle systems (models for traffic flow, queuing, mass transport,

Proceedings of the International Congress of Mathematicians, Seoul, 2014

driven gases, and shock-fronts), directed polymers in random media (models for competition interfaces, domain walls, and cracking interfaces), and parabolic Anderson models (models for population growth with migration). See the review [36] for further background and references.

The study of KPZ universality was initiated by Kardar-Parisi-Zhang [62] in 1986 and drew heavily on earlier work of Forster-Nelson-Stephens [50] in 1977. The ensuing decade of physical theories, numerics, and experiments produced strong physical evidence for universality of random interface fluctuations in their long-time and large-scale limits. The fluctuation scaling exponent and transversal correlation length was predicted to be $1/3$ and $2/3$ (meaning fluctuations of order $t^{1/3}$ correlated over distances $t^{2/3}$, with t measuring time). Certain statistics (e.g. skewness, kurtosis, tail decay) were also computed numerically during this period.

The involvement of mathematicians and the first (mathematically) rigorous results and exact formulas for fluctuation statistics came in 1999 with the work of Baik-Deift-Johansson [5] and Johansson [60]. Our first example below details some of Johansson's [60] asymptotic analysis results on TASEP. The methods of determinant point process or Schur measure / process (of which TASEP is a special limiting case) have driven many further advances in understanding KPZ class statistics (see [15, 27, 29, 80]). All those models analyzed by these determinantal methods are "totally asymmetric" or "zero temperature".

The first analysis (to the point of asymptotic statistics) of a non-determinantal KPZ class model was performed by Tracy and Widom [94–96] in 2009. Since then, a variety of methods have been developed to discover and analyze non-determinantal "partially asymmetric" or "positive temperature" models.

In this paper we describe some facets of the exact solvability of q-TASEP, the O'Connell-Yor semi-discrete directed polymer, ASEP, and the KPZ equation. We then develop two methods used in studying these examples: (1) the theory Macdonald processes, which is an algebraic framework for discovering and analyzing a variety of probabilistic system by leveraging the remarkable properties of Macdonald symmetric polynomials; (2) the theory of quantum integrable systems, which is based on the (coordinate / algebraic) Bethe ansatz and provides a means to diagonalize certain Hamiltonians, including some stochastic generators related to the processes with which we are concern. We develop both of these methods at a high combinatorial (or algebraic) level and thus avoid many of the analytic issues and demystify the apparent algebraic miracles which arise in various degenerations. For instance, our treatment of q-TASEP in Section 3 can be considered a mathematically rigorous version of the replica method for directed polymers [32, 45, 63].

There are many other exciting recent developments related to the KPZ which we will not discuss at any length. To name a few, these include: tropical combinatorics and directed polymers [40, 73, 74], line ensembles [38, 39, 77], coupling methods and second class particles [6, 7, 88], spectral methods [71, 72], experiments confirming KPZ statistics [92, 99], KPZ equation well-posedness [58, 59].

1.1. Example 1: TASEP. The totally asymmetric simple exclusion process (TASEP) is an interacting particle system on $\mathbb{Z}$. Particles inhabit sites of $\mathbb{Z}$ with only one particle per site at any given time. In continuous time, particles attempt to orchestrate independent random jumps according to rate one exponential clocks (in other words, according to exponential distributed waiting times of rate one) by one site to the right. If the destination site is occupied, the jump is suppressed. This process may be described in terms of occupation variables

which track of whether sites of $\mathbb{Z}$ are occupied, or particle location variables which tracks the location of indexed particles. We will, instead, appeal to a "height function" to describe this process. The TASEP height function is a piece-wise linear function made up of unit $+1$ or -1 slope line increments. Above every site of $\mathbb{Z}$ with a particle, there is a -1 slope and above every site without a particle there is a $+1$ sloped. The height function $h^{\mathrm{TASEP}}(t,x)$ pastes these increments together into a continuous function (uniquely defined up to an overall height shift). TASEP dynamics corresponds to replacing local minima $\vee$ by maxima $\wedge$ according to rate one exponential clocks. In this language of height functions, TASEP is equivalent to the *corner growth model*.

Johansson [60] computed an exact formula for the one-point distribution of the TASEP height function, when initialized from "step" or "wedge" initial data. This initial data corresponds to starting with every site to the left of the origin occupied, and all other sites empty. In terms of the height function this corresponds to starting with $h^{\mathrm{TASEP}}(0,x) = |x|$.

1.1.1. KPZ class asymptotics. Studying the long-time, large-scale fluctuation behavior of this height function revealed the first exact formulas for statistics of the KPZ universality class. The prediction coming from [50, 62] was that in large time L, the height function should be non-trivially correlated on a scale of order $L^{2/3}$ with fluctuations of order $L^{1/3}$. Define the scaled height function $h_L^{\mathrm{TASEP}}(t,x) := L^{-1/3}\big(h^{\mathrm{TASEP}}(Lt, L^{2/3}x) - \frac{Lt}{2}\big)$, where L is a large scaling parameter and the centering by $Lt/2$ follows from the hydrodynamic theory for TASEP. Johansson [60] showed the following:

Theorem 1.1. *For TASEP with step initial data* $\lim_{L\to\infty} \mathbb{P}\big(h_L^{\mathrm{TASEP}}(1,0) \geq -s\big) = F_{\mathrm{GUE}}(s)$ *where* $F_{\mathrm{GUE}}(s)$ *is the Tracy-Widom limit distribution [93] for the largest eigenvalue of a large Hermitian random matrix.*

This result provides an exact prediction for the limiting one-point behavior of a wide class of models which share general characteristics with TASEP (and which are started from step type initial data). In many ways, this F_{GUE} distribution is to $(1+1)$-dimensional random growth as the Gaussian distribution is to random walks. Asymptotic analysis of the remaining examples we discuss yield the same scaling and distributional limit, thus providing further evidence for KPZ universality.

Using methods of determinantal point processes, further exact statistics describing the KPZ class has been extracted through studying the large L limit of $h_L^{\mathrm{TASEP}}(t,x)$ (e.g. the Airy processes describing the fixed t and varying x limit for step and a few other types of initial data, see [36]). The connection between random matrix theory and growth processes will be alluded to further in our discussion of Macdonald processes in Section 2. TASEP is one of a handful of examples of particle systems and growth models which are analyzable via determinantal point processes (or equivalently Schur processes, free Fermions, non-intersecting paths). These other examples are discrete time TASEPs with sequential or parallel update rules, pushASEP or long range TASEP, directed last passage percolation in two dimension with geometric, exponential or Bernoulli weights, and the polynuclear growth process – see [15, 29] and references therein.

The examples which we address here are deformations (and limits of deformations) of TASEP. These examples are no longer determinantal, though there still turn out to be large families of observables whose averages are explicit (for determinantal systems correlation functions are written explicitly as determinants). Integrability is quite sensitive to perturbations and while these deformations are integrable, there are many simple models, closely

related to TASEP, which are not.

1.2. Example 2: q-TASEP. The q-deformed totally asymmetric simple exclusion process (q-TASEP) is a one parameter deformation of TASEP which was discovered and first studied in the context of Macdonald processes [17] (see also subsequent work [18, 20, 23, 26, 35, 48, 66, 76]). Fix $q \in (0,1)$ and let $x_i(t) \in \mathbb{Z}$ be the location of particle i at time t. We assume that $x_j(t) < x_i(t)$ for $j > i$ and that there is a right-most particle which we label $x_1(t)$ (for notational convenience fix $x_0(t) \equiv +\infty$). In continuous time, each particle x_i attempts to jump one site to the right according to an exponential clock of rate $1-q^{x_{i-1}(t)-x_i(t)-1}$. Here $x_{i-1}(t)-x_i(t)-1$ is the number of empty sites between particle i and the next-right particle $i-1$. This jump rate interpolates between rate zero when the gap is zero and rate one when the gap tends to infinity. This can be thought of as a traffic model in which cars (particles) slow down as they approach the car in front of them. A value of q near one represents a road with cautious drivers. When q goes to zero TASEP is recovered and cars move without caution, only yielding immediately before an accident (when two particles would occupy the same site). Note that x_1 always jumps to the right at rate one since the distance to $x_0 \equiv +\infty$ is infinite.

1.2.1. Moment formulas. The integrability of q-TASEP is partially captured in the following theorem, initially proved when all $n_i \equiv n$ in [17], and then for general n_i in [22, 26]. Sections 2 and 3 describe the two methods for proving this theorem.

Theorem 1.2. *For q-TASEP with step initial data ($x_n(0) = -n$, $n \geq 1$) and any $k \geq 1$, $n_1 \geq n_2 \geq \cdots \geq n_k \geq 1$,*

$$\mathbb{E}\Big[\prod_{j=1}^{k} q^{x_{n_j}(t)+n_j}\Big] = \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2\pi \mathrm{i})^k} \oint \cdots \oint \prod_{1\leq A<B\leq k} \frac{z_A - z_B}{z_A - q z_B} \prod_{j=1}^{k} \frac{e^{(q-1)tz_j}}{(1-z_j)^{n_j}} \frac{dz_j}{z_j}, \tag{1.1}$$

where, for each $A \in \{1,\ldots,k\}$ the contour of integration of z_A contains 1, as well as q times the contour of integration of z_B for $B > A$, but does not contain 0.

Step initial data means $x_n(t) + n = 0$ for all $n \geq 1$, and since particles only move to the right this implies that the random variables $q^{x_n(t)+n}$ are in $(0,1]$ for all $t \geq 0$. The knowledge of all joint moments uniquely identifies the joint distributions of this collection of random variables, and hence that of all $x_n(t)$ for fixed t and varying n. The challenge is to extract exact distributional formulas from the results of Theorem 1.2 in such that they are amenable to asymptotic analysis. So far, this has only been successfully implemented for the one-point distribution (i.e. distribution of $x_n(t)$ for a fixed n and t), as we now describe.

1.2.2. Fredholm determinant. Specializing Theorem 1.2 with all $n_k \equiv n \geq 1$ yields a nested integral formula for $\mathbb{E}\big[q^{k(x_n(t)+n)}\big]$. Such nesting become cumbersome as k grows (we will need to utilize these formulas for all $k \geq 1$), so it is natural to deform our formulas so the contours remain fixed as k varies. Such contour deformations can be made, though they necessarily involve deforming through poles. By keeping track of the residues from crossing these poles, the complexity of the nested contours is transferred into complexity of the integrand.

There are two ways to "un-nest" the contours. One way is to deform them sequentially (z_1 through z_k) to lie upon a large contour containing both 0 and 1. This deformation

crosses the simple pole of the integrand at $z_j = 0$ for all j. The other way is to deform them sequentially (z_k through z_1) to lie upon a small contour containing only 1. This deformation crosses the simple poles of the integrand coming from the denominator $z_A - qz_B$. Both deformations ultimately yield formulas for the distribution of $x_n(t)$. Though the second deformation is slightly more involved, it also yields a formula which ends up being more readily amenable to asymptotic analysis. We do not detail these residue considerations as they are explained at length in [17, 23, 26]. Consider the moment generating function $g(\zeta) = \sum_{k=0}^{\infty} \mathbb{E}\big[q^{k(x_n(t)+n)}\big] \frac{\zeta^k}{(1-q)\cdots(1-q^k)}$. For $|\zeta|$ small this is convergent. Using the second deformation described above this generating function is rewritten as the Fredholm determinant expansion

$$g(\zeta) = \det\big(I + K\big)_{L^2(C_1)} := 1 + \sum_{L=1}^{\infty} \frac{1}{L!} \oint_{C_1} \frac{dw_1}{2\pi\mathrm{i}} \cdots \oint_{C_1} \frac{dw_1}{2\pi\mathrm{i}} \det\big(K(w_i, w_j)\big)_{i,j=1}^{L}$$

where C_1 is a small circle around 1, and

$$\begin{aligned} K(w, w') &= \sum_{n=1}^{\infty} \frac{f(w)\cdots f(q^{n-1}w)}{q^n w - w'} \\ &= \int_{1/2-\mathrm{i}\infty}^{1/2+\mathrm{i}\infty} \frac{\pi}{\sin(-\pi s)} \frac{(w;q)_\infty^n}{(q^s w;q)_\infty^n} \frac{(-\zeta)^s e^{-(1-q^s)tw}}{q^s w - w'} \frac{ds}{2\pi\mathrm{i}}. \end{aligned} \tag{1.2}$$

Here, $f(w) = e^{(q-1)tw}(1-w)^{-n}$ and in the second equality the summation is replaced by a "Mellin-Barnes" contour integral. This replacement is important for studying asymptotics (such as those which yield Theorem 1.5). Though the summation formula for K becomes highly oscillatory (with no termwise limit), the integral involves a contour in which the integrand has clear and well-controlled asymptotic behavior.

Owing to the fact that $q^{x_n(t)+n} \in (0, 1]$, for $|\zeta|$ small enough we may interchange the expectation and the infinite summation over k and an application of the q-Binomial theorem yields:

Theorem 1.3. *For q-TASEP with step initial data and any $n \geq 1$, $t \geq 0$, $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$ $\mathbb{E}\big[\frac{1}{(\zeta q^{x_n(t)+n};q)_\infty}\big] = \det\big(I + K\big)_{L^2(C_1)}$, where $(a;q)_\infty := (1-a)(1-qa)(1-q^2a)\cdots$, the operator K is given by (1.2) and the contour C_1 is a small circle around* 1.

The expression on the left is known as the e_q-Laplace transform of $q^{x_n(t)+n}$ and dates back to 1949 work of Hahn [57]. Like the Laplace transform of a positive random variable, this e_q-Laplace transform can be inverted to compute the distribution of the random variable [17, Proposition 3.1.1].

1.2.3. KPZ class asymptotics. Asymptotic analysis of this formula yields a generalization of Theorem 1.1 (which corresponds with $q = 0$). This was performed by Ferrari-Vető [48] and Barraquand [8] who showed that for any $c > 0$, and suitable $c' = c'(c, q) > 0$ and $c'' = c''(c, q) > 0$, as $L \to \infty$, $c''L^{-1/3}\big(x_{cL}(L) - c'L\big)$ converges in distribution to F_{GUE} (just as in the case of TASEP). This demonstrates how the $L^{1/3}$ scaling and F_{GUE} limit theorem is not unique to TASEP, but rather extends to the whole family of q-TASEPs.

1.3. Example 3: O'Connell-Yor semi-discrete random polymer. A $q \to 1$ limit of q-TASEP leads to a systems of SDEs which can be thought of as a continuous space interacting particle systems. Exponentiating this system yields a semi-discrete version of the stochastic heat equation which is a special case of the parabolic Anderson model and also describes the evolution of the partition function for the semi-discrete random polymer model introduced by O'Connell-Yor [78].

1.3.1. Limit of q-TASEP as $q \to 1$. Recall that $q \in (0,1)$ controls the length scale on which particles in q-TASEP tend to separate. Let $q = e^{-\epsilon}$ with $\epsilon > 0$ a scaling parameter which will tend to zero. The behavior of $x_1(t)$ is quite simple. Since we have fixed $x_0(t) \equiv +\infty$, $x_1(t)$ orchestrates a simple Poisson jump process in which it increases its value by one according to an exponential rate one clock. Thus (regardless of q), the central limit theorem implies that $\epsilon\big(x_1(\epsilon^{-2}\tau) - \epsilon^{-2}\tau\big) \to B_1(\tau)$ where B_1 is a standard Brownian motion.

The behavior of $x_n(t)$ for $n > 1$ requires further consideration, and different scaling. Under the scaling $t = \epsilon^{-2}\tau$, and $x_n(t) = \epsilon^{-2}\tau - (n-1)\epsilon^{-1}\log\epsilon^{-1} - \epsilon^{-1}F_n^\epsilon(\tau)$ it is shown in [17, Theorem 4.1.26] (see also [26, Proposition 6.2]) that $\big\{F_n^\epsilon(\cdot)\big\}_{n\geq 1}$ converges to $\{F_n\}_{n\geq 1}$ which solves the systems of SDEs $dF_n(\tau) = e^{F_{n-1}(\tau)-F_n(\tau)}d\tau + dB_n(\tau)$ for independent Brownian motions $\{B_n\}_{n\geq 1}$ (with the convention that $F_0(\tau) \equiv -\infty$). Indeed, once x_n and x_{n-1} are separated by roughly $\epsilon^{-1}\log\epsilon^{-1}$, x_n jumps ahead at rate $1 - q^{x_n(\tau)-x_{n-1}(\tau)-1} \approx 1 - \epsilon e^{F^\epsilon_{n-1}(\tau)-F^\epsilon_n(\tau)}$. In time of order ϵ^{-2}, this ϵ correction to the jump rate only affects the overall drift, thus yielding the claimed SDEs. The initial data for this system corresponding to step initial data for q-TASEP can either be described via an entrance law, or through an exponential transform (as now done).

Define semi-discrete stochastic heat equation (SHE) with multiplicative noise as the system of SDEs $dz(\tau,n) = \nabla z(\tau,n)dt + z(\tau,n)dB_n(\tau)$ with $(\nabla f)(n) = f(n-1) - f(n)$, and independent Brownian motions $\{B_n\}_{n\geq 1}$ (with the convention that $z(\tau,0) \equiv 0$). By Itô's lemma, $z(\tau,n) = e^{-\frac{3}{2}\tau + F_{n(\tau)}}$. The semi-discrete SHE initial data which comes from step initial data for q-TASEP is $z(\tau,n) = \mathbf{1}_{n=1}$, i.e. the *fundamental solution*.

1.3.2. Parabolic Anderson model. The semi-discrete SHE in above arises in a simple model for population growth and migration in a random environment. Consider an ensemble of unit mass particles in $\mathbb{Z}$ that evolve according to the following rules: At each time $\tau \geq 0$, and location $n \in \mathbb{Z}$, each resident particle (1) splits into two identical unit mass particles, at exponential rate $r_+(\tau,n)$; (2) dies at an exponential rate $r_-(\tau,n)$; or (3) jumps to the right by one at an exponential rate 1.

The functions r_+ and r_- represent an environment in which the particles of this system evolve. Individual particles do not feel each other (and many can occupy the same site), as the exponential clocks controlling their splits, deaths and jumps are independent. A variant of the Feynman-Kac representation implies that the expected total mass $z(\tau,n)$ satisfies $\frac{d}{d\tau}z(\tau,n) = \nabla z(\tau,n) + z(\tau,n)\big(r_+(\tau,n) - r_-(\tau,n)\big)$. We have used z here since if the media is rapidly mixing in time and space, the environment $r_+(\tau,n) - r_-(\tau,n)$ may be modeled by independent white-noises $dB_n(\tau)$, in which case the above equation becomes the semi-discrete SHE. The fundamental solution corresponds to starting a cluster of particles at location 1, and nowhere else.

This population model is called a parabolic Anderson model and has been extensively studied within probability literature [33] (see also [19, 46, 53] and references therein). Since

the population will generally grow/die exponentially, it is natural to study $\log z(\tau, n)$. The spikes in this function record population explosions and can be studied in terms of the phenomenon called *intermittency*, while the typical fluctuations correspond to a semi-discrete variant of the KPZ equation (or a continuous space interacting particle system in which Brownian motions interact in an exponential potential with the next lowest index Brownian motion). We first investigate the atypical behavior of $\log z(\tau, n)$, and then the typical.

1.3.3. Intermittency and Lyapunov exponents. Systems with intermittency display large spikes distributed in time, space and magnitude in a certain multi-fractal manner. In the 1980's it was argued that such a phenomena arises in magnetic fields in turbulent flows, like those on the surface of the Sun [91]. The idea of intermittency seems to date back at least to [10, 64] (see [12, 19, 53] and references therein for more recent developments).

The mathematical definition of intermittency given in [33] captures a portion of this phenomenon (though not the full multi-fractal space-time structure). The p-th moment Lyapunov exponent γ_p ($p \geq 1$) and almost sure Lyapunov exponent $\tilde{\gamma}_1$ are defined as $\gamma_p(\nu) := \lim_{\tau\to\infty} \frac{1}{\tau} \log \mathbb{E}\left[z(\tau, \nu\tau)^p\right]$, and $\tilde{\gamma}_1(\nu) := \lim_{\tau\to\infty} \frac{1}{\tau} \log z(\tau, \nu\tau)$ where $\nu > 0$ determines the ratio of n/τ. We say that $z(\tau, n)$ displays *intermittency* if $\tilde{\gamma}_1 < \gamma_1 < \frac{\gamma_2}{2} < \frac{\gamma_3}{3} < \cdots$. Such an ordering has a clear interpretation. That $\tilde{\gamma}_1 < \gamma_1$ implies that the first moment of $z(\tau, n)$ is not determined by the typical behavior of $\log z(\tau, n)$, but rather by its uncommonly high peaks. In general, the growth of these moments reflects the fact that $\log z(\tau, n)$ has a sufficiently heavy upper tail so that moments are dominated by higher and higher peaks, of smaller and smaller probabilities. At a typical location n, these high peaks will not appear, however, over a wide range it is likely to see quite large peaks.

Under the scaling described in Section 1.3.1 the formulas Theorem 1.2 gave for $\mathbb{E}\left[q^{k(x_n(t)+n)}\right]$ converge to formulas for $\mathbb{E}[z(\tau, n)^p]$ (as shown in [19, Theorem 1.8]).

Proposition 1.4. *The moment Lyapunov exponents for the fundamental solution to the semi-discete SHE are given by $\gamma_p(\nu) = H_p(z_p^0)$, where $H_p(z) = \frac{p(p-3)}{2} + pz - \nu \log \prod_{i=0}^{p-1}(z+i)$ and z_p^0 is the unique solution to $H_p'(z) = 0$ with $z \in (0, \infty)$.*

The almost sure Lyapunov exponent was conjectured in [78] and proved in [75] (see also Theorem 1.6) to be $\tilde{\gamma}_1(\nu) = -\frac{3}{2} + \inf_{s>0}\left(s - \nu\Psi(s)\right)$ where $\Psi(s) := \left[\log\Gamma\right]'(s)$ is the digamma function. These formulas confirm explicitly the intermittency.

1.3.4. O'Connell-Yor semi-discrete directed polymer in random media. Whereas the atypical behavior of $\log z(\tau, n)$ is quite interesting through the lens of the parabolic Anderson model, it is the typical behavior which is most important when considering this as an interacting particle system or directed polymer model. The solution $z(\tau, n)$ to the semi-discrete SHE can be written in path integral form via the Feynman-Kac representation as $z(\tau, n) = \mathbb{E}_{x(\tau)=n}\left[\mathbf{1}_{x(0)=1} \exp\left\{\int_0^\tau dB_{x(s)}(s) - \frac{\tau}{2}\right\}\right]$ where the expectation $\mathbb{E}_{x(\tau)=n}$ is over Poisson jump processes (which increase value by one at exponential rate one) which are pinned to be n at time τ (in other words, $x(\cdot)$ is a Poisson jump process run backwards in time from n at time τ, decreasing by one at rate one in backwards time). The path integral formula for $z(\tau, n)$ shows that it equals the partition function for a particular semi-discrete directed polymer in a random Brownian environment, first studied by O'Connell-Yor [78]. In this interpretation, $\log z(\tau, n)$ is the *quenched free energy* of the model. See the reviews [34, 36] for some background on directed polymers.

1.3.5. Fredholm determinant. Since $q^{x_n(t)+n}$ converges (as $q \to 1$ and under appropriate scaling) to $z(\tau, n)$ the e_q-Laplace transform of $q^{x_n(t)+n}$ converges to the Laplace transform of $z(\tau, n)$. Thus, taking the $q \to 1$ limit of Theorem 1.3 yields the following result, first proved as [17, Theorem 5.2.11] (see also [20, Theorem 1.17]). An alternative route to proving this theorem utilizes O'Connell's work [73] on Whittaker measure in conjunction with an identity proved in [25].

Theorem 1.5. *For the fundamental solution to the semi-discrete SHE and any $n \geq 1$, $\tau \geq 0$, $u \in \mathbb{C}$ with $\mathrm{Re}(u) > 0$, $\mathbb{E}\big[e^{-ue^{\frac{3}{2}\tau} z(\tau,n)}\big] = \det\big(I + K\big)_{L^2(C_0)}$, where C_0 is a small contour around 0 and*

$$K(v, v') = \int_{1/2-\mathrm{i}\infty}^{1/2+\mathrm{i}\infty} \frac{\pi}{\sin(-\pi s)} \frac{\Gamma(v-1)^n}{\Gamma(s+v-1)^n} \frac{u^s e^{v\tau s + \frac{s^2\tau}{2}}}{s+v-v'} \frac{ds}{2\pi\mathrm{i}}.$$

It is natural to wonder whether this theorem could be proved directly in an analogous manner to the proof of Theorem 1.3. It is possible to compute similar moment formulas for $\mathbb{E}\big[z(\tau, n_1) \cdots z(\tau, n_k)\big]$ (see [17, Proposition 5.2.9] or [26, Section 6.2]). A natural route to compute the Laplace transform of $z(\tau, n)$ would be to write $\mathbb{E}\left[e^{\zeta z(\tau,n)}\right] = \mathbb{E}\left[\sum_{k=0}^{\infty} z(\tau, n)^k \frac{\zeta^k}{k!}\right] = \sum_{k=0}^{\infty} \mathbb{E}\left[z(\tau, n)^k\right] \frac{\zeta^k}{k!}$, and use the formulas for $\mathbb{E}\left[z(\tau, n)^k\right]$. Unfortunately, the last equality above is not true (the first is true as it just amounts to the Taylor expansion of the exponential). It is not always possible to interchange expectations and infinite summations. The moments of $z(\tau, n)$ grow super-exponentially (as we have already seen from the discussion on intermittency). Therefore, the right-hand series is divergent for all ζ despite the fact that the left-hand side is necessarily convergent for ζ with negative real part. This issue of moment indeterminacy is alleviated by lifting up to the level of q-TASEP, where the algebra and analysis work hand-in-hand.

1.3.6. KPZ class asymptotics. Theorem 1.5 is amenable to asymptotic analysis as was performed in [17, Theorem 5.2.13] and [20, Theorem 1.3] yielding:

Theorem 1.6. *For all $\nu > 0$, $\lim_{\tau\to\infty} \mathbb{P}\left(\frac{\log z(\tau,\nu\tau) - \tau\tilde{\gamma}_1(\nu)}{d(\nu)\tau^{1/3}} \leq s\right) = F_{\mathrm{GUE}}(s)$ where $d(\nu)$ $= \left(-\nu\Psi''(s(\nu))/2\right)^{1/3}$ with $s(\nu) = \arg\inf_{s>0}\big(s - \nu\Psi(s)\big)$.*

Earlier, [89] proved an upper bound on the variance of $\log z(\tau, \nu\tau)$ of order $\tau^{2/3}$, consistent with the $\tau^{1/3}$ scale of fluctuations. This theorem provides a matching lower bound as well as the exact limiting distribution.

1.4. Example 4: ASEP. The asymmetric simple exclusion process (ASEP) is a one-parameter deformation of TASEP in which particles can move both left and right. Let $x_i(t) \in \mathbb{Z}$ represent the location of particle i at time t. We assume that $x_i(t) < x_j(t)$ for $i > j$. The state space of ASEP is the set of all such ordered x_i, and the dynamics can be described as follows: each particle x_i has an exponential alarm clock (ringing after exponential waiting time, independent of all other particle clocks). When the alarm rings, the particle flips a coin and with probability p attempts to jump left, and with probability $q = 1 - p$ attempts to jump right. The jump is achieved only if the destination site is unoccupied at that time. Regardless of the outcome, the particle's clock is immediately reset. We will assume that

$0 < p < q < 1$ and $p + q = 1$ so that there is a drift in the positive direction (like for TASEP in which $p = 0$ and $q = 1$). ASEP can also be interpreted in terms of a growing (and shrinking) height function: each $\vee$ is replaced by $\wedge$ at rate q and each $\wedge$ is replaced by $\vee$ at rate p. As a measure of the asymmetry define the parameter $\tau = p/q < 1$ which will play a role akin to q from q-TASEP.

Just as for TASEP and q-TASEP, we will work with step initial data in which $x_i(0) = -i$ for all $i \geq 1$ (and there are no other particles with lower labels).

1.4.1. Moment formulas. An observable of interest is the number of particles to have crossed a given site y. For $y \in \mathbb{Z}$, let $N_y(t) = |\{m \geq 1 : x_m(t) \geq y\}|$. We would like to understand the behavior, in particular asymptotically, of this observable (which is closely related to the height in the growth interpretation of the model). Define $Q_y(t) = \tau^{N_y(t)}$ and its τ-derivative $\tilde{Q}_y(t) = (\tau - 1)^{-1}(Q_y(t) - Q_{y-1}(t))$.

Theorem 1.7. *For step initial data ASEP, any $k \geq 1$ and integers $y_1 > \cdots > y_k$,*

$$\mathbb{E}\Big[\prod_{j=1}^{k} \tilde{Q}_{y_j}(t)\Big] = \frac{\tau^{\frac{k(k-1)}{2}}}{(2\pi \mathrm{i})^k} \oint \cdots \oint \prod_{1 \leq A < B \leq k} \frac{z_A - z_B}{z_A - \tau z_B} \prod_{j=1}^{k} e^{-\frac{z_j (p-q)^2 t}{(1+z_j)(p+qz_j)}} \left(\frac{1 + \frac{z_j}{\tau}}{1 + z_j}\right)^{y_j + 1} \frac{dz_j}{\tau + z_j}$$

where the contours of integration are all along a small circle centered at $-\tau$ but not containing -1 or $-\tau^2$.

This result can, in turn, be used prove a similar integral formula for $\mathbb{E}\left[\tau^{kN_y(t)}\right]$.

1.4.2. Fredholm determinant. Utilizing the methods described in Section 1.2.2 for q-TASEP, it is possible to turn the integral formula for $\mathbb{E}\left[\tau^{kN_y(t)}\right]$ into a Fredholm determinant formula for the e_τ-Laplace transform of $\tau^{N_y(t)}$ which first appeared as [26, Theorem 5.3].

Theorem 1.8. *For step initial data ASEP, and any $y \in \mathbb{Z}$,*

$$\mathbb{E}\left[\frac{1}{\left(\zeta \tau^{N_y(t)}; \tau\right)_\infty}\right] = \det\left(I + K\right)_{L^2(C)}$$

where the contours C and D can be found from the statement of [26, Theorem 5.3] and

$$K(w, w') = \int_D \frac{\pi}{\sin(-\pi s)} \frac{e^{(q-p)t\frac{\tau}{z+\tau}} \left(\frac{\tau}{z+\tau}\right)^y}{e^{(q-p)t\frac{\tau}{\tau^s z+\tau}} \left(\frac{\tau}{\tau^s z+\tau}\right)^y} \frac{-(-\zeta)^s}{q^s w - w'} \frac{ds}{2\pi \mathrm{i}}.$$

1.4.3. KPZ class asymptotics. Theorem 1.8 characterizes the distribution of $N_y(t)$ and can be used to study its asymptotic behavior. There is another type of Fredholm determinant formula which can also be reached from the moment formulas given earlier. That Fredholm determinant (known in [26] as Cauchy-type) was essentially discovered earlier by Tracy-Widom [94–96] using a different approach (for a comparison of methods see [37]). Asymptotic analysis performed in [96] (and alternatively described in [26, Section 9]) yields:

Theorem 1.9. *For step initial data ASEP,*

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{N_0\big(t/(q-p)\big) - t/4}{t^{1/3}} \geq -r \right) = F_{\mathrm{GUE}}(2^{4/3}r).$$

1.5. Example 5: KPZ equation. The Kardar-Parisi-Zhang (KPZ) equation was introduced in 1986 [62] by the eponymous trio of physicists as a continuous (in space and time) model of random interface growth. The height function $h : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ satisfies the stochastic PDE $\frac{\partial}{\partial t}h(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}h(t,x) + \frac{1}{2}\left(\frac{\partial}{\partial x}h(t,x)\right)^2 + \xi(t,x)$ where $\xi(t,x)$ is space-time Gaussian white noise. In this continuous setting the Laplacian serves as a smoothing mechanism, the gradient squared serves as a mechanism for growth in the normal direction to the local slope, and the white noise inserts space-time uncorrelated randomness into the system. These three factors underly the KPZ universality class.

Making direct sense of this equation is challenging due to the non-linearity and the roughness of the spatial trajectories of h (see [12, 58, 59]). It has been understood since the work of [13] that the physically relevant notion of solution is to define $h(t,x) := \log z(t,x)$ where $z : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ solves the well-posed stochastic heat equation (SHE) with multiplicative noise $\frac{\partial}{\partial t}z(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}z(t,x) + \xi(t,x)z(t,x)$. The fundamental solution to the SHE has $z(0,x) = \delta_{x=0}$ and corresponds (under the weak scalings described in Section 1.5.1) to step initial data. (For more about this definition, see [4, 36].) A variant of this *Hopf-Cole transform* between growth process and stochastic heat equation was already present in the context of the semi-discrete polymer in Section 1.3.1. Similar transforms also hold for q-TASEP and ASEP, amounting to the $k = 1$ case of the dualities discussed later in Section 3.3 and 3.6, respectively.

The SHE has a directed polymer and parabolic Anderson model interpretation, though both require some care in making precise. Essentially, $z(t,x)$ can be interpreted as the partition function for a directed polymer model in which Brownian motion moves through a potential given by ξ (see more in [3, 4]) and can also be interpreted as the average mass density of a system of particles moving through $\mathbb{R}$ according to (independent) Brownian motions and splitting into two unit masses as well as dying according to the sign and amplitude of ξ.

1.5.1. Weak scaling universality of the KPZ equation. Rescale the solution to the KPZ equation by setting $h_\epsilon(t,x) = \epsilon^b h(\epsilon^{-z}t, \epsilon^{-1}x)$ where $b, z \in \mathbb{R}$. Then h_ϵ satisfies $\frac{\partial}{\partial t}h_\epsilon(t,x) = \frac{1}{2}\epsilon^{2-z}\frac{\partial^2}{\partial x^2}h_\epsilon(t,x) + \frac{1}{2}\epsilon^{2-z-b}\left(\frac{\partial}{\partial x}h_\epsilon(t,x)\right) + \epsilon^{b-z/2+1/2}\xi(t,x)$. Each term on the right-hand side rescaled differently.

Consider $b = 1/2$ and $z = 2$. Under this choice, the coefficients in front of the Laplacian and noise stay fixed as ϵ varies, however the one in front of the squared gradient grows like $\epsilon^{-1/2}$. If we inserted a parameter λ in front of the squared gradient in the original KPZ equation, and simultaneously scaled $\lambda = \epsilon^{1/2}$, then this would cancel the $\epsilon^{-1/2}$ and the KPZ equation would remain invariant as ϵ varied. We will call this *weak non-linearity scaling*.

Consider instead setting $b = 0$ and $z = 2$. Now, the coefficients in front of the Laplacian and squared gradient stay fixed as ϵ varies, while the one in front of the white noise grows like $\epsilon^{-1/2}$. Just as above, if we inserted a parameter β in front of the white noise in the original KPZ equation, and scaled it as $\beta = \epsilon^{1/2}$, then this would cancel the $\epsilon^{1/2}$ and the KPZ equation would remain invariant. We call this *weak noise scaling*.

These weak scalings are proxies for finding approximation schemes for the KPZ equation. Consider a model whose microscopic dynamics are characterized by a form of smooth-

ing, a non-linear dependence of the growth rate on the local slope, and space-time uncorrelated noise. If either the non-linearity or the noise have tunable parameters, then applying the above weak scalings may yield convergence of the model to the KPZ equation. It is important to note that it is only under these special weak scalings that growth models are expected to converge to the KPZ equation. The KPZ universality class scaling demonstrated through the examples we have studied has $b = 1/2$ and $z = 3/2$, and does not involve a parameter scaling. One may be misled in taking a formal $\epsilon \to 0$ limit of the rescaled KPZ equation with these choices of b and z. It would seem that the (deterministic) inviscid Burgers equation arises as the limit, but this cannot be (for instance, we know the limit remains random). The non-linearity seems to enhance the noise, which formally disappears as $\epsilon \to 0$. The KPZ-fixed point is the proposed [42] space-time limit of $h_\epsilon(t, x)$ (and any KPZ class model under the same scaling). The F_{GUE} distribution is just a one-point marginal distribution for the fundamental solution to this fixed point evolution.

Returning to the weak scalings, q-TASEP, the semi-discrete SHE and ASEP all have tunable parameters which control either the strength of the non-linearity or the noise. They also all admit Hopf-Cole type transform to the form of SHEs (of course the semi-discrete SHE is already in such a form). Since the KPZ equation is defined via such a transform, this reduces the problem to proving convergence (under suitable weak scaling) of discrete SHEs to the continuous one. This was first achieved for ASEP in 1997 work of Bertini-Giacomin [13], and subsequently has been extended to the other examples in [4, 70] (and to discrete polymers in [2]). The only weak universality result which has not utilized an exact Hopf-Cole transform is that of [43] which deals with finite (jumps up to distance three) exclusion. That result still proceeds through a discrete SHE which is shown to closely approximate a Hopf-Cole type transformed height function.

1.5.2. Moment formulas. Limits of the moment formulas for q-TASEP, the semi-discrete random polymer and ASEP under weak scaling from Section 1.5.1 yield the following moment formula for the fundamental solution to the SHE.

Theorem 1.10. *For the fundamental solution to the SHE $z(t, x)$ and any $k \geq 1$, $x_1 \leq \cdots \leq x_k$*

$$\mathbb{E}\Big[\prod_{j=1}^{k} z(t, x_j)\Big] = \frac{1}{(2\pi \mathrm{i})^k} \int \cdots \int \prod_{1 \leq A < B \leq k} \frac{z_A - z_B}{z_A - z_B - 1} \prod_{j=1}^{k} e^{\frac{t}{2} z_j^2 + x_j z_j} dz_j$$

where the z_j integration is over $\alpha_j + \mathrm{i}\mathbb{R}$ with $\alpha_1 > \alpha_2 + 1 > \alpha_3 + 2 > \cdots$.

The moment Lyapunov exponents for $z(t, 0)$ are easily computed from the above formula as $\gamma_k = \frac{k^3 - k}{24}$ (these were first computed by Kardar [63] and proved in [12]).

1.5.3. Fredholm determinant. Just as for the semi-discrete SHE, the moments of the SHE grow far to quickly to characterize the distribution of $z(t, x)$. However, we may use any of Theorems 1.3, 1.5, or 1.8 to prove the below Laplace transform formula for $z(t, x)$.

Theorem 1.11. *For the fundamental solution to the SHE and any $\zeta \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$*

$$\mathbb{E}\big[e^{-\zeta e^{\frac{t}{24}} z(t,0)}\big] = \det\big(I - K\big)_{L^2(\mathbb{R}_+)},$$

with

$$K(\eta,\eta')=\int_{\mathbb{R}}\frac{\zeta}{\zeta+e^{-s(t/2)^{1/3}}}\mathrm{Ai}(s+\eta)\,\mathrm{Ai}(s+\ \eta')\,ds.$$

This Fredholm determinant can also be written in the same form as that of the earlier theorems. This formula (in fact the inversion of it giving the distribution of $z(t,0)$) was discovered independently and in parallel by Sasamoto-Spohn [85] and Amir-Corwin-Quastel [4] in 2010 based on asymptotic analysis of Tracy-Widom's ASEP formulas [96]. The rigorous (mathematically) proof of the formula was provided by [4], and another subsequent proof in [20]. Soon after the work of [4, 85], this formula was re-derived by Dotsenko [45] and Calabrese-Le Doussal-Rosso [32] via the mathematically non-rigorous replica method (i.e. using moments to try to recover the Laplace transform, despite the aforementioned impediments). For more details, consult [36].

1.5.4. KPZ class asymptotics. A corollary of Theorem 1.11 is that the KPZ equation is in the KPZ universality class. The below result was first proved in [4, Corollary 1.3]. For stationary (i.e. $z(0,x)=B(x)$ a two-sided Brownian motion) initial data, the $t^{1/3}$ scale of fluctuations was demonstrated earlier in [7]. Recently, using the KPZ line ensemble, [39, Theorem 1.4] show that this $t^{1/3}$ scale holds true for all KPZ initial data.

Theorem 1.12. *For the fundamental solution to SHE,* $\lim_{t\to\infty}\mathbb{P}\left(\frac{\log z(t,0)+\frac{t}{24}}{(t/2)^{1/3}}\le r\right)=F_{\mathrm{GUE}}(r)$.

1.6. Further examples. The list of (non-determinantal) integrable probabilistic systems in the KPZ universality class continues to grow. Besides those models we have already discussed in the examples, the q-Hahn (or (q,μ,ν)) TASEP has been studied in [35, 83], the discrete time q-TASEPs in [18], the q-PushASEP in [30, 41], and the log-gamma polymer in [40, 74, 88]. It seems likely that the methods we now turn to will yield the discovery and analysis of further examples beyond these.

2. Macdonald processes

A high point of modern representation theory and symmetric function theory, Macdonald symmetric polynomials have found many diverse applications throughout mathematics. The canonical reference for their properties is the book [67] (see also the review material in [17, Section 2], and the historical perspective at the end of [15]). In this section we present a probabilistic application of these remarkable polynomials.

2.1. Defining Macdonald symmetric polynomials. Macdonald symmetric polynomials in N variables $x_1,\dots,x_N$ are indexed by non-negative integer partitions $\lambda=(\lambda_1\ge\cdots\ge\lambda_N\ge 0)$ and written as $P_\lambda(x_1,\dots,x_N)$. They are invariant under the action of the symmetric group S_N on the N variables, and have coefficients which are rational functions of two additional parameters q,t (i.e. coefficients in $\mathbb{Q}(q,t)$) which we assume are in $[0,1)$. The P_λ (as λ varies) form a linear basis in symmetric polynomials in N variables over $\mathbb{Q}(q,t)$. They can be defined in the following (rather inexplicit) manner (which will, however, suffice for our purposes). Define the *Macdonald first difference operator* D_1^N on the space of N variable symmetric functions f as $\big(D_1^N f\big)(x_1,\dots,x_N)=$

$\sum_{i=1}^N \prod_{\substack{j=1\\ j\neq i}}^N \frac{tx_i-x_j}{x_i-x_j} f(x_1,\dots,qx_i,\dots,x_N)$. It is not a priori clear (due to the denominator $x_i - x_j$), but this operator preserves the class of symmetric polynomials. This operator is self-adjoint (with respect to a natural inner product on symmetric polynomials with coefficients in $\mathbb{Q}(q,t)$) and the Macdonald symmetric polynomials are the eigenfunctions of D_1^N labeled via their (generically) pairwise different eigenvalues $\big(D_1^N P_\lambda\big)(x_1,\dots,x_N) = (q^{\lambda_1}t^{N-1} + q^{\lambda_2}t^{N-2} + \cdots + q^{\lambda_N})P_\lambda(x_1,\dots,x_N)$. The polynomials have many striking properties. They are orthogonal (as eigenfunctions of D_1^N) with respect to the earlier mentioned inner product, and the Macdonald Q_λ polynomials are defined as $P_\lambda/\langle P_\lambda, P_\lambda\rangle$ and form a dual basis to the P_λ. There is a Cauchy type identity providing a simple reproducing kernel: for variables $a_1,\dots,a_N$ and $b_1,\dots,b_M$ with $|a_ib_j| < 1$ for all i,j, $\sum_\lambda P_\lambda(a_1,\dots,a_N)Q_\lambda(b_1,\dots,b_M) = \prod_{i,j}\frac{(ta_ib_j;q)_\infty}{(a_ib_j;q)_\infty} =: \Pi(a_1,\dots,a_N;b_1,\dots,b_M)$. They satisfy Pieri and branching rules: the first describes the coefficients which result from multiplying Macdonald symmetric polynomials by elementary (or (q,t)-complete homogeneous) symmetric polynomials and reexpressing the answer in terms of other Macdonald symmetric polynomials; the second will be described below in Section 2.4. In the results explained below, these are essentially the only properties of these polynomials utilized. Other noteworthy properties are index/variable duality, and the existence of $N-1$ other difference operators which commute with D_1^N (and also are diagonalized by the P_λ).

2.2. Defining Macdonald processes. The *(ascending) Macdonald process* is a probability measures on interlacing partitions $\lambda^{(N)} \succeq \lambda^{(N-1)} \succeq \cdots \succeq \lambda^{(1)}$ where the number of non-zero elements in $\lambda^{(m)}$ is at most m, and the symbol $\succeq$ implies interlacing (so $\lambda_{j+1}^{(m)} \leq \lambda_j^{(m)} \leq \lambda_j^{(m+1)}$ for all meaningful inequalities). Such an interlacing triangular arrays of non-negative integers is also known as a *Gelfand-Tsetlin pattern*.

Measures on interlacing triangular arrays arise in many contexts. Before defining Macdonald processes, we consider a simpler example which comes from random matrix theory. Consider an $N\times N$ Gaussian Hermitian matrix drawn from the Gaussian unitary ensemble. For any $m \leq N$ let $\lambda_1^{(m)},\dots,\lambda_m^{(m)}$ be the ordered (largest to smallest) eigenvalues of the $m\times m$ upper-left corner of the matrix. By Rayleigh's Theorem the eigenvalues at level m interlace with those at level $m-1$. Thus, the eigenvalues form an interlacing triangular array, though the constitute elements are reals now instead of non-negative integers. The measure on this array inherited from the GUE measure is called the GUE-corner (or sometimes minor) process [9, 61] and has a very nice form. At level N, the measure on the eigenvalues $\lambda^{(N)}$ is the GUE measure written as (up to normalizations) $\prod_{i\neq j}^N(\lambda_i^{(N)} - \lambda_j^{(N)})^2 \prod_{i=1}^N e^{-(\lambda_i^{(N)})^2/2}$. Given the eigenvalues at level N, the distribution of $\lambda^{(N-1)},\dots,\lambda^{(1)}$ is uniform over the Euclidean simplex such that the interlacing inequalities are all satisfied [9, 52, 54].

The Macdonald process is a far reaching generalization of the GUE-corner process. In order to describe it we will start by describing the Macdonald analog of the GUE measure on level N. This single level measure is called the Macdonald measure and defined as $\mathbb{M}_{(N;a,b)}\big(\lambda^{(N)}\big) := \frac{P_{\lambda^{(N)}}(a_1,\dots,a_N)Q_{\lambda^{(N)}}(b_1,\dots,b_M)}{\Pi(a_1,\dots,a_N;b_1,\dots,b_M)}$. Here $a = (a_1,\dots,a_N)$ and $b = (b_1,\dots,b_M)$ for some $M \geq 0$ (one can work with more general Macdonald non-negative specializations – see [17, 22]). From the Cauchy type identity, it is clear that summing over all $\lambda^{(N)}$ yields one. If the a_i and b_j are all non-negative, then, due to a combinatorial expansion formula for the P_λ and Q_λ, the numerator (and thus the measure) is also non-negative. Besides the dependence on the a and b parameters, the measure also depends on the Mac-

donald q, t parameters. We will hold off defining the Macdonald process until Section 2.4.

Macdonald process generalizes a number of other measures (see the figure at the end of [15]). The GUE measure / GUE-corner process is a continuous space degeneration of the Schur measure / process [61, 79–81]. Macdonald measure seems to have first studied by Fulman in 1997 [51], and subsequently by [49, 97]. Until recently there were few examples of interesting probabilistic systems related to the Macdonald measure / process and there was a lack of ways to compute with them. In short, we were generally missing the answers to the questions of why and how to study Macdonald processes.

In 2011, Borodin-Corwin [17] provided partial answers to these two questions by: (1) constructing explicit Markov operators that map Macdonald processes to Macdonald processes (with updated parameters); (2) evaluating averages of a rich class of observables of the measures. In both cases, the integrable structure of Macdonald polynomials translates directly into probabilistic content. Since the work of [17], there has been a flurry of activity in these directions (see Section 2.7). We will only touch on the simplest example of how both of these answers work.

2.3. Computing expectations. Within statistical mechanics it is desirable to find explicit formulas for ensemble partition functions. For example, for the Ising model (at inverse temperature β in magnetic field h) the partition function is $Z(\beta, h) = \sum_{\sigma} e^{\beta \sum_{i \sim j} \sigma_i \sigma_j + h \sum_i \sigma_i}$. Taking derivatives of $\log Z(\beta, h)$ in h and β give (respectively) the expected magnetization, and expected product of spin over neighboring sites. The key here is that the Boltzmann weight (inside the sum over spin configurations σ) is an eigenfunction for the operators of differentiation in h and in β.

In our present case $\Pi(a; b)$ is like the partition function and $P_\lambda(a)Q_\lambda(b)$ the Boltzmann weight. Let D be any linear operator which is diagonalized by the Macdonald polynomials (e.g. a product of the Macdonald difference operators) with eigenvalue d_λ, so that $\big(DP_\lambda\big)(a) = d_\lambda P_\lambda(a)$. Since $\sum_\lambda P_\lambda(a)Q_\lambda(b) = \Pi(a; b)$, it follows that (with $D^{(a)}$ meaning to apply D on the a variables) $D^{(a)}\Pi(a; b) = \sum_\lambda D^{(a)} P_\lambda(a)Q_\lambda(b) = \sum_\lambda d_\lambda P_\lambda(a)Q_\lambda(b)$. Dividing both sides by $\Pi(a; b)$ yields $\mathbb{E}\big[d_\lambda\big] = \frac{D^{(a)}\Pi(a;b)}{\Pi(a;b)}$. If all of the ingredients are explicit (as they are for products of Macdonald difference operators), then we obtain meaningful and explicit probabilistic information without ever needing to know explicit formulas for the Macdonald measure itself. In fact, the eigenvalues of the commuting family of Macdonald difference operators provide explicit formulas for expectations of enough observables to entirely characterize the Macdonald measure. In this way, the Macdonald measure is a completely integrable probabilistic system.

We will return to one such explicit formula (with Macdonald parameter $t = 0$) in Section 2.6, and refer readers to [17, Section 2.2.3] and [22] for a more general discussion of developments here.

2.4. Constructing dynamics. The construction of dynamics on Gelfand-Tsetlin patterns which we present comes from an idea of Diaconis-Fill [44] in 1990 and was developed in the case of Schur processes by Borodin-Ferrari [27] in 2008 (see all [16]). Before describing this construction we explain how the full Macdonald process is defined (we have so far only defined the Macdonald measure on a given level N). The branching rule for $P_{\lambda^{(N)}}$ is $P_{\lambda^{(N)}}(a_1, \ldots, a_N) = \sum_{\lambda^{(N-1)} \preceq \lambda^{(N)}} P_{\lambda^{(N-1)}}(a_1, \ldots, a_{N-1}) P_{\lambda^{(N)}/\lambda^{(N-1)}}(a_N)$ where the sum is over all partitions $\lambda^{(N-1)}$ which interlace with $\lambda^{(N)}$ and where the skew Macdonald

polynomial $P_{\lambda/\mu}(u)$ is zero unless $\lambda \preceq \mu$ and $\psi_{\lambda/\mu} u^{|\lambda|-|\mu|}$ otherwise (with $\psi_{\lambda/\mu} \in \mathbb{Q}(q,t)$ explicit and not dependent on u).

It follows from the branching rule that the Markov kernel (or stochastic link) Λ^N_{N-1} from level N to level $N-1$ given by $\Lambda^N_{N-1}\big(\lambda^{(N)}, \lambda^{(N-1)}\big) := \frac{P_{\lambda^{(N-1)}}(a_1,\ldots,a_{N-1}) P_{\lambda^{(N)}/\lambda^{(N-1)}}(a_N)}{P_{\lambda^{(N)}}(a_1,\ldots,a_N)}$ maps the Macdonald measure $\mathbb{M}_{(N;a_1,\ldots,a_{N-1},a_N;b)}$ on level N to the Macdonald measure $\mathbb{M}_{(N;a_1,\ldots,a_{N-1};b)}$ on level $N-1$ (note that the a_N has been removed). The law of the trajectory of the Markov chain defined by these kernels started from Macdonald measure on level N is the Macdonald process: i.e., the Macdonald process on $\lambda^{(N)} \succeq \cdots \succeq \lambda^{(1)}$ specified by parameters $a_1,\ldots,a_N$ and $b_1,\ldots,b_M$ is written as $\mathbb{M}_{([1,N];a;b)}$ and defined as $\mathbb{M}_{([1,N];a;b)}\big(\lambda^{(N)},\ldots,\lambda^{(1)}\big) := \mathbb{M}_{(N;a;b)}\big(\lambda^{(N)}\big)\Lambda^N_{N-1}\big(\lambda^{(N)},\lambda^{(N-1)}\big)\cdots\Lambda^{(2)}_1\big(\lambda^{(2)},\lambda^{(1)}\big)$.

In the GUE-corner process, the stochastic link Λ^N_{N-1} is given by the indicator function that $\lambda^{(N-1)}$ interlaces with $\lambda^{(N)}$ times the ratio of the volume of the simplex of triangular arrays with top level $\lambda^{(N-1)}$ to that with top level $\lambda^{(N)}$ (there volumes are given by Vandermonde determinants).

There is another natural Markov chain which maps Macdonald measure to Macdonald measure on a single level N. For $u \geq 0$, the Markov kernel $\pi^{(u)}_N\big(\lambda^{(N)},\nu^{(N)}\big) := \frac{P_{\nu^{(N)}}(a)}{P_{\lambda^{(N)}}(a)} \cdot \frac{Q_{\nu^{(N)}/\lambda^{(N)}}(u)}{\Pi(a;u)}$ maps the Macdonald measure $\mathbb{M}_{(N;a;b_1,\ldots,b_M)}$ on level N to the Macdonald measure $\mathbb{M}_{(N;a;b_1,\ldots,b_M,u)}$ on level N (note that u has been appended to the b-list). This Markov kernel has the interpretation as the Doob-h transform of the sub-Markov kernel given by $\frac{Q_{\nu^{(N)}/\lambda^{(N)}}(u)}{\Pi(a;u)}$. Due to the explicit formula for the skew Macdonald polynomial, this sub-Markov kernel acts on $\lambda^{(N)}$ by increasing each element by independent geometrically distributed (with parameter u) amounts, and then killing all configurations which violate interlacing, and energetically penalizing all other configurations based on the value of $Q_{\nu^{(N)}/\lambda^{(N)}}(u)$. A generalized Cauchy type identity implies $\sum_{\nu^{(N)}} \frac{Q_{\nu^{(N)}/\lambda^{(N)}}(u)}{\Pi(a;u)} P_{\nu^{(N)}}(a) = P_{\lambda^{(N)}}(a)$ hence $P_{\nu^{(N)}}(a)$ has eigenvalue one for this sub-Markov kernel and is positive inside and zero outside the support of the kernel. The Markov kernel $\pi^{(u)}_N$ therefore corresponds to conditioning the sub-Markov chain to survive forever.

In the GUE setting, and in continuous time, this Markov chain is replaced by Dyson's Brownian motion (which can be thought of as conditioning N Brownian motions to never intersect). Therefore, the Markov chain corresponding to $\pi^{(u)}_N$ is a discrete time (q,t) -deformation of Dyson's Brownian motion.

We have defined two Markov chains. One chain goes from level N to level $N-1$ with kernel Λ^N_{N-1} and the other goes from level N to level N with kernel $\pi^{(u)}_N$. We will introduce a multivariate Markov chain with state space given by the entire Gelfand-Tsetlin pattern that 'stitches' these two chains together.

The key input into this construction is an intertwining relation of the two Markov chains. Specifically, for $u \geq 0$, and any $m \geq 2$, $\Lambda^m_{m-1}\pi^{(u)}_{m-1} = \pi^{(u)}_m \Lambda^m_{m-1}$.

For $u \geq 0$ define the Markov kernel

$$P^{(u)}\big((\lambda^{(1)},\ldots,\lambda^{(N)}),(\nu^{(1)},\ldots,\nu^{(N)})\big)$$
$$:= \pi^{(u)}_1\big(\lambda^{(1)},\nu^{(1)}\big)\prod_{k=2}^{N}\frac{\pi^{(u)}_k\big(\lambda^{(k)},\nu^{(k)}\big)\Lambda^k_{k-1}\big(\nu^{(k)},\nu^{(k-1)}\big)}{\big(\pi^{(u)}_k\Lambda^k_{k-1}\big)\big(\lambda^{(k)},\nu^{(k-1)}\big)}.$$

Then $P^{(u)}$ maps $\mathbb{M}_{([1,N];a;b_1,\ldots,b_M)}$ to $\mathbb{M}_{([1,N];a;b_1,\ldots,b_M,u)}$. The important property of this

construction is that each level m marginally evolves according to $\pi_m^{(u)}$, while the entire chain preserves the structure of the Macdonald process. $P^{(u)}$ first updates $\lambda^{(1)}$ to $\nu^{(1)}$ based on $\pi_1^{(u)}$, then updates $\lambda^{(2)}$ to $\nu^{(2)}$ according to the conditional law of $\nu^{(2)}$ given that the Λ_1^2 transition should bring $\nu^{(2)}$ to the previously determined $\nu^{(1)}$. The update proceeds similarly on each sequential pair of levels. These dynamics are constructed in [17, Section 2.3] and further constructions of dynamics which preserve the class of Macdonald processes (or their degenerations) are given in [30, 40, 73, 76].

In the GUE setting, and in continuous time, the limit (cf. [55, 56]) of this construction yields Warren's process [98] in which $\lambda_1^{(1)}$ evolves as a Brownian motion, $\lambda_1^{(2)}$ and $\lambda_2^{(2)}$ evolve according to independent Brownian motions which are reflected above and below (respectively) $\lambda_1^{(1)}$, and in general $\lambda_j^{(m)}$ evolves as a Brownian motion reflect to be above $\lambda_j^{(m-1)}$ and below $\lambda_{j-1}^{(m-1)}$. These dynamics preserve the class of GUE corner processes and have GUE Dyson's Brownian motion marginally on each level.

2.5. Example of dynamics. The dynamics constructed in Section 2.4 becomes simpler when we set the Macdonald parameter $t = 0$ and move into a continuous time setting. Since from here on out the Macdonald parameter t is fixed to be zero, we will abuse notation and use t for time. This transition to continuous time is achieved through setting the parameter u in the construction equal to $\epsilon(1-q)$ and running the discrete time Markov dynamics for $\epsilon^{-1}t$ steps (the factor of $1-q$ makes formulas nicer). Taking $\epsilon \to 0$ yields the following continuous time (measure by t) dynamics.

Treat the $\lambda_k^{(m)}$ as coordinates of particles where m is the level on which they live and k is their horizontal location. Each particle $\lambda_k^{(m)}$ jumps by one horizontally to the right independent of the others according to an exponential clock of rate

$$\text{rate}(\lambda_k^{(m)}) = a_m \frac{\left(1-q^{\lambda_{k-1}^{(m-1)}-\lambda_k^{(m)}}\right)\left(1-q^{\lambda_k^{(m)}-\lambda_{k+1}^{(m)}+1}\right)}{\left(1-q^{\lambda_k^{(m)}-\lambda_k^{(m-1)}+1}\right)}. \tag{2.1}$$

Those of the three terms above which refer to particles labeled with $m = 0$, $k = m+1$ or $k = 0$ are simply left out of the formula.

This is a $(2+1)$-dimensional interacting particle system with a local (in terms of particle labels) update rule. The particle $\lambda_k^{(m)}$ is influenced by the horizontal distance to three of its neighbors. As it gets closer to $\lambda_{k-1}^{(m-1)}$, its jump rate slows to zero (preventing jumps out of the interlacing condition). As it gets closer to $\lambda_k^{(m-1)}$ the jump rate increases to infinity (so as to immediately force a jump if $\lambda_k^{(m-1)}$ has overtaken the particle). These two interactions are the strongest, however there is also a slow down as $\lambda_{k+1}^{(m)}$ gets closer to the particle.

By virtue of (2.1), the set of coordinates $\left\{\lambda_m^{(m)} - m\right\}_{m\geq 1}$ evolves autonomously of the rest of the Gelfand-Tsetlin pattern. This $(1+1)$-dimensional interacting particle system is q-TASEP where $x_m(t) = \lambda_m^{(m)}$ at time t and where the jump rate of x_m is given by $a_m(1 - q^{x_{m-1}(t)-x_m(\tau)-1})$. We have been led to this particle system by virtue of the properties of Macdonald polynomials.

Step initial data for q-TASEP is achieved by running the above dynamics on Gelfand-Tsetlin patterns from initial data given by $\mathbb{M}_{([1,N];a;0)}$. By setting all b_j to be zero, this measure is entirely supported on the configuration where all $\lambda_k^{(m)} \equiv 0$. After performing the above affine shift to x_m coordinates, this corresponds with setting $x_m(0) = -m$ for $m \geq 1$.

2.6. Example of expectations. Running the continuous time (Macdonald parameter $t = 0$) dynamics for time t (abusing of notation) from initial data given $\mathbb{M}_{([1,N];a;0)}$ yields another Macdonald process, which can be thought of as the $\epsilon \to 0$ limit of $\mathbb{M}_{([1,N];a;\epsilon(1-q),\cdots\epsilon(1-q))}$ where there are $\epsilon^{-1}t$ entries of $\epsilon(1-q)$. This limit is called the *Plancherel specialization* and denoted by ρ_t so that the limiting measure becomes $\mathbb{M}_{([1,N];a;\rho_t)}$. Under this limit $\Pi(a;b)$ becomes $\Pi(a;\rho_t) = \prod_{i=1}^{N} e^{a_i t}$.

We will now utilize the receipe of Section 2.3 to compute observables of this Macdonald process (and hence also of q-TASEP started from step initial data). As we have fixed the parameter $t = 0$, the eigenvalue of the first difference operator becomes $D_1^N P_{\lambda^{(N)}}(a) = q^{\lambda_N^{(N)}} P_{\lambda^{(N)}}(a)$ so that $\mathbb{E}\big[q^{k(x_N(t)+N)}\big] = \mathbb{E}\big[q^{k\lambda_N^{(N)}}\big] = \frac{(D_1^N)^k \Pi(a;\rho_t)}{\Pi(a;\rho_t)}$ where the first expectation is over q-TASEP started from step initial data and the second expectation is over the Macdonald process $\mathbb{M}_{([1,N];a;\rho_t)}$. This can be generalized [22] to any $n_1 \geq \cdots \geq n_k \geq 1$ as $\mathbb{E}\big[\prod_{j=1}^{k} q^{x_{n_j}(\tau)+n_j}\big] = \mathbb{E}\big[\prod_{j=1}^{k} q^{\lambda_{n_j}^{(n_j)}}\big] = \frac{D_1^{n_k}\cdots D_1^{n_1}\Pi(a;\rho_t)}{\Pi(a;\rho_t)}$ where D_1^n represents the Macdonald first difference operator applied only to the variables $a_1, \ldots, a_n$. Theorem 1.2 follows (in fact a general a_i version of it) via encoding the application of these difference operators in terms of residues from contour integrals. To state the general a_i formula, assume (for simplicity of the choice of contours) that all a_i are very close to 1. Then using the multiplicative form of $\Pi(a;\rho_t)$ we find that

$$\frac{D_1^{n_k}\cdots D_1^{n_1}\Pi(a;\rho_t)}{\Pi(a;\rho_t)} = \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2\pi \mathrm{i})^k} \oint \cdots \oint \prod_{1\leq A<B\leq k} \frac{z_A - z_B}{z_A - q z_B} \prod_{j=1}^{k}\prod_{m=1}^{n_j} \frac{a_m}{a_m - z_j} \frac{e^{(q-1)tz_j}dz_j}{z_j}$$

where, for each $A \in \{1, \ldots, k\}$ the contour of integration of z_A contains the set of all a_i, as well as q times the contour of integration of z_B for $B > A$, but does not contain 0. Taking residues as the z_k through z_1 contours shrink provides a link to the difference operators.

2.7. Further developments. We record (without description) some further developments related to the theory of Macdonald processes. The figure at the of [15] highlights and organizes some of the probabilistic systems related to Macdonald processes and their degenerations (limits under special choices of Macdonald q, t parameters). These degenerations mimics those of Macdonald symmetric polynomials. The Schur process degeneration has been well studied during the past decade (see the review [15] or [29]) so we forego further discussion below. Note, in [1] the Macdonald process technology is utilized to rederive the Schur process determinantal structure. We also forego discussion of Kingman partition structures and refer the interested reader to [82] and references therein. Further dynamics have been constructed which preserve Macdonald processes (or their degenerations) [17, 28, 30, 31, 40, 56, 73, 76]. Further probabilistic systems have been connected to Macdonald processes (or their degenerations) or discovered via the above dynamics [17, 18, 28, 30, 31, 40, 41, 73, 74]. Exact and concise formulas have been found for expectations for a rich class of observables [17, 20–22, 25, 28, 40, 73]. A formal power series treatment of Macdonald processes and observable formulas has been developed [22]. Asymptotic analysis has been performed on some of the systems related to Macdonald processes [17, 19–21, 25, 28, 48]. Some of the structure related to Macdonald processes has been recast in the probabilistic language of Gibbsian line ensembles and used to prove some

universality results beyond exact solvable situations [39]. Formulas for expectations as well as dynamics preserving Macdonald process have begun to be connected to the Bethe ansatz and theory of quantum integrable systems [18, 26, 41]. It is this last point, the connection to Bethe ansatz and quantum integrable systems, which we expand upon in Section 3.

3. Quantum integrable systems

We will not define a quantum integrable system or go into any depth as to their algebraic origins (see [11, 47, 65, 84] for some references in this direction). Instead, we will study a few systems which arise in relation to the probabilistic analysis of models in the KPZ universality class.

3.1. Delta Bose gas. The first connection between the KPZ universality class and a quantum integrable system came from independent work of Kardar [63] and Molchanov [69] in 1987. For the SHE $z(t,x)$ (recall from Section 1.5) joint moments $\mathbb{E}\big[z(t,x_1)\cdots z(t,x_k)\big]$ are solutions to the quantum delta Bose gas, or Lieb-Liniger model (in imaginary time and with attractive delta interaction): $\frac{\partial}{\partial t}\mathbb{E}\big[z(t,x_1)\cdots z(t,x_k)\big] = \frac{1}{2}\big(\sum_{i=1}^k \frac{\partial^2}{\partial x_i^2} + \prod_{i\neq j}^k \delta(x_i - x_j)\big)\mathbb{E}\big[z(t,x_1)\cdots z(t,x_k)\big]$. In 1963 Lieb-Liniger solved (i.e. computed eigenfunctions for) the Hamiltonian on the right-hand side (i.e. the operator in the parentheses) via the Bethe ansatz (see also [68] expanding on this initial work). This was the second instance of a model being solved via this method, the first being Bethe's original solutions to the spin $1/2$ XXX Heisenberg chain. Lieb-Liniger's work marked the beginning of the development of the theory of quantum integrable systems. Besides computing eigenfunctions, for many purposes it is necessary to prove the completeness of the Bethe ansatz and determine the norms of the eigenfunctions. Such results go under the general title of *Plancherel theorems* and we will return to discuss these as well as the Bethe ansatz in Section 3.4.

Using the eigenfunctions for the delta Bose gas and the Plancherel theorem it is possible to solve the above differential equation for any initial data. For delta initial data the solution can be simplified considerably so as to take the form of Theorem 1.10. As we observed in Section 1.5.3, it is not possible to utilize the moments of the SHE to recover the distribution of, for instance, $z(t,x)$ for fixed t and x. Nevertheless, Dotsenko [45] and Calabrese-Le Doussal-Rosso [32] reconstructed the known one-point distribution for $z(t,x)$ via the (mathematically non-rigorous) replica method using these moments.

3.2. Be wise, discretize. What is a possible mathematical interpretation for this replica method calculations of [32, 45]? To answer this, we are drawn deeper into developing connections between the KPZ class and quantum integrable systems. The basic idea is that instead of working with the KPZ equation and delta Bose gas, we should first find an integrable discretization of the KPZ equation which converges to the equation under some scaling limit (such those in Section 1.5.1). Second, we should identify some observes whose expectations (analogous to moments of the SHE) solve a quantum integrable system. Third, we should solve this system via the Bethe ansatz (developing the Plancherel theory as necessary) for general initial data. And fourth, we should utilize the resulting expectation formulas to compute distributional information about the model and take the limit to KPZ/SHE. Steps one through three work for q-TASEP and ASEP (as well as a few other systems [18, 35, 41]).

Step four has only been accomplished for some special types of initial data, including step (which we saw earlier is the discrete precursor to the fundamental solution to the SHE).

We will focus on this for q-TASEP and only briefly mention the case of ASEP which is treated analogously. Our aim is to provide an alternative proof (than that of Macdonald processes) to Theorem 1.2.

3.3. Duality between q-TASEP and the q-Boson process. The q-Boson process was introduced by Sasamoto-Wadati [86] in 1998. It is a continuous time Markov process (a totally asymmetric zero range process) in which each site $j \in \mathbb{Z}$ has a non-negative number of particles y_j sitting above it. In continuous time the top particle at each location j jumps to the left by one site with a rate given by $1 - q^{y_j}$. The process is illustrated in Figure 3.1 along with the notation k, $\vec{y} = \{y_j\}_{j\in\mathbb{Z}}$, $\vec{n}$, $\vec{c}$ and m. Assuming there are $k \geq 1$ particles in the system (particle count is preserved in time) it is also natural to record the state $\vec{y}$ as a vector $\vec{n} = (n_1 \geq \cdots \geq n_k)$ of the ordered locations of the particles. Let $\vec{c} = (c_1, \ldots, c_m)$ represent the sizes of clusters in $\vec{n}$ and m be the total number of such clusters.

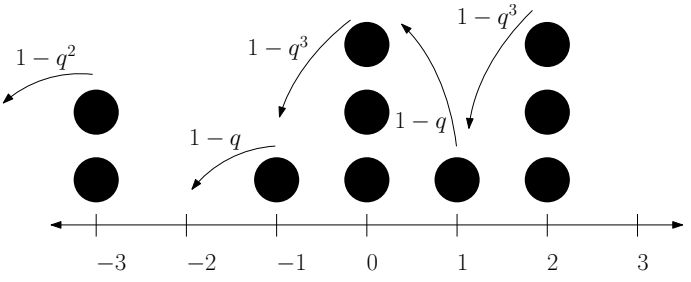


Figure 3.1. The q-Boson process with $k = 10$ particles at state $y_{-3} = 2, y_{-1} = 1, y_0 = 3, y_1 = 1, y_2 = 3$ and all other $y_j \equiv 0$. Equivalently, particles at ordered locations $\vec{n} = (2, 2, 2, 1, 0, 0, 0, -1, -3, -3)$ with $m = 5$ clusters of sizes $\vec{c} = (3, 1, 3, 1, 2)$.

The generator for the q-Boson process is $\big(Hf\big)(\vec{n}) = \sum_{i=1}^{m} \big(1 - q^{c_i}\big)\big(f(\vec{n}^{-}_{c_1+\cdots+c_i}) - f(\vec{n})\big)$ where f is a function of the ordered locations $\vec{n}$ and $\vec{n}^{-}_j = (n_1, \ldots, n_j - 1, \ldots, n_k)$.

There is an obvious relationship between q-TASEP and the q-Boson process since the gaps of q-TASEP evolve according to the same zero range jumping rates as the q-Boson process. A less apparent, but quite useful (and simple to prove – see [26, Theorem 2.2]) relationship is the *Markov duality* of these two processes. As q-TASEP is a discretization of the KPZ equation, this shows that the q-Boson process is a discretization of the delta Bose gas.

Proposition 3.1. *For q-TASEP $x_n(t)$, $f(t, \vec{n}) := \mathbb{E}\left[\prod_{j=1}^{k} q^{x_{n_j}(t)+n_j}\right]$ is the unique solution of $\frac{\partial}{\partial t} f(t, \vec{n}) = \big(Hf\big)(t, \vec{n})$, with $f(0, \vec{n}) = \mathbb{E}\big[\prod_{j=1}^{k} q^{x_{n_j}(0)+n_j}\big]$.*

3.4. Coordinate integrability of the q-Boson process. Define the *free generator* $\mathcal{L}$ via its action $\big(\mathcal{L}u\big)(\vec{n}) = (1-q)\sum_{i=1}^{k} \big(\nabla_i u\big)(\vec{n})$ where $u : \mathbb{Z}^k \to \mathbb{C}$, $\big(\nabla f\big)(n) = f(n-1) - f(n)$, and ∇_i acts as ∇ on coordinate i of u. When all elements of $\vec{n}$ are unique (no clusters of n's) the action of $\mathcal{L}$ matches that of H. The actions differ when clustering occurs. To repair this difference, we say that u satisfies the *boundary conditions* if for all $1 \leq i \leq k-1$ $\big(\nabla_i - q\nabla_{i+1}\big)u\big|_{n_i=n_{i+1}} = 0$. The boundary conditions involve arguments $\vec{n}$ outside of the

set of ordered n_i. The following result is proved as [26, Proposition 2.7].

Proposition 3.2. *If $u : \mathbb{R}_+ \times \mathbb{Z}^k \to \mathbb{C}$ satisfies the free evolution equation $\frac{\partial}{\partial t}u(t,\vec{n}) = \big(\mathcal{L}u\big)(t,\vec{n})$ and boundary conditions, then its restriction to $\{n_1 \geq \cdots \geq n_k\}$ satisfies the q-Boson process evolution equation $\frac{\partial}{\partial t}u(t,\vec{n}) = \big(Hu\big)(t,\vec{n})$.*

Using Propositions 3.1 and 3.2 we can provide another proof of Theorem 1.2. Let $u(t,\vec{n})$ be given by the right-hand side of (1.1). That u satisfies the free evolution equation follows from the equality $\frac{\partial}{\partial t}\frac{e^{(q-1)tz}}{(1-z)^n} = (1-q)\nabla\frac{e^{(q-1)tz}}{(1-z)^n}$ and Leibnitz rule. To check the boundary condition, observe that applying $\nabla_i - q\nabla_{i+1}$ to the integrand with $n_i = n_{i+1}$ results in a factor $z_i - qz_{i+1}$. This factor cancels the corresponding term in the denominator and allows the z_i and z_{i+1} contours to be freely deformed together thus showing that the remaining integral is zero by anti-symmetry. It remains to check initial data. Step initial data has $x_n(0) + n = 0$ for all $n \geq 1$ and hence we must check that $u(0,\vec{n}) \equiv \prod_{i=1}^k \mathbf{1}_{n_i \geq 1}$. This initial data is easily checked via residue calculus and comes from the poles of the product $\frac{1}{z_j}$ at zero.

The role that each term on the right-hand side of (1.1) plays in solving the q-Boson process evolution equation suggests that one should look to generalize the $\frac{1}{z_i}$ product in order to study general initial data (which in turn corresponds to general q-TASEP initial data). In order to do this we develop the Plancherel theory necessary to diagonalize the q-Boson process generator via Bethe ansatz.

3.4.1. Coordinate Bethe ansatz. Consider a space X, an operator L which acts on functions $f : X \to \mathbb{C}$, and an operator B which acts on functions $g : X^2 \to \mathbb{C}$. Let $\vec{x} = (x_1,\ldots,x_k) \in X^k$, L_i act as L on coordinate i of functions $\Psi : X^k \to \mathbb{C}$, and $B_{i,i+1}$ act as B on coordinates i and $i+1$ of functions $\Psi : X^k \to \mathbb{C}$.

Algebraic eigenfunctions for an operator $\mathcal{L}$ acting on $\Psi : X^k \to \mathbb{C}$ as $\big(\mathcal{L}\Psi\big)(\vec{x}) = \sum_{i=1}^k \big(L_i\Psi\big)(\vec{x})$ that satisfy boundary conditions $B_{i,i+1}\Psi\big|_{x_i=x_{i+1}} = 0$ for $1 \leq i \leq k-1$ can be diagonalized the following Bethe ansatz. First, diagonalize the one dimensional operator $\big(L\psi_z\big)(x) = \lambda_z\psi_z(x)$ where $\psi_z : X \to C$ and $z \in \mathbb{C}$ indexes the eigenfunctions. Then consider linear combinations of products of these one dimensional eigenfunctions $\Psi_{\vec{z}}(\vec{x}) := \sum_{\sigma\in S_k} A_\sigma(\vec{z})\prod_{i=1}^k \psi_{z_{\sigma(i)}}(x_i)$. For arbitrary $\vec{z} \in \mathbb{C}^k$ and functions $A_\sigma(\vec{z})$ we must have $\big(\mathcal{L}\Psi_{\vec{z}}\big)(\vec{x}) = \big(\sum_{i=1}^k \lambda_{z_i}\big)\Psi_{\vec{z}}(\vec{x})$. Finally, choose

$$A_\sigma(\vec{z}) := \operatorname{sgn}(\sigma)\prod_{k\geq A>B\geq 1}\frac{S(z_{\sigma(A)},z_{\sigma(B)})}{S(z_A,z_B)} \text{ where } S(z_1,z_2) := \frac{B(\psi_{z_1}\otimes\psi_{z_2})(x,x)}{\psi_{z_1}(x)\psi_{z_2}(x)}$$

Then, for any $\vec{z} \in \mathbb{C}^k$ the corresponding $\Psi_{\vec{z}}(\vec{x})$ will be eigenfunctions of $\mathcal{L}$ which satisfy the boundary conditions. Since instead of working on a finite or periodic domain (often the setting of Bethe ansatz) we are working on $\mathbb{Z}$, there is no quantization of the spectrum (Bethe equations).

3.4.2. Left and right eigenfunctions. We apply Bethe ansatz to the q-Boson process Hamiltonian with $L = (1-q)\nabla$ and $B_{1,2} = \nabla_1 - q\nabla_2$ to compute the left eigenfunctions for H (see [23, Proposition 2.10]). While H is not self-adjoint, it does enjoy a PT-invariance which immediately also yields right eigenfunctions.

Proposition 3.3. *Let* $\Psi^{\ell}_{\vec{z}}(\vec{n}) := \sum_{\sigma\in S_k}\prod_{k\geq A>B\geq 1}\frac{z_{\sigma(A)}-qz_{\sigma(B)}}{z_{\sigma(A)}-z_{\sigma(B)}}\prod_{j=1}^{k}\frac{1}{(1-z_{\sigma(j)})^{n_j}}$, *and* $\Psi^{r}_{\vec{z}}(\vec{n})=c_q^{-1}(\vec{n})\Psi^{\ell}_{\vec{z}}(R\vec{n})$ *where* $\vec{z}\in\big(\mathbb{C}\setminus\{1\}\big)^k$, $c_q(\vec{n})=(-1)^k q^{-k(k-1)/2}(c_1)!_q(c_2)!_q\cdots$ *(the* c_i *are the cluster sizes for* $\vec{n}$*) and* $R\vec{n}=(-n_k,\dots,-n_1)$. *Then* $H\Psi^{\ell}_{\vec{z}}=(1-q)(z_1+\cdots+z_k)\Psi^{\ell}_{\vec{z}}$, *and* $H^t\Psi^{r}_{\vec{z}}=(1-q)(z_1+\cdots+z_k)\Psi^{r}_{\vec{z}}$ *where* H^t *is the transpose of* H.

3.4.3. Direct and inverse Fourier type transforms. Proposition 3.3 gives algebraic eigenfunctions for H corresponding to every $\vec{z}\in\big(\mathbb{C}\setminus\{1\}\big)^k$. That does not mean, however, that all of these eigenfunctions participate in diagonalizing H. For example, the Laplacian (acting in x variables) has algebraic eigenfunctions e^{zx} for all $z\in\mathbb{C}$. However, the decomposition of $L^2(\mathbb{R})$ only involves those $z\in \mathrm{i}\mathbb{R}$. This fact is proved through the Plancherel theorem in Fourier analysis.

We define a direct and inverse Fourier type transform with respect to the q-Boson eigenfunctions. Let $\mathcal{W}^k$ be functions $f:\{n_1\geq\cdots\geq n_k | n_j\in\mathbb{Z}\}\to\mathbb{C}$ of compact support and $\mathcal{C}^k$ be symmetric Laurent polynomials in the variables (z_j-1), $1\leq j\leq k$. The *direct transform* $\mathcal{F}:\mathcal{W}^k\to\mathcal{C}^k$ acts on $f\in\mathcal{W}^k$ as $\big(\mathcal{F}f\big)(\vec{z}):=\sum_{n_1\geq\cdots\geq n_k}f(\vec{n})\Psi^{r}_{\vec{z}}(\vec{n})$. The *inverse transform* $\mathcal{J}:\mathcal{C}^k\to\mathcal{W}^k$ acts on $G\in\mathcal{C}^k$ as

$$\big(\mathcal{J}G\big)(\vec{n}):=\frac{(q-1)^k q^{-\frac{k(k-1)}{2}}}{(2\pi\mathrm{i})^k k!}\oint\cdots\oint\det\left[\frac{1}{qw_i-w_j}\right]_{i,j=1}^{k}\prod_{j=1}^{k}\frac{w_j}{1-w_j}\Psi^{\ell}_{\vec{w}}(\vec{n})G(\vec{w})d\vec{w},$$

$$=\frac{1}{(2\pi\mathrm{i})^k}\oint\cdots\oint\prod_{1\leq A<B\leq k}\frac{z_A-z_B}{z_A-qz_B}\prod_{j=1}^{k}\frac{1}{(1-z_j)^{n_j+1}}G(\vec{z})d\vec{z},$$

where in the first line, the contours are all along large circles around zero and in the second line for each $A\in\{1,\dots,k\}$ the contour of integration of z_A contains 1, as well as q times the contour of integration of z_B for $B>A$, but does not contain 0. This second express is familiar from Theorem 1.2.

3.4.4. Plancherel isomorphism theorem. The following results are from [23, Section 3].

Theorem 3.4. *On the spaces* $\mathcal{W}^k$ *and* $\mathcal{C}^k$, *the operators* $\mathcal{F}$ *and* $\mathcal{J}$ *are mutual inverses of each other, and biorthogonal (in appropriate bilinear pairings).*

This theorem diagonalizes the generator of the q-Boson process, proves completeness of the Bethe ansatz for it, and demonstrates remarkable biorthogonality properties of the eigenfunctions.

3.4.5. Back to the q-Boson process. An immediate corollary of Theorem 3.4 (see [23, Section 4]) is that for all initial data $f_0\in\mathcal{W}^k$, the unique solution to the q-Boson evolution equation $\frac{\partial}{\partial t}f(t,\vec{n})=\big(Hf\big)(t,\vec{n})$ with $f(0,\vec{n})=f_0(\vec{n})$ is

$$\mathcal{J}\big(e^{t(q-1)(z_1+\cdots+z_k)}\mathcal{F}f_0\big)(\vec{n})$$

$$=\frac{1}{(2\pi\mathrm{i})^k}\oint\cdots\oint\prod_{1\leq A<B\leq k}\frac{z_A-z_B}{z_A-qz_B}\prod_{j=1}^{k}\frac{e^{t(q-1)z_j}}{(1-z_j)^{n_j+1}}\big(\mathcal{F}f_0\big)(\vec{z})d\vec{z},$$

where integration is along nested contours.

The limitation that $f_0\in\mathcal{W}^k$ can be relaxed (with some additional work). For instance,

the above result can be extended to $f_0(\vec{n}) = \prod_{j=1}^k \mathbf{1}_{n_j \geq 1}$ which is the initial data corresponding to step initial data for q-TASEP (via the duality of Proposition 3.1).

However, the computation of $\mathcal{F} f_0$ can still be difficult (it involves an infinite summation over weakly ordered n_j). If there is some $G \in \mathcal{C}^k$ for which $f_0 = \mathcal{J} G$, then Theorem 3.4 implies that $\big(\mathcal{F} f_0\big)(\vec{z}) = G(\vec{z})$. One easily checks that $G(\vec{z}) = q^{\frac{k(k-1)}{2}} \prod_{j=1}^k \frac{z_j - 1}{z_j}$ yields $f_0(\vec{n}) = \prod_{j=1}^k \mathbf{1}_{n_j \geq 1}$. This (of course) agrees with our earlier solution to the q-Boson evolution equation.

3.5. Algebraic integrability of the q-Boson system. In 1998, Sasamoto-Wadati [86] first studied the q-Boson system (generalizing a similar system studied earlier in [14]) via the language of algebraic Bethe ansatz.

The q-Boson algebra is generated by $B_j, B_j^\dagger, N_j$, $1 \leq j \leq M$ subject to the relations (usually q would be replaced by q^{-2}, but the below parameterization is more convenient presently) $[B_i, B_j^\dagger] = q^{N_i} \mathbf{1}_{i=j}$, $[N_i, B_j] = -B_i \mathbf{1}_{i=j}$, $[N_i, B_j^\dagger] = B_i^\dagger \mathbf{1}_{i=j}$. The period (size M lattice) version of the q-Boson generator H is the image of the q-Boson Hamiltonian $\mathcal{H} = -(1-q) \sum_{j=1}^M \big(B_{j-1}^\dagger - B_j^\dagger\big) B_j$ under the representation in which $B_j, B_j^\dagger, N_j$ act on functions $f : (\mathbb{Z}_{\geq 0})^M \to \mathbb{C}$ as $\big(B_j f\big)(\vec{y}) = \frac{1-q^{y_i}}{1-q} f(\cdots, y_j - 1, \cdots)$, $\big(B_j^\dagger f\big)(\vec{y}) = f(\cdots, y_j + 1, \cdots)$, and $\big(N_j f\big)(\vec{y}) = y_j f(\vec{y})$. In [86], $\mathcal{H}$ arises from the monodromy matrix of a quantum integrable system with trigonometric R-matrix, the same as in the XXZ and six-vertex model (as well as in ASEP). There are many questions which remain to be investigated regarding the use of the algebraic Bethe ansatz (of which this is an application) in producing interesting integrable probabilistic system.

In a different direction, the q-Boson generator H also arises (see [18, Lemma 6.1]) from certain commutation relations for Macdonald first difference operators at Macdonald parameter $t = 0$. Recall from Section 2.1 that D_1^n is the Macdonald first difference operator acting on the variables $x_1, \ldots, x_n$.

Proposition 3.5. *Assume the Macdonald parameter $t = 0$, then*

$$\Big[\big(D_1^n\big)^k, p\Big] = (1 - q^k) x_n \big(D_1^{n-1} - D_1^n\big)\big(D_1^n\big)^{k-1}$$

where p is the operator of multiplication by $(x_1 + \cdots + x_n)$.

An immediate corollary of this is that for a symmetric, analytic function $F(x_1, \ldots, x_n)$, the function $\tilde{f}(t, \vec{y}) = e^{-tp} \big(D_1^1\big)^{y_1} \cdots \big(D_1^n\big)^{y_n} e^{tp} F(\vec{x})\big|_{x_1 = \cdots = x_n = 1}$ (if any $y_{-j} > 0$ for $j \geq 0$ set $\tilde{f}(t, \vec{y}) = 0$) solves the q-Boson evolution equation, in that $f(t, \vec{n}) = \tilde{f}(t, \vec{y}(\vec{n}))$ satisfies $\frac{\partial}{\partial t} f(t, \vec{n}) = \big(Hf\big)(t, \vec{n})$ where $\vec{y} = \vec{y}(\vec{n})$ is defined via $y_j = \big|\{i : n_i = j\}\big|$. Setting $F(x_1, \ldots, x_n) \equiv 1$ corresponds to step initial data for q-TASEP.

This provides one link between Macdonald processes and quantum integrable systems. Whether there is a deeper algebraic relationship between these two realms remains unclear.

3.6. ASEP and beyond. There is a parallel development for ASEP, as that explained above for q-TASEP. ASEP displays a non-trivial (self) duality [26, 87] through which (recalling the notation of Section 1.4.1) $f(t, \vec{y}) := \mathbb{E}\big[\prod_{j=1}^k \tilde{Q}_{y_j}(t)\big]$ solves the ASEP backward equation (with p and q interchanged). This provides a route to checking the result of Theorem 1.7.

The ASEP generator can likewise be diagonalized via Bethe ansatz (see [26, Section 4 and 5]), and a Plancherel theorem provides for the completeness of the ansatz and biorthogo-

nality of the eigenfunctions. In fact, the ASEP and q-Boson Plancherel theorems are unified [24] in terms of a theorem for the q-Hahn Boson process from [35, 83]. This Plancherel theorem also specializes to the general spin-s XXZ model and six vertex model on $\mathbb{Z}$.

These Plancherel theorems as well as the algebraic Bethe ansatz provide tools for further development of a theory of stochastic quantum integrable systems.

References

[1] A. Aggarwal, *Correlation functions of the Schur process through Macdonald difference operators*, arXiv:1401.6979.

[2] T. Alberts, K. Khanin, and J. Quastel, *Intermediate disorder regime for* $1+1$ *dimensional directed polymers*, arXiv:1202.4398.

[3] T. Alberts, K. Khanin, and T. Alberts, *The continuum directed random polymer*, J. Stat. Phys. **154** (2014), 305–326.

[4] G. Amir, I. Corwin, and J. Quastel, *Probability distribution of the free energy of the continuum directed random polymer in* $1+1$ *dimensions*, Commun. Pure Appl. Math. **64** (2011), 466–537.

[5] J. Baik, P. Deift, and K. Johansson, *On the distribution of the length of the longest increasing subsequence of random permutations*, J. Amer. Math. Soc. **12** (1999), 1119–1178.

[6] M. Balázs, J. Komjáthy, and T. Seppäläinen, *Microscopic concavity and fluctuation bounds in a class of deposition processes*, Ann. Inst. H. Poincaré B **48** (2012), 151–187.

[7] M. Balázs, J. Quastel, and T. Seppäläinen, *Scaling exponent for the Hopf-Cole solution of KPZ/stochastic Burgers*, J. Amer. Math. Soc. **24** (2011), 683–708.

[8] G. Barraquand, *A phase transition for q-TASEP with a few slower particles*, arXiv: 1404.7409.

[9] Y. Baryshnikov, *GUEs and queues*, Probab. Theory Rel. Fields **119** (2001), 256–274.

[10] G. Batchelor and A. Townsend, *The nature of turbulent motion at large wave-numbers*, Proc. R. Soc. London, A **199** (1949), 238–255.

[11] R. Baxter, *Exactly solved models in statistical mechanics*, Dover Publications, 2007.

[12] L. Bertini and N. Cancrini, *The stochastic heat equation: Feynman-Kac formula and intermittence*, J. Stat. Phys. **78** (1995), 1377–1401.

[13] L. Bertini and G. Giacomin, *Stochastic Burgers and KPZ equations from particle systems*, Commun. Math. Phys. **183** (1997), 571–607.

[14] N. M. Bogoliubov, R. K. Bullough, and J. Timonen, *Critical behavior for correlated strongly coupled Boson systems in* $1+1$ *dimensions*, Phys. Rev. Lett. **25** (1994), 3933–3936.

[15] A. Borodin, *Integrable probability*, Proceedings of the 2014 International Congress of Mathematicians.

[16] ———, *Schur dynamics of the Schur processes*, Adv. Math. **228** (2011), 2268–2291.

[17] A. Borodin and I. Corwin, *Macdonald processes*, Probab. Theor. Rel. Fields **158** (2014), 225–400.

[18] ———, *Discrete time q-TASEPs*, Int. Math. Res. Not., to appear. arXiv:1305.2972.

[19] ———, *On moments of the parabolic Anderson model*, Ann. Appl. Probab. **24** (2014), 1171–1197.

[20] A. Borodin, I. Corwin, and P. L. Ferrari, *Free energy fluctuations for directed polymers in random media in* $1 + 1$ *dimension*, Commun. Pure Appl. Math., to appear. arXiv:1204.1024.

[21] A. Borodin, I. Corwin, P. L. Ferrari, and B. Vető, *Height fluctuations for the stationary KPZ equation*, In preparation.

[22] A. Borodin, I. Corwin, V. Gorin, and S. Shakirov, *Observables of Macdonald processes*, Trans. Amer. Math. Soc., to appear. arXiv:1306.0659.

[23] A. Borodin, I. Corwin, L. Petrov, and T. Sasamoto, *Spectral theory for the q-Boson particle system*, Compositio Math., to appear. arXiv:1308.3475.

[24] ———, In preparation.

[25] A. Borodin, I. Corwin, and D. Remenik, *Log-Gamma polymer free energy fluctuations via a Fredholm determinant identity*, Commun. Math. Phys. **324** (2013), 215–232.

[26] A. Borodin, I. Corwin, and T. Sasamoto, *From duality to determinants for q-TASEP and ASEP*, Ann. Probab., to appear. arXiv:1207.5035.

[27] A. Borodin and P.L. Ferrari, *Anisotropic growth of random surfaces in* $2+1$ *dimensions*, Commun. Math. Phys., to appear. arXiv:0804.3035.

[28] A. Borodin and V. Gorin, General β Jacobi corners process and the Gaussian free field, arXiv:1305.3627.

[29] ———, Lectures on integrable probability, arXiv:1212.3351.

[30] A. Borodin and L. Petrov, *Nearest neighbor Markov dynamics on Macdonald processes*, arXiv:1305.5501.

[31] ———, *Law of large numbers for infinite random matrices over a finite field*, arXiv: 1402.1772.

[32] P. Calabrese, P. Le Doussal, and A. Rosso, *Free-energy distribution of the directed polymer at high temperature*, Euro. Phys. Lett. **90** (2010), 20002.

[33] R. Carmona and S. Molchanov, *Parabolic Anderson problem and intermittency*, Memoirs of AMS **518** (1994).

[34] F. Comets, T. Shiga, and N. Yoshida, *Probabilistic analysis of directed polymers in a random environment: A review* (ed. T. Funaki and H. Osada), Stochastic Analysis on Large Scale Interacting Systems 115–142. Math. Soc. Japan, Tokyo (2004).

[35] I. Corwin, *The (q, μ, ν)-Boson process and (q, μ, ν)-TASEP*, arXiv:1401.3321.

[36] ———, *The Kardar-Parisi-Zhang equation and universality class*, Random Matrices Theory Appl. **1** (2012).

[37] ———, Two ways to solve ASEP. In *Pan-American Summer Institute: Topics in Percolative and Disordered Systems*, Springer. Editors: Gérard Ben Arous, Chuck Newman, Alejandro Ramirez, Vladas Sidoravicius, and Maria Eulalia Vares.

[38] I. Corwin and A. Hammond, *Brownian Gibbs property for Airy line ensembles*, Invent. Math. **195** (2014), 441–508.

[39] ———, *KPZ line ensemble*, arXiv:1312.2600.

[40] I. Corwin, N. O'Connell, T. Seppäläinen, and N. Zygouras, *Tropical combinatorics and Whittaker functions*, Duke. Math. J. **163** (2014), 465–663.

[41] I. Corwin and L. Petrov, *The q-PushASEP: A new integrable traffic model in $1 + 1$ dimension*, arXiv:1308.3124.

[42] I. Corwin and J. Quastel, *The renormalization fixed point of the Kardar-Parisi-Zhang universality class*, arXiv:1103.3422.

[43] A. Dembo and L. Tsai, *Weakly asymmetric non-simple exclusion process and the Kardar-Parisi-Zhang equation*, arXiv:1302.5760.

[44] P. Diaconis and J. A. Fill, *Strong stationary times via a new form of duality*, Ann. Probab. **18** (1990), 1483–1522.

[45] V. Dotsenko, *Bethe ansatz derivation of the Tracy-Widom distribution for one-dimensional directed polymers*, Euro. Phys. Lett. **90** (2010), 20003.

[46] *Probability in Complex Physical Systems*, In honour of Erwin Bolthausen and Jurgen Gartner (eds. J.-D. Deuschel, B. Gentz, W. Konig, M.-K. van Renesse, M. Scheutzow, U. Schmock). Springer Proceedings in Mathematics 11, Springer, 2012, Berlin.

[47] L. Faddeev, *How algebraic Bethe ansatz works for integrable model*, In Symetries quantiques (Les Houches), 1996.

[48] P. L. Ferrari and B. Vető, *Tracy-Widom asymptotics for q-TASEP*, Ann. Inst. Henri Poinc., to appear.

[49] P. J. Forrester and E. Rains, *Interpretations of some parameter dependent generalizations of classical matrix ensembles*, Probab. Theo. Rel. Fields **131** (2005), 1–61.

[50] D. Forster, D.R. Nelson, and M.J. Stephen, *Large-distance and long-time properties of a randomly stirred fluid*, Phys. Rev. A **16** (1977), 732–749.

[51] J. Fulman, *Probabilistic measures and algorithms arising from the Macdonald symmetric functions*, arXiv:math/9712237.

[52] I. Gelfand and M. Naimark, *Unitary representations of the classical groups*, Trudy Mat. Inst. Steklov, Leningrad, Moscow, 1950.

[53] A. Greven and F. den Hollander, *Phase transitions for the long-time behavior of interacting diffusions*, Ann. Probab. **35** (2007), 1250–1306.

[54] Y. Neretin, *Rayleigh triangles and non-matrix interpolation of matrix beta integrals*, Sbornik: Mathematics **194** (2003), 515–540.

[55] V. Gorin and M. Shkolnikov, *Limits of multilevel TASEP and similar processes*, arXiv: 1206.3817.

[56] ———, *Multilevel Dyson Brownian motions via Jack polynomials*, arXiv:1401.5595.

[57] W. Hahn, *Beiträge zur Theorie der Heineschen Reihen. Die 24 Integrale der hypergeometrischen q-Differenzengleichung. Das q-Analogon der Laplace-Transformation*, Mathematische Nachrichten **2** (1949), 340–379.

[58] M. Hairer, *Solving the KPZ equation*, Ann. Math. **178** (2013), 559–664.

[59] ———, *A theory of regularity structures*, arXiv:1303.5113.

[60] K. Johansson, *Shape fluctuations and random matrices*, Commun. Math. Phys. **209** (2000), 437–476.

[61] K. Johansson and E. Nordenstam, *Eigenvalues of GUE minors*, Electron. J. Probab. **11** (2006), 1342–1371.

[62] K. Kardar, G. Parisi, and Y.Z. Zhang, *Dynamic scaling of growing interfaces*, Phys. Rev. Lett. **56** (1986), 889–892.

[63] M. Kardar, *Replica-Bethe Ansatz studies of two-dimensional interfaces with quenched random impurities*, Nucl. Phys. B **290** (1987), 582–602.

[64] A. N. Kolmogorov, *A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high Reynolds number*, J. Fluid Mech. **13** (1962), 82–85.

[65] V. Korepin, N. Bogoliubov, and A. Izergin, *Quantum inverse scattering method and correlation functions*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, 1993.

[66] M. Korhonen and E. Lee, *The transition probability and the probability for the left-most particle's position of the q-TAZRP*, arXiv:1308.4769.

[67] I.G. Macdonald, *Symmetric Functions and Hall Polynomials*, 2nd ed., Oxford University Press, New York, 1999.

[68] J. B. McGuire, *Study of exactly soluble one-dimensional N-body problems*, J. Math. Phys. **5** (1964), 622.

[69] S. Molchanov, *Lectures on random media*, Lecture Notes in Math. 1581, Springer, 1994, pp. 242-411.

[70] G. Moreno Flores, J. Quastel, and D. Remenik. In preparation.

[71] J. Quastel and B. Valko, *$t^{1/3}$ superdiffusivity of finite-range asymmetric exclusion processes on $\mathbb{Z}$*, Commun. Math. Phys. **273** (2007), 379–394.

[72] ———, *Diffusivity of lattice gases*, arXiv:1211.3716.

[73] N. O'Connell, *Directed polymers and the quantum Toda lattice*, Ann. Probab. **40** (2012), 437–458.

[74] N. O'Connell, T. Seppäläinen, and N. Zygouras, *Geometric RSK correspondence, Whittaker functions and symmetrized random polymers*, Inventiones Mathematicae, to appear.

[75] J. Moriarty and N. O'Connell, *On the free energy of a directed polymer in a Brownian environment*, Markov Process. Related Fields **13** (2007), 251–266.

[76] N. O'Connell and Y. Pei, *A q-weighted version of the Robinson-Schensted algorithm*, arXiv:1212.6716.

[77] N. O'Connell and J. Warren, *A multi-layer extension of the stochastic heat equation*, arXiv:1104.3509.

[78] N. O'Connell and M. Yor, *Brownian analogues of Burke's theorem*, Stoch. Proc. Appl. **96** (2001), 285–304.

[79] A. Okounkov, *Infinite wedge and random partitions*, Selecta Math. **7** (2001), 57–81.

[80] A. Okounkov and N. Reshetikhin, *Correlation function of Schur process with application to local geometry of a random 3-dimensional Young diagram*, J. Amer. Math. Soc. **16** (2003), 581–603.

[81] ———, *The birth of a random matrix*, Moscow Math. J. **6** (2006), 553–566.

[82] L. Petrov, *A two-parameter family of infinite-dimensional diffusions in the Kingman simplex*, Funct. Analys. Appl. **43** (2009), 279–296.

[83] A.M. Povolotsky, *On integrability of zero-range chipping models with factorized steady state*, J. Phys. A, **46** (2013), 465205.

[84] N. Reshetikhin, *Lecrtures on the integrability of the* 6*-vertex model*, arXiv:1010.5031.

[85] T. Sasamoto and H. Spohn, *One-dimensional KPZ equation: an exact solution and its universality*, Phys. Rev. Lett. **104** (2010), 23.

[86] T. Sasamoto and M. Wadati, *Exact results for one-dimensional totally asymmetric diffusion models*, J. Phys. A **31** (1998), 6057–6071.

[87] G. M. Schütz, *Exact solution of the master equation for the asymmetric exclusion process*, J. Stat. Phys. **88** (1997), 427–445.

[88] T. Seppäläinen, *Scaling for a one-dimensional directed polymer with boundary conditions*, Ann. Probab. **40** (2012), 19–73.

[89] T. Seppäläinen and B. Valko, *Bounds for scaling exponents for a 1+1 dimensional directed polymer in a Brownian environment*, ALEA, to appear.

[90] H. Spohn, *KPZ scaling theory and the semi-discrete directed polymer model*, arXiv: 1201.0645.

[91] S. Shandarin and Y. Zeldovich, *The large-scale structure of the universe: Turbulence, intermittency, structures in a self-gravitating medium*, Rev. Modern Physics **61** (1989), 185–220.

[92] K. Takeuchi and M. Sano, *Universal fluctuations of growing interfaces: evidence in turbulent liquid crystals*, Phys. Rev. Lett. **104** (2010), 230601.

[93] C. Tracy and H. Widom, *Level-spacing distributions and the Airy kernel*, Commun. Math. Phys. **159** (1994), 151–174.

[94] ———, *Integral formulas for the asymmetric simple exclusion process*, Commun. Math. Phys. **279** (2008), 815–844. Erratum: Commun. Math. Phys. **304** (2011), 875–878.

[95] ———, *A Fredholm determinant representation in ASEP*, J. Stat. Phys. **132** (2008), 291–300.

[96] ———, *Asymptotics in ASEP with step initial condition*, Commun. Math. Phys. **290** (2009), 129–154.

[97] M. Vuletic, *A generalization of MacMahon's formula*, Trans. Amer. Math. Soc. **361** (2009), 2789–2804.

[98] J. Warren, *Dyson's Brownian motions, intertwining and interlacing*, Elect. J. Probab. **12** (2007), 573–590.

[99] P. Yunker, M. Lohr, T. Still, A. Borodin, D. Durian, and A. Yodh, *Effects of particle shape on growth dynamics at edges of evaporating drops of colloidal suspensions*, Phys. Rev. Lett. **110** (2013), 035501.

Columbia University, Department of Mathematics, 2990 Broadway, New York, NY 10027, USA;
Clay Mathematics Institute, 10 Memorial Blvd. Suite 902, Providence, RI 02903, USA;
MIT, Department of Mathematics, 77 Massachusetts Av., Cambridge, MA 02139-4307, USA;
Institut Henri Poincaré, 11 Rue Pierre et Marie Curie, 75005 Paris, France
E-mail: ivan.corwin@gmail.com

Liouville Quantum Gravity, KPZ and Schramm–Loewner Evolution

Bertrand Duplantier

Abstract. We describe a canonical model of random surfaces, Liouville quantum gravity, its relation to the Gaussian free field (GFF) and to the canonical model of conformally invariant random curves in the plane, the Schramm-Loewner evolution (SLE). The Liouville random measure is formally written as $\mu_\gamma(dz) = e^{\gamma h(z)}\, dz$, where dz is the planar Lebesgue measure, h an instance of the GFF, and $\gamma \in [0, 2)$. We outline a probabilistic and geometrical proof of the Knizhnik–Polyakov–Zamolodchikov (KPZ) relation between the scaling exponents of a fractal with respect to the Euclidean and Liouville measures, including the boundary geometry case. The Liouville quantum measure in the $\gamma = 2$ critical case is defined after a further logarithmic renormalization, yielding an atom-free measure satisfying the KPZ relation. When $\gamma > 2$, the measure is purely atomic, and is related to a dual quantum measure $\mu_{\gamma'}$ by $\gamma\gamma' = 4$. For $\gamma < 2$, the conformal welding of boundary arcs of a γ-Liouville quantum gravity surface (in a quantum boundary length-preserving way) produces an SLE_κ curve, with $\kappa = \gamma^2$. This allows one to develop a theory of quantum fractal measures on the SLE curve itself (consistent with the KPZ relation) and analyze their evolution under conformal welding maps.

Mathematics Subject Classification (2010). Primary 81T40; Secondary 60J67.

Keywords. Liouville quantum gravity, Gaussian free field, Knizhnik-Polyakov-Zamolodchikov relation, duality, Schramm-Loewner evolution, conformal welding.

1. Introduction

1.1. Historical perspective. The study of certain natural probability measures on the space of two dimensional Riemannian manifolds (and singular limits of these manifolds) is often called "two-dimensional quantum gravity." These models have been very thoroughly studied in the physics literature (for reviews, see [37, 56]), starting in the mid-eighties with the modeling of discrete random surfaces [5, 24, 33, 63], followed by that of geometrical statistical models on discrete random surfaces (see [32, 40, 44, 52, 62, 68] for some early references, and [50] for a more complete list). These studies were motivated in part by connections to string theory and conformal field theory [34, 38, 66, 90], as well as to random matrix theory. More recently, a purely combinatorial and probabilistic (and mathematically rigorous) approach to the so-called random planar maps has been actively pursued [4, 7, 15, 22, 26, 28, 30, 78, 80, 81, 83, 94], allowing in particular a description of random geodesics [6, 21, 23, 79], of higher genus maps [25, 53, 82], and of some statistical models [17, 19, 20].

Proceedings of the International Congress of Mathematicians, Seoul, 2014

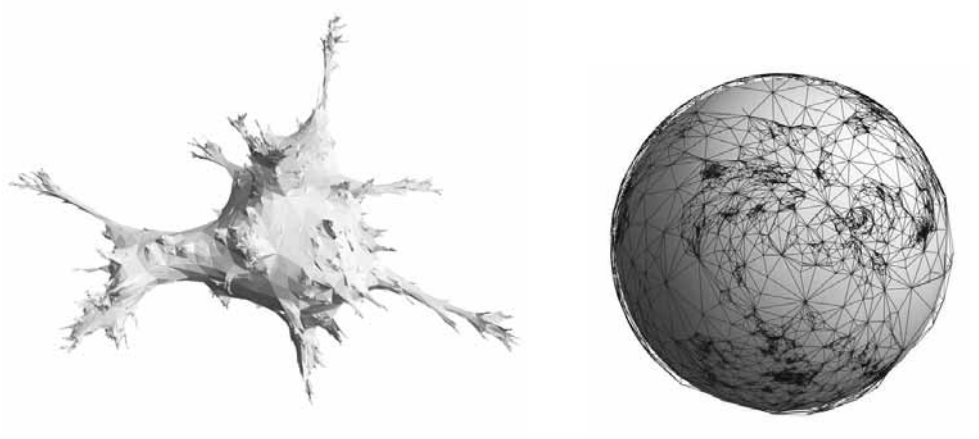


Figure 1.1. Left: A random triangulation of the sphere. Right: When viewing each face as an equilateral triangle with unit side length, the resulting Riemann surface is canonically embedded into the sphere $\mathbb{S}^2$ via the uniformization theorem, up to Möbius transformations. *(Courtesy of N. Curien.)*

Discrete quantum gravity models, e.g., random planar maps, are expected to converge in the continuum limit to some universal random metric spaces. As an illustrative example (Figure 1.1), consider a random planar triangulation M_n on the sphere, chosen uniformly with a given number n of triangles. It can be viewed as a Riemann surface by endowing each face with the metric of a unit equilateral triangle. The resulting manifold is singular, with negative (resp. positive) curvature localized at vertices where more (resp. fewer) than six triangles meet. This Riemann surface is homeomorphic to the sphere and the uniformization theorem yields a canonical embedding (modulo Möbius transformations), with conformally distorted triangles (Figure 1.1). One expects the existence in the $n \to +\infty$ limit, of a *universal random metric*, here on the sphere. Alternatively, one can take the limit $n \to +\infty$ first and obtain a (rooted) infinite triangulation M_∞ of the plane (see [7] for a precise description of this construction); when viewed as a Riemannian manifold, it can be conformally mapped to the plane [55], yielding the local structure of the uniformized triangulation on the sphere (Figure 1.1). In the scaling limit, one also expects to get *universal random measures* of areas and lengths of curves. Establishing mathematically their existence and their relation to the Liouville theory of quantum gravity, remains an outstanding open problem. (See J.-F. Le Gall in these Proceedings and [14]; see also [29, 36, 46, 99] for recent approaches.)

Polyakov [90] suggested in 1981 that the summation over random Riemannian metrics involved could be represented canonically by the now celebrated *Liouville theory of quantum gravity*. Consider a planar domain $D \subset \mathbb{C}$ as the parameter domain of the random surface, and h an instance of the (zero boundary for now) massless *Gaussian free field* (GFF), a random distribution on D, associated with the Dirichlet energy $(h,h)_\nabla := (2\pi)^{-1}\int_D [\nabla h(z)]^2 dz$, and whose two point correlations are given by Green's function on D. (Critical) Liouville quantum gravity consists of changing the Lebesgue area measure dz on D to the *quantum area measure*, formally defined as $\mu_\gamma(dz) := e^{\gamma h(z)} dz$, where γ is a real parameter. The GFF h is a random distribution, not a function, but the measure μ_γ can be constructed, for $\gamma \in [0,2)$, as the limit of regularized quantities (Section 1.2).

For instance, the pullback, via the conformal map to the plane, of the random area measure of the infinite triangulation M_∞ (rooted at the origin, and viewed as a Riemann surface) is precisely conjectured in [50] to take in the scaling limit the Liouville form, $\exp[\gamma(h(z) - \gamma \log|z|)]dz$, where h is an instance of the whole plane GFF, and $\gamma^2 = 8/3$.

The complete quantum Liouville action is $S(h) = \frac{1}{2}(h,h)_\nabla + \lambda\,\mu_\gamma(D)$, where the so-called "cosmological constant", $\lambda \geq 0$, weights the partition function according to the area of the random manifold. The corresponding (Boltzmann-Gibbs) statistical weight, $\exp[-S(h)]$, is to be integrated over with a "flat" uniform functional measure $\mathcal{D}h$ on h (which makes sense *a priori* for finite-dimensional approximations to h). In this review, we study the $\lambda = 0$ critical case, i.e., the *GFF case*.

Various values of γ are expected when weighting the random map by the partition function of a statistical physical model defined on that map (e.g., Ising model, $O(n)$ or Potts model). By the usual conformal invariance *ansatz* in physics, it is natural to expect that if one conditions on the infinite map, and then samples the loops or clusters in these critical models (as mapped into the plane, say), their law, in the scaling limit, will be *independent* of the random measure. This independence in turn leads to the famous *KPZ formula*.

One of the most influential papers in this field is indeed a 1988 one of Knizhnik, Polyakov, and Zamolodchikov [66] (see also [34, 38]). They derive a relationship between scaling dimensions (i.e., conformal weights x) of scaling fields defined using Euclidean geometry and analogous dimensions (Δ) defined via the Liouville quantum gravity measure μ_γ,

$$x = \frac{\gamma^2}{4}\Delta^2 + \left(1 - \frac{\gamma^2}{4}\right)\Delta. \tag{1.1}$$

The positive inverse to relation (1.1) is

$$\Delta_\gamma := \frac{1}{\gamma}\left(\sqrt{4x + a_\gamma^2} - a_\gamma\right), \quad a_\gamma := 2/\gamma - \gamma/2. \tag{1.2}$$

In the continuum limit, the statistical system is described at criticality by a *conformal field theory* (CFT) [89], whose universality class is characterized by the so-called *central charge* c ($c \leq 1$). KPZ determine γ as [34, 38, 66]

$$\gamma = \frac{1}{\sqrt{6}}\left(\sqrt{25 - c} - \sqrt{1 - c}\right) \leq 2, \quad c \leq 1. \tag{1.3}$$

With this parametrization, one recovers the usual form of the KPZ relation

$$\Delta = \frac{\sqrt{24x + 1 - c} - \sqrt{1 - c}}{\sqrt{25 - c} - \sqrt{1 - c}}. \tag{1.4}$$

The heuristic value of this formalism was checked against manifold instances of exactly solved lattice models, using the random matrix theory approach (e.g., in [32, 44, 52, 62, 68]); some direct comparison to correlation functions calculated in Liouville field theory was also possible (see [69] and references therein). The KPZ relation was further used to predict Brownian intersection exponents [40, 45, 75] (rigorously proved using the Schramm–Loewner Evolution in [71, 72]), or multifractal properties of SLE [41, 42].

A mathematical proof of the KPZ relation, based on the stochastic properties of the GFF, first appeared in [50]; it was then also proved for multiplicative cascades [16] and in the framework of Gaussian multiplicative chaos [91]. (See also [35].) One important

consequence is that KPZ appears to hold in a broader context than the original CFT realm (which relates γ to c), i.e., for any fractal structure sampled *independently* of the GFF, and measured with the quantum random measure μ_γ, and for any $0 \leq \gamma < 2$.

Note that Eq. (1.3) gives only values of γ in the range $\gamma \leq 2$. A different probabilistic approach is needed to construct measures for $\gamma > 2$ [9, 43, 49]; it is based on the *duality* property of Liouville quantum gravity: for $\gamma > 2$, a singular (purely atomic) quantum measure can be properly defined in terms of the γ'-quantum measure, for the dual value $\gamma' = 4/\gamma < 2$. This establishes the existence of the "dual version" of the KPZ relation, as argued long ago in Ref. [64].

For the critical value $\gamma = 2$, the construction of the corresponding Liouville measure $\mu_{\gamma=2}$ involves a so-called derivative martingale [47], or, equivalently, a logarithmic renormalization [48]. The resulting measure is atom-free, and a KPZ relation can be proved for it. Similar results hold for critical Mandelbrot cascades [10].

While Liouville quantum gravity is expected to offer a canonical description of the scaling limit of random maps, *Schramm-Loewner evolution* (SLE$_\kappa$), introduced by Schramm in 1999 [95], provides a canonical mathematical construction of conformally invariant random curves in the plane, depending on a real parameter κ. Its invention is on par with Wiener's 1923 mathematical construction of continuous Brownian motion. It gives the universal continuous scaling limit of 2D critical curves; of particular physical interest are the loop-erased random walk ($\kappa = 2$) [73], the self-avoiding walk ($\kappa = 8/3$), the Ising model interface ($\kappa = 3$ or $16/3$) [27, 101], the GFF contour lines ($\kappa = 4$) [96], and the percolation interface ($\kappa = 6$) [100]. Critical phenomena in the plane were earlier well-known to be related to CFT [13], a discovery anticipated in the so-called Coulomb gas approach to critical 2D statistical models (see, e.g., [89]), and now including SLE [11, 54, 61].

In Ref. [98], S. Sheffield established the first direct and rigorous connection between SLE and Liouville quantum gravity: gluing random surfaces (with the same $\gamma < 2$) along parts of their boundaries – and conformally mapping the combined surface to the half plane – produces an SLE curve with parameter $\kappa = \gamma^2, \kappa < 4$, as a random seam, a.k.a. a *conformal welding*. This in turn rigorously establishes the relation (1.3) between γ and c in the Liouville-CFT correspondence, given the function $c(\kappa)$ [11, 54, 61]. Note also that the aforementioned Liouville duality $\gamma\gamma' = 4$ then corresponds (here formally) to the SLE duality $\kappa\kappa' = 16$ [39, 41, 102]. A variant, i.e., the conformal welding of Euclidean and quantum surfaces, first conjectured by P. Jones, is constructed in [8]. (See also [93].) The generalization to the conformal welding of so-called *quantum wedges* is also possible [46].

One can also define *quantum fractal measures*, consistent with the KPZ relation; in particular one can construct quantum length and boundary intersection measures on the SLE curve itself, and analyze their evolution under welding [51]. This provides a rigorous probabilistic analog of the *"gravitational dressing"* of scaling fields in Liouville theory coupled to CFT [34, 37, 38, 56, 66]. (See also [70]).

The aim of this contribution is to present these recent developments relating the three acronyms, GFF, KPZ and SLE in a concise, yet (almost) rigorous way.

1.2. Quantum surface. Let h be an instance of a centered GFF on a bounded simply connected domain D with Dirichlet boundary conditions. This means that $h = \sum_n \alpha_n f_n$, where the α_n are i.i.d. zero mean unit variance normal random variables and the f_n are an orthonormal basis with respect to the Dirichlet inner product,

$$(f_1, f_2)_\nabla := \frac{1}{2\pi} \int_D \nabla f_1(z) \cdot \nabla f_2(z) dz, \tag{1.5}$$

of the Hilbert space closure $H(D)$ of the space $H_s(D)$ of C^∞ real-valued functions compactly supported on D. This sum diverges pointwise a.s., but it does converge a.s. in the space of distributions on D, and one can also make sense of the mean value of h on various sets [97]. The random variables $(h, f)_\nabla$ are zero mean Gaussian random variables for each f, with the covariance property:

$$\mathrm{Cov}\big((h, f_1)_\nabla, (h, f_2)_\nabla\big) = (f_1, f_2)_\nabla. \tag{1.6}$$

Special care is then required to make sense of the quantum gravity measure, because the GFF is a distribution and not a function (it typically oscillates between $\pm\infty$) [Figure 1.2].

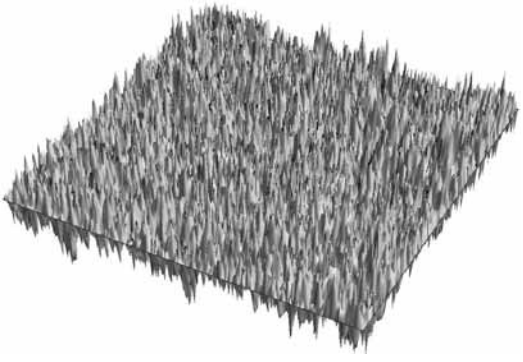


Figure 1.2. GFF with Dirichlet boundary conditions. *(Courtesy of J. Miller.)*

Given an instance h of the Gaussian free field on D, let $h_\varepsilon(z)$ denote the mean value of h on the circle of radius ε centered at z (where $h(z)$ is defined to be zero for $z \in \mathbb{C} \setminus D$). The regularized Liouville quantum measure is defined as [50]

$$\mu_{\gamma,\varepsilon}(dz) := \varepsilon^{\gamma^2/2} e^{\gamma h_\varepsilon(z)} dz, \tag{1.7}$$

in a way reminiscent of so-called Wick normal ordering (see also Ref. [57] for earlier work on the so-called Høegh-Krohn model).

One can show that for $\gamma \in [0, 2)$ the limit of this regularized measure exists as $\varepsilon \to 0$, which mathematically defines Liouville quantum gravity:

Proposition 1.1 ([50]). *Fix $\gamma \in [0, 2)$ and define h and D as above. Then it is (almost surely) the case that as $\varepsilon \to 0$ (along powers of two), the measures $\mu_{\gamma,\varepsilon}(dz) = \varepsilon^{\gamma^2/2} e^{\gamma h_\varepsilon(z)} dz$ converge weakly to a limiting non-degenerate measure, which we denote by $\mu_\gamma(dz) := e^{\gamma h(z)} dz$.*

For each $z \in D$, denote now by $C(z; D)$ the *conformal radius* of D viewed from z. That is, $C(z; D) = |\phi'(z)|^{-1}$ where $\phi : D \to \mathbb{D}$ is the conformal map to the unit disc with $\phi(z) = 0$. A geometrical analysis of GFF properties (Section 2) shows that the variance $\mathrm{Var}\, h_\varepsilon(z) = \log[C(z; D)/\varepsilon]$. By standard expectation of the exponential of a (centered) Gaussian variable,

$$\mathbb{E}\, e^{\gamma h_\varepsilon(z)} = e^{\gamma^2 \mathrm{Var}[h_\varepsilon(z)]/2} = [C(z; D)/\varepsilon]^{\gamma^2/2}. \tag{1.8}$$

The expectation of the quantum measure (1.7) is therefore

$$\mathbb{E}\,\mu_{\gamma,\varepsilon}(A) = \int_A C(z;D)^{\frac{\gamma^2}{2}}\,dz = \mathbb{E}\,\mu_\gamma(A),$$

independently of ε, and for each measurable subset $A \subset D$.

We interpret the pair (D,h) as describing a **quantum surface** conformally parameterized by D, and μ_γ as the area measure of this random surface. Similarly, as we shall see in Section 4, a **boundary length measure** ν_γ can also be defined on the quantum surface (D,h) [50].

One can parameterize the same quantum surface with a different domain $\widetilde{D}$, and the regularization procedure implies a simple rule for changing coordinates. For ψ a conformal map from $\widetilde{D}$ to D, consider the transformations

$$(D,h) \to (\widetilde{D},\widetilde{h}) = \psi^{-1}(D,h) := (\psi^{-1}(D), h\circ\psi + Q\log|\psi'|),\ \ Q := \gamma/2 + 2/\gamma. \quad (1.9)$$

Then μ_γ is a.s. the image under ψ of the measure $\widetilde{\mu}_\gamma$ associated with $\widetilde{h}$: $\widetilde{\mu}_\gamma(A) = \mu_\gamma(\psi(A))$ for $A \subset \widetilde{D}$. Similarly, ν_γ is a.s. the image under ψ of the measure $\widetilde{\nu}_\gamma$.

1.3. Euclidean and quantum scaling exponents.

Definition 1.2. For the quantum area measure μ_γ on D, we let $B^\delta(z)$ be the Euclidean ball centered at z whose radius is chosen so that $\mu_\gamma(B^\delta(z)) = \delta$. (If not unique, take the radius to be $\sup\{\varepsilon : \mu_\gamma(B_\varepsilon(z)) \le \delta\}$.) We refer to $B^\delta(z)$ as the **quantum ball** of area δ centered at z. In particular, if $\gamma = 0$ then μ_0 is Lebesgue measure and $B^\delta(z)$ is $B_\varepsilon(z)$ where $\delta = \pi\varepsilon^2$.

Given a subset $X \subset D$, we denote the ε neighborhood of X by $B_\varepsilon(X) = \{z : B_\varepsilon(z) \cap X \neq \emptyset\}$. We also define the **quantum** δ **neighborhood** of X by $B^\delta(X) = \{z : B^\delta(z) \cap X \neq \emptyset\}$.

Translated into probability language, the **KPZ formula** is a quadratic relationship between the fractal expectation dimension of a random subset of D defined in terms of Euclidean measure ($\gamma = 0$) and the corresponding fractal expectation dimension of X defined in terms of Liouville quantum measure with $\gamma \neq 0$.

We say that a (deterministic or random) fractal subset X of D has **Euclidean expectation dimension** $2-2x$ and **Euclidean scaling exponent** x if the expected area of $B_\varepsilon(X)$ decays like ε^{2x}, i.e.,

$$\lim_{\varepsilon\to 0} \frac{\log \mathbb{E}\,\mu_0(B_\varepsilon(X))}{\log \varepsilon^2} = x.$$

Fix $\gamma \in [0,2)$. We say that X has **quantum scaling exponent** Δ if when X and μ_γ are chosen independently we have

$$\lim_{\delta\to 0} \frac{\log \mathbb{E}\,\mu_\gamma(B^\delta(X))}{\log \delta} = \Delta.$$

Note that the expectation in this equation is with respect to both random variables, X and μ_γ.

1.4. Box formulation of Liouville quantum gravity. One can alternatively define quantum scaling dimensions using boxes instead of balls. Define a dyadic square to be a closed

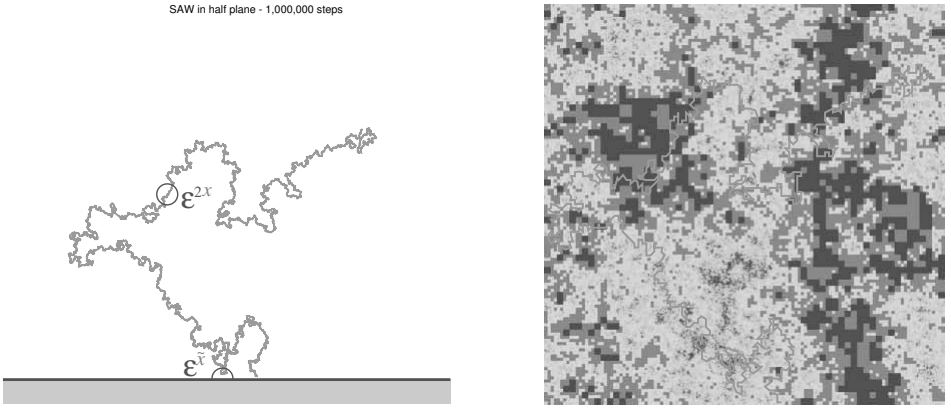


Figure 1.3. Fractal set X (here a self-avoiding walk) intersecting: [left] a Euclidean ball, [right] a quantum grid made of $(\mu_{\gamma=1}, \delta)$ boxes. *(Courtesy of T. Kennedy and J. Miller.)*

square (including its interior) of one of the grids $2^{-k}\mathbb{Z}^2$ for some integer k. For $\delta > 0$, we define a (μ_γ, δ) **quantum box** S to be a dyadic square S with $\mu_\gamma(S) < \delta$ and $\mu_\gamma(S') \geq \delta$ where S' is the dyadic parent of S. These boxes form a tiling of $\mathbb{R}^2$.

Let $N_\gamma(\delta, X)$ be the number of (μ_γ, δ) boxes intersected by a fractal X (Figure 1.3) and $N_0(\varepsilon^2, X)$ the number of dyadic squares intersecting X that have edge length ε (a power of 2), i.e., Euclidean area ε^2. Then we have the equivalent scaling dimension definitions, for the Euclidean exponent x, and for the quantum scaling exponent Δ,

$$\lim_{\varepsilon\to 0} \log \mathbb{E}[N_0(\varepsilon^2, X)]/\log \varepsilon^2 = x - 1, \quad \lim_{\delta\to 0} \log \mathbb{E}[N_\gamma(\delta, X)]/\log \delta = \Delta - 1,$$

respectively.

1.5. Statement of KPZ. The following is the KPZ scaling exponent relation. Its proof, based on GFF properties, is sketched in the next section [50]. To avoid boundary technicalities, we restrict attention here to a compact subset of D.

Theorem 1.3 ([50]). *Fix $\gamma \in [0, 2)$ and a compact subset $\widetilde{D}$ of D. If $X \cap \widetilde{D}$ has Euclidean scaling exponent $x \geq 0$ then it has quantum scaling exponent $\Delta = \Delta_\gamma$, where Δ_γ is the non-negative solution (1.2) to*

$$x = \frac{\gamma^2}{4}\Delta_\gamma^2 + \left(1 - \frac{\gamma^2}{4}\right)\Delta_\gamma. \tag{1.10}$$

2. Stochastic properties of the Liouville measure

2.1. GFF average and Brownian motion. Let h be a centered Gaussian free field on a bounded simply connected domain D with Dirichlet boundary conditions. For each $z \in D$, define the ball $B_\varepsilon(z) := \{w : |w - z| < \varepsilon\}$, and let ε_0^z be the largest ε such that $B_\varepsilon(z) \subset D$. When $\varepsilon \leq \varepsilon_0^z$, write $h_\varepsilon(z)$ for the average value of h on the circle $\partial B_\varepsilon(z) := \{w : |w - z| = \varepsilon\}$. Denote by $\rho_\varepsilon^z(y)$ the uniform density (of total mass one) localized on the circle

$\partial B_\varepsilon(z)$, such that one can write $h_\varepsilon(z)$ as the scalar product on D: $h_\varepsilon(z) = (h, \rho_\varepsilon^z) := \int_D h(y)\rho_\varepsilon^z(y)dy$. To this density ρ_ε^z is naturally associated a Newtonian potential,

$$f_\varepsilon^z(y) := -\log(\varepsilon \vee |y-z|) - \widetilde{G}_z(y), \tag{2.1}$$

where $\widetilde{G}_z(y)$ is the *harmonic extension* of $-\log|y-z|$ to D, i.e., the harmonic function of $y \in D$ with boundary value equal to the restriction of $-\log|y-z|$ to $y \in \partial D$. It satisfies Dirichlet boundary conditions and the Poisson equation $-\Delta f_\varepsilon^z = 2\pi\rho_\varepsilon^z$. In terms of the Dirichlet inner product (1.5), this gives

$$h_\varepsilon(z) = (h, f_\varepsilon^z)_\nabla. \tag{2.2}$$

The covariance (1.6) of (2.2) on nested circles centered at z is

$$\mathrm{Cov}\big(h_\varepsilon(z), h_{\varepsilon'}(z)\big) = (f_\varepsilon^z, f_{\varepsilon'}^z)_\nabla, \tag{2.3}$$

i.e., the *Newtonian interaction energy* of the two concentric circles. It is easily evaluated by using Gauss's theorem, the fact that the function $\widetilde{G}_z$ is harmonic, and that $\widetilde{G}_z(z) = -\log C(z;D)$, in terms of the conformal radius C of D viewed from z, a smooth function of z. One finds the covariances

$$\mathrm{Cov}\big(h_\varepsilon(z), h_{\varepsilon'}(z)\big) = \mathbb{E}\big(h_\varepsilon(z)h_{\varepsilon'}(z)\big) = -\log(\varepsilon \vee \varepsilon') + \log C(z;D), \tag{2.4}$$

where $\mathbb{E}\, h_\varepsilon(z) = 0$ for Dirichlet boundary conditions. From (2.4) we thus get two important variances,

$$\mathrm{Var}\, h_\varepsilon(z) = -\log\varepsilon + \log C(z;D), \tag{2.5}$$

$$\mathrm{Var}\big[h_\varepsilon(z) - h_{\varepsilon'}(z)\big] = |\log\varepsilon - \log\varepsilon'|. \tag{2.6}$$

Because $h_\varepsilon(z)$ is a Gaussian random variable, (2.6) is the *Lévy characterization* of Brownian motion in time parameter $t = -\log\varepsilon$. Define the reference radius $\varepsilon_0 := \varepsilon_0^z$. For fixed z, the Gaussian random variable $h_\varepsilon(z) - h_{\varepsilon_0}(z)$ is a *one-dimensional standard Brownian motion $\mathcal{B}_t$ when parameterized by time $t := -\log(\varepsilon/\varepsilon_0)$:*

$$h_\varepsilon(z) - h_{\varepsilon_0}(z) \overset{(\text{in law})}{=} \mathcal{B}_t, \ \ \mathcal{B}_0 = 0, \tag{2.7}$$

$$-\log(\varepsilon/\varepsilon_0) = t \in [0,\infty), \ \varepsilon \le \varepsilon_0. \tag{2.8}$$

This Brownian property of the GFF circular average is the first key to the KPZ relation [50].

2.2. Rooted measure and Brownian drift. Let us consider the so-called *rooted* measure [50] (sometimes also called Peyrière measure), on pairs (z,h) where h is a GFF, and given h, the point z is chosen from the regularized quantum area measure $\mu_{\gamma,\varepsilon}(dz)$ (1.7). Such a measure has the form (up to normalization)

$$e^{\gamma h_\varepsilon(z)}dhdz = \exp\left[-\frac{1}{2}(h,h)_\nabla + \gamma h_\varepsilon(z)\right]\mathcal{D}hdz, \tag{2.9}$$

where dh represents the whole GFF measure and $\mathcal{D}h$ the flat functional measure. The total action of the GFF *Liouville weighted measure* is thus the quadratic combination $S_\gamma(h) := \frac{1}{2}(h,h)_\nabla - \gamma h_\varepsilon(z)$. Thanks to (2.2), it can be rewritten as,

$$S_\gamma(h) = \frac{1}{2}(\underline{h},\underline{h})_\nabla - \frac{\gamma^2}{2}\mathrm{Var}\, h_\varepsilon(z), \qquad \underline{h} := h - \gamma f_\varepsilon^z, \tag{2.10}$$

where we used (2.3). In the substitution (2.10), $\underline{h}$, because of its quadratic weight, is now a *standard* GFF. The probability weight involved in the random measure (2.9) can finally be written, thanks to (1.8), as

$$\exp\left[-S_\gamma(h)\right] = \exp\left[-\frac{1}{2}(\underline{h},\underline{h})_\nabla\right] \mathbb{E}\, e^{\gamma h_\varepsilon(z)}, \tag{2.11}$$

where the second factor on the right hand side is the *marginal distribution* of z. The meaning of (2.11) is that, after sampling z from its marginal distribution, the law of h *weighted* by $e^{\gamma h_\varepsilon(z)}$ is identical to that of the original GFF $\underline{h}$ *plus* the deterministic function γf_ε^z (2.1):

$$h \overset{\text{(in law)}}{=} \underline{h} + \gamma f_\varepsilon^z. \tag{2.12}$$

Proposition 2.1 ([50]). *For fixed z, when the law of h is weighted by the Liouville conformal factor $e^{\gamma h_\varepsilon(z)}$ as in (2.9) or (2.11), the law of Gaussian random variable $h_\varepsilon(z) - h_{\varepsilon_0}(z)$ is identical to that of one-dimensional standard Brownian motion $\mathcal{B}_t$ with drift γt, when parameterized by time $t = -\log(\varepsilon/\varepsilon_0)$.*

Proof. For fixed z, take the average of the identity in law (2.12) on $\partial B_\varepsilon(z)$

$$h_\varepsilon(z) \overset{\text{(in law)}}{=} \underline{h}_\varepsilon(z) + \gamma f_\varepsilon^z(z) = \underline{h}_\varepsilon(z) - \gamma \log[\varepsilon/C(z;D)], \tag{2.13}$$

where use was made of the properties of the potential f_ε^z. Because $\underline{h}$ is a standard GFF, apply to $\underline{h}_\varepsilon(z)$ the identity in law (2.7) and rewrite (2.13) as

$$h_\varepsilon(z) - h_{\varepsilon_0}(z) \overset{\text{(in law)}}{=} \mathcal{B}_t + \gamma t,\ \mathcal{B}_0 = 0. \tag{2.14}$$

□

3. KPZ Proof

3.1. Quantum balls. We carry out the proof of Theorem 1.3 in several steps [50]. When ε is small, the limiting quantum measure of Proposition (1.1), $\mu_\gamma(B_\varepsilon(z))$ of the Euclidean ball $B_\varepsilon(z)$, is very well approximated by the simple form

$$\mu_{\gamma\odot}(B_\varepsilon(z)) := \varepsilon^{2+\gamma^2/2} e^{\gamma h_\varepsilon(z)}, \tag{3.1}$$

so that, in expectation, $\mathbb{E}[\mu_\gamma(B_\varepsilon(z))|h_\varepsilon(z)] \sim \pi\mu_{\gamma\odot}(B_\varepsilon(z))$ for $\varepsilon \to 0$. In this simplified perspective, one views $\mu_{\gamma\odot}$ as a function on balls, defined by (3.1), rather than the fully defined measure on D. It proves convenient to consider the normalized quantity

$$\mu_{\gamma\odot}(B_\varepsilon(z))/\mu_{\gamma\odot}(B_{\varepsilon_0}(z)) = \exp\left[\gamma\big(h_\varepsilon(z) - h_{\varepsilon_0}(z)\big) + \gamma\, Q \log(\varepsilon/\varepsilon_0)\right]. \tag{3.2}$$

where $Q = 2/\gamma + \gamma/2$. (The reference radius ε_0 can be taken as the largest radius such that $B_{\varepsilon_0}(\widetilde{D}) \subset D$ in the notation of Theorem 1.3.)

By similarity to definition 1.2 of quantum balls B^δ, one defines the quantum ball $\widetilde{B}^\delta(z)$ centered at z as the (largest) Euclidean ball $B_\varepsilon(z)$ such that,

$$\mu_{\gamma\odot}(\widetilde{B}^\delta(z))/\mu_{\gamma\odot}(B_{\varepsilon_0}(z)) = \delta. \tag{3.3}$$

One then first proves the analog of Theorem 1.3, where the quantum ball $B^\delta(z)$ is replaced by $\widetilde{B}^\delta(z)$, to evaluate how $\mathbb{E}[\mu_\gamma(z : \widetilde{B}^\delta(z) \cap X \neq \emptyset]$ scales with δ; this will yield the most straightforward form of KPZ. This expectation can be recast as the total probability under the law (2.9), (2.11) for pairs (z, h) to satisfy $\widetilde{B}^\delta(z) \cap X \neq \emptyset$.

Recall that a (deterministic or random) fractal subset X of D has Euclidean scaling exponent x (and Euclidean dimension $2 - 2x$) if, for z chosen uniformly in D and independently of X, the probability $\mathbb{P}\{B_\varepsilon(z) \cap X \neq \emptyset\} \asymp \varepsilon^{2x}$, in the sense that $\lim_{\varepsilon\to 0} \log \mathbb{P}/\log \varepsilon = 2x$. The same fractal X has *quantum scaling exponent* Δ if when X and (z, h), sampled with weight (2.9), (2.11), are chosen independently, we have

$$\mathbb{P}\{\widetilde{B}^\delta(z) \cap X \neq \emptyset\} \asymp \delta^\Delta. \tag{3.4}$$

3.2. KPZ as a Brownian martingale property. Using Proposition 2.1, we can rewrite (3.2) in the form,

$$\mu_{\gamma\odot}(B_\varepsilon(z))/\mu_{\gamma\odot}(B_{\varepsilon_0}(z)) \overset{\text{(in law)}}{=} \exp\left[\gamma(\mathcal{B}_t - a_\gamma t)\right], \tag{3.5}$$

where $\mathcal{B}_t$ is *independent* of z, and where we used the drift parameter

$$a_\gamma := Q - \gamma = 2/\gamma - \gamma/2 > 0\,;\ \gamma \in (0, 2). \tag{3.6}$$

The equality of (3.3) and (3.5) then relates *stochastically* the Euclidean radius ε to the given quantum area δ in terms of the *stopping time*

$$\begin{aligned} -\log(\varepsilon_A/\varepsilon_0) := T_A := \inf\{t : -\mathcal{B}_t + a_\gamma t = A\}, \\ A := -(\log\delta)/\gamma > 0. \end{aligned} \tag{3.7}$$

The probability that the Euclidean ball $B_{\varepsilon_A}(z) = \widetilde{B}^\delta(z)$ intersects X scales as $\varepsilon_A^{2x} \propto e^{-2xT_A}$. Computing the expectation $\mathbb{E}\left[\exp\left(-2xT_A\right)\right]$ with respect to the random stopping time T_A will give the "quantum" probability (3.4).

This is a classic computation in Brownian motion. Consider for any β the standard exponential martingale $\exp(-\beta\mathcal{B}_t - \beta^2 t/2)$. Because $a_\gamma > 0$ for $\gamma \in [0, 2)$, the stopping time T_A is finite a.s. For $t \in [0, T_A]$, the argument of the martingale stays bounded, for $\beta \geq 0$, by βA, hence by a fixed constant. One can then apply Doob's optional stopping theorem to the martingale at the stopping time T_A, so that $\mathbb{E}\left[\exp[\beta\mathcal{B}_{T_A} - \beta^2 T_A/2)\right] = 1$. By definition, $\mathcal{B}_{T_A} = a_\gamma T_A - A$, and we obtain $\mathbb{E}[\exp(-2xT_A)] = \exp(-\beta A)$, if we identify

$$2x := \beta a_\gamma + \beta^2/2, \tag{3.8}$$

with β the positive root of this equation. Thanks to the definition (3.7) of A, we finally obtain the scaling behavior,

$$\mathbb{E}[\exp(-2xT_A)] = \exp(-\beta A) = \delta^{\Delta_\gamma}, \tag{3.9}$$

$$\Delta_\gamma := \beta/\gamma = [(a_\gamma^2 + 4x)^{1/2} - a_\gamma]/\gamma. \tag{3.10}$$

We thus observe in (3.9) the expected quantum scaling behavior (3.4), together with the quantum exponent $\Delta = \Delta_\gamma$, which is the inverse KPZ function (1.2). Naturally, substituting $\beta = \gamma\Delta_\gamma$ into (3.8) gives the KPZ relation of Theorem 1.3.

It is interesting to note that in Eq. (3.8), which is in essence KPZ, the term linear in β (i.e., Δ in (1.10)) results, via the martingale representation, from the scaling of the regularized Liouville measure (1.7), including the logarithmic shift (2.12). The term $\beta^2/2$ in (3.8), i.e., the distinctive quadratic term in the KPZ formula (1.10), is generated by the intrinsic Brownian fluctuations brought in by the GFF.

In this first step, we used the function $\mu_{\gamma\odot}$ (3.1) (not a measure) to define quantum balls $\widetilde{B}^\delta$ as in (3.3), which gave the Brownian interpretation of the version above of the KPZ relation. However, the complete proof of the KPZ Theorem 1.3 requires using instead the *full Liouville measure* μ_γ, and to define quantum balls B^δ via the condition $\mu_\gamma\big(B^\delta(z)\big) = \delta$. A detailed comparison of the quantum measure μ_γ to $\mu_{\gamma\odot}$, carried out in [50], is then needed. Define a stopping time $\overline{T}_A := -\log\big(\overline{\varepsilon}_A/\varepsilon_0\big)$, where $\overline{\varepsilon}_A$ is the Euclidean radius of the ball $B_{\overline{\varepsilon}_A}(z) = B^\delta(z)$ such that $\mu_\gamma\big(B_{\overline{\varepsilon}_A}(z)\big) = \delta$; one can then show that

$$\lim_{A\to\infty} \frac{\log \mathbb{E}\left[\exp(-2x\overline{T}_A)\right]}{\log \mathbb{E}\left[\exp(-2xT_A)\right]} = 1,$$

which ultimately yields the proof of Theorem 1.3 [50].

3.3. GFF thick points. The probability density of the stopping time T_A, such that $P_A(t)dt := \mathbb{P}\left(T_A \in [t, t+dt]\right)$, with $t = -\log(\varepsilon/\varepsilon_0)$, gives access to some geometrical properties of the Liouville quantum measure and of its coupling to a given fractal X. An inverse Laplace transform with respect to $2x$ of result (3.9) gives [49, 50]:

$$P_A(t) = \frac{A}{\sqrt{2\pi t^3}} \exp\left[-\frac{1}{2t}\left(A - a_\gamma t\right)^2\right], \tag{3.11}$$

which characterizes, in logarithmic coordinates, the distribution of the Euclidean radius ε of a ball of given quantum area δ.

For $\delta \to 0$, the distribution (3.11) gets localized and gives the typical scaling of quantum balls in γ-Liouville quantum gravity, $\delta \asymp \varepsilon^{\gamma a_\gamma} = \varepsilon^{2-\gamma^2/2}$ (see also Ref. [58]). For $x \neq 0$, a saddle-point analysis of the Laplace transform of P_A yields:

Proposition 3.1 ([49, 50]). *A point z that is typical with respect to the quantum measure is an α-thick point* [58] *of the Gaussian free field h:*

$$\alpha := \lim_{\varepsilon\to 0} h_\varepsilon(z)/\log\varepsilon^{-1},$$

with the value $\alpha = \gamma$ for the quantum surface, and $\alpha = \gamma - \gamma\Delta_\gamma$ for a fractal of quantum scaling exponent Δ_γ.

4. Boundary KPZ

4.1. Boundary quantum measure. A boundary length measure can also be defined on a quantum surface (D, h) [49, 50], and most of the results for the quantum measure on D have straightforward boundary analogs.

Suppose that D is a domain with piecewise linear boundary or a domain with a smooth boundary containing a linear piece $\underline{\partial D} \subset \partial D$, and that h is an instance of the GFF on D

with *free* boundary conditions, normalized to have mean zero. One can also consider a GFF h with *mixed* boundary conditions, i.e., *free* boundary conditions on the linear component $\underline{\partial D} \subset \mathbb{R}$ and *Dirichlet* ones on $\partial D \setminus \underline{\partial D}$, by using a reflection principle w.r.t. the real line [50].

For a point $z \in \underline{\partial D}$, let $h_\varepsilon(z)$ be the mean value of $h(z)$ on the *semicircle* $\partial B_\varepsilon(z) \cap D$ of radius ε centered at z, for $\varepsilon \leq \varepsilon_0$, where ε_0 is small enough so that exactly one semi-disc of $B_{\varepsilon_0}(z)$ lies in D. One then has

Proposition 4.1. *For fixed $z \in \underline{\partial D}$, the Gaussian variable $h_\varepsilon(z) - h_{\varepsilon_0}(z)$ is identical in law to standard Brownian motion $\mathcal{B}_{2t}$ in time $t = -\log(\varepsilon/\varepsilon_0)$ for $\varepsilon \leq \varepsilon_0$, with $\mathcal{B}_0 = 0$.*

Thus at a boundary point in $\underline{\partial D}$, the variance of $h_\varepsilon(z)$ diverges as $-2\log\varepsilon$, which is twice that at interior points in D. We may therefore define a boundary length measure by

$$\nu_\gamma(dz) = e^{\gamma h(z)/2} dz := \lim_{\varepsilon \to 0} \varepsilon^{\gamma^2/4} e^{\gamma h_\varepsilon(z)/2} dz, \ \ \gamma \in [0,2), \tag{4.1}$$

where dz is Lebesgue measure on $\underline{\partial D}$ [50]. The existence of the weak limit (4.1) when $\gamma \in [0,2)$ is insured by the analog of Proposition 1.1 for the bulk case [50].

Like the bulk measure μ_γ, the boundary one ν_γ is conformally invariant under the transformations (1.9); this actually yields a definition of the quantum boundary length measure ν_γ when the boundary of D is not piecewise linear.

4.2. Boundary scaling and KPZ. For $z \in \underline{\partial D}$, write $\widehat{B}_\varepsilon(z) := B_\varepsilon(z) \cap \partial D$ and define $\widehat{B}^\delta(z)$ to be the (largest) boundary set $\widehat{B}_\varepsilon(z)$ whose ν_γ measure is δ. Likewise define the neighborhoods, $\widehat{B}_\varepsilon(X) = \{z \in \underline{\partial D} : \widehat{B}_\varepsilon(z) \cap X \neq \emptyset\}$ and $\widehat{B}^\delta(X) = \{z \in \underline{\partial D} : \widehat{B}^\delta(z) \cap X \neq \emptyset\}$.

We say that a (deterministic or random) fractal subset X of the boundary of D has **Euclidean expectation dimension** $1 - \widetilde{x}$ and **Euclidean scaling exponent** $\widetilde{x}$ in the boundary sense if $\lim_{\varepsilon \to 0} \log \mathbb{E}\left[\nu_0(\widehat{B}_\varepsilon(X))\right]/\log\varepsilon = \widetilde{x}$. We say that X has **boundary quantum scaling exponent** $\widetilde{\Delta}$ if, when X and ν_γ (as defined above) are chosen independently, we have $\lim_{\delta \to 0} \log \mathbb{E}\left[\nu_\gamma(\widehat{B}^\delta(X))\right]/\log\delta = \widetilde{\Delta}$.

The validity of KPZ relation (1.1) for the pair $(\widetilde{x}, \widetilde{\Delta})$, as anticipated in [42], is insured by the following theorem:

Theorem 4.2 ([50]). *Fix $\gamma \in [0,2)$ and $\underline{\partial D}$ (a closed subinterval of) the boundary line segment of ∂D. If $X \cap \underline{\partial D}$ has Euclidean scaling exponent $\widetilde{x} \geq 0$ then it has quantum scaling exponent $\widetilde{\Delta} = \widetilde{\Delta}_\gamma$, where $\widetilde{\Delta}_\gamma$ is the non-negative solution to*

$$\widetilde{x} = \frac{\gamma^2}{4}\widetilde{\Delta}_\gamma^2 + \left(1 - \frac{\gamma^2}{4}\right)\widetilde{\Delta}_\gamma. \tag{4.2}$$

Proof. The proofs in the boundary case proceed exactly the same as in the interior point case, and are based on Proposition 4.1, up to factors of 2 in various places, which recombine and give the same KPZ relation as in the bulk case [50]. □

5. Liouville quantum duality

5.1. $\gamma\gamma' = 4$ duality. Liouville quantum gravity is expected to describe, for $\gamma > 2$, random surfaces meant to be the scaling limit of random simply connected maps with large

amounts of area cut off by small bottlenecks, as first constructed with discrete matrix models in Refs. [31, 67]. The corresponding continuous surface is a tree-like foam of Liouville quantum "bubbles" (also called "baby universes") of *dual* parameter $\gamma' := 4/\gamma$, $\gamma' < 2 < \gamma$, connected to each other at "pinch points" and rooted at a "principal bubble"conformally parameterized by a domain D.

The Liouville measure was first formally considered for $\gamma > 2$ in Refs. [64, 65], in the so-called "other gravitational dressing" of the Liouville potential, or "dual branch of gravity", leading in particular to a *dual* version of the KPZ relation. However, the limit of the regularized measure (1.7) actually *vanishes* for $\gamma \geq 2$: $\lim_{\varepsilon\to 0} \mu_{\gamma\geq 2,\varepsilon}(z) = 0$. This is a quite general phenomenon, first observed by Kahane in the eighties [60] in the case of the so-called *Gaussian multiplicative chaos*, inspired by Mandelbrot's cascade model. We first relate γ to γ' *heuristically*, following Ref. [49], by using the regularized measure (1.7).

The definitions of the (pseudo) measure (3.1) on balls and of quantum balls (3.3) still make sense for $\gamma > 2$. The quantity (3.1) satisfies the duality relation

$$\mu_{\gamma'\odot}^{1/\gamma'} = \mu_{\gamma\odot}^{1/\gamma}, \ \ \gamma\gamma' = 4, \tag{5.1}$$

i.e., a γ-quantum ball $\widetilde{B}^\delta(z)$ (3.3) of size δ has γ'-quantum size $\delta' = \delta^{4/\gamma^2}$. (Intuitively, the ball contains about a δ' fraction of the total γ'-quantum area, but only a $\delta < \delta'$ fraction of the γ-quantum area because the latter also includes points on non-principal bubbles.) The expected number of γ'-quantum size-δ' balls (*i.e.*, γ-quantum size-δ balls) needed to cover the principal bubble D thus scales as $N_{\gamma'}(\delta', D) \asymp {\delta'}^{-1} = \delta^{-4/\gamma^2}$.

If a fractal random subset $X \subset D$ has Euclidean scaling exponent x, it has by KPZ Theorem 1.3 a quantum scaling exponent $\Delta_{\gamma'}$ for the *standard* $\gamma' < 2$ Liouville measure. This essentially says that the expected number $N_{\gamma'}(\delta', X)$ of γ'-quantum size-δ' balls (*i.e.*, number $N_\gamma(\delta, X)$ of γ-quantum size-δ balls) required to cover X, scales as $N_{\gamma'}(\delta', X) \asymp {\delta'}^{\Delta_{\gamma'}-1} = \delta^{\Delta_\gamma - 1} \asymp N_\gamma(\delta, X)$, where we define for $\gamma > 2$ the *dual* quantum exponent Δ_γ by

$$\gamma(\Delta_\gamma - 1) := \gamma'(\Delta_{\gamma'} - 1), \ \ \gamma\gamma' = 4. \tag{5.2}$$

By hypothesis, the triple $(\gamma', x, \Delta_{\gamma'})$ satisfies the KPZ relation (1.1), valid for $\gamma' < 2$, and it is easy to see that the *dual* triple $(\gamma, x, \Delta_\gamma)$ does also for $\gamma > 2$, with the further duality property $x = \Delta_\gamma \Delta_{\gamma'}$. The dual quantum exponent Δ_γ coincides with the continuation of (1.2), (3.10) to the range $\gamma > 2$. Notice, however, that for $\gamma > 2$ the coefficient a_γ in (1.2) is *negative*. As a consequence, the dual versions of the KPZ expressions (1.3) and (1.4), in terms of the central charge c, are obtained by changing the signs in front of all of the $\sqrt{1-c}$ terms there. These dual quantum exponents appear in Refs. [64, 65]. They also emerge in the natural context of *boundary* exponents associated with the *dense phase* of the $O(n)$ model [42, 68], or with the Stochastic Loewner Evolution SLE_κ for $\kappa > 4$.

A *Brownian approach to duality* is also possible [49]. If one chooses the pair (z, h) from the weighted measure $\mu_{\gamma,\varepsilon}(dz)dh$ as in Eq. (2.9), one can repeat for $\gamma > 2$ the whole Brownian description of Section 3.2 of the $\widetilde{B}^\delta(z)$ balls (3.3); however one finds that since $a_\gamma < 0$ for $\gamma > 2$ in Eq. (3.6), the drift term in Eq. (3.7) runs in a direction opposite to $A > 0$, so that $-\log(\varepsilon_A/\varepsilon_0) = T_A = +\infty$ a.s. for large $A = -(\log\delta)/\gamma$. The weighted measure is thus *singular*; i.e., there is a quantum area of at least δ *localized* at point z for small enough δ. The dual quantum exponents can then be obtained by conditioning in Eq. (3.9), with $\mathbb{E}[\exp(-2xT_A)1_{T_A<+\infty}] = \delta^{\Delta_\gamma}$ (see [43, 49]).

5.2. Dual quantum measure. A way to rigorously construct the singular quantum measures $\mu_{\gamma>2}$, which would reproduce the scaling properties seen above, has been presented in Refs. [9, 43, 49]. In the first approach [43, 49], one uses the dual measure $\mu_{\gamma'<2}$ and the additional randomness of sets of point masses where *finite* amounts of quantum area are *localized*. More precisely, conditionally on the quantum measure $\mu_{\gamma'}$, with $\gamma' = 4/\gamma < 2$, we consider a Poisson random measure $\mathcal{N}_\gamma(dz, d\eta)$ distributed on $D \times (0, \infty)$, of intensity $\mu_{\gamma'}(dz) \times \Lambda^\alpha(d\eta)$, where $\Lambda^\alpha(d\eta) := d\eta/\eta^{1+\alpha}$, $\alpha := 4/\gamma^2 \in (0, 1)$, letting each point (z, η) represent an atom of size η located at z. The dual measure for $\gamma > 2$ is then *purely atomic*,

$$\mu_\gamma(dz) := \int_0^\infty \eta\, \mathcal{N}_\gamma(dz, d\eta),\ \gamma > 2. \tag{5.3}$$

From this follows, for any Borelian $A \subset D$, the characteristic Laplace transform,

$$\mathbb{E} \exp[-u\, \mu_\gamma(A)] = \mathbb{E} \exp\left[\Gamma(-\alpha) u^\alpha \mu_{\gamma'}(A)\right], \tag{5.4}$$

a Lévy-Khintchine formula valid for all $u \in \mathbb{R}_+$, where $\Gamma(-\alpha) = -\Gamma(1-\alpha)/\alpha < 0$ is the usual Euler Γ-function.

From (5.4) follows the relation between moments of dual measures

$$\mathbb{E}\left[\left(\mu_\gamma(A)\right)^p\right] = \frac{\Gamma(1-p/\alpha)\Gamma(-\alpha)^{p/\alpha}}{\Gamma(1-p)} \mathbb{E}\left[\left(\mu_{\gamma'}(A)\right)^{p/\alpha}\right],\ p < \alpha = 4/\gamma^2,$$

showing the characteristic scaling of μ_γ as $\mu_{\gamma'}^{1/\alpha}$, in agreement with (5.1).

In Ref.[9], a slightly different (and more general) construction was proposed, which the authors called "atomic Gaussian multiplicative chaos". For $\alpha \in (0, 1)$, they consider a Poisson random measure N_α distributed on $\mathbb{R}^d \times \mathbb{R}_+^*$, with intensity $dz\Lambda^\alpha(d\eta)$, and introduce the random measure $n_\alpha(dz) = \int_0^\infty \eta\, N_\alpha(dz, d\eta)$, as an independently scattered α-stable random process. For $X(z)$ a (stationary, positive type kernel) log-correlated Gaussian distribution on $\mathbb{R}^d$, they introduce a random measure, formally written as,

$$\overline{M}^\gamma(dz) := e^{\gamma X(z) - \alpha\gamma^2 \mathbb{E}[X(z)^2]/2}\, n_\alpha(dz),\ \alpha\gamma < \sqrt{2d}. \tag{5.5}$$

It is shown in Ref. [9] that the measure (5.5) can be obtained as a limit of suitable cutoff approximated measures, that is *a.s. purely atomic*. For $\gamma' := \alpha\gamma < \sqrt{2d}$, let $M^{\gamma'}$ be the standard Gaussian multiplicative chaos [9]. The Laplace transform of (5.5) obeys the same relation (5.4) to $M^{\gamma'}$. When specifying to the duality case $\alpha = 2d/\gamma^2$ and to $d = 2$, $M^{\gamma'<2}$ essentially corresponds to the standard Liouville measure $\mu_{\gamma'}$. The random measure $\overline{M}^{\gamma>2}$ (5.5) and the dual measure $\mu_{\gamma>2}$ (5.3) then obey the same law (5.4) in terms of $\mu_{\gamma'}$, thus coincide, even though the two constructions differed a priori. A form of the KPZ relation (in the Hausdorff dimension sense) is also proved in Ref. [9] for the atomic Gaussian multiplicative chaos and for dual exponents defined exactly as in Eq. (5.2).

6. Critical $\gamma = 2$ case

As mentioned above, for the critical value $\gamma = 2$ (and above it), the limit of the Liouville regularized measure (1.7), $\lim_{\varepsilon\to 0} \mu_{\gamma\geq 2,\varepsilon}(dz) = 0$, vanishes, giving rise to the issue of

constructing non trivial measures for $\gamma = 2$. This question was addressed recently [47, 48] for Gaussian multiplicative chaos and for the GFF.

Following work on similar questions for branching random walks [1, 18], it was first shown in [47] that the natural object at criticality is a so-called *derivative martingale*, leading to the construction of a derivative Gaussian multiplicative chaos, formally written (in the GFF case) as

$$\mu'(dz) = (2\,\mathbb{E}[h(z)^2] - h(z))\, e^{2h(z)-2\mathbb{E}[h(z)^2]}\, dz.$$

It is shown to be a *positive, atomless* random measure. A proof of the similar non-atomicity of critical Mandelbrot cascades was obtained in Ref. [10].

The second approach involves using a modified renormalization scheme [48] inspired by works on branching random walks [1, 59], which shows that the derivative martingale can be recovered via a renormalization of the standard martingale, a procedure sometimes called Seneta-Heyde norming. This equivalence turns out to be crucial for applying Kahane's convexity inequalities at criticality. One can then study the moments and the power-law spectrum, and achieve a proof of the KPZ relation at criticality. This is done first for so-called $\star$-scale invariant correlation kernels [3], which are compact, and in a second step for the GFF which has long-range correlations, and requires some technical adjustments [48].

From a conformal field-theoretic view point, the critical value $\gamma = 2$ corresponds to a central charge $c = 1$ in (1.3). The critical measure thus gives a rigorous construction of the so-called "tachyon field" in $c = 1$ Liouville CFT. (See also [70].) We refer to Refs. [47, 48] and the references therein, for an extensive discussion of the interpretation of this measure in various $c = 1$ models of theoretical physics.

6.1. Derivative chaos - renormalization. One considers [47] centered Gaussian processes $(X_t(x))_{x\in\mathbb{R}^d}$ with covariance $\mathbb{E}[X_t(0)X_t(x)] = \int_1^{e^t} k(ux)\, du/u$, for some kernel k satisfying $k(0) = 1$, and vanishing outside a compact set (a condition to be removed for the GFF case in Section 6.2). For each $x \in \mathbb{R}^d$, the process $t \mapsto X_t(x)$ is a Brownian motion. For $\gamma \geq 0$, one considers the approximate Gaussian multiplicative chaos

$$M_t^\gamma(dx) = e^{\gamma X_t(x) - \frac{\gamma^2}{2}\mathbb{E}[X_t(x)^2]} dx. \tag{6.1}$$

It is well-known [3, 60] that the family of random measures $(M_t^\gamma)_{t>0}$, which are positive martingales, weakly converges a.s. as $t \to +\infty$ towards a log-normal $\star$-scale invariant random measure M^γ, which is non-trivial only for $\gamma^2 < 2d$. This construction is rather universal: the limiting measure M^γ is insensitive to the choice of the cut-off family $(X_t)_t$ made to approximate X.

One introduces the approximate derivative multiplicative chaos $M_t'(dx)$:

$$M_t'(dx) = (\sqrt{2d}\, t - X_t(x))\, e^{\sqrt{2d}X_t(x) - d\mathbb{E}[X_t(x)^2]} dx. \tag{6.2}$$

For each open bounded set $A \subset \mathbb{R}^d$, $M_t'(A)$ is a martingale, neither nonnegative nor regular. Nevertheless, the following theorems hold:

Theorem 6.1 ([47]). *Almost surely, the (locally signed) random measures $M_t'(dx)$ converge weakly as $t \to +\infty$ towards a positive random measure $M'(dx)$. This limiting measure has full support and is atomless.*

Theorem 6.2 ([48]). *The family $(\sqrt{t}M_t^{\sqrt{2d}})_t$ converges in probability as $t \to +\infty$ towards a non degenerate limit, which is the same, up to multiplicative constant, as the derivative multiplicative chaos: $\sqrt{t}M_t^{\sqrt{2d}}(dx) \to \sqrt{2/\pi}M'(dx)$, in probability as $t \to +\infty$.*

The measure M' is a non-trivial solution to the so-called $\star$-equation [3] at $\gamma = \sqrt{2d}$. The proofs, which rely in part on similar results for multiplicative cascades in [1], use critically the fact that under a rooted (Peyrière) measure, similar to (2.9), the law of the random process $\sqrt{2d}\,t - X_t(x) + \beta$, with $\beta > 0$ and conditioned to stay positive, is that of a 3-dimensional Bessel process started at level β. The main contributions to $M'_t(dx)$ (6.2) come from locations x where the factor $\sqrt{2d}\,t - X_t(x)$ is positive but close to zero (of the order of $\sqrt{t}$), and where $X_t(x)$ is nearly maximal, a scaling in agreement with Theorem 6.2.

6.2. GFF and critical Liouville measure. Analogous results can be proved for the GFF h, e.g., with Dirichlet boundary conditions on a bounded $D \subset \mathbb{R}^2$. A cut-off family of the GFF on D is then defined from a white noise W distributed on $D \times \mathbb{R}_+$ as: $h_\varepsilon(z) := \sqrt{\pi}\int_{D\times[\varepsilon^2,\infty[} p_D(\frac{s}{2}, z, y)W(dy, ds)$, where p_D is the heat kernel on D. (We keep here the same notation as for GFF circle averages, the two regularizations having the same scaling.) Define then the approximating measures

$$\mu_{(\gamma=2,\varepsilon)}(dz) := e^{2h_\varepsilon(z)-2\mathrm{Var}[h_\varepsilon(z)]}\,dz$$
$$\mu'_\varepsilon(dz) := (2\mathrm{Var}[h_\varepsilon(z)] - h_\varepsilon(z))\,e^{2h_\varepsilon(z)-2\mathrm{Var}[h_\varepsilon(z)]}\,dz.$$

Theorem 6.3 ([48]). *The random measure $\mu'_\varepsilon(dz)$ almost surely weakly converges as $\varepsilon \to 0$ towards a positive and atomless random measure $\mu'(dz)$, which can also be obtained as the limit $\sqrt{\log(1/\varepsilon)}\,\mu_{(\gamma=2,\varepsilon)}(dz) \to \sqrt{\frac{2}{\pi}}\,\mu'(dz)$, in probability as $\varepsilon \to 0$.*

Theorem 6.4 (**Conformal invariance** [48]). *The critical Liouville measure is defined as $\mu_{\gamma=2}(dz) := C(z;D)^2\mu'(dz)$, with $C(z;D)$ the conformal radius of D at z; it is conformally invariant as in* (1.9)*.*

From Kahane's convexity inequalities and similar results for the Gaussian multiplicative chaos, one further deduces

Corollary 6.5 ([48]). *For $A \subset D$ non intersecting ∂D, and for all $p < 1$, the random variable $\mu_{\gamma=2}(A)$ has finite moment of order p; its power law spectrum is given for $0 \le p < 1$ by $\xi(p) = 4p - 2p^2$, so that, for $p < 1$ and $z \in D$, one may find a constant C_p such that $\mathbb{E}[\mu_{\gamma=2}(B(z,\varepsilon))^p] \simeq C_p\varepsilon^{\xi(p)}$ as $\varepsilon \to 0$.*

This finally leads to a KPZ theorem, here in a Hausdorff dimension sense:

Theorem 6.6 (**KPZ at criticality**, [48]). *Let X be a compact (possibly random) subset of D of dimension d. Almost surely, $d = 4\dim_{\gamma=2}(X) - 2\dim^2_{\gamma=2}(X)$.*

It coincides with (1.1) for $\gamma = 2$, $x = \Delta^2$, with $d = 2 - 2x$ and $\dim_{\gamma=2}(X) = 1 - \Delta$.

7. SLE and Liouville quantum gravity

The aim of this section is to describe a fundamental connection between SLE and Liouville quantum gravity [98]: conformally welding two γ-quantum surfaces produces SLE_κ with

parameter $\kappa = \gamma^2 < 4$. We also give an example of a *quantum fractal measure*, here the SLE quantum length measure, using the KPZ relation; it gives a quantum gravity interpretation to a family of related SLE exponential martingales, which generalize the Liouville quantum measure [51].

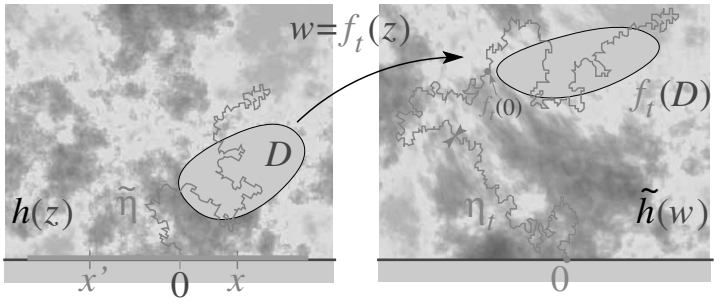


Figure 7.1. Chordal "zipping-up" SLE$_\kappa$ map $w = f_t(z)$ with curve η_t in $\mathbb{H}$. Conformal welding: the quantum boundary lengths, $\nu_\gamma([0,x]) = \nu_\gamma([x',0])$, are equal for any real segments $[0,x]$ and $[x',0]$ such that $f_t(x) = f_t(x') \in \eta_t$ [98]. The SLE$_\kappa$ $\widetilde{\eta}$ on the left is h-independent.

7.1. SLE martingale. Chordal SLE [95] is a random non self-crossing path in the complex half plane $\mathbb{H}$; we mainly use here a (time-reversed) version defined at time $t \geq 0$ by a *"zipping up"* conformal map $w := f_t(z)$, from the complex half-plane $\mathbb{H}$ to the slit domain $\mathbb{H} \setminus \eta_t$, with the SLE segment $\eta_t := f_t(\mathbb{R}) \setminus \mathbb{R}$ (or its external envelope) from 0 to the tip $f_t(0)$ (Figure 7.1). It satisfies the SDE,

$$df_t(z) = -2dt/f_t(z) - \sqrt{\kappa}dB_t$$

(with $f_0(z) = z$), where B_t is standard Brownian motion with $B_0 = 0$, and $\kappa \geq 0$. Let us recall that if $0 \leq \kappa \leq 4$, then SLE$_\kappa$ is a simple curve, while for $4 < \kappa < 8$ it develops double points and becomes space-filling for $\kappa \geq 8$ [92]. Define the real stochastic process for $t \geq 0$ and $z \in \mathbb{H}$,

$$\mathfrak{h}_0(z) := (2/\sqrt{\kappa}) \log|z|, \tag{7.1}$$

$$\mathfrak{h}_t(z) := \mathfrak{h}_0 \circ f_t(z) + Q \log|f_t'(z)|. \tag{7.2}$$

By stochastic Itô calculus, the particular choice,

$$Q = \sqrt{\kappa}/2 + 2/\sqrt{\kappa}, \tag{7.3}$$

gives a (local) martingale $d\mathfrak{h}_t(z) = -R_t(z)dB_t$, with $R_t(z) := \Re[2/f_t(z)]$. It has the further martingale property $\mathbb{E}\,\mathfrak{h}_t(z) = \mathfrak{h}_0(z)$.

Consider now the *Neumann Green function* in $\mathbb{H}$, $G_0(y,z) := -\log(|y-z||y-\overline{z}|)$, and define the *time-dependent* $G_t(y,z) := G_0\big(f_t(y), f_t(z)\big)$, i.e., G_0 taken at image points under f_t. A simple calculation of the Green function's variation shows that $-dG_t(y,z) = R_t(y)R_t(z)dt = d\langle \mathfrak{h}_t(y), \mathfrak{h}_t(z)\rangle$ (Hadamard's formula). Integrating w.r.t. t yields the covariation $\langle \mathfrak{h}_t(y), \mathfrak{h}_t(z)\rangle = G_0(y,z) - G_t(y,z)$. Taking the limit $y \to z$ in the latter, one obtains

$$\langle \mathfrak{h}_t(z), \mathfrak{h}_t(z)\rangle = C_0(z) - C_t(z), \tag{7.4}$$

where $C_t(z) := -\log\left[\Im f_t(z)|f_t'(z)|\right]$.

7.2. SLE-GFF coupling. Consider $h := \widetilde{h} + \mathfrak{h}_0$, sum of an instance $\widetilde{h}$ of the Gaussian free field on domain $D = \mathbb{H}$ with *free boundary conditions* (f.b.c.) on $\mathbb{R}$ (up to additive constant), and of the deterministic function $\mathfrak{h}_0$ (7.1). This h can be coupled [98] with the reverse Loewner evolution f_t described above so that, *given* f_t, the conditional law of h (denoted by $h|f_t$) is (Figure 7.1)

$$h(z)|f_t \overset{\text{(law)}}{=} \widetilde{h}\circ f_t(z) + \mathfrak{h}_t(z), \tag{7.5}$$

where $\widetilde{h}\circ f_t$ is the pullback of the free boundary GFF $\widetilde{h}$ in the image half-plane, and where $\mathfrak{h}_t$ is the martingale (7.2). This means that to sample h, one can first sample the B_t process (which determines f_t), then sample independently the f.b.c. GFF $\widetilde{h}$ and take (7.5). Its conditional expectation w.r.t. $\widetilde{h}$ is the martingale $\mathbb{E}\big[h(z)|f_t\big] = \mathfrak{h}_t(z)$. To understand (7.5), recall that the Green's function $G_0(y,z) = \mathrm{Cov}[\widetilde{h}(y),\widetilde{h}(z)]$, thus $G_t = \mathrm{Cov}[\widetilde{h}\circ f_t, \widetilde{h}\circ f_t]$. The random distribution $\widetilde{h}\circ f_t$ and the set of (time changed) Brownian motions $\mathfrak{h}_t$ are Gaussian processes, whose respective covariance G_t and covariation $\langle \mathfrak{h}_t, \mathfrak{h}_t\rangle$ thus add to the constant covariance G_0; this in essence yields (7.5) [98].

7.2.1. Liouville conformal invariance. Thanks to (7.2), we observe that the r.h.s. of (7.5) is of the form $h\circ f_t + Q\log|f_t'|$. For Q given by (1.9), this is the transformation law in (1.9) of the GFF h under the conformal map f_t^{-1}. Then the pair $(\mathbb{H}, \widetilde{h}\circ f_t + \mathfrak{h}_t) = f_t^{-1}(\mathbb{H}\setminus\eta_t, h)$ describes the same quantum surface as the pair $(\mathbb{H}\setminus\eta_t, h)$: Given f_t, the image under f_t of the Liouville measure $\mu_\gamma(dz)$ in $\mathbb{H}$ is a random measure whose law is the *a priori* (unconditioned) law of $\mu_\gamma(dw)$ in $\mathbb{H}\setminus\eta_t$.

Identifying (1.9) and (7.3), we find two dual solutions

$$\gamma = \sqrt{\kappa \wedge (16/\kappa)},\ \ \gamma' = 4/\gamma. \tag{7.6}$$

The first solution $\gamma \le 2$ corresponds precisely to the famous KPZ relation (1.3) [34, 38, 66] between the parameter γ in Liouville theory and the central charge c; here $c = \frac{1}{4}(6-\kappa)(6-16/\kappa)$ for the SLE's CFT [11, 54, 61]. The second solution $\gamma' = 4/\gamma \ge 2$ corresponds to the *dual* phase of Liouville quantum gravity [64], as described in Section 5, where the quantum measure develops atoms.

7.2.2. Quantum conformal welding. In the particular coupling (7.5) of h and f_t, the two strands of the boundary to be matched along η_t when "zipping up" by the reverse Schramm-Loewner map f_t have the *same quantum length* ν_γ (4.1) (at least for $\kappa < 4$) (Figure 7.1). This property defines *a quantum conformal welding*, and actually determines f_t as a function of h [98].

Let now $\widetilde{\eta}$ be an (infinite) SLE_κ, independent of h (Figure 7.1). For each time $t \ge 0$, the *forward, "zipping down"* SLE flow map f_{-t}, which obeys the same SDE as f_t, but for $dt \to -dt$, maps $\mathbb{H}\setminus\widetilde{\eta}_t \to \mathbb{H}$, where $\widetilde{\eta}_t$ is the SLE curve segment up to time t. When $\kappa < 4$, $\widetilde{\eta}$ divides $\mathbb{H}$ into a pair of welded quantum surfaces that is *stationary* w.r.t. zipping up or down via the transformations f_t ($t \in \mathbb{R}$) [98]. *The relation* (7.6) *between* γ *and* κ *is now rigorously clear: conformally welding two* γ*-quantum surfaces produces* SLE_κ.

7.3. Exponential martingales. Let us introduce the conditional expectations of exponentials of (7.5), $\mathcal{M}_t^\alpha(z) := \mathbb{E}\big[e^{\alpha h(z)}|f_t\big]$, depending on a real parameter α, which are fundamental objects describing quantum gravity coupled to the SLE process. They can be calculated explicitly in terms of (7.2) and (7.4):

$$\begin{aligned}\mathcal{M}_t^\alpha(z) := \mathbb{E}\big[e^{\alpha h(z)}|f_t\big] &= \exp\big[\alpha \mathfrak{h}_t(z) + (\alpha^2/2)C_t(z)\big] \\ &= |f_t'(z)|^d\,|w|^{2\alpha/\sqrt{\kappa}}(\Im w)^{-\alpha^2/2},\end{aligned} \tag{7.7}$$

with $w = f_t(z)$ and d given by $d := \alpha Q - \alpha^2/2$. This *is* the *KPZ relation* (1.1), after setting $d = 2 - 2x$, $\alpha = \gamma(1-\Delta)$ [51]. (See also Proposition 3.1.)

By definition, (7.7) is an *exponential martingale* with respect to the Brownian motion driving the reverse SLE process, so that

$$\mathbb{E}\mathcal{M}_t^\alpha(z) = \mathcal{M}_0^\alpha(z) = |z|^{2\alpha/\sqrt{\kappa}}(\Im z)^{-\alpha^2/2}. \tag{7.8}$$

A stronger statement is the identity in law of the conditional exponential measures

$$\big(e^{\alpha h(z)}|f_t\big)\,dz \overset{\text{(law)}}{=} |f_t'(z)|^{d-2}e^{\alpha h(w)}dw,$$

with $dw = |f_t'(z)|^2 dz$, and whose expectations (7.7) agree.

7.3.1. Expected quantum measure. For $\alpha = \gamma$, $d = 2$, we recover the conformal invariance of the Liouville quantum measure, $\big(\mu_\gamma(dz)|f_t\big) \overset{\text{(law)}}{=} \mu_\gamma\,(dw)$, with

$$\begin{aligned}\mathbb{E}[\mu_\gamma(dz)|f_t] &= |w|^{2-\kappa/2}(\sin\varphi)^{-\kappa/2}dw,\ \ \kappa \le 4 \\ &= (\sin\varphi)^{-8/\kappa}dw,\ \ \kappa \ge 4; \varphi := \arg w.\end{aligned}$$

We give now an explicit example of an invariant SLE quantum measure, using the martingales (7.7) for $\alpha \neq \gamma$.

7.3.2. SLE quantum length. An SLE measure recently introduced in the context of the so-called *natural parametrization* of SLE [74, 77] describes the "fractal length" of the intersection $\widetilde{\eta} \cap D$ of the SLE_κ path $\widetilde{\eta}$ (from 0 to ∞) with an arbitrary domain $D \subset \mathbb{H}$ (Figure 7.1). Its expectation with respect to the $\mathrm{SLE}_{\kappa\in[0,8]}$ law is finite for any bounded D, and given by

$$\nu(D) := \int_D G(z)dz,\ \ G(z) = |z|^a|\Im z|^b,$$

where G is the SLE Green's function in $\mathbb{H}$, with $a = 1 - 8/\kappa$, $b = 8/\kappa + \kappa/8 - 2$.

Under the forward direction SLE flow f_{-t} that generates $\widetilde{\eta}$, the quantity $M_t := (G \circ f_{-t})|f_{-t}'|^{2-d}$, where $d := 1 + \kappa/8$ is the SLE_κ Hausdorff dimension [12], describes the density of expected Euclidean fractal length of $\widetilde{\eta} \setminus \widetilde{\eta}_t$, *given* the segment $\widetilde{\eta}_t$ [74]. This M_t is a local martingale w.r.t. the forward SLE flow f_{-t} [74]. Geometrically, $\int_D M_t(z)dz$ is the expected length of $\widetilde{\eta} \cap D$ given f_{-t} (a martingale), minus the length of the segment $\widetilde{\eta}_t \cap D$ (an increasing process); this *Doob-Meyer decomposition* determines the latter length as a stochastic process [74].

We extend this construction to the quantum case by defining the expected (w.r.t. $\widetilde{\eta}$, given h) Liouville *quantum length* $\nu_{\mathcal{Q}}$ of an infinite SLE path in D

$$\nu_{\mathcal{Q}}(D,h) := \int_D e^{\alpha h(z)}G(z)dz, \tag{7.9}$$

where $\alpha = \sqrt{\kappa}/2$ ($= \gamma/2$ for $\kappa \leq 4$, and $\gamma'/2$ for $\kappa > 4$) is chosen to satisfy KPZ for the SLE dimension $d = 1 + \kappa/8$. Under the forward SLE flow f_{-t}, the corresponding integral $\int_D e^{\alpha h(z)} M_t(z) dz$ yields, by Doob-Meyer, an implicit construction of the quantum length measure. It exists by [76, 77] since the second moment $\mathbb{E}[e^{\alpha h(y)+\alpha h(z)} M_t(y) M_t(z)]$ is bounded by $|y-z|^{\mathfrak{d}-2}$, with $\mathfrak{d} = d - \alpha^2 = 1 - \kappa/8$, thus integrable for $\mathfrak{d} > 0$, i.e., $\kappa < 8$. By unzipping via f_{-t}, it must naturally coincide with the *Liouville boundary measure* ν_γ defined on $\mathbb{R}$; this follows rigorously from [74] (under a finite expectation assumption).

Alternatively, we can condition (7.9) on the *reverse* SLE flow f_t, and get from (7.7) its expectation w.r.t. h, conditioned on f_t,

$$\mathbb{E}[\nu_{\mathcal{Q}} | f_t] = \int_D \mathcal{M}_t^\alpha(z) G(z)\, dz.$$

Finally, taking expectation w.r.t. f_t via (7.8) gives the expected SLE quantum length in D (here $\vartheta := \arg z$):

$$\mathbb{E}\, \nu_{\mathcal{Q}}(D) = \int_D \mathcal{M}_0^\alpha(z)\, G(z)\, dz = \int_D (\sin\vartheta)^{8/\kappa - 2} dz,$$

which is finite for $\kappa \in [0, 8)$, and coincides with the *Euclidean area* of D for $\kappa = 4$.

SLE *boundary* quantum measures can be similarly constructed [51]. They use the SLE boundary fractal measure, supported on the intersection for $\kappa \in (4, 8)$ of a chordal SLE_κ curve with the real axis, as constructed in Ref. [2].

7.4. Quantum wedges. Roughly speaking, a quantum wedge $\mathcal{W}$ is a quantum surface with two marked boundary points that is obtained by taking h as the free boundary GFF on an infinite wedge $\{z : 0 < \arg z < \theta\}$ for some θ [46]. If we conformally map the half plane to the wedge via the map $\psi_\theta(\tilde{z}) = \tilde{z}^{\,\theta/\pi}$, then the coordinate change rule (1.9) gives $\widetilde{D} = \mathbb{H}$ and $\widetilde{h} = h - \alpha \log|\tilde{z}|$, where h is a free boundary GFF on $\mathbb{H}$ and $\alpha := Q(1 - \theta/\pi)$. The **weight** of a quantum wedge is defined as $W := \gamma\,(\gamma + 2/\gamma - \alpha)$, with $\gamma < 2$ and $W > \gamma^2/2$ for $\alpha < Q$ (i.e., $\theta > 0$).

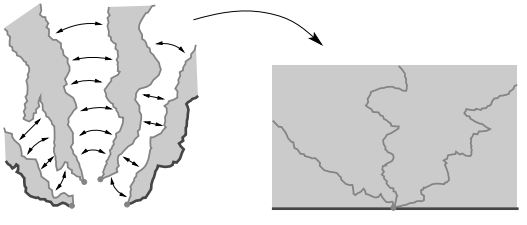


Figure 7.2. Four quantum wedges conformally welded along boundaries and conformally mapped to $\mathbb{H}$. The images of the interfaces are coupled $\mathrm{SLE}_\kappa(\rho_1; \rho_2)$ processes [46].

Theorem 7.1 ([46]). *Choose a quantum wedge $\mathcal{W}$ of weight $W > 0$, represented by some $(D, h, (z_1, z_2))$, with $z_1, z_2 \in \partial D$. Suppose $W = W_1 + W_2$ for some $W_i > \gamma^2/2$ and then independently choose an $\mathrm{SLE}_\kappa(\rho_1; \rho_2)$, for $\rho_i = W_i - 2$ and $\kappa = \gamma^2$, from z_1 to z_2. Let η denote the SLE curve and let $D_i, i = 1, 2$, denote left and right components of $D \setminus \eta$. Then the quantum surfaces $\mathcal{W}_i = (D_i, h, (z_1, z_2))$ (with h restricted to D_i), are independent.*

Each $\mathcal{W}_i$ is a quantum wedge of weight W_i, and $\mathcal{W}$ is uniquely determined by the $\mathcal{W}_i$ and may be obtained by a conformal welding *of the right side of $\mathcal{W}_1$ to the left side of $\mathcal{W}_2$, where each is parameterized by quantum length.*

Theorem 7.1 generalizes the welding results of Ref. [98] (Section 7.2), for which $W_i = 2$. In fact, it also holds for quantum wedges of positive weight less than $\gamma^2/2$; these wedges are not topologically homeomorphic to a disc (their left side hits their right side at a random fractal set of points) [46]. If one uses the boundary analog of the thick point Proposition 3.1 [50] to relate a boundary quantum exponent $\widetilde{\Delta}$ to $W = 2 + \gamma^2\widetilde{\Delta}$, one finds that Theorem 7.1 gives precisely the additivity of boundary quantum exponents predicted in Refs. [41, 42, 75].

We presented here a foundational relationship between SLE, KPZ and Liouville quantum gravity. It is hoped that it will help solve the outstanding open problem of its rigorous relation to discrete models and random planar maps, in line with the approach of Ref. [99], and Miller and Sheffield's *"Imaginary Geometry"* tetralogy [84–87]. In this perspective, the recent introduction by these authors of the so-called Quantum Loewner Evolution (QLE) in [88] might finally pave the way to endow Liouville quantum surfaces with a canonical metric space structure (for $\gamma^2 = 8/3$), isomorphic in law to the universal Brownian map of Refs. [80, 83].

Acknowledgements. I would like to thank my co-authors: JASON MILLER, RÉMI RHODES, SCOTT SHEFFIELD, and VINCENT VARGAS. I also thank the MIT Mathematics Department and Center for Theoretical Physics for their hospitality over the years, suppported by the MISTI MIT-France Seed Fund. This work was also partially supported by an Agence Nationale de la Recherche Grant, and by a CNRS PEPS PTI grant.

References

[1] E. Aïdékon and Z. Shi, *The Seneta-Heyde scaling for the branching random walk*, Ann. Probab. **42** (2014), no. 3, 959–993.

[2] T. Alberts and S. Sheffield, *The covariant measure of SLE on the boundary*, Probab. Th. Rel. Fields **149** (2011), 331–371.

[3] R. Allez, R. Rhodes, and V. Vargas, *Lognormal $\star$-scale invariant random measures*, Probab. Th. Rel. Fields **155** (2013), no. 3-4, 751–788.

[4] J. Ambjørn and T. G. Budd, *Trees and spatial topology change in causal dynamical triangulations*, J. Phys. A: Math. Theor. **46** (2013), no. 31, 315201.

[5] J. Ambjørn, B. Durhuus, and J. Fröhlich, *Diseases of triangulated random surface models, and possible cures*, Nucl. Phys. B **257** (1985), 433–449.

[6] J. Ambjørn and Y. Watabiki, *Scaling in quantum gravity*, Nucl. Phys. B **445** (1995), no. 1, 129–142.

[7] O. Angel and O. Schramm, *Uniform infinite planar triangulations*, Commun. Math. Phys. **241** (2003), no. 2-3, 191–213.

[8] K. Astala, P. Jones, A. Kupiainen, and E. Saksman, *Random conformal weldings*, Acta Math. **207** (2011), no. 2, 203–254.

[9] J. Barral, X. Jin, R. Rhodes, and V. Vargas, *Gaussian multiplicative chaos and KPZ duality*, Commun. Math. Phys. **323** (2013), no. 2, 451–485.

[10] J. Barral, A. Kupiainen, M. Nikula, E. Saksman, and C. Webb, *Critical Mandelbrot cascades*, Commun. Math. Phys. **325** (2014), no. 2, 685–711.

[11] M. Bauer and D. Bernard, *Conformal Field Theories of Stochastic Loewner Evolutions*, Commun. Math. Phys. **239** (2003), 493–521.

[12] V. Beffara, *The dimension of the SLE curves*, Ann. Probab. **36** (2008), no. 4, 1421–1452.

[13] A.A. Belavin, A.M. Polyakov, and A.B. Zamolodchikov, *Infinite conformal symmetry in two-dimensional quantum field theory*, Nucl. Phys. B **241** (1984), 333–380.

[14] I. Benjamini, *Random planar metrics*, Bhatia, Rajendra (ed.) et al., Proceedings of the International Congress of Mathematicians (ICM 2010), Hyderabad, India, August 19–27, 2010. Vol. IV. Hindustan Book Agency. 2177-2187, 2011.

[15] I. Benjamini and N. Curien, *Simple random walk on the uniform infinite planar quadrangulation: Subdiffusivity via pioneer points*, Geom. Funct. Anal. **23** (2013), no. 2, 501–531.

[16] I. Benjamini and O. Schramm, *KPZ in One Dimensional Random Geometry of Multiplicative Cascades*, Commun. Math. Phys. **289** (2009), 46–56.

[17] O. Bernardi and M. Bousquet-Mélou, *Counting colored planar maps: Algebraicity results*, J. Comb. Theory Ser. B **101** (2011), no. 5, 315–377.

[18] J.D. Biggins and A.E. Kyprianou, *Measure change in multitype branching*, Adv. Appl. Probab. **36** (2004), no. 2, 544–581.

[19] G. Borot, J. Bouttier, and E. Guitter, *Loop models on random maps via nested loops: the case of domain symmetry breaking and application to the Potts model*, J. Phys. A: Math. Theor. **45** (2012), 494017.

[20] M. Bousquet-Mélou and G. Schaeffer, *The degree distribution in bipartite planar maps: applications to the Ising model*, Proceedings of *FPSAC* 03, Vadstena, Sweden, June 2003, K. Eriksson and S. Linusson eds., 2003, pp. 312–323.

[21] J. Bouttier, P. Di Francesco, and E. Guitter, *Geodesic distance in planar graphs*, Nucl. Phys. B **663** (2003), 535–567.

[22] ———, *Planar maps as labeled mobiles*, Electron. J. Comb. **11** (2004), no. 1, #R69.

[23] J. Bouttier and E. Guitter, *Confluence of geodesic paths and separating loops in large planar quadrangulations*, J. Stat. Mech. **3** (2009), P03001.

[24] E. Brézin, C. Itzykson, G. Parisi, and J. B. Zuber, *Planar diagrams*, Commun. Math. Phys. **59** (1978), 35–51.

[25] G. Chapuy, M. Marcus, and G. Schaeffer, *A bijection for rooted maps on orientable surfaces*, SIAM J. Discrete Math. **23** (2009), no. 3, 1587–1611.

[26] P. Chassaing and G. Schaeffer, *Random planar lattices and integrated superbrownian excursion*, Probab. Th. Rel. Fields **128** (2004), 161–212.

[27] D. Chelkak and S. Smirnov, *Universality in the 2D Ising model and conformal invariance of fermionic observables*, Invent. Math. **189** (2012), no. 3, 515–580.

[28] R. Cori and B. Vauquelin, *Planar maps are well labeled trees*, Canad. J. Math. **33** (1981), no. 5, 1023–1042.

[29] N. Curien, *A glimpse of the conformal structure of random planar maps*, arXiv: 1308.1807 (2013).

[30] N. Curien and J.-F. Le Gall, *The Brownian plane*, J. Theoret. Probab., To appear.

[31] S. R. Das, A. Dhar, A. M. Sengupta, and S. R. Wadia, *New critical behavior in* $d = 0$ *large-*N *matrix models*, Mod. Phys. Lett. A **5** (1990), 1041–1056.

[32] J.-M. Daul, *Q-States Potts model on a random planar lattice*, arXiv:hep-th/9502014 (1995).

[33] F. David, *Planar diagrams, two-dimensional lattice gravity and surface models*, Nucl. Phys. B **257** (1985), 45 – 58.

[34] ———, *Conformal field theories coupled to* 2*-D gravity in the conformal gauge*, Mod. Phys. Lett. A **3** (1988), no. 17, 1651–1656.

[35] F. David and M. Bauer, *Another derivation of the geometrical KPZ relations*, J. Stat. Mech. **3** (2009), P03004.

[36] F. David and B. Eynard, *Planar maps, circle patterns and 2d gravity*, arXiv:1307.3123 (2013), to appear in Ann. Inst. Henri Poincaré Comb. Phys. Interact.

[37] P. Di Francesco, P. Ginsparg, and J. Zinn-Justin, 2*D gravity and random matrices*, Phys. Rep. **254** (1995), 1–133.

[38] J. Distler and H. Kawai, *Conformal field theory and* 2*D quantum gravity*, Nucl. Phys. B **321** (1989), 509–527.

[39] J. Dubédat, *Duality of Schramm-Loewner evolutions*, Ann. Sci. Éc. Norm. Supér. (4) **42** (2009), no. 5, 697–724.

[40] B. Duplantier, *Random walks and quantum gravity in two dimensions*, Phys. Rev. Lett. **81** (1998), 5489–5492.

[41] ———, *Conformally invariant fractals and potential theory*, Phys. Rev. Lett. **84** (2000), 1363–1367.

[42] ———, *Conformal fractal geometry & boundary quantum gravity*, Fractal geometry and applications: a jubilee of Benoît Mandelbrot, Part 2, Proc. Sympos. Pure Math., vol. 72, Amer. Math. Soc., Providence, RI, 2004, pp. 365–482.

[43] ———, *A rigorous perspective on Liouville quantum gravity and the KPZ relation*, Exact methods in low-dimensional statistical physics and quantum theory (Les Houches Summer School LXXXIX, 2008) (S. Ouvry, J. Jacobsen, V. Pasquier, D. Serban, and L. Cugliandolo, eds.), Oxford University Press, 2009, pp. 529–561.

[44] B. Duplantier and I. Kostov, *Conformal spectra of polymers on a random surface*, Phys. Rev. Lett. **61** (1988), 1433–1437.

[45] B. Duplantier and K.-H. Kwon, *Conformal invariance and intersections of random walks*, Phys. Rev. Lett. **61** (1988), 2514–2517.

[46] B. Duplantier, J. Miller, and S. Sheffield, *Liouville quantum gravity as a scaling limit*, In preparation.

[47] B. Duplantier, R. Rhodes, S. Sheffield, and V. Vargas, *Critical Gaussian Multiplicative Chaos: Convergence of the Derivative Martingale*, arXiv:1206.1671 (2012). To appear in Ann. Probab.

[48] ———, *Renormalization of Critical Gaussian Multiplicative Chaos and KPZ formula*, arXiv:1212.0529 (2012), To appear in Commun. Math. Phys., doi:10.1007/s00220-014-2000-6.

[49] B. Duplantier and S. Sheffield, *Duality and KPZ in Liouville Quantum Gravity*, Phys. Rev. Lett. **102** (2009), 150603.

[50] ———, *Liouville quantum gravity and KPZ*, Invent. Math. **185** (2011), 333–393.

[51] ———, *Schramm-Loewner Evolution and Liouville Quantum Gravity*, Phys. Rev. Lett. **107** (2011), 131305.

[52] B. Eynard and G. Bonnet, *The Potts-q random matrix model: loop equations, critical exponents, and rational case*, Phys. Lett. B **463** (1999), 273–279.

[53] B. Eynard and N. Orantin, *Invariants of algebraic curves and topological expansion*, Commun. Number Theory Phys. **1** (2007), no. 2, 347–452.

[54] R. Friedrich and W. Werner, *Conformal Restriction, Highest-Weight Representations and SLE*, Commun. Math. Phys. **243** (2003), 105–122.

[55] J. T. Gill and S. Rohde, *A note on quasiconformal maps with Hölder-continuous dilatation*, arXiv:1302.3899 (2013).

[56] P. Ginsparg and G. Moore, *Lectures on 2D gravity and 2D string theory (TASI 1992)*, Recent direction in particle theory, Proceedings of the 1992 TASI (J. Harvey and J. Polchinski, eds.), World Scientific, Singapore, 1993.

[57] R. Høegh-Krohn, *A general class of quantum fields without cut-offs in two space-time dimensions*, Commun. Math. Phys. **21** (1971), 244–255.

[58] X. Hu, J. Miller, and Y. Peres, *Thick points of the Gaussian free field*, Ann. Probab. **38** (2010), no. 2, 896–926.

[59] Y. Hu and Z. Shi, *Minimal position and critical martingale convergence in branching random walks, and directed polymers on disordered trees*, Ann. Probab. **37** (2009), no. 2, 742–789.

[60] J.-P. Kahane, *Sur le chaos multiplicatif*, Ann. Sci. Math. Québec **9** (1985), no. 2, 105–150.

[61] N. Kang and N. G. Makarov, *Gaussian free field and conformal field theory*, Astérisque 353. Paris: Société Mathématique de France (SMF), 2013.

[62] V. A. Kazakov, *Ising model on a dynamical planar random lattice: Exact solution*, Phys. Lett. A **119** (1986), 140–144.

[63] V. A. Kazakov, I. K. Kostov, and A. A. Migdal, *Critical properties of randomly triangulated planar random surfaces*, Phys. Lett. B **157** (1985), 295–300.

[64] I. R. Klebanov, *Touching random surfaces and Liouville gravity*, Phys. Rev. D **51** (1995), 1836–1841.

[65] I. R. Klebanov and A. Hashimoto, *Non-perturbative solution of matrix models modified by trace-squared terms*, Nucl. Phys. B **434** (1995), 264–282.

[66] V. G. Knizhnik, A. M. Polyakov, and A. B. Zamolodchikov, *Fractal structure of* 2*D-quantum gravity*, Mod. Phys. Lett. A **3** (1988), 819–826.

[67] G. P. Korchemsky, *Matrix model perturbed by higher order curvature terms*, Mod. Phys. Lett. A **7** (1992), 3081–3100.

[68] I. K. Kostov, $O(n)$ *vector model on a planar random lattice: Spectrum of anomalous dimensions*, Mod. Phys. Lett. A **4** (1989), 217–226.

[69] I. K. Kostov, B. Ponsot, and D. Serban, *Boundary Liouville theory and 2D quantum gravity*, Nucl. Phys. B **683** (2004), 309–362.

[70] H. Lacoin, R. Rhodes, and V. Vargas, *Complex Gaussian multiplicative chaos*, arXiv: 1307.6117 (2013).

[71] G. F. Lawler, O. Schramm, and W. Werner, *Values of Brownian intersection exponents. I. Half-plane exponents*, Acta Math. **187** (2001), no. 2, 237–273.

[72] ———, *Values of Brownian intersection exponents. II. Plane exponents*, Acta Math. **187** (2001), no. 2, 275–308.

[73] ———, *Conformal invariance of planar loop-erased random walks and uniform spanning trees*, Ann. Probab. **32** (2004), no. 1B, 939–995.

[74] G. F. Lawler and S. Sheffield, *A natural parametrization for the Schramm-Loewner evolution*, Ann. Probab. **39** (2011), no. 5, 1896–1937.

[75] G. F. Lawler and W. Werner, *Intersection exponents for planar Brownian motion*, Ann. Probab. **27** (1999), no. 4, 1601–1642.

[76] G. F. Lawler and B. M. Werness, *Multi-point Green's functions for SLE and an estimate of Beffara*, Ann. Probab. **41** (2013), no. 3A, 1513–1555.

[77] G. F. Lawler and W. Zhou, *SLE curves and natural parametrization*, Ann. Probab. **41** (2013), no. 3A, 1556–1584.

[78] J.-F. Le Gall, *The topological structure of scaling limits of large planar maps*, Invent. Math. **169** (2007), no. 3, 621–670.

[79] ———, *Geodesics in large planar maps and in the Brownian map*, Acta Math. **205** (2010), no. 2, 287–360.

[80] ———, *Uniqueness and universality of the Brownian map*, Ann. Probab. **41** (2013), no. 4, 2880–2960.

[81] G. Miermont, *On the sphericity of scaling limits of random planar quadrangulations*, Electron. Commun. Probab. **13** (2008), 248–257.

[82] ———, *Tessellations of random maps of arbitrary genus*, Ann. Sci. Éc. Norm. Supér. **42** (2009), no. 5, 725–781.

[83] ———, *The Brownian map is the scaling limit of uniform random plane quadrangulations*, Acta Math. **210** (2013), no. 2, 319–401.

[84] J. Miller and S. Sheffield, *Imaginary Geometry I: Interacting SLEs*, arXiv:1201.1496 (2012).

[85] ———, *Imaginary geometry II: Reversibility of* $SLE_\kappa(\rho_1;\rho_2)$ *for* $\kappa \in (0,4)$, arXiv: 1201.1497 (2012). To appear in Ann. Probab.

[86] ———, *Imaginary geometry III: Reversibility of* SLE_κ *for* $\kappa \in (4,8)$, arXiv: 1201.1498 (2012).

[87] ———, *Imaginary geometry IV: Interior rays, whole-plane reversibility, and space-filling trees*, arXiv:1302.4738 (2013).

[88] ———, *Quantum Loewner Evolution*, arXiv:1312.5745 (2013).

[89] B. Nienhuis, *Critical behavior of two-dimensional spin models and charge asymmetry in the Coulomb gas*, J. Stat. Phys. **34** (1984), 731–761.

[90] A. M. Polyakov, *Quantum geometry of bosonic strings*, Phys. Lett. B **103** (1981), 207–210.

[91] R. Rhodes and V. Vargas, *KPZ formula for log-infinitely divisible multifractal random measures*, ESAIM: Probability and Statistics **15** (2011), 358–371.

[92] S. Rohde and O. Schramm, *Basic properties of SLE*, Ann. of Math. **161** (2005), 883–924.

[93] S. Rohde and D. Zhan, *Backward SLE and the symmetry of the welding*, arXiv: 1307.2532 (2013).

[94] G. Schaeffer, *Conjugaison d'arbres et cartes combinatoires aléatoires*, PhD Thesis, Univ. Bordeaux I, Talence (1998).

[95] O. Schramm, *Scaling limits of loop-erased random walks and uniform spanning trees*, Israel J. Math. **118** (2000), 221–288.

[96] O. Schramm and S. Sheffield, *Contour lines of the two-dimensional discrete Gaussian free field*, Acta Math. **202** (2009), no. 1, 21–137.

[97] S. Sheffield, *Gaussian free fields for mathematicians*, Probab. Th. Rel. Fields **139** (2007), 521–541.

[98] ———, *Conformal weldings of random surfaces: SLE and the quantum gravity zipper*, arXiv:1012.4797 (2010).

[99] ———, *Quantum gravity and inventory accumulation*, arXiv:1108.2241 (2011).

[100] S. Smirnov, *Critical percolation in the plane: conformal invariance, Cardy's formula, scaling limits*, C. R. Acad. Sci. Paris Sér. I Math. **333** (2001), no. 3, 239–244.

[101] ———, *Conformal invariance in random cluster models. I. Holomorphic fermions in the Ising model*, Ann. of Math. (2) **172** (2010), no. 2, 1435–1467.

[102] D. Zhan, *Duality of chordal SLE*, Invent. Math. **174** (2008), no. 2, 309–353.

Institut de Physique Théorique, CEA/Saclay, 91191 Gif-sur-Yvette, France
E-mail: bertrand.duplantier@cea.fr

An overview of the topological recursion

Bertrand Eynard

Abstract. We recall how computing large size asymptotics in random matrices, has allowed to discover some fascinating and ubiquitous geometric invariants. Specializations of this method recover many classical invariants, like Gromov–Witten invariants, or knot polynomials (Jones, HOMFLY,...). In this short overview for the ICM 2014 in Seoul Korea, we give some examples, give definitions, and review some properties and applications of the formalism.

Mathematics Subject Classification (2010). Primary 14H70; Secondary 14H60.

Keywords. Topological recursion, invariants.

1. Introduction

The "topological recursion" is a recursive definition (introduced in [1]), which associates a double family (indexed by two non–negative integers g and n) of differential forms $\omega_{g,n}$, to a "spectral curve" $\mathcal{S}$ (a plane analytical curve with some additional structure, see definition below) called the "invariants" of $\mathcal{S}$.

$$\text{Topological Recursion:} \qquad \text{spectral curve } \mathcal{S} \longrightarrow \text{invariants } \omega_{g,n}(\mathcal{S})$$

The initial terms $\omega_{0,1}$ and $\omega_{0,2}$ are some canonical 1-form and 2-form on the spectral curve $\mathcal{S}$, the other $\omega_{g,n}$'s defined by a universal recursion on $(2g+n-2)$ are symmetric $n-$forms on $\mathcal{S}^n$, and the $n=0$ invariants, customarily denoted $F_g(\mathcal{S}) = \omega_{g,0}(\mathcal{S})$, are numbers $F_g(\mathcal{S}) \in \mathbb{C}$ (or in fact elements of the field over which $\mathcal{S}$ is defined).

Those invariants have fascinating mathematical properties, they are "symplectic invariants" (invariants under some symplectic transformations of the spectral curve), they are almost modular forms (under the modular $Sp_{2\mathfrak{g}}(\mathbb{Z})$ group when the spectral curve has genus $\mathfrak{g}$), they satisfy Hirota-like equations, they satisfy some form-cycle duality deformation relations (generalization of Seiberg-Witten relations), they are stable under many singular limits, and enjoy many other fascinating properties...

Moreover, specializations of those invariants recover many known invariants, including volumes of moduli spaces, Hurwitz numbers, intersection numbers, Gromov–Witten invariants, numbers of maps (Tutte's enumeration of maps), or asymptotics of random matrices expectation values. And since very recently, it is conjectured that they also include knot polynomials (Jones, HOMFLY, super polynomials...), which provides an extension of the volume conjecture.

The purpose of this short article is only a small glimpse of the fast evolving mathematics of those invariants. We shall present here a few examples, then mention how these invariants

Proceedings of the International Congress of Mathematicians, Seoul, 2014

were first discovered in random matrix theory, and then observed or conjectured in many other areas of maths and physics.

2. Examples of topological recursions

2.1. Mirzakhani's recursion for hyperbolic volumes. See short definition in appendix A.Let g, n be non–negative integers such that $2g-2+n > 0$ (i.e. $(g,n) = (0,0), (0,1), (0,2), (1,0)$ are excluded). Let $\mathcal{V}_{g,n}(L_1, \dots, L_n)$ be the hyperbolic volume (called "Weil-Petersson volume" [2]) of the moduli-space $\mathcal{M}_{g,n}$ of genus g bordered Riemann surfaces with n geodesic boundaries of respective lengths $L_1, \dots, L_n$

$$\mathcal{V}_{g,n}(L_1, \dots, L_n) = \int_{\mathcal{M}_{g,n}\ ,\ \ell(\partial_i)=L_i} w,$$

where $w =$ Well-Petersson form, and let its Laplace transform:

$$W_{g,n}(z_1, \dots, z_n) = \int_0^\infty \dots \int_0^\infty \mathcal{V}_{g,n}(L_1, \dots, L_n) \prod_{i=1}^n \mathrm{e}^{-z_i L_i} L_i dL_i$$

Those hyperbolic volumes, are not easy to compute with hyperbolic geometry. Only smallest values of g and n had been computed directly by hyperbolic geometry, for example:

$$\mathcal{V}_{0,3}(L_1, L_2, L_3) = 1, \qquad W_{0,3}(z_1, z_2, z_3) = \frac{1}{z_1^2\, z_2^2\, z_3^2}$$

$$\mathcal{V}_{1,1}(L) = \frac{1}{48}\,(L^2 + 4\pi^2), \qquad W_{1,1}(z) = \frac{1}{8z^4} + \frac{\pi^2}{12\, z^2}.$$

In 2004, M. Mirzakhani discovered a beautiful recursion relation [2], which computes all volumes $\mathcal{V}_{g,n}$ for all g and n, by recursion on $2g + n$. We shall not write Mirzakhani's relation among the $\mathcal{V}_{g,n}$'s, but we shall consider here its Laplace transformed version:

Theorem 2.1 (Recursion for Weil-Petersson volumes, Mirzakhani's recursion [2], Laplace transformed[6]). *For any (g,n) such that $2g - 2 + n > 0$, one has:*

$$W_{g,n}(z_1, \overbrace{z_2, \dots, z_n}^{J}) = \operatorname*{Res}_{z\to 0} \frac{1}{(z_1^2 - z^2)} \frac{\pi}{\sin(2\pi z)} \Big[W_{g-1,n+1}(z, -z, J) + {\sum_{I\uplus I'=J; h+h'=g}}' W_{h,1+\#I}(z, I)\, W_{h',1+\#I'}(-z, I') \Big]\, dz$$

where $\sum'$ means that we exclude from the sum the two cases $(I = J, h = g)$ and $(I' = J, h' = g)$, and we have denoted:

$$W_{0,2}(z_1, z_2) = \frac{1}{(z_1 - z_2)^2}.$$

This theorem [6, 62–64] is very efficient at actually computing the volumes. It is a recursion on the Euler–characteristics $\chi = 2-2g-n$, at each step, the absolute value of the Euler characteristics in the Left Hand Side, is one more than the total Euler characteristics of every Right Hand Side terms:

$$|2-2(g-1)-(n+1)| = |2-2h-(1+m)+2-2(g-h)-(1+n-1-m)| = |2-2g-n|-1.$$

This explains the name "**topological recursion**".

2.2. Hurwitz numbers. See short definition in appendix B. Let $H_{g,n}(\mu)$ be the simple Hurwitz number of genus g with ramification profile μ, i.e. the number of connected ramified coverings of the Riemann sphere, of genus g and with only one multiply ramified point whose ramification profile is a partition $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n)$ of length n (we denote $|\mu| = \sum_i \mu_i$), and all other ramification points are simple (and Riemann Hurwitz formula says that there are $b = 2g-2+n+|\mu|$ simple ramification points).

For example, when $n = 1$ and $g = 0$, one has that $H_{0,1}(\mu_1)$ is the number of ways of gluing μ_1 sheets together, at one fully ramified point, and at $\mu_1 - 1$ simple ramification points, which make a surface of genus 0, i.e. planar. Such a ramified covering is thus the data of μ_1 sheets (represented by μ_1 points) linked by $\mu_1 - 1$ ramification points (represented by $\mu_1 - 1$ edges), connected and without loops. Therefore this is the same as counting the number of covering trees which can be drawn on the complete graph with μ_1 points, and is given by Cayley's formula:

$$H_{0,1}(\mu_1) = \mu_1^{\mu_1-2}$$

With genus 0 and partitions of length 2, one finds (though not easily [3, 5]):

$$H_{0,2}(\mu_1,\mu_2) = (\mu_1+\mu_2-1)!\,\frac{\mu_1^{\mu_1+1}\,\mu_2^{\mu_2+1}}{\mu_1!\,\mu_2!}.$$

We define the generating functions, as discrete Laplace transforms of the $H_{g,n}$'s:

$$W_{g,n}(x_1,\dots,x_n) = \sum_{\mu,\,\ell(\mu)=n} \frac{H_{g,n}(\mu)}{(2g-2+n+|\mu|)!} \sum_{\sigma\in\mathfrak{S}_n} \prod_{i=1}^{n} \mathrm{e}^{\mu_i x_{\sigma(i)}}.$$

For example,

$$W_{0,1}(x) = \sum_{k=1}^{\infty} \frac{k^{k-2}}{(k-1)!}\,\mathrm{e}^{kx} = L(\mathrm{e}^{x})$$

where L is the Lambert function, i.e. solution of $\mathrm{e}^{x} = L\,\mathrm{e}^{-L}$. Similarly

$$\begin{aligned} W_{0,2}(x_1,x_2) &= \sum_{k,k'=1}^{\infty} \frac{k^{k+1}\,k'^{k'+1}}{(k+k')\,k!\,k'!}\;\mathrm{e}^{kx_1}\;\;\mathrm{e}^{k'x_2} \\ &= \frac{L(\mathrm{e}^{x_1})}{1-L(\mathrm{e}^{x_1})}\,\frac{L(\mathrm{e}^{x_2})}{1-L(\mathrm{e}^{x_2})}\,\frac{1}{(L(\mathrm{e}^{x_1})-L(\mathrm{e}^{x_2}))^2} - \frac{\mathrm{e}^{x_1}\,\mathrm{e}^{x_2}}{(\mathrm{e}^{x_1}-\mathrm{e}^{x_2})^2}. \end{aligned}$$

One finds that it is easier to make a change of variable and work with $z_i = L(\mathrm{e}^{x_i})$ rather than x_i, and thus define the following differential forms:

$$\omega_{g,n}(z_1,\dots,z_n) = \left(W_{g,n}(x_1,\dots,x_n) + \delta_{g,0}\,\delta_{n,2}\,\frac{\mathrm{e}^{x_1}\,\mathrm{e}^{x_2}}{(\mathrm{e}^{x_1}-\mathrm{e}^{x_2})^2}\right) dx_1\dots dx_n.$$

For example,

$$\omega_{0,1}(z) = (1-z)\,dz,\ \omega_{0,2}(z_1,z_2) = \frac{dz_1\,dz_2}{(z_1-z_2)^2}.$$

Goulden, Jackson, Vainshtein [3] derived a recursion formula (called "cut and join equation") satisfied by those Hurwitz numbers, and after Laplace transform [4], one finds (not so easily [4]) the topological recursion formula, which was first conjectured by Bouchard and Mariño [5]:

Theorem 2.2 (recursion for Hurwitz numbers = Bouchard-Mariño conjecture(first proof in [4])). *The forms $\omega_{g,n}$'s satisfy the following recursion:*

$$\omega_{g,n}(z_1,z_2,\dots,z_n) = \operatorname*{Res}_{z\to 1} K(z_1,z)\ \Big[\omega_{g-1,n+1}(z,s(z),z_2,\dots,z_n) + {\sum_{I\uplus I'=\{z_2,\dots,z_n\};h+h'=g}}' \omega_{h,1+\#I}(z,I)\,\omega_{h',1+\#I'}(s(z),I')\Big]$$

where $\sum'$ means that we exclude from the sum the two cases $(I=\{z_2,\dots,z_n\}, h=g)$ and $(I=\emptyset, h=0)$, and where the recursion kernel K is:

$$K(z_1,z) = \frac{dz_1}{2}\,\frac{\frac{1}{z_1-z}-\frac{1}{z_1-s(z)}}{(z-s(z))}\,\frac{z}{(1-z)\,dz}$$

and where the map $s: z\mapsto s(z)$ defined in a vicinity of $z=1$, is the involution such that $s\neq \mathrm{Id}$ and solution of

$$s(z)\ \mathrm{e}^{-s(z)} = z\ \mathrm{e}^{-z}\quad,\ s(1)=1.$$

Locally near $z=1$, its Taylor series expansion starts with:

$$s(z) = 1-(z-1)+\frac{2}{3}(z-1)^2-\frac{4}{9}(z-1)^3+\frac{44}{135}(z-1)^4-\frac{104}{405}(z-1)^5+\dots$$

This recursion is very efficient at computing, for instance it easily gives:

$$\omega_{0,3}(z_1,z_2,z_3) = \frac{dz_1\,dz_2\,dz_3}{(1-z_1)^2\,(1-z_2)^2\,(1-z_3)^2}\,,$$
$$\omega_{1,1}(z) = \frac{1}{24}\left(\frac{1+2z}{(1-z)^4}-\frac{1}{(1-z)^2}\right)dz.$$

Again, this theorem is an illustration of the universal "**topological recursion**", as we shall see below.

3. How it arose

The purpose of the present section is to recall how the topological recursion (which is mostly a geometric notion) was initially discovered from the study of large random matrices, and then happened to have a much broader reach in a geometric setting.

3.1. Random matrices. In random matrices, one is interested in the statistical properties of the spectrum, especially in the large size limit. The density of eigenvalues converges (in most cases) towards a continuous density function, often called the "**equilibrium measure**". Very often (with reasonable choices of a random matrix probability law), the equilibrium measure is found to have a compact support (not necessarily connected), and happens to be an algebraic function. This means that there is an algebraic curve related to the random matrix model.

For example, the equilibrium measure for eigenvalues of a Gaussian random matrix, is the famous "Wigner's semi-circle"

$$\rho(x)dx = \frac{1}{2\pi}\sqrt{4-x^2}\ \ \mathbf{1}_{[-2,2]}\ dx$$

it is described by the algebraic curve $y^2 = x^2 - 4$, where $y = 2\mathrm{i}\pi\rho(x)$ is the equilibrium density, supported on the segment $[-2, 2]$.

3.1.1. Large size expansions. Around 2004 it was observed [7, 9, 10] that the knowledge of the equilibrium measure, is sufficient to recover the asymptotic expansion of every expectation value, and **to all orders** in the asymptotic expansion ! In other words, if $\mathcal{S}$ is the plane algebraic curve of the equilibrium measure, then all correlations can be obtained as universal (the same for every matrix model) functionals of $\mathcal{S}$ only.

For example, a particularly interesting quantity is the "partition function". Let $d\mu(M)$ be a (family depending on N of) un–normalized measure on the set of Hermitian matrices of size N, the partition function is defined as

$$Z = \int_{H_N} d\mu(M).$$

Under "good assumptions" on the measure $d\mu$, the partition function has a large N asymptotic expansion of the form

$$\ln Z \sim \sum_{g=0}^{\infty} N^{2-2g}\, F_g.$$

A main question in random matrix theory, is to compute the coefficients F_g ?

In [1, 8, 9] it was discovered that there exists a universal functional $\mathcal{F}_g : \mathcal{S} \mapsto \mathcal{F}_g(\mathcal{S})$, such that, for many classes of random matrices:

$$F_g = \mathcal{F}_g(\mathcal{S}).$$

The functional $\mathcal{F}_g : \mathcal{S} \mapsto \mathcal{F}_g(\mathcal{S})$ is defined only in terms of the Riemannian geometry of the curve $\mathcal{S}$, it is often called "**topological recursion**":

$$\overset{\text{spectral curve}}{\mathcal{S}} \quad \xrightarrow{\qquad\qquad\text{topological recursion}\qquad\qquad} \mathcal{F}_g\left(\mathcal{S}\right)$$

It was also discovered in [1, 7, 10] that there are universal functionals $\omega_{g,n} : \mathcal{S} \mapsto \omega_{g,n}(\mathcal{S})$ which compute the g^{th} order in the large N expansion of the joint probability of $n-$eigenvalues (more precisely the cumulants of correlations of n resolvents)

$$\mathbb{E}\left(\prod_{i=1}^{n} \mathrm{Tr}\,(x_i - M)^{-1}\right)_{\text{cumulant}} \quad \underset{N\to\infty}{\sim} \quad \sum_{g=0}^{\infty} N^{2-2g-n}\,\omega_{g,n}.$$

Therefore, for random matrices, there exist some functionals $\omega_{g,n}$ (and we call $\mathcal{F}_g = \omega_{g,0}$) which compute all correlation functions from the geometry of the spectral curve alone. The functionals $\omega_{g,n}$ are defined by a recursion on g and n (we postpone the explicit writing of this recursion to section 4), or more precisely a recursion on $(2g+n-2)$. This is called the **topological recursion**:

$$\omega_{g,n} = \text{computed from } \omega_{g',n'} \quad \text{with} \quad 2g'+n'-2 < 2g+n-2.$$

Eventually, this means that the knowledge of $\mathcal{S}$ (which is the large N equilibrium density of eigenvalues) allows to recover all correlation functions of the random matrix law, i.e. recover the random matrix probability law itself:

$$\overset{\text{spectral curve}}{\mathcal{S}} \xrightarrow{\text{topological recursion}} \omega_{g,n}(\mathcal{S}) \longrightarrow \overset{\text{probability law}}{d\mu(M)} .$$

3.1.2. How to use random matrices for geometry? Since the functionals $\omega_{g,n}$ which give expectation values are universal and do not require anything but the curve $\mathcal{S}$, one may try to apply these functionals $\omega_{g,n}$ to any arbitrary algebraic curve, independently of whether that algebraic curve $\mathcal{S}$ was related to a random matrix law or not. This is the idea proposed in [1].

In some sense, **the topological recursion defines a "pseudo-random matrix law" associated to any plane curve $\mathcal{S}$.**

$$\overset{\text{plane curve}}{\mathcal{S}} \xrightarrow{\text{topological recursion}} \omega_{g,n}(\mathcal{S}) \longrightarrow \overset{\text{pseudo probability law}}{d\mu(M)} .$$

Since expectation values or correlations are numbers, we get a collection of functionals, which associate "numbers" to a curve $\mathcal{S}$, this defines **"invariants of a curve"**

$$\overset{\text{curve}}{\mathcal{S}} \xrightarrow{\text{topological recursion}} \omega_{g,n}(\mathcal{S}) = \text{invariants of } \mathcal{S}.$$

We thus have a definition of a family $\omega_{g,n}(\mathcal{S})$ of invariants of a plane curve $\mathcal{S}$. We shall call them the **"symplectic invariants of $\mathcal{S}$"** or the **"TR (topological recursion) invariants"** of $\mathcal{S}$.

Remark 3.1. We emphasize that not all algebraic curves can come from probabilities of random matrices, because probabilities have some real and positivity properties. However, since the functional relations are analytical, they also apply to curves which don't have any positivity properties.

For example, the function $y = \frac{1}{4\pi}\sin 2\pi\sqrt{x}$ appearing as the spectral curve for the Weil-Petersson volumes, can never be the density of eigenvalues of a random matrix.

3.2. Some applications of TR invariants. By definition, when $\mathcal{S}$ is the large N spectral curve of a random matrix law, $\omega_{g,n}(\mathcal{S})$ computes the g^{th} large N order of the $n-$point correlation function of resolvants:

$$\mathbb{E}\left(\prod_{i=1}^{n} \operatorname{Tr}\frac{dx_i}{x_i - M}\right)_{\text{connected}} = \sum_{g=0}^{\infty} N^{2-2g-n}\,\omega_{g,n}.$$

Is there other plane curves $\mathcal{S}$ for which those invariants compute something interesting? The answer is YES: many classical geometric invariants, including Gromov-Witten invariants, or knot polynomials, can be obtained as the invariants of a plane curve $\mathcal{S}$ closely related to the geometry.

• Let $\mathfrak{X}$ be a local toric Calabi-Yau 3-fold [59, 60], and let $\mathcal{W}_{g,n}(\mathfrak{X})$ be the genus g and n boundary open Gromov–Witten invariant of $\mathfrak{X}$ (i.e. roughly speaking, the formal series whose coefficients count the number of holomorphic immersions of a genus g Riemann surface with n boundaries, such that the boundaries are mapped into a given Lagrangian submanifold [61]). It is well known that the mirror [46] of $\mathfrak{X}$ is another Calabi-Yau 3-fold, of the form

$$\{(x, y, u, v) \in \mathbb{C}^4 \mid H(\mathrm{e}^{\,x}, \mathrm{e}^{\,y}) = uv\}$$

where H is some polynomial found from the moment map of $\mathfrak{X}$. This is an hyperbolic bundle over $\mathbb{C}^* \times \mathbb{C}^*$. The fibers are singular over the plane curve $H(\mathrm{e}^{\,x}, \mathrm{e}^{\,y}) = 0$. We call that plane curve $\mathcal{S} = \hat{\mathfrak{X}}$, and we call it the mirror curve of $\mathfrak{X}$.

Then Mariño and co conjectured in [11, 12], and it was proved in [13, 14] that:

Theorem 3.2 (Topological recursion for toric CY 3folds). *(called BKMP conjecture* [11, 12]*, first proved in* [13]*, and for CY orbifolds in* [14]*).*

The Gromov–Witten invariants $\mathcal{W}_{g,n}(\mathfrak{X})$ are the topological recursion invariants of $\mathcal{S} = \hat{\mathfrak{X}}$ the mirror curve of $\mathfrak{X}$:

$$\mathcal{W}_{g,n}(\mathfrak{X}) = \omega_{g,n}(\hat{\mathfrak{X}}).$$

In fact, special cases of this theorem were first proved in [51–54]. The idea of the proof of [13, 14], is that the recursive structure of the topological recursion can be encoded as graphs (see def. 4.8 below), and thus the $\omega_{g,n}(\hat{\mathfrak{X}})$ can be written as sums of weighted graphs. Those graphs, up to some combinatorial manipulations, happen to coincide with the "localization graphs" of Gromov-Witten invariants [55, 56]. It is mostly a combinatorial proof.

• Another famous example (still conjectured) concerns knot polynomials.
Let $\mathfrak{K}$ be a knot embedded in the 3-sphere S^3. The character variety of $\mathfrak{K}$ is the locus of eigenvalues of holonomies of a flat $SL_2(\mathbb{C})$ connection on the knot complement $S^3 \setminus \mathfrak{K}$. This character variety is algebraic and defines an algebraic curve , called the A-polybomial of $\mathfrak{K}$:

$$A(X, Y) = 0.$$

The colored-Jones polynomial $J_N(q) \in \mathbb{C}[q]$, of color N, is defined as the Wilson loop [15] of a flat $SL_2(\mathbb{C})$ connection on $S^3 \setminus \mathfrak{K}$, in the spin $N-1$ representation of $SL_2(\mathbb{C})$. Let us denote

$$\hbar = \ln q, \qquad x = N \ln q.$$

Then, it is conjectured [16, 18–20] that in the limit where $\hbar \to 0$ and $x = O(1)$, one has the asymptotic expansion:

$$\ln J_N(q) \sim \sum_{k=-1}^{\infty} \hbar^k \, S_k(x), \quad S_k(x) = \sum_{2g-2+n=k} \frac{1}{n!} \int_{\gamma_x} \dots \int_{\gamma_x} \omega_{g,n}(\mathcal{S})$$

where $\mathcal{S}$ is the character variety of $\mathfrak{K}$, of equation $A(\mathrm{e}^x, \mathrm{e}^y) = 0$, and γ_x is some path on $\mathcal{S}$ ending at x (the more precise statement can be found in [20]). In other words:

Conjecture. *The Jones polynomial of a knot, is a series in $\hbar$ whose coefficients are the principal TR invariants of its A-polynomial*

If true (which is of course expected), this conjecture would be an extension of the famous Kashaev's "Volume conjecture" [35–37], and would imply a new understanding of what Jones polynomials are, in particular that Jones polynomials are Tau-functions of some integrable systems [16, 16, 18–20].

4. Definition of topological recursion and invariants

4.1. Spectral curves. The topological recursion associates invariants $\omega_{g,n}$ to a spectral curve. There exists many definitions of what a spectral curve is, they are more or less equivalent, but formulated in rather different languages.

Let us adopt the following definition here, close to the one in [1].

Definition 4.1 (Spectral curve). A spectral curve $\mathcal{S} = (\mathcal{C}, x, y, B)$ is:

- a Riemann surface $\mathcal{C}$ not necessarily compact nor connected,
- a meromorphic function $x : \mathcal{C} \to \mathbb{C}$. The zeroes of dx are called the branchpoints. We assume that there is a finite number of them on $\mathcal{C}$.
- the germ of a meromorphic function at each branchpoint. We denote it collectively y. In other words near a branchpoint a of order r_a

$$y = \{\tilde{t}_{a,k}\}_{a\in\text{branchpoints},\, k\in\mathbb{N}} \quad \Longleftrightarrow \quad y(p) \underset{p\to a}{\sim} \sum_{k=0}^{\infty} \tilde{t}_{a,k}(x(p)-x(a))^{k/r_a}$$

- a symmetric 2-form B on $\mathcal{C}\times\mathcal{C}$, having a double pole on the diagonal and analytical elsewhere, normalized such that, with any local parameter:

$$B(p,q) \underset{p\to q}{\sim} \frac{dz(p)\otimes dz(q)}{(z(p)-z(q))^2} + \text{analytical at } q$$

  again, in fact all what is needed is that B is the germ of some analytical form at the branchpoints.

Remark 4.2. Since the topological recursion computes residues, in fact all what is needed to run the recursion, is that $\mathcal{C}$ is a union of "formal neighbourhoods of branchpoints", with y and B germs of analytical functions.

However, in most practical examples, these neighbourhoods form an actual Riemann surface, on which y and B are globally analytical, and the geometric structure of that Riemann surface impacts a lot the properties enjoyed by the invariants. In other words, the invariants are always well defined, but they enjoy more properties if in addition the Riemann surface has structure, for instance if it is connected and/or compact, and for instance if B is globally meromorphic.

Since all what is needed are germs at the branchpoints, we may define the spectral curve from the data of its Taylor (or Laurent) expansion coefficients:

Definition 4.3 (Spectral curve, bis). A spectral curve $\mathcal{S} = (\{\tilde{t}_{a,k}\}, \{\hat{B}_{a,k;b,j}\})$ is a collection of

- a set of "branchpoints" $\mathbf{a} = \{a_1, a_2, \dots, a_N\}$.
- a family of times $\tilde{t}_{a,k}$ for each $a \in \mathbf{a}$. They are related to y by

$$y(p) \underset{p\to a}{\sim} \sum_{k=0}^{\infty} \tilde{t}_{a,k} (x(p) - x(a))^{k/r_a}.$$

- the times $\hat{B}_{a,k;b,j}$ for each $(a,b) \in \mathbf{a} \times \mathbf{a}$. They are related to B by

$$B(p,q) \underset{p\to a, q\to b}{\sim} \delta_{a,b} \mathring{B}_a(p,q) + \sum_{k,l} \hat{B}_{a,k;b,l} \zeta_a(p)^k \, \zeta_b(q)^l \; d\zeta_a(p)\, d\zeta_b(q)$$

where $\zeta_a(p) = (x(p) - x(a))^{1/r_a}$, and

$$\mathring{B}_a(p,q) = \frac{d\zeta_a(p)\, d\zeta_a(q)}{(\zeta_a(p) - \zeta_a(q))^2}$$

4.2. Definition of TR invariants by recursion. For simplicity in this definition below, we assume all branchpoints to be simple, i.e. $r_a = 2$, the general case is done in [21–23]. We define $\sigma_a : U_a \to U_a$ the involution in a small neighbourhood U_a of a, that exchanges the two sheets of x^{-1} that meet at a, i.e. such that

$$x \circ \sigma_a = x.$$

σ_a is called the **local Galois involution** of x (it permutes the roots of $x(p) - \mathrm{x}$).

Definition 4.4. We define by recursion on $\chi = 2g + n - 2$, the following forms on $\mathcal{C}^n$:

$$\omega_{0,1}(p) = y(p)dx(p), \qquad \omega_{0,2}(p,q) = B(p,q)$$

and for $2g + n - 2 \geq 0$:

$$\omega_{g,n+1}(p_1, \dots, p_{n+1}) = \sum_{a \in \text{branchpoints}} \operatorname*{Res}_{q\to a} K_a(p_1, q) \Big[\omega_{g-1,n+2}(q, \sigma_a(q), p_2, \dots, p_{n+1}) + {\sum_{h+h'=g,\, I\uplus I' = \{p_2,\dots,p_{n+1}\}}}' \omega_{h,1+\#I}(q, I)\, \omega_{h',1+\#I'}(\sigma_a(q), I') \Big]$$

with the recursion kernel

$$K_a(p_1, q) = -\frac{1}{2} \frac{\int_{\sigma_a(q)}^{q} \omega_{0,2}(p_1, .)}{\omega_{0,1}(q) - \omega_{0,1}(\sigma_a(q))}$$

Remark 4.5. It is not obvious from the definition, but an important property (which can be proved by recursion, see [1]) is that $\omega_{g,n}$ is always a symmetric n-form on $\mathcal{C}^n$. The definition gives a special role to p_1, but the result of the sum of residues is in fact symmetric in all p_i's.

Remark 4.6. When the branchpoints are not simple, if $r_a > 2$, the general definition can be found in [21–23]. In fact, branchpoints of higher order $r_a > 2$ can be obtained by taking a limit of several simple branchpoints merging smoothly. It was proved in [23] that the limit of the definition with simple branchpoints, indeed converges to that of [21–23]. In other words, higher order branchpoints, can be recovered from simple branchpoints. This is why, for simplicity, we shall focus on simple branchpoints here.

Examples of applications of the definition for $(g,n) = (0,3)$:

$$\omega_{0,3}(p_1,p_2,p_3) = \sum_a \operatorname*{Res}_{q\to a} K_a(p_1,q)\left[B(q,p_2)B(\sigma_a(q),p_2) + B(q,p_1)B(\sigma_a(q),p_1)\right] \tag{4.1}$$

and for $(g,n) = (1,1)$ $$\omega_{1,1}(p_1) = \sum_a \operatorname*{Res}_{q\to a} K_a(p_1,q)\, B(q,\sigma_a(q)). \tag{4.2}$$

Definition 4.7. When $n = 0$ we define $\omega_{g,0}$ (denoted $F_g \equiv \omega_{g,0}$) by:

$$g \geq 2\,, \qquad F_g = \omega_{g,0} = \frac{1}{2-2g}\ \sum_a \operatorname*{Res}_{q\to a}\ \omega_{g,1}(q)\,\Phi(q)$$

where $d\Phi = \omega_{0,1}$ (F_g is independent of a choice of integration constant for Φ).

The definition of F_1 and F_0 is given in [1], but we shall not write it in this short review.

4.3. Definition as graphs. The recursive definition above can conveniently be rewritten in a graphical way.

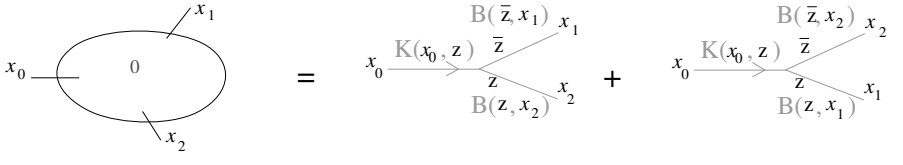
For example expression (4.1) or (4.2) are easily written in terms of graphs:

- associate to each $B(p,q)$ factor, a non-oriented line from p to q,
- associate to each $K_a(p,q)$ factor, an oriented line from p to q, whose end q has a "color" a,
- associate to each Residue $\operatorname{Res}_{q\to a}$ a tri-valent planar vertex of "color" a, with one ingoing edge (it must be oriented) and two outgoing edges (not necessarily oriented) the left one labeled with the point q and the right one labeled with the point $\sigma_a(q)$.
- The value of a graph is then obtained by computing residues at the vertices of the product of B's and K's of edges.

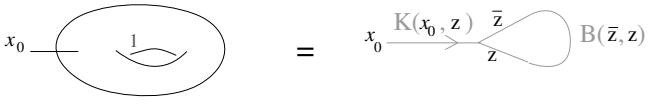
For example (4.1)

$$\omega_{0,3}(p_1,p_2,p_3) = \sum_a \operatorname*{Res}_{q\to a} K_a(p_1,q)\left[B(q,p_2)B(\sigma_a(q),p_2) + B(q,p_1)B(\sigma_a(q),p_1)\right]$$

is represented by:



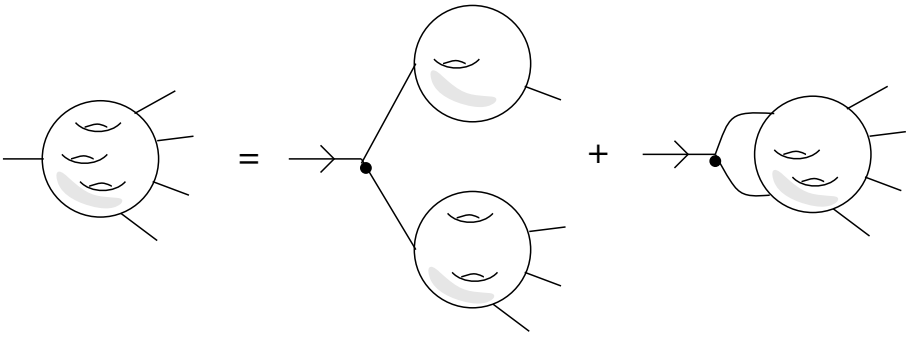
and $\omega_{1,1}(p_1) = \sum_a \operatorname{Res}_{q\to a} K_a(p_1,q)\, B(q,\sigma_a(q))$ is represented by



Therefore, following [1] we define the following set of graphs:

Definition 4.8. For any $k \geq 0$ and $g \geq 0$ such that $k + 2g - 2 > 0$, we define:

Let $\mathcal{G}^{(g)}_{k+1}(p, p_1, \ldots, p_k)$ be the set of connected trivalent graphs with $2g + k - 1$ trivalent vertices carrying a color $a \in \mathbf{a}$, $k + 1$ legs, $2g + k - 1$ arrowed edges forming a covering oriented planar binary tree of the graph and $k + g$ non-arrowed edges, and obtained by the recursion:



Then, we define the weight of a graph as:

$$w(G) = \prod_{v \in \{\text{vertices}\}} \operatorname*{Res}_{q_v \to a_v} \prod_{e=(p,q) \in \{\text{unarrowed edges}\}} B(p,q) \prod_{e=(p \mapsto q) \in \{\text{arrowed edges}\}} K_{a_p}(p,q)$$

where the order of taking the residues is by following the arrows from leaves to root (deeper vertices are integrated first).

Then, the definition of $\omega_{g,n}(\mathcal{S})$ is:

$$\omega_{g,n}(p_1, \ldots, p_n) = \sum_{G \in \mathcal{G}_{g,n}(p_1,\ldots,p_n)} w(G).$$

Those graphs are merely a convenient mnemotechnic notation for the recursive definition 4.4, they provide a good support for intuition and are very useful for proving some theorems.

4.4. A-model side definition. Let $\mathcal{S} = (\mathcal{C}, x, y, B)$ a spectral curve, with branchpoints $\mathbf{a} = \{a\}$. Near $a \in \mathbf{a}$, we define the local Laplace transforms

Definition 4.9 (Laplace transforms).

$$\mathrm{e}^{-f_{(a,j)}(u)} = \frac{u^{3/2}\ \mathrm{e}^{ux(a)}}{2\sqrt{\pi}} \int_{\gamma_{(a,j)}} y\,dx\ \ \mathrm{e}^{-ux}$$

where $\gamma_{(a,j)}$ is a "steepest descent path", i.e. in a neighbourhood U_a of a it is an arc included in $x^{-1}(x(a) + \mathbb{R}_+)$ (if dx vanishes to order $r_a - 1$ at a (x is locally $r_a : 1$ at a), then there

are $r_a - 1$ such steepest descent paths, i.e. $j \in [1, \dots, r_a - 1]$. For a simple branchpoint $r_a = 2$, and there is only $j = 1$, so we may drop the j index).

Its large u expansion doesn't depend on the neighbourhood U_a and defines the "times":

$$f_{(a,j)}(u) \sim \sum_{k=0}^{\infty} t_{(a,j),k}\, u^{-k}.$$

Similarly we Laplace transform B:

Definition 4.10.

$$\hat{B}_{(a,j);(b,l)}(u,v) = \frac{\sqrt{uv}}{\pi} \int_{\gamma_{(a,j)} \times \gamma_{(b,l)}} \Big(B(z_1,z_2) - \delta_{a,b}\, \mathring{B}_a(z_1,z_2) \Big) \mathrm{e}^{-u(x(z_1)-x(a))}\, \mathrm{e}^{-v(x(z_2)-x(b))}$$

where $\mathring{B}_a(z_1,z_2) = \dfrac{d\zeta_a(z_1)\, d\zeta_a(z_2)}{(\zeta_a(z_1) - \zeta_a(z_2))^2}$, with $\zeta_a(z) = (x(z) - x(a))^{1/r_a}$.

The large u and v expansion define the "times"

$$\hat{B}_{(a,j);(b,l)} \sim \sum_{m,n} \hat{B}_{(a,j),n;(b,l),m} u^{-n}\, v^{-m}.$$

We shall also define the half Laplace transform

$$\check{B}_{(a,j)}(u,z) = \frac{\sqrt{u}}{\sqrt{\pi}} \int_{z' \in \gamma_{(a,j)}} B(z',z)\ \mathrm{e}^{-u(x(z')-x(a))}$$

whose large u expansion defines a basis of meromorphic 1-forms having a pole at a:

$$\check{B}_{(a,j)}(u,z) = \sum_k u^{-k}\, d\xi_{(a,j),k}(z)$$

All this gives another definition of the notion of spectral curve:

Definition 4.11 (Spectral curve, ter). A spectral curve $\mathcal{S} = \{\, \{t_{\alpha,k}\}, \{\hat{B}_{\alpha,k;\beta,l}\}, \{d\xi_{\alpha,k}\}\}$ is the data of all the times.

This definition encodes in a slightly different way compared to def4.1, the Taylor expansions of all germs of analytical functions needed to run the recursion, which are much better encoded through Laplace transforms, as remarked in [24–26, 57, 58].

We shall now use the spectral curve data to define a tautological cohomology class in the cohomological ring of some moduli space, and thus define an A-model potential.

First, we define the moduli space. Let us first assume that all branchpoints are simple, i.e. $r_a = 2$ and thus the local Galois group is $\mathbb{Z}_2$:

Definition 4.12 (Colored moduli space (simple branchpoints)). Let

$$\mathbf{a} = \{a\}_{a=\text{branchpoints}}, \quad N = \#\mathbf{a}.$$

We start by defining the following moduli space (not compact):

$$\mathcal{M}_{g,n}(\mathbf{a}) = \{(\Sigma, p_1, \dots, p_n, s)\}$$

where Σ is a genus g nodal surface with n smooth marked points $p_1, \dots, p_n$, and $s : \Sigma \to \mathbf{a}$ be a map constant in each component of Σ.

In fact $\mathcal{M}_{g,n}(\mathbf{a})$ is merely a convenient notation for a union of smaller moduli spaces:

$$\mathcal{M}_{g,n}(\mathbf{a}) = \cup_{G=\text{dual graphs, } N \text{ colored}} \prod_{v \in \text{vertices}} \overline{\mathcal{M}}_{g_v,n_v}^{(a_v)}$$

where $\overline{\mathcal{M}}_{g,n}^{(a)}$ are N copies of $\overline{\mathcal{M}}_{g,n}$ labeled by the branchpoints a. The graphs G are dual graphs of stable nodal surfaces, of total genus g and n smooth marked points. Vertices v of G carry a genus g_v, a number of marked or nodal points n_v, and a color $s_v \in \mathbf{a}$. We must have:

$$\forall\, v,\ 2 - 2g_v - n_v < 0, \quad \text{and} \quad \sum_{v \in \text{vertices of } G} (2 - 2g_v - n_v) = 2 - 2g - n.$$

In fact this definition can be extended to multiple branchpoints $r_a > 2$, with a local Galois group $\mathfrak{G}_a$ (most often $\mathbb{Z}_{r_a}$), by replacing each $\overline{\mathcal{M}}_{g_v,n_v}$ by $B\mathfrak{G}_{a_v}\overline{\mathcal{M}}_{g_v,n_v}^{(a_v)}$ where $B\mathfrak{G}_a$ is the classifying space of the local Galois group $\mathfrak{G}_a$ and $B\mathfrak{G}_a\overline{\mathcal{M}}_{g,n}$ is defined by the Chen-Ruan cohomology of $\mathbb{C}^3/\mathfrak{G}_a$.

Then, we define the following tautological classes in the cohomological ring of the moduli space $\mathcal{M}_{g,n}(\mathbf{a})$. We present only the case of simple branchpoints for simplicity:

Definition 4.13 (Tautological class of a spectral curve).

$$\Lambda(\mathcal{S}) = \mathrm{e}^{\sum_k t_{s_*,k}\kappa_k + \frac{1}{2}\sum_{k,l} \hat{B}_{s_*,k;s_*,l} \sum_{\delta \in \partial\mathcal{M}_{g,n}(\mathbf{a})} l_\delta * (\tau_k \tau_l)}$$

where

- $\tau_k = c_1(T_p^*)^k = \psi(p)^k$ is the k^{th} power of the 1st Chern class of the cotangent bundle at the marked or nodal point p over $\overline{\mathcal{M}}_{g,n}^{(a_{s(p)})}$
- κ_k is the k^{th} Mumford class [65], that is the pushforward $\pi * \psi(p_{n+1})^{k+1}$ of the $(k+1)^{\text{th}}$ power of the 1st Chern class of the cotangent bundle at the $(n+1)^{\text{th}}$ marked point $p_{n+1} \in \overline{\mathcal{M}}_{g,n+1}$, under the forgetful map $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$.
- $\partial\mathcal{M}_{g,n}(\mathbf{a})$ is the set of boundary divisors of $\mathcal{M}_{g,n}(\mathbf{a})$, or in other words it is the set of nodal points. If $\delta \in \partial\mathcal{M}_{g,n}(\mathbf{a})$, then δ is a nodal point, i.e. it is a pair of points $\delta = (p, p')$ with p and p' in two components (possibly the same) of Σ, corresponding to two vertices v, v' (possibly the same) of the graph G. $l_\delta * (\tau_k \tau_l)$ denotes the class $\psi(p)^k \psi(p')^l$ pushed in $\overline{\mathcal{M}}_{g_v,n_v}^{(\sigma(p))} \times \overline{\mathcal{M}}_{g_{v'},n_{v'}}^{(\sigma(p'))}$.

Then the invariants $\omega_{g,n}(\mathcal{S})$ are given by:

Theorem 4.14 (A-model Invariants). *(proved in* [25]*, see also* [26]*).*

$$\begin{aligned}\omega_{g,n}(z_1, \dots, z_n) &= \int_{\mathcal{M}_{g,n}(\mathbf{a})} \Lambda(\mathcal{S}) \prod_{i=1}^{n} \check{B}_{s(p_i)}(1/\psi(p_i), z_i) \\ &= \sum_{d_1,\dots,d_n} \int_{\mathcal{M}_{g,n}(\mathbf{a})} \Lambda(\mathcal{S}) \prod_{i=1}^{n} \psi(p_i)^{d_i}\, d\xi_{s(p_i),d_i}(z_i)\end{aligned}$$

In fact this theorem and the definition of $\mathcal{M}_{g,n}(\mathbf{a})$ means a sum over graphs of products at vertices of usual intersection numbers in some $\overline{\mathcal{M}}_{g_v,n_v}^{(a_v)}$'s, it is merely a short hand notation for the following sum:

$$\omega_{g,n}(z_1,\dots,z_n) = 2^{3g-3+n} \sum_{\text{graphs } G} \sum_{\{d_h\}\in\mathbb{Z}^{\{\text{half-edges}(G)\}}} \frac{2^{-\#\text{edges}(G)}}{\#\mathrm{Aut}(G)} \prod_{e=(v,v')\in\text{edges}(G)} \hat{B}_{a_v,d_{(v,e)};a_{v'},d_{(v',e)}} \prod_{i=1}^{n} d\xi_{s(p_i),d_i}(z_i) \prod_{v\in\text{vertices}(G)} \int_{\overline{\mathcal{M}}_{g_v,n_v}^{(a_v)}} \mathrm{e}^{\sum_k t_{a_v,k}\kappa_k} \prod_{h\in\text{half-edges}(G)\text{ adjacent to }v} \psi(p_h)^{d_h}$$

This theorem is thus a mirror symmetry statement [46]. It was first proved in [24] for a single branchpoint, and then in [25] for the general case, and see also [26].

Idea of the Proof. Using the graphical definition def.4.8 of the $\omega_{g,n}$'s, by a recombination of vertices with the same colors, one finds that $\omega_{g,n}$ can be written as a sum over graphs of a Wick theorem [27–29], where the edge weights are the $\hat{B}_{a,k;b,l}$'s, and it remains to compute the weights of vertices.

Since vertices are independent of the $\hat{B}_{a,k;b,l}$, they can be found from the case where all $\hat{B}_{a,k;b,l}$ vanish, and when there is only one branch point. This can be achieved by chosing the spectral curve $\mathcal{S} = (\mathbb{C}P^1, x : z \mapsto z^2, y : z \mapsto z, B(z,z') = \frac{dzdz'}{(z-z')^2})$, and shows that the weights of vertices [24, 58] are the Witten Kontsevich intersection numbers [42, 43]. Therefore, this theorem is mostly of combinatorial nature. □

Remark 4.15. In fact this theorem is very similar to Givental's formalism [45]. The only difference with Givental's formalism, is that it applies to more general situations. Givental's formalism applies to Gromov–Witten's theories, and thus applies only if the coefficients $\hat{B}$'s and t's satisfy certain relationships which we don't assume here. All this is explained in [26].

Examples of applications of theorem 4.14:

• **Weil-Petersson volumes.** Chose $\mathcal{S} = (\mathbb{C}P^1, x : z \mapsto z^2, y : z \mapsto \frac{1}{4\pi}\sin(2\pi z), B(z,z') = \frac{dzdz'}{(z-z')^2})$. In that case, there is only one branchpoint at $z = 0$. An easy computation yields $\hat{B}_{a,k;a,l} = 0$, and the Laplace transform of ydx yields:

$$\mathrm{e}^{-f(u)} = \frac{u^{3/2}}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\sin(2\pi z)}{4\pi} 2zdz\ \mathrm{e}^{-uz^2} = \frac{1}{4}\,\mathrm{e}^{-\pi^2/u}$$

and we also find

$$d\xi_d(z) = \frac{(2d+1)!!}{2^d}\ \frac{dz}{z^{2d+2}}$$

Definition 4.13 gives

$$\Lambda(\mathcal{S}) = \mathrm{e}^{\pi^2\kappa_1},$$

and the theorem 4.14 gives

$$\begin{aligned}\omega_{g,n}(z_1,\dots,z_n) &= 2^{5g-5+2n}\sum_{d_1,\dots,d_n}\prod_{i=1}^n d\xi_{d_i}(z_i)\ \int_{\overline{\mathcal{M}}_{g,n}} \mathrm{e}^{\,\pi^2\kappa_1}\ \prod_{i=1}^n \psi_i^{d_i}\\ &= 2^{2g-2+n}\sum_{d_1,\dots,d_n}\prod_{i=1}^n \frac{(2d_i+1)!!\,dz_i}{z_i^{2d_i+2}}\ \int_{\overline{\mathcal{M}}_{g,n}} \mathrm{e}^{\,2\pi^2\kappa_1}\ \prod_{i=1}^n \psi_i^{d_i}\end{aligned}$$

which are indeed the Weil-Petersson volumes of moduli spaces [6, 62–64]. In other words, this theorem gives a very easy proof of Mirzakhani's recursion, by simply computing the Laplace transform of $y=\sin(2\pi\sqrt{x})/4\pi$.

• **Kontsevich-Witten.** Chose $\mathcal{S}=(\mathbb{C}P^1, x: z\mapsto z^2, y: z\mapsto z, B(z,z')=\frac{dzdz'}{(z-z')^2})$. In that case, there is only one branchpoint at $z=0$. An easy computation yields $\hat{B}_{a,k;a,l}=0$, and the Laplace transform of ydx yields:

$$\mathrm{e}^{-f(u)}=\frac{u^{3/2}}{2\sqrt{\pi}}\int_{-\infty}^{\infty} z\ 2zdz\ \mathrm{e}^{-uz^2}=\frac{1}{2},\qquad d\xi_d(z)=\frac{(2d+1)!!}{2^d}\ \frac{dz}{z^{2d+2}}$$

Definition 4.13 gives

$$\Lambda(\mathcal{S})=2^{\kappa_0}=2^{2g-2+n}.$$

The theorem 4.14 thus gives

$$\omega_{g,n}(z_1,\dots,z_n)=2^{2g-2+n}\sum_{d_1,\dots,d_n}\prod_{i=1}^n \frac{(2d_i+1)!!\,dz_i}{z_i^{2d_i+2}}\ \int_{\overline{\mathcal{M}}_{g,n}}\prod_{i=1}^n \psi_i^{d_i},$$

which are the Kontsevich-Witten [42, 43] intersection numbers [24, 58, 62–64].

• **ELSV formula.** Chose $\mathcal{S}=(\mathbb{C}P^1, x: z\mapsto -z+\ln z, y: z\mapsto z, B(z,z')=\frac{dzdz'}{(z-z')^2})$. Again there is only one branchpoint at $z=1$. The Laplace transform of ydx yields:

$$\begin{aligned}\mathrm{e}^{-f(u)} &= \frac{u^{3/2}\,\mathrm{e}^{-u}}{2\sqrt{\pi}}\int_{\gamma_1} z\,\frac{(1-z)\,dz}{z}\ z^{-u}\ \mathrm{e}^{uz}\\ &= \frac{\mathrm{i}\sqrt{\pi}\,u^u\,\mathrm{e}^{-u}}{\sqrt{u}\,\Gamma(u)}=\frac{\mathrm{i}}{\sqrt{2}}\ \mathrm{e}^{-\sum_k \frac{B_{2k}}{2k(2k-1)}\,u^{1-2k}},\end{aligned}$$

where B_k are the Bernoulli numbers. We leave the reader an exercise to compute the $\hat{B}_{a,k;a,l}$ and the $d\xi_{a,k}(z)$, and we just mention that:

$$\Lambda_{\text{Hodge}}=\ \mathrm{e}^{\sum_k \frac{B_{2k}}{2k(2k-1)}\left(\kappa_{2k-1}-\sum_i \psi_i^{2k-1}+\frac{1}{2}\sum_\delta \sum_{l=0}^{2k-2}(-1)^l\ l_\delta * \tau_{2k-2-l}\tau_l\right)}$$

is the Hodge class [24, 66]. The theorem 4.14 easily gives the ELSV formula [34], but we refer the reader to [24, 25] for a more detailed computation.

5. Main properties

Let us make a brief summary of some of the properties enjoyed by those invariants.

Symplectic invariance.

$$\hat{\mathcal{F}}_g = \mathcal{F}_g - \frac{1}{2-2g}\sum_{\alpha}(\operatorname*{Res}_{\alpha} y dx)(\int_o^{\alpha}\omega_{g,1})$$

is invariant under symplectomorphisms. This means that if $\phi : \mathbb{C}P^1 \times \mathbb{C}P^1 \to \mathbb{C}P^1 \times \mathbb{C}P^1$ is a symplectomorphism (conserves $dx \wedge dy$), then

Theorem 5.1. *If $\mathcal{S} = (\mathcal{C}, x, y, B)$ is such that $\mathcal{C}$ is a compact Riemann surface, x and y are globally meromorphic functions on $\mathcal{C}$, and B is the fundamental 2nd kind form* [44] *on $\mathcal{C}\times\mathcal{C}$, normalized on a symplectic basis of cycles, then if ϕ is a symplectomorphism of $\mathbb{C}P^1 \times \mathbb{C}P^1$ then*

$$\hat{\mathcal{F}}_g(\phi * \mathcal{S}) = \hat{\mathcal{F}}_g(\mathcal{S}).$$

This theorem [1, 30, 31] is extremely powerful and useful. It allows to compare very easily some apparently unrelated enumerative problems, just by comparing their spectral curves, it allows to find dualities. A special case is $\phi : (x, y) \mapsto (y, -x)$, i.e.

$$\hat{\mathcal{F}}_g(\mathcal{C}, y, -x, B) = \hat{\mathcal{F}}_g(\mathcal{C}, x, y, B).$$

Let us mention that the proof of that theorem is highly non–trivial, and it was proved so far only for algebraic spectral curves (x and y meromorphic on a compact $\mathcal{C}$), but it is believed to be valid in more general cases for example when dx and dy are meromorphic 1-forms on a compact $\mathcal{C}$, see [22, 47].

Modular invariance and BCOV [67]. Let $\mathcal{S} = (\mathcal{C}, x, y, B)$ a spectral curve such that $\mathcal{C}$ is a compact Riemann surface of genus $\mathfrak{g}$, and B is the fundamental 2nd kind form on $\mathcal{C}\times\mathcal{C}$, normalized on a symplectic basis of cycles [44, 48]. The modular group $Sp_{2\mathfrak{g}}(\mathbb{Z})$ acts on B, namely if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp_{2\mathfrak{g}}(\mathbb{Z})$ (a, b, c, d are $\mathfrak{g}\times\mathfrak{g}$ integer matrices), the period matrix is changed to $\tau \mapsto (d-\tau b)^{-1}(\tau a - c)$, and B is changed to

$$B(p,p') \mapsto B(p,p') + 2\pi\mathrm{i}\sum_{i,j=1}^{\mathfrak{g}} \omega_i(p)\,(b\,(d-\tau b)^{-1})_{i,j}\,\omega_j(p'),$$

where ω_i are the normalized holomorphic 1-forms on the $\mathcal{A}$-cylces of $\mathcal{C}$, $\oint_{\mathcal{A}_i}\omega_j = \delta_{i,j}$. Then

Theorem 5.2. *$\mathcal{F}_g(\mathcal{C}, x, t, B)$ is an almost modular form under the modular group $Sp_{2\mathfrak{g}}(\mathbb{Z})$ acting on B.*

The proof of this theorem appeared in [1] and [32], and follows easily from the graphical decomposition of def 4.8. Indeed, B appears only in edges of the graphs, and eq.(5) amounts to cutting edges. The effect of a modular transformation thus produces dual graphs of degenerate Riemann surfaces, with factors of $b(d-\tau b)^{-1}$ at degeneracies, and thus coincides with the transformations of almost modular forms.

As a consequence $\mathcal{F}_g(\mathcal{S})$'s satisfy BCOV's holomorphic anomaly equations [32, 67].

Deformations and Form cycle duality (Seiberg-Witten). The tangent space to the space of spectral curves at $\mathcal{S} = (\mathcal{C}, x, y, B)$ is the space $\mathcal{M}^1(\mathcal{C})$ of meromorphic 1-forms on $\mathcal{C}$. Lat us chose B to be the fundamental 2nd kind differential on $\mathcal{C}$, see [1, 44, 48]. Then B provides the kernel for a form–cycle duality pairing, namely the meromorphic form Ω dual to a cycle Ω^* is

$$\Omega(p) = \oint_{p' \in \Omega^*} B(p, p').$$

(here we call cycle Ω^* any linear form on the space of meromorphic forms, i.e. an element of the dual of $\mathcal{M}^1(\mathcal{C})$). Then we have

Theorem 5.3. *Let Ω be a tangent vector to the space of spectral curves, i.e. a meromorphic 1-form on $\mathcal{C}$, and ∂_Ω be the derivative in the direction of Ω, then we have*

$$\partial_\Omega \, \omega_{g,n} = \oint_{\Omega^*} \omega_{g,n+1}.$$

This theorem first proved in [1, 9] follows easily from the graphical decomposition of def 4.8. Indeed, one just has to see how ∂_Ω acts on the kernels K and B, i.e. on the edges of the graphs, and it produces exactly the graphs of $\omega_{g,n+1}$.

Special cases of that theorem are:

$$\partial_\Omega \, ydx = \partial_\Omega \, \omega_{0,1} = \oint_{\Omega^*} \omega_{0,2} = \oint_{\Omega^*} B = \Omega.$$

we thus recover that Ω is the derivative of $\omega_{0,1} = ydx$, i.e. the definition of Ω. Another example is

$$\partial_\Omega \, \mathcal{F}_0 = \oint_{\Omega^*} \omega_{0,1} = \oint_{\Omega^*} ydx,$$

which means that $\mathcal{F}_0$ is the prepotential, this relation is Seiberg-Witten's duality. Yet another example is

$$\partial_\Omega \, B(p_1, p_2) = \oint_{p_3 \in \Omega^*} \omega_{0,3}(p_1, p_2, p_3) = \sum_a \operatorname*{Res}_{q \to a} \frac{B(q, p_1) B(q, p_2) \, \Omega(q)}{dx(q) dy(q)}$$

which is known as the Rauch variational formula [48] for the fundamental 2nd kind form B. Another example is:

$$\partial_\Omega \, \mathcal{F}_1 = \oint_{\Omega^*} \omega_{1,1}$$

which means that $\mathcal{F}_1$ is (up to some details which we don't enter here) the Bergman Tau function of Kokotov–Korortkin [49], i.e. $\mathcal{F}_1$ is the log of the determinant of some canonical Laplacian on $\mathcal{S}$.

Again, this theorem is very powerful, it is sometimes called "special geometry", or it can also be viewed as a generalization of Seiberg-Witten.

Dilaton equation. This is an equation saying that

Theorem 5.4. *For any (g, n) such that $2g - 2 + n > 0$ we have*

$$\sum_a \operatorname*{Res}_{q \to a} \omega_{g,n+1}(p_1, \ldots, p_n, q) \, \Phi(q) = (2 - 2g - n) \, \omega_{g,n}(p_1, \ldots, p_n)$$

where Φ is such that $d\Phi = ydx = \omega_{0,1}$.

Notice that this theorem was used to define $\mathcal{F}_g$ in def 4.7.

There are many other properties. For instance the $\omega_{g,n}$'s behave well under taking limits of singular spectral curves, in some sense they commute with taking limit. See [1] for details.

They are also deeply related to integrable systems [1, 19, 33, 40, 41, 50], for instance to Hitchin systems [39] and they have many other beautiful properties.

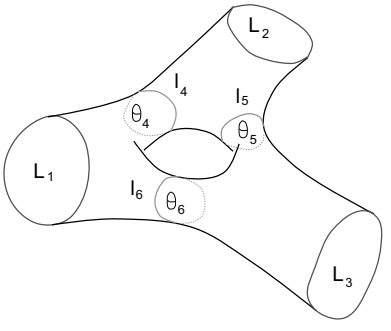
6. Conclusion

We hope to have shown the reader that topological recursion is a beautiful and powerful piece of mathematics. It defines new invariants associated to "spectral curves".

Topological recursion has found a large number of (sometimes unexpected) applications, ranging from combinatorics (maps, plane partitions) to Gromov–Witten invariants and knot theory, or integrable systems.

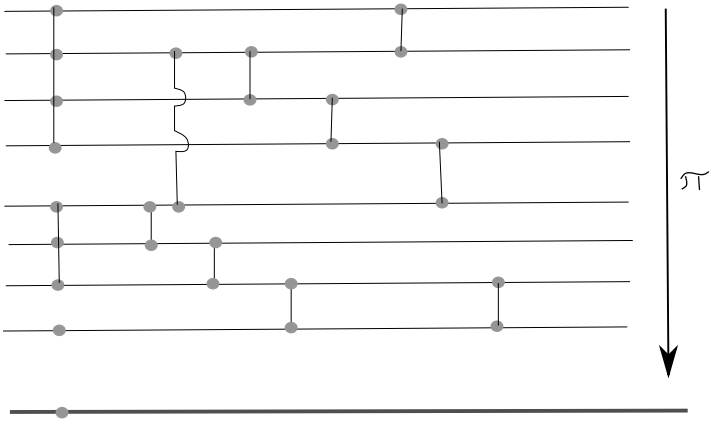
However, for many cases, the fact that a given enumerative geometry problem satisfies the topological recursion, is most often only conjectured, not yet proved, and finding proofs is a challenge. Even in proved cases, the proofs are always very technical and not natural, almost never bijective, so unsatisfactory. Finding a good deep geometric reason (in fact an A-model proof) for the topological recursion is a challenging open problem.

A. Weil-Petersson volumes in a nutshell



The Weil-Petersson metrics and Fenchel-Nielsen coordinates on $\mathcal{M}_{g,n}$ are obtained as follows: let $2g-2+n>0$, and let $\Sigma \in \mathcal{M}_{g,n}$. The Poincaré metrics on Σ is the unique metrics of constant negative curvature -1, such that the boundaries of Σ are geodesics of prescribed lengths $L_1, \dots, L_n$. Then, we may cut Σ into $2g-2+n$ pairs of pants, all of whose boundaries are geodesics. This cutting is not unique. Vice versa, a connected gluing of $2g-2+n$ hyperbolic pairs of pants along their geodesic boundaries, gives a unique Riemann surface in $\mathcal{M}_{g,n}$. Boundaries of pairs of pants can be glued together provided that the glued geodesics have the same lengths, and they can be rotated by some angle. The $3g-3+n$ lengths ℓ_i of the glued boundaries and the $3g-3+n$ gluing angles ϑ_i, are called the Fenchel Nielsen coordinates. They are local coordinates on $\mathcal{M}_{g,n}$. They are not defined globally because of non-unicity of the cutting. However, the form $w = \prod_i d\ell_i \wedge d\vartheta_i$, called the Weil-Petersson form, is globally defined. The Weil–Petersson volume is $\mathcal{V}_{g,n}(L_1, \dots, L_n) = \int_{\mathcal{M}_{g,n};\, \ell(\partial_i \Sigma)=L_i} w$ where one fixes the boundary lengths $L_1, \dots, L_n$.

B. Hurwitz numbers in a nutshell



A ramified covering (Σ, π) of the Riemann sphere $\mathbb{C}P^1$, is the data of a Riemann surface Σ, and an analytical map $\pi : \Sigma \to \mathbb{C}P^1$ of some degree d. For all generic points $x \in \mathbb{C}P^1$, the preimage $\pi^{-1}(x) \subset \Sigma$ consists of d points $\#\pi^{-1}(x) = d$. Branchpoints are the points $x \in \mathbb{C}P^1$ such that $\#\pi^{-1}(x) < d$. Ramification points are the preimages of branchpoints.

Ramification points are the points near which the map $\pi : \Sigma \to \mathbb{C}P^1$ is analytical but not locally invertible. A ramification point a is said of degree $r = \deg(a)$, if locally near a, the map π behaves like (in any choice of local coordinate)

$$\pi : p \mapsto \pi(a) + c_a(p-a)^r + O((p-a)^{r+1}), \;\; c_a \neq 0$$

Let x be a branchpoint, and $\{a_1, \dots, a_l\} = \pi^{-1}(x)$ be its preimages on Σ, and let $r_i = \deg a_i$ be its degrees. We assume that we have ordered the points a_i's such that $r_1 \geq r_2 \geq \cdots \geq r_l$. Then $(r_1, \dots, r_l)$ is called the ramification profile of x.

A regular branchpoint x is of degree 2, and such that $\#\pi^{-1}(x) = d-1$, its ramification profile is $(2, \overbrace{1, \dots, 1}^{d-2})$.

The Simple Hurwitz numbers $H_g(\mu)$ count the number of (equivalence homotopy classes) ramified coverings (Σ, π) such that Σ is a connected surface of genus g, and π has only one non–regular branchpoint, whose profile is given by the partition $\mu = (\mu_1, \dots, \mu_l)$.

The Hurwitz formula implies that such a covering must have $b = 2g - 2 + \sum_i (\mu_i + 1))$ regular branchpoints.

Acknowledgements. The author is grateful to the CRM in Montréal, and to Piotr Sułkowski and the ERC Fields-Knots, during the writing of this paper. The author thanks the CRM Centre de Recherches Mathématiques de Montréal, the FQRNT grant from the Quebec government, Piotr Sułkowski and the ERC starting grant Fiels-Knots, and the ICM and Korea.

References

[1] B. Eynard and N. Orantin, *Invariants of algebraic curves and topological expansion*, Commun. Number Theory Phys. **1**(2) (2007), 347-452, arXiv:math-ph/0702045.

[2] M. Mirzakhani, *Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces*, Invent. Math. **167** (2007), 179-222.

[3] I.P. Goulden, D.M. Jackson, and A. Vainshtein, *The number of ramified coverings of the sphere by the torus and surfaces of higher genera*, Annals of Combinatorics, **4** (2000), 27–46, 10.1007/PL00001274.

[4] Bertrand Eynard, Motohico Mulase, and Brad Safnuk. *The Laplace transform of the cut-and-join equation and the Bouchard-Marino conjecture on Hurwitz numbers*.

[5] Vincent Bouchard and Marcos Marino, *Hurwitz numbers, matrix models and enumerative geometry*. 21p, September 2007.

[6] B. Eynard and N. Orantin, *Weil-Petersson volume of moduli spaces, Mirzakhani's recursion and matrix models*, math-ph: arXiv:0705.3600v1.

[7] A. Alexandrov, A. Mironov, and A. Morozov, *Solving Virasoro constraints in matrix models*, Fortschritte der Physik, **53**(5-6) (May 2005), 512–521.

[8] L. Chekhov, B. Eynard, and N. Orantin, *Free energy topological expansion for the 2-matrix model*, J. High Energy Phys. **JHEP12** (2006) 053, arXiv:math-ph/0603003.

[9] L. Chekhov and B. Eynard, *Hermitian matrix model free energy: Feynman graph technique for all genera*, J. High Energy Phys. **JHEP03** (2006) 014, arXiv:hep-th/0504116.

[10] B. Eynard, *Topological expansion for the 1-hermitian matrix model correlation functions*, JHEP/024A/0904, arXiv:hep-th/0407261.

[11] Vincent Bouchard, Albrecht Klemm, Marcos Mariño, and Sara Pasquetti, *Remodeling the B-Model*, Communications in Mathematical Physics, **287**(1) (September 2008), 117–178.

[12] Marcos Mariño, *Open string amplitudes and large order behavior in topological string theory*, Journal of High Energy Physics, 2008(03):060–060, March 2008.

[13] B. Eynard and N. Orantin, *Computation of open Gromov-Witten invariants for toric Calabi-Yau 3-folds by topological recursion, a proof of the BKMP conjecture*, arXiv:1205.1103v1 [math-ph].

[14] Bohan Fang, Chiu-Chu Melissa Liu, and Zhengyu Zong, *All Genus Open-Closed Mirror Symmetry for Affine Toric Calabi-Yau 3-Orbifolds*, arXiv:1310.4818.

[15] E. Witten, *Quantum field theory and the Jones polynomial*, Commun. Math. Phys. **121** (1989), 351.

[16] R. Dijkgraaf and H. Fuji, *The volume conjecture and topological strings*, Fortsch. Phys. **57** (2009), 825–856, arXiv:0903.2084.

[17] R. Dijkgraaf, H. Fuji, and M. Manabe, *The volume conjecture, perturbative knot invariants, and recursion relations for topological strings*, Nucl. Phys. B **849** (2011), 166–211, arXiv:1010.4542.

[18] H. Fuji, S. Gukov, and P. Sułkowski, Volume conjecture: refined and categorified (2012), arXiv:1203.2182.

[19] S. Gukov and P. Sułkowski, *A-polynomial, B-model, and quantization*, JHEP **1202** (2012), 070 [arXiv:1108.0002v1 [hep-th]].

[20] G. Borot and B. Eynard, *All-order asymptotics of hyperbolic knot invariants from non-perturbative topological recursion of A-polynomials*, arXiv:1205.2261v1 [math-ph].

[21] A. Prats Ferrer, *New recursive residue formulas for the topological expansion of the Cauchy Matrix Model*, JHEP **1010** (2010), 090 [arXiv:0912.2984v4 [math-ph]].

[22] V. Bouchard, J. Hutchinson, P. Loliencar, M. Meiers and M. Rupert, *A generalized topological recursion for arbitrary ramification*, arXiv:1208.6035v1 [math-ph].

[23] V. Bouchard and B. Eynard, *Think globally, compute locally*, JHEP **02** (2013) 143 [arXiv:1211.2302v2 [math-ph]].

[24] B. Eynard, *Intersection numbers of spectral curves*, math-ph: arXiv:1104.0176.

[25] ———, *Invariants of spectral curves and intersection theory of moduli spaces of complex curves*, math-ph: arXiv:1110.2949.

[26] P. Dunin-Barkowski, N. Orantin, S. Shadrin, and L. Spitz, *Identification of the Givental formula with the spectral curves topological recursion procedure*, arXiv:1211.4021.

[27] I.K. Kostov, *Conformal field theory techniques in random matrix models*, arXiv:hep-th/9907060.

[28] Orantin, N., *Symplectic invariants, Virasoro constraints and Givental decomposition*, eprint arXiv:0808.0635, 2008.

[29] Ivan Kostov and Nicolas Orantin, *CFT and topological recursion*, Journal of High Energy Physics, 2010(11), November 2010.

[30] B. Eynard, N. Orantin, *Topological expansion of mixed correlations in the hermitian 2 Matrix Model and $x - y$ symmetry of the F_g algebraic invariants*, arXiv:0705.0958v1.

[31] ———, *About the x-y symmetry of the F_g algebraic invariants*, arXiv:1311.4993.

[32] B. Eynard, N. Orantin and M. Mariño, *Holomorphic anomaly and matrix models*, JHEP, 2007.

[33] G. Borot and B. Eynard, *Geometry of spectral curves and all order dispersive integrable system* (2011), arXiv:1110.4936.

[34] T. Ekedahl, S. Lando, M. Shapiro, and A. Vainshtein, *On Hurwitz numbers and Hodge integrals*, Comptes Rendus de l'Académie des Sciences - Series I - Mathematics **328** (1999), 1175-1180, arXiv:math/9902104.

[35] R. Kashaev, *The hyperbolic volume of knots from the quantum dilogarithm*, Lett. Math. Phys. **39** (1997), 269–275, arXiv:q-alg/9601025.

[36] S. Gukov and H. Murakami, $\mathrm{SL}_2(\mathbb{C})$ *Chern-Simons theory and the asymptotic behavior of the colored Jones polynomial*, Lett. Math. Phys. **86** (2008), 79–98, arXiv:math/0608324.

[37] K. Hikami, *Generalized volume conjecture and the A-polynomials: the Neumann-Zagier potential function as a classical limit of the partition function*, J. Geom. Phys. **57** (2007), 1895–1940.

[38] V.F.R. Jones, *Polynomial invariants of knots via von Neumann algebras*, Bull. Amer. Math. Soc. **12** (1985), 103–111.

[39] M. Mulase and P. Sułkowski, *Spectral curves and the Schroedinger equations for the Eynard-Orantin recursion*, arXiv:1210.3006 [math-ph].

[40] M. Bergère and B. Eynard, *Determinantal formulae and loop equations* (2009), arXiv: 0901.3273.

[41] M. Bergère, G. Borot, and B. Eynard, *Rational differential systems, loop equations, and application to the q-th reduction of KP*, arXiv:1312.4237.

[42] M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function*, Comm. Math. Phys. **147** (1992), no. 1, 1–23.

[43] E. Witten, *Two-dimensional gravity and intersection theory on moduli space*, Surveys in Differential Geometry **1** (1991), 243–310.

[44] S. Bergman and M. Schiffer, *Kernel functions and elliptic differential equations in mathematical physics*, Academic Press Inc., Publishers, New York, NY, 1953.

[45] Alexander B. Givental, *Gromov - witten invariants and quantization of quadratic hamiltonians*, 2001.

[46] Rahul Pandharipande and Richard Thomas, *Cumrun Vafa, Ravi Vakil, Eric Zaslow, Kentaro Hori, Albrecht Klemm, and Sheldon Katz*, *Mirror Symmetry*, American Mathematical Society, November 2003.

[47] V. Bouchard and P. Sułkowski, *Topological recursion and mirror curves*, arXiv:1105. 2052v1 [hep-th].

[48] J. Fay, *Theta Functions on Riemann Surfaces*, Number 352 in Lecture Notes in Mathematics. Springer–Verlag, 1970.

[49] A.Kokotov, D.Korotkin, *Tau functions on Hurwitz spaces*, Math. Phys., Analysis and Geom. **7** (2004), no. 1, 47-96, arXiv:math-ph/0202034.

[50] Bertrand Eynard and Nicolas Orantin, *Algebraic methods in random matrices and enumerative geometry*, 139p, November 2008.

[51] Chen, Lin, *Bouchard-Klemm-Marino-Pasquetti Conjecture for* $\mathbb{C}^3$, eprint arXiv:0910. 3739, 2009.

[52] Zhou, Jian, *Local Mirror Symmetry for One-Legged Topological Vertex*, eprint arXiv: 0910.4320, 2009.

[53] P. Norbury and N. Scott, *Polynomials representing Eynard-Orantin invariants*, Q J Math (2012) [arXiv:1001.0449v1 [math.AG]].

[54] P. Norbury and N. Scott, *Gromov-Witten invariants of $\mathbb{P}^1$ and Eynard-Orantin invariants*, arXiv:1106.1337v2 [math.AG].

[55] Duiliu-Emanuel Diaconescu and Bogdan Florea, *Localization and gluing of topological amplitudes*, September 2003.

[56] Chiu-Chu M. Liu, *Localization in Gromov-Witten theory and orbifold Gromov-Witten theory*, February 2012.

[57] O. Dumitrescu, M. Mulase, B. Safnuk and A. Sorkin, *The spectral curve of the Eynard-Orantin recursion via the Laplace transform*, arXiv:1202.1159v1 [math.AG].

[58] B. Eynard, *Recursion between Mumford volumes of moduli spaces*, Annales de l'IHP Volume 12, Issue 8 (2011), 1431–1447, Springer Link. arXiv:0706.4403.

[59] Vincent Bouchard, *Lectures on complex geometry, Calabi-Yau manifolds and toric geometry*, February 2007.

[60] W. Fulton, *Introduction to toric varieties*, Princeton University Press, 1993.

[61] Katz, Sheldon and Liu, Chiu-Chu Melissa, *Enumerative geometry of stable maps with Lagrangian boundary conditions and multiple covers of the disc*, eprint arXiv:math/0103074, 2001.

[62] Motohico Mulase and Brad Safnuk, *Mirzakhani's recursion relations, Virasoro constraints and the KdV hierarchy*, 21p, January 2006.

[63] K. Liu and H. Xu, *A simple proof of Mirzakhani's recursion formula of Weil-Petersson volumes*, arXiv:0705.2086.

[64] M. Mulase and B. Safnuk, *Mirzakhani's recursion relations, Virasoro constraints and the KdV hierarchy*, arXiv:math.AG/0101147.

[65] E Arbarello and M Cornalba, *Combinatorial and algebro-geometric cohomology classes on the moduli space of curves*, J. Alg. Geom **5** (1996), 705–709.

[66] D. Mumford, *Towards an enumerative geometry of the moduli space of curves*, in Arithmetic and Geometry (M Artin, J. Tate, editors), vol. 2, Birkhauser, Boston, 1983, pp. 271–328.

[67] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, *Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes*, Commun.Math.Phys. **165** (1994), 311–428.

IPHT CEA Saclay, France; CRM, Montréal, Canada.
E-mail: bertrand.eynard@cea.fr

Cluster varieties and integrable systems

Vladimir V. Fock

Abstract. In this lecture we present a combinatorial approach to integrable systems on affine Poisson-Lie groups using cluster technique and use it to study their properties such as discrete flows and explicit solutions.

Mathematics Subject Classification (2010). Primary 14H70; Secondary 13F60.

Keywords. Integrable systems, cluster varieties.

1. Introduction

Integrable systems have been in the focus of mathematical community for about thirty years and shown to be related to various branches of mathematics. The most fruitful approach to integrable systems, classical and quantum, are related to their interpretation in the context of Poisson-Lie groups. In this approach the phase space of an integrable system is interpreted as a symplectic leaf of a certain Lie group provided with a Poisson structure and the commuting Hamiltonians are Ad-invariant functions on these groups. The advantage of this approach is that since Poisson-Lie groups can be quantized to quantum groups the integrable systems can be also quantized using this technique.

However this approach meets certain technical difficulties. First of all, for groups of higher rank the explicit solutions and quantization turn out to be rather complicated, even though in principle it can be done. Due to these difficulties some structures of integrable systems become rather hard to discover and study.

Recently A.Goncharov and R.Kenyon [1] suggested a description of a class of integrable systems *à-priori* having nothing in common with Poisson-Lie groups coming out of a study of partition function of perfect matchings of bipartite graphs on a 2D torus. Their construction is rather elementary and easy to handle. As we have shown [6], their class of integrable systems is a particular, but a very large subclass of the integrable systems on Poisson-Lie groups, namely the one related to the affine Poisson-Lie groups of type $A_n^{(1)}$. The phase space of GK integrable systems are cluster varieties, and the integrable system structure is compatible with the cluster one. This allows to use such features of cluster varieties as discrete group action (and thus describe discrete flows commuting with continuous ones), to quantize the system cobinatorially without use of quantum groups, to identify textità-priori different integrable systems coming from different Poisson-Lie groups. But perhaps the main advantage of the Goncharov-Kenyon approach is its simplicity. In particular the formulas for for classical solutions of the systems are rather simple and hopefully they will facilitate the study of quantum solutions.

Proceedings of the International Congress of Mathematicians, Seoul, 2014

One should mention also some difficulties of the theory. First of all not all integrable systems admit such description. The largest class of systems which does not so far has direct relations to the GK systems are Hitchin systems. We hope however, that the construction will admit a generalization to incorporate them. Such rather popular systems as Toda and KdV are not in the class either, but in this case the relation is more straightforward. Namely, the so-called relativistic Toda system [7] is in the class, and the ordinary Toda system is a certain limit of the latter. Similar situation arises with the KdV and its generalizations. Though they are infinite-dimensional systems, they can be obtained as a certain continuous limit of the systems of GK class due to an observation by V.Ovsienko, R.Schwartz and S.Tabachnikov[4].

2. Cluster varieties

Cluster varieties is a class of varieties appeared as a generalization of the properties of Teichmüller space and the space of simple Lie group local systems on Riemann surfaces [12, 14, 15]. On the other hand it is a closely related to the notion of cluster algebra by S.Fomin and A.Zelevinsky [3] used to describe canonical basis of functions on simple Lie groups. A cluster variety (strictly speaking, the simplest version of it — of x-type with skew-symmetric exchange matrix) is a smooth algebraic variety admitting an atlas of charts such that each of them is a split algebraic torus $\{(x_1, \dots, x_n)\}$ provided with a Poisson structure $\sum_{ij} \varepsilon_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_i}$. The matrix ε_{ij} is a skew-symmetric matrix with integral entries called *exchange matrix*. A *mutation* is a Poisson birational map between two such algebraic tori given by an explicit formula

$$x_i' = \begin{cases} x_i^{-1} & \text{if} \quad i = k \\ x_i(1+x_k)^{\varepsilon_{ik}} & \text{if} \quad \varepsilon_{ik} \geqslant 0 \\ x_i(1+x_k^{-1})^{\varepsilon_{ik}} & \text{if} \quad \varepsilon_{ik} < 0 \end{cases}$$

for some $k \in \{1, \dots, n\}$. On can easily check that inverse to a mutation is also a mutation. We require that any chart be glued to n other charts by mutations and that one can pass from any chart to any other by a sequence of them.

The entire cluster variety is thus defined by any of its charts, i.e., by an exchange matrix of any of the charts.

It is often convenient to encode an exchange matrix ε_{ij} by a graph with n vertices and ε_{ij} oriented edges form vertex i to vertex j each time than $\varepsilon_{ij} > 0$.

If two charts are isomorphic as a Poisson split algebraic tori, the gluing map between them extends to a Poisson automorphism of the whole cluster variety. The group of such automorphisms is called the (generalized) *mapping class group* of the variety. The name comes from the Teichmüller space where this group indeed coincide with the mapping class group of the corresponding surface.

Since the formulas for the mutation do not involve subtraction, a cluster variety is defined not only over a field, but also over a semi-field. For example over positive real numbers or over integers with tropical operations.

A regular function on a cluster variety is a function which is given by a Laurent polynomial in any chart. Among such functions there is a cone of function taking only positive values over positive reals and having integral coefficients in any chart. Though it is not true

in general, for a large class of cluster varieties such positive and integral functions is a free cone and thus its generators give a canonical basis in the space of functions.

Any cluster variety admits thus a Poisson structure which can be quantized in different senses of this word. In particular, [11, 13] the algebra of functions on a cluster variety over positive reals admits a canonical deformation compatible with the mapping class group action. One of the features of this quantization is that it is invariant under inversion of the quantization parameter $\hbar \to 1/\hbar$.

3. Planar algebraic curves and Newton polygons

Denote by $\mathcal{C}$ the space of algebraic curves in $(\mathbb{C}^\times)^2$ considered up to the multiplicative shift of the ambient torus. This space decomposes into strata denoted by $\mathcal{C}_\Delta$ corresponding to curves given by zero locus of polynomials $P(\lambda, \mu)$ with a given Newton polygon Δ. Denote by I_Δ, B_Δ, C_Δ and S_Δ the number of integral points strictly inside, on the boundary and the number of corners of the polygon Δ, respectively. Denote also by its area by S_Δ. The curves from $\mathcal{C}_\Delta$ have geometric genus I_Δ and can be compactified by adding B_Δ points at infinity (counted with multiplicity). Two polynomials define the same curve if they differ by a transformation $P(\lambda, \mu) \to aP(b\lambda, c\mu)$ with nonvanishing a, b, c. One can use this degree of freedom to make three of the coefficients in the corner of Δ to be equal to one. The remaining coefficients are natural coordinates on $\mathcal{C}_\Delta$. Therefore the space $\mathcal{C}_\Delta$ is a toric variety of dimension $I_\Delta + B_\Delta - 3$. The space $\mathcal{C}_\Delta$ also does not change if we act on it by any integral lattice preserving affine transformation of the plane.

4. Integrable systems on Poisson-Lie groups

The starting observation is that on a Poisson-Lie group, with the Poisson bracket defined by classical r-matrix, the Ad-invariant functions Poisson commute with each other. For a finite dimensional simple group the number of independent Ad-invariant functions is equal to the rank of the group, and thus the corresponding integrable system can arise on a symplectic leaf of dimension not more than twice the rank. However, for loop groups the number of independent Ad-invariant functions is infinite though all Poisson submanifolds are still finite dimensional, and thus the set of integrable systems one gets in this way is much larger. For simplicity we restrict ourselves to the systems on the groups of type $\hat{A}_n^{(1)}$, namely the groups $\widehat{PGL}(N)$ of matrix-valued Laurent polynomials $A(\lambda)$ of a single variable λ with nonzero constant determinant and considered up to multiplication by a nonzero constant. For a given $\mathcal{A}(\lambda)$ the zero locus of the Laurent polynomial of two variables $P(\lambda, \mu) = \det(\mathcal{A}(\lambda) - \mu) = \sum_{ij} \mathcal{H}_{ij} \lambda^i \mu^j$ defines an algebraic curve in $\mathcal{C}$ called *spectral*.

This curve obviously is a conjugacy class invariant. It comes together with a line bundle, given by the kernel of $\mathcal{A}(\lambda) - \mu \cdot \mathrm{Id}$. The map from the group $\widehat{PGL}(N)$ to the space of $\mathcal{C}$ of curves is called the action map, and it is a Poisson map if we take a trivial Poisson bracket on the space of curves $\mathcal{C}$. The functions $\mathcal{H}_{ij}$ themselves are well defined if we require three corner coefficients to be equal to unities. With this condition $\mathcal{H}_{ij}$ Poisson-commute. Moreover $B_\Delta - 3$ nontrivial coefficients corresponding to the boundary of the polygon generate the center of the Poisson algebra. The map to the pair (curve, line bundle

on it) is called the action-angle map. The flows generated by the I_Δ nontrivial Poisson commuting Hamiltonians $\mathcal{H}_{ij}$ amount to the constant flow of the line bundle along the Picard variety of the spectral curve.

A loop group $\widehat{PGL}(N)$ does not have a cluster variety structure. However it is embedded as a Poisson submanifold into a central coextension $\widehat{PGL}(N)^\sharp$, which admits a standard decomposition into disjoint union of Poisson submanifolds $\mathcal{X}_u$, called double Bruhat cells [10],[6] and which are cluster varieties. The elements of the group $\widehat{PGL}(N)^\sharp$ can be presented as an expressions $A(\lambda)z^{\lambda\frac{\partial}{\partial\lambda}}$, where $A(\lambda)$ is an $N\times N$ matrix valued Laurent polynomial with constant determinant and defined up to a constant factor.

The double Bruhat cells C^u of this group are enumerated by the elements u of a coextension $(\widehat{W}\times\widehat{W})^\sharp$ of the square of the Weyl group $\widehat{W}$ of $\hat{G}$ by the automorphism group of the Dynkin diagram. For the group $\hat{G}^\sharp=\widehat{PGL}(N)^\sharp$ the group $(\widehat{W}\times\widehat{W})^\sharp$ is a group generated by n generators s_i, N generators $\bar{s}_i$ with $i\in\mathbb{Z}/N\mathbb{Z}$ and one generator Λ satisfying relations

$$\begin{aligned}
&s_is_{i+1}s_i=s_{i+1}s_is_{i+1},\\
&\bar{s}_i\bar{s}_{i+1}\bar{s}_i=\bar{s}_{i+1}\bar{s}_i\bar{s}_{i+1},\\
&s_i\Lambda=\Lambda s_{i+1},\\
&\bar{s}_i\Lambda=\Lambda\bar{s}_{i+1},\\
&\Lambda^N=1
\end{aligned}$$

and all other pairs of generators commute.

Quotients of the double Bruhat cells of the group by conjugation by the finite dimensional Cartan subgroup H, (we call them also the double Bruhat cells and denote by $\mathcal{X}_u$) are also cluster varieties of dimension equal to the length of u. Given a presentation of u as a reduced word of the standard generators, one can define the cluster coordinates $\boldsymbol{x}=\{x_i\}$, enumerated by the letters of the word (except for the generator Λ). The image of the group $\widehat{PGL}(N)$ (without coextension) in these cells is just a codimension one Poisson subvariety of $\mathcal{X}_u$ given by the condition $\prod x_i=1$ in any chart. For a given u the Laurent polynomial $P(\lambda,\mu)=\det(\mathcal{A}(\boldsymbol{x},\lambda,\mu)=\sum\mathcal{H}_{ij}(\boldsymbol{x})\lambda^i\mu^j$ defines a curve in $\mathcal{C}_\Delta(u)$ for some fixed polygon $\Delta(u)$. In order for this system to be integrable the Hamiltonians must be independent, and their number should be maximal possible, namely $(\dim\mathcal{X}_u^1+\frac{1}{2}\text{corank }\mathcal{X}_u^1)$ for the given dimension of the cell and given rank of the Poisson bracket). This condition is satisfied on double Bruhat cells, corresponding to u having minimal length in its conjugacy class (such elements are also called *cyclically irreducible*).

Many of such integrable system admit a discrete flow commuting with continuous one described in [9]. Namely define a map $\tau:\mathcal{X}_u^1\to\mathcal{X}_u^1$ as follows. Let $A(\lambda)$ be a representative of a point $\boldsymbol{x}\in\mathcal{X}_u^1$ and let $A(\lambda)=A^-(\lambda)A^+(\lambda)$ be its Gauss decomposition into lower- and upper-triangular parts with respect to the finite-dimensional Cartan subgroup H. Then the point $\tau(\boldsymbol{x})$ will be represented by $A^+(\lambda)A^-(\lambda)$. This transformation obviously does not change the spectral curve and thus commutes with continuous flows.

5. Goncharov-Kenyon integrable systems

The scheme of the construction of the GK integrable system is as follows. The starting point is a bipartite graph (i.e. a graph with vertices colored in black and white and with edges connecting only vertices of different colors) on a closed surface Σ, satisfying certain

minimality and non-triviality conditions, which will be explained below. In this exposition we follow [1] and consider the surface Σ to be of genus one.

A line bundle on a bipartite graph Γ is an association of a one dimensional vector space V_v to every vertex v. A discrete connection on this bundle is just a collection of isomorphisms $A_e : V_{b(e)} \to V_{b(e)}$ for every edge e of the graph. Here we denoted by $b(e)$ and $w(e)$ the white and the black ends of the edge e, respectively.

Consider the space of discrete connections on the graph Γ with values in the multiplicative group (to be specific we assume it to be the multiplicative group of non-zero complex numbers $\mathbb{C}^\times$). Since every edge of a bipartite graph can be canonically oriented, say from white to black vertex, we can interpret this space as the multiplicative cohomology group $H^1(\Gamma, \mathbb{C}^\times)$. We have the following exact sequence

$$1 \to H^1(\Sigma, \mathbb{C}^\times) \to H^1(\Gamma, \mathbb{C}^\times) \to H^2(\Sigma/\Gamma, \mathbb{C}^\times) \to H^2(\Sigma, \mathbb{C}^\times) \to 1.$$

Denote by $\mathcal{X} = H^2(\Sigma/\Gamma, \mathbb{C}^\times)$ the space of association of nonzero complex numbers x_i to every face i of the graph and by $\mathcal{X}^1$ the kernel of the map to $H^2(\Sigma, \mathbb{C}^\times)$ which is just determined by the condition $\prod_i x_i = 1$. The exact sequence implies that the space $H^1(\Gamma, \mathbb{C}^\times)$ is a principal $H^1(\Sigma, \mathbb{C}^\times)$-bundle over $\mathcal{X}^1$.

The algebraic torus $\mathcal{X}$ has a natural exchange matrix given by a graph obtained from the one dual to Γ by eliminating pairs of parallel oppositely oriented edges.

A Kasteleyn orientation $\boldsymbol{K} = \{K_e\}$ on a graph Γ is a class in $H^1(\Gamma, \pm 1)$. such that its image in $H^2(\Sigma/\Gamma)$ is equal to $(-1)^{l/2+1}$ on any face with l sides.

For every discrete connection $\boldsymbol{A} = \{A_e\} \in H^1(\Gamma, \mathbb{C}^\times)$ and for every Kasteleyn orientation $\boldsymbol{K}$ define a discrete Dirac operator $\mathfrak{D}_{\boldsymbol{K}}(\boldsymbol{A}) : \oplus_b V_b \to \oplus_w V_w$ given by the formula

$$\mathfrak{D}_{\boldsymbol{K}}(\boldsymbol{A})|_{V_b} = \oplus_{e|b(e)=b} A_e K_e.$$

This operator degenerates on a subvariety of $H^1(\Gamma, \mathbb{C}^\times)$ which is the vanishing locus of the determinant of $\mathfrak{D}_{\boldsymbol{K}}(\boldsymbol{A})$. Intersection of this locus with a fiber over a point $\boldsymbol{x} = \{x_i\} \in \mathcal{X}^1$ gives an algebraic curve (defined up to a shift of the torus since the identification of the fiber with the torus $H^1(\Sigma, \mathbb{C}^\times)$ is not canonical) with a line bundle on it given by the kernel of $\mathfrak{D}_{\boldsymbol{K}}(\boldsymbol{A})$. Observe, that the determinant of $\mathfrak{D}_{\boldsymbol{K}}(\boldsymbol{A})$ written in any trivialization is a sum of monomials over the perfect matchings of the white and a black vertices, and this is how the dimer configurations on the graph Γ come into play. The Kasteleyn orientation structure permits to control signs of the monomials.

This construction defines therefore the action map of the phase space of our integrable system $\mathcal{X}^1$ to the space of planar algebraic curves in $\mathcal{C}_\Delta$ for a Newton polygon Δ determined by the graph (see below). It also defines the action-angle map to the pairs (plane curve, line bundle on it) which is a birational isomorphism.

The Poisson structure on $\mathcal{X}$ has a geometric interpretation suggested in [1] as follows. Embedding the graph Γ into the surface Σ induces cyclic order of ends of the edges at every vertex (a fat graph structure). Consider a surface $\check{\Sigma}$ corresponding to the same bipartite graph Γ, but with the cyclic order changed to the opposite in white vertices and kept unchanged in the black ones. Since the graph Γ is embedded into $\check{\Sigma}$ consider the composition of the maps $H^1(\check{\Sigma}, \mathbb{C}^\times) \to H^1(\Gamma, \mathbb{C}^\times) \to \mathcal{X}$. The space $H^1(\check{\Sigma})$ has a canonical Poisson structure, coming from the intersection index on $\check{\Sigma}$ and the map onto $\mathcal{X}^1$ induces the Poisson bracket on the latter and a Poisson bracket on the whole $\mathcal{X}$ if we require its shift invariance.

In order for the action map to provide maximal number of comuting Hamiltonians we need to impose on on the graph Γ an additional condition called *minimality*. It amounts to

the requirement that the number of faces of Γ be equal to $2S_\Delta$. As it will be clear below, any polygon Δ admits a minimal bipartite graph with this property.

In [1] it is observed that graphs admit elementary transformations called *spider moves* such that integrable systems corresponding to them are isomorphic provided the phase spaces are related by a cluster mutation. Equivalence classes of integrable systems under such transformations are enumerated by Newton polygons Δ.

In [6] it is shown that the subgroup $\mathcal{G}_\Delta$ of the mapping class group generated by such transformations admits the following description. Let $\mathbb{Z}^{C_\Delta}$ be a group of integer valued functions on the vertices of the Newton polygon and let R be a subgroup of functions extendible to an affine function on the plane and taking integral values at integral points. Then the group $\mathcal{G}_\Delta$ is isomorphic to the quotient $\mathbb{Z}^{C_\Delta}/R$. The rank of this group is $C_\Delta - 3$ but in general its torsion is also nontrivial.

6. Relations between two approaches

We claim that the GK integrable systems coincide with the integrable systems on the Poisson-Lie loop groups $\widehat{PGL}(N)$. The isomorphism identifies not only their phase spaces and commuting flows, but also the discrete group action and the canonical cluster coordinates.

In both constructions the spectral curve of an integrable system is given by degeneracy condition of some matrix operator $A(\boldsymbol{x},\lambda) - \mu \cdot \mathrm{Id}$ in the group-theory approach and the Dirac operator $\mathfrak{D}_{\boldsymbol{K}}(\boldsymbol{A}(\boldsymbol{x},\lambda,\mu))$ respectively). Though the matrices do not coincide, their determinants do. The correspondence goes roughly as follows. The determinant of any matrix $\mathfrak{D}(\boldsymbol{A})$ can be written as a Grassmann integral:

$$S_\Gamma(\boldsymbol{A}) = \det \mathfrak{D}(\boldsymbol{A}) = \int \exp\Big(\sum \mathfrak{D}(\boldsymbol{A})^b_w \xi_b \eta^w\Big) \prod_b d\xi_b \prod_w d\eta^w$$

Therefore $\det \mathfrak{D}(\boldsymbol{A})$ can be interpreted as a partition function $S_\Gamma(\boldsymbol{A})$ of some lattice fermions in the background gauge field $\boldsymbol{A}$. Cutting torus into a cylinder corresponds to rewriting this partition function as a trace of the evolution operator from one boundary circle to another. This evolution operator is given by the matrix $A(\lambda)$ acting in the external algebra of the N-dimensional space. Moreover cutting further this cylinder into a set smaller cylinders, one can present the evolution operator as a product of elementary steps, each depending on no more than one variable x_i and exactly coinciding with elementary matrices, used to parametrize the double Bruhat cells, thus establishing the coincidence of spectral curves.

The first part of this program is establishing correspondence between words in generators of $(\hat{W} \times \hat{W})^\sharp$, enumerating cluster coordinate systems in the first approach, with the bipartite graphs, enumerating coordinates on the second approach. Moreover this correspondence should identify the letters of the word with the faces of the corresponding bipartite graphs, drawn on torus Σ, since these both sets correspond to the cluster coordinates in corresponding cases.

In order to do this we use the third combinatorial object, suggested by Dylan Thurston in unpublished paper [8] (and already used in [1] in our context), which we call the *Thurston diagrams*. A Thurston diagram is an isotopy class of a collection of curves on a surface, either closed or connecting two boundary points with only triple intersection points and such, that the connected components of the complement (faces) are colored in white and

gray with any two faces sharing a segment of a curve having different colors (chessboard coloring). Such diagrams admit elementary modifications called Thurston moves. As it was already observed by D. Thurston and A. Henriques, every Thurston diagram defines a cluster seed (a chart on a cluster manifold) with cluster variables attached to the white faces. Thurston moves correspond to mutations (passing from one chart to another). Having Thurston diagrams on open surfaces one can glue together boundary components respecting their coloring, and thus obtain a new surface with a Thurston diagram.

In order to construct a Thurston diagram out of a reduced decomposition of an element $u \in (\widehat{W} \times \widehat{W})^\sharp$, we first associate a Thurston diagram on a cylinder with a single triple point and with N gray (and white) segments on every boundary circle to every generator of $(\widehat{W} \times \widehat{W})^\sharp$ (except the cocentral one). Then we glue the cylinders together according to the order of the generators in the reduced decomposition, and finally we glue both ends of the resulting cylinder together with a twist, given by the power of the cocentral generator Λ.

In order to construct a bipartite graph out of a Thurston diagram we put a black vertex at every triple point and a white vertex at every gray face. Then we draw three edges from each black vertex inside the three gray sectors, meeting at this vertex, to the respective white vertices. It is easy to see, that the set of letters of the reduced word is in a canonical bijection with the set of white faces of the Thurston diagram and the latter are in bijection with faces of the bipartite graph. Next observation, almost as simple, is that this bijection induces a bijection between the cluster seeds, i.e. the Poisson bracket between the coordinates coincide.

The Newton polygon of the corresponding integrable system can be easily read off the Thurston diagram. Since every curve of the diagram posesses a canonical orientation it represents a homology class in $H_1(\Sigma, \mathbb{Z})$ where Σ is the torus. The sum of all such classes vanishes since it is a boundary of the union of grey areas. Therefore they are sides of a unique convex polygon Δ with integral vertices.

Thus the Newton polygons constructed out of cyclically reduced words u corresponging to the cells of the group $\widehat{PGL}(N)$ have width N. Since the width of a polygon depends on the coordinate system on the plane the same itegrable system can be obtained on cells coming from different groups. Thisis a generalization of the known phenomenon that the N-particle Toda system can be described both using $\widehat{PGL}(N)$ and $\widehat{PGL}(2)$.

Finally, we need to show that equations $\det(A(\boldsymbol{x}, \lambda) - \mu) = 0$ and $\det \mathfrak{D}_{\boldsymbol{K}}(\boldsymbol{A}(\boldsymbol{x}, \lambda, \mu)) = 0$ define the same curve.

For this purpose we extend the lattice fermion partition functions on a bipartite graph to surfaces with boundary. Graphs on such surfaces are allowed to have vertices of the third type, terminating on the boundary and which can be connected to both white and black vertices, but not to each other. Denote the set of such vertices by T. The Dirac operator now acts as $\mathfrak{D}(\boldsymbol{A}) : \mathbb{C}^{B \cup T} \to \mathbb{C}^{W \cup T}$, where W and B denote the sets of white and black vertices, respectively. For extra Grassmann variables $\boldsymbol{\zeta} = \{\zeta_t | t \in T\}$ we define

$$S(\boldsymbol{A}, \boldsymbol{\zeta}) = \int \exp\left(\sum \mathfrak{D}(\boldsymbol{A})^b_w \xi_b \eta^w + \mathfrak{D}(\boldsymbol{A})^b_t \xi_b \zeta^t + \mathfrak{D}(\boldsymbol{A})^t_w \zeta_t \eta^w\right) \prod_b d\xi_b \prod_w d\eta^w$$

Gluing two boundary components of Σ with a bipartite graph Γ on it in a way, that terminal vertices are glued to terminal vertices, one gets a bipartite graph $\overline{\Gamma}$ on the glued surface $\overline{\Sigma}$. The connection $\boldsymbol{A}$ on Γ induces a connection $\overline{\boldsymbol{A}}$ on $\overline{\Gamma}$: we just multiply the numbers of two halves of a glued edges. If $S_\Gamma(\boldsymbol{A}, \boldsymbol{\zeta})$ is a partition function for Γ then the partition function

for $\overline{\Gamma}$ is given by:

$$S_{\overline{\Gamma}}(\overline{\boldsymbol{\zeta}},\overline{\boldsymbol{A}})=\int S_\Gamma(\boldsymbol{\zeta},\boldsymbol{A})e^{\sum\zeta_t\zeta_{\sigma(t)}}\prod d\zeta_t d\zeta_{\sigma(t)},$$

where the index t runs over terminal edges on one side, σ is a map sending a terminal vertex to the one it is glued to and $\overline{\boldsymbol{\lambda}}$ is $\boldsymbol{\lambda}$ with entries corresponding to glued vertices removed.

On the other hand observe that for any $N\times N$ matrix M one can associate a function of $2N$ Grassmann variables $\boldsymbol{\xi}=\{\xi_i\}$ and $\boldsymbol{\eta}=\{\eta_i\}$ given by $S_M(\boldsymbol{\xi},\boldsymbol{\eta})=\exp M^i_j\xi_i\eta^j$. Matrix product corresponds to convolution of the corresponding functions

$$S_{M_1M_2}(\boldsymbol{\xi},\boldsymbol{\eta})=\int S_{M_1}(\boldsymbol{\xi},\boldsymbol{\eta}')S_{M_2}(\boldsymbol{\xi}',\boldsymbol{\eta})e^{\sum\xi'_i\eta'_i}\prod d\xi'_i d\eta'_i$$

Thus if a partition function on a graph on a cylinder coincides with a partition function of a matrix, the partition function of several cylinders glued together corresponds to product of the matrices. Therefore in order to show the coincidence of the curves we need to cut the torus into small cylinders and verify for each of them the coincidence of partition functions.

7. Solution of the integrable systems

The GK integrable systems (and thus the ones on $\widehat{PGL}(N)$) admit explicit solution in terms of theta functions. By a solution here we mean just the map inverse to the action-angle one. Namely, for a given nonsingular planar curve $\Sigma'\in\mathcal{C}_\Delta$, any graph Γ corresponding to Δ and any point $p\in Pic^{g-1}(\Sigma')$ of the Picard variety, where $g=I_\Delta$ is the genus of the curve, we can find a point of the cluster variety $\boldsymbol{x}=\{x_i\}\in\mathcal{X}^1$ such that the action-angle map of x would give (Σ',p).

The main observation allowing to find this map is that the algebraic curve Σ' and the dual curve $\check{\Sigma}$ obtained by changing cyclic order in white vertices of the bipartite graph Γ are very closely related. More precisely, points at infinity of the spectral curve correspond to the sides of the Newton polygon Δ in such a way that the number of points corresponding to a given side is equal to its length (the number of segments between integral points). The same correspondence exists from the faces of the dual surface and the sides. Moreover the genera of the curves coincide. Therefore one can identify both surfaces in a way that every face of $\check{\Sigma}$ contains exactly one point at infinity of the spectral curve corresponding to the same side of Δ.

Denote by F the space of faces of the graph $\Gamma\subset\Sigma$ and by $\check{F}$ the set of faces of the graph $\Gamma\subset\check{\Sigma}$. The set $\check{F}$ is simultaneously the set of points at infinity of Σ'. For every point at infinity $\check{\imath}\in\check{F}$ denote by $[\check{\imath}]$ a lift of this point to the universal cover of the Picard variety $Pic^1(\Sigma')$. Let Z be a universal cover of the Jacobian $Jac(\check{\Sigma})$. The fundamental group of $Jac(\Sigma')$ can be identified with the group $H^1(\Sigma,\mathbb{Z})$. Choose two complimentary Lagrangian sublattices $L_a,L_b\subset H^1(\Sigma,\mathbb{Z})$ (make a choice of a- and b-cycles on $\check{\Sigma}$). Define a map associating a point $\boldsymbol{z}_i\in Z$ for every face $i\in F$ satisfying the condition that for any path γ_{ij} connecting the points at infinity i and j we have

$$\boldsymbol{z}_i-\boldsymbol{z}_j=\sum_{\check{\imath}\in\check{F}}\langle\gamma_{ij},\partial\check{\imath}\rangle[\check{\imath}]\text{ modulo }L_a.$$

The pairing $\langle , \rangle$ is the intersection index of oriented paths on Σ. One can check that such map exists and is unique up to a simultaneous shift $\boldsymbol{z}_i \to \boldsymbol{z}_i + t$ with $t \in Jac(\Sigma')$. Finally choose an odd nonsingular theta-characteristic $q \in Pic^{g-1}(\check{\Sigma})$.

After these choices made the inverse map is given by an explicit formula

$$x_i = \prod_j \left(\frac{\theta_q(\boldsymbol{z}_j - \boldsymbol{z}_i)}{\theta_q(p + \boldsymbol{z}_j - q)} \right)^{\varepsilon_{ij}}$$

Here θ_q is the Riemann theta function.

The action of the discrete group $\mathcal{G}_\Delta$ has also a very simple expression in these terms. Namely for every side of the polygon Δ consider the sum in $Pic(\Sigma')$ of the corresponding points at infinity. Such sums generate a discrete subgroup in $Pic(\Sigma')$. The group $\mathcal{G}_\Delta$ is isomorphic to the intersection of this subgroup with $Jac(\Sigma')$ and it obviously acts on $Jac(\Sigma')$ by shifts.

References

[1] A.B. Goncharov and R. Kenyon, *Dimers and cluster integrable systems*, arXiv:1107.5588.

[2] V. V. Fock and A. B. Goncharov, *Moduli spaces of local systems and higher Teichmüller theory*, Publ. Math. Inst. Hautes Etudes Sci. **103** (2006),1–211.

[3] S.V. Fomin and A.B. Zelevinsky, *Cluster algebras. I. Foundations*, J. Amer. Math. Soc. **15** (2002), no. 2, 497–529.

[4] V. Ovsienko, R. Schwartz, and S. Tabachnikov, *The Pentagram map: a discrete integrable system*, Comm. Math. Phys. **299** (2010), no. 2, 409–446, arXiv:0810.5605.

[5] V. V. Fock and A. B. Goncharov, *Cluster X-varieties, amalgamation and Poisson-Lie groups*. In *Algebraic Geometry Theory and Number Theory*, pp. 27–68, Progr. Math., 253, Birkhäuser Boston, Boston, MA, 2006. arXiv:math.RT/0508408.

[6] V. V. Fock and A. Marshakov, *Loop groups, Clusters, Dimers and Integrable systems*, arXiv:1401.1606.

[7] S.N.M. Ruijsenaars, *Relativistic Toda systems,* Comm. Math. Phys. **133** (1990), 217.

[8] D. Thurston, *From Dominoes to Hexagons,* arXiv:math/0405482.

[9] T. Hoffman, J. Kellendonk, N. Kuntz, and N. Reshetikhin, *Factorization dynamics and Coxeter-Toda lattices.*, Comm. Math. Phys. **212** (2000), no. 2, 297–321, arXiv:solv-int/9906013.

[10] M. Kogan and A. Zelevinsky, *On symplectic leaves and integrable systems in standard complex semisimple Poisson-Lie groups.*, Int. Math. Res. Not. (2002), no. 32, 1685–1702, arXiv:math/0203069.

[11] L.O. Chekhov and V.V. Fock, *Quantum Teichmüller space*, Preprint CPT-99/P.3825, Teor.Mat.Fiz. **120** (1999) 3, 511, arXiv:math.QA/9908165.

[12] V.V. Fock and A.B. Goncharov, *Moduli spaces of local systems and higher Teichmüller theory*, Publ.IHES **vol. 103** (2006), no.1, 1–221, arXiv:math.AG/0311149.

[13] ———, *The quantum dilogarithm and representations of quantum cluster varieties.*, Inv. Math. **175** (2009) 223–286.

[14] V.V. Fock, *Dual Teichmüller spaces*, Preprint ITEP-TH-5-96, arXiv:dg-ga/9702087.

[15] V.V. Fock and A.B. Goncharov, *Dual Teichmuller and lamination spaces*, in Handbook of Teichmüller Theory, Volume I, (ed. A.Papadopoulos) IRMA Lectures in Mathematics and Theoretical Physics Vol. 11, arXiv:math.DG/0510312.

IRMA, 7 rue René Descartes, 67084 Strasbourg Cedex, France
E-mail: fock@math.unistra.fr

Archimedean Langlands duality and exactly solvable quantum systems

Anton A. Gerasimov

Abstract. We demonstrate that Baxter operators for $\mathfrak{gl}_{\ell+1}$-Toda chains understood as elements of spherical Archimedean Hecke algebras provide a concise formulation of a special case of the local Archimedean Langlands correspondence. Categorification of the class one eigenfunctions of the q-deformed $\mathfrak{gl}_{\ell+1}$-Toda chain supplies a q-analog of the Shintani-Casselman-Shalika formula over non-Archimedean fields, thus revealing a q-version of the local Langlands correspondence. In the non-deformed case the q-analog of the formula turns into an expression of a matrix element of $GL_{\ell+1}(\mathbb{R})$ principal series spherical representation as the equivariant volume of an infinite-dimensional symplectic manifold. This provides another manifestation of the local Archimedean Langlands correspondence. Reformulation in terms of two-dimensional topological field theories allows identification of the considered instance of the Archimedean Langlands correspondence as mirror symmetry in two-dimensional quantum field theories.

Mathematics Subject Classification (2010). Primary 11F66, 81R12; Secondary 81T45, 11F70.

Keywords. Langlands duality, Baxter operator, L-factors, Toda chain, Whittaker function, Mirror symmetry.

1. Introduction

The local Archimedean Langlands correspondence for $GL_{\ell+1}(\mathbb{R})$ is a correspondence between (packages of) isomorphism classes of admissible representations of $GL_{\ell+1}(\mathbb{R})$ and isomorphism classes of admissible $\ell+1$-dimensional representations of the Weil group $W_{\mathbb{R}}$ (see e.g. [11], [1]). It is a very spacial case of a larger picture relating representation theory of reductive groups with the arithmetic properties of the base fields captured by slightly modified versions of the Galois groups called the Weil groups. The Langlands correspondence over the Archimedean field $\mathbb{R}$ of real numbers occupies a peculiar place in this picture. One of the reasons is that in contrast with their non-Archimedean counterparts the corresponding Weil group $W_{\mathbb{R}}$ is much larger then the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \mathbb{Z}_2$ of real numbers. Moreover various relevant objects such as local L-factors attached to admissible representations of reductive groups have a rather simple structure for non-Archimedean base fields and are given by transcendental functions in the Archimedean case. This implies that a proper understanding of the Archimedean completions of number fields is yet to come and new ideas are needed to be invoked.

Some steps in this direction were made in a series of papers [12]-[21], [22]-[24] using methods of the theory of quantum integrable systems and quantum fields. Below we

Proceedings of the International Congress of Mathematicians, Seoul, 2014

summarize the main results in the case of the Langlands correspondence for general linear groups over $\mathbb{R}$. Traditionally the local Archimedean Langlands correspondence attaches local Archimedean L-factors to admissible representations of $GL_{\ell+1}(\mathbb{R})$ using as input data the characters of the action of the center of the universal enveloping algebra $\mathcal{U}(\mathrm{Lie}(GL_{\ell+1}))$. This shall be contrasted with the non-Archimedean case where the action of the spherical Hecke algebra $\mathcal{H}(GL_{\ell+1}(\mathbb{Q}_p), GL_{\ell+1}(\mathbb{Z}_p))$ is used. As was shown in [12] this asymmetry can be removed. An element of the Archimedean spherical Hecke algebra $\mathcal{H}(GL_{\ell+1}(\mathbb{R}), O_{\ell+1}(\mathbb{R}))$ was constructed such that its action in the principal series spherical representations of $GL_{\ell+1}(\mathbb{R})$ is given by multiplication on the corresponding local Archimedean L-factors. The corresponding element of the spherical Hecke algebra in the Archimedean case turns out to be a universal form of the Baxter operator for the $\mathfrak{gl}_{\ell+1}$-Toda chain constructed in [12]. The construction of local L-factors as eigenvalues of appropriate elements of the spherical Hecke algebra holds in the non-Archimedean case too, thus providing a unified formulation of the local Langlands correspondence in terms of Baxter operators. The Baxter operator formalism [2] is one the most powerful approaches to solve quantum integrable systems, and the fact that the transcendental form of the Archimedean L-factor arises naturally in this setup seems suggests a relevance of quantum integrable systems to arithmetic problems.

There is also a direct way to formulate the local Langlands correspondence over $\mathbb{Q}_p$ as an identification of non-Archimedean Whittaker functions given by matrix elements of principal series spherical representations of $GL_{\ell+1}(\mathbb{Q}_p)$ with characters of finite dimensional irreducible representations of the dual complex group $GL_{\ell+1}(\mathbb{C})$ (the Shintani-Casselman-Shalika formula [3, 44]). The characters are evaluated on semisimple conjugacy classes of the homomorphisms $W_{\mathbb{Q}_p} \to GL_{\ell+1}(\mathbb{C})$ of the Weil group. Thus the Shintani-Casselman-Shalika formula provides a categorification of the Whittaker function in the sense of function-sheaf correspondence. This form of the local Langlands correspondence eludes any use of the spherical Hecke algebra in the non-Archimedean case.

To establish the Archimedean analog of the Shintani-Casselman-Shalika formula the use of a q-lattice version of the quantum Toda chains (also known as relativistic Toda chain [40]) was proposed [13–15]. The common eigenfunction of q-lattice quantum Toda Hamiltonians provides a q-analog of the classical Whittaker functions interpolating Whittaker functions over $\mathbb{R}$ and $\mathbb{Q}_p$. This interpolation allows the transfer of structures typical for non-Archimedean fields into the Archimedean world. This indeed gives rise to a q-version of the Shintani-Casselman-Shalika formula representing q-lattice Whittaker functions as traces over auxiliary vector spaces. These vector spaces allow two essentially different cohomological interpretations. Thus for instance for $\ell = 1$ one has a realization of the vector space as a limit $d \to \infty$ of zero cohomology groups of holomorphic line bundles over a quasi-map compactification of the space of degree d holomorphic maps $\mathbb{P}^1 \to \mathbb{P}^1$. Alternatively there exists a realization in terms of the de Rham cohomology of the formal sum of products of classifying spaces for unitary groups. These two representations (after slight modification) appear to be related with Demazure modules for the affine Lie algebra and the de Rham cohomologies of the Nakajima graded quivers. In a sense the existence of these two realizations of the underlying vector space in the q-analog of the Shintani-Casselman-Shalika formula captures the basic underlying structure behind local Langlands duality.

Existence of the q-lattice interpolation of the Whittaker function over $\mathbb{Q}_p$ and $\mathbb{R}$ implies that both fields have some common underlying structure. This hidden structure can be recovered via common degeneration of $\mathbb{Q}_p$- and $\mathbb{R}$-Whittaker functions to kinds of elementary

Whittaker functions solving eigenfunction problems of quantum billiard systems. The corresponding degeneration of the local L-factors leads to elementary L-functions first introduced in [32] with similar purpose to find a common underlying field behind $\mathbb{Q}_p$ and $\mathbb{R}$ with the associated residual field being the mysterious field $\mathbb{F}_1$ [45] (see also e.g. [5]). In the considered setup this enigmatic arithmetic structure manifests itself in the form of the tropical semifield $\mathcal{R}$ playing the role of the valuation map domain of a hypothetical non-Archimedean field with residual field $\mathbb{F}_1$. There is an analog of the Shintani-Casselman-Shalika type formula realizing a local Langlands correspondence for this field. It identifies elementary Whittaker functions given by matrix elements of irreducible representations of monoids $GL_{\ell+1}(\mathcal{R})$ with $U_{\ell+1}$-equivariant symplectic volumes of flag spaces $GL_{\ell+1}(\mathbb{C})/B$. In this formulation the equivariant volume plays the role of the character in the standard Shintani-Casselman-Shalika formula thus providing a symplectification of the elementary Whittaker function.

The main purpose of q-lattice interpolation is to construct an analog of the Shintani-Casselman-Shalika formula for real numbers. The corresponding representation of the Whittaker functions associated with $GL_{\ell+1}(\mathbb{R})$ generalizes the elementary Whittaker function as equivariant volume. Thus for instance the $\mathfrak{gl}_2$-Whittaker function can be identified with the $S^1 \times U_2$-equivariant volume of the limit $d \to \infty$ of the compactified spaces of degree d holomorphic maps $\mathbb{P}^1 \to \mathbb{P}^1$. This limiting infinite-dimensional space can be realized directly as the space of holomorphic maps of a two-dimensional holomorphic disk D thus providing an opportunity to formulate the Archimedean analog of the Shintani-Casselman-Shalika formula in terms of a two-dimensional topological quantum field theory. Indeed the classical $\mathfrak{gl}_2$-Whittaker function can be represented as a particular correlation function in the $S^1 \times U_2$-equivariant type A topological sigma model on D (in the sense of [46]) with target space $\mathbb{P}^1$ [18]. In turn the local Archimedean L-factors can be realized as correlation functions in the equivariant type A topological sigma model on the disk D with complex linear target spaces [16] (this shall be compared with earlier realizations [6] of the local Archimedean L-factors as infinite determinants).

The representation of classical Whittaker functions as correlation functions in two-dimensional quantum field theories prompts application of known quantum field theory constructions to the problem. The most notable case gives the use of mirror duality in two-dimensional sigma models. It was shown in [17, 18] that the mirror dual Landau-Ginzburg type description of the corresponding type A topological sigma models provides a representation of the $\mathfrak{gl}_2$-Whittaker function as a finite-dimensional integral. This integral representation coincides with an integral form arising in the representation theoretic construction of the Whittaker function as a matrix element of the principal series spherical representation. Thus the considered local Archimedean Langlands correspondence appears as an instance of mirror symmetry in two-dimensional topological quantum field theory.

2. Whittaker functions and quantum Toda chains

The exactly solvable quantum theory of Toda type associated with the Lie algebra $\mathfrak{gl}_{\ell+1} = \mathrm{Lie}(GL_{\ell+1})$ has the quadratic Hamiltonian,

$$H_2(x,\partial_x) = -\frac{\hbar^2}{2}\sum_{i=1}^{\ell+1}\frac{\partial^2}{\partial {x_i}^2} + \sum_{i=1}^{\ell} e^{x_i - x_{i+1}}, \tag{2.1}$$

where $\hbar$ is a quantization parameter. Integrability implies that this differential operator is an element of a commutative ring of differential operators generated by differential operators $\mathcal{H}_j(x,\partial_x), j=1,\dots,\ell+1$ with principle symbols given by elementary symmetric functions of $\ell+1$ variables. It is useful to introduce a generating function for these higher $\mathfrak{gl}_{\ell+1}$-Toda chain Hamiltonians

$$T(s)=\sum_{k=0}^{\ell+1}(-\imath)^k s^{\ell+1-k}\,\mathcal{H}_k(x,\partial_x). \tag{2.2}$$

The common eigenfunction of the $\mathfrak{gl}_{\ell+1}$-Toda chain Hamiltonians solves the equation

$$T(s)\;\;\Psi_{\lambda_1,\dots,\lambda_{\ell+1}}(x_1,\dots,x_{\ell+1})=\prod_{j=1}^{\ell+1}(s-\lambda_j)\;\;\Psi_{\lambda_1,\cdots,\lambda_{\ell+1}}(x_1,\dots,x_{\ell+1}), \tag{2.3}$$

and satisfies the fast decay conditions in appropriate asymptotic regions

$$\Psi_{\lambda_1,\dots,\lambda_{\ell+1}}(x_1,\dots,x_{\ell+1})\to 0,\qquad (x_i-x_{i+1})\to+\infty,\quad i=1,\dots,\ell. \tag{2.4}$$

In the following we will call the unique solution of (2.3), (2.4) the class one $\mathfrak{gl}_{\ell+1}$-Whittaker function.

The Toda chain permits explicit solution by expressing the common eigenfunctions (2.3) via a particular matrix elements of the principal series spherical representations of $GL_{\ell+1}(\mathbb{R})$ (see e.g. [10, 39]). Let E_{ij}, $i,j=1,\dots\ell+1$ be the standard basis of the Lie algebra $\mathfrak{gl}_{\ell+1}=\mathrm{Lie}(GL_{\ell+1})$. Let $\mathcal{Z}\subset\mathcal{U}\mathfrak{gl}_{\ell+1}$ be the center of the universal enveloping algebra $\mathcal{U}\mathfrak{gl}_{\ell+1}$, $B, B_-\subset GL_{\ell+1}(\mathbb{C})$ be upper-triangular and lower-triangular Borel subgroups and N, N_- be upper-triangular and lower-triangular unipotent subgroups. Let $\mathfrak{h}\subset\mathfrak{gl}_{\ell+1}$ be a diagonal maximal commutative subalgebra and we identify the corresponding Weyl group with the permutation group $\mathfrak{S}_{\ell+1}$. Using the Harish-Chandra isomorphism of $\mathcal{Z}$ with $\mathfrak{S}_{\ell+1}$-invariant subalgebra of the symmetric algebra $S^*\mathfrak{h}$ we identify central characters of $\mathcal{Z}$ with homomorphisms $c:\mathbb{C}[h_1,\cdots,h_{\ell+1}^{\mathfrak{S}_{\ell+1}}\to\mathbb{C}$. Let $\pi_\lambda: GL_{\ell+1}(\mathbb{R})\to\mathrm{End}(\mathcal{V}_\lambda)$, $\mathcal{V}_\lambda=\mathrm{Ind}_B^{GL_{\ell+1}}\chi_\lambda$ be a family of principal series spherical representations of $GL_{\ell+1}(\mathbb{R})$ induced from one-dimensional representations $\chi_\lambda(b)=\prod_{j=1}^{\ell+1}|b_{jj}|^{\lambda_k/\hbar+\rho_k}$ of B where $\rho_k=(\ell-2k+2)/2$, $k=1,\dots,\ell+1$ are components of a vector $\rho\in\mathbb{R}^{\ell+1}$. Let $\langle\,,\,\rangle$ be an invariant pairing $\mathcal{V}_{-\bar\lambda}\times\mathcal{V}_\lambda\to\mathbb{C}$. For unitary principal series representations the pairing $\langle\,,\,\rangle$ defines an inner product on $\mathcal{V}_\lambda$ and thus provides a notion of the matrix element of $GL_{\ell+1}(\mathbb{R})$. In the following we use the term matrix element for the pairing $\langle\, v_1,\pi_\lambda(g)\,v_2,\,\rangle$, $v_1\in\mathcal{V}_{-\bar\lambda}$, $v_2\in\mathcal{V}_\lambda$ in general case. We have the Iwasawa decomposition $GL_{\ell+1}(\mathbb{R})=KAN_-$, where $K=O_{\ell+1}(\mathbb{R})\subset GL_{\ell+1}(\mathbb{R})$ is a maximal compact subgroup, A is the group of diagonal matrices with positive diagonal elements. By the multiplicity one theorem [43], there is a unique smooth K-spherical vector ϕ_K in the representation $\mathcal{V}_{-\bar\lambda}$ i.e. there exists a unique smooth vector ϕ_K invariant with respect to K. Let ψ be the Whittaker vector in $\mathcal{V}_\lambda$ providing a one-dimensional representation N with character $\chi_N(n)=\exp(-2\hbar^{-1}i\sum_{j=1}^{\ell}n_{j,j+1})$. Consider the following matrix element in the representation $\mathcal{V}_\lambda$ of $GL_{\ell+1}(\mathbb{R})$

$$\Phi_\lambda(g)=\langle\phi_K,\pi_\lambda(g)\,\psi\rangle,\qquad g\in GL_{\ell+1}(\mathbb{R}), \tag{2.5}$$

where $\lambda=(\lambda_1,\dots,\lambda_{\ell+1})$. Here we normalize the spherical vector in such a way that the resulting matrix element is invariant under permutations of λ_j, $j=1,\dots,\ell+1$. The obvious

functional equation :

$$\Phi_\lambda(kgn) = \chi_{N_-}(n_-)\,\Phi_\lambda(g), \qquad k \in K,\, n \in N_-, \tag{2.6}$$

allows projection of the matrix element to a function on the space A of the diagonal matrices of the form $a = \mathrm{diag}(e^{x_1}, \dots, e^{x_{\ell+1}})$. Now the $\mathfrak{gl}_{\ell+1}$-Whittaker can be expressed through the matrix element as follows:

$$\Psi_{\lambda_1,\dots,\lambda_{\ell+1}}(x_1,\dots,x_{\ell+1}) = e^{-1/2\langle\rho,x\rangle}\Phi_{2\lambda_1,\dots,2\lambda_{\ell+1}}(x_1/2,\dots,x_{\ell+1}/2), \tag{2.7}$$

where $x = (x_1, \dots, x_{\ell+1})$. Standard considerations (see e.g. [41]) show that the matrix element (2.5) is a common eigenfunction of a family of commuting differential operators descending from the action of the center $\mathcal{Z}$. These differential operators can be identified with the elements of the ring of quantum $\mathfrak{gl}_{\ell+1}$-Toda chain Hamiltonians generated by (2.2), (2.3).

The representation (2.7) of the Whittaker function as a matrix element implies the existence of various natural integral representations arising from functional space realizations of $\mathcal{V}_\lambda$. An example of such an integral representation (obtained first in [27] by other means) is given by

$$\Psi_{\lambda_1,\dots,\lambda_{\ell+1}}(x_1,\dots,x_{\ell+1}) = \int_{\mathcal{C}} \prod_{k=1}^{\ell}\prod_{i=1}^{k} dx_{k,i}\; e^{\mathcal{F}(x)/\hbar}, \tag{2.8}$$

$$\mathcal{F}(x) = \sum_{k=1}^{\ell+1}\lambda_k\Big(\sum_{i=1}^{k} x_{k,i} - \sum_{i=1}^{k-1} x_{k-1,i}\Big) - \sum_{k=1}^{\ell}\sum_{i=1}^{k}\Big(e^{x_{k+1,i}-x_{k,i}} + e^{x_{k,i}-x_{k+1,i+1}}\Big),$$

where $x_i := x_{\ell+1,i},\ \ i = 1, \dots, \ell+1$. Note that the integral representation (2.8) has obvious recursive structure with respect to the rank ℓ. This recursive structure can be understood in representation theory terms as construction of the principal series representation $\mathcal{V}_{\lambda_1,\cdots\lambda_{\ell+1}}$ of $GL_{\ell+1}$ via induction $\mathrm{Ind}_{P_{\ell,1}}^{GL_{\ell+1}}\mathcal{V}_{\lambda_1,\cdots\lambda_\ell}\otimes\mathcal{V}_{\lambda_{\ell+1}}$ from the representation $\mathcal{V}_{\lambda_1,\cdots\lambda_\ell}\otimes\mathcal{V}_{\lambda_{\ell+1}}$ of the associated parabolic subgroup $P_{\ell,1}\subset GL_{\ell+1}$.

The $\mathfrak{gl}_{\ell+1}$-Whittaker function being a common eigenfunction of a family of mutually commuting differential operators turns out to be simultaneously a common eigenfunction of a specific one-parameter family of integral operators. These integral operators provide an instance of the family of the Baxter Q-operators in the theory of quantum integrable systems. Operators of this kind were first introduced by Baxter as a key tool to solve quantum integrable theories [2]. Such operators for the affine Toda chain were constructed in [38] and the integral operators for the $\mathfrak{gl}_{\ell+1}$-Toda chain are their direct analogs. The Baxter operators for the $\mathfrak{gl}_{\ell+1}$-Toda chain are defined as a family $\mathcal{Q}(s)$ of integral operators parameterized by $s\in\mathbb{C}$ with the integral kernel

$$\begin{aligned}\mathcal{Q}(x,y|s) = \exp\Bigg\{&\hbar^{-1}\sum_{j=1}^{\ell+1} s(x_j-y_j) - \hbar^{-1}\sum_{k=1}^{\ell}\Big(e^{x_k-y_k}+e^{y_k-x_{k+1}}\Big)\\ &-\hbar^{-1}e^{x_{\ell+1}-y_{\ell+1}}\Bigg\}.\end{aligned} \tag{2.9}$$

Then the following relations hold

$$\mathcal{Q}(s)\cdot\Psi_\lambda(x) = Q(s|\lambda)\,\Psi_\lambda(x), \tag{2.10}$$

where the eigenvalue of the $\mathcal{Q}$-operator is given by

$$Q(s|\lambda) = \prod_{j=1}^{\ell+1} \hbar^{\frac{s-\lambda_j}{\hbar}} \Gamma\Big(\frac{s-\lambda_j}{\hbar}\Big). \tag{2.11}$$

The operators $\mathcal{Q}(s)$ satisfy the commutativity relations

$$\mathcal{Q}(s)\cdot\mathcal{Q}(s') = \mathcal{Q}(s')\cdot\mathcal{Q}(s), \qquad \mathcal{Q}(s)\cdot T(s) = T(s)\cdot\mathcal{Q}(s), \tag{2.12}$$

and the $\mathfrak{gl}_{\ell+1}$-Toda chain version of the Baxter equations

$$\mathcal{Q}(s+\hbar) = T(s)\ \mathcal{Q}(s), \tag{2.13}$$

where $T(s)$ is given by (2.2).

The interpretation of the $\mathfrak{gl}_{\ell+1}$-Toda chain eigenfunctions as matrix elements of principal series spherical representations (2.7) makes it natural to look for an interpretation of the Baxter operator (2.9) in representation theory terms. An interpretation of the Baxter operators in terms of spherical Hecke algebras $\mathcal{H}(GL_{\ell+1}(\mathbb{R}), K)$ was given in [12] . Recall that the spherical Hecke algebra $\mathcal{H}(GL_{\ell+1}(\mathbb{R}), K)$ is an algebra of a certain class of K-biinvariant functions on G, $\phi(g) = \phi(k_1 g k_2)$, $k_1, k_2 \in K$ acting by the following convolution on the space of K left-invariant functions on G:

$$\phi * f(g) = \int_{GL_{\ell+1}(\mathbb{R})} \phi(\tilde{g}^{-1})\, f(\tilde{g}g) d\tilde{g}, \tag{2.14}$$

under condition that the integral in (2.14) is defined. Here we fix normalization of the Haar measure on $GL_{\ell+1}(\mathbb{R})$ so that $\mathrm{Vol}(O_{\ell+1}(\mathbb{R})) = 1$. The matrix element (2.5) is left K-invariant and thus allows the action (2.14) of the Hecke algebra $\mathcal{H}(GL_{\ell+1}(\mathbb{R}), K)$. By the multiplicity one theorem [43] the smooth K-spherical vector in the principal series irreducible representation $\mathcal{V}_\lambda = \mathrm{Ind}_{B_-}^{GL_{\ell+1}} \chi_\lambda$ is unique and thus any element $\phi(g)$ of the spherical Hecke algebra acts on the Whittaker function (2.7) via multiplication by a character of $\mathcal{H}(GL_{\ell+1}(\mathbb{R}), K)$. Let $M \subset K$, $|M| = 2^{\ell+1}$ be a subgroup of diagonal matrices.

Theorem 2.1. *Let $\mathcal{Q}_0(g; s)$ be the following family of K-biinvariant function on $GL_{\ell+1}(\mathbb{R})$:*

$$\mathcal{Q}_0(g; s) = |M||\det g|^{s+\frac{\ell}{2}} e^{-\pi \mathrm{Tr} g^t g}. \tag{2.15}$$

Then, its action on the matrix element (2.5) *considered as a function on $A = \mathbb{R}_+^{\ell+1}$ descends to the action of the integral operator with the kernel*

$$\mathcal{Q}_0(x, y; s) = 2^{\ell+1}\ \exp\left\{\sum_{j=1}^{\ell+1}(s+\rho_j)(x_j - y_j) - \pi\sum_{k=1}^{\ell}\left(e^{2(x_k - y_k)} + e^{2(y_k - x_{k+1})}\right) - \pi e^{2(x_{\ell+1} - y_{\ell+1})}\right\}. \tag{2.16}$$

The matrix element (2.5) *is an eigenfunction of the operator* (2.16)

$$\mathcal{Q}_0(s) * \Phi_\lambda(g) = Q_0(s|\lambda)\ \Phi_\lambda(g), \tag{2.17}$$

with eigenvalue

$$Q_0(s|\lambda) = \prod_{j=1}^{\ell+1} \pi^{-\frac{s-\lambda_j}{2}} \Gamma\Big(\frac{s-\lambda_j}{2}\Big). \tag{2.18}$$

The relation (2.10) now follows from Theorem 2.1 and the identification (2.7). Let us remark that the integral relation (2.17) can be also complimented by the action of the following integral operators $\mathcal{T}_g$, $g \in GL_{\ell+1}(\mathbb{R})$ multiplying one Whittaker function by another Whittaker function:

$$\mathcal{T}_g * \Phi_\lambda(\tilde{g}) = \int_{K\times N_-} dk\, dn\ \chi_{N_-}(n)\ \Phi_\lambda(gnk\tilde{g}) = \Phi_\lambda(g)\Phi_\lambda(\tilde{g}), \tag{2.19}$$

where appropriate normalization of the Whittaker functions is implied.

3. Arithmetic perspective

To understand better the meaning of the integral Baxter operator (2.15) it is useful to look at a larger picture including along with the field of real numbers $\mathbb{R}$ the other, non-Archimedean, completions of rational numbers $\mathbb{Q}$. Given a prime p the corresponding non-Archimedean field $\mathbb{Q}_p$ is obtained by completion of $\mathbb{Q}$ with respect to an extension to $\mathbb{Q}$ of the non-Archimedean norm $|\ |_p : \mathbb{Z} \to \mathbb{R}_+$

$$|a|_p = p^{-n} \quad \text{iff} \quad a = p^n a_0, \qquad (p, a_0) = 1. \tag{3.1}$$

The non-Archimedean norms satisfy the relations

$$|x\, y|_p = |x|_p\, |y|_p, \qquad |x + y|_p \le \max(|x|_p, |y_p|), \tag{3.2}$$

and one takes $|x|_p = 0$ for $x = 0$. The last property in (3.2) (the non-Archimedean triangle inequality) is a counterpart of the triangle inequality $|x + y| \le |x| + |y|$ holding for the standard absolute value Archimedean norm on $\mathbb{Q}$. The elements of $\mathbb{Q}_p$, p-adic numbers, can be represented by possibly infinite series $a = p^n(a_0 + a_1 p + a_2 p^2 + \cdots)$, $a_i \in \{0, 1, \cdots, p-1\}$. One specifies a subring of p-adic integer numbers $\mathbb{Z}_p$ having $n \ge 0$. The fact that p-adic norms together with the standard absolute value norm exhaust essentially different norms on $\mathbb{Q}$ manifested for example in the following relation:

$$|a| \cdot \prod_{p\in\mathcal{P}} |a|_p = 1, \quad a \in \mathbb{Q}, \tag{3.3}$$

which is an arithmetic analog of the geometric property: the sum of residues of a meromorphic one-form $d \ln f(z)$ on a compact Riemann surface is equal to zero.

A much more complicated example of the global relation associated with the set of norms on $\mathbb{Q}$ is given by analytic continuation of the Riemann ζ-function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\in\mathcal{P}} \frac{1}{1 - p^{-s}}, \qquad \mathrm{Re}(s) > 1, \tag{3.4}$$

where the product goes over the set $\mathcal{P}$ of all primes. The analytic continuation satisfies the functional equation which can be most symmetrically formulated in terms of modified function $\zeta^*(s)$

$$\zeta^*(s) = \zeta^*(1-s), \qquad \zeta^*(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s). \tag{3.5}$$

In the region $\mathrm{Re}(s) > 1$ this modified ζ-function is represented as a product

$$\zeta^*(s) = \zeta_{\mathbb{R}}(s)\prod_{p\in\mathcal{P}}\zeta_p(s), \tag{3.6}$$

of local contributions

$$\zeta_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right), \qquad \zeta_p(s) = \frac{1}{1-p^{-s}}, \tag{3.7}$$

of all essentially different norms on $\mathbb{Q}$. Thus the transcendental factor $\zeta_{\mathbb{R}}(s)$ is attributed to the contribution of the Archimedean norm and hence to field of real numbers. Taking into account the form of the eigenvalues (2.18) of Baxter operators, the appearance of the Γ-function in the completed Riemann ζ-function is a sign of the relevance of the Baxter operator formalism to the theory of (generalized) ζ-functions.

The Riemann zeta-function (3.4) is a simple instance of a large family of L-functions allowing an analog of the product decomposition (3.6) indexed by completions of $\mathbb{Q}$ and having analytic continuation governed by a functional equation. We will consider only the L-functions associated with general linear groups. The corresponding collection of local L-factors $L_p(s)$ allowing an appropriate analytic continuation of their product over all primes can be constructed from the data provided by $GL_{\ell+1}(\mathbb{Z})$- automorphic forms. For a fixed p one can equivalently define the corresponding local factor $L_p(s)$ in terms of the representation theory of $GL_{\ell+1}(\mathbb{Q}_p)$. Recall that the principal series spherical representations of $GL_{\ell+1}(\mathbb{Q}_p)$ can be realized as subrepresentations of the $GL_{\ell+1}(\mathbb{Q}_p)$-representation given by the space of functions $\mathcal{F}un^{(p)}_{\ell+1}$ on $GL_{\ell+1}(\mathbb{Z}_p)\backslash GL_{\ell+1}(\mathbb{Q}_p)$ where $GL_{\ell+1}(\mathbb{Z}_p)$ is a maximal compact subgroup of $GL_{\ell+1}(\mathbb{Q}_p)$. Besides $GL_{\ell+1}(\mathbb{Q}_p)$ there is a commuting action of the non-Archimedean spherical Hecke algebra $\mathcal{H}(GL_{\ell+1}(\mathbb{Q}_p), GL_{\ell+1}(\mathbb{Z}_p))$ on $\mathcal{F}un^{(p)}_{\ell+1}$. Precisely, the elements of the Hecke algebra realized by $GL_{\ell+1}(\mathbb{Z}_p)$-biinvariant locally constant functions on $GL_{\ell+1}(\mathbb{Q}_p)$ act on the space $\mathcal{F}un^{(p)}_{\ell+1}$ of functions on $GL_{\ell+1}(\mathbb{Z}_p)\backslash GL_{\ell+1}(\mathbb{Q}_p)$ by convolution.

The non-Archimedean Hecke algebra $\mathcal{H}(GL_{\ell+1}(\mathbb{Q}_p), GL_{\ell+1}(\mathbb{Z}_p))$ is a commutative algebra isomorphic to the ring of characters of finite-dimensional representations of the complex Lie group $GL_{\ell+1}(\mathbb{C})$. For principal series spherical representations of $GL_{\ell+1}$ the corresponding representation of $\mathcal{H}(GL_{\ell+1}(\mathbb{Q}_p), GL_{\ell+1}(\mathbb{Z}_p))$ is one-dimensional and defines a semisimple conjugacy classes $\mathcal{O}_p$ in $GL_{\ell+1}(\mathbb{C})$. The associated local non-Archimedean L-factor is given by

$$L_p(s) = \det{}_{\mathbb{C}^{\ell+1}}(1-g_p p^{-s})^{-1}, \tag{3.8}$$

where $g_p \in \mathcal{O}_p$ is an arbitrary representative of the conjugacy class acting in $\mathbb{C}^{\ell+1}$ via the standard representation of $GL_{\ell+1}(\mathbb{C})$.

Local non-Archimedean L-functions allow a totally different construction via $\ell+1$-dimensional complex semisimple representations of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ or more specifically of its factor, the Galois group $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ of the residual field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. Recall that the Galois extensions $\mathbb{F}_{p^n}$ of $\mathbb{F}_p$ are generated over $\mathbb{F}_p$ by solutions of the equations

$x^{p^k} = x$ and the group of automorphisms of $\mathbb{F}_{p^n}$ over $\mathbb{F}_p$ is a cyclic group generated by the Frobenius homomorphism $Fr_p : x \to x^p$. To avoid some topological issues one replaces profinite group $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ with the closely related Weil group $W_{\mathbb{Q}_p} = \mathbb{Z}$ whose generator will be also denoted Fr_p.

To define local non-Archimedean L-factors let us consider isomorphism classes of homomorphisms $W_{\mathbb{Q}_p} \to GL_{\ell+1}(\mathbb{C})$ such that the image of Fr_p is a semisimple conjugacy class $\mathcal{O}_p$ in $GL_{\ell+1}$. Then the corresponding L-factor is again given by the formula (3.8) for the thus constructed conjugacy class $\mathcal{O}_p$. The fact that these two ways to construct local non-Archimedean L-factors provides the same output is a manifestation of the local non-Archimedean Langlands duality.

There exists the non-Archimedean analog of the Whittaker functions given by matrix elements of principal series spherical representations of $GL_{\ell+1}(\mathbb{Q}_p)$. Given an array $\lambda = (\lambda_1, \ldots, \lambda_{\ell+1})$ of complex numbers the corresponding principal series spherical representation $\mathcal{V}_\lambda^{(p)}$ of $GL_{\ell+1}(\mathbb{Q}_p)$ is induced from the one-dimensional representation of the Borel subgroup $B(\mathbb{Q}_p)$

$$\chi_{(\lambda_1,\ldots,\lambda_{\ell+1})}(g) = \prod_{j=1}^{\ell+1} |g_{jj}|_p^{-\lambda_j+\rho_j}, \qquad g \in B(\mathbb{Q}_p). \tag{3.9}$$

The non-Archimedean analog $\Phi^{(p)}$ of the Whittaker function matrix element (2.7) solves the functional relation (2.6) where now $K = GL_{\ell+1}(\mathbb{Z}_p)$ and $\chi_{N_-}(n)$ is a generic character of $N_- \in GL_{\ell+1}(\mathbb{Q}_p)$ trivial on $[N_-, N_-]$. The Whittaker function descends to a function on the double coset $GL_{\ell+1}(\mathbb{Z}_p)\backslash GL_{\ell+1}(\mathbb{Q}_p)/N_-(\mathbb{Q}_p)$ identified with the set of diagonal matrices $\mathrm{diag}(p^{n_1}, \cdots, p^{n_{\ell+1}})$, $n_i \in \mathbb{Z}$.

The action of the non-Archimedean spherical Hecke algebra on the Whittaker functions has a simple form. Consider generators $T_p^{(i)}$, $i = 1, \ldots, (\ell+1)$ of the Hecke algebra given by the characteristic functions of the following subsets:

$$\mathcal{O}_i = GL_{\ell+1}(\mathbb{Z}_p) \cdot \mathrm{diag}(\underbrace{p, \cdots, p}_{i}, 1 \cdots, 1) \cdot GL_{\ell+1}(\mathbb{Z}_p) \ \subset \ GL_{\ell+1}(\mathbb{Q}_p). \tag{3.10}$$

The operators $T_p^{(i)}$ commute for different i and act on functions $f \in \mathcal{F}_{\ell+1}^{(p)}$ as follows:

$$T_p^{(i)} f(g) = \int_{\mathcal{O}_i} f(gh) dh. \tag{3.11}$$

Their action on the non-Archimedean Whittaker function is given by

$$T_p^{(i)} \Phi_\lambda^{(p)}(n_1, \ldots, n_{\ell+1}) = p^{-i(i-1)/2} \mathrm{Tr}_{V_{\omega_i}} \pi_i(\hat{z}) \ \Phi_\lambda^{(p)}(n_1, \ldots, n_{\ell+1}), \tag{3.12}$$

where $\pi_i : GL_{\ell+1}(\mathbb{C}) \to \mathrm{End}(V_{\omega_i}, \mathbb{C})$, $V_{\omega_i} = \wedge^i \mathbb{C}^{\ell+1}$ is a representation of $GL_{\ell+1}(\mathbb{C})$ corresponding to the fundamental weight ω_i and $\hat{z} = \mathrm{diag}(p^{+\lambda_1}, \cdots, p^{+\lambda_{\ell+1}})$ is a diagonal representative of a semisimple conjugacy class in $GL_{\ell+1}(\mathbb{C})$ corresponding to the principal series representation involved.

Association of the local non-Archimedean L-factors with principal series spherical representations of $GL_{\ell+1}(\mathbb{Q}_p)$ can be made more straightforward if we introduce suitable generating functions of non-Archimedean Hecke algebra elements. There exist elements $T_p^{(V)}$

of the Hecke algebra associated with an arbitrary finite dimensional representations $\pi_V : GL_{\ell+1}(\mathbb{C}) \to \mathrm{End}(V,\mathbb{C})$ such that

$$T_p^{(V)}\,\Phi_\lambda^{(p)}(n_1,\dots,n_{\ell+1}) = \mathrm{Tr}_V\,\pi_V(\hat z)\;\Phi_\lambda^{(p)}(n_1,\dots,n_{\ell+1}). \tag{3.13}$$

Let us define the generating series

$$T_p(s) = \sum_{j=1}^{\ell+1}(-1)^j p^{-(\ell+1-j)s+j(j-1)/2}\,T_p^{(j)}, \tag{3.14}$$

$$\mathcal{Q}_p(s) = \sum_{n=0}^{\infty} p^{-ns}\;T_p^{(S^n\mathbb{C}^{\ell+1})}, \tag{3.15}$$

satisfying the following relations:

$$\mathcal{Q}_p(s)\cdot\mathcal{Q}_p(s') = \mathcal{Q}_p(s')\cdot\mathcal{Q}_p(s), \qquad \mathcal{Q}(s)\cdot T_p(s') = T_p(s')\cdot\mathcal{Q}_p(s), \tag{3.16}$$

$$T_p(s)\cdot\mathcal{Q}_p(s) = 1. \tag{3.17}$$

The operators $T_p(s)$ and $\mathcal{Q}_p(s)$ act on the non-Archimedean Whittaker function as

$$T_p(s)\;\Phi_\lambda^{(p)}(n_1,\dots,n_{\ell+1}) = \det{}_{\mathbb{C}^{\ell+1}}(1-p^{-s}\hat z)\,\Phi_\lambda^{(p)}(n_1,\dots,n_{\ell+1}), \tag{3.18}$$

$$\mathcal{Q}_p(s)\;\Phi_\lambda^{(p)}(n_1,\dots,n_{\ell+1}) = \det{}_{\mathbb{C}^{\ell+1}}(1-p^{-s}\hat z)^{-1}\,\Phi_\lambda^{(p)}(n_1,\dots,n_{\ell+1}). \tag{3.19}$$

Comparing with the construction of (3.8) we derive that the one-parameter family $\mathcal{Q}_p(s)$ of the spherical Hecke algebra elements acts on the Whittaker function associated with a principal series spherical representation of $GL_{\ell+1}(\mathbb{Q}_p)$ by multiplication by the corresponding local non-Archimedean L-factor

$$L_p(s) = \det{}_{\mathbb{C}^{\ell+1}}(1-p^{-s}\hat z)^{-1} = \prod_{j=1}^{\ell+1}(1-p^{-s+\lambda_k}), \tag{3.20}$$

where $\hat z$ represents an image of the Weil group $W_{\mathbb{Q}_p}$ generator in $GL_{\ell+1}(\mathbb{C})$. Thus the operators $\mathcal{Q}_p(s)$ provide a transparent realization of the local non-Archimedean Langlands duality for principal series spherical representations.

There exists a more concise realization of the local Langlands duality formulated purely in terms of non-Archimedean Whittaker functions. According to the Shintani-Casselman-Shalika formula [44], [3] for $GL_{\ell+1}$ the non-Archimedean Whittaker function defined as a matrix element of the principal series spherical representation of $GL_{\ell+1}(\mathbb{Q}_p)$ can also be represented as a character of finite-dimensional irreducible representations of $GL_{\ell+1}(\mathbb{C})$

$$\Phi_\lambda^{(p)}(\mathrm{diag}(p^{n_1},\dots,p^{n_{\ell+1}})) = p^{<(\ell/2-\rho,u)>}\,\mathrm{Tr}_{V_n}\,\pi(\mathrm{diag}(p^{+\lambda_1},\dots,p^{+\lambda_{\ell+1}})), \tag{3.21}$$

where $\pi : GL_{\ell+1}(\mathbb{C}) \to \mathrm{End}(V_n)$ is a finite-dimensional irreducible representation of $GL_{\ell+1}(\mathbb{C})$ corresponding to the partition $n = (n_1,\dots,n_{\ell+1})$, $n_1 \le \dots \le n_{\ell+1}$ and the right hand side of (3.21) is supposed to be zero otherwise. Note that the fact that the $\mathbb{Q}_p$-Whittaker function is zero outside the dominant domain $n_1 \le \dots \le n_{\ell+1}$ is a non-Archimedean counterpart of the condition (2.4). An irreducible finite-dimensional representation of $GL_{\ell+1}(\mathbb{C})$ can be realized as a vector space with the basis enumerated by the

Gelfand-Zetlin patterns. This leads to the following explicit expression for the $GL_{\ell+1}(\mathbb{Q}_p)$-Whittaker function

$$\Phi_{\lambda}^{(p)}(n) \;=\; \sum_{q_{k,i}\in\mathcal{P}^{\ell+1}} \prod_{k=1}^{\ell+1} p^{+\lambda_k\,(\sum_{i=1}^{k} q_{k,i}-\sum_{i=1}^{k-1} q_{k-1,i})}\,, \tag{3.22}$$

where $\mathcal{P}^{\ell+1}$ is a set of collections $\underline{q}=\{q_{i,j}\}$, $i=1,\dots,\ell+1$, $j=1,\dots,j$ of integers satisfying the conditions $q_{i+1,j}\le q_{i,j}\le q_{i+1,j+1}$ and $q_{\ell+1,i}=n_i$, $i=1,\dots,\ell+1$.

Let us now return to the representation theory of $GL_{\ell+1}(\mathbb{R})$. The traditional construction of local L-factors $L_{\mathbb{R}}(s)$ corresponding to Archimedean completion of $\mathbb{Q}$ differs from its non-Archimedean case counterpart. Instead of the corresponding spherical Hecke algebra $\mathcal{H}(GL_{\ell+1}(\mathbb{R}),K)$ one considers an action of the center $\mathcal{Z}$ of the universal enveloping algebra $\mathcal{U}\mathfrak{gl}_{\ell+1}$ in a principal series spherical representation $\mathcal{V}_\lambda$. This allows us to recover a semisimple conjugacy class in the Lie algebra $\mathfrak{gl}_{\ell+1}$ with the diagonal representative $\lambda=(\lambda_1,\dots,\lambda_{\ell+1})$. This must be contrasted with the non-Archimedean case where the action of the spherical Hecke algebra allows recovering of a conjugacy class in the Lie group $GL_{\ell+1}(\mathbb{C})$. The local Archimedean L-factor associated with $GL_{\ell+1}(\mathbb{R})$-representation $\mathcal{V}_\lambda$ is then given by

$$L_{\mathbb{R}}(s)=\prod_{j=1}^{\ell+1}\;\pi^{-\frac{s-\lambda_j}{2}}\,\Gamma\Big(\frac{s-\lambda_j}{2}\Big). \tag{3.23}$$

This expression obviously generalizes the expression for the local Archimedean factor $\zeta_{\mathbb{R}}(s)$ in (3.7).

It was shown in [12] that there exists a full Archimedean analog of the non-Archimedean construction of local L-factors via representation theory of the spherical Hecke algebra. Comparison of (2.17), (2.18) with (3.19), (3.20) reveals that the Baxter operator (2.15) considered as an element of the Archimedean spherical Hecke algebra $\mathcal{H}(GL_{\ell+1}(\mathbb{R}),K)$ provides the complete analog of (3.15) and in particular its eigenvalues on the matrix elements (2.5) in principal series spherical representation are given by the corresponding local Archimedean L-factors (3.23). The recursive structure of the Whittaker function integral representation (2.8) also has a counterpart in the non-Archimedean case [13], [14] and can be described in terms of the local Langlands corresponds as follows. Given a direct sum $V_1\oplus V_2$, $V_i=\mathbb{C}^{n_i}$ of the representations $\pi_i: W_{\mathbb{Q}_p}\to GL_{n_i}(\mathbb{C})$ the local Langlands correspondence associates with $V_1\oplus V_2$ a representation of $GL_{n_1+n_2}(\mathbb{Q}_p)$. This representation is obtained from representations $\mathcal{V}_i$ of $GL_{n_i}(\mathbb{Q}_p)$ corresponding to π_i via the parabolic induction $\mathrm{Ind}_{P_{n_1,n_2}}^{GL_{n_1+n_2}(\mathbb{Q}_p)}\mathcal{V}_1\otimes\mathcal{V}_2$ where $P_{n_1,n_2}\subset GL_{n_1+n_2}$ is the parabolic subgroup with Levi factor $GL_{n_1}\times GL_{n_2}$. The Archimedean analog for $n_1=\ell$ and $n_2=1$ leads to recursive relations for the Whittaker function integral representation (2.8).

Upon establishing a unified way to associate local L-factors to principal series spherical representations in Archimedean and non-Archimedean cases it is natural to look at the second construction of L-factors in terms of representations of Weil groups. Naively the Weil group $W_{\mathbb{R}}$ of $\mathbb{R}$ shall be close to the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})=\mathbb{Z}_2$. Actually $W_{\mathbb{R}}$ is given by the twisted product of $\mathbb{C}^*$ and a group generated by an element j such that $jxj^{-1}=\overline{x}$, $j^2=-1\in\mathbb{C}^*$. The appearance of this group is enigmatic and the actual nature of its representations relevant to Archimedean Langlands correspondence is unclear. As a preliminary step to a better understanding of the Archimedean Weil group we construct the Archimedean analog of the Shintani-Casselman-Shalika formula (3.21) essentially relying

on the Weil group representations. Actually we will consider only the factor of $W_{\mathbb{R}}$ relevant to the description of principal series spherical representations of $GL_{\ell+1}(\mathbb{R})$ via homomorphisms $W_{\mathbb{R}} \to \mathrm{End}(\mathbb{C}^{\ell+1})$. This factor is obtained by factorization of the abelianized Weil group $W_{\mathbb{R}}^{ab} = W_{\mathbb{R}}/[W_{\mathbb{R}}, W_{\mathbb{R}} = \mathbb{R}^*$ over the kernel of the absolute value map $\mathbb{R}^* \to \mathbb{R}_+$. Indeed the data to construct local Archimedean L-factors are basically isomorphism classes of semisimple homomorphisms $\mathbb{R}_+ \to GL_{\ell+1}(\mathbb{C})$ equivalently described by conjugacy classes of diagonal elements $\lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_{\ell+1}) \subset \mathfrak{gl}_{\ell+1}(\mathbb{C})$. The Archimedean analog of the Shintani-Casselman-Shalika formula will be constructed by using q-lattice interpolations of the Archimedean and non-Archimedean Whittaker functions [13].

4. Lattice $\mathfrak{gl}_{\ell+1}$-Whittaker function

A lattice version of the $\mathfrak{gl}_{\ell+1}$-Toda chain is defined by the commutative algebras of the quantum Hamiltonians generated by the following difference operators [40]:

$$\mathcal{H}_r \,=\, \sum_{I_r} \prod_{k=1}^{r} \left(1 - q^{p_{i_k+1} - p_{i_k} + 1}\right)^{1 - \delta_{i_{k+1} - i_k, 1}} \mathcal{S}_{I_r}\,, \qquad r = 1, \ldots, \ell+1. \tag{4.1}$$

Here the sum is over ordered subsets $I_r = \{i_1 < i_2 < \ldots < i_r\} \subset \{1, 2, \ldots, \ell+1\}$ and we assume $q \in \mathbb{C}^*, |q| < 1$. The operators $\mathcal{S}_{I_r} = \prod_{i \in I_r} \mathcal{S}_i$ are products of the elementary shift operators $\mathcal{S}_i$ acting on functions $f(\underline{p}) = f(p_1, \ldots, p_{\ell+1})$ on the lattice $(p_1, \ldots, p_{\ell+1}) \in \mathbb{Z}^{\ell+1}$ as follows

$$\mathcal{S}_i\, f(p_1, \ldots, p_{\ell+1}) = f(\widetilde{p}_1, \ldots, \widetilde{p}_{\ell+1}), \qquad \widetilde{p}_k = p_k + \delta_{k,i}, \tag{4.2}$$

for $i, k = 1, \ldots, \ell+1$. For example, the first Hamiltonian has the form

$$\mathcal{H}_1 \,=\, \sum_{i=1}^{\ell} (1 - q^{p_{i+1} - p_i + 1}) \mathcal{S}_i \,+\, \mathcal{S}_{\ell+1}. \tag{4.3}$$

In the limit $q \to 1$ the quantum integrable system defined by the quantum Hamiltonians (4.1) reduces to the $\mathfrak{gl}_{\ell+1}$-Toda chain.

The q-lattice version of the class one $\mathfrak{gl}_{\ell+1}$-Whittaker function is defined as the function $\Psi_z^{(q)}(\underline{p})$ on the lattice $\mathbb{Z}^{\ell+1}$ satisfying eigenfunction equations with respect to quantum Hamiltonians (4.1)

$$\mathcal{H}_r(\underline{p}) \Psi_z^{(q)}(\underline{p}) \,=\, \Big(\sum_{I_r} \prod_{i \in I_r} z_i\Big)\; \Psi_z^{(q)}(\underline{p}), \qquad z = (z_1, \ldots, z_{\ell+1}), \tag{4.4}$$

and the class one condition

$$\Psi_z^{(q)}(\underline{p}) = 0, \tag{4.5}$$

outside the dominant domain $p_{\ell+1,1} \leq \ldots \leq p_{\ell+1,\ell+1}$. In the following we will call such a function a q-Whittaker function.

An explicit expression for the q-Whittaker function was constructed [13]. Let $\mathcal{P}^{(\ell+1)}$ be a set of collections of integers $p_{i,j} \in \mathbb{Z}$, $i = 1, \ldots, \ell+1$, $j = 1, \ldots, i$ satisfying the conditions $p_{i+1,j} \leq p_{i,j} \leq p_{i+1,j+1}$ with fixed $p_{\ell+1,i} := p_i$, $i = 1, \ldots, \ell+1$. Thus $\mathcal{P}^{(\ell+1)}$ is

a set of the Gelfand-Zetlin patterns corresponding to the irreducible finite-dimensional representation of $GL_{\ell+1}(\mathbb{C})$ associated with the partition $(p_1,\dots,p_{\ell+1})$. Below we will use the standard notation $(n)_q! = (1-q)...(1-q^n)$.

Theorem 4.1. *The q-lattice $\mathfrak{gl}_{\ell+1}$-Whittaker function solving the eigenfunction problem* (4.4), (4.5) *can be represented as follows:*

$$\Psi^{(q)}_z(\underline{p}) \;=\; \sum_{p_{k,i}\in\mathcal{P}^{(\ell+1)}} \prod_{k=1}^{\ell+1} z_k^{\sum_{i=1}^{k} p_{k,i}-\sum_{i=1}^{k-1} p_{k-1,i}} \tag{4.6}$$

$$\times \frac{\prod\limits_{k=2}^{\ell}\prod\limits_{i=1}^{k-1}(p_{k,i+1}-p_{k,i})_q!}{\prod\limits_{k=1}^{\ell}\prod\limits_{i=1}^{k}(p_{k+1,i+1}-p_{k,i})_q!\;(p_{k,i}-p_{k+1,i})_q!}, \qquad p_{\ell+1,1}\le\dots\le p_{\ell+1,\ell+1},$$

$$\Psi^{(q)}_z(\underline{p}) \;=\; 0, \qquad \textit{otherwise}.$$

The representation (4.6) is a q-analog of the integral representation (2.8) of the $\mathfrak{gl}_{\ell+1}$-Whittaker function and turns into (2.8) after taking an appropriate limit $q\to 1$ [19]. In the particular case $\ell = 1$ we have the following simple expression for q-lattice $\mathfrak{gl}_2$-Whittaker function:

$$\Psi^{(q)}_{z_1,z_2}(p_1,p_2) = \sum_{p_1\le p_{1,1}\le p_2} \frac{z_1^{p_{1,1}} z_2^{p_1+p_2-p_{1,1}}}{(p_{1,1}-p_1)_q!(p_2-p_{1,1})_q!}, \qquad p_1\le p_2\,, \tag{4.7}$$

$$\Psi^{(q)}_{z_1,z_2}(p_1,p_2) = 0, \qquad p_1 > p_2\,.$$

Note that the q-lattice Toda chain allows a representation theory interpretation. The commutative ring of the Hamiltonians can be identified with the center of the quantum group $\mathcal{U}_q\mathfrak{gl}_{\ell+1}$ [42], [7] and the q-Whittaker function is given by a matrix element of the q-version of the principal series representation (see [15] for explicit example). The Baxter operator (2.9) for $\mathfrak{gl}_{\ell+1}$-Whittaker functions also has a q-lattice analog.

Theorem 4.2. *The family of operators*

$$\mathcal{Q}^{(q)}(w)\cdot f(\underline{p}) \;=\; \sum_{\underline{\tilde p}\in\mathbb{Z}^{\ell+1}} \Delta(\underline{\tilde p})\,\mathcal{Q}^{(q)}(\underline{\tilde p},\underline{p};w)\,f(\underline{\tilde p})\,, \tag{4.8}$$

with kernel

$$\mathcal{Q}^{(q)}(\underline{\tilde p},\underline{p};w) \;=\; \frac{\Theta(\tilde p_1-p_1)}{(\tilde p_1-p_1)_q!}\prod_{i=1}^{\ell}\frac{\Theta(p_{i+1}-\tilde p_i)}{(p_{i+1}-\tilde p_i)_q!}\frac{\Theta(\tilde p_i-p_i)}{(\tilde p_i-p_i)_q!}\,w^{\tilde p_i-p_i}\,, \tag{4.9}$$

acts on the q-Whittaker functions (4.6) *as follows:*

$$\mathcal{Q}^{(q)}(w)\cdot\Psi^{(q)}_z(\underline{p}) \;=\; L^{(q)}(z;w)\,\Psi^{(q)}_z(\underline{p})\,, \tag{4.10}$$

$$L^{(q)}(z;w) \;=\; \prod_{i=1}^{\ell+1}\Gamma_q(wz_i)\,, \tag{4.11}$$

where

$$\Gamma_q(w) \;=\; \prod_{n=0}^{\infty} \frac{1}{1-wq^n} = \sum_{n=0}^{\infty} \frac{w^n}{(n)_q!}. \tag{4.12}$$

and $\Delta(\underline{p}) = \prod_{k=1}^{\ell}(p_{k+1}-p_k)_q!$, $\underline{p} = (p_1,\dots,p_{\ell+1})$.

The q-lattice version of the $\mathfrak{gl}_{\ell+1}$-Toda chain has another but related interpretation via representation theory of the affine Lie algebra $\widehat{gl}_{\ell+1}$, for the level k such that $q = \exp\left(\frac{2\pi\imath}{(k+\ell+1)}\right)$. Therefore one can consider the eigenvalue (4.11) of the Baxter operator (4.8) as a kind of local L-factor for loop groups.

An important property of the explicit representation (4.6) is a possibility to recast it in a form generalizing the Shintani-Casselman-Shalika formula (3.21) to the case of q-Whittaker functions. Precisely, the common eigenfunction (4.6) of the q-lattice Toda chain allows the following two representations for $p_{\ell+1,1} \le p_{\ell+1,2} \le \dots p_{\ell+1,\ell+1}$:

$$\Psi_z(\underline{p}) \;=\; \mathrm{Tr}_V\; q^D \prod_{i=1}^{\ell+1} z_i^{E_{i,i}}, \tag{4.13}$$

$$\Psi_z(\underline{p}) = \Delta(\underline{p})^{-1}\widetilde{\Psi}_{\underline{z}}(\underline{p}) \;=\; \mathrm{Tr}_{V_f}\; q^D \prod_{i=1}^{\ell+1} z_i^{E_{i,i}}, \tag{4.14}$$

where V and V_f are $\mathbb{C}^*\times GL_{\ell+1}(\mathbb{C})$-module, $E_{i,i}$, $i=1,\dots\ell+1$ are generators of the diagonal subalgebra of $\mathfrak{gl}_{\ell+1}$, D is a generator of $\mathrm{Lie}(\mathbb{C}^*)$. Similarly the q-version (4.11) of the local L-factor has a representation as an inverse determinant

$$L^{(q)}(w) = \det{}_{\mathbb{C}^{\ell+1}[\xi]}(1-q^D\prod_{i=1}^{\ell+1} w z_i^{E_{i,i}})^{-1}, \tag{4.15}$$

where $D=\xi\partial_\xi$. The Shintani-Casselman-Shalika formula (3.21) in the form (3.22) is recovered in the limit $q\to 0$ with $z_i = p^{+\lambda_i}$. Thus the q-Whittaker function interpolates classical $\mathfrak{gl}_{\ell+1}$-Whittaker functions and their non-Archimedean analogs.

Let us describe the structure of the modules V and V_f [15]. Both in (4.13) and (4.14) the underlying vector spaces allow pairs of cohomology group realizations. The $\mathbb{C}^*\times GL_{\ell+1}$-module V_f can be identified with a the standard module of the level zero quantum affine Lie algebra $\mathcal{U}_q\widehat{\mathfrak{gl}}_{\ell+1}$ obtained by fusion of the elementary modules associated with fundamental weights ω_i of $\mathfrak{gl}_{\ell+1}$ (see [37] for details)

$$\begin{aligned} M_{(p_{\ell+1,1},\dots,p_{\ell+1,1})} \;&= V_{\omega_1}^{\otimes(p_{\ell+1,\ell+1}-p_{\ell+1,\ell})}(a)\otimes\cdots \\ &\otimes V_{\omega_\ell}^{\otimes(p_{\ell+1,2}-p_{\ell+1,1})}(a)\otimes V_{\omega_{\ell+1}}^{\otimes p_{\ell+1,1}}(a), \qquad a\in\mathbb{C}, \end{aligned} \tag{4.16}$$

such that the corresponding Drinfeld polynomials are given by $P_i(u) = (1-au)^{p_{\ell+1,i+1}-p_{\ell+1,i}}$, $i=1,\dots,\ell$. There are two types of cohomological realization of V_f. The first identifies V_f with a Demazure module of the affine Lie algebra $\widehat{\mathfrak{gl}}_{\ell+1}$ and thus realizes V_f as the zero cohomology of line bundles on desingularized compactifications of affine Schubert cells. Another interpretation of the space V_f is in terms of the de Rham cohomology groups of appropriate collection of algebraic varieties [37]

$$\widetilde{\Psi}_z^{(q)}(\underline{p}_{\ell+1}) = \sum_{(m_1,\dots,m_{\ell+1})\in\mathbb{Z}^{\ell+1}} z_1^{m_1}\cdots z_{\ell+1}^{m_{\ell+1}} P_q(\mathfrak{L}(\mathcal{V}_{\underline{m}},\mathcal{W}_{\underline{p}_{\ell+1}})), \tag{4.17}$$

where $P_q(X) := \sum_k (-q^{1/2})^k \dim H^k(X,\mathbb{C})$ is the Poincare polynomial. Thus the cohomology of the moduli space $\mathfrak{L}(\mathcal{V}_{\underline{m}}, \mathcal{W}_{\underline{p}_{\ell+1}})$ of the $A_{\ell+1}$-graded quiver representations provide a realization of the standard modules (4.16) via geometric representation theory. For example in the simplest non-trivial case of $\ell = 1$ this gives an interpretation of the modified version of the q-deformed $\mathfrak{gl}_2$-Whittaker function (4.7) via cohomology of the Grassmannians

$$\widetilde{\Psi}^{(q)}_{z_1,z_2}(p_1,p_2) = \sum_{p_1 \le p_{1,1} \le p_2} z_1^{p_{1,1}} z_2^{p_1+p_2-p_{1,1}} P_q(\mathrm{Gr}(p_{1,1}-p_1, p_2-p_1)), \tag{4.18}$$

where we take into account

$$P_q(\mathrm{Gr}(m,n)) = \frac{(n)!_q}{(m)!_q (n-m)!_q}. \tag{4.19}$$

A similar dual pair of interpretations exists for the vector space V. Thus in the case $\ell = 1$ one can realize V as a limit of spaces of holomorphic sections of line bundles over a sequence of finite-dimensional manifolds. Consider the space $\mathcal{X}_d^{(0)}$ of degree d holomorphic maps of $\mathbb{P}^1$ to $\mathbb{P}^1$ which can be described as the space of pairs of degree $\le d$ mutually prime polynomials up to a common constant factor. There is an action of $\mathbb{C}^* \times GL_2(\mathbb{C})$ on the space of maps, the action of $GL_2(\mathbb{C})$ is induced by the standard action on the target $\mathbb{P}^1$ and $\mathbb{C}^*$ acts on the source $\mathbb{P}^1$ with two fixed points (0) and (∞). The space $\mathcal{X}_d^{(0)}$ is non-compact and its compactification $\mathcal{X}_d = \mathbb{P}^{2d+1}$ is obtained by omitting the condition on polynomials to be mutually prime. Let $\mathcal{L}_k$ be a one-dimensional determinant $GL_2(\mathbb{C})$-module such that $E_{i,i}\mathcal{L}_k = k\mathcal{L}_k$. We use the same symbol $\mathcal{L}_k$ to denote the corresponding trivial line bundle on $\mathbb{P}^{2d+1}$. Cohomology groups $H^0(\mathcal{X}_d, \mathcal{L}_k \otimes \mathcal{O}(n))$ have a natural structure of $GL_2(\mathbb{C})$-modules. For the $\mathbb{C}^* \times GL_2$-character of the module $H^0(\mathcal{X}_d, \mathcal{L}_k \otimes \mathcal{O}(n))$ the following integral representation holds

$$\mathrm{Tr}_{H^0(\mathcal{X}_d, \mathcal{L}_k \otimes \mathcal{O}(n))}\, q^D \prod_{i=1}^2 z_i^{E_{i,i}} = (z_1 z_2)^k \oint_{t=0} \frac{dt}{2\pi\imath\, t}\, t^{-n} \prod_{m=1}^2 \prod_{j=0}^d \frac{1}{(1-tq^j z_m)}, \tag{4.20}$$

for $n \ge 0$ and zero otherwise. Here $z_m = e^{\lambda_m}$ and D is a generator of $\mathrm{Lie}(\mathbb{C}^*)$. On the other hand the q-deformed $\mathfrak{gl}_2$-Whittaker function has also the following representation for $p_1 \le p_2$:

$$\Psi^{(q)}_{z_1,z_2}(p_1,p_2) = (z_1 z_2)^{p_1} \oint_{t=0} \frac{dt}{2\pi\imath t}\, \frac{1}{t^{p_2-p_1}} \prod_{n=0}^\infty \frac{1}{(1-z_1 t q^n)\,(1-z_2 t q^n)}, \tag{4.21}$$

which easily follows from (4.6) for $\ell = 1$ and the classical identity (4.12). Thus taking the limit $d \to \infty$ we obtain the identity

$$\Psi^{(q)}_{z_1,z_2}(k,k+n) = \lim_{d\to\infty} \mathrm{Tr}_{H^0(\mathcal{X}_d, \mathcal{L}_k \otimes \mathcal{O}(n))}\, q^D \prod_{i=1}^2 z_i^{E_{i,i}}. \tag{4.22}$$

This gives representation of the q-deformed $\mathfrak{gl}_2$-Whittaker function as a character of the limit $d \to \infty$ of space of holomorphic section spaces of line bundles on $\mathcal{X}_d$. Note that the space

$\mathcal{X}_d$ for $d \to \infty$ might be considered as successive algebraic approximations of the space of holomorphic disks in $\mathbb{P}^1$ [28] and V can be identify with the formal limit

$$H^0(\mathcal{X}_\infty, \mathcal{L}_k \otimes \mathcal{O}(n)).$$

It is possible to give another interpretation of the space V this time using the de Rham cohomologies. Recall that the cohomology ring of the classifying space $BU_m = \mathrm{Gr}(m, \infty)$ of the unitary group U_m is isomorphic to the polynomial ring

$$H^*(BU_m, \mathbb{C}) = \mathbb{C}[c_1, c_2, \cdots, c_m, \qquad \deg(c_i) = 2i. \tag{4.23}$$

Hence the Poincare polynomial of BU_m is given by

$$P_q(BU_m) = \sum_{i=0}^{\infty} (-q^{1/2})^i \dim H^i(BU_m) = \frac{1}{\prod_{k=1}^m (1-q^k)} = \frac{1}{[m]_q!}.$$

Taking into account (4.7) we obtain a representation of the $\mathfrak{gl}_2$-Whittaker function as a trace over the direct sum of the de Rham cohomology of products of classifying spaces

$$\Psi^{(q)}_{z_1,z_2}(p_1,p_2) \;=\; \sum_{m_1+m_2=p_2-p_1} z_1^{m_1+p_1} z_2^{m_2+p_1} P_q(BU_{m_1} \times BU_{m_2}). \tag{4.24}$$

For example comparison of the representations (4.22) and (4.24) leads to an identification of the $k+1$-dimensional irreducible representation of $GL_2(\mathbb{C})$ realized in the space $H^0(\mathbb{P}^1, \mathcal{O}(k)) = \mathbb{C}^{k+1}$, $k \geq 0$ with cohomologies $H^*(\mathbb{P}^k, \mathbb{C})$.

Let us stress that the q-deformed Whittaker function provides an interpolation between the Whittaker function (4.6) corresponding to representation theory of $GL_{\ell+1}(\mathbb{Q}_p)$ and the classical Whittaker function (2.8) corresponding to representation theory of $GL_{\ell+1}(\mathbb{R})$. This interpolation reveals non-obvious structures in the Archimedean case via interpolation of the relevant structures over non-Archimedean fields. Thus taking the appropriate limit $q \to 1$ one can infer the Archimedean substitute of the trace type representation of the q-deformed Whittaker functions and non-Archimedean Whittaker functions i.e. to recover the Archimedean analog of the Shintani-Casselman-Shalika formula (3.21).

5. Whittaker functions as equivariant symplectic volumes

To define an Archimedean analog of the Shintani-Casselman-Shalika formula we start with a simplified situation which has an interesting interpretation of its own. Let us note that both Whittaker functions (3.21) and (2.5) interpolated by the q-lattice $\mathfrak{gl}_{\ell+1}$-Whittaker function (4.6) allow a common degeneration. Let $n_i(p, x_j)$ be integer parts of $x_j/\ln p$, $x_j \in \mathbb{R}$. In the following limit:

$$\Psi^{(0)}_\lambda(x) = \lim_{p\to 1} (\ln p)^{\ell(\ell+1)/2} \; \Psi^{(p)}_\lambda(n_1(p,x_1), \cdots, n_{\ell+1}(p, x_{\ell+1})), \tag{5.1}$$

the non-Archimedean Whittaker function reduces to the function given by the integral

$$\Psi^{(0)}_\lambda(x) \;=\; \int\limits_{\mathcal{D}_{\ell+1}} e^{\sum\limits_{k=1}^{\ell+1} \lambda_k \left(\sum_{i=1}^k x_{k,i} - \sum_{i=1}^{k-1} x_{k-1,i}\right)} \prod_{k=1}^{\ell} \prod_{i=1}^{k} dx_{k,i}, \tag{5.2}$$

for $x_1 \leq x_2 \leq \cdots \leq x_{\ell+1}$ and zero otherwise. Here $x_i = x_{\ell+1,i}, i = 1, \ldots, \ell+1$ and the integration domain $\mathcal{D}_{\ell+1}$ is defined by the inequalities $x_{j+1,k} \leq x_{j,k} \leq x_{j+1,k+1}$, $j = 1, \ldots, \ell+1$, $k = 1, \ldots, j$. The same function can be obtained as a limit

$$\Psi_\lambda^{(0)}(x) = \lim_{\hbar \to \infty} (\hbar)^{-\ell(\ell+1)/2} \Psi_\lambda(\hbar x, \hbar), \tag{5.3}$$

of the Archimedean $\mathfrak{gl}_{\ell+1}$-Whittaker function (2.8). The $\mathfrak{gl}_{\ell+1}$-Toda chain equations (2.3) are reduced in this limit to equations of the quantum billiard [9] in the domain $D_{\ell+1} = \{x = (x_1, \ldots, x_{\ell+1}) \in \mathbb{R}^{\ell+1} | x_i \leq x_{i+1}\}$ and the elementary $\mathfrak{gl}_{\ell+1}$-Whittaker function (5.2) is a unique solution of the quantum billiard eigenfunction equations in $D_{\ell+1}$

$$P(\partial_x)\, \Psi_\lambda^{(0)}(x) = P(\lambda)\, \Psi_\lambda^{(0)}(x), \tag{5.4}$$

where $P(y) \in \mathbb{C}[y_1, \ldots, y_{\ell+1}^{\mathfrak{S}_{\ell+1}}$ is an arbitrary symmetric polynomial and Dirichlet boundary conditions

$$\Psi_\lambda^{(0)}(x)|_{x_j = x_{j+1}} = 0, \tag{5.5}$$

are imposed. The corresponding analog $\mathcal{Q}_0(s)$ of the Baxter integral kernel (2.9) has the following simple form:

$$\mathcal{Q}^{(0)}(x,\, y|\, s) = e^{\, s \sum_{i=1}^{\ell+1} (x_i - y_i)} \Theta(y_{\ell+1} - x_{\ell+1}) \prod_{i=1}^{\ell} \Theta(y_{i+1} - x_i) \Theta(x_i - y_i), \tag{5.6}$$

where $x := (x_1, \cdots, x_{\ell+1})$, $y := (y_1, \ldots, y_{\ell+1})$ and $\Theta(a) = (1 + \mathrm{sign}(a))/2$. The elementary $\mathfrak{gl}_{\ell+1}$-Whittaker function is an eigenfunction of the integral operators $\mathcal{Q}_0(s)$ with eigenvalue

$$L^{(0)}(s) = \prod_{j=1}^{\ell+1} \frac{1}{s - \lambda_j}. \tag{5.7}$$

The existence of a reasonable limit of the integral representation (2.8) is ultimately related with a deep positivity property of the classical Whittaker functions. One can demonstrate that the matrix element (2.5) can be represented as an integral over a subset $N^>$ of positive elements of the subgroup $N \subset GL_{\ell+1}(\mathbb{R})$. Recall that positive elements of $GL_{\ell+1}(\mathbb{R})$ are the elements realized in the standard matrix representation by positive matrices i.e. the matrices with all minors positive. The same definition applies to the subgroup N where only non-trivial minors are considered. The subset of positive elements $GL_{\ell+1}^>(\mathbb{R})$ is a monoid (for the monoid structure on the subset of positive element of general reductive Lie groups see [34]) and the $\mathfrak{gl}_{\ell+1}$-Whittaker function appears to be closely related to representation theory of $GL_{\ell+1}^>(\mathbb{R})$. A counterpart of this for the limiting elementary Whittaker function (5.2) is an interpretation of (5.2) in terms of representation theory of the monoid $GL_{\ell+1}(\mathcal{R})$ of matrices over the tropical semifield $\mathcal{R}$. The tropical semifield $\mathcal{R}$ (see e.g. [35], [36]) is defined as the set $\mathbb{R}$ of real numbers with two operations

$$\alpha \dot{\times} \beta = \alpha + \beta, \qquad \alpha \dot{+} \beta = \min(\alpha, \beta), \tag{5.8}$$

which can be understood as a limit $\hbar \to +\infty$ of the family of semifield structures on $\mathbb{R}_+$

$$a \times_\hbar b = a \times b, \qquad a +_\hbar b = (a^\hbar + b^\hbar)^{1/\hbar}, \tag{5.9}$$

where $a = e^{-\alpha}$, $b = e^{-\beta}$. Note that semifield structure for generic $\hbar \neq 0$ is isomorphic to the standard semifield structure of $\mathbb{R}_+$ via the map $a \to a^{\frac{1}{\hbar}}$. The matrix element representation of the classical $\mathfrak{gl}_{\ell+1}$-Whittaker function degenerates in the limit $\hbar \to +\infty$ into a representation of the elementary $\mathfrak{gl}_{\ell+1}$-Whittaker function as a matrix element of an analog of the principal series representation of $GL_{\ell+1}(\mathcal{R})$.

An advantage of the elementary Whittaker functions is an easy way to guess the corresponding analog of the Shintani-Casselman-Shalika formula (3.21). Precisely, the elementary Whittaker function (5.2) can be identified with $U_{\ell+1}$-equivariant volume of the flag manifold $\mathcal{B}_{\ell+1} = GL_{\ell+1}(\mathbb{C})/B$, B is a Borel subgroup. Consider the standard family of $U_{\ell+1}$-invariant symplectic structures $\omega(x)$ linearly parameterized by elements $x = (x_1, \ldots, x_{\ell+1})$ of the maximal commutative subalgebra $\mathfrak{h} \subset \mathfrak{gl}_{\ell+1}$. Identifying the equivariant cohomology $H^*_{U_{\ell+1}}(\mathrm{pt})$ with the polynomial central functions on the Lie algebra of $U_{\ell+1}$, the $U_{\ell+1}$-equivariant extension $\omega_{U_{\ell+1}}(x, \lambda) \in H^2_{U_{\ell+1}}(\mathcal{B}_{\ell+1})$ of $\omega(x) \in H^2(\mathcal{B}_{\ell+1})$ additionally depends linearly on an element λ of the Lie algebra of $U_{\ell+1}$. The $U_{\ell+1}$-equivariant integral over $\mathcal{B}_{\ell+1}$ we are going to consider depends only on an $Ad_{U_{\ell+1}}$-orbit of λ and thus we may pick a diagonal representative $\lambda = (\lambda_1, \ldots, \lambda_{\ell+1})$.

Theorem 5.1. *The $U_{\ell+1}$-equivariant volume of the flag space $\mathcal{B}_{\ell+1} = GL_{\ell+1}(\mathbb{C})/B$ is given by the elementary $\mathfrak{gl}_{\ell+1}$-Whittaker function* (5.2)

$$\Psi^{(0)}_{\lambda}(x) = \int_{\mathcal{B}_{\ell+1}} e^{\imath\omega_{U_{\ell+1}}(x,\lambda)}, \tag{5.10}$$

for $x_1 \leq \ldots \leq x_{\ell+1}$.

One should point out that the identification of the elementary $\mathfrak{gl}_{\ell+1}$-Whittaker functions as the eigenfunctions of the quantum billiard Hamiltonians with $U_{\ell+1}$-equivariant symplectic volumes of the flag spaces $\mathcal{B}_{\ell+1}$ is not surprising and can be understood via the canonical action on the equivariant cohomology $H^*_{U_{\ell+1}}(\mathcal{B}_{\ell+1})$ of the affine nil-Hecke algebra $\mathcal{H}^{nil}_{\ell+1}$ associated with $\mathfrak{gl}_{\ell+1}$ (see e.g. [26] and references therein). In turn the relation between quantum billiard eigenfunctions and representation theory of affine nil-Hecke algebras should be understood as an analog of the similar relation between affine Hecke algebras and many body integrable systems with δ-function potentials [30].

The elementary analog of the Shintani-Casselman-Shalika formula (3.21) can now be formulated as follows. The elementary $\mathfrak{gl}_{\ell+1}$-Whitaker function understood as a matrix element of the tropical monoid $GL_{\ell+1}(\mathcal{R})$ is equal to the $U_{\ell+1}$-equivariant symplectic volume of the flag space $\mathcal{B}_{\ell+1}$ (5.10). The reason why this relation should be considered as an analog of (3.21) is that the equivariant symplectic volumes are naturally limits of characters of irreducible finite-dimensional representations of $U_{\ell+1}$ obtained by geometric quantization of $\mathcal{B}_{\ell+1}$. Therefore while the classical Shintani-Casselman-Shalika formula (3.21) provides a categorification of the non-Archimedean Whittaker function i.e. a representation as the trace over an auxiliary vector space the elementary analog (5.10) provides a symplectification of the elementary Whittaker function (5.2) by representing it as the equivariant volume of an auxiliary symplectic space.

An equivariant symplectic volume representation exists not only for the limit (5.2) of the classical $\mathfrak{gl}_{\ell+1}$-Whittaker function but for the $\mathfrak{gl}_{\ell+1}$-Whittaker function itself. This can be directly inferred from the realization of the q-lattice version of the Whittaker function as a character of zero cohomology groups of line bundles over the compactified space of

holomorphic maps (4.22). In the limit $q \to 1$ the character turns into the equivariant symplectic volume thus providing an equivariant symplectic volume realization of the classical Whittaker functions. This was done explicitly in [14] for the first non-trivial case $\ell = 1$. Consider the compactification $\mathcal{X}_d = \mathbb{P}^{2d+1}$ of the space of degree d holomorphic maps $\mathbb{P}^1 \to \mathbb{P}^1$ supporting an action of $\mathbb{C}^* \times GL_2(\mathbb{C})$. Supply $\mathcal{X}_d$ with a symplectic structure $\Omega(y_1 - y_2)$ linearly depending on $(y_1 - y_2)$ and proportional to the Fubini-Studi two-form to obtain $S^1 \times U_2$-symplectic manifold. We will identify $H^*_{S^1 \times U_2}(\mathrm{pt})$ with $\mathbb{C}[\hbar] \times \mathbb{C}[\lambda_1, \lambda_2]^{\mathfrak{S}_2}$ where $\hbar$ corresponds to the generator of S^1 and λ_j, $j = 1, 2$ correspond to the generator of j-th factor in maximal abelian subgroup $U_1 \times U_1 \subset U_2$. Denote $\Omega_{S^1 \times U_2}(y_1, y_2, \lambda_1, \lambda_2)$ the $S^1 \times U_2$-equivariant extension of the symplectic structure on $\mathcal{X}_d$. Note that equivariant symplectic for depends on $y_1 + y_2$ via the constant shift normalization of the momentum map. The $S^1 \times U_2$-equivariant symplectic volume of $\mathcal{X}_d$ is defined as the following integral:

$$Z_d(y, \lambda) = \int_{\mathcal{X}_d} e^{\imath \Omega_{S^1 \times U_2}(y, \lambda)}. \tag{5.11}$$

Direct calculation gives for the equivariant symplectic volume of $\mathcal{X}_d$ the following integral representation (where we use analytic continuation $\hbar \to i\hbar$):

$$Z_d(y, \lambda) = \int_{\mathbb{R} - \imath\epsilon} d\gamma \; e^{(\lambda_1 + \lambda_2) y_1 + \imath \gamma (y_2 - y_1)} \prod_{j=1}^{2} \frac{1}{\prod_{m=0}^{d} (\imath\gamma - \lambda_j + \hbar m)}. \tag{5.12}$$

Following the example of the q-deformed Whittaker function (4.22) we expect to recover the $\mathfrak{gl}_2$-Whittaker function in the limit $d \to +\infty$ of (5.11). Indeed, taking the limit of the integral representation (5.12) and using the ζ-function regularization of the infinite products we obtain the following integral expression:

$$Z_\infty(x, \lambda) = \int_{\mathbb{R} - \imath\epsilon} d\gamma \, e^{(\lambda_1 + \lambda_2) y_1 + \imath \gamma (y_2 - y_1)} \prod_{j=1}^{2} \Gamma\left(\frac{\imath\gamma - \lambda_j}{\hbar} \right) \hbar^{(\imath\gamma - \lambda_i)/\hbar}. \tag{5.13}$$

Using the Euler integral representation for the Γ-function this integral can be easily transformed into the integral representation (2.8) of the $\mathfrak{gl}_2$-Whittaker function for $x_i = \hbar y_i$

$$\Psi_{\lambda_1, \lambda_2}(x_1, x_2) = \int_{\mathbb{R}} dx \; e^{\lambda_2 (x_1 + x_2 - x)/\hbar + \lambda_1 x/\hbar} \; e^{-\hbar^{-1} e^{x_1 - x} - \hbar^{-1} e^{x - x_2}}. \tag{5.14}$$

Recall that the compactification $\mathcal{X}_d$ of the space of holomorphic maps $\mathbb{P}^1 \to \mathbb{P}^1$ of degree $d \to +\infty$ may be considered as an approximation of the space of holomorphic maps of a two-dimensional disk $D = \{z \,|\, |z| \leq 1\}$ into $\mathbb{P}^1$. Actually one can replace the limiting procedure by considering directly the $S^1 \times U_{\ell+1}$-equivariant volume of the space $\mathcal{X}_D$ of holomorphic maps $D \to \mathbb{P}^1$ to obtain

$$\Psi_{\lambda_1, \lambda_2}(x_1, x_2) = \int_{\mathcal{X}_D} e^{\imath \Omega_{S^1 \times U_2}(\hbar^{-1} x, \lambda)}. \tag{5.15}$$

Here we have extended the definition of the equivariant volume to infinite-dimensional manifolds. Although this leads to potential difficulties with integration over infinite-dimensional spaces all encountered integrals are reduced to combinations of finite-dimensional integrals

and infinite-dimensional Gaussian integrals which can be defined via ζ-function regularization [18].

The identification of the finite-dimensional integral (5.14) representing the matrix element (2.7) of the principal series spherical representation of $GL_2(\mathbb{R})$ with the analytic continuation of the equivariant volume (5.15) of the infinite-dimensional space $\mathcal{X}_D$ provides an instance of the Archimedean counterpart of the Shintani-Casselman-Shalika formula (3.21). This should be compared with the identification (5.10) of the elementary Whittaker functions with equivariant symplectic volumes of the flag spaces. Thus (5.15) provides a symplectification of the classical Whittaker function.

The infinite-dimensional integral representation (5.11) implies that the proper framework for the Archimedean Shintani-Casselman-Shalika formula is two-dimensional quantum field theory. Indeed the equivariant symplectic volumes (5.11) can be identified with a correlation function on a disk D in $S^1 \times U_{\ell+1}$-equivariant type A topological sigma models with target spaces $\mathbb{P}^1$ and the compactification $\mathcal{X}_d$ of the space of degree d holomorphic maps $\mathbb{P}^1 \to \mathbb{P}^1$ appears naturally in the linear gauged sigma-model description of the Gromov-Witten invariants of projective spaces [47]. Similarly, an equivariant volume representation exists for the local Archimedean L-factors (3.23) based on the expression of the $S^1 \times U_{\ell+1}$-equivariant volume of the space $\mathcal{M}(D, \mathbb{C}^{\ell+1})$ of holomorphic maps $D \to \mathbb{C}^{\ell+1}$ in terms of Γ-function [18]. The infinite-dimensional integral can be identified with a correlation functions on a disk D in $S^1 \times U_{\ell+1}$-equivariant type A topological sigma models with target spaces $\mathbb{C}^{\ell+1}$.

Mirror symmetry provides an alternative interpretation of the type A topological sigma model correlation function as a correlation function in the dual type B topological sigma model with non-trivial superpotential. In contrast with correlation functions in type A topological field theories expressed via integrals over spaces of holomorphic maps into target spaces the correlation functions in type B topological theories are given by finite-dimensional integrals over middle-dimensional subspaces in the corresponding target space. In the case of the integral (5.15) the mirror dual description of the correlation function leads directly to a finite-dimensional integral representation (5.14) associated with the matrix element description (2.7) of the $\mathfrak{gl}_2$-Whittaker function. Recall that the Shintani-Casselman-Shalika formula concisely express local Langlands duality between representations of groups over local fields and representations of the Weil groups of the field in the non-Archimedean case. Taking into account that two sides of the Archimedean Shintani-Casselman-Shalika formula are related by mirror symmetry of the underlying two-dimensional topological theory we conclude that the Archimedean Langlands correspondence is an instance of the mirror symmetry in two-dimensional quantum field theory. The elementary analog identifying equivariant symplectic volume of $GL_{\ell+1}(\mathbb{C})/B$ with a matrix element of the monoid $GL_{\ell+1}(\mathcal{R})$ expressed by the integral (5.2) over the polytope $\mathcal{D}_{\ell+1}$ shall be considered as a counterpart of the mirror symmetry for zero-dimensional quantum field theories.

One should point out that the Toda integrable systems appear in the description of the Gromov-Witten invariants of flag spaces [27]. Precisely the counting functions of the $GL_{\ell+1} \times S^1$-equivariant volumes of the spaces of holomorphic maps of $\mathbb{P}^1$ with two punctures into the flag spaces $\mathcal{B}_{\ell+1} = GL_{\ell+1}(\mathbb{C})/B$ satisfy eigenfunction equations of the $\mathfrak{gl}_{\ell+1}$-Toda integrable system. The counting functions represent correlation functions in two-dimensional topological field theories with the target space $\mathcal{B}_{\ell+1}$ and depend on a choice of the observables at the punctures. The choice of observables leading to the Whittaker functions is rather complicated and actually equivalent to consideration of the topological field theory on a disk D thus recovering our interpretation of (5.15).

6. Conclusions and further directions

To conclude let us stress that the formalism of topological field theories seems to provide a good framework for proper description of the transcendental structures arising in the formulation of the local Archimedean correspondence. Realization of the local Archimedean L-factors as eigenvalues of elements of the Archimedean spherical Hecke algebra, the Archimedean Shintani-Casselman-Shalika formula and an interpretation of the local Archimedean Langlands duality as mirror symmetry in topological field theories are just a few of the details in a more general picture.

We have considered only the special matrix element of the spherical principal series representations of general linear groups leading to Toda chain eigenfunctions. Thus there are many obvious directions for generalizations to various series of representations of other reductive groups and related quantum integrable systems (some partial generalizations can be found e.g. in [4, 20, 24]). In particular, as the Baxter operator formalism is known for many quantum integrable systems it would be interesting to use the spherical Hecke algebra approach to more general systems and to identify the corresponding eigenvalues as generalized L-functions: for some very preliminary steps in this direction see [21].

The study of global number fields is another direction to pursue. It is natural to look for an adelic version of the Baxter operator as being an element of the global spherical Hecke algebra over $\mathbb{Z}$. Action of the Baxter operators on automorphic forms would then be the same as multiplication by the corresponding complete global L-functions. Recall that in the case of a quantum integrable system with a discrete spectrum, zeroes of the Baxter operator eigenvalues as functions of the auxiliary parameter are governed by a system of Bethe ansatz equations. This surprisingly links the Bethe ansatz technique with generalized Riemann hypotheses for global L-functions. In this regard it may be worth reviving an old approach to the Riemann hypotheses via scattering theory [8].

Identification of the eigenfunctions of quantum billiards with analytic continuation of the equivariant symplectic volumes of flag spaces, or those of Toda chains with moduli spaces of maps into flag spaces are both instances of a general relation between quantum many-body integrable systems, representation theory of the Hecke algebras and (generalized) equivariant cohomology rooted in the Lusztig philosophy of realization of Hecke algebra representations via generalized equivariant cohomology (see e.g. [26] and references therein). The discussed construction can be also lifted to higher dimensions to identify eigenfunctions of quantum integrable systems with equivariant symplectic volumes of appropriate infinite-dimensional spaces related with spaces of solutions of boundary value problems [23]. The corresponding version of the Langlands duality will be a manifestation of higher-dimensional analogs of mirror duality/S-dualities in quantum field theories. Note that identification of topological quantum field theory correlation functions on a two-dimensional disk with solutions of quantum integrable systems was proposed in [25] as a rather general phenomena.

Appearance of the tropical semifield $\mathcal{R}$ in the elementary version of the Archimedean Langlands correspondence deserves further clarification. The tropical semifield $\mathcal{R}$ is a tentative substitute for a more complicated arithmetic object. Note that the non-Archimedean valuation $\nu_p(x) = -\log_p |x|_p$ satisfies relations closely resembling the basic operations (5.8) of the tropical semifield $\mathcal{R}$. The image of the valuation map ν_p is a lattice $\ln p\,\mathbb{Z} \subset \mathbb{R}$ and the semifield $\mathcal{R}$ can be understood as a formal limit $p \to 1$ of the image of the non-Archimedean valuation map with the semifield structure compatible with limiting properties of the valuation map ν_p. It is tempting to suggest the existence of a non-Archimedean field

$\mathbb{Q}_*$ providing a kind of a limit of $\mathbb{Q}_p$ for $p \to 1$ such that the corresponding valuation ν_* maps $\mathbb{Q}_*$ surjectively onto $\mathbb{R}$. The elementary Whittaker functions will then be considered as Whittaker functions over $\mathbb{Q}_*$. The same reasoning will work for elementary L-factors (compare with the association of the same L-factors with the residue field $\mathbb{F}_1$ in [32]). Note that the norm map $\nu : \mathbb{Q}_p \to \mathbb{R}$ has a large kernel of invertible p-adic numbers $\mathbb{Z}_p^*$ and thus the field $\mathbb{Q}_*$ will be much larger then $\mathcal{R}$. It is easy to construct a non-Archimedean field with the valuation surjectively mapping onto $\mathbb{R}$ using functions of one real variable and indeed such field naturally appears in the description of asymptotic expansions in quantum theories (see e.g. [33, 35]). However a connection, if any, with the hypothetical fields $\mathbb{Q}_*$ and $\mathbb{F}_1$ is yet to find. Note also that $\mathbb{R}$ should be considered as an extension of $\mathbb{Q}_*$.

Finally one should point out that we did not discuss a relation (of some results of Section 3) to the geometric Langlands correspondence. May be more important we did not touch the problem of proper interpretation of the Archimedean Weil group $W_{\mathbb{R}}$ and the corresponding more deep level of understanding of the local Archimedean Langlands correspondence (even in the non-Archimedean case the situation is not quite satisfactory as we shall change the topology of the Galois group to fit the Langlands correspondence). One of the directions to pursue in this regard is proper understanding of the Archimedean limit of the de Rham type representation of the q-Whittaker function via the Poincare polynomials (4.24). Presumably this leads to a relation with String theory.

Acknowledgements. Author is grateful to Max-Planck-Institut für Mathematik, Bonn, Germany for its hospitality during the writing of this paper.

References

[1] Adams, J., Barbash, D., and Vogan Jr., D.A., *The Langlands Classification and Irreducible Characters of Real Reductive Groups*, Progr. Math., 104, Birkhäuser, 1992.

[2] Baxter, R.J., *Exactly solved models in statistical mechanics.* London: Academic Press, 1982.

[3] Casselman, W. and Shalika, J., *The unramified principal series of p-adic groups II. The Whittaker function*, Comp. Math. **41** (1980), 207–231 .

[4] Cherednik, I., *Whittaker limits of difference spherical functions*, Int. Math. Res. Notes, **20** (2009), 3793–3842, arXiv:0807.2155.

[5] Consani, C. and Connes, A., *Noncommutative geometry, arithmetic, and related topics*, Proceedings of the 21st meeting of the Japan-U.S. Mathematics Institute (JAMI), Baltimore, MD: Johns Hopkins University Press, 2011.

[6] Deninger, C., *Local L-factors of motives and regularized determinants*, Invent. Math. **107** (1992), 135–150.

[7] Etingof, P.I., *Whittaker functions on quantum groups and q-deformed Toda operators, Differential topology, infinite-dimensional Lie algebras, and applications*, Amer. Math. Soc. Transl. Ser. 2, 194, Amer. Math. Soc., Providence, RI, (1999), 9–25; arXiv:math.QA/9901053.

[8] Faddeev, L.D. and Pavlov, B.S., *Scattering theory and automorphic functions* in Boundary-value problems of mathematical physics and related problems of function theory. Part 6, Zap. Nauchn. Sem. LOMI **27**, (1972), "Nauka", Leningrad, 161–193.

[9] Itzykson, C., *Simple Integrable Systems and Lie Algebras*, International Journal of Modern Physics A **1**, 01, (1986), 65–115.

[10] Jacquet, H., *Fonctions de Whittaker associées aux groupes de Chevalley*, Bull. Soc. Math. France **95** (1967), 243–309.

[11] Gelbart, S., *An elementary introduction to the Langlands program*, American Mathematical Society. Bulletin. New Series **10**(2), (1984), 177–219.

[12] Gerasimov, A., Lebedev, D., and Oblezin, S., *Baxter operator and Archimedean Hecke algebra*, Commun. Math. Phys. **284** (2008), 867–896; arXiv:math.RT/0706.3476.

[13] ———, *On q-defomed* $\mathfrak{gl}_{\ell+1}$*-Whittaker function I*, Commun. Math. Phys. **294** (2010), 97–119; arXiv:math.RT/0803.0145.

[14] ———, *On q-defomed* $\mathfrak{gl}_{\ell+1}$*-Whittaker function II*, Commun. Math. Phys. **294** (2010), 121–143; arXiv:math.RT/0803.0970.

[15] ———, *On q-defomed* $\mathfrak{gl}_{\ell+1}$*-Whittaker funciton III*, Lett. Math. Phys. **97** (2011), 1–24; arXiv:math.RT/0805.3754.

[16] ———, *Archimedean L-factors and topological field theories I*, Commun. Number Theory and Physics **5** (2011), 57–101; arXiv:math.NT/0906.1065.

[17] ———, *Archimedean L-factors and topological field theories II*, Commun. Number Theory and Physics **5** (2011), 102–134; arXiv:math.NT/0909.2016.

[18] ———, *Parabolic Whittaker functions and topological field theories I*, Commun. Number Theory and Physics **5** (2011), 135–202; arXiv:hep-th/1002.2622.

[19] ———, *On a classical limit of q-deformed Whittaker functions*, Lett. Math. Phys. **100**, 3 (2012), 279–290; arXiv:math.AG/1101.4567.

[20] ———, *New integral representations of Whittaker functions for classical groups*, Uspekhi Mat. Nauk, **67**, issue 1(403), (2012), 3–96; transl. Russian Math. Surveys **67**, issue 1, (2012), 1–92; arXiv:math.RT/0705.2886.

[21] ———, *Baxter Q-operators and their Arithmetic implications*, Lett. Math. Phys. 88:1-3 (2009), 3–30; arXiv:math.RT/0711.2812.

[22] Gerasimov, A. and Lebedev, D., *Representation Theory over Tropical Semifield and Langlands Duality*, Comm. Math. Phys. **320**(2) (2013), 301–346, arXiv:math.AG/1011.2462.

[23] ———, *On topological field theory representation of higher analogs of classical special functions*, JHEP v. 2011 no. 9 (2011), 1–23; arXiv:arXiv:1011.0403.

[24] ———, *On Universal Baxter Operator for Classical Groups*, arXiv:arXiv:1104.0420.

[25] Gerasimov A.A. and Shatashvili S.L., *Two-Dimensional Gauge Theories and Quantum Integrable Systems*, in From Hodge Theory to Integrability and TQFT: tt*-geometry, R. Donagi and K. Wendland, Eds., Proc. of Symposia in Pure Mathematics Vol. 78, AMS Providence, Rhode Island, (2008), 239–262; arXiv:arXiv:0711.1472.

[26] Ginzburg, V., *Geometric methods in the representation theory of Hecke algebras and quantum groups*, in Representation Theories and Algebraic Geometry, Springer Netherlands, 1998, 127–183.

[27] Givental, A., *Stationary Phase Integrals, Quantum Toda Lattices, Flag Manifolds and the Mirror Conjecture, Topics in Singularity Theory*, Amer. Math. Soc. Transl. Ser., 2 **180**, AMS, Providence, Rhode Island, (1997), 103–115; arXiv:alg-geom/9612001.

[28] _____, *Homological geometry I. Projective hypersurfaces*, Selecta Mathematica, New Series Volume 1, **2** (1995), 325–345.

[29] _____, Lee Y.-P., *Quantum K-theory on flag manifolds, finite-difference Toda lattices and quantum groups*, Invent. Math. **151** (2003), 193–219; arXiv:math.AG/0108105.

[30] Heckman, G.J. and Opdam, E.M., *Yang's system of particles and Hecke algebras*, Annals of Math. **145** (1997), 139–173.

[31] Kostant, B., *Quantization and representation theory*, in Representation theory of Lie groups **34,** London Math. Soc. Lecture Notes Series, (1979), 287–316.

[32] Kurokawa, N., *Multiple zeta functions: an example*, Adv. Stud. Pure Math. **21** (1992), 219–226.

[33] Leray, J., *Lagrangian Analysis and Quantum Mechanics*, The M.I.T. Press, Cambridge, London, 1981, (Analyse Lagrangienne, R.C.P. 25, Strasbourg, 1978; Collège de France 1976.

[34] Lusztig, G., *Total positivity in reductive groups*, in Lie Theory and Geometry: In Honor of B. Kostant, Progr. Math. **123**, Birkhäuser, (1994), 531–568.

[35] *Idempotent analysis*, Maslov V. P., (ed.) S.N. Samborskii (ed.), Amer. Math. Soc. (1992).

[36] Mikhalkin, G., *Amoebas of Algebraic Varieties and Tropical Geometry*, in Different Faces of Geometry, ed. Donaldson, S., Eliashberg, Y., Gromov, M., **3** (2004), 257–300; arXiv:math.AG/0403015.

[37] Nakajima, H., *Quiver varieties and t-analogs of q-characters of quantum affine algebras*, Annals of Mathematics **160** (2004), 1057–1097; arXiv:math/0105173.

[38] Pasquier, V. and Gaudin, M., *The periodic Toda chain and a matrix generalization of the Bessel function recursion relation*, J. Phys. A **25**, (1992), 5243–5252 .

[39] Hashizume, M., *Whittaker functions on semi-simple Lie groups*, Hiroshima Math.J. **12**, (1982), 259–293.

[40] Ruijsenaars, S. N. M., *Relativistic Toda system*, Comm. Math. Phys. **133** (1990), 217–247.

[41] Semenov-Tian-Shansky, M., *Quantization of open Toda lattice*, in Encyclopedia of Math. Sciences **16**, Springer Verlag (1994), 226–259.

[42] Sevostyanov, A., *Quantum deformation of Whittaker modules and Toda lattice*, Duke Math. J. **105** (2000), 211–238; arXiv:math.QA/9905128.

[43] Shalika, J.A., *The multiplicity one theorem for GL_n*, Ann. Math. **100:1** (1974), 171–193.

[44] Shintani, T., *On an explicit formula for class 1 Whittaker functions on GL_n over p-adic fields*, Proc. Japan Acad. **52** (1976), 180–182.

[45] Tits, J., *Sur les analogues algébriques des groupes semi-simples complexes*, Colloque dálgébre supérieure, Bruxelles, Centre Belge de Recherches Mathématiques Établissements Ceuterick, Louvain, Paris: Librairie Gauthier-Villars, (1957), 261–289.

[46] Witten, E., *Mirror Manifolds and Topological Field Theory*, in Essays on mirror manifolds, Internat. Press, Hong Kong (1992), 120–158; arXiv:hep-th/9112056.

[47] ———, *Phases of $N = 2$ theories in two dimensions*, Nucl. Phys. B403 (1993), 159–222, arXiv:hep-th/9301042.

Institute for Theoretical and Experimental Physics, 117259, Moscow, Russia
E-mail: anton.a.gerasimov@gmail.com

Lyapunov functional approach and collective dynamics of some interacting many-body systems

Seung-Yeal Ha

Abstract. A Lyapunov functional approach is a standard tool for studying the nonlinear stability of equilibria in the theory of dynamical systems. In this paper, we survey recent progress on the collective dynamics of interacting many-body systems and discuss how the Lyapunov functional approach can be used in the formation of collective motions, such as in flocking and the synchronization of many-body systems. We also propose some open questions in the mathematical theory of flocking and synchronization.

Mathematics Subject Classification (2010). Primary 34H05 ; Secondary 35B35.

Keywords. Complete synchronization, Cucker-Smale model, Lyapunov function, global flocking, Kuramoto model.

1. Introduction

Emergent flocking and synchronization phenomena appear in many biological and physical systems, e.g., the flocking of birds, swarming of fish, flashing of fireflies, and synchronous firing of a cardiac pacemaker [1, 5, 10, 29–32, 63, 67, 70, 72]. These are simple collective modes of complex systems, and have been an active area of research in applied mathematics, biology, sociology, and physics because of their engineering applications, e.g., sensor and power networks, formation control of robots and unmanned aerial vehicles [27, 54, 60, 62, 64–66, 68]. The works of Winfree, Kuramoto, and Vicsek et al. have led to the proposal of several agent-based models, which have been extensively studied both analytically and numerically. In this paper, our interest lies in two particle models. In [29], Cucker and Smale introduced a second-order Newton-type equation supplemented by weighted relaxation-type internal forces describing the dynamics of positions and velocities of particles. They also provided sufficient conditions for admissible initial configurations leading to global flocking. In [52], Kuramoto proposed an analytically treatable first-order phase model for limit-cycle oscillators from the complex Ginzburg–Landau system in the complex plane. He explicitly calculated the critical coupling strength, and showed that his model exhibits a phase-transition phenomenon from disorder to a partially ordered state as the coupling strength exceeds this critical value. Kuramoto's seminal work has generated research in the statistical physics community for the last 40 years, and has left many open questions [1, 67].

The purpose of this paper is to survey recent progress in the flocking and synchronization estimates for the aforementioned models using an analytical tool, namely the Lyapunov

Proceedings of the International Congress of Mathematicians, Seoul, 2014

functional approach.

The rest of this paper is organized as follows. In Section 2, we introduce models for flocking and synchronization, and their kinetic and hydrodynamic analogues. In Section 3, we describe the emergent large-time dynamics of the Cucker–Smale flocking model and flocking particle–fluid interaction problems. In Section 4, we present the emergent large-time dynamics of the Kuramoto model and its quantum counterparts. Finally, Section 5 gives a brief summary and addresses some open questions related to the topics discussed in the paper.

2. Preliminaries

In this section, we present two interacting many-body systems, "*the Cucker–Smale (C-S) model*" and "*the Kuramoto model*," exhibiting emergent collective dynamics.

2.1. The Cucker–Smale model. Let x_j and v_j be the position and velocity of the j-th C-S particle, and $\psi(|x_k - x_j|)$ be the communication weight between the k-th and j-th particles. In this situation, the C-S model reads as follows. For $j = 1, \cdots, N$,

$$\begin{aligned} &\frac{dx_j}{dt} = v_j, \quad x_i, v_i \in \mathbb{R}^d, \ \ t \in \mathbb{R}, \\ &\frac{dv_j}{dt} = \frac{K}{N}\sum_{k=1}^{N} \psi(|x_k - x_j|)(v_k - v_j), \end{aligned} \tag{2.1}$$

where ψ is assumed to be nonnegative, bounded, Lipschitz continuous and non-increasing in its argument, i.e.,

$$\psi \geq 0, \quad ||\psi||_{L^\infty} + ||\psi||_{\text{Lip}} < \infty, \quad (\psi(r_1) - \psi(r_2))(r_1 - r_2) \leq 0, \quad r_1, r_2 > 0. \tag{2.2}$$

We next provide the definition of (asymptotic) global flocking for (2.1).

Definition 2.1 ([27, 50]). Let $\mathcal{P} := \{(x_i, v_i)\}_{i=1}^N$ be the solution to (2.1). Then, the particle system $\mathcal{P}$ exhibits global flocking if and only if the following two conditions hold.

1. The spatial configuration has a diameter that is uniformly bounded in time:

$$\sup_{0\leq t<\infty} \max_{1\leq i,j\leq N} ||x_i(t) - x_j(t)|| < \infty.$$

2. The velocity diameter goes to zero asymptotically, meaning that all particles become asymptotically aligned.

$$\lim_{t\to\infty} \max_{1\leq i,j\leq N} ||v_i(t) - v_j(t)|| = 0,$$

where $||a|| := \left(\sum_{i=1}^d |a_i|^2\right)^{\frac{1}{2}}$ is the ℓ^2-norm of $a = (a_1, \cdots, a_d)$.

In [29], Cucker and Smale derived sufficient conditions for global flocking in terms of the initial configuration and the decay rate of ψ, and their results were immediately generalized

in several directions, e.g., stochastic noise effects [4, 28, 46], collision avoidance [3, 17], formation control [61], and the mean-field limit [11, 16, 36, 39, 48, 50]. The relation with a mechanical oscillator model [49] was also explored, and discrete-time C-S models with leaders were developed in [55, 56].

When the number of particles goes to infinity ($N \to \infty$) in (2.1), it is reasonable to introduce a one-particle distribution function $f = f(x, v, t)$. Then, the dynamics of f are governed by the Vlasov type equation:

$$\begin{aligned} &\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (F[f]f) = 0, \quad x, v \in \mathbb{R}^d,\ t \in \mathbb{R}, \\ &F[f](x, v, t) = -K \int_{\mathbb{R}^{2d}} \psi(|x - y|)(v - v_*) f(y, v_*) dv_* dy. \end{aligned} \tag{2.3}$$

System (2.3) admits a global $\mathcal{C}^1$-solution as long as the initial datum is $\mathcal{C}^1$-regular and compactly supported in x and v (see [50]). The global existence of measure-valued solutions to (2.3) when the initial datum is a Radon measure was studied in [48]. For a macroscopic description of the C-S model, we introduce the first three velocity moments of f. For given $(x, t) \in \mathbb{R}^d \times \mathbb{R}_+$, we set hydrodynamic mass, momentum, and energy densities:

$$\rho := \int_{\mathbb{R}^d} f dv, \quad \rho u := \int_{\mathbb{R}^d} v f dv, \quad \rho E := \rho e + \frac{1}{2}\rho |u|^2, \tag{2.4}$$

where $\rho e := \dfrac{1}{2} \displaystyle\int_{\mathbb{R}^d} |v - u(x)|^2 f dv$ is the internal energy. Then, the macroscopic observables in (2.4) satisfy the following hydrodynamic equations [50]:

$$\begin{aligned} &\partial_t \rho + \nabla_x \cdot (\rho u) = 0, \quad x \in \mathbb{R}^d,\ t \in \mathbb{R}, \\ &\partial_t (\rho u) + \nabla_x \cdot (\rho u \otimes u + P) = S^{(1)}, \\ &\partial_t (\rho E) + \nabla_x \cdot (\rho E u + P u + q) = S^{(2)}, \end{aligned} \tag{2.5}$$

where $P = (p_{ij})$ and $q = (q_1, \cdots, q_d)$ are the stress tensor and heat flow defined by the following moments:

$$p_{ij} := \int_{\mathbb{R}^d} (v_i - u_i)(v_j - u_j) f dv, \quad q_i := \int_{\mathbb{R}^d} (v_i - u_i)|v - u|^2 f dv,\ 1 \le i, j \le d. \tag{2.6}$$

The source terms are given by the following relations:

$$\begin{aligned} S^{(1)} &:= -K \int_{\mathbb{R}^d} \psi(|x - y|)(u(x) - u(y))\rho(x)\rho(y) dy, \\ S^{(2)} &:= -K \int_{\mathbb{R}^d} \psi(|x - y|)(E(x) + E(y) - u(x) \cdot u(y))\rho(x)\rho(y) dy. \end{aligned} \tag{2.7}$$

System (2.5) is not a closed system, because we need the third velocity moment of f to calculate the heat flux q in (2.6). When the configuration is close to collisionless global flocking state, we may assume the monokinetic ansatz for f:

$$f(x, v, t) = \rho(x, t)\delta(v - u(x, t)), \quad x, v \in \mathbb{R}^d,\ t \in \mathbb{R}. \tag{2.8}$$

Then, under assumption (2.8), the stress tensor P and heat flux q in (2.6) vanish:

$$p_{ij} = 0, \qquad q_i = 0, \quad 1 \le i, j \le d.$$

Thus, system (2.5) with (2.7) becomes

$$\begin{aligned} &\partial_t \rho + \nabla_x \cdot (\rho u) = 0, \quad x \in \mathbb{R}^d, \ \ t > 0, \\ &\partial_t (\rho u) + \nabla_x \cdot (\rho u \otimes u) = -K\rho \int_{\mathbb{R}^d} \psi(|x-y|)\rho(y)(u(x)-u(y))dy. \end{aligned} \tag{2.9}$$

For the one-dimensional case with all-to-all coupling ($\psi = 1$), the global existence of an entropic weak solution to (2.9) has been studied in [41], whereas the global existence of a classical solution to (2.9) on the periodic domain was treated in [44]. In the absence of coupling, i.e., $K = 0$, system (2.9) describes pressureless gas dynamics, which has been extensively studied in [13, 14, 38, 51, 71].

2.2. The Kuramoto model. Let $z_j \in \mathbb{C}$ be the position of the j-th oscillator, whose dynamics is governed by the complex Ginzburg–Landau model in the complex plane $\mathbb{C}$:

$$\frac{dz_j}{dt} = (1 - |z_j|^2 + i\Omega_j)z_j + \frac{K}{N}\sum_{i=1}^{N}(z_i - z_j), \quad t \in \mathbb{R}, \tag{2.10}$$

where K is the positive coupling strength and Ω_j is the natural frequency of the j-th oscillator. Then, z_j approaches the limit cycle $\mathbb{S}^1$ formally. Thus, on the limit cycle $\mathbb{S}^1$, the system (2.10) becomes

$$\frac{dz_j}{dt} = i\Omega_j z_j + \frac{K}{N}\sum_{i=1}^{N}(z_i - z_j). \tag{2.11}$$

We now substitute the ansatz $z_j = e^{i\theta_j}$ into (2.11) to obtain the Kuramoto model:

$$\frac{d\theta_j}{dt} = \Omega_j + \frac{K}{N}\sum_{k=1}^{N}\sin(\theta_k - \theta_j), \quad t \in \mathbb{R}. \tag{2.12}$$

The Kuramoto model (2.12) has been extensively studied in relation to phase-transition-like phenomena from the disordered state to partially ordered state as the coupling strength K increases from zero. For a detailed survey, we refer to [1, 10, 67]. For a later use, we present the definition of complete synchronization as follows.

Definition 2.2. [27, 50] Let $\Theta := (\theta_1, \cdots, \theta_N)$ be the solution to (2.12). Then, the phase configuration Θ exhibits a complete (frequency) synchronization if and only if the following relation holds:

$$\lim_{t\to\infty} \max_{1\le i,j\le N} |\dot{\theta}_i(t) - \dot{\theta}_j(t)| = 0.$$

Remark 2.3. Note that if we add the Kuramoto model (2.12) over j, we have

$$\sum_{j=1}^{N}\dot{\theta}_j = \sum_{j=1}^{N}\Omega_j + \frac{K}{N}\sum_{j,k}\sin(\theta_k - \theta_j) = \sum_{j=1}^{N}\Omega_j.$$

If all phase velocities approach the same value, say ω, then the above relation implies that the common frequency ω should be the average of the natural frequencies $\frac{1}{N}\sum_{j=1}^{N}\Omega_j$.

We briefly recall known results for the complete synchronization in the followings. Ermentrout [37] found a critical coupling at which all oscillators become phase-locked, independent of their number. The linear stability of this phase-locked state has been studied in several papers [2, 12, 33, 69] using tools such as Lyapunov functionals, spectral graph theory, and control theory. The studies most closely related to this paper are those of Chopra and Spong [27], Choi et al. [21], and Dórfler and Bullo [35]. These papers use the phase-diameter $D(\Theta) := \max_{1\le i,j\le N} |\theta_i - \theta_j|$ as a Lyapunov functional, and study its temporal evolution via Gronwall's inequality.

In the mean-field limit $N \to \infty$, system (2.12) becomes the kinetic model [19, 53]:

$$\begin{aligned} &\partial_t f + \partial_\theta(\omega[f]f) = 0, \qquad (\theta,\Omega) \in \mathbb{T}\times\mathbb{R}, \ \ t>0, \\ &\omega[f](\theta,\Omega,t) := \Omega - K\int_{\mathbb{T}\times\mathbb{R}} \sin(\theta-\theta_*) f(\theta_*,\Omega_*,t) d\Omega_* d\theta_*. \end{aligned} \tag{2.13}$$

For the nonlinear stability of the phase-locked state of (2.13), we refer to [15].

2.3. Relation between the C-S and Kuramoto models. As the C-S model is a second-order system, whereas the Kuramoto model is first-order, the two cannot be compared directly. Thus, we introduce an auxiliary phase velocity (frequency) in the Kuramoto model (2.12) so that it can be considered as a second-order model:

$$\begin{aligned} \dot\theta_j &= \omega_j, \quad t>0, \\ \dot\omega_j &= \frac{K}{N}\sum_{k=1}^N \cos(\theta_k-\theta_j)(\omega_k-\omega_j), \end{aligned}$$

where the initial $(\theta_{j0},\omega_{j0})$ is constrained by

$$\omega_{j0} = \Omega_j + \frac{K}{N}\sum_{k=1}^N \sin(\theta_{k0}-\theta_{j0}).$$

Note that the following relation $(x_j,v_j) \iff (\theta_j,\omega_j)$ bridges the C-S model to the Kuramoto model.

3. The Cucker–Smale flocking model

In this section, we discuss asymptotic flocking estimates for the C-S model and the coupled Cucker–Smale-Navier–Stokes system.

3.1. Asymptotic flocking estimates. In this subsection, we discuss the asymptotic flocking estimate for the C-S model. We now introduce a Lyapunov functional approach for the flocking estimate. For a given configuration $(x,v) \in \mathbb{R}^{2dN}$ with zero sum

$$\sum_{i=1}^N x_i(t) = 0, \qquad \sum_{i=1}^N v_i(t) = 0, \quad t\ge 0,$$

we set

$$||x||_\infty := \max_{1\le i\le N} ||x_i||, \qquad ||v||_\infty := \max_{1\le i\le N} ||v_i||.$$

Then, $||x||_\infty$ and $||v||_\infty$ are Lipschitz continuous functions, and it is easy to see that

$$\begin{aligned} &\Big|\frac{d}{dt}||x||_\infty\Big| \le ||v||_\infty \quad \text{a.e. } t \in (0,\infty), \\ &\frac{d}{dt}||v||_\infty \le -K\psi(2||x||_\infty)||v||_\infty. \end{aligned} \tag{3.1}$$

We next introduce Lyapunov-type functionals $\mathcal{E}^0_\pm(t) \equiv \mathcal{E}^0_\pm(x(t), v(t))$:

$$\mathcal{E}^0_\pm(t) := ||v(t)||_\infty \pm \frac{K}{2}\int_0^{2||x(t)||_\infty} \psi(s)ds, \quad t \ge 0.$$

Then, it is easy to see the non-increasing property of $\mathcal{E}^0_\pm$ using (3.1):

$$\mathcal{E}^0_\pm(t) \le \mathcal{E}^0_\pm(0), \quad t \ge 0,$$

which leads to the stability estimate of $\mathcal{E}^0_\pm(t)$:

$$||v(t)||_\infty + \frac{K}{2}\Big|\int_{2||x_0||_\infty}^{2||x(t)||_\infty} \psi(s)ds\Big| \le ||v_0||_\infty, \quad t \ge 0.$$

The above stability estimate yields the following flocking estimate.

Theorem 3.1 ([3]). *Let (x, v) be the smooth global solution to system* (2.1)–(2.2) *with initial data (x_0, v_0) satisfying*

$$||x_0||_\infty > 0, \quad ||v_0||_\infty < \frac{K}{2}\int_{2||x_0||_\infty}^\infty \psi(r)dr.$$

Then, there exists a positive constant x_M satisfying

$$||x(t)||_\infty \le x_M, \quad ||v(t)||_\infty \le ||v_0||_\infty e^{-K\psi(2x_M)t}, \quad t \ge 0.$$

Remark 3.2. Note that the result of Theorem 3.1 is independent of N so that it can be lifted to the kinetic regime. In contrast, the flocking estimates for the C-S model and its modified system have been studied in the ℓ^2-norm framework [29, 48, 50, 59] so that the resulting flocking estimate depend on the number of particles N.

3.2. Flocking particle–fluid interactions. In this subsection, we consider an ensemble of C-S particles moving in a highly viscous incompressible fluid on the periodic domain $\mathbb{T}^3 := \Big(\mathbb{R}/\mathbb{Z}\Big)^3$. In this case, the dynamics between C-S particles and incompressible fluid can be described by the coupled system of a Vlasov-type equation and the incompressible Navier–Stokes equations. Let $f = f(x, v, t)$ be the one-particle distribution function of the C-S particles with velocity $v \in \mathbb{R}^3$ at position $x \in \mathbb{T}^3$ at time $t > 0$, and $u = u(x, t)$ be the bulk velocity of the incompressible fluid. Then, (f, u) is governed by the coupled Cucker–Smale-Navier–Stokes system [9]:

$$\begin{aligned} &\partial_t f + \nabla_x \cdot (vf) + \nabla_v \cdot \big(\big(F_a[f] + F_d\big)f\big) = 0, \quad (x, v) \in \mathbb{T}^3 \times \mathbb{R}^3, \ t > 0, \\ &\partial_t u + (u \cdot \nabla_x)u + \nabla_x p - \mu\Delta_x u = -\int_{\mathbb{R}^3} F_d f dv, \quad \nabla_x \cdot u = 0, \end{aligned} \tag{3.2}$$

where F_a and F_d represent the alignment (flocking) force and the drag force per unit mass, respectively:

$$\begin{aligned} F_a[f](x,v,t) &:= \int_{\mathbb{T}^3\times\mathbb{R}^3} \psi(x,y)(v_*-v)f(y,v_*,t)dv_*dy, \\ F_d(x,v,t) &:= u(x,t)-v. \end{aligned}$$

The kernel function $\psi : \mathbb{T}^3\times\mathbb{T}^3 \to \mathbb{R}_+$ is a $\mathcal{C}^1$-function that satisfies the following conditions of symmetry and boundedness:

$$\psi(x,y)=\psi(y,x), \quad m_\psi \le \psi(x,y) \le M_\psi, \quad M_\psi - m_\psi < 1, \quad x,y\in\mathbb{T}^3,$$

where m_ψ and M_ψ are positive constants.

For the asymptotic flocking estimate of system (3.2), we introduce a Lyapunov functional $\mathcal{E}^1$ measuring the local velocity fluctuations and the distance between local velocity averages:

$$\mathcal{E}^1(t) := 2\int_{\mathbb{T}^3\times\mathbb{R}^3} |v-v_c(t)|^2 f(t)dvdx + 2\int_{\mathbb{T}^3} |u(t)-u_c(t)|^2dx + |u_c(t)-v_c(t)|^2, \ t\ge 0,$$

where u_c and v_c are local velocity averages of the fluid and particles:

$$u_c(t) := \int_{\mathbb{T}^3} udx \quad \text{and} \quad v_c(t) := \frac{\int_{\mathbb{T}^3\times\mathbb{R}^3} vf dvdx}{\int_{\mathbb{T}^3\times\mathbb{R}^3} f dvdx}.$$

Theorem 3.3 ([8]). *Suppose that initial data $[f_0,u_0]$, and μ satisfy*

$$\begin{aligned} &\int_{\mathbb{T}^3\times\mathbb{R}^3} f_0 dvdx = 1, \quad \textit{supp}_v f_0 \textit{ is bounded for each } x\in\mathbb{T}^3, \quad \mathcal{E}^1(0)<\infty \\ &\sup_{0\le t<\infty} ||\rho_p(t)||_{L^\infty} < \infty, \quad \mu > \frac{\sup_{0\le t<\infty}||\rho_p(t)||_{L^\infty}}{\pi_3}, \end{aligned}$$

where π_3 is a positive constant appearing in Poincare's inequality for the torus $\mathbb{T}^3$, and ρ_p is the local particle density:

$$\rho_p(x,t) := \int_{\mathbb{R}^3} f(x,v,t)dv.$$

Then, for any classical solution (f,u) to (3.2) *in the time-interval $[0,\infty)$, the following estimate of exponential convergence holds:*

$$\mathcal{E}^1(t) \le \mathcal{E}^1(0)\exp\Big(-\min\{2m_\psi+1, K, 2\}t\Big), \quad 0\le t<\infty,$$

where K is a positive constant given by $K := 2\mu\pi_3 - 2\sup_{0\le t\le\infty}||\rho_p(t)||_{L^\infty} > 0$.

Remark 3.4. The existence of weak and strong solutions to system (3.2) has been studied in [8, 9], and the generalization of the system (3.2) for two species C-S flocking particles interacting with the fluid has been investigated in [24]. In the whole space $\mathbb{R}^2$, the global well-posedness for the system (3.2) has also been treated in [25]. Concerning the couplings with other fluids, such as a compressible Navier–Stokes system and Stokes system, have been discussed in [6, 7].

4. The Kuramoto model

In this section, we discuss the emergent synchronous dynamics of the Kuramoto model. First, we present frameworks for complete synchronization, whereby the phase velocities (frequencies) of oscillators converge to the same value, namely the average of their natural frequencies. We also provide two generalizations of the Kuramoto model in the realm of quantum synchronization in [20, 57, 58].

4.1. Complete synchronization. In the subsection, we briefly discuss the state-of-the-art result and methodology for complete synchronization given in Definition 2.2. So far, most works only deal with initial configurations whose phase-diameter is less than π. To date, π is the best upper bound; if we could extend this upper bound to 2π, it would be possible to rigorously justify the independence of initial configurations observed in numerical simulations. To describe the results in [21, 27, 40], we introduce the following notation.

For a phase configuration $\Theta = (\theta_1, \cdots, \theta_N)$ and natural frequencies Ω_j, we introduce several diameters that play a key role as Lyapunov functionals:

$$D(\Theta) := \max_{1\le i,j\le N} |\theta_i - \theta_j|, \quad D(\omega) := \max_{1\le i,j\le N} |\dot\theta_i - \dot\theta_j|, \quad D(\Omega) := \max_{1\le i,j\le N} |\Omega_i - \Omega_j|.$$

Lemma 4.1 ([40]). *Suppose that $K, D(\Omega)$, and the initial phase configuration Θ_0 satisfy*

$$0 < D(\Omega) < K, \quad D(\Theta_0) < D_\theta^\infty.$$

Then, we have

$$\sup_{t\ge 0} D(\Theta(t)) \le D_\theta^\infty,$$

where the constant $D_\theta^\infty \in (0, \frac{\pi}{2})$ is the root in the first quadrant of the following trigonometric equation:

$$\sin\theta = \frac{D(\Omega)}{K}.$$

This lemma yields the following complete synchronization.

Theorem 4.2 ([40]). *Suppose that $K, D(\Omega)$, and the initial phase configuration Θ_0 satisfy*

$$0 < D(\Omega) < K, \quad D(\Theta_0) < D_\theta^\infty,$$

and let Θ be the solution to (2.12)*. Then, we have*

$$D(\omega(t)) \le D(\omega_0) \exp\Big[- K(\cos D_\theta^\infty)t\Big], \quad t \ge 0.$$

Proof. By direct estimate, we can derive Gronwall's inequality for the frequency diameter $D(\omega)$ (see [40])

$$\frac{dD(\omega)}{dt} \le -K(\cos D_\theta^\infty) D(\omega), \quad t \ge 0,$$

and then derive the desired complete synchronization estimate. A detailed proof can be found in [40]. □

Remark 4.3.

(1) Note that the arguments employed in Theorem 4.2 are valid as long as the phase-diameter $D(\Theta)$ is smaller than $\frac{\pi}{2}$. Recently, the author and his collaborators [21] extended the admissible class of initial data as follows. For any given initial data $D(\Theta_0) < \pi$, once we choose a coupling strength $K > \frac{D(\Omega)}{\sin D(\Theta_0)}$, we can show that the phase-configuration shrinks to a configuration whose diameter is smaller than $\frac{\pi}{2}$ in finite time. More precisely, there exists $T_* \in (0, \infty)$ such that

$$D(\Theta(T_*)) < D_\theta^\infty.$$

Then, we can apply Theorem 4.2 with a new initial configuration $D(\Theta(T_*))$ to derive complete synchronization.

(2) For identical oscillators with $D(\Omega) = 0$, complete synchronization was obtained in [34] for an arbitrary initial configuration using the gradient flow structure of the Kuramoto model and uniform boundedness of the phase diameter.

(3) For a locally interacting case, complete synchronization has been studied in [47]. The effects of inertia and frustrations in the synchronization were discussed in [22, 23, 26, 42, 43].

We next discuss a recent result [45] extending Theorem 4.2 to a configuration Θ_0 with $D(\Theta_0) > \pi$. Our main idea for this is as follows. For a given initial configuration Θ_0 that is larger than π, we choose a sufficiently large coupling strength to squeeze the initial configuration to a smaller configuration whose diameter is less than π in finite time. Then, we can use Theorem 4.2 to obtain complete synchronization. To do this, we need to estimate the dynamics of the real Kuramoto order parameters. For the phase configuration $\Theta = (\theta_1, \cdots, \theta_N)$, we define the real order parameters r and ϕ as follows:

$$re^{i\phi} := \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}.$$

Note that r satisfies $0 \le r \le 1$, and we have

$$\begin{aligned} \dot{r} &= -\frac{1}{N}\sum_{j=1}^{N} \sin(\theta_j - \phi)\Big(\Omega_j - Kr\sin(\theta_j - \phi)\Big), \\ \dot{\phi} &= \frac{1}{rN}\sum_{j=1}^{N} \cos(\theta_j - \phi)\Big(\Omega_j - Kr\sin(\theta_j - \phi)\Big). \end{aligned} \tag{4.1}$$

For positive constants $\alpha, \delta < \frac{1}{2}$, we set r_*, r^*, and ε:

$$\begin{aligned} &\beta_\delta := (1-\delta)\pi, \qquad r_* := \frac{\max_j |\Omega_j|}{\sqrt{\alpha}K\sin\beta_\delta}, \qquad r^* := 1 - \alpha(2 + \sin^2\beta_\delta), \\ &\varepsilon(r_0, \delta, K, \Omega_1, \cdots, \Omega_N) := \frac{\beta_\delta - \frac{\pi}{2}}{1 + \frac{K(1-r_0) + \left(1 + \frac{1}{r_0}\right)\max_j |\Omega_j|}{\frac{\pi K r_0 \sin\beta_\delta}{2\beta_\delta} - \frac{D(\Omega)}{2}}}. \end{aligned}$$

A detailed estimate of (4.1) under some conditions on K and Θ_0 yields that the configuration shrinks to some configuration whose diameter is strictly smaller than π in finite time. Then, we again apply the result of Theorem 4.2 to bring about complete synchronization.

Theorem 4.4 ([45]). *Suppose that the initial configuration Θ_0 and coupling strength K satisfy*

$$\text{(i) } r_* \le r_0 \le r^*, \qquad \max_{1\le i\le N}|\theta_{i0}-\phi_0| < \frac{\pi}{2}+\varepsilon.$$
$$\text{(ii) } K > \max\left\{\frac{\max_j|\Omega_j|}{[1-\alpha(2+\sin^2\beta_\delta)][\sqrt{\alpha}\sin\beta_\delta]}, \frac{\beta_\delta D(\Omega)}{\pi r_0\sin\beta_\delta}, D(\Omega)\right\}.$$

Then, complete synchronization occurs asymptotically.

4.2. Quantum Kuramoto models. Most analytical studies on synchronization have mainly focused on classical Kuramoto oscillators [1]. Even for quantum mechanical phenomena appearing in the Josephson junction array for superconductors, classical Kuramoto-like systems have been used instead of quantum models. Quantum synchronization for quantum mechanical systems is an emerging topic, and has been studied numerically and experimentally for possible applications in the control and stability of quantum devices and quantum computation [57, 58]. Recently, the physicist M. Lohe introduced a non-abelian quantum model that generalizes the Kuramoto model under certain conditions. Let $U_j = U_j(t)$ be a $d\times d$ unitary matrix encoding the quantum information of the j-th quantum oscillator. The Lohe model reads as follows:

$$i\dot{U}_jU_j^\dagger = H_j + \frac{iK}{2N}\sum_{k=1}^N\Big(U_kU_j^+ - U_jU_k^+\Big), \quad 1\le j\le N, \tag{4.2}$$

where $U_j^\dagger$ is the Hermitian conjugate of U_j, H_j is the prescribed constant $d\times d$ Hermitian matrix whose eigenvalues correspond to the natural frequencies of the quantum oscillator at node j, and K is a uniform nonnegative coupling strength. For $d=1$, by the ansatzs $U_j = e^{-i\theta_j}$ and $H_j = \Omega_j$, it is easy to see that system (4.2) becomes the Kuramoto model (2.12). On the other hand, for $d=2$ corresponding to an interacting qubit system, the unitary matrix U_j can be written as a linear combination of Pauli matrices $\sigma_k,\ k=1,2,3$ and the identity matrix I_2 up to a phase factor θ_j:

$$U_j = e^{-i\theta_j}\begin{pmatrix} x_j^4+ix_j^3 & x_j^2+ix_j^1\\ -x_j^2+ix_j^1 & x_j^4-ix_j^3\end{pmatrix}, \qquad H_j = \begin{pmatrix}\omega_j^3+\nu_j & \omega_i^1-j\omega_j^2\\ \omega_j^1+i\omega_j^2 & -\omega_j^3+\nu_j\end{pmatrix}.$$

In this setting, the dynamics of U_j are completely determined by the state $(\theta_j, x_j)\in\mathbb{T}\times\mathbb{S}^3$ governed by the coupled ODE system:

$$\begin{aligned} ||x_j||^2\dot{\theta}_j &= \nu_j + \frac{K}{N}\sum_{k=1}^N\sin(\theta_k-\theta_j)\langle x_j,x_k\rangle, \quad t\in\mathbb{R},\\ ||x_j||^2\dot{x}_j &= \Omega_jx_j + \frac{K}{N}\sum_{k=1}^N\cos(\theta_k-\theta_j)\Big(||x_j||^2x_k - \langle x_k,x_j\rangle x_j\Big),\end{aligned} \tag{4.3}$$

where Ω_i and ν_i are 4×4 skew-symmetric real matrices and real numbers, respectively, and $\langle\cdot,\cdot\rangle$ denotes the standard inner product in $\mathbb{R}^4$. Lohe showed numerically that, for qubit nodes ($d=2$) with different natural frequencies and a sufficiently large coupling strength, complete synchronization of quantum nodes occurs [57, 58].

In the sequel, we consider the following simplified situation:

$$\theta_j \equiv 0, \quad \nu_j \equiv 0, \quad 1 \le j \le N.$$

In this case, system (4.3) reduces to the following system on the 3-sphere $\mathbb{S}^3$:

$$||x_j||^2 \dot{x}_j = \Omega_j x_j + \frac{K}{N} \sum_{k=1}^{N} \Big(||x_j||^2 x_k - \langle x_k, x_j \rangle x_j \Big), \quad t \in \mathbb{R}. \tag{4.4}$$

We next introduce distinct concepts "*complete synchronization*" and "*practical synchronization*" as follows.

Definition 4.5 ([18]). Let $\mathcal{X} := \{x_i\}_{i=1}^N$ be a position configuration governed by system (4.4).

(1) The ensemble $\mathcal{X}(t)$ approaches to *asymptotic complete flocking* if and only if

$$\lim_{t \to \infty} \sup_{1 \le i,j \le N} ||x_i(t) - x_j(t)|| = 0.$$

(2) The ensemble $\mathcal{X}(t)$ exhibits practical flocking if and only if for a given $\varepsilon > 0$, there exists a lower bound of coupling strength $K(\varepsilon) > 0$ and $T = T(\varepsilon) > 0$ such that

$$K > K(\varepsilon) \quad \Longrightarrow \quad \sup_{T \le t < \infty} \max_{1 \le i,j \le N} ||x_i(t) - x_j(t)|| < \varepsilon.$$

Remark 4.6. Note that complete synchronization implies practical synchronization.

For a rigorous analysis of Lohe's numerical results, we introduce a Lyapunov functional $D(x)$ representing the diameter of the configuration x:

$$D(x(t)) := \max_{1 \le i,j \le N} ||x_i(t) - x_j(t)||, \quad t \ge 0.$$

Theorem 4.7 ([18]). *Suppose that the coupling strength K, Ω_i, and initial data x_0 satisfy*

$$D(\Omega) := \max_{1 \le i,j \le N} |\Omega_j - \Omega_i| < \frac{K}{32}, \quad 0 < D(x_0) < \frac{1}{4}, \quad ||x_{i0}|| = 1, \quad 1 \le i \le N.$$

Then, for any solution $x = (x_1, \cdots, x_N)$ to system (4.4) *with initial data x_0, we have a practical synchronization in the sense of Definition* 4.5*:*

$$\limsup_{t \to \infty} D(x(t)) \le \frac{16}{K} D(\Omega).$$

We next discuss the PDE analogue of the Lohe model (4.2). Let $\psi_j = \psi_j(x,t)$ be the wave function of an identical quantum oscillator at the j-th node. Then, the wave function $\psi = \psi(x_1, \cdots, x_N, t)$ for the whole system is given by the tensor product $\otimes_{i=1}^N \psi_i$ of one-body wave functions. In the absence of potential force, the Schrödinger-Lohe (S-L) model [57] for a one-body wave function reads as follows:

$$i \partial_t \psi_j = -\Delta \psi_j + \frac{iK}{N} \sum_{k=1}^{N} \Big(\psi_k - \frac{\langle \psi_k, \psi_j \rangle}{\langle \psi_j, \psi_j \rangle} \psi_j \Big), \quad j = 1, \cdots, N. \tag{4.5}$$

Here, $\langle\cdot,\cdot\rangle$ is the standard L^2-inner product in an infinite-dimensional Hilbert space, K is the positive coupling strength, and the Planck constant $\hbar$ and mass are assumed to be unity.

Given an ordered N-tuple of wave functions $\Psi = (\psi_1, \cdots, \psi_N)$, we set

$$D(\Psi) := \max_{1\le i,j\le N} ||\psi_i - \psi_j||_{L^2_x}.$$

Definition 4.8. Let $\Psi = (\psi_1, \cdots, \psi_N)$ be a solution to system (4.5). Then, system (4.5) exhibits quantum synchronization if and only if the following estimate holds:

$$\lim_{t\to\infty} D(\Psi(t)) = 0.$$

Then, we can obtain similar complete synchronization estimates as in Theorem 4.7.

Theorem 4.9 ([20]). *Suppose that the coupling strength and initial data $\Psi_0 = (\psi_{10}, \cdots, \psi_{N0})$ satisfy*

$$K > 0, \qquad ||\psi_{j0}||_{L^2_x} = 1,\ 1 \le j \le N, \qquad D(\Psi_0) < \frac{1}{2}.$$

Then, for any solution $\Psi = (\psi_1, \cdots, \psi_N)$ to (4.5)*, the diameter $D(\Psi)$ satisfies*

$$D(\Psi(t)) \le \frac{D(\Psi_0)}{D(\Psi_0) + (1 - 2D(\Psi_0))e^{Kt}}, \quad t \ge 0.$$

Proof. By direct calculation, we can show that $D(\Psi)$ satisfies

$$\dot{D}(\Psi) \le K\Big(- D(\Psi) + 2D(\Psi)^2\Big).$$

This yields

$$D(\Psi(t)) \le \frac{D(\Psi_0)}{D(\Psi_0) + (1 - 2D(\Psi_0))e^{Kt}}.$$

□

5. Conclusion

We briefly discussed how a Lyapunov functional approach can be used to describe the emergence of collective motions in the Cucker–Smale and Kuramoto models and their variants. There are still many open questions to be investigated for these naive-looking models. Before we finish this paper, we address some interesting open questions for the aforementioned models.

5.1. The Cucker-Smale model. In this subsection, we propose some open problems related to the flocking dynamics of the C-S model.

Question 1. The large-time dynamics of the Cucker–Smale model has been extensively studied in previous literature, but these works mainly focus on global flocking, i.e., the formation of a uni-group. When the condition on initial data in Theorem 3.1 is violated, numerical results and the explicit solution for a two-body system illustrate the formation of a multi-flocking scenario known as local flocking, i.e., several local flocking groups emerge

asymptotically over time. To the best of our knowledge, these numerical results have never been confirmed by a rigorous analysis. It might be very difficult to verify the emergence of local flocking from a given initial configuration. However, a stability-type analysis of local flocking configurations might be plausible. Thus, our first question is

> "Can we prove the emergence of multi-clusters using the C-S model"?

Question 2. When global flocking emerges, the asymptotic flocking is the average of the initial velocities, which is independent of the communication network, i.e., the detailed structure of the communication weight. However, the communication network will affect the convergence rate and spatial flocking configuration. Thus, it would be interesting to know whether the asymptotic spatial configuration depends on the communication weight continuously or not. Our second question is

> "Can we prove the structural stability of the C-S model in terms of communication weights"?

Question 3. To date, most analytical studies on Cucker–Smale flocking have been restricted to the flat space, i.e., Euclidean space. Thus, the curvature of the underlying ambient manifolds in which the flocking particles are located does not affect the particle dynamics. However, from an application viewpoint, flocking on non-Euclidean manifolds will be important. Thus, the third question is

> "Can we develop a mathematical theory for flocking on Riemannian manifolds"?

5.2. The Kuramoto model. As explained in Section 2.3, the Kuramoto model can be regarded as a special case of the Cucker–Smale model on the N-tori. Thus, we can ask similar questions as those posed in the previous part for the C-S model.

Question 4. Due to the geometric constraint, numerical results for the Kuramoto model always suggest that complete synchronization occurs for any initial configuration, as long as the coupling strength is sufficiently large. However, as discussed in Section 4.1, the analytical justification of this result is still far from completeness. Thus, the forth question is

> "Can we verify the complete synchronization for any initial configuration for sufficiently large coupling strength"?

Question 5. The Lohe model (4.2) has only been analytically investigated for the special cases of $d = 1$ and $d = 2$. Even for these special cases, the range of admissible initial configurations leading to complete synchronization is rather restrictive. Hence, an interesting question is

> "Can we find a framework for complete synchronization of the Lohe model in any dimension and arbitrary initial configuration, and prove the convergence in the large quantum particle limit"?

Acknowledgements. The work of S.-Y. Ha is partially supported by NRF-2009-0083521 (SRC).

References

[1] Acebron, J. A., Bonilla, L. L., Pérez Vicente, C. J. P., Ritort, F., and Spigler, R., *The Kuramoto model: A simple paradigm for synchronization phenomena*, Rev. Mod. Phys. **77** (2005), 137–185.

[2] Aeyels, D. and Rogge, J., *Stability of phase locking and existence of frequency in networks of globally coupled oscillators*, Prog. Theor. Phys. **112** (2004), 921–941.

[3] Ahn, S. Choi, H., Ha, S.-Y., and Lee, H., *On collision-avoiding initial configurations to Cucker-Smale type flocking models*, Commun. Math. Sci. **10** (2012), 625–643.

[4] Ahn, S. and Ha, S.-Y., *Stochastic flocking dynamics of the Cucker-Smale model with multiplicative white noises*, J. Math. Phys. **51** (2010), 103301.

[5] Buck, J. and Buck, E., *Biology of synchronous flashing of fireflies*, Nature **211** (1996), 562.

[6] Bae, H., Choi, Y.-P., Ha, S.-Y., and Kang, M., *Global existence of strong solutions to the Cucker-Smale-Stokes system*, Submitted.

[7] ———, *Asymptotic flocking dynamics of Cucker-Smale particles immersed in compressible fluids*, To appear in Discrete and Continuous Dynamical Systems.

[8] ———, *Global existence of strong solutions for the Cucker-Smale-Navier-Stokes system*, Submitted.

[9] ———, *Time-asymptotic interaction of flocking particles and an incompressible viscous fluid*, Nonlinearity **25** (2012), 1155–1177.

[10] Balmforth, N. J. and Sassi, R., *A shocking display of synchrony*, Physica D **143** (2000), 21–55.

[11] Bolley, F., Canizo, J. A., and Carrillo, J. A., *Stochastic mean-field limit: non-Lipschitz forces and swarming*, Math. Models Methods Appl. Sci. **21** (2011), 2179–2210.

[12] Bonilla, L. L., Neu, J. C., and Spigler, R., *Nonlinear stability of incoherence and collective synchronization in a population of coupled oscillators*, J. Stat. Phy. **67** (1992), 313–330.

[13] Bouchut, F., *On zero pressure gas dynamics*, Advances in kinetic theory and computing, Series on Advances in Mathematics for Applied Sciences, World Scientific, **22** (1994), 171–190.

[14] Brenier, Y. and Grenier, E., *Sticky particles and scalar conservation laws*, SIAM. J. Num. Anal. **35** (1998), 2317–2328.

[15] Carrillo, J. A., Choi, Y.-P., Ha, S.-Y., Kang, M.-J., and Kim, Y., *Contractivity of the Wasserstein metric for the kinetic Kuramoto equation*, To appear in J. Stat. Phys.

[16] Carrillo, J. A., Fornasier, M., Rosado, J., and Toscani, G., *Asymptotic flocking dynamics for the kinetic Cucker-Smale model*, SIAM J. Math. Anal. **42** (2010), 218–236.

[17] Cucker, F. and Dong, J.-G., *Avoiding collisions in flocks*, IEEE Trans. Automatic Control **55** (2010), 1238–1243.

[18] Chi, D., Choi, S.-H., and Ha, S.-Y., *Emergent behaviors of a holonomic particle system on a sphere*, To appear in J. Math. Phys.

[19] Chiba, H., *Continuous limit of the moments system for the globally coupled phase oscillators*, Submitted.

[20] Choi, S.-H. and Ha, S.-Y., *Quantum synchronization of the Schrödinger-Lohe model*, Submitted.

[21] Choi, Y.-P., Ha, S.-Y., Jung, S., and Kim, Y., *Asymptotic formation and orbital stability of phase-locked states for the Kuramoto model*, Physica D **241** (2012), 735–754.

[22] Choi, Y.-P., Ha, S.-Y., and Noh, S. E., *On the relaxation dynamics of the Kuramoto oscillators with small inertia*, J. Math. Phys. **54** (2013), 072701.

[23] Choi, Y.-P., Ha, S.-Y., and Yun, S.-B., *Complete synchronization of Kuramoto oscillators with finite inertia*, Physica D **240** (2011), 32–44.

[24] Choi, Y.-P. and Kwon, B., *Two-species flocking particles immersed in a fluid*, To appear in Commun. Info. System.

[25] Choi, Y.-P. and Lee, J., *Global existence of weak and strong solutions to the Cucker-Smale-Navier-Stokes equations in* $\mathbb{R}^2$, Submitted.

[26] Choi, Y.-P., Li, Z., Ha, S.-Y., Xue, X., and Yun, S.-B., *Complete entrainment of Kuramoto oscillators with inertia on networks via gradient-like flow*, To appear in J. Differential Equations.

[27] Chopra, N. and Spong, M. W., *On exponential synchronization of Kuramoto oscillators*, IEEE Trans. Automatic Control **54** (2009), 353–357.

[28] Cucker, F. Cucker, and Mordecki, E., *Flocking in noisy environments*, J. Math. Pures Appl. **89** (2008), 278–296.

[29] Cucker, F. and Smale, S., *Emergent behavior in flocks*, IEEE Trans. Automat. Control **52** (2007), 852–862.

[30] Degond, P. and Motsch, S., *A macroscopic model for a system of swarming agents using curvature control*, J. Stat. Phys. **143** (2011), 685–714.

[31] _____, *Continuum limit of self-driven particles with orientation interaction*, Math. Models Methods Appl. Sci. **18** (2008), 1193–1215.

[32] _____, *Large scale dynamics of the Persistent Turning Walker model of fish behavior*, J. Stat. Phys. **131** (2008), 989–1021.

[33] De Smet, F. and Aeyels, D., *Partial frequency in finite Kuramoto-Sakaguchi model*, Physica D **234** (2007), 81–89.

[34] Dong, J.-G. and Xue, X., *Synchronization analysis of Kuramoto oscillators*, Commun. Math. Sci. **11** (2013), 465–480.

[35] Dörfler, F. and Bullo, F., *On the critical coupling for Kuramoto oscillators*, SIAM. J. Appl. Dyn. Syst. **10** (2011), 1070–1099.

[36] Duan, R., Fornasier, M., and Toscani, G., *A kinetic flocking model with diffusion*, Commun. Math. Phys. **300** (2010), 95–145.

[37] Ermentrout, G. B., *Synchronization in a pool of mutually coupled oscillators with random frequencies*, J. Math. Biol. **22** (1985), 1–9.

[38] E, W., Rykov, Y., and Sinai, Y., *Generalized variational principles, global weak solutions and behavior with random initial data for systems of conservation laws arising in adhesion particle dynamics*, Commun. Math. Phys. **177** (1996), 349–380.

[39] Fornasier, M., Haskovec, J., and Toscani, G., *Fluid dynamic description of flocking via Povzner-Boltzmann equation*, Physica D **240** (2011), 21–31.

[40] Ha, S.-Y., Ha, T. Y., and Kim, J.-H., *On the complete synchronization for the globally coupled Kuramoto model*, Physica D **239** (2010), 1692–1700.

[41] Ha, S.-Y., Huang, F., and Wang, Y., *A global unique solvability of entropic weak solution to the one-dimensional pressureless Euler system with a flocking dissipation*, Submitted.

[42] Ha, S.-Y., Kim, Y., and Li, Z., *Asymptotic synchronous behavior of Kuramoto type models with frustrations*, To appear in Networks and Heterogeneous Media.

[43] _____, *Large-time dynamics of Kuramoto oscillators under the effects of inertia and frustration*, To appear in SIAM J. Applied Dynamical Systems.

[44] Ha, S.-Y., Kang, M.-J., and Kwon, B., *A hydrodynamic model for the interaction of Cucker-Smale particles and incompressible fluid*, To appear in M3AS.

[45] Ha, S.-Y., Kim, H., and Park, J., *Remarks on the complete synchronization of Kuramoto oscillators*, Submitted.

[46] Ha, S.-Y., Lee, K., and Levy, D., *Emergence of time-asymptotic flocking in a stochastic Cucker-Smale system*, Commun. Math. Sci. **7** (2010), 453–469.

[47] Ha, S.-Y., Li, Z., and Xue, X., *Formation of phase-locked states in a population of locally interacting Kuramoto oscillators*, J. Diff. Equat. **255** (2013), 3053–3070.

[48] Ha, S.-Y. and Liu, J.-G., *A simple proof of Cucker-Smale flocking dynamics and mean field limit*, Commun. Math. Sci. **7** (2009), 297–325.

[49] Ha, S.-Y. and Slemrod, M., *Flocking dynamics of a singularly perturbed oscillator chain and the Cucker-Smale system*, J. Dyn. Diff. Equat. **22** (2010), 325–330.

[50] Ha, S.-Y. and Tadmor, E., *From particle to kinetic and hydrodynamic description of flocking*, Kinetic and Related Models **1** (2008), 415–435.

[51] Huang, F. and Wang, Z., *Well posedness for pressureless flow*, Commun. Math. Phys. **222** (2001), 117–146.

[52] Kuramoto, Y., *International symposium on mathematical problems in mathematical physics*, Lecture Notes in Theoretical Physics **30** (1975), 420.

[53] Lancellotti, C., *On the vlasov limit for systems of nonlinearly coupled oscillators without noise*, Transport Theory and Statistical Physics **34** (2005), 523–535.

[54] Leonard, N. E., Lekien Paley, D. A., Sepulchre, R., Fratantoni, D. M., and Davis, R. E., *Collective motion, sensor networks and ocean sampling*, Proc. IEEE **95** (2007), 48–74.

[55] Li, Z. and Ha, S.-Y., *On the Cucker-Smale model with alternating leaders*, To appear in Q. Appl. Math.

[56] Li, Z. and Xue, X., *Cucker-Smale flocking under rooted leadership with fixed and switching topologies*, SIAM J. Appl. Math. **70** (2010), 3156–3174.

[57] Lohe, M. A., *Quantum synchronization over quantum networks*, J. Phys. A: Math. Theor. **43** (2010), 465301.

[58] ———, *Non-abelian Kuramoto model and synchronization*, J. Phys. A: Math. Theor. **42** (2009), 395101–395126.

[59] Motsch, S. and Tadmor, E., *A new model for self-organized dynamics and its flocking behavior*, J. Stat. Phys. **144** (2011), 923–947.

[60] Perea, L., Gómez, G., and Elosegui, P., *Extension of the Cucker-Smale control law to space flight formation*, J. Guidance, Control and Dynamics **32** (2009), 526–536.

[61] Park, J., Kim, H., and Ha, S.-Y., *Cucker-Smale flocking with inter-particle bonding forces*, IEEE Tran. Automatic Control **55** (2010), 2617–2623.

[62] Paley, D. A., Leonard, N. E., Sepulchre, R., Grunbaum, D., and Parrish, J. K., *Oscillator models and collective motion: Spatial patterns in the dynamics of engineered and biological networks*, IEEE Control Systems Magazine **27** (2007), 89–105.

[63] Pikovsky, A., Rosenblum, M., and Kurths, J., *Synchronization: A universal concept in nonlinear sciences*, Cambridge University Press, Cambridge, 2001.

[64] Ren. W. and Beard, R. W., *Consensus seeking in multi-agent systems under dynamically changing interaction topologies*, IEEE Trans. Automatic Control **50** (2005), 655–661.

[65] Saber, R. O., Fax, J. A., and Murray, R. M., *Consensus and cooperation in networked multi- agent systems*, Proc. IEEE **95** (2007), 215–233.

[66] Saber, R. O. and Murray, R. M., *Consensus problems in networks of agents with switching topology and time-delays*, IEEE Trans. Automatic Control **49** (2004), 1520–1533.

[67] Strogatz, S. H., *From Kuramoto to Crawford: exploring the onset of synchronization in populations of coupled oscillator*, Physica D **143** (2000), 1.

[68] Tanner, H. G., Jadbabaie, A., and Pappas, G. J., *Flocking in fixed and switching networks*, IEEE Trans. Automatic Control **52** (2007), 863–868.

[69] van Hemmen, J. L. and Wreszinski, W. F., *Lyapunov function for the Kuramoto model of nonlinearly coupled oscillators*, J. Stat. Phys. **72** (1993), 145–166.

[70] Vicsek, T., Czirok, A., Ben Jacob, E., Cohen, I., and Schochet, O., *Novel type of phase transitions in a system of self-driven particles*, Phys. Rev. Lett. **75** (1995), 1226–1229.

[71] Wang, Z., Huang, F., and Ding, X., *On the Cauchy problem of transportation equation*, Acta Math. Appl. Sinica **13** (1997), 113–122.

[72] Winfree, A. T., *Biological rhythms and the behavior of populations of coupled oscillators*, J. Theor. Biol. **16** (1967), 15–42.

Department of Mathematical Sciences and Research Institute of Mathematics, Seoul National University, Seoul 151-747, Korea (Republic of)

E-mail: syha@snu.ac.kr

Fermionic spectra in integrable models

Rinat Kedem

Abstract. This is a brief review of several algebraic constructions related to generalized fermionic spectra, of the type which appear in integrable quantum spin chains and integrable quantum field theories. We discuss the connection between fermionic formulas for the graded dimensions of the spaces of conformal blocks of WZW theories, quantum cluster algebras, discrete integrable noncommutative evolutions and difference equations.

Mathematics Subject Classification (2010). Primary 81R10; Secondary 82B23, 05E10.

Keywords. Fermionic character formulas, Fusion products, discrete integrable systems.

1. Partition functions in statistical mechanics and conformal field theory

In statistical mechanics, a fundamental object of interest is the partition function, the sum over the space of configurations $\mathcal{C}$ of the Boltzmann weight $e^{-E/kT}$, where E is the energy of a configuration:

$$Z = \sum_{c \in \mathcal{C}} e^{-E(c)/kT}.$$

If the lattice is two-dimensional, the standard test for integrability is the existence of a commuting family of transfer matrices. For a system with periodic boundary conditions, the partition function can be written as the trace of the product of transfer matrices. These are operators on the Hilbert space of a one-dimensional slice of the lattice, which depend on a spectral parameter. The coefficients in expansion of this operator as a series in the spectral parameter gives commuting integrals of the motion, hence integrability.

The Hamiltonian associated with the one-dimensional system is one of those integrals. For example, the six-vertex model transfer matrix is associated with the XXZ Heisenberg spin chain Hamiltonian [2].

The two-dimensional integrable lattice model may undergo a second-order phase transition at certain critical points, in the infinite-lattice limit. At the critical point, the behavior of the model may be described by an effective conformal field theory. The correspondence includes the identification of the critical exponents, given by the conformal dimensions, and the specific heat, given by the central charge of the family of Virasoro representations which make up the Hilbert space of the quantum field theory. It was shown in [26, 30] that the massless part of the spectrum – that is, order 1 excitations in the statistical model, and quasiparticles in the quantum field theory – are also related, and an identification can be made via the partition functions.

Proceedings of the International Congress of Mathematicians, Seoul, 2014

For the lattice model the spectrum is computed from the Bethe ansatz. The "order one", or massless, excitations, which contribute to the conformal partition function, have a "quasi-particle-like" behavior. For small momenta, their energy is a linear function of the momentum. We call this the linearized spectrum.

In conformal field theory, the chiral part of the partition function is given by the specialized characters of certain (not necessarily irreducible) Virasoro modules. The full partition function is a modular invariant sesquilinear combination of these and includes both the chiral and anti-chiral parts.

In the original work of the author and collaborators [26, 30], it was shown that, starting from the Bethe ansatz, linearizing the spectrum and considering only massless excitations above the vacuum, the resulting partition function is equal to the chiral part of the partition function in the conformal field theory, given by Virasoro characters.

The spectrum obtained from the Bethe ansatz is invariably of fermionic nature. At the time when this work was done, few fermionic constructions of Virasoro modules were known. For example, the Feigin-Fuks construction of the most interesting Virasoro modules involves a resolution of the Verma module using the singular vectors, and is thus given by an inclusion-exclusion principle, or (in general) an infinite alternating sum.

A fermionic construction is a basis of the representation given by the action of skew-commuting operators on the vacuum. This gives rise to fermionic statistics: Identical fermions cannot occupy the same point in phase space. One type of generalization of fermionic statistics will be given below. These rules are combinatorial and this is reflected in the expression for the partition function.

There are various ways of constructing bases for any given Virasoro module. The idea of fermionic constructions is that physically meaningful ones reflect the spectrum away from criticality of the integrable quantum field theory. The particle content is some reflection of the form of the primary fields of the conformal field theory. These fields are the generalized fermions. See [24] for a recent example of this.

This note is organized as follows. In Section 2 we will give a few examples of fermionic partition functions related to WZW models. In Section 3, we will relate the general fermionic formulas for graded dimensions of the space of conformal blocks to cluster algebras and quantum cluster algebras. In Section 4, we will show how the integrability of the resulting discrete difference equations (Q-systems and their quantized version) can be used to give difference equations satisfied by generating functions for the graded dimensions of the space of conformal blocks. These are variants of quantum difference Toda equations. These dimensions are the dimensions of the moduli space of holomorphic vector bundles on the sphere with prescribed punctures, and their graded analogs.

2. Generalized fermionic formulas

Let us be specific about what we mean by a generalized fermion and the resulting fermionic formula for the partition function. This phenomenon occurs in finite or infinite systems. The natural finite-dimensional system to start from is a solvable model on the finite lattice with a spectrum governed by the statistics of the Bethe ansatz equations. The eigenstates of the Hamiltonian with a Bethe ansatz solution are in bijection with solutions of a coupled set of algebraic equations. The solutions are specified by a set of integers chosen distinctly on certain finite intervals. We interpret a choice of one integer as a quasi-particle, and a choice

of m integers as m quasi-particles. The corresponding choice of integers is proportional to their momentum, one of the conserved quantities. The fact that the integers should be distinct is what gives them a fermionic nature.

The resulting combinatorics is as follows. We approximate the energy of each quasi-particle as a linear function of the momentum (they are massless) and hence the Bethe integers. This is a reasonable assumption in the conformal, infinite-size limit. Suppose the Hilbert space with m quasi-particles has m integers chosen distinctly from the set $[1, p+m]$ for some integer $p \geq 0$. Let $q = e^{-\alpha}$, where α is the proportionality constant between the energy and the Bethe integers. Then the partition function of m quasi-particles is

$$q^{m(m+1)/2} \left[\begin{array}{c} p+m \\ m \end{array} \right]_q$$

where the q-binomial coefficient is defined as

$$\left[\begin{array}{c} p+m \\ m \end{array} \right]_q = \prod_{j=1}^{m} \frac{(1-q^{p+j})}{(1-q^j)}, \quad p \geq m,$$

and is defined to be zero if $p < m$. The partition function of fermions on the interval $[1, p+m]$ is

$$Z_p(q) = \sum_{m \geq 0} q^{m(m+1)/2} \left[\begin{array}{c} p+m \\ m \end{array} \right]_q.$$

In the limit $p \to \infty$, this formula becomes

$$Z(q) = \sum_{m \geq 0} q^{m(m+1)/2} \frac{1}{\prod_{j=1}^{m}(1-q^j)}.$$

The important characteristics to note are

1. There is a quadratic function of the particle number m in the exponent. This is the "ground state energy" of a fermionic system with m particles.
2. There is a q-binomial coefficient, or its $p \to \infty$ limit, which is just the weighted sum over configurations above the ground state of m fermions.

A slight generalization of fermionic statistics always occurs in the Bethe ansatz solution: The integer p is a linear function of m itself, in addition to the external parameters of the system (such as size).

Moreover, there is in general more than one "color" of quasi-particle, and these have available energy ranges for each color separately. Again, these are free fermions, except for the generalized statistic which hides in the integers p_i for each color: Each p_i is a function of the number m_j of quasi-particles of type j in the system.

Thus a fermionic formula for the (conformal, linearized version of the) partition on the finite lattice might has the form

$$Z(q) = \sum_{\mathbf{m}}^{(1)} q^{Q(\mathbf{m})} \prod_i \left[\begin{array}{c} p_i + m_i \\ m_i \end{array} \right]_q. \tag{2.1}$$

Here, $\mathbf{m} \in \mathbb{Z}_+^k$ for some k, The ground state energy $Q(\mathbf{m})$ is a quadratic function of the particle content $\mathbf{m}$ which depends on the model, as are p_i, which are in general linear functions of $\mathbf{m}$, and may tend to infinity as the size of the system becomes infinite. Here, the

superscript (2.1) on the summation indicates possible restrictions on the summation variables corresponding to symmetry sectors of the Hamiltonian. A finite system will have only a finite number of terms in the summation. Moreover there may be several different symmetry sectors of the Hamiltonian, in which case the partition function can be projected to the different sectors separately.

If the model has a conformal limit (the size of the system is infinite while the spectrum remains linearized, that is, the system remains critical), the partition function – properly normalized and restricted – tends in the limit to the graded character of some Virasoro module. That is, $Z(q)$ is proportional to the trace of q^{L_0} over the module, where L_0 is the grading element of the Virasoro algebra. A conformal field theory is built out of such modules. This gives a direct connection between the spectrum of the lattice model and the conformal field theory in certain cases.

2.1. The Fock space as a limit of the reduced wedge product. It is well known that the basic representation of the affine algebra $\widehat{\mathfrak{sl}}_n$ can be realized as a quotient of the Fock space of free fermions by a Heisenberg algebra [25]. It is possible to give a finite-dimensional version of this construction [27]. As it is closely connected to the graded tensor product construction introduced below in Section 2.4, we briefly summarize it. This finite-dimensional fermionic space gives – in the inductive limit – the Frenkel Kac construction of the level-1 modules.

Let $V = V(\omega_1) \simeq \mathbb{C}^n$ be the defining representation of $\mathfrak{g} = \mathfrak{sl}_n$, and $V(z) = V \otimes \mathbb{C}[z]$ a representation on which $\mathfrak{g}^- := \mathfrak{g} \otimes \mathbb{C}[t^{-1}] \subset \widehat{\mathfrak{sl}}_n$ acts as $x \otimes f(t^{-1})v = f(z)xv$ with $x \in \mathfrak{g}$, $f(t) \in \mathbb{C}[t]$, and $v \in V(z)$.

Consider the N-fold tensor product

$$V_N(z_1, ..., z_N) = V(z_1) \otimes \cdots \otimes V(z_N) \simeq V^{\otimes N} \otimes \mathbb{C}[z_1, ..., z_N]$$

on which $\mathfrak{g}^-$ acts by the usual co-product:

$$\Delta_{\mathbf{z}}(x \otimes f(t)) = \sum_{i=1}^{N} x_{(i)} f(z_i^{-1})$$

where $x_{(i)}$ indicates x acting on the ith factor in the tensor product. Obviously, this action commutes with the diagonal action of the symmetric group S_N, simultaneously permuting factors in the tensor product and variables z_i.

It also commutes with the action of the negative part of the Heisenberg algebra $\mathcal{H}_-$, acting on the space by multiplication by symmetric polynomials in $z_1, ..., z_N$. (Operators of the form $\mathrm{id} \otimes t^{-n}, n > 0$). Thus, we have three commuting actions. We quotient by the action of the Heisenberg, and project onto the alternating representation of S_N, and the result is called the reduced wedge space. It is a finite dimensional space described explicitly as follows.

The quotient by the Heisenberg action is the quotient of $\mathbb{C}[z_1, ..., z_N]$ by symmetric polynomials of positive degree I_N. That is,

$$V_N[\mathbf{z}]/\mathrm{Im}\mathcal{H}^- = V^{\otimes N} \otimes \mathbb{C}[z_1, ..., z_N]/I_N := V^{\otimes N} \otimes R_N$$

The space R_N is isomorphic to the cohomology ring of the Flag variety and to the regular representation of S_N. In particular, it is finite-dimensional. It is a graded by the homogeneous degree in z_i and the action of the symmetric group preserves the graded components.

Thus,

$$R_N \simeq \bigoplus_{\lambda \vdash N} W_\lambda \otimes M_{\lambda,N},$$

where W_λ are the irreducible representations of S_N and $M_{\lambda,N}$ is a graded multiplicity space. We also have the decomposition

$$V^{\otimes N} \underset{\mathfrak{g}\times S_N}{\simeq} \bigoplus_{\nu \vdash N, l(\nu)\leq n} V(\overline{\nu}) \boxtimes W_\nu$$

where $\overline{\nu}$ is the partition ν stripped of its columns of length n, and $V(\lambda)$ are irreducible finite-dimensional representations of $\mathfrak{g}$.

Taking the tensor product with R_N and projecting onto the alternating representation with respect to the diagonal action of S_N, we identify $\nu = \lambda^t$. Thus, the reduced wedge space is isomorphic to

$$\mathcal{F}_N \simeq \oplus V(\overline{\lambda}) \otimes M_{\lambda^t,N},$$

where λ^t is the transpose of λ. The hilbert polynomial of $M_{\lambda^t,N}$ is a Kostka polynomial. In the limit as $N \to \infty$, the properly normalized coefficient of $V(\lambda)$ is a character of the W-algebra, which is the centralizer of $\mathfrak{g}$ acting on the level-1 module of $\widehat{\mathfrak{g}}$, and the character of F_N tends to the character of the basic representation of the affine algebra. Thus the reduced wedge product is a truncation of this space, a Demazure module.

We will give fermionic formulas for the generalizations of this Kostka polynomial below.

2.2. The Hilbert space of the generalized Heisenberg model. We now give a very general setting which gives rise to fermionic partition functions. The wedge space in the previous section is a special case of this construction.

The fermionic formula of the type (2.1) appears in particular in the generalized Heisenberg spin chain with periodic boundary conditions. This is a quantum spin chain, whose Hamiltonian is derived via the R-matrix which intertwines tensor products of Yangian modules $Y(\mathfrak{g})$. The simplest case of this is known as the XXX spin chain, which was the subject of Bethe's original ansatz [4].

To define this spin chain, choose a the following data:

1. Any finite-dimensional Yangian module V_0. This is known as the auxiliary space.
2. A sequence of N Yangian modules $\{V_1, ..., V_N\}$ of KR-type (see below).

The choice of non-isomorphic representations $V_i, i > 0$ is the anisotropy of the model.

Let $R_{ij} : V_i \otimes V_j \mapsto V_j \otimes V_i$ be the intertwiner of finite-dimensional representations, known as (a rational) R-matrix. For generic spectral parameters, the tensor product is irreducicble and R is unique, up to scalar multiple. The transfer matrix of the generalized anisotropic Heisenberg model with periodic boundary conditions is the trace over V_0 of the matrix $M = R_{0,1}R_{0,2}\cdots R_{0,N}$. The transfer matrix T_{V_0} is an operator on the space $V_1 \otimes V_2 \otimes \cdots \otimes V_N$, which is the Hilbert space of the spin chain, also known as the quantum space.

Since the R-matrix satisfies the Yang-Baxter equation, it follows easily that the transfer matrices corresponding to different auxiliary spaces commute. Expanding the transfer matrix as a series in in the spectral parameter of V_0, each of the coefficients in the expansion – an element in an algebra acting on the Hilbert space – commutes with the other coefficients. These coefficients therefore form a family of commuting integrals of motion. The spin chain is a quantum integrable system. The quantum spin chain Hamiltonian is one of the integrals.

This model has a Bethe ansatz solution, at least when the modules $\{V_i\}$ are of Kirillov-Reshetikhin (KR)-type [5, 31]. Such modules are parameterized by a highest weight with respect to the Cartan subalgebra of $\mathfrak{g} \subset Y(\mathfrak{g})$ and a spectral parameter. The highest weight of a KR-module is a multiple of one of the fundamental weights of $\mathfrak{g}$.

The eigenvectors and eigenvalues of the Hamiltonian are given by solutions of the Bethe equations. Solutions are parameterized in terms of sets of distinct integers in the same manner described above. The linearized spectrum is proportional to the sum of these integers.

For this particular model, there is an arbitrary number of quasi-particle species or "colors" for each root of the Lie algebra $\mathfrak{g}$, which obey generalized fermionic statistics. The statistics depends only on the Cartan matrix and the highest weights of $\{V_i\}$. We will write down this function explicitly, as it is key to the rest of the paper (we restrict our attention here to simply-laced $\mathfrak{g}$ here for simplicity; The other cases are explained in [1, 18, 23]).

The Hilbert space of the model is $V_1 \otimes \cdots \otimes V_N$, so its dimension is $\prod_i |V_i|$. (The ordering of these representations does not effect the spectrum.) The partition function of the linearized spectrum gives a graded version of this dimension, and we will provide a representation theoretical interpretation of this grading.

Let $\lambda_1, ..., \lambda_N$ be the highest weights of $V_1, ..., V_N$ respectively. Each λ_i is a multiple of one of the fundamental weights, and therefore the choice of highest weights is parameterized by a multi-partition

$$\boldsymbol{\nu} = (\nu^{(1)}, \dots, \nu^{(r)}), \quad \nu^{(a)} \vdash n_a, \tag{2.2}$$

where the non-negative integers $\mathbf{n} = (n_1, ..., n_r)$ are defined by $\sum_{i=1}^{N} \lambda_i = \sum_{a=1}^{r} n_a \omega_a$ and r is the rank of the algebra.

Define a set of integers $\mathbf{m} = (m_1, ..., m_r)$ as follows:

$$C\mathbf{m} = \mathbf{n} - \boldsymbol{\ell}, \quad \ell_a = \langle \alpha_a, \lambda \rangle.$$

for any choice of a dominant integral weight λ such that $\mathbf{m} \in \mathbb{Z}_+^r$. The evaluation of the character $\mathrm{ch}_{\mathbf{z}} V(\lambda)$ of the irreducible $\mathfrak{g}$-module $V(\lambda)$ at $\mathbf{z} = (1, ..., 1)$ is the dimension of $V(\lambda)$.

Theorem 2.1. *The linearized partition function $Z_{\boldsymbol{\nu}}(q)$ is the evaluation at $\mathbf{z} = (1, ..., 1)$ of*

$$M_{\boldsymbol{\nu}}(q; \mathbf{z}) = \sum_{\lambda} M_{\boldsymbol{\nu},\lambda}(q) \mathrm{ch}_{\mathbf{z}}(V_\lambda),$$

where

$$M_{\boldsymbol{\nu},\lambda}(q) = \sum_{\boldsymbol{\mu} \vdash \mathbf{m}} q^{Q(\boldsymbol{\mu})} \prod_{a=1}^{r} \prod_{j \geq 1} \begin{bmatrix} p_j^{(a)} + \mu_j^{(a)} - \mu_{j+1}^{(a)} \\ \mu_j^{(a)} - \mu_{j+1}^{(a)} \end{bmatrix}_q. \tag{2.3}$$

The sum extends over all multipartitions $\mu^{(a)}$ of m_a, and

- *The integers $p^{(a)}$ are the sum over the first j rows of the integer sequence $\pi^{(a)} = \nu^{(a)} - \sum_b C_{a,b} \mu^{(b)}$;*
- *The quadratic function in the exponent is*

$$Q(\boldsymbol{\mu}) = \frac{1}{2} \sum_{a,b=1}^{r} \sum_{i \geq 1} \mu_i^{(a)} C_{a,b} \mu_i^{(b)}.$$

Theorem 2.2 (Combinatorial Kirillov-Reshetikhin conjecture, [18]). *The sets of Bethe ansatz integers correctly count the dimension of the Hilbert space of the anisotropic Heisenberg model.*

That is, when evaluated at $q = 1$, Equation (2.3) gives an expression for the dimension of the space of $\mathfrak{g}$-linear homomorphisms from the tensor product of KR-modules to the irreducible representation $V(\lambda)$. This was known as the Kirillov-Reshetikhin conjecture [23].

Remark 2.3. The sum in (2.3) is known as the "M-sum" in the language of [23]. There is a similar sum called the "N-sum", where the definition of the q-binomial coefficient is continued to values of $p < 0$ by

$$\left[\begin{array}{c} p+m \\ m \end{array} \right]_q = \frac{(q^{p+1};q)_\infty (q^{m+1};q)_\infty}{(q;q)_\infty (q^{p+m+1};q)_\infty}, \qquad (a;q)_\infty = \prod_{i \geq 0} (1 - aq^i).$$

The fact that $N(q) = M(q)$ is highly non-trivial; it was first conjectured by [23], who showed that the N-sum gave the correct dimension of the tensor product. It was later proven in [18, 19] and shown to be closely tied with the Laurent property [15] of the quantum cluster algebra [3] associated with the Q-system, defined below.

Remark 2.4. The sum (2.3) is a generating function for certain Betti numbers of quiver varieties in special cases [34], see also more recent work giving a geometric context [32].

2.3. Space of conformal blocks in WZW theory. The formula for the linearized partition function of the Heisenberg spin chain is of interest for several reasons.

First, it is known that, in special stabilized infinite limits, its conformal limit is the Wess-Zumino-Witten model at a level which depends on the representations V_i.

Example 2.5. Let $\mathfrak{g} = \mathfrak{sl}_2$, set $V_i = V(k\omega_1)$ for all $i = 1, ..., N$, and consider the limit as the number of representations, $N = 2M$ becomes infinite. Then the limit $M \to \infty$ of the normalized partition function $\lim_{M\to\infty} \widetilde{Z}_{2M}(q; \mathbf{z})$ is the character of the level-k module of the affine Lie algebra $\widehat{\mathfrak{sl}}_2$ with highest weight $k\Lambda_0$.

Example 2.6. Let $\mathfrak{g} = \mathfrak{sl}_n$ and $V_i = V(\omega_1) \simeq \mathbb{C}^n$. Then in the limit $N \to \infty$ the normalized, linearized partition function (2.3) is a Kostka polynomial [27]. In the conformal limit, this gives a character of the W_n-algebra which centralizes the action of $\mathfrak{g}$ when acting on the level-1 modules.

Another important role of the linearized partition function of the Heisenberg model is that it gives the dimension (at $q = 1$) of the space of conformal blocks of WZW theory (when $k \gg 1$ is an integer). This is the dimension of the moduli space of holomorphic vector bundles on a Riemann surface with N punctures, with specified monodromy given by the representations V_i, which are taken to be arbitrary $\widehat{\mathfrak{g}}$-modules induced from KR-modules, localized at distinct points. It is also known as the space of coinvariants.

Remark 2.7. The reason we take k to be integer is that the integrality property of the representations is used in the proof of the statement. The reason we require $k \gg 1$ is that for finite k, one has the Verlinde coefficients rather than the Littlewood Richardson coefficients for multiplicities of the irreducibles in the tensor product of integrable modules affine algebra modules. We did not take this into account in (2.3). A separate conjecture for the

fermionic formula of the linearized partition function of this space can be found in [11]. If k is sufficiently large, the multiplicity is just as a sum of products of Littlewood Richardson coefficients or their generalization.

We have a graded version of the dimension of the moduli space, meaning we keep track of a certain grading or a refinement of the space. It is known that, in special cases, this corresponds to keeping track of the Betti numbers for a certain quiver variety, giving a geometric meaning to the graded dimensions.

2.4. A grading on the tensor product. It is known that the Hilbert space of the Heisenberg model, together with the linearized spectrum of the Hamiltonian, in the limit when the number of representations V_i becomes infinite (taking all $V_i \simeq V$, the defining representation, for example), gives the characters of affine algebras in the limit as the (chiral) conformal partition function. The relevant conformal field theory is the WZW model at level 1.

Remark 2.8. There is also an explicit construction of this infinite-dimensional Hilbert space for the XXZ model, using the quantum affine algebra, using a stabilized semi-infinite tensor product [7]. In this case it is possible to construct the transfer matrix in terms of intertwining operators which gives a direct connection with the deformed primary fields of the conformal field theory.

Moreover, we identify the *dimension* of the Hilbert space of the finite, inhomogeneous Heisenberg model with dimension of the space of conformal blocks (for level k sufficiently large).

These two facts form the motivation for the following definition of a graded tensor product [11]. Whereas there are other definitions of an "energy function" on the tensor product which defines a grading on the tensor product in the case of quantum affine algebras (these correspond to the XXZ model, or the limit $q \to 0$ in the case of the crystal basis), the definition here refers only to the undeformed current algebra.

Remark 2.9. KR-modules are defined for three algebras: For the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$, the Yangian $Y(\mathfrak{g})$, and the current algebra $\widehat{\mathfrak{g}}$. [5, 31]. One of the consequences of the theorems of [1, 18] is that these all have the same structure under restriction to the underlying finite dimensional algebra, $\mathfrak{g}$ or $U_q(\mathfrak{g})$ [29]. Here we use only the current algebra version.

Definition 2.10. Let V be a cyclic $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$-module, defined by the representation π. We define the representation π_ζ on V as follows. Given $x \otimes f(t) \in \mathfrak{g}[t]$ and $w \in V$, $\pi_\zeta(x \otimes f(t))w = \pi(x \otimes f(t+\zeta))w$, for some $\zeta \in \mathbb{C}^*$.

That is, the localization takes place at ζ. We use the shorthand $V(\zeta)$ for the module with the action π_ζ, even though the vector space itself is simply V.

Now pick $V_i(\zeta_i)$ to be KR-modules of $\mathfrak{g}[t]$, with $1 \le i \le N$, with $\zeta_i \neq \zeta_j$ for all $i \neq j$. Let v_i be the cyclic, highest weight vector of V_i. We have $V_i(\zeta_i) = U(\mathfrak{g}[t])v_i$, and the tensor product is also cyclic (as long as the localization parameters are distinct):

$$V_1(\zeta_1) \otimes \cdots \otimes V_N(\zeta_N) = U(\mathfrak{g}[t])v_1 \otimes \cdots \otimes v_N.$$

(The assumption that we have KR-modules is not essential at this point, only that each of the modules V_i is cyclic.)

The algebra $\mathfrak{g} \otimes \mathbb{C}[t]$ is graded by degree in t, and so is its universal enveloping algebra. Let U_i denote the graded component. The action of U_i on the tensor product of cyclic vectors

inherits this filtration, and therefore we have a filtration of the tensor product itself. The associated graded space of this filtered space is called the Feigin-Loktev "fusion" product, $\mathcal{F}^*_{V_1,\dots,V_N}$.

Theorem 2.11 ([1, 18, 29]). *The associated graded space is isomorphic to the tensor product of KR-modules as a $\mathfrak{g}$-module. That is, it is independent of the localization parameters ζ_i.*

For the proof of this theorem, it is essential that V_i are of KR-type. The graded $\mathcal{F}^*_{V_1,\dots,V_N}$ is defined as a quotient space, so in general, its dimension may be greater than the dimension of the tensor product itself. It corresponds to the "collision" of all the points ζ_i.

The theorem about the dimension of this space was proven using a function space realization for the space of conformal blocks (coinvariants), and the use of the Kirillov-Reshetikhin conjecture about the explicit fermionic formula for the dimension of this space [1]. The final step in this proof uses a theorem of [23] and the proof of the "$M = N$" conjecture at $q = 1$ in [18].

Let $\boldsymbol{\nu}$ be the parameterization of the collection of the KR-modules as in (2.2), and let Let $\mathcal{F}^*_{\boldsymbol{\nu},\lambda}[n] = \mathrm{Hom}_{\mathfrak{g}}(\mathcal{F}^*_{\boldsymbol{\nu}}[n], V(\lambda))$. and consider the Hilbert polynomial $\sum_{n\geq 0} q^n \dim \mathcal{F}^*_{\boldsymbol{\nu},\lambda}[n]$.

The following strong version of Theorem 2.11 is proven in [19]:

Theorem 2.12 ([19]). *The Hilbert polynomial of the Feigin-Loktev graded tensor product is equal to the conformal partition function* (2.3)*.*

We will introduce an expression for the partition function (2.3) as a constant term in the product of solutions of the Q-system, a discrete recursion relation, in the next section. At the same time, we will identify the Q-system as a mutation in a cluster algebra, which therefore has a natural q-deformation. The proof of Theorem 2.12 will uses the methods of [18] applied to this quantum cluster algebra.

Remark 2.13. In special cases, the FL graded tensor product is an affine Demazure module [17], which has a grading by the Cartan element d of the affine algebra. This grading is essentially the same as the FL-grading. Therefore we are guaranteed that the appropriate semi-infinite graded tensor product is the full affine algebra module.

By definition [1], the idea of an associated graded space is equivalent to taking all the spectral parameters $\zeta_i \to 0$. The sum over the multipartitions μ in equation (2.3) can be viewed as a sum over all possible desingularizations of this degeneracy (this is evident from the derivation using functional space realization in [1], see also [36]).

3. Difference equations and the fermionic formulas

It was originally observed in the context of the completeness conjecture of the Bethe ansatz, and later by the original attempt at proving the combinatorial Kirillov-Reshetikhin conjecture [23], the fermionic sum $M_{\boldsymbol{\nu}}(q; z)$ in Equation (2.3) is closely related to a difference equation called the Q-system.

Let $\chi_{a,k}$ be the character of the KR-module with highest weight $k\omega_a$, restricted to $\mathfrak{g} \subset \mathfrak{g}[t]$.

Example 3.1. If $\mathfrak{g} = \mathfrak{sl}_n$ then the KR-modules are irreducible under the restriction to $\mathfrak{g}$, and are the modules with "rectangular highest weights". In that case, $\chi_{a,k}$ is a Schur function $S_{(a)^k}(z_1, ..., z_n)$ with $\prod z_i = 1$.

For any Lie algebra, the functions $\chi_{a,k}$ satisfy a simple difference equation: In the case where $\mathfrak{g}$ is simply-laced, this is a two-step recursion relation. Consider the system

$$Q_{k+1}^{(a)}Q_{k-1}^{(a)} = (Q_k^{(a)})^2 - \prod_{b\neq a}(Q_k^{(b)})^{-C_{ab}}. \tag{3.1}$$

(The relation is only slightly more cumbersome for non-simply laced algebras, and has a generalization for $\mathfrak{g}$ an affine algebra.)

A two-step recursion relation has a unique solution given initial data. The natural initial data for the Q-system is

1. The character of the trivial representation is equal to 1, $\chi_{a,0} = 1$. Therefore, set $Q_0^{(a)} = 1$ for all $1 \leq a \leq r$ where r is the rank of the algebra.
2. Identify $Q_1^{(a)}$ with the character of the fundamental KR-modules, $\chi_{a,1}$.

Theorem 3.2 ([35]). *The characters of the Kirillov-Reshetikhin are solutions of the Q-system* (3.1) *with the initial data* (2.1) *and* (2.2).

The Q-system is a specialization of the T-system, satisfied by the transfer matrices of the XXZ spin chain, or by the q-characters [20] of the KR-modules.

Remark 3.3. For any given quantum spin chain, one can derive Bethe ansatz equations from different functional relations, obtaining a different set of coupled algebraic equations (the Bethe equations). Although it is standard procedure to use Baxter's equation to derive Bethe equations, it is also possible to use the T-system, see e.g. [6]. The resulting equations, their solution and linearized spectrum, take a different form depending on the original functional relation. This reflects the fact that in the degenerate case of a massless spectrum (the critical point) there may be several different descriptions of the spectrum as a quasi-particle spectrum. This degeneracy is resolved when a massive integrable perturbation is considered.

The transfer matrices satisfy T-system relation, a conjecture of Kirillov and Reshetikhin proved (for finite, simply-laced Lie algebras) by Nakajima [35], using the realization of the representation theory of the quantum affine algebra in terms of his quiver varities. The T-system is satisfied by q-characters of the KR-modules [20]. This is the algebraic Kirillov-Reshetikhin conjecture. Nakajima even proved a deformed version of the T-system which holds for twisted tensor products of KR-modules. Theorem 3.2 follows from this work.

The relation of the Q-system to the multiplicity formulas starts as follows. Recall the definition of the "N-sum" in Remark 2.3. Then there is a constant term identity for the N-sum in terms of solutions of the Q-system. Define

$$Z_{\boldsymbol{\nu},\lambda}(\mathbf{Q}_0,\mathbf{Q}_1)^{(k)} = \prod_{a=1}^{r} Q_1^{(a)}(Q_0^{(a)})^{-1}\left(\prod_{i\geq 1}(Q_i^{(a)})^{\nu_i^{(a)}-\nu_{i+1}^{(a)}}\right)(Q_k^{(a)}(Q_{k+1}^{(a)})^{-1})^{\langle\alpha_a,\lambda\rangle+1}, \tag{3.2}$$

where $Q_i^{(a)}$ are solutions of the Q-system (3.1). Define $\langle Z\rangle$ to be the constant term of Z in $\{Q_1^{(a)}\}_a$, evaluated at $\{Q_0^{(a)} = 1\}_a$.

Theorem 3.4. *Let*

$$N_{\boldsymbol{\nu},\lambda}^{(k)}(1) = \langle Z_{\boldsymbol{\nu},\lambda}(\mathbf{Q}_0,\mathbf{Q}_1)^{(k)}\rangle.$$

Then there exists an integer J such that whenever $k > J$, $N_{\boldsymbol{\nu},\lambda}^{(k)}(1)$ is independent of k, and is equal to $N_{\boldsymbol{\nu},\lambda}(1)$.

The proof is by induction, using direct computation, starting from the fermionic formula (2.3).

We still need to show that $N = M$, however. Moreover, we need an identity for the q-multiplicities themselves. For this, we do not need to use the representation theoretical interpretation of the Q-system. Instead, we will use the Laurent property of cluster algebras.

3.1. Q-systems as mutations in a cluster algebra. Here, we give an interpretation of the variables $Q_k^{(a)}$ as cluster variables in a cluster algebra.

A cluster algebra is the commutative algebra generated by the union of *cluster variables*, defined recursively. It was originally introduced by Fomin and Zelevinsky [15] in a representation theoretical context, but has been shown to have applications far beyond the original motivation. We refer to Fomin's ICM lecture notes for a good overview [14].

We use only the simplest version. Let B be a skew symmetric $n \times n$ integer matrix (equivalently, a quiver with no 1- or 2- cycles), called the exchange matrix. Vertices of the quiver are numbered from 1 to n and the integer B_{ij} is the number of arrows from j to i. Let $\mathbf{x} = (x_1, ..., x_n)$ be formal (commutative) variables associated with the vertices. Fix $1 \leq j \leq n$ and define x'_j

$$x'_j = \frac{\prod\limits_{i:j\to i} x_i + \prod\limits_{i:i\to j} x_i}{x_j}. \tag{3.3}$$

This is called a mutation of $\mathbf{x}$ in the direction j, denoted by the operation μ_j. If $i \neq j$, $\mu_j(x_i) = x_i$. However, the quiver itself changes under the mutation as follows:

- For any sequence $i \to j \to k$, add an arrow $i \to k$.
- Reverse any arrows incident to i.
- Erase any resulting 2-cycles.

The collection of generators of the cluster algebra is the result of all possible sequences of mutations of $\mathbf{x}$. The pair $(B, \mathbf{x})$ is called the seed data.

Any Q-system (that is, Equation (3.1) and its generalizations), can be shown to be a mutation in a cluster algebra [28]. In the case of (3.1), it is the cluster algebra defined by the seed data $(\mathbf{x}_0, B)$ where

$$B = \begin{pmatrix} 0 & -C \\ C & 0 \end{pmatrix}, \quad \mathbf{x}_0 = (Q_0^{(1)}, \dots, Q_0^{(r)}; Q_1^{(1)}, \dots, Q_1^{(r)}). \tag{3.4}$$

Note that we do not impose $Q_0^{(a)} = 1$ at this stage.

The Q-system equations are a special subset of the mutations of the cluster algebra[1] . They can be shown to be the equations a discrete integrable system [9], with the integrals of motion given by those of the Toda system [22]. This is due to the existence of an integrable Poisson structure compatible with the cluster algebra structure. Such a Poisson structure can always be deformed to give a quantum system, which in the case of cluster algebras is called a quantum cluster algebra [3, 13] (see below).

[1] Although traditionally, the coefficients in a cluster algebra are taken to be $+1$, we keep the minus sign in the current context. This can be dealt with by (1) renormalizing the Q-variables or (2) introducing coefficients [8]. However this is irrelevant in the current context.

Any cluster algebra (and a much larger class of discrete rational evolution equations) can be shown to have a *Laurent property*. The transformation (3.3) is a rational transformation. Although it is obvious after a single mutation, it is not at all obvious after several steps of mutations that the rational function is, in fact, a Laurent polynomial in the seed data, because the term in the denominator is itself a polynomial in the initial seed data.

Theorem 3.5 (Laurent property [16]). *Any cluster variable in a cluster algebra is a Laurent polynomial in the cluster variables of any other seed in the cluster algebra.*

Taking the Q-system with the initial seed data consistent with the character interpretation, this implies the following:

Theorem 3.6 ([8]). *Any cluster variable (not just solutions of the Q-system) in the cluster algebra with seed data* (3.4) *is a* polynomial *in the variables $(Q_1^{(1)}, \dots, Q_1^{(r)})$ after evaluation at $Q_0^{(a)} = 1$.*

Proof. This is a consequence of the Laurent phenomenon and the fact that the right hand side (the numerator in the exchange relation) of (3.1) vanishes at $k = 0$.

We illustrate this for the case of $\mathfrak{g} = \mathfrak{sl}_2$. The generalization is clear. Let x be a cluster variable in the cluster algebra. Then $x(Q_{-1}; Q_0) = Q_0^{-m} \sum_{n\in\mathbb{Z}} p_n(Q_0) Q_1^n$, where p_n is a polynomial. Performing the exchange $Q_1 = N(Q_0)/Q_{-1}$, we have $x = Q_0^{-m} \sum_{n\in\mathbb{Z}} p_n(Q_0)$ $Q_1^{-n} N(Q_0)^n$. If $n < 0$, since $N(Q_0)^n$ in the denominator is a polynomial, it must cancel with the term $p_n(Q_0)$ in the numerator, because the result must be a Laurent polynomial. That is, for any $n < 0$, $p_n(Q_0)$ is divisible by $N(Q_0)$. (Up until this point, the argument holds for any bi-partite cluster algebra of any rank.) Therefore, $p_n(1) = 0$ for any $n < 0$. Therefore, the cluster variable x under this evaluation has only terms Q_1^n with $n \geq 0$. The generalization to arbitrary rank relies on the identical argument: All variables and indices should be changed to multi-variables and multi-indices. □

When applied to the Q-system, the theorem implies that the KR-modules are generated as Groethendieck ring by the fundamental KR-modules.

The importance of polynomiality is in the proof of the "$M = N$ conjecture" [23] which is the final step in the proof of the combinatorial KR-conjecture [18, 29] and hence the Feigin-Loktev conjectures [1].

Theorem 3.7 ([18]). *The constant term in $Q_1^{(a)}$ of $Z_{\boldsymbol{\nu},\lambda}(\mathbf{Q}_0 = 1, \mathbf{Q}_1)$ has no contributions from terms in the summation in which any of the integers $p_{a,i} < 0$. That is, $M_{\boldsymbol{\nu},\lambda}(1) = N_{\boldsymbol{\nu},\lambda}(1)$.*

The Laurent phenomenon and the polynomiality theorem generalize to the quantum Q-system.

3.2. The quantum Q-system from quantum cluster algebras. We are interested in the q-graded version of Theorem 3.7. This is obtained by using a q-deformation of the Q-system. There is a constant term identity for the graded partition function (2.3) in terms of the solutions of the quantum Q-system. Aside from enabling us to prove that "$N(q) = M(q)$", it gives yet another interpretation of the grading of the multiplicity.

Given any skew-symmetric exchange matrix, one can define a quantum cluster algebra [3, 13], a deformation of the compatible Poisson structure of the cluster algebra [21]. A

quantum cluster algbra is a non-commutative algebra generated by the seed data obeying q-commutation relations, together with all its mutations. The combinatorial data is the same as in the classical case, and the exchange matrix is still the same matrix B.

Performing this deformation for the cluster algebra of the Q-system, one obtains a quantum Q-system:

$$t^{\Lambda_{a,a}}\mathcal{Q}_{k+1}^{(a)}\mathcal{Q}_{k-1}^{(a)} = (\mathcal{Q}_k^{(a)})^2 - \prod_{b\neq a}(\mathcal{Q}_k^{(a)})^{-C_{ab}}, \tag{3.5}$$

where $\mathcal{Q}_k^{(a)}$ generate a non-commutative algebra defined by (3.5) and the commutation relations

$$\mathcal{Q}_n^{(a)}\mathcal{Q}_{n+1}^{(b)} = t^{\Lambda_{ab}}\mathcal{Q}_{n+1}^{(b)}\mathcal{Q}_n^{(a)}, \tag{3.6}$$

with $\Lambda = |C|C^{-1}$. The variables $\{\mathcal{Q}_n^{(1)}, ..., \mathcal{Q}_n^{(r)}\}$ commute.

We will eventually identify $q = t^{-|C|}$ in our derivation of the M-sums below.

Given initial seed data $\mathbf{x}_0 = (\mathcal{Q}_0^{(a)}, \mathcal{Q}_1^{(a)})_{a\in[1,r]}$, any cluster variable can be expressed as a Laurent polynomial in the initial seed data, with coefficients in $\mathbb{Z}[t, t^{-1}]$ (the Laurent phenomenon for quantum cluster algebras was proven in [3]). Therefore, any Laurent polynomial M of cluster variables can be expressed as a Laurent polynomial in terms of any initial cluster seed, for example, $\{\mathcal{Q}_0^{(a)}, \mathcal{Q}_1^{(a)}\}$. By using the commutation relations (3.6), this Laurent polynomial can be written in a normal ordered form, as a finite sum

$$M = \sum_{\mathbf{n},\mathbf{m}\in\mathbb{Z}^r}\prod_{a=1}^{r}(\mathcal{Q}_0^{(a)})^{m_a}\prod_{b=1}^{r}(\mathcal{Q}_1^{(b)})^{n_b} f_{\mathbf{m},\mathbf{n}}(t) \tag{3.7}$$

where $f_{\mathbf{n},\mathbf{m}}(t) \in \mathbb{Z}[t, t^{-1}]$.

We define the analogue of "a constant term" identity in the quantum case by taking the constant term of this expression in $Q_1^{(a)}$, and by evaluating at $Q_0^{(a)} = 1$.

Definition 3.8. Given a Laurent polynomial M in $\{\mathcal{Q}_0^{(a)}, \mathcal{Q}_1^{(b)}\}_{a,b}$, define its constant term evaluated at $\mathcal{Q}_0^{(a)} = 1$ by first, defining the coefficients $f_{\mathbf{m},\mathbf{n}}(t) \in \mathbb{Z}[t, t^{-1}]$ as in (3.7), then defining

$$\langle M\rangle = \sum_{\mathbf{m}} f_{\mathbf{m},0}(t).$$

(Note that it is important to perform the evaluation *after* normal ordering the expression, otherwise, we miss out on the t-grading.)

The quantum Laurent property can be shown to imply that for the quantum Q-system, any cluster variable, after evaluation at $\mathcal{Q}_0^{(a)} = 1$, is in fact a polynomial in $\{\mathcal{Q}_1^{(a)}\}_a$ with coefficients in $\mathbb{Z}[t, t^{-1}]$ (the analog of theorem 3.6).

For a given finite sequence $\boldsymbol{\nu}$ and a fixed k, define

$$M_{\boldsymbol{\nu},\lambda}^{(k)} = \prod_{a=1}^{r}\left(\mathcal{Q}_1^{(a)}(\mathcal{Q}_0^{(b)})^{-1}\right)\overrightarrow{\prod_{i\geq 1}}\prod_{a=1}^{r}(\mathcal{Q}_i^{(a)})^{\nu_i^{(a)}-\nu_{i+1}^{(a)}}\prod_{a=1}^{r}(\mathcal{Q}_k^{(a)}(\mathcal{Q}_{k+1}^{(a)})^{-1})^{\langle\omega_a,\lambda\rangle+1} \tag{3.8}$$

Again, when k is sufficiently large, $\langle M_{\boldsymbol{\nu},\lambda}^{(k)}\rangle$ is independent of k.

Upon multiplying by an appropriate power of q and identifying the deformation parameter t of the cluster algebra as $q = t^{-|C|}$, we have

Theorem 3.9 (Constant term identity [19]).

$$M_{\boldsymbol{\nu},\lambda}(q^{-1}) = q^{h(\boldsymbol{\nu},\lambda)}\langle M_{\boldsymbol{\nu},\lambda}\rangle.$$

for k sufficiently large. Here the normalization factor is

$$h(\boldsymbol{\nu},\lambda) = -\frac{1}{2}\sum_{a,b=1}^{r}\sum_{i\geq 1}\nu_i^{(a)}C^{-1}_{ab}\nu_i^{(b)} - \frac{1}{2}\sum_{a=1}^{r}C^{-1}_{aa}\ell_a - \sum_{a,b=1}^{r}C^{-1}_{ab}\nu_1^{(b)}.$$

The polynomiality property, which follows from the Laurent property for the quantum Q-system, implies

Lemma 3.10 ([19]). *The cluster variables in the quantum cluster algebra corresponding to the Q-system, after normal ordering and evaluation at $\mathcal{Q}_0^{(a)} = 1$ for all a, are polynomials in $\{\mathcal{Q}_1^{(a)}\}_a$.*

Thus, we have the graded version of the $M = N$ identity:

Theorem 3.11 ([19]). *In the summation in Equation* (2.3)*, terms with $p_{a,i} < 0$ do not contribute to the sum in the q-graded version of the identity. That is, $M_{\boldsymbol{\nu},\lambda}(q) = N_{\boldsymbol{\nu},\lambda}(q)$.*

4. Difference equations

So far, we have said nothing about the integrability of the Q-system and its Q-deformed version. But in fact, this is a two-step recursion relation of rank r, and it has r integrals of the discrete evolution (which are in involution with each other with respect to the Poisson structure of the cluster algebra, or the commutation relations of the quantum cluster algebra). In type A for example, the solutions $Q_k^{(a)}$ satisfy linear recursion relations with $r+2$ terms, and with coefficients which are integrals of the motion (or constants).

These integrals of the motion can be used to find differential/difference equations satisfied by generating functions for partition functions (characters of graded tensor products). This derivation is analogous to the construction of the Whittaker functions, which are solutions of the quantum Toda equations in the case of classical Lie groups, where the integrals of the motion are the Casimir elements of the algebra [33]. More recently there has been a certain interest in the so-called Gaiotto vector, which is the analog of the Whittaker vector for Virasoro algebras, or some degenerate version thereof.

Since certain stabilized limits of the graded tensor products tend to various Virasoro modules or integrable affine algebra modules, it is useful to first write these equations for the finite tensor product. The result are Toda-like equations satisfied by the generating function (the relation of fermionic character formulas and Toda equations was noted in, e.g. [12]). This can be used to derive difference equations satisfied by the stabilized limits of the graded tensor products, and even solve them in special cases. One can obtain the character formulas of Feigin and Stoyanovskii, or of spinon type, by analyzing these difference equations.

The analog of the Whittaker function in the case of the graded tensor product is the generating function [10]

$$G(q;\mathbf{z},\mathbf{y}) = \sum_{\boldsymbol{\nu},\lambda} q^{f_1(\boldsymbol{\nu})}\mathrm{ch}_{\mathbf{z}}V(\lambda)M_{\boldsymbol{\nu},\lambda}(q)\prod_{a,i}(y_{a,i})^{\nu_i^{(a)}-\nu_{i+1}^{(a)}},$$

with $f_1(\boldsymbol{\nu}) = \frac{1}{2}\sum_{a,b,i}\nu_i^{(a)}C^{-1}_{a,b}\nu_i^{(b)} + \sum_{a,b}C^{-1}_{ab}\nu_1^{(b)}$. Using the factorization formula of Section 3.2, this has a very simple form:

$$G(q;\mathbf{z},\mathbf{y}) = \langle \prod_{a=1}^{r} \mathcal{Q}_1^{(a)}(\mathcal{Q}_0^{(a)})^{-1} \prod_{a=1}^{r}\prod_{j\geq 1}(1-y_j^{(a)}Q_j^{(a)})^{-1}\tau(\mathbf{z})\rangle$$

where

$$\tau(\mathbf{z}) = \lim_{k\to\infty}\sum_{\lambda} q^{f_2(\lambda)}\mathrm{ch}_{\mathbf{z}}V(\lambda)\prod_{a=1}^{r}(Q_k^{(a)}(Q_{k+1}^{(a)})^{-1})^{\langle\alpha_a,\lambda\rangle+1},$$

where $f_2(\lambda) = -\frac{1}{2}\sum_a C^{-1}_{aa}\ell_a$.

We claim that $\tau(\mathbf{z})$ plays the role of the Whittaker vector, with the role of the Casimir elements played by the discrete integrals of motion of the Q-system. These act by scalars on this function, whereas they act as q-difference operators on the product of $\mathcal{Q}$'s to the left. This is the origin of the difference equations satisfied by the partition functions.

5. Summary

We reviewed here the role played by the fermionic formulas for the characters of graded tensor products of current algebra modules, their close connection with discrete integrable equations called Q-systems, and their q-deformations. In the process, we used the formulation of these systems in terms of (quantum) cluster algebras, and found that the grading coming from the affine algebra action on the tensor product can be identified with the grading coming from the q-deformation of the cluster algebra, hence from the natural Poisson structure satisfied by the cluster algebra variables. The resulting graded tensor products, in the stabilized, semi-infinite limit, give a constrution of affine algebra or Virasoro modules. The integrability of the quantum Q-system is closely connected with difference equations satisfied by the characters of these modules.

Acknowledgements. The author would like to express her gratitude to her advisor B.M. McCoy, with whom the original formulation of the physical interpretation of conformal partition functions and their fermionic expressions was made; To M. Jimbo, T. Miwa for their patient mentorship over many years; to them as well as B. Feigin and E. Ardonne, with whom she first worked on fusion products; and most especially to P. Di Francesco for an ongoing illuminating collaboration. The author thanks S. Fomin, H. Nakajima, N. Reshetikhin for their kindness and for helpful discussions related to this work over the years. This work has been supported by the National Science foundation through several grants, most recently NSF DMS grant 1100929.

References

[1] Eddy Ardonne and Rinat Kedem, *Fusion products of Kirillov-Reshetikhin modules and fermionic multiplicity formulas*, J. Algebra **308**(1) (2007), 270–294.

[2] Rodney J. Baxter, *Exactly solved models in statistical mechanics*, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London, 1989. Reprint of the 1982 original.

[3] Arkady Berenstein and Andrei Zelevinsky, *Quantum cluster algebras*, Adv. Math., **195**(2) (2005), 405–455.

[4] H. Bethe, *Theorie der metalle i. eigenwerte und eigenfunktionen der linearen atomkette*, Zeitschrift für Physik (71) (1931), 2005–2006.

[5] Vyjayanthi Chari, *On the fermionic formula and the Kirillov-Reshetikhin conjecture*, Internat. Math. Res. Notices (12) (2001), 629–654.

[6] Srinandan Dasmahapatra, Rinat Kedem, and Barry M. McCoy, *Spectrum and completeness of the three state superintegrable chiral potts model*, Nucl.Phys. **B396** (1993), 506–540.

[7] Brian Davies, Omar Foda, Michio Jimbo, Tetsuji Miwa, and Atsushi Nakayashiki, *Diagonalization of the XXZ Hamiltonian by vertex operators*, Comm. Math. Phys. **151**(1) (1993), 89–153.

[8] Philippe Di Francesco and Rinat Kedem, *Q-systems as cluster algebras. II. Cartan matrix of finite type and the polynomial property*, Lett. Math. Phys. **89**(3) (2009), 183–216.

[9] ———, *Q-system cluster algebras, paths and total positivity*, SIGMA Symmetry Integrability Geom. Methods Appl., 6:Paper 014, 36, 2010.

[10] Philippe Di Francesco and Rinat Kedem. unpublished, 2014.

[11] B. Feigin and S. Loktev, *On generalized Kostka polynomials and the quantum Verlinde rule*, In Differential topology, infinite-dimensional Lie algebras, and applications, volume 194 of Amer. Math. Soc. Transl. Ser. 2, pp. 61–79. Amer. Math. Soc., Providence, RI, 1999.

[12] Boris Feigin, Evgeny Feigin, Michio Jimbo, Tetsuji Miwa, and Evgeny Mukhin, *Fermionic formulas for eigenfunctions of the difference Toda Hamiltonian*, Lett. Math. Phys. **88**(1-3) (2009), 39–77.

[13] V. V. Fock and A. B. Goncharov, *Cluster X-varieties, amalgamation, and Poisson-Lie groups*, In Algebraic geometry and number theory, volume 253 of Progr. Math., pp. 27–68. Birkhäuser Boston, Boston, MA, 2006.

[14] Sergey Fomin, *Total positivity and cluster algebras*, In Proceedings of the International Congress of Mathematicians. Volume II, pp. 125–145. Hindustan Book Agency, New Delhi, 2010.

[15] Sergey Fomin and Andrei Zelevinsky, *Cluster algebras. I. Foundations*, J. Amer. Math. Soc. **15**(2) (2002), 497–529 (electronic).

[16] ———, *The Laurent phenomenon*, Adv. in Appl. Math. **28**(2) (2002), 119–144.

[17] G. Fourier and P. Littelmann, *Weyl modules, Demazure modules, KR-modules, crystals, fusion products and limit constructions*, Adv. Math. **211**(2) (2007), 566–593.

[18] Philippe Di Francesco and Rinat Kedem, *Proof of the combinatorial Kirillov-Reshetikhin conjecture*, Int. Math. Res. Not. IMRN, (7):Art. ID rnn006, 57, 2008.

[19] ———, *Quantum cluster algebras and fusion products*, Int. Math. Res. Not. IMRN, (doi: 10.1093/imrn/rnt004), 2013.

[20] Edward Frenkel and Nicolai Reshetikhin, *The q-characters of representations of quantum affine algebras and deformations of w-algebras*, In Recent developments in quantum affine algebras and related topics (Raleigh, NC, 1998), volume 248 of *Contemp. Math.*, pp. 163–205. Amer. Math. Soc., Providence, RI, 1999.

[21] Michael Gekhtman, Michael Shapiro, and Alek Vainshtein, *Cluster algebras and Poisson geometry*, Mosc. Math. J., 3(3):899–934, 1199, 2003. {Dedicated to Vladimir Igorevich Arnold on the occasion of his 65th birthday}.

[22] ———, *Cluster algebras and Poisson geometry*, volume 167 of Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 2010.

[23] G. Hatayama, A. Kuniba, M. Okado, T. Takagi, and Y. Yamada, *Remarks on fermionic formula*, In Recent developments in quantum affine algebras and related topics (Raleigh, NC, 1998), volume 248 of Contemp. Math., pp. 243–291. Amer. Math. Soc., Providence, RI, 1999.

[24] Michio Jimbo, Tetsuji Miwa, and Feodor Smirnov, *Fermions acting on quasi-local operators in the XXZ model*, In Symmetries, integrable systems and representations, volume 40 of Springer Proc. Math. Stat., pp. 243–261. Springer, Heidelberg, 2013.

[25] V. G. Kac and A. K. Raina, *Bombay lectures on highest weight representations of infinite-dimensional Lie algebras*, volume 2 of Advanced Series in Mathematical Physics, World Scientific Publishing Co., Inc., Teaneck, NJ, 1987.

[26] R. Kedem, T. R. Klassen, B. M. McCoy, and E. Melzer, *Fermionic sum representations for conformal field theory characters*, Phys. Lett. B **307**(1-2) (1993), 68–76.

[27] Rinat Kedem, *Fusion products, cohomology of* GL_N *flag manifolds, and Kostka polynomials*, Int. Math. Res. Not. (25) (2004), 1273–1298.

[28] ———, *Q-systems as cluster algebras*, J. Phys. A **41**(19):194011, 14, 2008.

[29] ———, *A pentagon of identities, graded tensor products, and the Kirillov-Reshetikhin conjecture*, In New trends in quantum integrable systems, pp. 173–193, World Sci. Publ., Hackensack, NJ, 2011.

[30] Rinat Kedem and Barry M. McCoy, *Construction of modular branching functions from Bethe's equations in the* 3*-state Potts chain*, J. Statist. Phys. **71**(5-6) (1993), 865–901.

[31] A. N. Kirillov and N. Yu. Reshetikhin, *Formulas for the multiplicities of the occurrence of irreducible components in the tensor product of representations of simple Lie algebras*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI), **205**(Differentsialnaya Geom. Gruppy Li i Mekh. 13):30–37, **179**, 1993.

[32] Ryosuke Kodera and Katsuyuki Naoi, *Loewy series of Weyl modules and the Poincaré polynomials of quiver varieties*, Publ. Res. Inst. Math. Sci. **48**(3) (2012), 477–500.

[33] Bertram Kostant, *On Whittaker vectors and representation theory*, Invent. Math. **48**(2) (1978), 101–184.

[34] G. Lusztig, *Fermionic form and betti numbers*, ArXiv:math/0005010.

[35] Hiraku Nakajima, *t-analogs of q-characters of Kirillov-Reshetikhin modules of quantum affine algebras*, Represent. Theory **7** (2003), 259–274 (electronic).

[36] A. V. Stoyanovskii and B. L. Feigin, *Functional models of the representations of current algebras, and semi-infinite Schubert cells*, Funktsional. Anal. i Prilozhen. **28**(1):68–90, 96, 1994.

Department of Mathematics MC-382, University of Illinois, Urbana, IL 61801 USA
E-mail: rinat@illinois.edu

Deformed ensembles of random matrices

Sandrine Péché

Abstract. In this article, we review recent results in the study of asymptotic spectral properties of some perturbation of large random matrices. Deformed models have arisen in random matrix theory in Baik, J.; Ben Arous, G.; Péché S. *Phase transition of the largest eigenvalue for nonnull complex sample covariance matrices.* Ann. Probab. 33 (2005), no. 5, 1643-1697. In this review, we consider additive or multiplicative deformations of standard Wigner or sample covariance matrices. We consider the phenomenon of separation of extreme eigenvalues and the question of universality of their asymptotic distribution for random matrices with a non necessarily Gaussian distribution.

Mathematics Subject Classification (2010). Primary 60B20; Secondary 15B52, 62H20.

Keywords. Probability, random matrices, separation of eigenvalues, universality.

1. Introduction

Random matrix theory developed during the last two decades in numerous fields of mathematics and physics. In particular, connections with mathematical statistics, mathematical physics and the KPZ equation, random graphs, combinatorics , number theory and operator algebra theory have now made Random Matrix Theory some kind of paradigm. One reason for this outstanding development is that limiting random matrix quantities have been proved to be universal objects, in the same vein as the Gaussian distribution is universal. Hereafter, we propose to review the known results about some special ensembles of random matrices, known as *deformed random matrices*, as opposed to *standard random matrices.* These models have proved to be relevant to many fields of mathematics, such as statistics, mathematical physics and random graphs as we propose to briefly expose in these proceedings.

1.1. Models of random matrices.

Standard random matrices. There are two fundamental classes of standard random matrices, namely Wigner matrices and sample covariance matrices.

A real symmetric (resp. complex Hermitian) Wigner random matrix is a matrix $W = (W_{ij})$ of size $N \times N$ with independent real (resp. complex) entries above the diagonal W_{ij}, $1 \leq i \leq j \leq N$. These entries have mean 0 ($m = 0$) and have the same variance $\mathbb{E}|W_{ij}|^2 = \sigma^2$. Since the pioneering work of Wigner [58], the spectrum of large Wigner matrices is now quite well known. In particular, denoting by

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$$

Proceedings of the International Congress of Mathematicians, Seoul, 2014

the ordered eigenvalues of $W_N := \frac{1}{\sqrt{N}} W$, the spectral measure $\frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i}$ a.s. converges weakly to the famous semi-circle distribution with density

$$\frac{\sqrt{4\sigma^2 - x^2}}{2\pi\sigma^2} \mathbb{1}_{[-2\sigma,2\sigma]}(x).$$

This is the first demonstration of a *universality* phenomenon, as this property holds true regardless of the details of the distribution of the matrix elements W_{ij}, apart from the variance σ^2.

In addition, provided the fourth moment of the entries is uniformly bounded, the whole spectrum lies in the interval $[-2\sigma, 2\sigma]$ with probability arbitrarily close to 1 as shown in [4]. The local properties of the spectrum of Wigner random matrices are now quite well understood thanks to the fantastic work of Erdös-Schlein-Yau (see [30, 31] and references therein) and Tao-Vu [54]. In particular, it is known (under some assumption on the tail distribution of the matrix elements) that the largest eigenvalue of a standard Wigner matrix has universal fluctuations. More precisely [30, 52, 54] consider the case where the W_{ij}'s have sub-exponential tails:

$$\exists C, c > 0,\ \mathbb{P}(|W_{ij}| \geq x) \leq Ce^{-cx^\eta}, \forall\, 1 \leq i, j \leq N, \tag{1.1}$$

and vanishing third moments $m_3 := \mathbb{E}W_{ij}^3 = 0, \forall i, j = 1, \ldots, N$. One can show that for any integer k, the random vector

$$N^{2/3}(\lambda_1 - 2\sigma, \ldots, \lambda_k - 2\sigma)$$

converges in distribution as $N \to \infty$ to the famous Tracy-Widom distribution for the k largest eigenvalues defined in [55].

Random sample covariance matrices are formed from a matrix $X = (X_{ij})$ of size $N \times p$ with independent real (resp. complex) entries X_{ij}, $1 \leq i \leq N; 1 \leq j \leq p$ such that $\mathbb{E}X_{ij} = 0$ and $\mathbb{E}|X_{ij}|^2 = 1$. In the high dimensional setting, it is assumed that $p/N \to \gamma \in (0, \infty)$. In the large dimension limit, the associated rescaled sample covariance matrix $M_N = \frac{1}{N} XX^*$ exhibits properties which are similar to those of Wigner matrices. Its spectral measure converges to the so-called Marcenko-Pastur [44] distribution with density

$$\frac{1}{2\pi x}\sqrt{(x - u_-)(u_+ - x)}, \quad u_\pm = (1 \pm \sqrt{\gamma})^2.$$

Similarly, under the same moment assumptions on the X_{ij}'s as for the Wigner case, the vector of k largest eigenvalues

$$N^{2/3}\,(\lambda_1 - u_+, \ldots, \lambda_k - u_+)$$

converges in distribution as $N, p \to \infty, p/N \to \gamma$ to the Tracy-Widom distribution.

Deformed ensembles. A *deformation* of a standard random matrix can be more or less understood as the modifications of the distribution of some of the entries of W or X. The set of possible deformations is non exhaustive (one can force some of the entries to be zero such as for sparse matrices) but we here restrict our attention to some particular deformations. More precisely, we consider a matrix A of size N, which can be deterministic or random. We consider the deformed matrices

$$W_N + A \quad \text{or} \quad (I + A)^{1/2} M_N (I + A)^{1/2}.$$

The question is to understand the asymptotic properties of the eigenvalues and eigenvectors of the deformed matrix, knowing that of A and W_N (or M_N). This question actually has a long history in random matrix theory and applications, as we recall hereafter. In full generality, when the rank of the perturbation is $r << N$, the global behavior of the spectrum is not impacted by the deformation matrix A. Indeed, consider the empirical eigenvalue distribution

$$\mu_N := \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i(W_N+A)}.$$

Then μ_N still weakly converges a.s. to the semi-circle distribution. This is an easy consequence of the Weyl interlacing property of eigenvalues. The same result also holds true for sample covariance matrices. Throughout the article, we assume that the eigenvalues of A do not depend on N (unless explicitly stating this dependence). However we allow their multiplicity and thus the rank of A to depend on N. This will ease the exposition, without impacting the generality of the statements.

Deformed Ensembles essentially provide modifications at the edges of the spectrum. Therefore, the rest of the review will essentially be concerned with the behavior of extreme eigenvalues. We consider both convergence and fluctuations of extreme eigenvalues, focusing on the largest. Similar statements can be made for the smallest eigenvalues in the Wigner case and in the sample covariance matrix case when $\gamma > 1$.

1.2. Motivations. The first field of applications comes from mathematical statistics. In [59], Wishart initiated the first rudiments of random matrix theory. Wishart indeed determines the distribution of the covariance matrix XX^* of a p-sample of multivariate normal N-dimensional vectors $\mathcal{N}(0, \Sigma)$, where Σ is a given non negative matrix of size N. The idea beyond was to use the data XX^* to estimate the true covariance Σ. Using central limit theorem arguments, the study of the Gaussian case shall then extend to more general distributions. This idea proved to be fruitful and has led to the definition of multiple statistical procedures (including principal component analysis) in the case where the dimension N is much smaller than the sample size p. However, due to the improvement of storage capacities, usual statistical schemes have now to be considered in the *high dimensional* setting, i.e. when both N and p grow in a comparable way. This is the regime where random matrix theory, in particular *random sample covariance matrices* enters the scene, because usual statistical procedures fail. In a much more general context, this question of reconstructing Σ (partially) from the data can be understood in the following way: the information Σ is blurred into random noise yielding the data matrix XX^*. This question is of interest nowadays in digital communications [26] , finance [18], genetics [46] and can be summarized as follows: can we detect (or estimate some parameters in) the true signal Σ given the data matrix XX^*?

Random matrix theory has also deep connections with mathematical physics, starting with Wigner [58] who was motivated by problems from quantum physics, on the energy levels of heavy nuclei and initiated the investigation of spectra of large random matrices. Since then, limiting distributions of random matrix ensembles have been proved to determine the asymptotic fluctuations around limiting shapes of many random growth models (see e.g. [27] for a recent review). We mention hereafter a basic example making an exact connection between random matrices and growth models.

Consider the following directed percolation model (which is also related to the directed

polymer problem in (1 + 1) dimension): assume that at each of the lattice points $(i, j) \in \mathbb{N}^2$ a random waiting time is attached $w_{i,j} \geq 0$. The waiting times are independent random variables. A path from (i, j) to (i', j') is said to be north-east (N-E) if it has only up $(0, 1)$ or right $(1, 0)$ increments. A passage time from $(1, 1)$ to (N, p) is the sum of waiting times collected along a N-E path from $(1, 1)$ to (N, p). The last passage time from $(1, 1)$ to (N, p), denoted by $L(N, p)$ is then the maximum of passage times over the set of N-E paths. K. Johansson proves the following connection in [37]. Let $\pi_1, \ldots, \pi_N$ be positive real numbers.

In the special case where the $w'_{i,j}s$ are exponential random variables with mean π_i, $L(N, p)$ has the same distribution as the largest eigenvalue of the complex Wishart matrix with covariance $\Sigma = \text{diag}(\pi_1^{-1}, \ldots, \pi_N^{-1})$. However it is believed that the asymptotic distribution of $L(N, p)$ as $N, p \to \infty$ shall be the same for any distribution of the waiting times with subexponential tails (see [7, 16]). The directed percolation model described hereabove has a natural interpretation in terms of a randomly growing Young diagram (see [37]). It is also related (asymptotically) to the description of some models of Totally Asymmetric Simple Exclusion Process -Tasep- which is a non reversible process of interacting particles on $\mathbb{Z}$ (see e.g. [49]). In this model, a configuration of particles on $\mathbb{Z}$ starts with some given (possibly random) initial configuration and obeys the following dynamics. Each particle has its own independent clock, which rings after an exponential waiting time (whose rate possibly depends on the particle) and then resets. Each time its clock rings, the particle attempts to jump to the right neighboring site. Jumps are however allowed only if the neighboring site is free of particles. Obviously, the large time asymptotic behavior of the configuration of particles depends on the initial distribution of particles on $\mathbb{Z}$. The most complete connection between Tasep and (deformed) random matrices has been obtained in [9]. Therein, the authors consider the two-sided Bernoulli initial condition, that is when all the sites to the left (resp. right) of the origin are initially occupied independently with probability ρ_- (resp. ρ_+). Such initial conditions give rise to different interesting phenomenons in Tasep (including shocks for example) depending on ρ_- and ρ_+. The asymptotic behavior of two-sided Tasep can then be characterized using the distribution of the largest eigenvalue of some deformed random matrix ensemble. We refer the reader to [9] for more detail.

It happens that all the above models (LPP, growth models of Young diagram and Tasep) are discrete models which are believed to fall in the so-called KPZ universality class. The KPZ [39] equation describes the evolution of the height of the random surface above x at time t :

$$\frac{\partial h(\vec{x}, t)}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 + \eta(\vec{x}, t) \ ,$$

where $\eta(\vec{x}, t)$ is space-time Gaussian white noise. Here ν, λ, and D are parameters of the model and d is the dimension. It is believed that the height function appropriately rescaled (with universal spatial and temporal exponents) fluctuates around some limiting shape $\bar{h}$ and that the limiting distribution is also universal. In addition, in dimension 1, the limiting distribution shall be that of the largest eigenvalue of a random matrix (deformed or standard). The expected corresponding random matrix ensemble is actually described in [27] depending on $\bar{h}$ and the initial condition only.

Another kind of application would concern linear algebra theory. Given the spectrum of two real symmetric (resp. complex Hermitian) matrices W and A, what can be said about the spectrum of their sum $W + A$? This fantastic problem has been first considered by Weyl [57] in the algebraic setting, and since then by many authors (see the recent review by Knutson and Tao [42] e.g.). This algebraic question turns out to be easier, but very rich,

in the randomized version. One can indeed partially answer this question in the context of (large) random matrix theory, as we now explain.
Consider the deformation of a Wigner matrix W_N by the addition of a deterministic matrix A. This question has been first considered in the seminal article [32]. Therein, Komlós and Füredi consider the rank-one perturbation

$$M_N = W_N + \frac{m}{\sqrt{N}} J,\ J_{ij} = 1, \forall\, 1 \leq i, j \leq N.$$

This model corresponds to a large real symmetric random matrix with non centered entries. Their motivation comes from random graph theory, more precisely the spectral analysis of the adjacency matrix of the Erdös-Rényi graph. Lidskii's theorem (see [1] e.g.) ensures that the asymptotic eigenvalue empirical distribution is unchanged by the perturbation. However [32] prove that the largest eigenvalue λ_1 of M separates from the rest of the spectrum in the large size limit. More precisely, when $m \neq 0$ they show that

$$\frac{\lambda_1 - Nm - \dfrac{\sigma^2}{m}}{\sqrt{N}} \underset{N\to\infty}{\overset{d}{\to}} \mathcal{N}(0, 2\sigma^2).$$

The rest of the spectrum is shown to lie in the interval $[-2\sigma\sqrt{N}, 2\sigma\sqrt{N}]$ with probability arbitrarily close to 1. In the above context, only the behavior of the largest eigenvalue differs from that of a standard random matrix (after rescaling). This is again a consequence of the interlacing property of eigenvalues due to Weyl. One can also note that the "size" of the deformation, being measured by the non trivial eigenvalue $m\sqrt{N}$ of the matrix $\frac{mJ}{\sqrt{N}}$ overwhelms the spectrum of the Wigner matrix W_N. This result, which can be understood using perturbation theory, can actually be refined to the more challenging case where the size of the deformation is of order 1, that is in the same order as the spectral norm of W_N. Depending on the size (and associated eigenvectors) and the rank of the perturbation, the spectrum of a deformed Wigner matrix exhibits various asymptotic properties, which we develop hereafter. For ease of exposition, we focus on Wigner matrices. Similar results can be stated for sample covariance matrices.

2. The Baik-Ben Arous-Péché (BBP) phase transition

The BBP phase transition [5] theorem below establishes the threshold for the size of the deformation at which an eigenvalue separates from the bulk of the spectrum. The BBP phase transition has first been exhibited for complex spiked Wishart ensembles in [5]. Matrices considered therein are complex Wishart matrices with covariance matrix $\Sigma = \mathrm{Id} + A$ for some finite rank positive matrix A. This is some kind of multiplicative (in place of additive) perturbation. Complex deformed Gaussian ensembles (GUE or Wishart) have been first introduced by [19, 20, 36]. Indeed many eigenvalues statistics (such as the joint eigenvalue density and the distribution of the largest eigenvalues e.g.) can be explicitly computed for these ensembles thanks to the Itzykson-Zuber [35]-Harisch-Chandra [34] integral. Results have then extended to much more general ensembles by Baik and Silverstein[6] as we now review.

We focus on the top edge of the spectrum of deformed Wigner random matrices. It is however worth mentioning that the same phenomenon arises at the lower edge. Let W_N be

a standard Wigner matrix s.t.

$$\sup_{1\leq i,j\leq N} \mathbb{E}|H_{ij}|^4 < \infty.$$

We consider a low rank perturbation of a Wigner matrix, whose rank r may depend on N in such a way that $r << N$. Let A be a deterministic matrix of rank r and non trivial eigenvalues $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_r > 0$.

We denote by

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$$

the ordered eigenvalues of the deformed matrix $W_N + A$.

Theorem 2.1 ([5, 6, 22, 33, 47]). *Consider an integer $1 \leq i \leq r$. One has the following phase transition: If $\theta_i \leq \sigma$, $\lambda_i \to 2\sigma$ a.s. If $\theta_i > \sigma$, $\lambda_i \to \theta_i + \dfrac{\sigma^2}{\theta_i}$ a.s.*

We may emphasize that $\theta_i + \dfrac{\sigma^2}{\theta_i} > 2\sigma$ as soon as $\theta_i > \sigma$. Therefore, each eigenvalue θ_i greater than σ, called a *spike*, forces a number of eigenvalues (equal to the multiplicity of θ_i) to separate from the rest of the spectrum. This phenomenon has been named *exact separation* of the eigenvalues after [6]. As a consequence the phenomenon identified by Komlós and Füredi [32] then extends to the whole interval (σ, ∞) for the size of the perturbation. Theorem 2.1 thus shows that the local statistics of eigenvalues of large random matrices at the edges of the spectrum are more sensitive to the distribution of the matrix elements. Modifying e.g. a single diagonal entry of a Wigner matrix can result into an *outlier* in the spectrum, that is an eigenvalue separating from the bulk of the spectrum.

Theorem 2.1 is important to many fields of applied mathematics such as mathematical statistics, mathematical finance or wireless communications. Indeed, in these contexts, consider the perturbation matrix as the signal with significant parameters while the non-perturbed random matrix is the noise. The observation of the data, and in particular of the largest eigenvalues λ_i can give access to the significant parameters (the θ_i's for instance). We however emphasize that this is possible only if θ_i is large enough. In a few words, the above theorem thus states that re-construction of a signal is possible provided the power of the signal is strong enough. However for the construction of statistical tests, the convergence of the largest eigenvalues outside the support of the semi-circle distribution is not sufficient. The fluctuations of the largest eigenvalues are needed. This is the object of the next subsection.

3. Fluctuations of the largest eigenvalues

Fluctuations of eigenvalues separating or not from the bulk have been first computed in [5] for complex Wishart matrices and [47] for the deformed GUE. This has then been extended to more general ensembles of Wigner matrices by Capitaine, Donati-Martin and Féral [22, 23] first when there are spikes and Knowles and Yin [40, 41] in full generality.

For ease of exposition, we assume that A has a single non trivial eigenvalue (but we allow the multiplicity to be greater than one). We explain below the modifications in the general case. Denote by $\theta_1 > 0$ the non trivial eigenvalue of A and by k_1 its multiplicity, which is the rank of A here.

We also need some assumptions on the tail of the matrix elements W_{ij} to extend the result of the Gaussian case to more general distributions. We make the following assumptions.

When $\theta_1 > \sigma$, we assume that the entries W_{ij} have a common third and fourth moment:

$$\widetilde{m_3} := \mathbb{E}|W_{ij}|^2 W_{ij} \text{ is independent of } i,j \text{ and finite,}$$
$$m_3 := \mathbb{E}(W_{ij}^3) < \infty, \quad m_4 := \mathbb{E}|W_{ij}|^4 < \infty \text{ are independent of } i,j. \tag{3.1}$$

We also define $\rho_{\theta_1} := \theta_1 + \dfrac{\sigma^2}{\theta_1}$ and $c_{\theta_1} := \dfrac{\theta_1^2}{\theta_1^2 - \sigma^2}$ for $j = 1, \ldots, l$.

When $\theta_1 \leq \sigma$, we assume that there exist constants $C, c, \eta > 0$, such that

$$\mathbb{P}\left(|W_{ij}| \geq x\right) \leq Ce^{-cx^\eta}, \forall\, 1 \leq i,j \leq N.$$

It is believed that this assumption could be relaxed to (3.1) but this is still a conjecture. We also assume that the third moment of W_{ij} vanishes (as for a Gaussian random variable)

$$\mathbb{E}\Re H_{ij}^3 = \mathbb{E}\Im H_{ij}^3 = 0.$$

We can now state the following Theorem which proves that the BBP phase transition also arises at the level of fluctuations.

Theorem 3.1 ([5, 10, 22, 40, 41, 48, 50]). *Assume that $\theta_1 < \sigma$. Then for any integer l, as $N \to \infty$, the vector*

$$N^{2/3}(\lambda_1 - 2\sigma, \lambda_2 - 2\sigma, \ldots, \lambda_l - 2\sigma)$$

converges in distribution to the Tracy-Widom law of the l largest eigenvalues of a GOE (resp. GUE) F^{TW} defined in [55]. Assume that $\theta_1 = \sigma$, then for any integer l, as $N \to \infty$,

$$N^{2/3}(\lambda_1 - 2\sigma, \lambda_2 - 2\sigma, \ldots, \lambda_l - 2\sigma)$$

converges in distribution to a probability measure (which can be expressed in terms of the Tracy-Widom law) depending on k_1. Assume that $\theta_1 > \sigma$. Then, as $N \to \infty$, the vector

$$c_{\theta_1}\sqrt{N}(\lambda_1 - \rho_{\theta_1}, \ldots, \lambda_{k_1} - \rho_{\theta_1})$$

converges in distribution to the vector of eigenvalues of a random matrix M of size k_1.

A few comments are in order here.

One can first observe that the scale of the fluctuations completely changes when there is a spike. Thus the phase transition also arises at the level of fluctuations.

In the case where there is no outlier, $\theta_1 < \sigma$, the limiting distribution of λ_1 is the same as in the non deformed case. The first statement is thus a universality result in the same vein as the results of [52], [30], [54] obtained for standard Wigner matrices. From a statistical point of view, the signal is overwhelmed by the noise and no information on the signal can be recovered from the data using extreme eigenvalues. In the case where there is an outlier, the distribution of the matrix M depends on more parameters (especially eigenvectors) of the deformation matrix A. This question is discussed in detail in the next paragraph. However, we can mention the extension to the case of multiple spikes. Each spike $\theta_i > \sigma$ of multiplicity k_i causes a small bulk of k_i eigenvalues to separate and fluctuate around ρ_{θ_i} as the eigenvalues of a small random matrix M_i of size k_i. The limiting matrix M_i shares "some similarity" (to be precised hereafter) with the matrix M. A natural question is the transitional regimes between the "small random matrix" asymptotics and those of the Tracy-Widom type. It has been shown in [17] (Gaussian matrices) and [40] (general case) that the transition occurs when $\theta_1 - 2\sigma = O(N^{-1/3})$ yielding a family of possible limiting distributions at the edge of the spectrum of large random matrices.

The role of the eigenvectors of the deformation when there is a spike. In the case where there is spike, the fluctuations can be understood as some kind of Central Limit Theorem. However, the limiting distribution may not be universal (as a Central Limit Theorem would yield). Let us explain this in a few words. To explain the role of the eigenvectors, we introduce the resolvent of random matrices. Given a real symmetric or complex Hermitian matrix M_N and a non-real complex number z, we define the resolvent $R_{M_N}(z)$ by $R_{M_N}(z) = (M_N - zI)^{-1}$. This definition extends to real numbers z lying outside the support of the spectrum of M_N. Using the resolvent equation, it is possible to check that a real number λ , which does not belong to the spectrum of W_N with probability 1, is an eigenvalue of $M_N = W_N + A$ iff

$$\det\left(V^* R_{W_N}(\lambda) V + D^{-1}\right) = 0,$$

where V is a $N \times r$ matrix of eigenvectors of A and $D = \mathrm{diag}(\theta_1, \theta_2, \ldots, \theta_r)$. The important feature of eigenvectors is their *localization*. A normalized vector $v = (v_1, \ldots, v_N)^t$ of dimension N is said to be localized if there exists a subset $I \subset \{1, \ldots, N)$ such that $\sum_{i \in I} v_i^2 \approx 1$ with $|I| << N$. A non-localized vector is delocalized. Typically a delocalized vector is spread out in an approximately uniform way on its entries.
To give a flavor of the results, consider the rank-one deformation $A = \theta v v^*$ for some $\theta > \sigma$ and a normalized vector v. In this case, λ_1 is characterized by the fact that $1 + \theta v^* R_{W_N} v = 0$. It is then obvious that when v is delocalized, some kind of averaging of the resolvent entries shall make the distribution of the largest eigenvalue "more universal" than in the localized case.

More precisely, one has the following statement. Again we expose here for ease the case where A has a single non trivial eigenvalue $\theta_1 > \sigma$ with multiplicity k_1. Denote by $v_1, v_2, \ldots, v_{k_1}$ an orthonormal family of eigenvectors of A associated to θ_1.

Theorem 3.2 ([22, 23, 50]).
Case 1: *Assume that the family $\{v_j, j = 1, \ldots, k_1\}$ is spanned by a fixed number K of the canonical basis vectors of $\mathbb{C}^N$. Then the vector $c_{\theta_1}(\lambda_1 - \rho_{\theta_1}, \ldots, \lambda_{k_1} - \rho_{\theta_1})$ converges in distribution to the distribution of the ordered eigenvalues of the $k_1 \times k_1$ random matrix M defined as*

$$M = U^*(G + H)U,$$

where G is a Wigner random matrix of size K whose entries are distributed as the W_{ij}, H is a Hermitian Gaussian matrix of size K, independent of G (the entries are centered, with variance depending on m_4 and σ^2) and U is a $K \times k_1$ matrix such that its columns are written from the first K coordinates of the v_j's, $j = 1, \ldots, k_1$.

Case 2: *Assume that for each j $||v_j||_\infty \to 0$ as N grows to infinity. Then the vector $c_{\theta_1}(\lambda_1 - \rho_{\theta_1}, \ldots, \lambda_{k_1} - \rho_{\theta_1})$ converges in probability to the vector of ordered eigenvalues of the sum of a GUE (with variance depending on θ_1 and σ^2 only) and the matrix $\left(\widetilde{m_3} v_j^*(J - I)v_i\right)_{i,j}$, where J is the matrix whose entries are all 1.*

Note that if W is Gaussian, the rotational invariance of the Gaussian distribution makes both cases indistinguishable. The resulting matrix M is then a GUE. For non Gaussian distributions, the situation is completely different. For instance, when $k_1 = 1$ and $A = \theta_1 e_1 e_1^*$, the limiting distribution of the largest eigenvalue is the convolution of a Gaussian and the law μ of W_{11}. This limiting distribution is non universal in essence, compared to the case where the rank one matrix A would have a delocalized eigenvector.

In the above theorem, one goes from fully localized (Case 1) to completely delocalized eigenvectors (Case 2). The gap has been filled very recently by Knowles and Yin [40], [41], where generic eigenvectors of the perturbation A are considered. In particular, the limiting distribution of outliers is identified provided the entries of W have sub-exponential tails. The basic idea for the proof is an isotropic local semicircle law outside the spectrum (established in [40]), that is a (complete) understanding of the resolvent R_{W_N} outside the support of the limiting semi-circle distribution. This isotropic law states that for a complex number z such that $|\Re z| > 2\sigma$ and any deterministic vectors v, w, one has that

$$\left|\langle v, R_{W_N}(z)w\rangle - m_{sc}(z)\langle v, w\rangle\right| \leq (\ln N)^{C \ln\ln N} \frac{\Im m_{sc}(z)}{\sqrt{N\Im z}} ||v||\,||w||,$$

where $m_{sc}(z)$ is the Stieltjes transform of the limiting semi-circle distribution:

$$m_{sc}(z) := \int_{\mathbb{R}} \frac{1}{y-z} \frac{\sqrt{4\sigma^2 - y^2}}{2\pi\sigma^2} 1_{[-2\sigma,2\sigma]}(y)dy.$$

This estimates holds uniformly on compact subsets of $\mathbb{C} \cap \{|\Re z| \geq 2\sigma\}$. Interestingly, therein the authors also consider the joint distribution of distinct outliers. In particular, they prove that macroscopically distinct outliers are not asymptotically independent (unless W is a GUE).

At this point, it is worth mentioning that the situation is different for sample covariance matrices as proved in [2]. In the large size limit, the fluctuations of outliers are necessarily those of the eigenvalues of a Gaussian matrix M (with non necessarily independent entries and whose covariance may depend on the third and fourth moments of the X_{ij}'s). This can be explained by the fact that there is some more averaging in the entries of sample covariance matrices. Another approach based on the isotropic Marcenko-Pastur law is derived in [14] and [15].

4. Extensions to full rank deformations of Wigner matrices

The study of deformed ensembles extends to the case where the matrix A has low rank $r_N << N, r_N \to \infty$ or full rank i.e. when $r_N = O(N)$. In this case, it is natural to assume that the empirical eigenvalue distribution of A has a weak limit as $N \to \infty$, which is possibly δ_0. Denote by $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_N$ the ordered eigenvalues of A. Let $\mu_N(A) = \frac{1}{N}\sum_{i=1}^N \delta_{\theta_i}$. We assume that there exists a probability distribution ν on $\mathbb{R}$ such that

$$\mu_N(A) \underset{N\to\infty}{\overset{w}{\to}} \nu.$$

Let us diagonalize A through $A = V\text{diag}(\theta_1, \ldots, \theta_n)V^*$. Roughly speaking the deformed model is now understood in the sense that A is a "small" perturbation of the matrix $W_N + VA_0V^*$ where A_0 would be a diagonal matrix made up with quantiles of the probability ν.

The asymptotic global behavior of the spectrum is well-known in this case. Indeed, let μ_{W_N+A} be the empirical eigenvalue distribution of $W_N + A$. Its Stieltjes transform is

$$m_N(z) := \int \frac{1}{z-y} d\mu_{W_N+A}(y),\ \Im z \neq 0.$$

It is proved by [45] (see also [1]) that m_N converges as $N \to \infty$ to the Stieltjes transform m_ρ of a probability distribution ρ, called the free convolution of ν and the semi-circle distribution. This probability distribution ρ has been studied in detail in [13]. It is uniquely characterized by a fixed point equation satisfied by m_ρ. Similarly the Marcenko-Pastur theorem [44] gives the limiting distribution of $(I+A)^{1/2}M_N(I+A)^{1/2}$.

Let us first consider the case where A is deterministic. The question of separation of extreme eigenvalues naturally arises in this setting also. This question has been considered in [24], under the assumption that the entries satisfy a Poincaré inequality. In particular, spikes are characterized using a free probability approach. At the level of fluctuations, this question has been much less investigated actually. So far, only the case where W_N is a GUE has been investigated.

In [51], the author considers the case where $\mu_N(A)$ concentrates quite fast around the measure ν. In particular, there are no spikes. When W_N is GUE, she proves that the largest eigenvalues have a universal asymptotic behavior, characterized by the Tracy-Widom distribution.

In [25], the authors consider the general case, removing in particular the concentration assumption on $\mu_N(A)$. Therein the asymptotic behavior at the edges of the spectrum is described using a deterministic equivalent (depending on N) of the spectral measure of the deformed ensemble. This deterministic equivalent, arising from free probability, still converges to the free convolution of ν and the semi-circle distribution but is much closer to the true spectral measure. The authors derive the asymptotic distribution of eigenvalues close to the edges of the support of ρ (which may be disconnected) and also at possible outliers. The picture is then essentially the same as for the finite rank deformation of a GUE. Eigenvalues close to the edge of a connected component are shown to exhibit Tracy-Widom fluctuations with the usual rescalings, provided the connected component is well separated from the others. It can also be shown that an eigenvalue θ_i is a spike iff

$$\int \frac{1}{(\theta_i - y)^2} d\nu(y) < \sigma^{-2}.$$

A spike creates an outlier at the location

$$\alpha(\theta_i) := \theta_i + \sigma^2 \sum_{j:\theta_j \neq \theta_i} \frac{1}{N(\theta_i - \theta_j)}.$$

A spike θ_i with multiplicity k_i will cause k_i outliers fluctuate around $\alpha(\theta_i)$ as the eigenvalues of a k_i GUE. Interestingly, as the limiting measure ρ may have a non connected support, spikes can arise not only at the top/bottom edges of the support but also at intermediate edges. The limiting statistics of eigenvalues at spikes between connected components are of the same GUE types than at the edges. The main difference from the finite rank deformation case actually comes from the fact that there may exist points where two connected components of the support of ρ merge. Eigenvalues close to such a merging point are proved in [25] to exhibit fluctuations which are governed by the so-called Pearcey distribution studied in e.g.[56].

The case of sample covariance matrices has been considered in [28] for complex Wishart matrices and extended to arbitrary distribution (with suitable moment assumptions) in [3] and [8]. The extension to arbitrary Wigner matrices is a challenging open problem. Actually quite few is known on the resolvent of such a deformed Wigner matrix $W_N + A$ for a generic full rank matrix A.

We now consider the case where $A = A_N$ is random. It is technically easier to consider the case where A_N is diagonal with i.i.d. random variables. Assume in the following that $V_N = \mathrm{diag}(v_1, \ldots, v_N)$ where the v_i's are i.i.d. random variables with distribution ν. The extreme eigenvalues of V can then be described as extremes of a N sample of distribution ν. In particular the asymptotic distribution of the largest eigenvalue v_{max} is well known and is either the Gumbel, or Fréchet or Weibull distribution depending on the characteristics of ν. This follows from the Fisher-Tippett-Gnedenko theorem. Consider the deformed ensemble

$$H_N = W_N + \lambda V_N,$$

where λ is a real parameter and W_N is a Wigner random matrix. In view of the fixed rank case, one would expect a transition from the Tracy-Widom distribution to some limiting statistics of extremes by allowing the parameter λ to grow from 0 to infinity. This is actually not always the case as we explain hereafter.
In [38], Johansson considers the case where the v_i's are i.i.d. centered random variables with distribution ν in the domain of attraction of the Gumbel distribution and W_N is a GUE. By the choice of ν, the largest eigenvalue v_{max} has asymptotic Gumbel fluctuations: more precisely there exist sequences a_N and b_N (which depend on ν) such that

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{v_{max} - a_N}{b_N} \leq x\right) = e^{-e^{-x}}, \forall x \in \mathbb{R}.$$

Then [38] proves the following fact: the fluctuations of the largest eigenvalues are asymptotically governed by the Tracy-Widom distribution as long as $\lambda << N^{-1/6}$. In the transitional regime where $\lambda = \alpha N^{-1/6}$, for a given positive number α, the asymptotic distribution of the (suitably rescaled) largest eigenvalue of H_N is given by the convolution of a Gaussian distribution and the Tracy-Widom distribution. When $\lambda >> N^{-1/6}$ the asymptotic distribution of the largest eigenvalue of H_N is Gaussian. This follows from the fact that the top edge of the limiting semi-circle fluctuates as $1/N \sum_{i=1}^{N} y_i$ i.e. like a Gaussian. Thus there is no transition in this model from the Tracy-Widom to the Gumbel distribution. In [43], the authors consider in place a diagonal matrix A where the v_i's are i.i.d. with density

$$\frac{1}{Z}(1+v)^a(1-v)^b \mathbb{1}_{[-1,1]}(v). \tag{4.1}$$

In this case, the largest eigenvalue of A after suitable rescaling has asymptotically Weibull fluctuations, with cumulative distribution function $G_{b+1}(x) = 1 - \mathrm{e}^{-(x/c)^{b+1}}$ for some parameter c depending on ν. The authors also consider the general case where W_N is a Wigner matrix whose entries satisfy the usual exponential decay assumption (1.1). In this case, the authors prove the following phase transition:

Theorem 4.1. *Assume that* $b > 1$ *in* (4.1). There exist constants $\lambda_0 = \lambda_0(a, b)$ and $L = L(a, b, \lambda)$ such that

- if $\lambda > \lambda_0$, then $\lim_{N\to\infty} \mathbb{P}\left(N^{1/(b+1)}(L - \lambda_{max}) \leq x\right) = G_{b+1}(x)$.
- If $\lambda < \lambda_0$, then $\lim_{N\to\infty} \mathbb{P}\left(N^{1/2}(L - \lambda_{max}) \leq x\right) = G(x)$, where G is the cumulative distribution function of a centered Gaussian variable with (explicit) variance.

The Gaussian fluctuations can be explained as in the case studied by Johansson and are due to the fluctuations of the v_i's. The value λ_0 has a nice interpretation in terms of free

probability, which can be inferred from [25]. When the matrix W_N is rescaled so that its entries are of variance 1, λ_0 is such that

$$\int \frac{1}{(1-x)^2} d\nu(x) = \lambda_0^2.$$

In the case where $\lambda > \lambda_0$, the largest eigenvalues of A_N are to be considered as "internal" spikes, due to the fact that the density ν vanishes fast at the edge. The fluctuations of spikes then transfer to the largest eigenvalues of H_N. Note that the same phenomenon arises in the deterministic case when ν has a density vanishing sufficiently fast at the edge of the support. The largest eigenvalues then exhibit Gaussian fluctuations in place. In this random perturbation case, almost nothing is known for an arbitrary Wigner matrix when A is not diagonal.

5. Concluding remarks

To conclude this state of the art about deformed ensembles of random matrices, we should add a word about the asymptotic behavior of eigenvectors. In the context where the deformation is the signal whereas the random matrix is seen as noise, the re-construction of the signal requires some knowledge of the relationship between the eigenvectors of A and those of the deformed matrix H_N. We emphasize that this question makes sense only when the signal can be detected because of outliers. In this context, the most comprehensive study of eigenvectors has been achieved in [21], showing that eigenvectors associated to outliers can help in the reconstruction of eigenvectors associated to spikes in the deformation. We refer the reader to the articles [11, 21, 41, 43] for more detail and references about the behavior of eigenvectors of deformed ensembles.

We are thus at the point where our understanding of small rank perturbation of random matrices is achieved, due to the enormous progress accomplished in the recent last years. However, in the full rank deformation case, the picture is much less clear. Essentially the very special cases of the deformed GUE and that of diagonal deformations are understood. So far, the road map to extend the results to arbitrary random matrices is not fully clear. This probably needs some new ideas to be brought into the field.

In addition, we would also like to mention that deformed ensembles can be considered for much more general ensembles of random matrices than Hermitian or real symmetric random matrices. In particular, [53] (see also the recent article [12]) considers $N \times N$ random matrices with i.i.d. entries (with finite fourth moment) perturbed by the addition of a fixed rank matrix A. This time no symmetry assumption is made on the matrices. The limiting spectral measure is then the so-called circular distribution. [53] proves that outliers can be created by the deformation matrix A provided its eigenvalues are sufficiently large in modulus. In addition, outliers are proved to be very close to their associated spikes, which is quite different from the symmetric case. This is quite surprising as eigenvalues of such random matrices are believed to be less stable than those from a symmetric random matrix. Thus, so far there does not seem to be a unified answer to the questions about the spectrum of the sum of two (possibly random) matrices.

This unified answer may actually come from free probability, using subordination equations. These equations relate the Stieltjes transform of the limiting spectral distribution of the deformed matrix to that of the asymptotic distribution of the deformation. It turns out

that the phenomenon of outliers of classical deformed models (including complex matrices with no symmetry) can be fully described in terms of the subordination functions related to the free additive or multiplicative convolution. However understanding in detail the specificities of Wigner random matrices, random sample covariance matrices or matrices with i.i.d. entries and their impact on the deformed ensembles is still to be achieved.
Lastly, the statistics of extreme eigenvalues of random matrices have been proved to be relevant to many fields of mathematics and physics (including statistics, random growth models, KPZ universality class, queuing theory, random permutations ...). Deformed ensembles have proved to exhibit a very rich family of possible limiting distributions at the edges of the spectrum, including the paradigmatic Tracy-Widom distribution. As we said before, the set of possible deformations of random matrices is not exhaustive. We still have quite limited knowledge of the family of possible limiting distributions of extreme eigenvalues of deformations of random matrices.

Acknowledgements. Université Paris Diderot, LPMA, 5 Rue Thomas Mann, 75013 Paris, France.

References

[1] Anderson, G., Guionnet, A., and Zeitouni, O., *An introduction to random matrices*, Cambridge Studies in Advanced Mathematics, 118. Cambridge University Press, Cambridge, 2010.

[2] Bai, Z. D. and Yao, Y. F., *Central limit theorems for eigenvalues in a spiked population model*, Ann. Inst. H. Poincaré Probab. Statist. **44** (2008), no. 3, 447–474.

[3] _____, *On sample eigenvalues in a generalized spiked population model*, J. Multivariate Anal. **106** (2012), 167–177.

[4] Bai, Z. D. and Yin, Y. Q., *Necessary and sufficient conditions for almost sure convergence of the largest eigenvalue of a Wigner matrix*, Ann. Probab. **16** (1988), no. 4, 1729–1741.

[5] Baik, J., Ben Arous, G., and Péché, S., *Phase transition of the largest eigenvalue for nonnull complex sample covariance matrices*, Ann. Probab. **33** (2005), no. 5, 1643–1697.

[6] Baik, J. and Silverstein, J., *Eigenvalues of large sample covariance matrices of spiked population models*, J. Multivariate Anal. **97** (2006), no. 6, 1382–1408.

[7] Baik, J. and Suidan, T., *A GUE central limit theorem and universality of directed first and last passage site percolation*, Int. Math. Res. Not. (2005), no. 6, 325–337.

[8] Bao, Z., Pan, Z., and Zhou, W., *Universality for the largest eigenvalue of a class of sample covariance matrices*, arXiv:1304.5690.

[9] Ben Arous, G. and Corwin, I., *Current fluctuations for TASEP: A proof of the Praehofer-Spohn conjecture*, Ann. Probab. **39** (2011), no. 1, 104–138.

[10] Benaych-Georges, F., Guionnet, A., and Maida, M. *Fluctuations of the Extreme Eigenvalues of Finite Rank Deformations of Random Matrices* Elec. Journ. Probab. **16** (2011) 1621-1662.

[11] Benaych-Georges, F. and Rao, R., *The eigenvalues and eigenvectors of finite, low rank perturbations of large random matrices*, Advan. Math. **227** (2011), no. 1, 494–452.

[12] Benaych-Georges, F. and Rochet, J., *Outliers in the Single Ring Theorem*, Preprint arXiv:1308.3064 (2013).

[13] Biane, P., *On the free convolution with a semi-circular distribution*, Indiana Univ. Math. J., **46** (1997), no. 3, 705–718.

[14] Bloemendal A., Erdös L., Knowles A., Yau H.T., and Yin J., *Isotropic Local Laws for Sample Covariance and Generalized Wigner Matrices*, arXiv:1308.5729, to appear in Elec. J. Probab.

[15] Bloemendal A., Knowles A., Yau H.T., and Yin J., *Eigenvectors of deformed random matrices*, In preparation.

[16] Bodineau, T. and Martin, J., *A universality property for last-passage percolation paths close to the axis*, Electron. Comm. Probab. **10** (2005), 105–112.

[17] Borodin, A. and Péché, S., *Airy kernel with two sets of parameters in directed percolation and random matrix theory*, J. Stat. Phys. **132** (2008), no. 2, 275–290.

[18] Bouchaud, J.P. and Potters, M., *Financial Applications of Random Matrix Theory: a short review*, The Oxford Handbook of Random Matrix Theory: Chapter 40, Oxford University Press, 2011.

[19] Brézin, E. and Hikami, S., *Spectral form factor in a random matrix theory*, Phys. Rev. E (3) **55** (1997), no. 4, 4067–4083.

[20] _____, *Correlations of nearby levels induced by a random potential*, Nuclear Phys. B **479** (1996), no. 3, 697–706.

[21] Capitaine, M., *Additive/multiplicative free subordination property and limiting eigenvectors of spiked additive deformations of Wigner matrices and spiked sample covariance matrices*, Journal of Theoretical Probability, 1-54 (2012), doi:10.1007/s10959-012-0416-5.

[22] Capitaine, M., Donati-Martin, C., and Féral, D., *The largest eigenvalues of finite rank deformation of large Wigner matrices: convergence and non universality of the fluctuations*, Ann. Probab. **37** (2009), no. 1, 1–47.

[23] _____, *Central limit theorems for eigenvalues of deformations of Wigner matrices*, Ann. Inst. H. Poincaré Probab. Statist., **48** (2012), no. 1, 107–133.

[24] Capitaine, M., Donati-Martin, C., Féral, D., and Février, M., *Free convolution with a semicircular distribution and eigenvalues of spiked deformations of Wigner matrices*, Elec. J. Probab. **16** (2011), no. 64, 1750–1792.

[25] Capitaine, M. and Péché S., *Fluctuations at the edges of the spectrum of the full rank deformed GUE*, Preprint arXiv:1402.2262 (2014).

[26] Couillet R. and Debbah M., *Random Matrix Methods for Wireless Communications*, Cambridge University Press, 2011.

[27] Corwin, I., *The Kardar Parisi Zhang equation and universality class*, Random Matrices: Theory Appl., 01, 1130001 (2012), DOI:10.1142/S2010326311300014.

[28] El Karoui, N., *Tracy-Widom limit for the largest eigenvalue of a large class of complex sample covariance matrices*, Ann. Probab. **35** (2007), no. 2, 663–714.

[29] Erdös, L., Knowles, A., Yau, H.T., and Yin, J., *Spectral statistics of Erdòs-Rényi Graphs II: Eigenvalue spacing and the extreme eigenvalues*, Comm. Math. Phys. **314** (2012), no. 3, 587–640.

[30] Erdös, L., Yau, H.T., and Yin, J., *Rigidity of eigenvalues of generalized Wigner matrices*, Adv. Math. **229** (2012), no. 3, 1435–1515.

[31] Erdös, L., Yau, H.T., and Yin, J., *Universality of random matrices and local relaxation flow*, Invent. Math. **185** (2011), no. 1, 75–119.

[32] Füredi, Z. and Komlós, J., *The eigenvalues of random symmetric matrices*, Combinatorica **1** (1981), no. 3, 233–241.

[33] Féral, D. and Péché, S., *The largest eigenvalue of rank one deformation of large Wigner matrices*, Comm. Math. Phys. **272** (2007), 185–228.

[34] Harish-Chandra., *Differential operators on a semisimple Lie algebra.*, Amer. J. Math., **79** (1957), 87–120.

[35] Itzykson, C. and Zuber, J.B., *The planar approximation*, Journ. Math. Phys., **21** (1980), 411–421.

[36] Johansson, K., *Universality of the local spacing distribution in certain ensembles of Hermitian Wigner matrices*, Comm. Math. Phys. **215** (2001), no. 3, 683–705.

[37] _____, *Shape fluctuations and random matrices*, Comm. Math. Phys., **209** (2000), no. 2, 437–476.

[38] _____, *From Gumbel to Tracy-Widom*, Probab. Theory Relat. Fields, **138** (2007), 75–112.

[39] Kardar M., Parisi G., and Zhang Y. Z., *Dynamic scaling of growing interfaces,* Phys. Rev. Lett. **56** (1986), 889–892.

[40] Knowles, A. and Yin, J., *The Isotropic Semicircle Law and Deformation of Wigner Matrices*, Preprint arXiv:1110.6449 (2011), To appear in Comm. Math. Phys.

[41] _____, *The outliers of a deformed Wigner matrix*, Preprint arXiv:1207.5619. To appear in Ann. Prob.

[42] Knutson, A. and Tao, T., *Honeycombs and sums of Hermitian matrices*, Notices Amer. Math. Soc. **48** (2001), no. 2, 175–186.

[43] Lee, J.O. and Schnelli, K., *Extremal eigenvalues and eigenvectors of Deformed Wigner matrices*, Preprint arXiv:1310.7057 (2013).

[44] Marchenko,V. and Pastur, L., *Distribution of eigenvalues for some sets of random matrices*, Mat. Sb. (N.S.), **72** (114):4 (1967), 507–536; doi:10.1070/SM1967v001 n04ABEH001994.

[45] Pastur, L. and Vasilchuk, V., *On the Law of Addition of Random Matrices*, Comm. Math. Phys. **214** (2000), no. 2, 249–286.

[46] Patterson, N., Price, A. L., and Reich, D., *Population structure and eigenanalysis*, PLoS Genetics 2 (12): e190, doi:10.1371/journal.pgen.0020190, 2006.

[47] Péché, S., *The largest eigenvalue of small rank perturbations of Hermitian random matrices*, Probab. Theory Related Fields **134** (2006), no. 1, 127–173.

[48] Pizzo, A., Renfrew, D., and Soshnikov, A., *On finite rank deformations of Wigner matrices*, Ann. Inst. Henri Poincaré Probab. Stat. **49** (2013), no. 1, 64–94.

[49] Praehofer, M. and Spohn, H., *Current fluctuations for the totally asymmetric simple exclusion process*, In and Out of Equilibrium (Mambucaba, 2000) Progress in Probability, **51** (2002) 185-204, Birkhauser, Boston, MA.

[50] Renfrew, D. and Soshnikov, A., *On finite rank deformations of Wigner matrices II. Delocalized perturbations*, arXiv:1203-5130. (2013).

[51] Shcherbina, T., *On universality of local edge regime for the deformed Gaussian Unitary ensemble*, J. Stat. Phys. **143** (2011), no. 3, 455–481.

[52] Soshnikov, A., *Universality at the edge of the spectrum in Wigner random matrices*, Comm. Math. Phys. **207** (1999), no. 3, 697–733.

[53] Tao, T., *Outliers in the spectrum of iid matrices with bounded rank perturbations*, Probab. Theory Related Fields **155** (2013), 231–263.

[54] Tao, T. and Vu, V., *Random matrices: universality of local eigenvalue statistics up to the edge*, Comm. Math. Phys. **298** (2010), no. 2, 549–572.

[55] Tracy, C. and Widom, H., *Level-spacing distributions and the Airy kernel*, Comm. Math. Phys. **159** (1994), no. 1, 151–174.

[56] ———, *The Pearcey process*, Comm. Math. Phys. **263** (2006), no. 2, 381–400.

[57] Weyl, H., *Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung)*, Math. Ann. **71** (1912), 441–479.

[58] Wigner, E., *On the distribution of the roots of certain symmetric matrices*, Ann. of Math. **67** (1958), no. 2, 325–327.

[59] Wishart, J., *The generalised product moment distribution in samples from a normal multivariate population*, Biometrika 20A (1928), 32–52.

Université Paris Diderot, 5 Rue Thomas Mann, 75013 Paris, FRANCE

E-mail: sandrine.peche@math.univ-paris-diderot.fr

Structure of the excitation spectrum for many-body quantum systems

Robert Seiringer

Abstract. Many questions concerning models in quantum mechanics require a detailed analysis of the spectrum of the corresponding Hamiltonian, a linear operator on a suitable Hilbert space. Of particular relevance for an understanding of the low-temperature properties of a system is the structure of the excitation spectrum, which is the part of the spectrum close to the spectral bottom. We present recent progress on this question for bosonic many-body quantum systems with weak two-body interactions. Such system are currently of great interest, due to their experimental realization in ultra-cold atomic gases. We investigate the accuracy of the Bogoliubov approximations, which predicts that the low-energy spectrum is made up of sums of elementary excitations, with linear dispersion law at low momentum. The latter property is crucial for the superfluid behavior the system.

Mathematics Subject Classification (2010). Primary 82B10; Secondary 82-06, 46N50.

Keywords. Schrödinger equation, quantum statistical mechanics, Bose–Einstein condensation, dilute Bose gas, superfluidity, excitation spectrum.

1. Introduction

Many interesting effects in quantum mechanics result from the interactions among the fundamental particles that constitute the system. A famous example of such an effect is superconductivity, the vanishing of electrical resistance in certain materials at low enough temperature. The relevant fundamental particles in this case are the electrons, which are *fermions* and obey the Pauli exclusion principle, which demands that each quantum state can be occupied by at most one particle. *Bosons*, on the other hand, are particles in quantum mechanics that do not obey the Pauli principle. Examples of bosons include photons (the quanta of the electromagnetic field) or also composite particles like atoms, which themselves consists of fermions (electrons and nucleons) but behave as bosons if the number of fermionic constituents is even. There is no bound on the number of bosons occupying the same quantum state. This leads to the phenomenon of *Bose–Einstein condensation* (BEC), which occurs if a macroscopic fraction of all the particles occupy the same quantum state. The resulting state of matter displays various interesting phenomena, like superfluidity, for instance, where the viscosity of a fluid vanishes completely at low temperature.

BEC in cold atomic gases was first achieved experimentally in 1995 [1, 9]. After initial failed attempts with spin-polarized atomic hydrogen, the first successful demonstrations of this phenomenon used gases of rubidium and sodium atoms, respectively. In these experiments, a large number of (bosonic) atoms is confined to a trap and cooled to very low

Proceedings of the International Congress of Mathematicians, Seoul, 2014

temperatures; below a critical temperature condensation of a large fraction of particles into the same one-particle state occurs. Since then there has been a surge of activity in this field, with ingenious experiments putting forth more and more astonishing results about the behavior of matter at very cold temperatures. BEC has now been achieved by more than a dozen different research groups working with gases of different types of atoms. Literally thousands of scientific articles, concerning both theory and experiment, have been published in recent years. Various interesting quantum phenomena have been explored, like the appearance of quantized vortices in rotating systems and the property of superfluidity. The latter is related to the low-energy excitation spectrum of the system. We refer to [3, 6, 8, 13] for reviews of the recent developments in this field of physics.

The theoretical investigation of BEC goes back much further, and even pre-dates the modern formulation of quantum mechanics. It was investigated in two papers by Einstein [12] in 1924 and 1925, respectively, following up on a work by Bose [5] on the derivation of Planck's law for black-body radiation. Einstein's result, in its modern formulation, can be found in any textbook on quantum statistical mechanics, and was concerned with ideal, i.e., non-interacting gases. The understanding of BEC in the presence of interparticle interactions poses a formidable challenge to mathematical physics. One of the key contributions to the theory of weakly interacting Bose gases is Bogoliubov's 1947 paper [4], where he introduces an approximate model (now referred to as the Bogoliubov approximation) to explain its superfluid behavior. In this paper, we will summarize recent progress made towards a rigorous justification of this approximation.[1]

2. The Bose Gas: A quantum many-body problem

The quantum-mechanical description of a system of N bosons is given in terms of the Hamiltonian H_N, acting as a linear operator on a suitable Hilbert space $\mathscr{H}_N$. Typically, H_N is an unbounded operator, defined only on a dense subspace of $\mathscr{H}_N$, but it should be bounded from below in order to describe a stable physical system. For bosons interacting via a pair-interaction potential denoted by $v(x)$, the Hamiltonian is given, in appropriate units, by

$$H_N = -\sum_{i=1}^{N} \Delta_i + \sum_{1\leq i<j\leq N} v(x_i - x_j)\,. \tag{2.1}$$

The kinetic energy is described by Δ, the Laplacian on a suitable domain in $\mathbb{R}^3$, which we will typically take to be a cube of side length L, i.e., $[0, L]^3$. Suitable boundary conditions have to be imposed in order for Δ to define a self-adjoint operator, with periodic boundary conditions being a typical example. The subscript i indicates, as usual, that the second derivative is with respect to $x_i \in \mathbb{R}^3$.

As appropriate for bosons, the Hamiltonian H_N acts on the Hilbert space of *permutation-symmetric* wave functions $\Psi(x_1, \dots, x_N)$ in $\bigotimes^N L^2([0, L]^3)$, which we shall denote by $\mathscr{H}_N$:

$$\mathscr{H}_N = \bigotimes_{\mathrm{sym}}^{N} L^2([0, L]^3)\,. \tag{2.2}$$

[1] The paper is an updated and extended version of the manuscript published in [36].

The interaction v is a real-valued function $v : \mathbb{R}^3 \to \mathbb{R}$, which we assume to be bounded and symmetric, i.e., $v(x) = v(-x)$. It acts as a multiplication operator on $\mathscr{H}_N$.

Of fundamental importance is the *spectrum* of H_N, i.e., the complement of the subset of $\mathbb{C}$ where $z - H_N$ has a bounded inverse. For the Hamiltonian H_N acting on the Hilbert space $\mathscr{H}_N$, it is not difficult to see that the spectrum is discrete, i.e., it consists of eigenvalues of H_N of finite multiplicity, which are bounded from below and accumulate at $+\infty$. The corresponding eigenfunctions describe the stationary states of the system.

The following quantities, derived from the Hamiltonian H_N, will interest us here.

- **Ground state energy**, defined as the lowest value of the spectrum of the Hamiltonian,

$$E_0(N, L) = \inf \operatorname{spec} H_N \,. \tag{2.3}$$

- The ground state wave function Ψ_0 is the eigenfunction of H_N corresponding to eigenvalue $E_0(N, L)$, i.e.,

$$H_N \Psi_0 = E_0(N, L)\Psi_0 \,. \tag{2.4}$$

  For large particle number N, it is typically much too complicated to compute. Instead one considers the corresponding reduced density matrices of Ψ_0, the simplest of which is the **one-particle density matrix**, given by the integral kernel

$$\gamma_0(x, x') = N \int_{\mathbb{R}^{3(N-1)}} \Psi_0(x, x_2, \ldots, x_N)\overline{\Psi_0(x', x_2, \ldots, x_N)}\, dx_2 \cdots dx_N \,. \tag{2.5}$$

  It satisfies $0 \leq \gamma_0 \leq N$ as an operator, and its trace equals $\operatorname{Tr} \gamma_0 = N$. With the aid of creation and annihilation operators (to be reviewed in Section 4 below) one can also write

$$\gamma_0(x, x') = \left\langle a^\dagger(x')a(x) \right\rangle \,, \tag{2.6}$$

  and this definition generalizes to arbitrary mixed states as well.

- The diagonal of the one-particle density matrix is the **particle density**

$$\varrho_0(x) = \gamma_0(x, x) = N \int_{\mathbb{R}^{3(N-1)}} |\Psi_0(x, x_2, \ldots, x_N)|^2 dx_2 \cdots dx_N \,, \tag{2.7}$$

  with $\int \varrho_0(x)dx = N$. For translation invariant systems, ϱ_0 is a constant and does not depend on x, but for inhomogeneous systems the spatial variation of ϱ_0 represents a non-trivial question.

- By definition, **Bose–Einstein condensation** in a state Ψ_0 means that the one-particle density matrix γ_0 has an eigenvalue of order N, i.e., that $\|\gamma_0\| \geq cN$ for some $c > 0$ and all (large) N, with $\| \cdot \|$ denoting the operator norm. The corresponding eigenfunction is called the *condensate wave function*.

- Of particular interest to us will be the structure of the **excitation spectrum**, i.e., the spectrum of H_N above the ground state energy $E_0(N, L)$, and the relation of the corresponding eigenstates to the ground state. For translation invariant systems, H_N commutes with the total momentum operator

$$P = -i \sum_{j=1}^{N} \nabla_j \,, \tag{2.8}$$

and hence one can look at their joint spectrum. Of particular relevance is the infimum

$$E_q(N,L) = \inf \operatorname{spec} H_N \restriction_{P=q} \ . \tag{2.9}$$

In contrast to the non-interacting case, for interacting particles one expects a linear behavior of $E_q(N,L)$ in q for not too large values of $|q|$. For a review of various questions related to the excitation spectrum of Bose gases we refer to [7].

The particle number N is typically very large. This large number of variables involved in the problem is the main reason why the quantities above are very hard to compute. We will be interested in their behavior as $N \to \infty$.

3. The Ideal Bose Gas

For *non-interacting bosons*, i.e., in the case $v \equiv 0$, the ground state energy is simply N times the lowest eigenvalue of the Laplacian. In the case of periodic boundary conditions, i.e., the Laplacian on the flat torus $[0,L]^3$, this is simply zero:

$$E_0(N,L) = 0 \quad \text{for all } N \text{ and } L. \tag{3.1}$$

The corresponding ground state wave function Ψ_0 is the constant function in $\mathscr{H}_N$.

Also the excitation spectrum can easily be computed explicitly for the ideal gas. The spectrum of $-\Delta$ on the flat torus $[0,L]^3$ equals

$$\left\{ |p|^2 \,:\, p \in \left(\frac{2\pi}{L}\mathbb{Z}\right)^3 \right\}, \tag{3.2}$$

with corresponding eigenfunctions $L^{-3/2} e^{ip\cdot x}$. The spectrum of N bosons is then simply

$$\sum_p |p|^2 n_p \,, \tag{3.3}$$

where the sum is over $p \in (\frac{2\pi}{L}\mathbb{Z})^3$ and $n_p \in \{0,1,2,\dots\}$ for each p, with $\sum_p n_p = N$. The latter are called the *occupation numbers* of the corresponding momentum states. The eigenstate of H_N corresponding to an eigenvalue of the form (3.3) is given by

$$\mathcal{S} \prod_j \varphi_j(x_j) \tag{3.4}$$

where $\mathcal{S}$ denotes symmetrization with respect to permutations, and all the φ_j are eigenfunctions of the Laplacian, the one corresponding to momentum p appearing n_p times in the product. Note that for each set of occupations numbers $\{n_p\}$ there is exactly one such eigenstate in $\mathscr{H}_N$.

4. Second Quantization on Fock space

In the following, it will be convenient to regard $\mathscr{H}_N = \bigotimes_{\rm sym}^N L^2([0,L]^3)$ as a subspace of the bosonic *Fock space*

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathscr{H}_n \quad , \ \mathscr{H}_0 \equiv \mathbb{C} \,. \tag{4.1}$$

On this space, the particle number N is now an operator, which acts simply as multiplication by n on the subspace $\mathscr{H}_n$ of $\mathcal{F}$.

A basis of $L^2([0,L]^3)$ is given by the plane waves $L^{-3/2}e^{ip\cdot x}$ for $p \in (\frac{2\pi}{L}\mathbb{Z})^3$, and we introduce the corresponding *creation and annihilation operators*, which satisfy the canonical commutation relations (CCR)

$$\left[a_p, a_q\right] = \left[a_p^\dagger, a_q^\dagger\right] = 0\,, \quad \left[a_p, a_q^\dagger\right] = \delta_{p,q}\,. \tag{4.2}$$

Here, the creation operator $a_p^\dagger$ is the adjoint of the annihilation operator a_p. The latter maps $\mathscr{H}_n$ to $\mathscr{H}_{n-1}$ for $n \geq 1$ and acts as

$$\left(a_p\psi\right)(x_1,\ldots,x_{n-1}) = \sqrt{\frac{n}{L^3}}\int_{[0,L]^3} e^{-ip\cdot x_n}\psi(x_1,\ldots,x_n)dx_n\,. \tag{4.3}$$

We consider again the Hamiltonian H_N, with the particles moving on the flat torus $[0,L]^3$. It is then natural to assume that the interaction v in (2.1) is a periodic function on $\mathbb{R}^3$, with period L in all three coordinate directions. In other words,

$$v(x) = L^{-3}\sum_{p\in(\frac{2\pi}{L}\mathbb{Z})^3}\widehat{v}(p)e^{ip\cdot x} \tag{4.4}$$

where the

$$\widehat{v}(p) = \int_{[0,L]^3} v(x)e^{-ip\cdot x}dx \tag{4.5}$$

are the Fourier coefficients of v.

A simple calculation shows that the Hamiltonian H_N in (2.1) is equal to the restriction of

$$\mathbb{H} = \sum_p |p|^2 a_p^\dagger a_p + \frac{1}{2L^3}\sum_p \widehat{v}(p)\sum_{q,k} a_{q+p}^\dagger a_{k-p}^\dagger a_k a_q \tag{4.6}$$

to the subspace $\mathscr{H}_N \subset \mathcal{F}$. Here, all sums are over $(\frac{2\pi}{L}\mathbb{Z})^3$. The expression (4.6) for the Hamiltonian on Fock space serves as a basis for the approximation introduced by Bogoliubov in [4], which we shall discuss next.

5. The Bogoliubov approximation

At low energy, and for sufficiently weak interactions, one expects the occurrence of Bose–Einstein condensation. That is, the zero momentum mode is expected to be macroscopically occupied, meaning that $a_0^\dagger a_0 \sim N$. In particular, the $p = 0$ mode plays a special role.

The *Bogoliubov approximation* consists of

- dropping all terms in $\mathbb{H}$ higher than quadratic in $a_p^\dagger$ and a_p for $p \neq 0$;
- replacing $a_0^\dagger$ and a_0 in $\mathbb{H}$ by the number $\sqrt{N}$.

The resulting Hamiltonian is quadratic in the $a_p^\dagger$ and a_p, and equals[2]

$$\mathbb{H}^{\mathrm{Bog}} = \frac{N(N-1)}{2L^3}\widehat{v}(0)$$

[2] Note that the contribution of $p = 0$ to the second sum in (4.6) is exactly equal to $N(N-1)\widehat{v}(0)/(2L^3)$, hence the substitution of $a_0^\dagger$ and a_0 by $\sqrt{N}$ was not applied to this term.

$$+\sum_{p\neq 0}\left(\left(|p|^2+\varrho\widehat{v}(p)\right)a_p^\dagger a_p+\tfrac{1}{2}\varrho\widehat{v}(p)\left(a_p^\dagger a_{-p}^\dagger+a_p a_{-p}\right)\right), \tag{5.1}$$

with $\varrho = N/L^3$ the particle density. It can be explicitly diagonalized via a *Bogoliubov transformation*:

Let $b_p = \cosh(\alpha_p)a_p + \sinh(\alpha_p)a_{-p}^\dagger$, with

$$\tanh(\alpha_p) = \frac{|p|^2+\varrho\widehat{v}(p)-\sqrt{|p|^4+2|p|^2\varrho\widehat{v}(p)}}{\varrho\widehat{v}(p)}. \tag{5.2}$$

Here, the right side is interpreted as 0 if $\widehat{v}(p) = 0$. Moreover, we have to *assume* that $|p|^2 + 2\varrho\widehat{v}(p) \geq 0$ for all p in order for the square root to be well-defined. The b_p and $b_p^\dagger$ again satisfy CCR (for any choice of real numbers α_p, in fact). A simple calculation shows that

$$\mathbb{H}^{\mathrm{Bog}} = E_0^{\mathrm{Bog}} + \sum_{p\neq 0} e_p b_p^\dagger b_p\,, \tag{5.3}$$

where

$$E_0^{\mathrm{Bog}} = \frac{N(N-1)}{2L^3}\widehat{v}(0) - \frac{1}{2}\sum_{p\neq 0}\left(|p|^2+\varrho\widehat{v}(p)-\sqrt{|p|^4+2|p|^2\varrho\widehat{v}(p)}\right) \tag{5.4}$$

and

$$e_p = \sqrt{|p|^4+2|p|^2\varrho\widehat{v}(p)}\,. \tag{5.5}$$

Note that in contrast to the non-interacting case, where $e_p = p^2$, the function e_p in (5.5) behaves linearly in p for small p (assuming that $\widehat{v}(p)$ does not vanish near zero).

The Bogoliubov approximation thus predicts that the ground state energy equals the value E_0^{Bog} displayed in (5.4). Moreover, it also allows to compute the complete excitation spectrum. In fact, from (5.3) we see that the spectrum of $\mathbb{H}^{\mathrm{Bog}} - E_0^{\mathrm{Bog}}$ is given by

$$\sum_p e_p n_p \quad \text{with } n_p \in \{0,1,2,\dots\}\,, \tag{5.6}$$

with e_p defined in (5.5). It has the exact same structure as for non-interacting particles (3.3), except for the replacement of $|p|^2$ by e_p. Moreover, the corresponding eigenstates can be constructed out of the ground state $\Psi_0 \in \mathcal{F}$ of $\mathbb{H}^{\mathrm{Bog}}$ by *elementary excitations* of the form

$$b_{p_n}^\dagger \cdots b_{p_1}^\dagger \Psi_0\,, \tag{5.7}$$

with $b_p^\dagger = \cosh(\alpha_p)a_p^\dagger + \sinh(\alpha_p)a_{-p}$, as before.

One can also calculate the ground state energy E_q^{Bog} in a sector of total momentum q, and arrives at

$$\begin{aligned} E_q^{\mathrm{Bog}} - E_0^{\mathrm{Bog}} &= \text{subadditive hull of } e_p \\ &= \inf_{\sum_p pn_p = q}\sum_p e_p n_p\,. \end{aligned} \tag{5.8}$$

In particular, also $E_q^{\mathrm{Bog}} - E_0^{\mathrm{Bog}}$ behaves linearly in q for not too large $|q|$.

For a detailed discussion of variants of the Bogoliubov approximation we refer the interested reader to [40].

6. The mean-field (Hartree) limit

It is a major open problem to understand the regime of validity of the Bogoliubov approximation for many-body quantum systems of interacting particles. The progress made in recent years was mainly limited to giving bounds on the ground state energy of the system, and we refer to [14, 15, 26–28, 37–39] for various interesting results in this direction. Virtually nothing is known concerning the excitation spectrum of such systems in general, however.

A simple case where the analysis of the validity of the Bogoliubov approximation can be extended beyond the ground state energy is the Hartree limit. This is an extreme form of a *mean-field limit*, where the interaction potential extends over the whole size of the system, but the interaction is sufficiently weak (of order $1/N$) in order for the interaction energy to be of the same order as the kinetic energy.

We consider again a system of N bosons in a cubic box, with periodic boundary conditions. For simplicity, let us choose units such that the length of the box L equals 1. The Hamiltonian of the systems is thus given by

$$H_N = -\sum_{i=1}^{N} \Delta_i + \frac{1}{N-1} \sum_{1\leq i<j\leq N} v(x_i - x_j)\,, \tag{6.1}$$

and it acts on the Hilbert space

$$\mathscr{H}_N = \bigotimes_{\mathrm{sym}}^{N} L^2([0,1]^3)\,. \tag{6.2}$$

Here we wrote the interaction potential as $(N-1)^{-1}v(x)$, reflecting the weakness of the potential as mentioned above. The case of fixed, N-independent v corresponds to the mean-field or *Hartree limit*.

It is not difficult to see that the ground state energy is determined, to leading order in N for large N, by minimizing the energy $\langle\Psi|H_N|\Psi\rangle$ over product states of the form

$$\Psi(x_1,\dots,x_N) = \phi(x_1)\cdots\phi(x_N)\,. \tag{6.3}$$

This has been shown, in a much more general setting than what is discussed here, in [19]. For a constant ϕ, corresponding to a homogeneous system, the resulting Hartree energy is then simply equal to $\frac{1}{2}N\int v$.

It is also known that starting from a product state of the form (6.3), a solution to the time-dependent Schrödinger equation $i\partial_t\Psi = H_N\Psi$ stays roughly a product at later times, with the factors in the limit $N\to\infty$ determined by the time-dependent Hartree equation

$$i\partial_t\phi = -\Delta\phi + 2\big(|\phi|^2 * v\big)\,\phi\,, \tag{6.4}$$

where $*$ denotes convolution. For a history of this problem and a review of recent results, we refer to [31].

Going beyond the leading order, where the Hartree equation applies, we can ask the following questions.

- Given that the ground state energy $E_0(N) = \inf\operatorname{spec} H_N$ satisfies $E_0(N) = \frac{1}{2}N\widehat{v}(0) + o(N)$ for fixed (i.e., N-independent) v, what is the next order correction? It turns out that it is actually $O(1)$, and the $O(1)$-term can be explicitly computed and agrees with the prediction from the Bogoliubov approximation.

- What is the spectrum of $H_N - E_0(N)$, i.e., the excitation spectrum of the system? Does it converge as $N \to \infty$? Is the Bogoliubov approximation valid? The latter predicts a dispersion law for elementary excitations that is *linear* for small momentum, as discussed in Section 5.
- What fraction of particles are in a Bose–Einstein condensate? Recall that Bose–Einstein condensation concerns the largest eigenvalue of the one-particle density matrix γ of a many-body wave function Ψ, defined via the matrix elements

$$\begin{aligned}&\langle f|\gamma|g\rangle\\ =&N\int \overline{f(x)}\Psi(x,x_2,\dots,x_N)g(y)\overline{\Psi(y,x_2,\dots,x_N)}\,dx\,dy\,dx_2\cdots dx_N\,. \end{aligned}\tag{6.5}$$

  For fixed v, the Bogoliubov approximation predicts that $\|\gamma\| \geq N - O(1)$ in the ground state, and this can actually be proved to be correct.

6.1. Main results. For our analysis of the excitation spectrum, we assume that $v(x)$ is bounded and of positive type, i.e.,

$$v(x) = \sum_{p\in(2\pi\mathbb{Z})^3} \widehat{v}(p)e^{ip\cdot x} \tag{6.6}$$

with

$$\widehat{v}(p) \geq 0 \quad \forall p \in (2\pi\mathbb{Z})^3 \quad , \sum_{p\in(2\pi\mathbb{Z})^3} \widehat{v}(p) < \infty\,. \tag{6.7}$$

Under these assumptions, the following theorem holds.

Theorem 6.1. *The ground state energy $E_0(N)$ of H_N equals*

$$E_0(N) = \frac{N}{2}\widehat{v}(0) + E_0^{\mathrm{Bog}} + O(N^{-1/2}) \quad \textit{as } N\to\infty, \tag{6.8}$$

with

$$E_0^{\mathrm{Bog}} = -\frac{1}{2}\sum_{p\neq 0}\left(|p|^2 + \widehat{v}(p) - \sqrt{|p|^4 + 2|p|^2\widehat{v}(p)}\right)\,. \tag{6.9}$$

Moreover, the excitation spectrum of $H_N - E_0(N)$ below an energy ξ is equal to finite sums of the form

$$\sum_{p\in(2\pi\mathbb{Z})^3\setminus\{0\}} e_p\, n_p + O\left(\xi^{3/2}N^{-1/2}\right)\,, \tag{6.10}$$

where

$$e_p = \sqrt{|p|^4 + 2|p|^2\widehat{v}(p)} \tag{6.11}$$

and $n_p \in \{0,1,2,\dots\}$ for all $p \neq 0$.

Theorem 6.1 is proved in [34]. The proof consists of constructing a unitary operator U that makes $UH_NU^\dagger$ close to the operator

$$\frac{N}{2}\widehat{v}(0) + E_0^{\mathrm{Bog}} + \sum_{p\in(2\pi\mathbb{Z})^3\setminus\{0\}} e_p\, a_p^\dagger a_p\,. \tag{6.12}$$

In particular, the proof implies that the excited eigenfunctions can be (approximately) obtained by acting with products of $U^\dagger a_p^\dagger a_0 U$ on the ground state.

Let us comment on the error terms in (6.8) and (6.10). Both the ground state energy and all excited energy levels a distance $O(1)$ from the ground state agree with the prediction obtained via Bogoliubov's approximation up to errors of order $N^{-1/2}$ for large N. Moreover, an excitation energy a distance ξ from the ground state energy is necessarily of the form $\sum_p e_p n_p(1+o(1))$ as long as $\xi^{3/2}N^{-1/2} \ll \xi$, i.e., for $\xi \ll N$. That is, the Bogoliubov approximation gives the correct excitation energies to leading order in a very large window above the ground state energy, whose size has to be small compared with N. This restriction is presumably optimal. The existence of Bose–Einstein condensation is only guaranteed for excitation energies small compared to N, and the existence of BEC is one of the key assumptions entering the Bogoliubov approximation.

Theorem 6.1 implies the following corollary concerning the momentum dependence of the spectrum of H_N.

Corollary 6.2. *Let $E_P(N)$ denote the ground state energy of H_N in the sector of total momentum P. We have*

$$E_P(N) - E_0(N) = \min_{\{n_p\},\, \sum_p p\, n_p = P} \sum_{p\neq 0} e_p\, n_p \;+\; O\left(|P|^{3/2}N^{-1/2}\right). \tag{6.13}$$

In particular,

$$E_P(N) - E_0(N) \geq |P| \min_p \sqrt{2\widehat{v}(p) + |p|^2} + O(|P|^{3/2}N^{-1/2}). \tag{6.14}$$

The bound (6.14) implies that $E_P(N) - E_0(N)$ behaves linearly in P for not too large P (assuming that $\widehat{v}(p)$ does not vanish for small $|p|$). Note that this fact is caused by the interactions among the particles, non-interacting systems do not show this behavior. The linear behavior is very important physically and is responsible for the superfluid behavior of the system. According to Landau, the coefficient multiplying $|P|$ in (6.14) is, in fact, the critical velocity for frictionless flow. We refer to [7] for further details on this correspondence.

Note that under the unitary transformation

$$\widetilde{U} = \exp\left(-iq \cdot \sum_{j=1}^N x_j\right), \quad q \in (2\pi\mathbb{Z})^3, \tag{6.15}$$

the Hamiltonian H_N transforms as

$$\widetilde{U}^\dagger H_N \widetilde{U} = H_N + N|q|^2 - 2q \cdot P, \tag{6.16}$$

where $P = -i\sum_{j=1}^N \nabla_j$ denotes again the total momentum operator. Hence our results apply equally also to the parts of the spectrum of H_N with excitation energies close to $N|q|^2$, corresponding to *collective excitations* where the particles move uniformly with momentum q; cf. Figure 6.1.

6.2. Ideas in the proof. In the language of second quantization, the Hamiltonian H_N is the restriction of the operator

$$\mathbb{H} = \sum_{p\in(2\pi\mathbb{Z})^3} |p|^2 a_p^\dagger a_p + \frac{1}{2(N-1)} \sum_p \widehat{v}(p) \sum_{q,k} a_{q+p}^\dagger a_{k-p}^\dagger a_k a_q \tag{6.17}$$

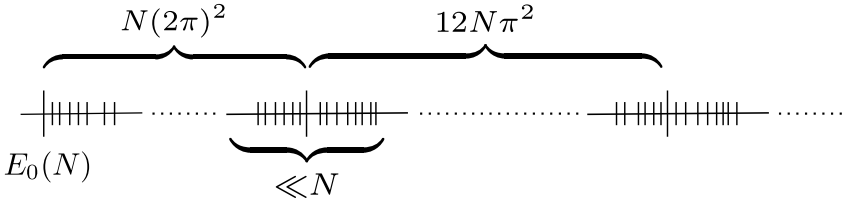


Figure 6.1. Sketch of the parts of the spectrum that are correctly determined by the Bogoliubov approximation in the Hartree limit.

to the N-particle subspace of the Fock space $\mathcal{F}$. Note that N has two different roles here. It determines the particle number, but also appears as a parameter in the Hamiltonian $\mathbb{H}$.

As discussed in Section 5, the Bogoliubov approximation consists of

- dropping all terms higher than quadratic in $a_p^\dagger$ and a_p, $p \neq 0$;
- replacing $a_0^\dagger$ and a_0 by $\sqrt{N}$.

The resulting quadratic Hamiltonian is $\frac{N}{2}\widehat{v}(0) + \mathbb{H}^{\mathrm{Bog}}$, where

$$\mathbb{H}^{\mathrm{Bog}} = \sum_{p\neq 0} \left(\left(|p|^2 + \widehat{v}(p) \right) a_p^\dagger a_p + \tfrac{1}{2}\widehat{v}(p) \left(a_p^\dagger a_{-p}^\dagger + a_p a_{-p} \right) \right) . \tag{6.18}$$

It is diagonalized via a Bogoliubov transformation of the form

$$b_p = \cosh(\alpha_p) a_p + \sinh(\alpha_p) a_{-p}^\dagger \,, \tag{6.19}$$

leading to

$$\mathbb{H}^{\mathrm{Bog}} = E_0^{\mathrm{Bog}} + \sum_{p\neq 0} e_p b_p^\dagger b_p \tag{6.20}$$

for an appropriate choice of α_p, with E_0^{Bog} and e_p defined in (6.9) and (6.11), respectively.

The proof of Theorem 6.1 consists of *two main steps*:

1. As a first step, one shows that H_N is well approximated by an operator similar to the Bogoliubov Hamiltonian $\mathbb{H}^{\mathrm{Bog}}$ in (6.18), but with a_p and $a_p^\dagger$ replaced by

$$a_p^\dagger \to c_p^\dagger := \frac{a_p^\dagger a_0}{\sqrt{N}} \quad , \quad a_p \to c_p := \frac{a_p a_0^\dagger}{\sqrt{N}} \,. \tag{6.21}$$

   Note that the operators c_p and $c_p^\dagger$ conserve the particle number. The resulting Hamiltonian is quadratic in $c_p^\dagger$ and c_p and is, in particular, also particle number conserving. Hence it has a chance of being close to H_N on the subspace of particle number N. The original Bogoliubov Hamiltonian (6.18) does not leave this subspace invariant, and hence can not be directly compared with H_N.

2. Mimicking the Bogoliubov transformation (6.19), we introduce the operators $d_p = \cosh(\alpha_p) c_p + \sinh(\alpha_p) c_{-p}^\dagger$. It turns out that the modified Hamiltonian from Step 1 is close to

$$E_0^{\mathrm{Bog}} + \sum_{p\neq 0} e_p \, d_p^\dagger d_p \,, \tag{6.22}$$

whose spectrum now has to be analyzed. This analysis is complicated by the fact that the operators d_p and $d_p^\dagger$ do *not* satisfy CCR. It turns out that they do, however, approximately on the subspace where $a_0^\dagger a_0$ is close to N, which is sufficient for our purpose.

In the following, we shall explain these two steps in greater detail. For further details, we refer to [34].

6.2.1. Step 1: Approximation by a quadratic Hamiltonian. Under our assumptions on the interaction potential v, it is not difficult to see that

$$N - a_0^\dagger a_0 \leq \text{const.}\ [1 + H_N - E_0(N)]\ . \tag{6.23}$$

This proves that the excitation energy dominates the condensate depletion. In particular, if the excitation energy is small compared with N, most particles occupy the zero momentum mode, i.e., Bose–Einstein condensation occurs.

To show that cubic and quartic terms in $a_p^\dagger$ and a_p, $p \neq 0$, in the Hamiltonian are negligible, one needs to prove a stronger bound of the form

$$\left(N - a_0^\dagger a_0\right)^2 \leq \text{const.}\ \left[1 + (H_N - E_0(N))^2\right]\ , \tag{6.24}$$

however. It implies that also the fluctuations in the number of particles outside the condensate are suitably small.

The first statement (6.23) follows easily from positivity of $\widehat{v}(p)$. Positivity implies that

$$\sum_{p \in (2\pi\mathbb{Z})^3 \setminus \{0\}} \widehat{v}(p) \left| \sum_{j=1}^N e^{ip\cdot x_j} \right|^2 \geq 0\,, \tag{6.25}$$

which can be rewritten as

$$\sum_{1 \leq i < j \leq N} v(x_i - x_j) \geq \frac{N^2}{2}\widehat{v}(0) - \frac{N}{2} v(0)\,. \tag{6.26}$$

Thus H_N is bounded from below as

$$H_N \geq \frac{N}{2}\widehat{v}(0) + T - \frac{N}{2(N-1)}\left(v(0) - \widehat{v}(0)\right)\ , \tag{6.27}$$

where T denotes the kinetic energy

$$T = -\sum_{i=1}^N \Delta_i\,. \tag{6.28}$$

The statement (6.23) follows from (6.27) since $T \geq (2\pi)^2 (N - a_0^\dagger a_0)$.

For the second statement (6.24) one has to work a bit more. It turns out to be useful to actually prove a slightly stronger bound, namely the inequality

$$\left(N - a_0^\dagger a_0\right) T \leq \text{const.}\ \left[1 + (H_N - E_0(N))^2\right]\ . \tag{6.29}$$

Since $T \geq (2\pi)^2(N - a_0^\dagger a_0)$ (and the two operators commute), this indeed implies the bound (6.24).

For the proof of (6.29), let us introduce the notation

$$N^> = N - a_0^\dagger a_0 = \sum_{i=1}^N Q_i \tag{6.30}$$

for the number of particles outside the condensate, where Q denotes the projection onto the subspace of $L^2([0,1]^3)$ of co-dimension one orthogonal to the constant function. For any bosonic (i.e., permutation-symmetric) wave function Ψ, we can write

$$\begin{aligned} \left\langle \Psi \left| N^> T \right| \Psi \right\rangle &= N \left\langle \Psi \left| Q_1 T \right| \Psi \right\rangle \\ &= N \left\langle \Psi \left| Q_1 S \right| \Psi \right\rangle + \left\langle \Psi \left| N^> \left(H_N - E_0(N) \right) \right| \Psi \right\rangle , \end{aligned} \tag{6.31}$$

where

$$\begin{aligned} S &= T - H_N + E_0(N) \\ &= E_0(N) - (N-1)^{-1} \sum_{i<j} v(x_i - x_j) . \end{aligned} \tag{6.32}$$

With the aid of the Cauchy–Schwarz inequality, the last term in (6.31) can be bounded as

$$\left\langle \Psi \left| N^> \left(H_N - E_0(N) \right) \right| \Psi \right\rangle \leq \left\langle \Psi \left| (N^>)^2 \right| \Psi \right\rangle^{1/2} \left\langle \Psi \left| \left(H_N - E_0(N) \right)^2 \right| \Psi \right\rangle^{1/2} . \tag{6.33}$$

We split S into two parts, $S = S_a + S_b$, with

$$S_a = E_0(N) - \frac{1}{N-1} \sum_{2 \leq i < j \leq N} v(x_i - x_j) \tag{6.34}$$

and

$$S_b = -\frac{1}{N-1} \sum_{j=2}^N v(x_1 - x_j) . \tag{6.35}$$

Note that S_a does not depend on x_1. By using positivity of $\widehat{v}(p)$ as in (6.25), but with the sum over j running from 2 to N only, as well as the simple upper bound $E_0(N) \leq \frac{N}{2} \widehat{v}(0)$ on the ground state energy, we see that

$$S_a \leq \frac{1}{2} \left(\widehat{v}(0) + v(0) \right) . \tag{6.36}$$

In particular, this implies that

$$N \left\langle \Psi \left| Q_1 S_a \right| \Psi \right\rangle \leq \frac{1}{2} \left(\widehat{v}(0) + v(0) \right) \left\langle \Psi \left| N^> \right| \Psi \right\rangle . \tag{6.37}$$

To bound the contribution of S_b, we use

$$\begin{aligned} -\langle \Psi | Q_1 S_b | \Psi \rangle = \left\langle \Psi \left| Q_1 v(x_1 - x_2) \right| \Psi \right\rangle &= \left\langle \Psi \left| Q_1 Q_2 v(x_1 - x_2) \right| \Psi \right\rangle \\ &\quad + \left\langle \Psi \left| Q_1 P_2 v(x_1 - x_2) P_2 \right| \Psi \right\rangle \end{aligned}$$

$$+ \langle \Psi | Q_1 P_2 v(x_1 - x_2) Q_2 | \Psi \rangle \,, \qquad (6.38)$$

where $P = 1 - Q$ denotes the rank-one projection onto the constant function in $L^2([0,1]^3)$. The second term on the right side of (6.38) is positive. For the first and the third, we use Schwarz's inequality and $\|v\|_\infty = v(0)$ to conclude that

$$\langle \Psi | Q_1 S_b | \Psi \rangle \leq v(0) \langle \Psi | Q_1 Q_2 | \Psi \rangle^{1/2} + v(0) \langle \Psi | Q_1 | \Psi \rangle \,. \qquad (6.39)$$

Since

$$\langle \Psi | Q_1 Q_2 | \Psi \rangle = \frac{\langle \Psi | N^>(N^> - 1) | \Psi \rangle}{N(N-1)} \leq \frac{\langle \Psi | (N^>)^2 | \Psi \rangle}{N^2} \,, \qquad (6.40)$$

we have thus shown that

$$\begin{aligned} \left\langle \Psi \left| N^> T \right| \Psi \right\rangle \leq \frac{1}{2} \left(\widehat{v}(0) + 3v(0) \right) \left\langle \Psi \left| N^> \right| \Psi \right\rangle \\ + \left(v(0) + \left\langle \Psi \left| \left(H_N - E_0(N) \right)^2 \right| \Psi \right\rangle^{1/2} \right) \left\langle \Psi \left| (N^>)^2 \right| \Psi \right\rangle^{1/2} \,. \end{aligned} \qquad (6.41)$$

Using that $N^> \leq (2\pi)^{-2} T$ in the last factor, this further implies that

$$\begin{aligned} \left\langle \Psi \left| N^> T \right| \Psi \right\rangle \leq \left(\frac{v(0) + \left\langle \Psi \left| \left(H_N - E_0(N) \right)^2 \right| \Psi \right\rangle^{1/2}}{2\pi} \right)^2 \\ + \left(3v(0) + \widehat{v}(0) \right) \left\langle \Psi \left| N^> \right| \Psi \right\rangle \,. \end{aligned} \qquad (6.42)$$

The desired result (6.29) then follows from (6.23).

6.2.2. An algebraic identity. The inequalities (6.23) and (6.29) allow us to conclude that $\mathbb{H}$ is, at low energy, well approximated by

$$\frac{N}{2} \widehat{v}(0) + \frac{1}{2} \sum_{p \neq 0} \left[A_p \left(c_p^\dagger c_p + c_{-p}^\dagger c_{-p} \right) + B_p \left(c_p^\dagger c_{-p}^\dagger + c_p c_{-p} \right) \right] \,, \qquad (6.43)$$

where $A_p = |p|^2 + \widehat{v}(p)$ and $B_p = \widehat{v}(p)$, and the operators c_p are defined in (6.21). A simple identity, which does *not* use the CCR, is

$$\begin{aligned} & A_p \left(c_p^\dagger c_p + c_{-p}^\dagger c_{-p} \right) + B_p \left(c_p^\dagger c_{-p}^\dagger + c_p c_{-p} \right) \\ & = \sqrt{A_p^2 - B_p^2} \left(\frac{\left(c_p^\dagger + \beta_p c_{-p} \right) \left(c_p + \beta_p c_{-p}^\dagger \right)}{1 - \beta_p^2} + \frac{\left(c_{-p}^\dagger + \beta_p c_p \right) \left(c_{-p} + \beta_p c_p^\dagger \right)}{1 - \beta_p^2} \right) \\ & \quad - \frac{1}{2} \left(A_p - \sqrt{A_p^2 - B_p^2} \right) \left([c_p, c_p^\dagger] + [c_{-p}, c_{-p}^\dagger] \right) \,, \end{aligned} \qquad (6.44)$$

where

$$\beta_p = \begin{cases} \frac{1}{B_p} \left(A_p - \sqrt{A_p^2 - B_p^2} \right) & \text{if } B_p > 0 \\ 0 & \text{if } B_p = 0. \end{cases} \qquad (6.45)$$

Note that if the operators c_p and $c_p^\dagger$ satisfied CCR, the term in the last line of (6.44) would be a constant. Its deviation from a constant can be controlled in terms of the condensate depletion, and the inequality (6.29) can be used to control the error made by simple replacing it by the value it would take in the case of CCR.

Introducing the operators

$$d_p = \frac{c_p + \beta_p c_{-p}^\dagger}{\sqrt{1-\beta_p^2}} \tag{6.46}$$

and their adjoints, we conclude that $\mathbb{H}$ is, in fact, close to the operator

$$\frac{N}{2}\widehat{v}(0) + E_0^{\mathrm{Bog}} + \sum_{p\neq 0} e_p\, d_p^\dagger d_p\,, \tag{6.47}$$

where we used that

$$E_0^{\mathrm{Bog}} = -\frac{1}{2}\sum_{p\neq 0}\left(A_p - \sqrt{A_p^2 - B_p^2}\right) \tag{6.48}$$

and

$$e_p = \sqrt{A_p^2 - B_p^2}\,. \tag{6.49}$$

6.2.3. Step 2: The spectrum of $d_p^\dagger d_p$. If the operators d_p and $d_p^\dagger$ satisfied CCR, we could immediately read off the spectrum of the operator in (6.47), and we would be done. However, without CCR we do not know the spectrum of $d_p^\dagger d_p$. Moreover, the various summands in (6.47) do not actually commute in our case.

The usual Bogoliubov transformation (6.19) is of the form

$$b_p = \cosh(\alpha_p)a_p + \sinh(\alpha_p)a_{-p}^\dagger = e^{-X}a_p e^X\,, \tag{6.50}$$

where X is the anti-hermitian operator

$$X = \frac{1}{2}\sum_{p\neq 0}\alpha_p\left(a_p^\dagger a_{-p}^\dagger - a_p a_{-p}\right)\,. \tag{6.51}$$

This identity can easily be verified using the CCR $[a_p, a_q^\dagger] = \delta_{p,q}$. Our operators $c_p = a_p a_0^\dagger/\sqrt{N}$, on the other hand, satisfy

$$\left[c_p, c_q^\dagger\right] = \delta_{p,q}\frac{a_0 a_0^\dagger}{N} - \frac{a_p a_q^\dagger}{N}\,. \tag{6.52}$$

We now define, in analogy to (6.51), the particle-number conserving anti-hermitian operator

$$\widetilde{X} = \frac{1}{2}\sum_{p\neq 0}\alpha_p\left(c_p^\dagger c_{-p}^\dagger - c_p c_{-p}\right)\,. \tag{6.53}$$

In order to compute the spectrum of $d_p^\dagger d_p$, we apply the unitary $e^{\widetilde{X}}$, and argue that the resulting operator is close $a_p^\dagger a_p$, at least in the subspace of low energy. More precisely, we show that

$$e^{-\widetilde{X}}a_p e^{\widetilde{X}} = \overbrace{\cosh(\alpha_p)c_p + \sinh(\alpha_p)c_{-p}^\dagger}^{d_p} + \mathrm{Error}_p \tag{6.54}$$

for suitable small error term. Here it is important that actually the sum over all error terms (depending on p) is still (relatively) small as long as $(N - a_0^\dagger a_0)^2 \ll N^2$. The proof of (6.54) is somewhat lengthy and will be skipped here. It proceeds by studying $e^{-t\widetilde{X}} a_p e^{t\widetilde{X}}$ as a function of $t \in [0, 1]$, using a Grönwall type estimate. The details are presented in [34].

6.3. Conclusions and generalizations. The mean-field or Hartree limit may be somewhat unphysical when it comes to the description of cold atomic gases. It can be used as a toy model, however, which is analytically much easier to handle than the Gross-Pitaevskii limit of dilute gases [23–25, 29, 35], for instance. The results reviewed in this paper are the first rigorous results concerning the excitation spectrum of an interacting Bose gas, in a suitable limit of weak, long-range interactions. With the notable exception of exactly solvable models in one dimension (like the Lieb–Liniger model [22], for instance), this is the only model where rigorous results on the excitation spectrum are available. The results verify Bogoliubov's prediction that the spectrum consists of sums of elementary excitations. In the translation invariant case, the excitation energy turns out to be linear in the momentum for small momentum. In particular, Landau's criterion for superfluidity is verified.

The methods presented in this paper can be generalized to inhomogeneous systems without translation invariance. This was shown in [16], where the excitation spectrum of the Hamiltonian

$$H_N = \sum_{i=1}^N (-\Delta_i + V(x_i)) + \frac{1}{N-1} \sum_{1 \leq i < j \leq N} v(x_i - x_j) \tag{6.55}$$

on the Hilbert space $\bigotimes_{\mathrm{sym}}^N L^2(\mathbb{R}^3)$ was studied, with a trap potential V that is locally bounded and tends to infinity at infinity, in order to ensure that all the particles are confined and cannot escape to infinity. Moreover, v is assumed to be non-negative, bounded, and of positive type. To leading order in N, the ground state energy of (6.55) is determined by minimizing the *Hartree functional*

$$\begin{aligned} \mathcal{E}^{\mathrm{H}}(\phi) = &\int_{\mathbb{R}^3} \left(|\nabla \phi(x)|^2 + V(x) |\phi(x)|^2 \right) dx \\ &+ \frac{1}{2} \int_{\mathbb{R}^6} |\phi(x)|^2 v(x-y) |\phi(y)|^2 dx dy \,, \end{aligned} \tag{6.56}$$

with minimal energy $E^{\mathrm{H}} = \inf\{\mathcal{E}^{\mathrm{H}}(\phi) : \int |\phi|^2 = 1\}$. Under the stated conditions on v and V, it is not difficult to see that there exists a unique minimizer ϕ_0 (up to a constant phase, of course, which we can choose such that ϕ_0 is positive) with $E^{\mathrm{H}} = \mathcal{E}^{\mathrm{H}}(\phi_0)$. The corresponding Euler-Lagrange equation for the minimizer ϕ_0 can be written as $K^{\mathrm{H}} \phi_0 = 0$, where K^{H} is the Hartree operator

$$K^{\mathrm{H}} = -\Delta + V(x) + v * |\phi_0|^2(x) - \varepsilon_0 \,, \tag{6.57}$$

with $\varepsilon_0 = E^{\mathrm{H}} + \frac{1}{2} \int_{\mathbb{R}^6} |\phi_0(x)|^2 v(x-y) |\phi_0(y)|^2 dx dy$ and $*$ denoting convolution.

The excitation spectrum of (6.55) turns out to have a similar structure as in (6.10), i.e., it consists of sums of elementary excitations. These are described by an effective one-body operator given by

$$E = \left(\sqrt{K^{\mathrm{H}}} \left(K^{\mathrm{H}} + 2W \right) \sqrt{K^{\mathrm{H}}} \right)^{1/2} , \tag{6.58}$$

where W denotes the operator with integral kernel $\phi_0(x)v(x-y)\phi_0(y)$. More precisely, to leading order in N the spectrum of $H_N - E_0(N)$ is of the form $\sum_i e_i n_i$, with $n_i \in \{0, 1, 2, \dots\}$ and e_i the (non-zero) eigenvalues of E. We refer to [16] for details.

By using different techniques, this result was further generalized in [21], where the validity of the Bogoliubov approximation in the Hartree limit was shown for a much larger class of Hamiltonians and interaction potentials. The method of [21] does not require that v has positive Fourier transform, for instance, one merely needs to assume that the corresponding Hartree functional has a unique minimizer and that its Hessian is strictly positive at the minimum. While the result of [21] applies to a much larger class of models, it does not yield so precise estimates on the error terms as the ones obtained in Theorem 6.1, and is restricted to studying the excitation spectrum in a smaller window above the ground state energy.

All the results in [10, 16, 21, 34] are limited to the case where the Hartree functional (6.56) has a unique minimizer (up to a constant phase). However, at least in the case of attractive interactions, uniqueness will not hold, in general (see, e.g. [2, 17]). Even with repulsive interactions, uniqueness can fail in the presence of magnetic fields or, equivalently, the case of rotating Bose gases [32, 33]. In this case, there can even be uncountably many minimizers. This happens, for instance, in rotating systems if the system is rotation invariant with respect to the axis of rotation, and the rotation speed is large enough for quantized vortices to form. If there is more than one such vortex, the rotation symmetry is necessarily broken in the minimizer, and hence there are infinitely many minimizers, which are all related via rotation.

An extension of the spectral analysis of H_N in the case when the corresponding Hartree functional has multiple minimizers was recently given in [30]. There it was shown that the excitation spectrum in the Hartree limit is given by the union of the excitation spectra of the Bogoliubov Hamiltonians constructed as above for the respective Hartree minimizers. In other words, all the low-energy eigenfunctions of H_N can be thought of as describing a BEC with condensate wave function given by one of the Hartree minimizers, and fluctuations around the pure condensate state described by Bogoliubov's theory.

In [30] also the question of existence of *collective excitations* was addressed, which are described by solutions of the Hartree equation

$$-\Delta\phi(x) + V(x)|\phi(x)|^2 + v * |\phi|^2(x)\phi(x) = \mu\phi(x) \tag{6.59}$$

for some $\mu \in \mathbb{R}$, that are not minimizers of $\mathcal{E}^{\mathrm{H}}$ and hence correspond to (non-linear) excited states of the Hartree functional. In the translation invariant case, collective excitations are related to the ground state via a Galileo transformation, as explained in Section 6.1. In the absence of translation invariance, there is no such symmetry, and the existence of such states is therefore a non-trivial question, in general. It was shown in [30] that corresponding to each solution of (6.59) there are eigenstates of the many-body Hamiltonian H_N which are approximately given by the eigenfunctions of the corresponding Bogoliubov Hamiltonian. These eigenfunctions are *high* energy excited states, their total energy differs by $O(N)$ from the ground state energy. We refer to [30] for details.

It remains to be seen to what extent the methods in [16, 34] or the ones in [21, 30] can be generalized to the study of less restrictive parameter regimes, away from the Hartree limit. A first step in this direction was recently taken in [10], where bounds were given on the maximally allowed rate at which the system size is allowed to grow with N in order for the Bogoliubov approximation to remain valid. Equivalently, one can let the interaction potential v depend on N and ask at what rate it is allowed to tend to a δ-function as $N \to \infty$. Since

all error terms in Theorem 6.1 are explicit, an estimate of this kind is actually contained in Theorem 6.1, but the dependence of the error terms on v was greatly improved in [10].

Finally, we mention that the validity of the Bogoliubov approximation in the Hartree limit can also be investigated concerning the dynamics generated by the Hamiltonian H_N. We refer to [20] and the references there for recent results in this direction.

7. Open problems

In this final section, we collect a list of open problems related to the Bogoliubov approximation for many-boson systems. Some of these problems have already been mentioned in the preceding sections.

- One of the key assumptions motivating the Bogoliubov approximation is the existence of *Bose–Einstein condensation*. While this property is easy to demonstrate in the Hartree limit discussed in the previous section, it is not known how to prove it in more general cases. In particular, the existence of BEC in the usual thermodynamic limit ($N \to \infty$, $L \to \infty$ with N/L^3 fixed) remains an open problem. The only model where the occurrence of BEC has been proved in the thermodynamic limit is the hard-core lattice gas at exactly half-filling, which is equivalent to the quantum XY spin model [11, 18]. BEC is also known to occur in the Gross-Pitaevskii limit of dilute trapped gases [23–25, 35].
- The results in Section 6 on the excitation spectrum concern the mean-field or Hartree limit, where the interaction among the particles is very weak and of long range. In fact, the range is of the same order as the system size. In view of applications to cold atomic gases, a physically more relevant limit would be the *Gross-Pitaevskii limit* [23–25, 35], where the interaction potential takes the form

$$v(x) = N^2 w(Nx) \tag{7.1}$$

  for some fixed, N-independent function w. As discussed in more detail in [16], one expects that in this limit the excitation spectrum is still of the form (6.10), but with $\widehat{v}(p)$ replaced by $8\pi a$, where a denotes the scattering length of w.
- An even more challenging problem concerns the low energy excitation spectrum in the *thermodynamic limit*, and to study its relation to the property of *superfluidity*. There are no rigorous results available up to now, not even rough bounds are known. In fact, not even the absence of a *spectral gap* in the thermodynamic limit of an interacting Bose gas is rigorously known. We refer to [7] for further discussion of this topic.

References

[1] M.H. Anderson, J.R. Ensher, M.R. Matthews, C.E. Wieman, and E.A. Cornell, *Observation of Bose–Einstein Condensation in a Dilute Atomic Vapor*, Science **269** (1995), 198–201.

[2] W.H. Aschbacher, J. Fröhlich, G.M. Graf, K. Schnee, and M. Troyer, *Symmetry breaking regime in the nonlinear Hartree equation*, J. Math. Phys. **43** (2002), 3879–3891.

[3] I. Bloch, J. Dalibard, and W. Zwerger, *Many-body physics with ultracold gases*, Rev. Mod. Phys. **80** (2008), 885–964.

[4] N. Bogolubov, *On the theory of superfluidity*, J. Phys. (U.S.S.R.) **11** (1947), 23–32.

[5] S.N. Bose, *Plancks Gesetz und Lichtquantenhypothese*, Z. Phys. **26** (1924), 178–181.

[6] N.R. Cooper, *Rapidly rotating atomic gases*, Adv. Phys. **57** (2008), 539–616.

[7] H.D. Cornean, J. Dereziński, and P. Zin, *On the infimum of the energy-momentum spectrum of a homogeneous Bose gas*, J. Math. Phys. **50** (2009), 062103.

[8] F. Dalfovo, S. Giorgini, L.P. Pitaevskii, and S. Stringari, *Theory of Bose–Einstein condensation in trapped gases*, Rev. Mod. Phys. **71** (1999), 463–512.

[9] K.B. Davis, M.O. Mewes, M.R. Andrews, N.J. van Druten, D.S. Durfee, D.M. Kurn, and W. Ketterle, *Bose–Einstein Condensation in a Gas of Sodium Atoms*, Phys. Rev. Lett. **75** (1995), 3969–3973.

[10] J. Dereziński, and M. Napiórkowski, *Excitation spectrum of interacting bosons in the mean-field infinite-volume limit*, preprint, arXiv:1305.3641, Ann. Henri Poincaré (in press).

[11] F.J. Dyson, E.H. Lieb, and B. Simon, *Phase Transitions in Quantum Spin Systems with Isotropic and Nonisotropic Interactions*, J. Stat. Phys. **18** (1978), 335–383.

[12] A. Einstein, *Quantentheorie des einatomigen idealen Gases*, Sitzber. Preuss. Akad. Wiss., phys.-math. Kl., 261–267 (1924); 3–14 (1925).

[13] A.L. Fetter, *Rotating trapped Bose–Einstein condensates*, Rev. Mod. Phys. **81** (2009), 647–691.

[14] J. Ginibre, *On the asymptotic exactness of the Bogoliubov approximation for many boson systems*, Commun. Math. Phys. **8** (1968), 26–51.

[15] A. Giuliani and R. Seiringer, *The Ground State Energy of the Weakly Interacting Bose Gas at High Density*, J. Stat. Phys. **135** (2009), 915–934.

[16] P. Grech and R. Seiringer, *The Excitation Spectrum for Weakly Interacting Bosons in a Trap*, Commun. Math. Phys. **322** (2013), 559–591.

[17] Y. Guo and R. Seiringer, *On the Mass Concentration for Bose–Einstein Condensates with Attractive Interactions*, Lett. Math. Phys. **104** (2014), 141–156.

[18] T. Kennedy, E.H. Lieb, B.S. Shastry, *The XY Model Has Long-Range Order for All Spins and All Dimensions Greater than One*, Phys. Rev. Lett. **61** (1988), 2582–2584.

[19] M. Lewin, P.T. Nam, and N. Rougerie, *Derivation of Hartree's theory for generic mean-field Bose systems*, Adv. Math. **254** (2014), 570–621.

[20] M. Lewin, P.T. Nam, and B. Schlein, *Fluctuations around Hartree states in the mean-field regime*, preprint arXiv:1307.0665.

[21] M. Lewin, P.T. Nam, S. Serfaty, and J.P. Solovej, *Bogoliubov spectrum of interacting Bose gases*, preprint, arXiv:1211.2778, Comm. Pure Appl. Math. (in press).

[22] E.H. Lieb and W. Liniger, *Exact Analysis of an Interacting Bose Gas. I. The General Solution and the Ground State*, Phys. Rev. **130** (1963), 1605–1616. E.H. Lieb, *Exact Analysis of an Interacting Bose Gas. II. The Excitation Spectrum*, Phys. Rev. **130** (1963), 1616–1624.

[23] E.H. Lieb and R. Seiringer, *Proof of Bose–Einstein Condensation for Dilute Trapped Gases*, Phys. Rev. Lett. **88** (2002), 170409.

[24] E.H. Lieb, R. Seiringer, J.P. Solovej, and J. Yngvason, *The Mathematics of the Bose Gas and its Condensation*, Oberwolfach Seminars, Vol. 34, Birkhäuser (2005); arXiv:cond-mat/0610117.

[25] E.H. Lieb, R. Seiringer, and J. Yngvason, *Bosons in a trap: A rigorous derivation of the Gross-Pitaevskii energy functional*, Phys. Rev. A **61** (2000), 043602.

[26] ———, *Justification of c-Number Substitutions in Bosonic Hamiltonians*, Phys. Rev. Lett. **94** (2005), 080401.

[27] E.H. Lieb and J.P. Solovej, *Ground State Energy of the One-Component Charged Bose Gas*, Commun. Math. Phys. **217** (2001), 127–163. Errata **225** (2002), 219–221.

[28] ———, *Ground State Energy of the Two-Component Charged Bose Gas*, Commun. Math. Phys. **252** (2004), 485–534.

[29] E.H. Lieb and J. Yngvason, *Ground state energy of the low density Bose gas*, Phys. Rev. Lett. **80** (1998), 2504–2507.

[30] P.T. Nam and R. Seiringer, *Collective excitations of Bose gases in the mean-field regime*, preprint, arXiv:1402.1153.

[31] B. Schlein, *Effective equations for quantum dynamics*, preprint, arXiv:1208.0185.

[32] R. Seiringer, *Gross-Pitaevskii Theory of the Rotating Bose Gas*, Commun. Math. Phys. **229** (2002), 491–509.

[33] ———, *Ground state asymptotics of a dilute, rotating gas*, J. Phys. A: Math. Gen. **36** (2003), 9755–9778.

[34] ———, *The Excitation Spectrum for Weakly Interacting Bosons*, Commun. Math. Phys. **306** (2011), 565–578.

[35] ———, *Hot topics in cold gases — A mathematical physics perspective*, Japan. J. Math. **8** (2013), 185–232.

[36] ———, *The excitation spectrum for Bose fluids with weak interactions*, Jahresber. Dtsch. Math.-Ver. **116** (2014), 21–41.

[37] J.P. Solovej, *Upper Bounds to the Ground State Energies of the One- and Two-Component Charged Bose Gases*, Commun. Math. Phys. **266** (2006), 797–818.

[38] A. Sütő, *Equivalence of Bose–Einstein condensation and symmetry breaking*, Phys. Rev. Lett. **94** (2005), 080402.

[39] H.-T. Yau and J. Yin, *The Second Order Upper Bound for the Ground Energy of a Bose Gas*, J. Stat. Phys. **136** (2009), 453–503.

[40] V.A. Zagrebnov and J.-B. Bru, *The Bogoliubov model of weakly imperfect Bose gas*, Phys. Rep. **350** (2001), 291–434.

IST Austria, Am Campus 1, 3400 Klosterneuburg, Austria
E-mail: robert.seiringer@ist.ac.at

Gauge theory angle at quantum integrability

Samson L. Shatashvili

Abstract. We review the relationship between supersymmetric gauge theories and quantum integrable systems. From the quantum integrability side this relation includes various spin chains, as well as many well-known quantum many body systems like elliptic Calogero-Moser system and generalisations. From the gauge theory side one has the supersymmetric gauge theory with four (and eight) supercharges in the space-time background which is a product of a d-dimensional torus, or a two dimensional cigar with Ω-deformation, and a flat space (with the total dimension of space-time being 2, 3, 4 or 5). The gauge theory perspective provides the exact energy spectrum of the corresponding quantum integrable system. Key notions, usually appearing in the topic of quantum integrability, such as Baxter equation, Yang-Yang function, Bethe equation, spectral curve, Yangian, quantum affine algebra, quantum elliptic algebra - all acquire meaning in these gauge theories.

Mathematics Subject Classification (2010). Primary 81T13; Secondary 14H70.

Keywords. Gauge theory, supersymmetry, quantum integrability, TQFT, CFT.

1. Introduction

Various types of relations between quantum integrable systems and gauge theories have been around for several decades, and appeared in many different contexts. This talk is dedicated to one such relation which was first discovered in [1].

Main motivation of [1] came from the previous works in physics literature on electric-magnetic duality in supersymmetric gauge theories [2, 3] and from the mathematics work [4]. In order to find the quantum field theory interpretation of [4] useful technique for integrating over Higgs branches in supersymmetric theories with 4 and 8 supercharges was developed in [1]. In particular - regularised (equivariant) volume for hyperkähler quotients was defined, evaluated for certain ALE and ALF spaces in terms of the hyperkähler periods, and reduced, for a large class of examples, to simpler integrals. Useful examples, studied in [1], were complex coadjoint orbits, instanton moduli spaces on R^4 and ALE manifolds, Hitchin spaces, and moduli spaces of parabolic Higgs bundles on Riemann surfaces. For particular case of Hitchin space it was found in [1] that such volume reduces to a summation over solutions of Bethe ansatz equations for the non-linear Schrödinger system for fixed particle number. This was later interpreted in [5, 6] as the correspondence between the quantum states of N-particle sector of non-linear Schrödinger system (which itself is described by the quantum many-body system of N-particles on a circle with pair-wise δ-function interaction) and the states in the topological quantum field theory associated with the particular supersymmetric gauge theory of [1]. Interestingly, this quantum many body system comes with so

Proceedings of the International Congress of Mathematicians, Seoul, 2014

called Yang-Yang (YY) function, a function whose critical points equation is a Bethe equation determining allowed energy spectrum (spectrum is discrete since we consider particles on a circle), and [5, 6] identified this YY-function with the effective twisted superpotential of the gauge theory.

Topologically twisted supersymmetric gauge theory describes only the vacuum sector of the physical gauge theory, and thus the sum over all states in the topologically twisted theory is the same as the sum over only vacuum states of the physical theory. This means that the correspondence can be restated as the correspondence between the quantum states of an integrable many-body system and the vacuum states of the supersymmetric gauge theory [7–10]. In [7–9] general 2d gauge theory with four supercharges on $R \times S^1$ (and with mass gap - necessary for the discrete spectrum of vacua) was shown to correspond to some quantum integrable system, and specific examples of such gauge theories were demonstrated to give the quantum spectrum of all known spin chains, like XXX, Habbard model, their generalisations, limits, and basically for any spin group. The lift of these gauge theories to 3d (on $R \times S^1 \times S^1$) and 4d (on $R \times S^1 \times T^2$) according to [7–9] in the vacuum sector relates to XXZ and XYZ spin chains correspondingly.

At the end of the day in all above cases one studies a 2d supersymmetric gauge theory on a cylinder $R \times S^1$ with finite or infinite number of fields (coming from reduction on torus from higher dimension, Kaluza-Klein (KK) modes). Another possibility [10] for generating effective 2d (3d) supersymmetric theory, originated also from [1] (and [11–16]), is to study 4d (5d) supersymmetric gauge theory in Ω-background [15, 16]. This background depends on two complex parameters ϵ_1, ϵ_2, and for generic values of these parameters it describes 0-dimensional theory. But, for $\epsilon_2 = 0, \epsilon_1 \neq 0$ it has 2d super-Poincare invariance and describes effective 2d supersymmetric gauge theory with infinite number of massive fields. The latter 2d gauge theory fits in the general picture of [7–9] and has twisted effective superpotential which can be explicitly calculated. The geometry of the moduli space of vacua for the theory with $\epsilon_1 = \epsilon_2 = 0$ has a structure of classical Algebraic Integrable System (AIS) [17–19]; it turns out the in the above gauge theory, for $\epsilon_1 \neq 0, \epsilon_2 = 0$, the critical points of the effective twisted superpotential describe the quantum spectrum of this AIS with $\epsilon = \epsilon_1$ being the Plank constant.

The rest of this paper is dedicated to the explanation of the previous paragraphs.

2. 2d SUSY Gauge theories and integrability

In this section we explain that: *essentially for every quantum integrable system solved by Bethe ansatz there is a supersymmetric gauge theory with 4 supercharges $Q_+, Q_-, \overline{Q}_+, \overline{Q}_-$ such that : a) exact Bethe eigenstates correspond to supersymmetric vacua, b) the ring of commuting Hamiltonians - to twisted chiral ring.*

In this correspondence the equations describing supersymmetric vacua in the gauge theory coincide with the Bethe equation of the integrable system. We think about the supersymmetric gauge theory in dimensions 2, 3, 4 and 5 as 2d theory with the infinitely many fields. In order to do this we need to compactify higher dimensional theories on $T^k, k = 1, 2, 3$ (Kaluza-Klein compactification), or study in the special Ω-background. Vacuum equations correspond to critical points of (shifted) effective twisted superpotential $\widetilde{W}^{eff}(\sigma)$ of this 2d theory. In the quantum integrable system usually one has the Yang-Yang (YY) function $Y(\lambda)$ whose critical point equations are Bethe equations (most of Bethe equations come

from YY-function with a few exceptions, for which YY-function is either unknown or might not exist). $\widetilde{W}^{eff}(\sigma)$, as a function of diagonalised complex scalar $\sigma_i, i = 1, ..., N$, of the vector multiplet (for $U(N)$ gauge group, but construction is valid for any group) coincides with $Y(\lambda)$ as a function of rapidities $\lambda_i, i = 1, ..., N$, under above correspondence with $\sigma_i = \lambda_i$. Importantly - the eigenvalues of quantum Hamiltonians in the integrable system, $\widehat{H}_k \Psi(\lambda) = E_k(\lambda)\Psi(\lambda)$, are vacuum expectation values of special operators in the gauge theory, chiral ring operators $O_k(\sigma)$: $< \sigma | O_k | \sigma >= E_k(\sigma)$.

The general $N = (2,2)$ supersymmetric gauge theory in 2d is defined by the gauge group $\mathbf{G}$ and its representation $\mathbf{R} = \bigoplus_{\mathbf{i}} \mathbf{M_i} \otimes \mathbf{R_i}$. $\mathbf{R_i}$ are irreducible representations of $\mathbf{G}$ and $\mathbf{M_i}$ are multiplicity spaces. The matter fields transform in the representation $\mathbf{R}$ and are in the chiral super-multiplet $\mathbf{X}(\mathbf{x}, \theta)$ (for fundamental, anti-fundamental and adjoint representation we use notations $\mathbf{Q}$, $\widetilde{\mathbf{Q}}$, and $\mathbf{\Phi}$). The gauge field $A(x)$ is in the vector multiplet $\mathbf{V}(\mathbf{x}, \theta)$ which also contains a complex scalar field $\sigma(x)$ in the adjoint representation, fermions and auxiliary fields. It is convenient to use the twisted chiral multiplet $\mathbf{\Sigma}(\mathbf{x}, \theta)$ (in the adjoint representation of $\mathbf{G}$) - the field strength $F(A)$ is in the twisted chiral multiplet $\mathbf{\Sigma} = D_+ \overline{D}_- \mathbf{V}$ ($D_\pm, \overline{D}_\pm$ - differential operators in the superspace, form the representation of 2d super-Poincare algebra), as well as complex scalar $\sigma(x)$, fermions and auxiliary fields. Superspace coordinates are x, θ; four supercharges - so four θ's, $\theta^+, \theta^-, \overline{\theta}^+, \overline{\theta}^-$; x are local coordinates in 2d space-time. The global group $H^{\max} = \times_{\mathbf{i}} U(\mathbf{M_i})$ acts on $\mathbf{R}$ and this action commutes with the gauge group action. The actual global symmetry group H of the theory may be smaller: $H \subset H^{\max}$.

The action functional is a sum of three terms - D ($d^4\theta$ integration), F and the twisted F terms (both have only $d^2\theta$ integration, but different combinations):

$$A = \int \mathrm{d}^2x\, \mathrm{d}^4\theta\, [\,\mathrm{tr}\, \left(\mathbf{\Sigma}\overline{\mathbf{\Sigma}}\right) + \mathbf{K}(e^{\mathbf{V}/2}\,\mathbf{X}\,,\, \overline{\mathbf{X}}\, e^{\mathbf{V}/2})] + \int \mathrm{d}^2x\, \mathrm{d}\theta^+ \mathrm{d}\theta^-\, [W(\mathbf{X}) \,+\, \mathrm{c.c.}]$$
$$+ \int \mathrm{d}^2x\, \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^-\, [\widetilde{W}(\mathbf{\Sigma}) \,+\, \mathrm{c.c.}] + \int \mathrm{d}^4\theta\, Tr_{\mathbf{R}}\, \mathbf{X}^\dagger \left(\sum_{\mathbf{i}} e^{\widetilde{V}_{\mathbf{i}}} \otimes \mathrm{Id}_{R_{\mathbf{i}}}\right) \mathbf{X} \qquad (2.1)$$

We included in (2.1) separately a "twisted mass term" ($\widetilde{V}_{\mathbf{i}} = \widetilde{m}_{\mathbf{i}}\, \theta^+ \overline{\theta}^-$ and $\widetilde{m}_{\mathbf{i}}$ are complex numbers - "twisted" masses) [20], [21] though it has the form of D-term for matter fields (or because of very special form of $\widetilde{V}$ it can be also viewed as a part of F-term). Here $\widetilde{m} = (\widetilde{m}_{\mathbf{i}})$, $\widetilde{m}_{\mathbf{i}} \in \mathrm{End}\,(\mathbf{M_i}) \cap H$. Action can have the field content corresponding to the higher, $N = 4$, supersymmetry and twisted masses can break it to $N = 2$; those twisted masses which preserve the $N = 4$ will be denoted by m, and the ones which break it down to the $N = 2$, by u.

The superpotential $W(X)$ has to be a holomorphic gauge invariant function of the chiral matter fields, such as $\Phi, Q, \widetilde{Q}$ and other representations. The twisted superpotential $\widetilde{W}(\Sigma)$ has to be a holomorphic function of twisted superfield Σ. $\widetilde{W}(\Sigma)$ may contain, for each $U(1)$ factor in $\mathbf{G}$, so-called "complexified" θ-term (θ and Fayet-Illiopoulos (FI) terms) which is linear in Σ. In order to write such terms one introduces non-dynamical twisted superfield Σ'_b (for every $U(1)$-factor labeled by b) whose scalar component is $t_b \equiv \widetilde{\sigma}_b = \frac{\vartheta_{\mathbf{b}}}{2\pi} + i r_{\mathbf{b}}$ and writes $\int d^2x d\theta^+ d\theta^- \Sigma'_{\mathbf{b}} tr \Sigma^{\mathbf{b}}$.

When $\mathbf{R}$ is a sum of a fundamental representation $\mathbf{R_f}$ with multiplicity $n_{\mathbf{f}}$ and anti-fundamental representation $\mathbf{R}_{\overline{\mathbf{f}}}$ with multiplicity $n_{\overline{\mathbf{f}}}$ - complex masses come from the super-

potential

$$W_{\text{complex mass}}(X) = \sum_{a,b} m_a^b \widetilde{Q}_b Q^a,$$

which breaks the $H^{max} = U(n_{\mathbf{f}}) \times U(n_{\overline{\mathbf{f}}})$ down to $U(1)^{\min(n_{\mathbf{f}}, n_{\overline{\mathbf{f}}})}$. When $\mathbf{R}$ contains one more representation, the adjoint $\mathbf{R}_{adj}$, one can consider more sophisticated superpotential:

$$W_{\widetilde{Q}\Phi Q} = \sum_{a,b} \widetilde{Q}^a m_a^b(\Phi) Q_b = \sum_{a,b;s} m_{a;s}^b \widetilde{Q}^a \Phi^{2s} Q_b \tag{2.2}$$

$n_{\mathbf{f}} = n_{\overline{\mathbf{f}}} = L$ we later call the **main example**.

The theory can be "abelianised". When the twisted masses are turned on in the generic fashion, the matter fields are massive and can be integrated out in the path integral. The theory becomes an effective pure $N = 2$ gauge theory with an infinite number of interaction terms in the Lagrangian, with the high derivative terms suppressed by the inverse masses of the fields we integrated out. Of all these terms the term without derivatives, the effective twisted superpotential $\widetilde{W}^{\text{eff}}(\Sigma)$ (this is the only one without derivatives), can be computed exactly. In fact, it receives only one-loop contribution and is mathematically well-formulated as a regularised determinant. This procedure, abelianisation, is formulated in a mathematically precise terms in the language of topologically twisted theory (which is equivalent to the vacuum sector of physical theory described in this section). Let $\widetilde{\mathbf{m}}$ denote collectively the set of the twisted masses of the fields we are integrating out. We get following contribution from matter fields to the effective twisted superpotential:

$$\widetilde{W^{\text{eff}}}_{\text{matter}}(\sigma) = \sum_{\mathbf{b}} 2\pi i \, t_{\mathbf{b}} tr_{\mathbf{b}} \sigma + tr_{\mathbf{R}} \left(\sigma + \widetilde{\mathbf{m}}\right) \left(\log\left(\sigma + \widetilde{\mathbf{m}}\right) - 1\right) \tag{2.3}$$

where for each $U(1)$ factor in $\mathbf{G}$ we have introduced by hand at tree-level a t_b term.

There are other massive fields which can be integrated out on the Coulomb branch in a similar fashion - $\mathbf{g}/\mathbf{t}$-components of the vector multiplets (where $\mathbf{g}$ denotes Lie algebra corresponding to Lie group $\mathbf{G}$ and $\mathbf{t}$ is its Cartan sub-algebra), the W-bosons and their superpartners. Their contribution is simple:

$$\widetilde{W}^{\text{eff}}_{\text{gauge}}(\sigma) = -\sum_{\alpha \in \Delta} \langle \alpha, \sigma \rangle \left[\log \langle \alpha, \sigma \rangle - 1 \right] = -2\pi i \langle \rho, \sigma \rangle \tag{2.4}$$

where $\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha$ is a half the sum of the positive roots of $\mathbf{g}$. The total effective twisted superpotential is, therefore: $\widetilde{W}^{\text{eff}}(\sigma) = \widetilde{W}^{\text{eff}}_{\text{matter}}(\sigma) + \widetilde{W}^{\text{eff}}_{\text{gauge}}(\sigma)$. It is a functional of twisted superfield Σ - we write it as $\widetilde{W}^{\text{eff}}(\sigma)$ for simplicity.

2.1. Examples. There are two classes of examples in 2d: a) the asymptotically free theories and b) the asymptotically conformal theories. The a) examples include the gauge theories which look at low energy as the $N = 2$ sigma models with various Kähler target spaces: the complex projective space $\mathbf{CP}^{L-1}$, the Grassmanian $\text{Gr}(N, L)$, or, more generally, the (partial) flag variery $F(n_1, n_2, \ldots, n_{\mathbf{r}}, n_{\mathbf{r}+1} \equiv L)$. The b) examples can also be identified at the low energy level with the sigma models. These sigma models typically have the hyperkähler target spaces, such as the cotangent bundles to the Kähler manifolds from the a) list. The b) examples turn out to include essentially all known quantum integrable models of

statistical physics. By taking an appropriate scaling limit one can get the $a)$ examples from the $b)$ examples, so it is sufficient to focus on the $b)$ cases. For example, the Grassmanian model is a limit of the $T^*\mathrm{Gr}(N,L)$ model in the limit where the twisted mass u corresponding to the rotations of the cotangent direction is sent to infinity, with the complexified Kähler class adjusted in such a way, that the effective mass scale $\Lambda_{\mathrm{Gr}} = ue^{\frac{2\pi i t}{L}}$ remains finite (this will correspond to a non-Hermitian deformation of Heisenberg magnet which itself is dual to original $T^*\mathrm{Gr}(N,L)$ theory).

In the above general construction 2d theory with matter fields in the representation $\mathbf{R}$ can have each irreducible representation $\mathbf{R_i}$ infinitely many times, but with different twisted mass. There are also two interesting types of such examples known, both originating in higher dimensions: I) 2d theory can come from 3d theory compactified on S^1 (or from 4d theory on T^2, 5d theory on T^3, ..., KK compactification) - 2d fields, labeled by the Fourier mode n_i, will be in the same representation $\mathbf{R}_i$ for all n_i; II) two-dimensional effective theory can be a 4d(5d) theory in a special Ω-background. We will give bellow the twisted effective superpotentials both for type I and type II theories (for $G = U(N)$, other groups and more examples can be found in [7–10]).

Main example in 2d. $\mathbf{R} \times \mathbf{S}^1$. In this example $H^{\max} = U(L) \times U(L) \times U(1)$ global symmetry group. Matter fields are: $\mathbf{Q}_a$, $a = 1, ..., L$, in the N-dimensional fundamental representation of $\mathbf{G} = U(N)$, $\widetilde{\mathbf{Q}}^a$ - L copies of anti-fundamental representation and one Φ in the adjoint representation. This theory has the matter content of the 4d $N_c = N$, $N_f = L$, $N = 2$ theory with fundamental hypermultiplets, however, the supersymmetry is half of the 4d theory. It has $2L + 1$ twisted mass parameters $\widetilde{m} = (m_a^{\mathrm{f}}, m_a^{\bar{\mathrm{f}}})_{a=1}^L$ and $\widetilde{m}^{\mathrm{adj}} \equiv -iu$. For $\widetilde{W}^{\mathrm{eff}}_{\widetilde{Q}\Phi Q}(\sigma)$ we have:

$$\sum_{i=1}^{N}\sum_{a=1}^{L}\left[\left(\sigma_i + m_a^{\mathrm{f}}\right)\left(\log\left(\sigma_i + m_a^{\mathrm{f}}\right) - 1\right) - \left(\sigma_i - m_a^{\bar{\mathrm{f}}}\right)\left(\log\left(-\sigma_i + m_a^{\bar{\mathrm{f}}}\right) - 1\right)\right]$$
$$+ \sum_{i,j=1}^{N}\left(\sigma_i - \sigma_j + m^{\mathrm{adj}}\right)\left(\log\left(\sigma_i - \sigma_j + m^{\mathrm{adj}}\right) - 1\right) - 2\pi i\sum_{i=1}^{N}\left(t + i - \tfrac{1}{2}(N+1)\right)\sigma_i \tag{2.5}$$

The generic twisted masses are incompatible with any tree level superpotential. However, for the special choice of the twisted masses, not affecting final (2.5), $m_a^{\mathrm{f}} = -\mu_a + is_a u$, $m_a^{\bar{\mathrm{f}}} = \mu_a + is_a u$, $m^{adj} = -iu$, one can turn on the tree level superpotential (2.2) with $m^b_{a;s} = \delta^b_a \alpha_a$ ((2.2) has $U(1)^{L+1}$ global symmetry).

Main example in 3d. This is example of Type I. We consider the theory on $\mathbf{R}^1 \times \mathbf{S}^1 \times \mathbf{S}^1$. It suffices to make all the fields depend on an extra coordinate $x^2 = y$, $y \sim y + 2\pi$. Since the translations in y are the global symmetry of the theory like for every global symmetry we can turn on the corresponding twisted mass $\widetilde{m}_{\mathbf{S}^1}$. The twisted mass corresponding to the translations is $\widetilde{m} = \frac{i}{R}$. Thus, the KK modes with momentum n, $n \in \mathbf{Z}$, have the twisted mass $\widetilde{m}_n = \frac{in}{R}$. To compute the effective twisted superpotential it is enough to enumerate the KK modes and sum up their contributions. One needs to use a kind of zeta-regularization, which can be justified by topological field theory methods. Let $\widetilde{\mathbf{m}}$ denote the ordinary two dimensional twisted mass, corresponding to the centralizer of $\mathbf{G}$ in $\mathbf{R}$ which preserves other couplings of the theory, such as the superpotential. We assume $\widetilde{\mathbf{m}}$ sufficiently generic so

that all the modes of the matter multiplet are massive. The sum over KK modes for matter multiplet and W's give:

$$\begin{aligned}\widetilde{W}^{\rm eff}_{\rm matter}(\sigma) &= {\rm tr}_{\mathbf{R}}\left[\sum_{n\in\mathbf{Z}}\left(\sigma+\widetilde{\mathbf{m}}+\frac{in}{R}\right)\left(\log\left(\sigma+\widetilde{\mathbf{m}}+\frac{in}{R}\right)-1\right)\right]\\ &\sim \frac{1}{2\pi R}\,{\rm tr}_{\mathbf{R}}\left[{\rm Li}_2\left(e^{-2\pi R(\sigma+\widetilde{\mathbf{m}})}\right)\right]\end{aligned} \tag{2.6}$$

$$\widetilde{W}^{\rm eff}_{\rm gauge} = -\,{\rm tr}_{\mathbf{g/t}}\left[\frac{1}{2\pi R}{\rm Li}_2\left(e^{-2\pi R\sigma}\right)\right] = \frac{\pi R}{2}\,{\rm tr}_{\rm adj}\left(\sigma^2\right)+2\pi i\,\langle\rho,\sigma\rangle \tag{2.7}$$

We used: ${\rm Li}_2(e^{-x})+{\rm Li}_2(e^{x}) = \frac{\pi^2}{3} - i\pi x - \frac{x^2}{2}$ and dropped an irrelevant constant. The quadratic term in (2.7) corresponds to the anomaly-induced Chern-Simons interaction [22, 23] in the three dimensional theory.

Main example in 4d. This is again the Type I example. We consider the theory on $\mathbf{R}^2\times\mathbf{T}^2$ - the lift to the $N=1$ supersymmetric Yang-Mills theory (with matter), compactified on a two-torus $\mathbf{T}^2$. Again, we can view the lift to 4d as the 2d theory with the infinite number of fields, which depend on the two additional coordinates (y,z), with $y\sim y+2\pi$, $z\sim z+2\pi$. The theory is regularised by the twisted masses corresponding to the translations along $\mathbf{T}^2$. We choose one of the masses to be $\frac{i}{R}$, then the other is $\frac{i\tau}{R}$. τ is the complex modulus of $\mathbf{T}^2$. The normalized holomorphic coordinate on $\mathbf{T}^2$ is: $w=\frac{1}{2\pi}\,(y+\tau z)$. The gauge theory is sensitive to the metric on the torus and a 2-form (B-field) via the coupling $\int_{\mathbf{R}^2\times\mathbf{T}^2} B\wedge{\rm tr}\,F$. The invariance under the large gauge transformations translates to the double-periodicity of the twisted superpotential: $\sigma\to\sigma+\frac{i}{R}\,(m+n\tau)$, $m,n\in\mathbf{Z}$. The effective twisted superpotential up to linear terms is ($\mathrm{q}=\exp 2\pi i\tau$):

$$\begin{aligned}\widetilde{W}^{\rm eff}(\sigma) &= \frac{\pi R}{2}\,{\rm tr}_{\mathbf{R}}(\sigma+\widetilde{\mathbf{m}})^2+\frac{\pi i\tau}{6}\,{\rm tr}_{\mathbf{R}}(\sigma)\\ &\quad+\frac{1}{2\pi R}\sum_{n=1}^{\infty}{\rm tr}_{\mathbf{R}}\left[{\rm Li}_2\left(\mathrm{q}^{n-1}e^{-2\pi R(\sigma+\widetilde{\mathbf{m}})}\right)-{\rm Li}_2\left(\mathrm{q}^{n}e^{2\pi R(\sigma+\widetilde{\mathbf{m}})}\right)\right]\\ &= \frac{\pi R}{2}\,{\rm tr}_{\mathbf{R}}(\sigma+\widetilde{\mathbf{m}})^2+\frac{\pi i\tau}{6}\,{\rm tr}_{\mathbf{R}}(\sigma)+\frac{1}{2\pi R}\sum_{n\in\mathbf{Z}_{\neq 0}}\frac{{\rm tr}_{\mathbf{R}}\left[e^{2\pi Rn(\sigma+\widetilde{\mathbf{m}})}\right]}{n^2(1-\mathrm{q}^n)}\end{aligned} \tag{2.8}$$

Ω-background in 4d. This is the Type II example. We consider special Ω-background: $\mathbf{R}^2\times\mathbf{R}^2_\epsilon$. The ordinary KK compactification on a torus $\mathbf{T}^k$ is equivalent to the theory with twisted masses corresponding to the translations in extra dimensions $\mathbf{R}^k$. Alternatively one can study the twisted masses corresponding to the rotational symmetry. Consider a four dimensional $N=2$ theory on a four manifold M^4 fibered over a 2d base Σ with the Ω-background along the fibers $\mathbf{R}^2$. Somewhat schematically we shall denote the fibers by $\mathbf{R}^2_\varepsilon$. The base Σ of the fibration could be $\mathbf{R}^{1,1}$, or $\mathbf{R}^1\times\mathbf{S}^1$, or arbitrary Riemann surface. Let us denote the coordinates on the fiber $\mathbf{R}^2_\varepsilon$ by (x^2,x^3), and the coordinates on the base Σ by (x^0,x^1). Introduce the vector field $U=x^2\partial_3-x^3\partial_2$ generating the $U(1)$ rotation in $\mathbf{R}^2$. Let $\varepsilon\in\mathbf{C}$ be a complex parameter and let $V=\varepsilon U, \overline{V}=\overline{\varepsilon}U$ be the complex vector fields on $\mathbf{R}^2$. The bosonic part of the pure $N=2$ super-Yang-Mills Lagrangian of the theory on

$\mathbf{R}^2 \times \mathbf{R}^2_\varepsilon$ is simply [15], [16]:

$$\begin{aligned} L = &- \frac{1}{4g_0^2}\,\mathrm{tr}\ F \wedge \star F + \mathrm{tr}\ \left(D_A\phi - \varepsilon\iota_U F\right) \wedge \star \left(D_A\overline{\phi} - \overline{\varepsilon}\iota_U F\right) \\ &+ \frac{1}{2}\,\mathrm{tr}\ \left([\phi,\overline{\phi}] + \iota_U D_A\left(\varepsilon\overline{\phi} - \overline{\varepsilon}\phi\right)\right)^2 + \frac{\theta_0}{2\pi}\,\mathrm{tr}\, F \wedge F \end{aligned} \tag{2.9}$$

It is clear that the Poincare invariance in the (x^0, x^1) directions is unbroken. It is possible to show that the whole 2d $N = 2$ super-Poincare invariance is preserved. Thus, the theory is effectively 2d and there are 4 supercharges, so we are in our original setup; like in KK case, we have infinitely many matter fields. Twisted effective superpotential is again exactly computable (we use notation a for complex scalar instead of σ since it is common notation in 4d). If we send ε back to zero then the low energy theory is fully 4d with a continuous moduli space M_{v} of vacua.

In [10] the superpotential $\widetilde{W}^{eff}(a,\epsilon)$ was computed for the several important examples of 4d $N = 2$ gauge theories on $R^2 \times R^2_\varepsilon$. One starts with the partition function $Z(a, \varepsilon_1, \varepsilon_2; q)$ of 4d theory in the general Ω-background $R^2_{\epsilon_1} \times R^2_{\epsilon_2}$, with both rotation parameters ε_1, ε_2 non-zero - SW-prepotential $F(a; q)$, which defines the two-derivative low-energy effective action of 4d $N = 2$ theory, is [15, 16]:

$$F(a;q) = \mathrm{Lim}_{\varepsilon_1\to 0,\varepsilon_2\to 0}\ \ [\varepsilon_1\varepsilon_2 \log Z(a,\varepsilon_1,\varepsilon_2;q)] = F^{\mathrm{pert}}(a;q) + F^{\mathrm{inst}}(a;q) \tag{2.10}$$

Perturbative part of prepotential (as in the case of abelianisation for 2d supersymmetric gauge theory and effective twisted superpotential) is one loop exact and can be mathematically well-formulated. The non-perturbative/instanton part is an infinite sum $F^{\mathrm{inst}}(a;q) = \sum_{n=1}^{\infty} q^n F_n(a)$ where q is an instanton counting parameter for a given theory - e. g. for $N = 2^*$ theory it is the complexified bare gauge coupling of the action functional (2.9): $q = \exp 2\pi i\tau; \quad \tau = \frac{\vartheta_0}{2\pi} + \frac{4\pi i}{{g_0}^2}$.

According to [10] the *universal* part of the twisted effective superpotential for the 2d theory corresponding to the special Ω-background with $\epsilon_2 = 0, \epsilon_1 = \epsilon$ is:

$$\widetilde{W}(a;\varepsilon;q) = \mathrm{Limit}_{\varepsilon_2\to 0}\ \ [\varepsilon_2 \log Z(a,\varepsilon_1 = \varepsilon,\varepsilon_2;q)] = W^{\mathrm{pert}}(a;\varepsilon;q) + W^{\mathrm{inst}}(a;\varepsilon;q). \tag{2.11}$$

The perturbative part has a tree level term, proportional to $\log(q)$ and the one-loop term, which is q-independent; the non-perturbative/instanton part has, again, infinite expansion: $W^{\mathrm{inst}}(a;\varepsilon;q) = \sum_{k=1}^{\infty} q^k W_k^{\mathrm{inst}}(a;\varepsilon)$. When we view 2d theory in a special Ω-background as 2d theory with infinitely many fields in order to have a well-defined effective theory we need to specify boundary condition at infinity; one may view the choice of the boundary condition at infinity as a choice of a 3d supersymmetric theory compactified on a circle, coupled to the original 4d theory on the product of a cigar-like 2d geometry and the 2d Minkowski world-sheet of the resulting theory. The 3d theory at infinity is compactified on a circle because of the asymptotics of the cigar-like geometry [24] (we will briefly return to this in **Section 3**), which looks like $\mathbf{R} \times \mathbf{S}^1$ at infinity; the contribution $\widetilde{W}^{\infty}(a,\epsilon,q)$ of the 3d theory at infinity of the cigar is purely perturbative (see 3d example above) and is given by a sum of dilogarithims. Total superpotential is [10, 25]:

$$\widetilde{W}^{eff}(a,\epsilon,q) = \widetilde{W}(a,\epsilon,q) + \widetilde{W}^{\infty}(a,\epsilon,q) \tag{2.12}$$

As we see, there is a certain ambiguity in the perturbative part. The instanton part is unambiguous. It follows from (2.10), (2.11), (2.12) that in the limit $\varepsilon \to 0$ the twisted superpotential $W(a,\varepsilon)$ behaves as:

$$W(a;\varepsilon;q) = \frac{F(a;q)}{\varepsilon} + regular\ in\ \epsilon\ terms \tag{2.13}$$

2.2. Supersymmetric vacua of $N = 2$ theories.
We return to the abelianised effective 2d theory where we know exact effective twisted superpotential. The only local gauge invariant of the abelian gauge field in 2d is the field strength F_{01} subject to the global constraint: $\frac{1}{2\pi i}\int_\Sigma F^i = m^i \in \mathbf{Z}$, the integrality of the magnetic flux. In addition, the global invariants of the $\mathbf{T}$-valued gauge field include the holonomies, which are irrelevant for our discussion at the moment. To minimize the potential energy and find the vacua we promote F^α_{01} to the independent fields, while adding at the same time the term $\sum_{i=1}^r n_i \int_\Sigma F^i$ to the action (cf. [26–28]). The latter shift is equivalent to the shift [13]:

$$\widetilde{W}^{\rm eff}(\sigma) \longrightarrow \widetilde{W}^{\rm eff}_{\vec{n}}(\sigma) = \widetilde{W}^{\rm eff}(\sigma) - 2\pi i \sum_{i=1}^r n_i \sigma^i \tag{2.14}$$

where now $H \pm iF_{01}$ (here H is a 2-form auxiliary field, the top component of twisted super field $\Sigma(x,\theta)$ in the θ expansion) are two independent auxiliary fields, which can be integrated out. Thus, the target space of the effective sigma model becomes, *a priori*, disconnected, with $\vec{n}$ labeling the connected components. In fact, the actual connected components are labeled by the equivalence classes of $\vec{n}$ up to the action of the monodromy group (the effective superpotential is not a univalent function of σ). After eliminating the auxiliary fields the exact potential (term in the effective action without derivatives) on the $\vec{n}$-th component is:

$$U_{\vec{n}} = \tfrac{1}{2}\mathrm{g}^{ij}(-2\pi i n_i + \frac{\partial\widetilde{W}^{\rm eff}(\sigma)}{\partial\sigma^i})(+2\pi i n_j + \frac{\partial\overline{\widetilde{W}}^{\rm eff}(\sigma)}{\partial\overline{\sigma}^j}) \tag{2.15}$$

where g^{ij} is a Kähler metric (from D-term - we mentioned above that D-term can not be computed exactly, only F^{tw}-term can be computed exactly so we need to assume that Kähler metric is non-degenerate). Thus, the minima are at:

$$\frac{1}{2\pi i}\frac{\partial\widetilde{W}^{\rm eff}(\sigma)}{\partial\sigma^i} = n_i \tag{2.16}$$

This equation is derived under very general conditions. Everything is hidden in $\widetilde{W}^{\rm eff}$. The n_i dependence in (2.16) can be eliminated by exponentiating both sides:

$$\exp\left(\frac{\partial\widetilde{W}^{\rm eff}(\sigma)}{\partial\sigma^i}\right) = 1 \tag{2.17}$$

The same exact equation can be derived if we remember that the topologically twisted theory is equivalent to the vacuum sector of the physical theory. After abelianisation the action of the topologically twisted theory can be brought to the simple form by the so-called quartet mechanism (one adds the anti-twisted superpotential $\bar{t}\,\mathrm{Tr}\bar{\sigma}^2$, and sends $\bar{t}\to\infty$):

$$S = \sum_{i=1}^r \int_\Sigma \frac{\partial\widetilde{W}^{\rm eff}(\sigma)}{\partial\sigma_i} F^i_A + \frac{1}{2}\sum_{i,j=1}^r \int_\Sigma \frac{\partial^2\widetilde{W}^{\rm eff}(\sigma)}{\partial\sigma_i\partial\sigma_j}\psi^i\wedge\psi^j \tag{2.18}$$

Here F^i_A is a curvature of abelian gauge field A^i and ψ^i is anti-commuting 1-form on Σ, the super-partner of A^i. In this form the path integral becomes a contour integral over the σ field. In fact (2.18) is a general form of abelianised action in any 2d topological gauge theory, the information about original theory is hidden in the form of $\widetilde{W}^{eff}$. The original supersymmetric theory contains information hidden in the D-terms, which ultimately leads to the wall-crossing phenomena [13].

The canonical quantization of (2.18) on the cylinder $\Sigma = \mathbf{R} \times \mathbf{S}^1$ is simple. The only physical degree of freedom is the monodromy $\exp i\vartheta^i = \exp \oint_{\mathbf{S}^1} A^i$ of the gauge field around the circle $\mathbf{S}^1$ and the momentum conjugate to ϑ^i, $I_i = \partial\widetilde{W}^{\rm eff}/\partial\sigma_i$. ϑ^i takes values on a circle (due to the large gauge transformations) so I_i is quantized, leading to the equations (2.16), (2.17). Now these equations describe all states of topological theory or the same - vacuum states of physical theory.

One can study the vacuum sector of the physical $N = 2$ theory or the topologically twisted version - ultimately one deals with same equations (2.16)*,* (2.17)*.*

The language of topological theory was adopted originally in [1, 5, 6]. In **Subsection 2.3**, we identify these vacuum equations as equations defining the Bethe states for many well-known quantum integrable systems (these identification can be made for all known, to the author, examples solved by Bethe ansatz [31]).

2.2.1. Twisted chiral ring and quantum integrability. The space of supersymmetric vacua of a theory with four supersymmetries carries a representation of a commutative associative algebra, the so-called (twisted) chiral ring, see e.g [29].

In 2d the $N = 2$ supersymmetry is generated by the fermionic charges $Q_\pm, \overline{Q}_\pm$:

$$\{Q_\pm, \overline{Q}_\pm\} = 2(H \pm P)\,,$$

$$\{Q_+, Q_-\} = \{\overline{Q}_+, \overline{Q}_-\} = 0; \quad \{Q_+, \overline{Q}_-\} = \{\overline{Q}_+, Q_-\} = 0 \tag{2.19}$$

the last two lines being valid in the absence of the central extension, induced, e.g. by some global symmetry charges.

One can define a nilpotent charge as a linear combination of fermonic charges $Q_\pm, \overline{Q}_\pm$. There are two independent combinations, Q_A and Q_B, which square to 0 and such that their anti-commutator with Q^+_A and Q^+_B respectively produces H:

$$Q_A = Q_+ + \overline{Q}_-\,, \quad Q_B = Q_+ + Q_-, \qquad Q^2_A = Q^2_B = 0 \tag{2.20}$$

$$\{Q_A, Q^\dagger_A\} = H, \qquad \{Q_B, Q^\dagger_B\} = H \tag{2.21}$$

The twisted chiral ring is generated by the operators $O_k(x)$, $k = 1, 2, \ldots$ which (anti)-commute with the operator Q_A : $\{Q_A, O_k(x)\} = 0$. Equivalently one can define a chiral ring whose generators (anti)-commute with the operator Q_B : $\{Q_B, O_k(x)\} = 0$. Here we concentrate on the Q_A-cohomology and assume that the possible central extension of (2.19) leaves Q_A nilpotent $Q^2_A = 0$. The local operators $O_k(x)$ are independent up to the Q_A-commutators of their location x. Their operator product expansion defines a commutative associative ring,

$$O_iO_j = c^k_{ij}O_k + \{Q_A, \ldots\} \tag{2.22}$$

Vacuum states $|0\rangle$, $H|0\rangle = 0$, are the Harmonic representatives in Q_A-cohomology: $Q|0\rangle = 0, |0\rangle \sim |0\rangle + Q|..\rangle$. Also, if $|0\rangle$ is a vacuum state of the Hamiltonian, $H|0\rangle = 0$, then so

is $O_i|0\rangle = |i\rangle$, and moreover the space of vacua is the representation of the twisted chiral ring. In this space $\{|i\rangle\}$ form a basis; so one has exactly same number of (anti)-commuting operators O_i's as the number of states in vacuum. Thus the space of supersymmetric vacua, which can be effectively studied using the cohomology of the operator Q_A (or Q_B), is the space of states of some quantum integrable system: $\mathbf{H}^{\rm quantum} = {\rm ker}Q_A/{\rm im}Q_A$. The operators O_k and more generally the functions of O_k's are the quantum Hamiltonians H_k's of some integrable system.

The operators O_k can be chosen *under the assumption of the absence of massless charged matter fields* (in order to justify integrating out massive fields - abelianisation) to be the gauge invariant polynomials of the complex scalar σ:

$$O_k = \frac{1}{k!(2\pi i)^k}{\rm Tr}\,\sigma^k \tag{2.23}$$

In integrable system one looks for the common eigenstates of these Hamiltonians:

$$\mathbf{\Psi}_\lambda \in \mathbf{H}^{\rm quantum}\,, \qquad H_k\mathbf{\Psi}_\lambda = E_k(\lambda)\mathbf{\Psi}_\lambda \tag{2.24}$$

where $E_k(\lambda)$ are the corresponding eigenvalues, and λ are some labels. In general they are complex, $E_k(\lambda) \in C$ (in case when quantum integrable system is solved with the use of Bethe ansatz λ's usually have the meaning of rapidities).

The important, or at least the interesting, problem is to identify the quantum integrable system given an $N = 2$ gauge theory, or to solve the converse problem – to find the $N = 2$ theory given a quantum integrable system. For a large class of models on both sides this problem has been solved in [1, 5–10].

In previous subsection we wrote the equations (2.16), (2.17) which count the vacua and thus after identification $H_k = O_k, \sigma_i = \lambda_i$ solve the eigenstates problem (2.24). Loosely speaking, given a vector of electric fluxes $(n_1, \ldots, n_r)$, with $n_i \in \mathbf{Z}$ the vacua are given by the critical points of the shifted superpotential $\widetilde{W}^{\rm eff}(\sigma) - 2\pi i\sum_{i=1}^r n_i\sigma^i$ (sometimes we loosely refer to the critical points of the superpotential without explicitly saying "shifted") or equivalently by (2.17).

2.2.2. Back to examples - vacuum equations.

$\mathbf{CP^{L-1}}$ model. $G = U(1)$, $R = R_{+1} \otimes \mathbf{C}^L$, where R_{+1} is a one-dimensional charge $+1$ representation of $U(1)$. This is asymptotically free theory. From (2.17) we get:

$$\prod_{a=1}^{L}(\sigma + \widetilde{m}_a) = q \equiv e^{2\pi i\,t} \tag{2.25}$$

This describes the $U(L)$-equivariant quantum cohomology $H^*_{U(L)}(\mathbf{CP}^{L-1})$ ring.

Grassmanian $\boldsymbol{Gr(N, L)}$ model. $G = U(N)$ and $R = \mathbf{C}^N \otimes \mathbf{C}^L$. This is also asymptotically free theory. Vacuum equations (2.17) in this case are ($q = e^{2\pi it}$):

$$\prod_{a=1}^{L}(\sigma_i + \widetilde{m}_a) = (-1)^{N+1}\,q\,,\ i = 1, \ldots, N \tag{2.26}$$

$N = 2^*$ theory. The example of the pure $N = 4$ theory broken down to $N = 2$ by the twisted mass term for the adjoint chiral multiplet is the first example where the supersymmetry is broken, for $N > 1$. Here $G = U(N)$ and $R = \mathbf{g} \otimes \mathbf{C}$, i.e. the adjoint representation. Using (2.3) and (2.4) we derive:

$$\prod_{j=1}^{N} \frac{\sigma_i - \sigma_j + m}{\sigma_i - \sigma_j - m} = -q \,, \tag{2.27}$$

This equation has no solutions for $N > 2$, or for $N = 1, q \neq 1$.

Hitchin theory. The model studied in [1, 5, 6] corresponds to the $N = 2^*$ theory, previous example, with the tree level *twisted* superpotential $\widetilde{W}(\sigma) = \frac{\lambda}{2} \operatorname{tr} \sigma^2$, which corresponds to the two-observable representing the Kähler form on the Hitchin's moduli space M_H. This leads to the change in the right hand side of (2.27):

$$\prod_{j=1}^{N} \frac{\sigma_i - \sigma_j + m}{\sigma_i - \sigma_j - m} = \exp 2\pi i \lambda \, \sigma_i \,, \tag{2.28}$$

and one now gets solutions for σ_i's for all N. The topological twist of this theory, introduced in [1] and was studied in detail in [5, 6].

Main example in two-dimensions. Equations for vacua (2.17) are (we shift t by $L/2$ to avoid extra phases in the right hand side):

$$\prod_{a=1}^{L} \frac{\sigma_i + m_a^{\mathrm{f}}}{\sigma_i - m_a^{\bar{\mathrm{f}}}} = -e^{2\pi i t} \prod_{j=1}^{N} \frac{\sigma_i - \sigma_j - m^{\mathrm{adj}}}{\sigma_i - \sigma_j + m^{\mathrm{adj}}} \tag{2.29}$$

This equation has solutions related by permuting σ_i's. We shall write it invariantly - supplement (2.29) with the condition that $\sigma_l \neq \sigma_m$ for $l \neq m$ and identify the solutions which differ by the permutations of σ_l's. So, we have to rewrite (2.29) as equations on the elementary symmetric functions: $c_l = \sum_{i_1 < \ldots < i_l} \sigma_{i_1} \sigma_{i_2} \ldots \sigma_{i_l}$ or, equivalently, via gauge invariant order parameter $\mathbf{Q}(x)$ (Baxter-Chern, BC, order parameter) $\mathbf{Q}(x) \equiv \det(x - \sigma) = \prod_{i=1}^{N} (x - \sigma_i) = x^N + \sum_{i=1}^{N} (-1)^i c_i x^{N-i}$:

$$a(x)\, \mathbf{Q}(x + m^{\mathrm{adj}}) \, + \, e^{2\pi i t}\, d(x) \mathbf{Q}(x - m^{\mathrm{adj}}) \, = \, t(x) \mathbf{Q}(x) \tag{2.30}$$

$$a(x) = \prod_{a=1}^{L} (x + m_a^{\mathrm{f}}) \,, \; d(x) = \prod_{a=1}^{L} (x - m_a^{\bar{\mathrm{f}}}) \tag{2.31}$$

and $t(x)$ is an unknown polynomial of degree L. This polynomial is uniquely fixed in terms of c_i's from the equation (2.30) by expanding both sides at $x = \infty$.

Main example in three and four dimensions. It is straightforward to generalise all above to three and four-dimensional lifts - with the use of (2.6), (2.7), (2.8) one obtains formulas analogous to (2.29), (2.30), (2.31). For example (2.29) becomes trigonometric in 3d and elliptic in 4d - for more details see [7, 8].

2.3. Vacuum states as Bethe states. We have already mentioned in the introduction that the vacuum equations (2.28) for the Hitchin example of [1, 5, 6] correspond to the Bethe equation for N-particle sector of non-linear Schrödinger system. This sector is described by N-particles on a circle, $x_i \in S^1$, with pairwise δ-function interaction:

$$H_{(2)} = -\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + c\sum_{i<j}\delta(x_i - x_j) \tag{2.32}$$

Higher Hamiltonians, corresponding to the operators (2.23), are written in terms of Dunkle operators D_i and permutation s_{ij}:

$$H_{(k)} = \sum_{i=1}^{N} D_i^k; \qquad D_i = -i\frac{\partial}{\partial x_i} + i\frac{c}{2}\sum_{j=i+1}^{N}(\epsilon(x_i - x_j) + 1)s_{ij}$$

Here $\epsilon(x)$ is a sign-function and $s_{ij} \in S_N$ is a transposition (ij). (D_i, s) provide a representation of the degenerate affine Hecke algebra $H_{N,c}$ for Lie algebra $gl(N)$. Image of the quadratic element of the centre is the Hamiltonian (2.32). Eigenvalues of this Hamiltonian are parametrized in terms of N-variables $\lambda_i, i = 1, ..., N$ as $E_i = \sum_i \lambda_i^2$. Eigenfunctions $\Psi_\lambda(x)$ for (2.32) where found in [32, 33] and condition of periodicity $x_i \sim x_i + 1$ is the restriction (2.28) on λ_i [34, 35] with $\sigma_i = \lambda_i$.

Let us now make this identification for our main 2d example (2.29). We already know that for special choice of twisted masses $m_a^{\mathrm{f}} = -\mu_a + is_a u, \quad m_a^{\bar{\mathrm{f}}} = \mu_a + is_a u, \quad m^{adj} = -iu$, one can turn on the tree level superpotential (2.2) with $m_{a;s}^b = \delta_a^b \alpha_a$ preserving $U(1)^{L+1}$ global symmetry. For this choice the vacua of main 2d example correspond to Bethe eigenstates of $SU(2)$ XXX_s spin chain with impurities $\mu_1, ..., \mu_L$ and $SU(2)$ spins s_a. Here we give few details only for $s = \frac{1}{2}$.

$XXX_{1/2}$ spin chain is defined on the one dimensional length L lattice. At each lattice point one has the spin $s = \frac{1}{2}$ representation of $SU(2)$, and the Hilbert space of the system is the tensor product $H_L = \mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \ldots \otimes \mathbf{C}^2$. The Hamiltonian H_{Heis} acts in H_L. It is written in terms of $SU(2)$ generators $\vec{S}_a$ where a denotes the position on the lattice of the spin $s = \frac{1}{2}$ representation of $SU(2)$ and has the nearest-neighbor interaction form:

$$H_{\mathrm{Heis}} = J\sum_{a=1}^{L}(S_a^x S_{a+1}^x + S_a^y S_{a+1}^y + S_a^z S_{a+1}^z) \tag{2.33}$$

The boundary conditions are quasi-periodic: $\vec{S}_{L+1} = e^{2\pi t \vec{S}^3_{tot}}\vec{S}_1 e^{-2\pi t \vec{S}^3_{tot}}$. The N-particle eigenstates/eigenvalues (N spins up, $L-N$ spins down), $H_{Heis}\Psi_\lambda = E(\lambda)\Psi$, are described with the help of Bethe ansatz and are:

$$e^{ip_j} = \frac{\lambda_j + \frac{i}{2}}{\lambda_j - \frac{i}{2}}; \quad E(\lambda) = J(L - 2N + 2\sum_{i=1}^{N} cos(p_i))$$

such that λ_i's solve Bethe equation (2.29) with identification $\lambda_i = \sigma_i$, and $s_a = \frac{1}{2}, \mu_a = 0$ (u is a scaling variable and can be rescaled to 1).

Similar equations exist for an arbitrary spin chain XXX_s when $\vec{S}_a$ is in the spin s_a representation of $SU(2)$ at every site and in addition the spin sites are (in some sense) displaced

from the symmetric round-the-clock configuration so one gets L additional parameters - impurities $\mu_1, ..., \mu_L$. The Hamiltonian for the general local spins is given by a polynomial in the neighbouring spins, which is more complicated than (2.33). Similarly - the vacuum sector for the main example in three dimensions from previous subsection is identified with XXZ_s spin chain and four-dimensional case with XYZ_s spin chain, for more details see [7, 8, 31]. Also, one can modify main example in such form to accommodate above spin chains for arbitrary spin group [7, 8, 31, 36]; corresponding models are based on quiver gauge groups. To conclude - we have following relations:

- Main example in two-dimensions — XXX spin chain,
- Main example in three-dimensions — XXZ spin chain,
- Main example in four-dimensions — XYZ spin chain.

3. Quantum many-body systems and SUSY vacua

Now we concentrate on Type II examples - special Ω-backgrounds. In these examples we deal with 4d(5d) $N = 2$ theories. Exact low energy effective action (in two-derivative approximation) was found by Seiberg and Witten (SW) [2]; this can be viewed as 4d analog of 2d exact result for effective twisted prepotential reviewed in the beginning of this paper. Later, it was realised that the SW moduli space of vacua is best described in terms of the geometry of classical Algebraic Integrable System (AIS) [17–19]. The key elements of SW solution - spectral curve, differential and prepotential - all appear in the classical AIS.

3.1. Algebraic integrable systems. Algebraic Integrable System is a complex analog of usual classical Hamiltonian integrable system - one is given a data $(M, \omega, \mathbf{H})$ consisting of the complex manifold M of complex dimension $2n$, the holomorphic non-degenerate closed $(2,0)$ form ω, and the holomorphic map $\mathbf{H} : M \to \mathbf{C}^n$ whose fibers $J_h = \mathbf{H}^{-1}(h)$ are Lagrangian polarized abelian varieties. The polarization is a Kähler form ϖ, whose restriction on each fiber is an integral class $[\varpi] \in H^2(J_h, \mathbf{Z}) \cap H^{1,1}(J_h)$. The image $B = \mathbf{H}(M)$ is an open domain in $\mathbf{C}^n$. It has a special Kähler geometry, with the metric $\mathrm{d}s^2 = \frac{1}{\pi}\sum_{i=1}^n \mathrm{Im}\,\left(\mathrm{d}a^i \otimes \mathrm{d}\overline{a}_{D,i}\right)$ where the *special coordinates* $a^i, a_{D,i}$ are given by the periods:

$$a^i = \frac{1}{2\pi}\oint_{A_i} \mathbf{p}\mathrm{d}\mathbf{q} \qquad a_{D,i} = \frac{1}{2\pi}\oint_{B_i} \mathbf{p}\mathrm{d}\mathbf{q} \tag{3.1}$$

over the A and B-cycles, which are the Lagrangian (with respect to the intersection form given by $[\varpi]$) subspaces in $H_1(J_h, \mathbf{Z})$. It follows that the 2-form $\sum_i \mathrm{d}a^i \wedge \mathrm{d}a_{D,i}$ vanishes on B thereby embedding the covering U of the complement $B\backslash\Sigma$ to the discriminant $\Sigma \subset B$ of the singular fibers to the first cohomlogy $H^1(J_{h_0}, \mathbf{C})$ of the fiber over some distinguished point $h_0 \in B$, as a Lagrangian submanifold L. As such, it comes with the function $F : L \to \mathbf{C}$, locally a function of a^i's, such that:

$$a_{D,i} = \frac{\partial F}{\partial a^i} \tag{3.2}$$

In analogy with *real* classical integrable system the equations (3.1) define the action variables - local complex coordinates on the base. Since the $2n$-dimensional symplectic manifold

has at most n functionally independent Poisson-commuting functions, there ought to be a relation between a^i and $a_{D,i}$'s. It is remarkable that this relation has a potential function. The action variables a^i come with the corresponding angle variables $\phi_i = \alpha_i + \tau_{ij}\beta^j$, $\tau_{ij} = \partial^2_{ij}F$, while $a_{D,i}$ correspond to $\phi^{D,i} = \left(\tau^{-1}\right)^{ij}\phi_j$: $\omega = \sum_i \mathrm{d}a^i \wedge \mathrm{d}\phi_i = \sum_i \mathrm{d}a_{D,i} \wedge \mathrm{d}\phi^{D,i}$ Here $\alpha_i, \beta^i \in \mathbf{R}/2\pi\mathbf{Z}$ are the real angular coordinates on the Liouville torus.

Main observation which led to the relation between the SW solution and the classical AIS is the existence of prepotential $F(a)$ in both and the explicit statement that these prepotentials coincide for pure super-Yang-Mills theory and N-particle periodic Toda (pToda) integrable system, as well as for $N = 2^*$ super-Yang-Mills theory and elliptic Calogero-Moser (eCM) system (many other examples discovered later and extended to large class of $N = 2$ theories in four-dimensions in [74], [46]).

In [10] it was shown that *the special Ω-deformation with one ϵ gives the quantisation of these AIS with ϵ being a (complexified) Planck constant $\hbar$.*

The quantization of the classical integrable system is a (possibly discrete) family $(A_\varepsilon, H_\varepsilon, \widehat{\mathbf{H}})$, of the associative algebras A_ε (ϵ is a Planck constant, can be any complex number), which deform the algebra of functions on the Poisson manifold (X, ω^{-1}), the (Hilbert) vector spaces H_ε, with the action of A_ε, and the mutually commuting operators $\widehat{\mathbf{H}} = (\widehat{H}_1, \ldots, \widehat{H}_n)$, $\widehat{H}_i \in A_\varepsilon; [\widehat{H}_i, \widehat{H}_j] = 0$, and generate H_ε in the following sense - the common spectral problem defines a basis in H_ε:

$$\widehat{H}_i \Psi = E_i \Psi \tag{3.3}$$

The construction of the common eigenstates and the spectrum of the operators $\widehat{H}_i$ is a problem of the coordinate Bethe ansatz, quantum inverse scattering method (ABA [39–41]), quantum separation of variables [42], Baxter equation [43], the spectral curve quantization [44] and various other versions of the Bethe ansatz.

Here we take a different route [1, 5–10]. Hint comes from the following observation: consider the Ω-deformation of gauge theory with one ϵ parameter and use the argument we gave above (from [10]) that resulting effective theory is two-dimensional with effective twisted superpotential (2.11); its vacuum equation (2.17) defines the spectrum of some quantum integrable system as already explained; but latter equation, due to the relation (2.13), in the "classical" limit $\hbar = \epsilon \to 0$, reduces to $a_{D,i} = 2\pi i \epsilon n_i + O(\epsilon^2)$ which is nothing but a Bohr-Sommerfeld quantisation of AIS with prepotential F. This suggests that quantum integrable system corresponding to the supersymmetric vacua in a special Ω-background is a quantisation of AIS describing the moduli space of vacua of the theory with $\epsilon = 0$.

The second deformation, leading yet to another ϵ, comes from the turning on the "times" - the quantum integrable system can be deformed by making $\widehat{H}_i$ depend on the additional parameters $\mathbf{t} = (t_1, \ldots, t_n)$, where $2n$ is a complex dimension of phase space, so they define a flat connection depending on a spectral parameter κ:

$$[\kappa \frac{\partial}{\partial t_i} - \widehat{H}_i(\mathbf{t}),\ \kappa \frac{\partial}{\partial t_j} - \widehat{H}_j(\mathbf{t})] = 0 \tag{3.4}$$

Applied to quantum Hitchin system (itself being AIS) it produces KZ connection for the WZW conformal field theory (the gauge theory meaning of the κ-parameter is ε_2 of the general Ω-background, or more precisely κ vanishes when $\epsilon_2 = 0$).

Following above logic of *quantisation via deformation of gauge theory* - we need to calculate the superpotential from (2.12) and compare the resulting spectrum (2.16) with independent calculations from integrable system side, so we can conclude:

The gauge theory allows to find the exact spectrum of (3.3) which is the invariant of the choice of polarization used in the quantization procedure for all examples when moduli space of vacua is described by classical AIS.

Hitchin integrable systems are examples of AIS. They are defined via the Riemann surface of genus g with n punctures and the gauge group G. First we will make such comparison for one particular example of Hitchin system - elliptic Calogero-Moser system corresponding to $g = n = 1, G = SU(N)$ and describing the vacua of four-dimensional $SU(N)$ $N = 2^*$ theory. Exact quantum spectrum of eCM system is not know in integrability literature but it has a simple limit, pToda system, for which it is known [47]; in this limit, from gauge theory side, $N = 2^*$ theory becomes pure $N = 2$ theory. Thus we need to calculate superpotential for $N = 2^*$ theory with one ϵ, get the spectrum for eCM system, take limit to pToda system and compare to [47]. This was done in [10], where also the spectrum of eCM system was calculated in certain perturbative expansion to sufficiently high order in order to compare with gauge theory answer before taking pToda limit.

3.1.1. The classical elliptic Calogero-Moser system. The elliptic Calogero-Moser system (eCM) is the Hamiltonian system of N particles $(x_1, x_2, \dots, x_N)$ on the circle of circumference $\beta, x_i \sim x_i + \beta$ which interact with the pair-wise potential

$$\begin{aligned} H_2^{eCM} =& \frac{1}{2}\sum_{i=i}^{N} p_i^2 + m^2 \sum_{i<j} u(x_i - x_j), \\ u(x) =& C(\beta) + \sum_{k\in\mathbf{Z}} \frac{1}{\sinh^2(x + k\beta)} = -\partial_x^2 \log\Theta(x) \end{aligned} \tag{3.5}$$

where Θ is the odd theta function on the elliptic curve E_τ with the modular parameter $\tau = \frac{i\mathbf{e}}{\pi}$, $\Theta(x) = -\sum_{k\in\mathbf{Z}+\frac{1}{2}}(-1)^k q^{\frac{k^2}{2}} e^{2kx}, q = \exp 2\pi i\tau$ and $C(\beta)$ is some constant. There exists a Lax representation [48] and a characteristic polynomial:

$$\Phi_{ij}(z) = p_i\delta_{ij} + m\frac{\Theta(z + x_i - x_j)\Theta'(0)}{\Theta(x_i - x_j)\Theta(z)}(1 - \delta_{ij}) \tag{3.6}$$

$$\mathrm{Det}(\Phi(z) - x) = 0.$$

Spectral curve $\mathrm{Det}(\Phi(z) - x) = 0$ defines the Hamiltonians H_i of eCM system as the coefficients of x^{N-i} in its expansion. This is an N-sheeted ramified cover of the elliptic curve E_τ where z lives, and it is the SW curve of $N = 2^*$ theory. The parameter m in (3.6) is the mass of the adjoint hypermultiplet, the complex structure of the curve E_τ is determined by the complexified bare gauge coupling of the ultraviolet theory (which is in fact the superconformal $N = 4$ theory). Prepotential calculated from (3.2) coincides with SW prepotential for this theory.

In the limit $\beta \to \infty$, $m \to \infty$, such that $\Lambda^{2N} = m^{2N}q$ is kept finite, the eCM system becomes the periodic Toda chain [49], where $x_i^{\mathrm{eCM}} = \frac{i}{N}\beta + x_i^{\mathrm{pToda}}$:

$$H_2^{pToda} = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \Lambda^2\left(\sum_{i=1}^{N-1} e^{x_i - x_{i+1}} + e^{x_N - x_1}\right) \tag{3.7}$$

Spectral curve (3.6) and prepotential (3.2) become those of pure $N = 2$ theory.

3.1.2. Relativistic model and five-dimensions. There exists [50, 51] a one-parametric relativistic deformation of the eCM integrable system, whose Hamiltonians are trigonometric (hyperbolic) functions of the momenta. In particular, the Hamiltonians $\widehat{P} = \widehat{H}_1, \widehat{E} = \widehat{H}_2$ become

$$\widehat{P} = \sum_{i=1}^{N} \sinh(\beta p_i) f_i(x), \quad \widehat{E} = \sum_{i=1}^{N} \cosh(\beta p_i) f_i(x), f_i(x) = \prod_{j \neq i} \sqrt{1 - \frac{\wp(x_i - x_j)}{\wp(\mathbf{e}m)}} \tag{3.8}$$

These systems (3.8) correspond to the 5d theory compactified on S^1 [30].

3.2. Quantisation. After replacement $p_i \to \epsilon \frac{\partial}{\partial x_i}, m^2 \to m(m+\epsilon)$ - H_2 is:

$$H_2 = \frac{\epsilon^2}{2} \sum_{i=i}^{N} \frac{\partial^2}{\partial {x_i}^2} + m(m+\epsilon) \sum_{i<j} u(x_i - x_j) \tag{3.9}$$

Usually eCM Hamiltonian is written in notations $\epsilon = -i\hbar, m = \nu\epsilon, \hbar, \nu, \beta \in \mathbf{R}_+$.

Now we turn on ϵ and calculate (2.11) for $N = 2^*$ gauge theory in order to find the expression for $\widetilde{W}^{eff} = \widetilde{W}^{pert}(a, \tau) + \widetilde{W}^{inst}(a, q)$. The perturbative part is:

$$\widetilde{W}^{\rm pert}(a, \tau) = \frac{1}{2\varepsilon} \tau \sum_{n=1}^{N} a_n^2 + \sum_{l,n=1}^{N} \left(\varpi_\varepsilon(a_l - a_n) - \varpi_\varepsilon(a_l - a_n - m - \varepsilon) \right) \tag{3.10}$$

$$\frac{d}{dx} \varpi_\varepsilon(x) = \log \Gamma \left(1 + \frac{x}{\varepsilon} \right) . \tag{3.11}$$

The instanton part can be written with the use of the "action functional" $A(a, q)$ of two fields $\varphi(x)$ and $\rho(x)$ on complex plane ($Q(x), G(x-y)$ are given bellow):

$$A(a; q) = \frac{1}{2} \int_{C \times C} \rho(x) \rho(y) G(x - y) + \int_C \left[\rho(x) \varphi(x) + \mathrm{Li}_2 \left(q Q(x) e^{-\varphi(x)} \right) \right] \tag{3.12}$$

$\widetilde{W}^{inst}$ is a classical action - one has to minimise with respect to fields $\varphi(x), \rho(x)$:

$$\widetilde{W}^{\rm inst}(a; q) = \int_C \left[-\frac{1}{2} \varphi(x) \log \left(1 - q Q(x) e^{-\varphi(x)} \right) + \mathrm{Li}_2 \left(q Q(x) e^{-\varphi(x)} \right) \right] \tag{3.13}$$

where $\varphi(x)$ solves a TBA-like equation:

$$\varphi(x) = \int_C G(x - y) \log \left(1 - q Q(y) e^{-\varphi(y)} \right) \tag{3.14}$$

$$Q(x) = \frac{P(x - m) P(x + m + \varepsilon)}{P(x) P(x + \varepsilon)}, \quad P(x) = \prod_{l=1}^{N} (x - a_l) \tag{3.15}$$

$$G(x) = \frac{d}{dx} \log \frac{(x + m + \varepsilon)(x - m)(x - \varepsilon)}{(x - m - \varepsilon)(x + m)(x + \varepsilon)},$$

The contour C in the complex plane comes from infinity, goes around the points $a_l + k\varepsilon$, $l = 1, \ldots, N$, $k = 0, 1, 2, \ldots$, and goes back to infinity. It separates these points and

the points $a_l + lm + k\varepsilon$, $l \in \mathbf{Z}$, $k = -1, -2, \ldots$, see [10] (pToda/pure $N = 2$ answers follow from the above limit; formulas for relativistic model mentioned above are very similar, see [10]). Given these data, it is straightforward to solve (3.14) recursively: $\varphi(x) = \sum_{k=1}^{\infty} q^k \varphi_k(x)$.

The perturbative Bethe equations are:

$$q^{\frac{a_i}{\varepsilon}} = \prod_{j \neq i} \Gamma\left(\frac{a_i - a_j}{\varepsilon}\right)\Gamma\left(\frac{a_i - a_j}{\varepsilon}\right)\Gamma\left(\frac{-m - a_i + a_j}{\varepsilon}\right)\Gamma\left(\frac{-m + a_i - a_j}{\varepsilon}\right) \quad (3.16)$$

and correspond to the system of N interacting particles on the circle of the size $\propto \log q$, with the factorisable S-matrix of the hyperbolic Sutherland model and a_i's having the meaning of asymptotic momenta. The energy is given by:

$$E = \epsilon q \frac{d}{dq} \widetilde{W}^{eff}(a; \epsilon; q) \quad (3.17)$$

This description of a spectrum via integral equation (3.14) reminds very much the Thermodynamic Bethe Ansatz approach initiated many decades ago in quantum integrability literature. Similarities are obvious, but it doesn't seem to correspond to any known TBA system so far. Alternative approach of computing $\widetilde{W}^{eff}$ was given in [36], but because of limited space we are not able to cover it here.

For eCM model prior to [10] the exact solution of proper spectrum was not known in quantum integrability literature. Thus, in order to make sure the above solution, obtained via supersymmetric gauge theory methods, is correct - one needs to compare the spectrum (2.17)), (3.10), (3.13), (3.14) to calculations directly in eCM quantum mechanics. It has been checked in [10] for the expansion of wave-function in q (Hamiltonian (3.9) also expanded in q around $\frac{1}{sinh^2 x}$ potential) to sufficiently high order that if the equations (2.17), (3.10), (3.13), (3.14) are satisfied (in the same order) - the wave-function is normalisable. Slightly different, and better, check has been done in [10] for pToda case - answers obtained by gauge theory methods coincide with the exact solution [47] via simple identification of their variables δ_i with a_i's here: $\delta_i = a_i$; exact comparison was made in [53].

3.3. Hitchin systems, Darboux coordinates and gauge theory. The existence of YY - function always was a bit magical phenomenon observed in the study of quantum integrable systems. There is no prove that such function should always exist, but in (almost) all examples of quantum integrable systems solved, by use of Bethe ansatz, there is such function. In the previous subsection we obtained YY-function, by methods of supersymmetric gauge theory, for several very important examples of quantum many-body systems. We also clearly saw that the classical version of YY-function always exists for all AIS and it coincides with the prepotential. In order to understand the geometric meaning of YY-function one may ask following two questions: 1. What is the geometric meaning of ϵ-deformation of the prepotential (without use of the relation with supersymmetric gauge theory)? 2. What is the geometric meaning of $\{a_i\}$'s appearing in the YY-function (since YY-function gives the quantum answer one expects the deformation of classical relations (3.1)) and what is special in this coordinate system? There are many other important questions, we concentrate here on these two for the special class of integral systems - Hitchin systems. This is sufficiently large class of non-relativistic integrable systems and for many of them we know supersymmetric gauge theory with the same prepotential.

Consider Riemann surface $\Sigma_{g,n}$ of genus g with n-punctures. Pick the gauge group G and the complex structure on $\Sigma_{g,n}$, denote by $A_z, A_{\overline{z}}$ components of gauge field, and by $\phi_z, \phi_{\overline{z}}$ components of $Lie(G)$ valued one-form. Moduli space M_H of solutions to Hitchin's equations [54] (modulo the gauge transformations):

$$F_{z\overline{z}}(A) + [\phi_z, \phi_{\overline{z}}] = 0, \qquad D_z(A)\phi_{\overline{z}} = 0, \qquad D_{\overline{z}}(A)\phi_z = 0, \tag{3.18}$$

is hyperkähler (see [55] for the detailed review of its properties). In the complex structure, denoted by I, where the components $A_{\overline{z}}, \phi_z$ are holomorphic, the space M_H has the structure of the algebraic integrable system [56], with the base being the space of holomorphic differentials of degrees $d_1, d_2, \ldots, d_r$, for $r = rk(G)$, $P_i(\phi_z) \in H^0(\Sigma, K_\Sigma^{d_i})$ for the degree d_i invariant polynomials on $Lie(G)$, and the fiber being a (complement to a divisor in a) Jacobian of the spectral curve $C \subset T^*\Sigma_{g,n}$, defined by the equation (for $G = SU(N)$, for other Lie groups see the review [57]): $\mathrm{Det}(\phi_z - \lambda) = 0$. In the complex structure J, where the holomorphic coordinates are the components $\mathbf{A}_z = A_z + i\phi_z, \quad \mathbf{A}_{\overline{z}} = A_{\overline{z}} + i\phi_{\overline{z}}$, M_H is identified (up to the usual stability issues) with the moduli space of the complex $G_{\mathbf{C}}$-flat connections - $M^{loc} : F(\mathbf{A}) = 0$ (modulo the $G_{\mathbf{C}}$ gauge transformations). Finally, $K = IJ$, and the K-holomorphic coordinates are $A_z + \phi_z, \quad A_{\overline{z}} - \phi_{\overline{z}}$. Note - the complex structure J is natural, if one thinks of the three dimensional theory as coming from the compactification of the six dimensional $N = 1$ gauge theory on a three manifold $\Sigma \times \mathbf{S}^1_{r'}$, in the limit $r' \to \infty$. To say that M_H is hyperkähler means - there exists the whole 2-sphere of complex structures, $\mathbf{I} = aI + bJ + cK, \quad \mathbf{I}^2 = -1$,, for any (a, b, c), s.t. $a^2 + b^2 + c^2 = 1$, and the 2-sphere of the corresponding Kähler forms, $\omega_{\mathbf{I}} = a\omega_I + b\omega_J + c\omega_K$ where

$$\omega_I = \int_\Sigma \mathrm{tr}\ (\delta A \wedge *\delta\phi)\,, \quad \omega_J = \int_\Sigma \mathrm{tr}\ (\delta\mathbf{A} \wedge \delta\mathbf{A})\,, \quad \omega_K = \int_\Sigma \mathrm{tr}\ (\delta A \wedge \delta\phi) \tag{3.19}$$

For the compact $\Sigma_{g,n}$ the form ω_I realizes a nontrivial cohomology class of M_H, while ω_J and ω_K are cohomologically trivial. We shall normalize these forms in such a way that ω_I realizes the integral cohomology class, the restriction of ω_I onto the subvariety Bun_G where $\phi = 0$ is, up to the $(2\pi i)$ multiple, the curvature of the canonical Hermitian connection on the determinant line bundle L over Bun_G: $[\omega_I]\Big|_{\mathrm{Bun}_G} = c_1(L)$. If the Riemann surface $\Sigma_{g,n}$ has $n > 0$ punctures, then all three symplectic forms $\omega_{I,J,K}$ on the moduli space of the solutions to Hitchin's equations with the sources are, in general, cohomologically non-trivial.

Now we would like to formulate the conjecture about the geometric meaning of YY-function [25]. In this part the reader also needs to consult [24], [25]. Suppose we study the four-dimensional theory on the manifold $X^4 = \mathbf{D} \times \mathbf{S}^1 \times \mathbf{R}^1$, where $\mathbf{D}$ is topologically a disk, with the cigar metric:

$$ds^2_{C^2} = dr^2 + f(r)d\varphi_1^2 \tag{3.20}$$

with $f(r) \sim r^2$ for $r \to 0$ and $f(r) \sim R^2$, for $r \to \infty$, for some constant R. Let φ_2 be the angular coordinate on the second $\mathbf{S}^1$. The base B^2 is, in this case, the half-plane $\mathbf{R}_+ \times \mathbf{R}$. Suppose we turn on the Ω-deformation corresponding to the isometry rotating $\mathbf{D}$, i.e. the isometry generated by $\partial/\partial\varphi_1$. We can view this situation in two different ways. 1. This is a four-dimensional $N = 2$ theory in Ω background which, as we argued above many times, is a two-dimensional theory with $N = (2, 2)$ supersymmetry on the world-sheet $R \times S^1$ with infinitely many massive fields - the Ω-deformation corresponds, in this two dimensional language, to turning on the twisted mass ε, corresponding to the global symmetry $U(1)$,

which is the rotation of $\mathbf{D}$. This two dimensional theory was called T_{2t} in [25]. 2. On the one hand, we can relate this theory to the sigma model with the worldsheet B^2, as in [24]. The sigma model brane corresponding to the boundary $r = 0$ is B^{cc} or its T-dual B_{O_τ} - brane of opers. The other "boundary", at $r \to \infty$, leads to some asymptotic boundary condition, which we view as the brane B^∞, or B^∞_γ if we want to specify the type γ of the boundary conditions. This was called the theory T_{rt} in [25]. The two dimensional theory has at low energies (in the sense of the theory on $\mathbf{R} \times \mathbf{S}^1$) a description of the sigma model on the complexified Cartan subalgebra $\mathbf{t_C}$ of the gauge group, with the effective twisted superpotential $\widetilde{W}^{\rm eff}(\sigma; \tau; m, \varepsilon)$. Here σ denotes the flat coordinates on $\mathbf{t_C}$, τ denotes the four dimensional complexified gauge couplings, which are identified, for the A_1 theories, with the complex moduli of $\Sigma_{g,n}$ in a suitable parametrization, m denotes the masses of the matter multiplets in four dimensions, and finally ε is the parameter of the Ω-background which is viewed as the two dimensional twisted mass. This twisted superpotential can be split as a sum of two contributions: a contribution of the fixed point in $\mathbf{D}$ and a contribution of the boundary at infinity:

$$\widetilde{W}^{\rm eff}(\sigma, \tau, m, \varepsilon, \gamma) = \quad \varepsilon \left(W_{O_\tau}(\sigma/\varepsilon, m/\varepsilon) - W_\infty(\sigma/\varepsilon, m/\varepsilon, \gamma) \right) \tag{3.21}$$

The contribution $W_{O_\tau}(\sigma/\varepsilon; m/\varepsilon)$ of the fixed point is given by (2.11). The contribution of the region at infinity $W_\infty(\alpha; \nu, \gamma)$ is independent of τ:

$$W_\infty(\alpha; \nu, \gamma) \sim \sum_\ell Li_2(e^{\ell(\alpha,\nu)}) \tag{3.22}$$

where the sum is over the charged matter fields, and ℓ is the linear function of the gauge multiplet scalar vevs and the masses. The degrees of freedom living at $\mathbf{S}^1_\infty$ depend on some combinatorial data γ - we shall not attempt to identify the boundary theory and the corresponding superpotential more precisely (for more details see [25]). Instead, we focus on W_{O_τ}.

The main **conjecture of [25]** is as follows (we explain this statement bellow for the A_1 case): *The effective twisted superpotential of the theory on $\mathbf{R} \times \mathbf{S}^1$ obtained by localizing at the fixed point in $\mathbf{D}$ is essentially the difference of the generating functions of the Lagrangian subvarieties O_τ and L_γ in M^{loc}, defined with respect to the appropriate Darboux coordinate system on M^{loc}. The supersymmetric vacua of the theory T_{2t} correspond to the intersection points $v \in O_\tau \cap L_\gamma$, which are also the vacua of the theory T_{rt} subject to the appropriate D-brane boundary conditions.*

This statement is the improvement on the result of A. Beilinson and V. Drinfeld. They show in [58] that upon the holomorphic quantization of the Hitchin system for the group G the spectrum (in the sense of commutative algebra) of twisted differential operators, e.g. the abstract quantum commuting Hamiltonians, identifies canonically with O_τ for the dual group ${}^LG$. Here we concentrate on $G = PGL(2, C)$, ${}^LG = SL(2, C)$. Claim is - *there exist Darboux coordinate system (α, β) such that the generating function $W_{O_\tau}(\alpha, \nu)$ of the variety of opers,*

$$\beta = \frac{\partial W_{O_\tau}(\alpha, \nu)}{\partial \alpha}, \tag{3.23}$$

is essentially the YY-function function of the quantum Hitchin system. More precisely, the Yang-Yang function $Y(\alpha, \nu, \tau, \gamma)$ of the quantum Hitchin system depends on the complex

structure τ of Σ (as does the Hitchin's integrable system) and on the choice of the "real slice", which defines the space of states H_γ:

$$Y(\alpha,\nu,\tau;\gamma) = W_{O_\tau}(\alpha,\nu) - W_\gamma(\alpha,\nu) \tag{3.24}$$

i.e. up to a τ-independent piece the superpotential $W_{O_\tau}(\alpha,\nu)$ computed by the four dimensional instanton calculus above is the YY-function. Indeed, the coordinates (α,β) are defined up to $2\pi i\mathbf{Z}$, so the *Bethe* equations define the spectrum $E_k(\vec{n})$:

$$E_k(\vec{n}) = \frac{\partial Y(\alpha,\nu,\tau,\gamma)}{\partial \tau_k}, \qquad \frac{\partial Y(\alpha,\nu,\tau,\gamma)}{\partial \alpha_k} = 2\pi i n_k,\ n_k \in \mathbf{Z} \tag{3.25}$$

Basically this answers the questions 1. and 2. formulated in the beginning of this Subection - α coordinates are geometric way to think about a's from the gauge theory, β's play the role of a_D's and YY-function is generating function in these coordinates. More precisely the questions are reduced to finding (α,β) coordinates, which was done in [25] for A_1 case, and explaining why they are so special.

Let us clarify the meaning of (3.25) for A_1 case. As it follows, from general description above via the characteristic polynomials, the Hamiltonians of the A_1 Hitchin system are the quadratic differentials with the second order poles at the punctures, with the prescribed leading singularity. Given a basis $\mu^z_{(k)\overline{z}}$, $k = 1,\dots,3g-3+n$, of the Beltrami differentials which correspond to the variations of the complex structure of the $\Sigma_{g,n}$, $\mu^z_{(k)\overline{z}} \leftrightarrow \frac{\partial}{\partial\tau_k}$:

$$H_k = \int_{\Sigma_{g,n}} \mu^z_{(k)\overline{z}} \mathrm{Tr}\phi_z^2 \tag{3.26}$$

Upon the deformation quantization H_k become the elements of the noncommutative ring (one has to talk about the sheaf of D-modules to see the noncommutative algebras, since the globally defined objects form a commutative subalgebra, [58]), whose spectrum we wish to determine. One has to specify the space of states on which we represent both the noncommutative algebra and its commutative subalgebra of the quantum integrals of motion. This is done (indirectly) by picking up a Lagrangian submanifold L_γ. To make this construction explicit one needs to involves the T-duality along the fibers of the Hitchin fibration (see [24, 59] for more details). The formula (3.25) makes sense even without specifying the space of states. It reflects the canonical identification [58] of the spectrum of the commutative algebra of the quantized Hitchin's Hamiltonians (the twisted differential operators on Bun_G) with the variety of opers. Indeed, O_τ is a Lagrangian submanifold in M^{loc}. As we vary the complex structure τ infinitesimally, the corresponding variation of the Lagrangian submanifold is described by a closed 1-form δ defined on O_τ, a holomorphic function, since the variety of opers is simply-connected.

In order to introduce the special Darboux coordinates (α,β) in M^{loc} it is sufficient to consider situation when $\Sigma_{g,n}$ is a sphere with four punctures (this corresponds to Gaudin integrable system, for any number of punctures) and the torus with one puncture (in case of $U(N)$ Hitchin gauge group this is N-particle eCM integrable system). Roughly speaking, a flat connection $\mathbf{A} = \mathbf{A}_z dz + \mathbf{A}_{\overline{z}} d\overline{z}$ is a G-oper, if the gauge equivalence class of $\overline{\mathbf{A}} = \mathbf{A}_{\overline{z}} d\overline{z}$ defines a particular holomorphic G-bundle on $\Sigma_{g,n}$, which is determined by the complex structure of $\Sigma_{g,n}$. For $G = SL(2,\mathbf{C})$ this bundle is such that the associated rank two vector bundle is the (unique up to isomorphism) nontrivial extension of the bundle $K_\Sigma^{-1/2}$ by $K_\Sigma^{1/2}$.

In this case of $\mathbf{G} = SL(2,\mathbf{C})$ the space of opers for varying complex structure of $\Sigma_{g,n}$ is the open subset in the moduli space of flat $SL(2,\mathbf{C})$-connections. Locally an $SL(2,C)$-oper is a second order (meromorphic) differential operator (projective connection) which acts on the $(-\frac{1}{2})$-differentials:

$$D = -\partial_z^2 + T(z) \tag{3.27}$$

$$\Sigma_{0,n}: \qquad T(z) = \sum_{a=1}^{n} \frac{\Delta_a}{(z-x_a)^2} + \frac{\epsilon_a}{z-x_a} \tag{3.28}$$

$$\Sigma_{1,n}: \qquad T(z) = u + \sum_{a=1}^{n} \Delta_a \wp(z-x_a) + \epsilon_a \zeta(z-x_a) \tag{3.29}$$

where the for (3.28) *accessory parameters* ϵ_a obey linear relations in order for (3.27) be non-singular at $z=\infty$:

$$\sum_{a=1}^{n} \epsilon_a = 0, \quad \sum_{a=1}^{n} (x_a\epsilon_a + \Delta_a) = 0, \quad \sum_{a=1}^{n} \left(x_a^2\epsilon_a + 2x_a\Delta_a\right) = 0; \tag{3.30}$$

and for (3.29) - $\sum_a \epsilon_a = 0$, u is a constant, and we used the Weierstrass ζ and $\wp = \zeta'$ functions. For $\Delta_a = \nu_a(\nu_a - 1)$ (ν_a's parametrize the traces of monodromy matrices of (3.28): $trg_a = 2\cos 2\pi\nu_a$) (3.27) defines the point in M^{loc}. The correspondence $\partial_{\tau_k} \leftrightarrow H_k$ (3.26) between the variations of the complex moduli of Σ and the functions on the variety of opers O_τ is provided by the one-form $\delta = \sum_a \epsilon_a dx_a$ for (3.28), $\delta = \sum_a \epsilon_a dx_a + ud\tau$ for (3.29). Note that ϵ_a's are in fact energy levels E_s's for quantised Hitchin Hamiltonians (3.26), see discussion after formula (3.26).

According the conjecture, formulated above, one needs to define the special topological Darboux variables on M^{loc} for symplectic form ω_J (3.19) and in these coordinates write the generating function of Lagrangian submanifold $O_\tau \subset M^{loc}_{\Sigma_{g,n}}$. In case of a sphere with four punctures there are three points of the maximal degeneration of the complex structure which correspond to the s, t, and u channel tree scattering graphs; one covers the moduli space of flat connections M^{loc} in this case by three coordinate charts U_i each with the coordinates (α_i, β_i), $i = s,t,u$. This moduli space, $M^{loc}_{0,4}$, can be identified with the space of $SL(2,C)$ matrices $(g_1, ..., g_4)$ (monodromies of (3.27)) with a fixed conjugacy classes $trg_i = 2\cos(2\pi\nu_i)$ obeying the condition $g_1g_2g_3g_4 = 1$ up to the simultaneous conjugation $(hg_1h^{-1}, ..., hg_4h^{-1})$; $\quad h \in SL(2,C)$. The geometric meaning of the special Darboux variables (α_i, β_i) in fact is simple - in the space of complex 2×2 matrices $Mat_2(\mathbf{C})$, endowed with the complex metric $< X,Y >= tr < X,Y > -trXtrY$, we can form the complex analog of hyperbolic tetrahedron with vertices $(v_1 = 1, v_2 = g_1, v_3 = g_2g_1, v_4 = g_3g_2g_1)$ (numbered clockwise). Since the conjugacy classes trg_i are fixed - the only freedom we have is to vary the length of the diagonal $< v_1, v_3 >$ (or $< v_2, v_4 >$) which defines the variable α_s (or α_t) and the angle between the planes spanned by the triangles (v_1, v_2, v_3) and (v_1, v_3, v_4) (or (v_1, v_2, v_4) and (v_2, v_3, v_4)) defining the variable β_s (or β_t). For example the expression for $\alpha_{s(t)}$ is simple: $\cosh(\alpha_s) =< v_1, v_3 >, \quad \cosh(\alpha_t) =< v_2, v_4 >$. There are obvious ambiguities in defining these hyperbolic angles, which need to be fixed since later they affect the definition of generating function of Lagrangian submanifold O_τ (for details see [25]). The check that these are canonical variables for symplectic form ω_J is simple - after fixing the conjugacy classes the remaining three invariants $A = trg_1g_2$, $B = trg_2g_3$ and $C = trg_1g_3$, with one polynomial relation $W_{0,4} = A^2+B^2+C^2+ABC+... = 0$, obey

the Poisson bracket relation $\{A, B\} = \frac{\partial W_{0,4}}{\partial C}$ in symplectic structure ω_J. $A = 2\cosh(\alpha)$ and with similar explicit formulas for B, C in terms of (α, β) one checks $\{\alpha, \beta\} = 1$. For torus with one puncture very similar constructions lead to (α, β) Darboux variables, [25]. Once Darboux variables are defined for sphere with four punctures and torus with one puncture - definition for general case of $\Sigma_{g,n}$ is straightforward. (α, β) coordinates can be viewed as the complexification of well-known Fenchel-Nielsen coordinates.

Now one can verify the conjecture - express the monodromies of (3.27) in terms of (α, β) coordinates (establish relation between (α, β) and (ϵ_a, x_a)) and check that in these special coordinates the generating function of Lagrangian submanifold for opers (3.27) (see also (3.23)) is given by the effective twisted superpotential computed in 4d $N = 2$ gauge theory in Ω-background $\mathbf{R}^2 \times \mathbf{R}^2_\epsilon$ (for $\Sigma_{0,4}$ this is $SU(2)$ $N = 2$ gauge theory with $N_f = 4$ flavours and for $\Sigma_{1,1}$ - $N = 2^*$ $SU(2)$ gauge theory). These checks have been performed in [25].

The Yang-Yang function, defined as a potential for Bethe equations, is unique up to a constant which could be function of the parameters of the system, such as the complex structure parameters τ in our case. This ambiguity can be partly fixed by requiring that the derivatives of the Yang-Yang function with respect to the parameters τ_k correspond to the suitably normalized operators Φ_k, forming a basis in the space of quantum integrals of motion. The eigenvalue $E_k(\vec{n})$ in (3.25) is calculated on the solutions of second equation in (3.25) and thus depends on the discrete parameters $\vec{n}$. Obviously, the ambiguity we referred to above does not affect the differences $E_k(\vec{n}) - E_k(\vec{n}')$ of levels. Once both α and β coordinates are fixed, the Yang-Yang function is determined by the Lagrangian submanifolds O_τ and L_γ, and the claim that the τ-dependent piece coincides with the localized four dimensional gauge theory twisted superpotential (2.11) becomes quite nontrivial. This completely solves the problem for A_1 case. Regretfully for the higher rank case the analogs of α, β coordinates are not known except for A_n and sphere with four punctures when two monodromies are maximally degenerate.

4. Conclusion

This review was written as a contribution to ICM in Seoul, Korea, August 2014. Due to very limited space we reviewed here only very basic concepts and relations. In a sense this review has to be considered as a guide to the published papers, co-written by the author on the topic of the relationship between quantum integrability and supersymmetric vacua. Many important details have been omitted, as well as many important contributions of other authors have not been reviewed. We didn't cover the reformulation of Ω-background and its use (together with coisotropic branes) [24] to the quantisation problems described here, though we did use it in **Subsection 3.3**. We didn't touch at all the AGT relation [60] and the literature dedicated to it, though the supersymmetric vacua reviewed here correspond to the $\epsilon_2 \to 0$ limit on the gauge theory side of AGT, thus to the classical conformal blocks on the CFT side of it. We didn't discuss here large amount of work done in regard to quantisation of spectral curve, matrix models and the relation to quantum integrability ([61], later work of these authors in various collaborations, [62]). We didn't cover here the work [63] (closely related to [25]), the verification of claims made in **Subsection 3.3** from the CFT approach, as well as the consequences for classical conformal blocks and further important relationships [64]. String theory interpretation of Ω-background was also omitted, see e.g. [65–68], though it is clearly required in order to make further progress; we also said nothing about 2d/4d

relations [73] and many other important recent developments. Instead we will use the remaining space to make a few comments.

YY-function. Existence of YY-function for all known quantum integrable models solved by Bethe ansatz is an experimental fact, and we still do not have good conceptual understanding. Here we saw the importance of YY-function for the entire topic of relation with supersymmetric vacua, but we were not able to give a satisfactory explanation of its existence from integrability side. Explanation given here and in [25] reduces the question to existence of a very special Darboux coordinate system and can not be considered fully satisfactory. If we look back into the history of the quantum integrability we see that the major advances into the large subject of modern mathematics were achieved after Drinfeld explained the algebraic structure appearing in the theory of quantum integrable systems known at that time [39, 40, 42] by introducing the notion of Quantum Groups. The proper understanding of the YY-function may lead to a similar breakthrough. There is some progress in this direction, but the complete picture is yet to emerge [69].

Knizhnik-Zamolodchikov (KZ) equation at critical level. In the 90's another approach to the quantum many body systems of Hitchin type was developed (see e.g. [70] for Gaudin model) based on the KZ equation at critical level. eCM system, reviewed above, was studied for the integer coupling in this approach in [71]; it corresponds to the Knizhnik-Zamolodchikov-Bernard equation at the critical level. YY-function, which follows from this approach, differs from ours conceptually - it depends on $k\frac{N(N-1)}{2}$ variables instead of N as in our approach (here N is number of particles and k - coupling constant, which is integer in [71]). In [10] a conjecture was made about how to relate these two sets of variables and corresponding YY-functions. Checking this conjecture and finding the relation between two seemingly different methods shall lead to an important breakthrough.

TBA, BPS states and wall crossing. TBA type equations, similar to (3.14), appeared in the study of BPS states for large class of 4d $N = 2$ supersymmetric gauge theories in [46]. Interestingly, the Darboux variables, different from those we used above, on the moduli space of Higgs bundles play a key role there also. It is not clear how these two different TBA equations are related and whether rewriting one in the coordinates of another can help establishing the conceptual relation, but certainly this is an important work that needs to be done, see recent paper [74].

$N = 4$ integrability. Can we find the gauge theory such that its supersymmetric vacua correspond to the quantum states of the integrable system discovered in the study of anomalous dimensions of $N = 4$ gauge theory in 4d (see the collection of review articles in [72])? If such identification is found - one will have a magical relation between the sectors of two different supersymmetric gauge theories. Such relation yet has to be found; important obstacle at the moment seems to be the absence of YY-function on the side of $N = 4$ integrability, in fact even more - it has been claimed that such YY-function doesn't exist. If so - this is a first example known to the author of the absence of YY-function for Bethe equations. Clearly more work needs to be done in this direction.

Acknowledgements. I thank N. Nekrasov for two decades of our remarkable collaboration in developing the subject covered in this review, A. Gerasimov for the collaboration on arti-

cles critically influencing this development, and A. Losev, G. Moore, A. Rosly and V. Pestun for their invaluable contribution to our joint work. This work is supported in part by the Science Foundation Ireland under the 2011 RFP program and by the ESF Research Networking Programme Low-Dimensional Topology and Geometry with Mathematical Physics (ITGP).

References

[1] Gregory W. Moore, *Nikita A Nekrasov and Samson L. Shatashvili*, Integrating over Higgs branches, Commun. Math. Phys. **209** (2000), 97, arXiv:hep-th/9712241.

[2] N. Seiberg and E. Witten, *Monopole Condensation, And Confinement In N=2 Supersymmetric Yang-Mills Theory*, Nucl. Phys. B **426** (1994), 19–52; Erratum-ibid. B **430** (1994), 485–486, arXiv:hep-th/9407087.

[3] C. Vafa and E. Witten, *A Strong Coupling Test of S-Duality*, Nucl. Phys. B **431** (1994), 3–77, arXiv:hep-th/9408074.

[4] Hiraku Nakajima, *Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras*, Duke Math. J. Volume **76** (1994), Number 2, 333–672.

[5] A. A. Gerasimov and S. L. Shatashvili, *Higgs bundles, gauge theories and quantum groups,* Commun. Math. Phys. **277** (2008), 323, arXiv:hep-th/0609024.

[6] ———, *Two-dimensional Gauge Theories and Quantum Integrable Systems*, in From Hodge Theory to Integrability and TQFT: tt*-geometry, 239–262, R. Donagi and K. Wendland, Eds., Proc. of Symposia in Pure Mathematics **Vol. 78** (2008), AMS; arXiv:0711.1472 [hep-th].

[7] N. A. Nekrasov and S. L. Shatashvili, *Supersymmetric vacua and Bethe ansatz*, Nucl. Phys. Proc. Suppl. **91** (2009), 192-193, arXiv:0901.4744 [hep-th].

[8] ———, *Bethe Ansatz And Supersymmetric Vacua*, AIP Conf. Proc. **1134**, 154 (2009).

[9] ———, *Quantum integrability and supersymmetric vacua*, Prog. Theor. Phys. Suppl. **177**, 105 (2009), arXiv:0901.4748 [hep-th].

[10] ———, *Quantization of Integrable Systems and Four Dimensional Gauge Theories*, in 16th International Congress on Mathematical Physics, Prague, August 2009, P. Exner, Editor, pp. 265-289, World Scientific 2010, arXiv:0908.4052 [hep-th].

[11] G. W. Moore, N. Nekrasov and S. Shatashvili, *D particle bound states and generalized instantons*, Comm. Math. Phys. **209**, 77 (2000), arXiv:hep-th/9803265.

[12] A. Losev, G. W. Moore, N. Nekrasov, and S. Shatashvili, *Four-dimensional avatars of two-dimensional RCFT*, Nucl. Phys. Proc. Suppl. **46** (1996), 130–145, arXiv:hep-th/9509151.

[13] A. Losev, N. Nekrasov, and S. L. Shatashvili, *Issues in topological gauge theory*, Nucl. Phys. **B534** (1998), 549–611, arXiv:hep-th/9711108.

[14] ———, *Testing Seiberg-Witten solution*, in Cargese 1997, arXiv:hep-th/9801061.

[15] N. A. Nekrasov, *Seiberg-Witten prepotential from instanton counting*, Adv. Theor. Math. Phys. **7** (2004), 831–864, arXiv:hep-th/0206 161.

[16] N. Nekrasov and A. Okounkov, *Seiberg-Witten theory and random partitions*, arXiv:hep-th/0306238.

[17] A. Gorsky, I. Krichever, A. Marshakov, A. Mironov, and A. Morozov, *Integrability and Seiberg-Witten exact solution*, Phys. Lett. **B355** (1995), 466-474. arXiv:hep-th/9505035.

[18] E. Martinec and N. Warner, *Integrable systems and supersymmetric gauge theory*, Nucl.Phys.B459:97-112,1996, arXiv:hep-th/9509161.

[19] R. Donagi, E. Witten, *Supersymmetric Yang-Mills theory and integrable systems*, Nucl. Phys. **B460** (1996), 299-334. arXiv:hep-th/9510101.

[20] L. Alvarez-Gaume, D. Freedman, Comm. Math. Phys. **91** (1983), 87.

[21] S.J. Gates, C.M. Hull, and M. Rocek, Nucl. Phys. **248** (1984), 157.

[22] A. Redlich, *Gauge Noninvariance and Parity Nonconservation of Three-Dimensional Fermions*, Phys. Ref Lett. **52** (1984), 18-21.

[23] A.J. Niemi and G.W. Semenoff, *Axial Anomaly Induced Fermion Fractionization and Effective Gauge Theory Actions in Odd Dimensional Space-Times*, Phys. Rev. Lett. [**51** (1983), 2077.]

[24] N. Nekrasov and E. Witten, *The Omega Deformation, Branes, Integrability, and Liouville Theory*, JHEP **1009**, 092 (2010) [arXiv:1002.0888 [hep-th]].

[25] N. Nekrasov, A. Rosly and S. Shatashvili, *Darboux coordinates, Yang-Yang functional, and gauge theory*, Nucl. Phys. Proc. Suppl. **216** (2011) 69–93, arXiv:1103.3919.

[26] M. Blau and G. Thompson, *Lectures on 2d GaugeTheories: Topological Aspects and Path Integral Techniques*, hep-th/9310144.

[27] ———, *Derivation of the Verlinde Formula from Chern-Simons Theory and the G/G model*, Nucl. Phys. **B408** (1993), 345–390.

[28] A. Gerasimov, *Localization in GWZW and Verlinde formula*, arXiv:hep-th/9305090.

[29] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil, and E. Zaslow (Eds.), *Mirror symmetry*, American Mathematical Society, Providence, 2003, 929p.

[30] N. Nekrasov, *Five dimensional gauge theories and relativistic integrable systems*, Nucl. Phys. B **531** (1998), 323–344, arXiv:hep-th/9609219.

[31] N. Nekrasov and S. Shatashvili, *Supersymmetric vacua and quantum integrability*, book in preparation, to appear.

[32] E. H. Lieb and W. Liniger, *Exact analysis of an interacting Bose gas I*, The general solution and the ground state, Phys. Rev. (2), **130** (1963), 1605.

[33] F. A. Berezin, G. P. Pohil, and V. M. Finkelberg, *Vestnik MGU* **1**, 1 (1964), 21.

[34] C. N. Yang, *Some exact results for the many-body problems in one dimension with repulsive delta-function interaction*, Phys. Rev. Lett. **19** (1967), 1312.

[35] C. N. Yang and C. P. Yang, *Thermodinamics of a one-dimensional system of bosons with repulsive delta-function interaction*, J. Math. Phys. **10** (1969), 1115.

[36] N. Nekrasov, V. Pestun, and S. Shatashvili, *Quantum geometry and quiver gauge theories*, arXiv:1312.6689.

[37] A. Gorsky, I. Krichever, A. Marshakov, A. Mironov, and A. Morozov, *Integrability and Seiberg-Witten exact solution,* Phys. Lett. B355:466-474,1995, arXiv:hep-th/9505035.

[38] R. Donagiv E. Witten, *Supersymmetric Yang-Mills theory and integrable systems,* Nucl. Phys. B460:299-334, 1996, arXiv:hep-th/9510101.

[39] L. Faddeev, E. Sklyanin, and L. Takhtajan, *Quantum inverse problem method*, Theor. Math. Phys. **40**:2 (1980), 688–706. Teor. Mat. Fiz. 40:194-220, 1979 (in Russian)

[40] L. Takhtajan and L. Faddeev, *The Quantum method of the inverse problem and the Heisenberg XYZ model,* Russ. Math. Surveys **34** (1979), 11–68. Usp. Mat. Nauk 34:13-63, 1979.

[41] L. Faddeev, *How algebraic Bethe ansatz works for integrable model*, arXiv:hepth/9605 187.

[42] E. Sklyanin, *Separation of variables—new trends*, Prog. Theor. Phys. Suppl. **118** (1995), 35–60, e-Print: solv-int/9504001

[43] R. Baxter, *Exactly solved models in statistical mechanics*, Academic Press, 1982.

[44] F. A. Smirnov, *Structure of Matrix Elements in Quantum Toda Chain,* J. Phys. A. 31 (1998) 8953, arXiv:math-ph/9805011.

[45] D. Gaiotto, $N = 2$ *dualities*, arXiv:0904.2715.

[46] D. Gaiotto, G. W. Moore, and A. Neitzke, *Wall-crossing, Hitchin Systems, and the WKB Approximation*, arXiv:0907.3987.

[47] S. Kharchev, D. Lebedev, *Integral representations for the eigenfunctions of quantum open and periodic Toda chains from QISM formalism*, J. Phys. **A34** (2001), 2247–2258, arXiv:hep-th/0007040.

[48] I. Krichever, *Elliptic solutions of the Kadomtsev Petviashvili equation and integrable systems of particles*, Funkts. Anal. Prilozh., 14:4 (1980), 45–54.

[49] V. Inozemtsev, *The Finite Toda Lattices*, Comm. Math. Phys. **121** (1989), 629–638.

[50] S. Ruijsenaars and H. Schneider, *A new class of integrable systems and its relation to solitons*, Ann. Phys. (NY) **170** (1986), 370–405.

[51] S. Ruijsenaars, *Complete integrability of relativistic Calogero-Moser systems and elliptic function identities*, Comm. Math. Phys. **110** (1987), 191–213.

[52] I. Macdonald, *Symmetric Functions and Hall Polynomials*, Oxford, 1998.

[53] K. K. Kozlowski and J. Teschner, arXiv:1006.2906.

[54] N. Hitchin, *The self-duality equations on a Riemann surface*, Proc. London Math. Soc. (3) **55** (1987), 59–126.

[55] A. Kapustin and E. Witten, *Electric-magnetic duality and the geometric Langlands program*, arXiv:hep-th/0604151.

[56] N. Hitchin, *Stable bundles and integrable systems*, Duke Math. J. **54** (1987) 91–114.

[57] R. Donagi, *Spectral Covers*, arXiv:alg-geom/9505009.

[58] A. Beilinson and V. Drinfeld, *Quantization of Hitchin's integrable system and Hecke eigensheaves*, preprint.

[59] S. Gukov and E. Witten, *Branes And Quantization*, arXiv:0809.0305v2 [hep-th].

[60] L. Alday, D. Gaiotto, and Y. Tachikawa, *Liouville Correlation Functions from Four-dimensional Gauge Theories*, Lett. Math. Phys. **91** (2010), arXiv:0906.3219.

[61] A. Mironov and A. Morozov, *Nekrasov Functions and Exact Bohr-Sommerfeld Integrals*, JHEP 04 (2010) 040, arXiv:0910.5670.

[62] M. Aganagic, M. C.N. Cheng, R. Dijkgraaf, D. Krefl, and C. Vafa, *Quantum Geometry of Refined Topological Strings*, arXiv:1105.0630.

[63] J. Teschner, *Quantization of the Hitchin moduli spaces, Liouville theory, and the geometric Langlands correspondence I*, arXiv:1005.2846.

[64] A. Litvinov, S. Lukyanov, N. Nekrasov, and A. Zamolodchikov, *Classical Conformal Blocks and Painleve VI*, arXiv:1309.4700.

[65] I. Antoniadis, S. Hohenegger, K.S. Narain, and T.R. Taylor, *Deformed Topological Partition Function and Nekrasov Backgrounds*, arXiv:1003.2832.

[66] Yu. Nakayama, H. Ooguri, *Comments on Worldsheet Description of the Omega Background*, Int. J. Mod. Phys. **A27** (2012), arXiv:1106.5503.

[67] S. Hellerman, D. Orlando, and S. Reffert, *String theory of the Omega deformation*, arXiv:1106.0279.

[68] C. Vafa, *Supersymmetric Partition Functions and a String Theory in 4 Dimensions*, arXiv:1209.2425.

[69] A. Gerasimov and S. L. Shatashvili, work in progress.

[70] B. Feigin, E. Frenkel and N. Reshetikhin, Comm. Math. Phys. **166** (1994), 27–62.

[71] G. Felder, A. Varchenko, arXiv:hep-th/9502165.

[72] Beisert et al, *Review of AdS/CFT Integrability*, arXiv:1012.3982.

[73] Nick Dorey, Timothy J. Hollowood, and Sungjay Lee, *Quantization of Integrable Systems and a 2d/4d Duality*, arXiv:1103.5726.

[74] D. Gaiotto, *Opers and TBA*, arXiv:1403.6137.

The Hamilton Mathematics Institute, Trinity College Dublin, Dublin 2, IRELAND;
The School of Mathematics, Trinity College Dublin, Dublin 2, IRELAND;
Simons Center for Geometry and Physics, Stony Brook, NY 11794-3636, USA;
Euler International Mathematical Institute, Pesochnaya nab. 10, St. Petersburg 197022, RUSSIA - on leave of absence; Institute for Information Transmission Problems, Lab. 5, Bolshoy Karetny per. 19, Moscow 127994, RUSSIA - on leave of absence
E-mail: samson@maths.tcd.ie

Quantization of moduli spaces of flat connections and Liouville theory

Jörg Teschner

Abstract. We review known results on the relations between conformal field theory, the quantization of moduli spaces of flat $\mathrm{PSL}(2,\mathbb{R})$-connections on Riemann surfaces, and the quantum Teichmüller theory.

Mathematics Subject Classification (2010). 81T40, 51P05.

Keywords. Liouville theory, Quantization of Teichmüller spaces, Moduli spaces of flat connections.

1. Introduction

The Teichmüller spaces $\mathcal{T}(C)$ are the spaces of deformations of the complex structures on Riemann surfaces C. The classical uniformization theorem gives an alternative picture as spaces of constant negative curvature metrics modulo diffeomorphisms. Such metrics naturally define flat $\mathrm{PSL}(2,\mathbb{R})$-connections on C, relating the Teichmüller spaces to the moduli spaces $\mathcal{M}^{\mathbb{R}}_{\mathrm{flat}}(C)$ of flat $\mathrm{PSL}(2,\mathbb{R})$-connections.

There are well-known connections between the Teichmüller spaces and the (complexified) Lie-algebra of smooth vector fields on the unit circle. Cutting out a disc from a Riemann surface C, and gluing it back after twisting by the flow generated by a given vector field may generate changes of the complex structure of C. Both the spaces of functions on the Teichmüller spaces and the dual to the space of vector fields on the unit circle have natural Poisson-structures which can be used to formulate quantisation problems. Quantisation of the dual to the space of vector fields on the unit circle gives the Virasoro algebra, the Lie algebra of symmetries of any conformal field theory. Despite the fact that the existence of a relation between the quantisation of the Teichmüller spaces and conformal field theory may seem natural from this point of view, it has turned out to be nontrivial to establish such connections more precisely. The goal in this article will be to outline what is currently known about the connections between quantized moduli spaces of flat $\mathrm{PSL}(2,\mathbb{R})$-connections, quantum Teichmüller theory, and conformal field theory.[1]

The resulting picture appears to be of certain mathematical interest. It can in particular be seen as a first example for non-compact generalisations of the known relations between conformal field theories, quantum groups and three-manifold invariants associated to compact (quantum-) groups. Indeed, many pieces of the resulting picture show deep analogies or even concrete relations to the harmonic analysis of non-compact groups. Relations with three-dimensional hyperbolic geometry appear naturally, coming from known

Proceedings of the International Congress of Mathematicians, Seoul, 2014

relations between three-dimensional hyperbolic geometry and Teichmüller theory. There are furthermore various connections with the theory of classical and quantum integrable models, including relations to the isomonodromic deformation problem. A unifying perspective was outlined in [55], embedding such relations into a diamond of relations between conformal field theory, the (classical and quantized) Hitchin moduli spaces, and the geometric Langlands correspondence.

There exist various applications of the mathematical results described here in theoretical physics. They include relations to two-dimensional quantum gravity and matrix models, (non-critical) string theory and various relations to integrable models. Most strikingly, there even exist relations to four-dimensional $\mathcal{N} = 2$-supersymmetric gauge theories. Most direct seem to be the relations to work of Alday, Gaiotto and Tachikawa [1], Gaiotto, Moore and Neitzke [21, 22], Nekrasov and Witten [42] and Nekrasov, Rosly and Shatashvili [40]. Some of these connections are described in [57].

This article concentrates on some mathematical aspects of the connections between quantization of moduli spaces of flat connections and conformal field theory.

2. Moduli of flat $\mathrm{PSL}(2,\mathbb{R})$-connections and Teichmüller theory

In this section we will briefly review the necessary background on the relevant moduli spaces $\mathcal{M}_{\rm flat}(C)$ of flat connections and Teichmüller theory. The main goal will be to describe the algebra $\mathcal{A}(C) \equiv \mathrm{Fun}^{\rm alg}(\mathcal{M}_{\rm flat}(C))$ of algebraic functions on $\mathcal{M}_{\rm flat}(C)$ in terms of generators and relations in a way that will be useful for the quantisation.

We will consider Riemann surfaces $C = C_{g,n}$ of genus g with n marked points called punctures. In this article we will consider connections having regular[2] singularities at the punctures only.

2.1. Flat connections and uniformization. Let $\mathcal{M}_{\rm flat}(C)$ be the moduli space of flat $\mathrm{PSL}(2,\mathbb{C})$-connections modulo gauge transformations. To each flat $\mathrm{PSL}(2,\mathbb{C})$-connection $\nabla = d + A$ we may associate its holonomies $\rho(\gamma)$ along closed curves γ as $\rho(\gamma) = \mathcal{P}\exp(\int_\gamma A)$. The map $\gamma \mapsto \rho(\gamma)$ defines a representation of the fundamental group $\pi_1(C)$ in $\mathrm{PSL}(2,\mathbb{C})$, defining a point in the character variety

$$\mathcal{M}^{\mathbb{C}}_{\rm char}(C) := \mathrm{Hom}(\pi_1(C), \mathrm{PSL}(2,\mathbb{C}))/\mathrm{PSL}(2,\mathbb{C})\,. \tag{2.1}$$

The space $\mathcal{M}^{\mathbb{C}}_{\rm char}(C)$ contains the real slice $\mathcal{M}^{\mathbb{R}}_{\rm char}(C)$ which is known to decompose into a finite set of connected components [25, 30].

The uniformisation theorem allows us to represent any Riemann surface C as a quotient of the upper half plane $\mathbb{U}$ by certain discrete subgroups Γ of $\mathrm{PSL}(2,\mathbb{R})$ called Fuchsian groups, $C \simeq \mathbb{U}/\Gamma$. The Fuchsian subgroups Γ define representations of $\pi_1(C)$ in $\mathrm{PSL}(2,\mathbb{R})$. There is a distinguished connected component $\mathcal{M}^{\mathbb{R},0}_{\rm char}(C)$ in $\mathcal{M}^{\mathbb{R}}_{\rm char}(C)$ containing all the Fuchsian groups Γ uniformising Riemann surfaces. This component corresponds to a connected component $\mathcal{M}^{\mathcal{T}}_{\rm flat}(C)$ in the moduli space $\mathcal{M}^{\mathbb{R}}_{\rm flat}(C)$ of flat $\mathrm{PSL}(2,\mathbb{R})$-connections on C. $\mathcal{M}^{\mathcal{T}}_{\rm flat}(C)$ is called Teichmüller component as it is isomorphic to the Teichmüller

[1] Here understood more precisely as representation theory of the Virasoro algebra with central charge $c > 1$, corresponding to what is often called Liouville theory in the physics literature.

[2] The connection is gauge equivalent to a meromorphic connection with simple poles at the punctures.

space $\mathcal{T}(C)$ [25, 30].

2.2. Coordinates associated to triangulations. There exist useful systems of coordinates for $\mathcal{M}_{\rm flat}(C)$ associated to triangulations of C if C has at least one puncture. Coordinates of this type were introduced for $\mathcal{T}(C)$ in [44]; the shear-coordinates introduced in [16] are closely related; there exists a natural complexification [17]; the following formulation is due to [21].

Let τ be a triangulation of the surface C such that all vertices coincide with marked points on C. An edge e of τ separates two triangles defining a quadrilateral Q_e with corners being the marked points $P_1, \ldots, P_4$. For a given connection $\nabla = d + A$, let us choose four sections s_i, $i = 1, 2, 3, 4$ that are horizontal in Q_e,

$$\nabla s_i = (d + A)\, s_i = 0\,. \tag{2.2}$$

We shall furthermore assume that the sections s_i are eigenvectors of the monodromy around P_i. Out of the sections s_i form

$$\mathcal{X}_e^\tau := -\frac{(s_1 \wedge s_2)(s_3 \wedge s_4)}{(s_2 \wedge s_3)(s_4 \wedge s_1)}, \tag{2.3}$$

where all s_i, $i = 1, 2, 3, 4$ are evaluated at the same point $P \in Q_e$. The ratio $\mathcal{X}_e^\tau$ does not depend on the choice of P.

There is a natural Poisson structure on $\mathcal{M}_{\rm flat}(C)$ induced by the symplectic form $\Omega_{\rm AB}$ introduced by Atiyah and Bott. The Poisson bracket of the coordinates $\mathcal{X}_e$ becomes very simple,

$$\{\mathcal{X}_e^\tau, \mathcal{X}_{e'}^\tau\} \;=\; n_{e,e'}\, \mathcal{X}_{e'}^\tau\, \mathcal{X}_e^\tau\,; \tag{2.4}$$

the definition of $n_{e,e'} \in \{-2, -1, 0, 1, 2\}$ is best described in terms of the fat graph $\hat{\mathfrak{t}}$ dual to the given triangulation $\mathfrak{t}$: It is the total intersection index of the edges $\hat{e}$ and $\hat{e}'$ dual to e and e', respectively.

There furthermore exists a simple description of the relations between the coordinates associated to different triangulations. If triangulation τ_e is obtained from τ by changing only the diagonal in the quadrangle containing e, we have

$$\mathcal{X}_{e'}^{\tau_e} \;=\; \begin{cases} \mathcal{X}_{e'}^\tau \left(1 + (\mathcal{X}_e^\tau)^{-{\rm sgn}(n_{e'e})}\right)^{-n_{e'e}} & \text{if } \; e' \neq e\,, \\ (\mathcal{X}_e^\tau)^{-1} & \text{if } \; e' = e\,. \end{cases} \tag{2.5}$$

Poisson bracket (2.4) and transformation law (2.5) reflect the cluster algebra structure that $\mathcal{M}_{\rm flat}(C)$ has [17].

2.3. Trace functions. Useful coordinate functions for $\mathcal{M}_{\rm flat}(C)$ are the trace functions

$$L_\gamma := \nu_\gamma {\rm tr}(\rho(\gamma))\,; \tag{2.6}$$

the signs $\nu_\gamma \in \{+1, -1\}$ will be chosen in such a way that the restriction of L_γ to $\mathcal{M}_{\rm flat}^{\mathcal{T}}(C)$ is positive and larger than two. It is possible to show that the length l_γ of the geodesic on $\mathbb{H}/\Gamma$ isotopic to γ is related to L_γ as $L_\gamma \;=\; 2\cosh(l_\gamma/2)$. If the curves γ_r encircle the punctures P_r on $C = C_{g,n}$ for $r = 1, \ldots, n$, we will identify the surface C with the surface with constant negative curvature metric obtained by cutting out n discs having the geodesics isotopic to γ_r as boundaries.

There exists a natural complex structure on $\mathcal{M}_{\rm flat}(C)$ which is such that the trace functions L_γ defined above are *complex analytic*.

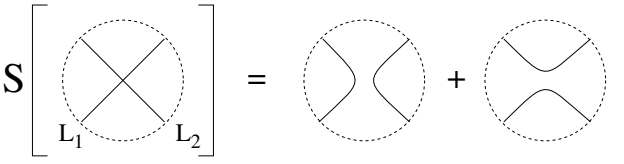


Figure 2.1. The symmetric smoothing operation

2.3.1. Skein algebra. Let $\mathcal{A}(C) \simeq \mathrm{Fun}^{\mathrm{alg}}(\mathcal{M}_{\mathrm{flat}}(C))$ be the commutative algebra of functions on $\mathcal{M}_{\mathrm{flat}}(C)$ generated by the coordinate functions L_γ. We will explain how to describe $\mathcal{A}(C)$ in terms of generators and relations

The well-known relation $\mathrm{tr}(g)\mathrm{tr}(h) = \mathrm{tr}(gh) + \mathrm{tr}(gh^{-1})$ valid for any pair of $SL(2)$-matrices g, h implies that the trace functions satisfy the skein relations,

$$L_{\gamma_1} L_{\gamma_2} \;=\; L_{S(\gamma_1,\gamma_2)}\,, \tag{2.7}$$

where $S(\gamma_1, \gamma_2)$ is the curve obtained from γ_1, γ_2 by means of the smoothing operation, defined as follows. The application of S to a single intersection point of γ_1, γ_2 is depicted in Figure 2.1. The general result is obtained by applying this rule at each intersection point, and summing the results.

2.3.2. Topological classification of closed curves. A Riemann surface C of genus g with n punctures may be cut into pairs of pants by cutting along $h := 3g-3+n$ non-intersecting simple closed curves $\gamma = \{\gamma_1, \ldots, \gamma_h\}$ on C. It will be useful to supplement the collection of curves γ specifying a pants decomposition by a three-valent graph Γ on C which has exactly one vertex inside each pair of pants, and the three edges emanating from a given vertex each intersect exactly one of the boundaries of the pair of pants. The pair of data $\sigma = (\gamma, \Gamma)$ will be called a pants decomposition[3] .

With the help of pants decompositions one may conveniently classify all non-selfintersecting closed curves on C up to homotopy [11]. Recall that there is a unique curve $\gamma_e \in \gamma$ that intersects a given edge e on Γ exactly once, and which does not intersect any other edge. To a curve $\gamma_e \in \gamma$ let us associate the integers (r_e, s_e) defined as follows.The integer r_e is defined as the number of intersections between γ and the curve γ_e. Having chosen an orientation for the edge e we will define s_e to be the intersection index between e and γ.

Dehn's theorem [11] ensures that the curve γ is up to homotopy uniquely classified by the collection of integers $(r, s) : e \mapsto (r_e, s_e)$, subject to the restrictions

(i) $r_e \geq 0$,

(ii) if $r_e = 0$, then $s_e \geq 0$,

(iii) $r_{e_1} + r_{e_2} + r_{e_3} \in 2\mathbb{Z}$ whenever $\gamma_{e_1}, \gamma_{e_2}, \gamma_{e_3}$ bound the same pair of pants.

We will use the notation $\gamma_{(r,s)}$ for the geodesic which has parameters $(r, s) : e \mapsto (r_e, s_e)$.

2.3.3. Generators. As set of generators for $\mathcal{A}(C)$ one may take the functions $L_{(r,s)} \equiv L_{\gamma_{(r,s)}}$. The skein relations allow us to express arbitrary $L_{(r,s)}$ in terms of a finite subset of

[3] The graph Γ allows us to distinguish pants decompositions related by Dehn-twists, the operation to cut open along a curve $\gamma_e \in \gamma$, twisting by 2π, and gluing back.

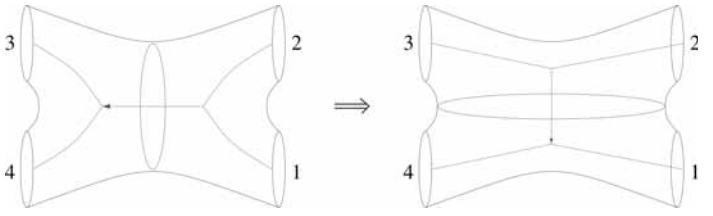


Figure 2.2. The geodesics γ_s^e and γ_t^e are the red curves on the left and right pieces of the figure. The change of pants decomposition from left to right is called F-move

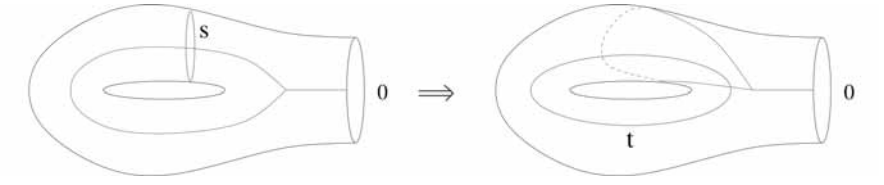


Figure 2.3. The geodesics γ_s^e and γ_t^e are the red curves on the left and right pieces of the figure. The change of pants decomposition from left to right is called S-move

the set of $L_{(r,s)}$. We shall now describe convenient choices for sets of generators.

Let us note that to each internal[4] edge e of the graph Γ of $\sigma = (\gamma, \Gamma)$ there corresponds a unique curve γ_e in the cut system $\mathcal{C}_\sigma$. There is a unique subsurface $C_e \hookrightarrow C$ isomorphic to either $C_{0,4}$ or $C_{1,1}$ that contains γ_e in the interior of C_e. The subsurface C_e has boundary components labeled by numbers $1, 2, 3, 4$ if $C_e \simeq C_{0,4}$, and if $C_e \simeq C_{1,1}$ we will assign to the single boundary component the label 0.

For each edge e let us introduce the geodesics γ_t^e which have Dehn parameters $(r^e, 0)$, where $r^e_{e'} = 2\delta_{e,e'}$ if $C_e \simeq C_{0,4}$ and $r^e_{e'} = \delta_{e,e'}$ if $C_e \simeq C_{1,1}$. The geodesics γ_s^e and γ_t^e are depicted as red curves on the left and right halfs of Figures 2.2 and 2.3, respectively. There furthermore exist unique geodesics γ_u^e with Dehn parameters (r^e, s^e), where $s^e_{e'} = \delta_{e,e'}$. We will denote the trace functions associated to γ_k^e by L_k^e, where $k \in \{s, t, u\}$. The set $\{L_s^e, L_t^e, L_u^e\,;\, \gamma_e \in \gamma\}$ generates $\mathcal{A}(C)$.

2.3.4. Relations. These coordinates are not independent, though. Further relations follow from the relations in $\pi_1(C)$. It can be shown (see e.g. [26] for a review) that any triple of coordinate functions L_s^e, L_t^e and L_u^e satisfies an algebraic relation of the form

$$P_e(L_s^e, L_t^e, L_u^e) \,=\, 0\,. \tag{2.8}$$

The polynomial P_e in (2.8) is for $C_e \simeq C_{0,4}$ explicitly given as

$$\begin{aligned} P_e(L_s, L_t, L_u) := {} & L_1L_2L_3L_4 + L_s^2 + L_t^2 + L_u^2 + L_1^2 + L_2^2 + L_3^2 + L_4^2 - 4 \\ & + L_s(L_3L_4 + L_1L_2) + L_t(L_2L_3 + L_1L_4) + L_u(L_1L_3 + L_2L_4) - L_sL_tL_u\,, \end{aligned} \tag{2.9}$$

while for $C_e \simeq C_{1,1}$ we take P to be

$$P_e(L_s, L_t, L_u) := L_s^2 + L_t^2 + L_u^2 - L_sL_tL_u + L_0 - 2\,. \tag{2.10}$$

[4] An internal edge does not end in a boundary component of C.

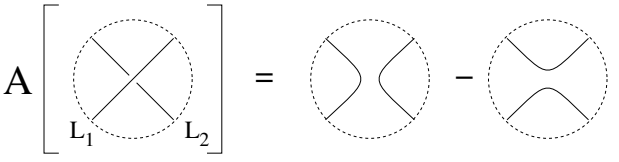


Figure 2.4. The anti-symmetric smoothing operation

In the expressions above we have denoted $L_i := \nu_{\gamma_i}\mathrm{Tr}(\rho(\gamma_i))$, $i = 0,1,2,3,4$, where γ_0 is the geodesic representing the boundary of $C_{1,1}$, while γ_i, $i = 1,2,3,4$ represent the boundary components of $C_{0,4}$. Generators L^e_k, $k \in \{s,t,u\}$, and relations (2.8) for all edges e of Γ describe $\mathcal{M}_{\mathrm{flat}}(C)$ as an algebraic variety.

2.4. Poisson structure. There is also a natural Poisson bracket on $\mathcal{A}(C)$ [24], defined such that

$$\{\, L_{\gamma_1}\,,\, L_{\gamma_2}\,\} \,=\, L_{A(\gamma_1,\gamma_2)}\,, \tag{2.11}$$

where $A(\gamma_1,\gamma_2)$ is the curve obtained from γ_1, γ_2 by means of the anti-symmetric smoothing operation, defined as above, but replacing the rule depicted in Figure 2.1 by the one depicted in Figure 2.4. The Poisson structure (2.11) coincides with the one induced from the symplectic form introduced by Atiyah and Bott.

The Poisson bracket $\{\, L^e_s\,, L^e_t\,\}$ can be written elegantly in the form [40]

$$\{\, L^e_s\,, L^e_t\,\} \,=\, \frac{\partial}{\partial L^e_u} P_e(L^e_s, L^e_t, L^e_u)\,. \tag{2.12}$$

It is remarkable that the same polynomial appears both in (2.8) and in (2.12), which indicates that the symplectic structure on $\mathcal{M}_{\mathrm{flat}}(C)$ is compatible with its structure as algebraic variety.

It is sometimes useful to introduce Darboux-coordinates $e \mapsto (l_e, k_e)$ such that $\{l_e, k_{e'}\} = \delta_{ee'}$ and $L^e_s = 2\cosh(l_e/2)$. The Fenchel-Nielsen coordinates for $\mathcal{T}(C)$ are such coordinates. There is a natural complexification of the Fenchel-Nielsen coordinates discussed in [40].

3. Quantization of $\mathcal{M}^{\mathcal{T}}_{\mathrm{flat}}(C)$

We shall next review the quantization of the moduli spaces $\mathcal{M}^{\mathcal{T}}_{\mathrm{flat}}(C)$ that was constructed in [53, 57] based on the pioneering works [9, 16, 35].

3.1. Quantization of coordinates associated to triangulations. The simplicity of the Poisson brackets (2.4) of the coordinates $\mathcal{X}^{\mathfrak{t}}_e$ makes part of the quantization quite simple. To each edge e of a triangulation $\mathfrak{t}$ of a Riemann surface $C_{g,n}$ associate the generator $\mathcal{X}^{\mathfrak{t}}_e$ of a non-commutative algebra $\mathcal{B}_{\mathfrak{t}}$ which has generators $\mathcal{X}^{\mathfrak{t}}_e$ and relations

$$\mathcal{X}^{\mathfrak{t}}_e \mathcal{X}^{\mathfrak{t}}_{e'} = e^{2\pi i b^2 n_{ee'}}\, \mathcal{X}^{\mathfrak{t}}_{e'}\, \mathcal{X}^{\mathfrak{t}}_e\,, \tag{3.1}$$

the integers $n_{ee'}$ coincide with the structure constants of the Poisson algebra (2.4), and we have introduced the notation b^2 for the deformation parameter traditionally denoted $\hbar$.

Note furthermore that the variables $\mathcal{X}_e$ are positive for the Teichmüller component. This motivates us to consider representations $\pi_{\mathfrak{t}}$ of $\mathcal{B}_{\mathfrak{t}}$ in which the operators $\mathsf{X}_e^{\mathfrak{t}} := \pi_{\mathfrak{t}}(\mathcal{X}_e^{\mathfrak{t}})$ are *positive* self-adjoint. By choosing a polarization one may define representations $\pi_{\mathfrak{t}}$ in terms of multiplication and finite difference operators on suitable dense subspaces of the Hilbert space $\mathcal{H}_{\mathfrak{t}} \simeq L^2(\mathbb{R}^{3g-3+n})$.

There exists a family of automorphisms which describe the relations between the quantized coordinate functions associated to different triangulations [9, 16, 35]. If triangulation $\mathfrak{t}_e$ is obtained from $\mathfrak{t}$ by changing only the diagonal in the quadrangle containing e, we have

$$\mathsf{X}_{e'}^{\mathfrak{t}_e} \;=\; \begin{cases} \mathsf{X}_{e'}^{\mathfrak{t}} \prod\limits_{a=1}^{|n_{e'e}|} \big(1 + e^{\pi i(2a-1)b^2} (\mathsf{X}_e^{\mathfrak{t}})^{-\mathrm{sgn}(n_{e'e})}\big)^{-\mathrm{sgn}(n_{e'e})} & \text{if } \; e' \neq e\,, \\ (\mathsf{X}_e^{\mathfrak{t}})^{-1} & \text{if } \; e' = e\,. \end{cases} \tag{3.2}$$

Any two two triangulations $\mathfrak{t}_1$ and $\mathfrak{t}_2$ can be connected by a sequence of changes of diagonals in quadrilaterals. It follows that the quantum theory of $\mathcal{M}_{\rm flat}^{\mathcal{T}}(C)$ has the structure of a quantum cluster algebra [18].

It is possible to construct [35, 36] unitary operators $\mathsf{T}_{\mathfrak{t}_2\mathfrak{t}_1} : \mathcal{H}_{\mathfrak{t}_1} \to \mathcal{H}_{\mathfrak{t}_2}$ that represent the quantum cluster transformations (3.2) in the sense that

$$\mathsf{X}_e^{\mathfrak{t}_2} \;=\; \mathsf{T}_{\mathfrak{t}_1\mathfrak{t}_2}^{-1} \cdot \mathsf{X}_e^{\mathfrak{t}_1} \cdot \mathsf{T}_{\mathfrak{t}_1\mathfrak{t}_2}\,. \tag{3.3}$$

The operator $\mathsf{T}_{\mathfrak{t}_2\mathfrak{t}_1}$ describes the change of representation when passing from the quantum theory associated to triangulation $\mathfrak{t}_1$ to the one associated to $\mathfrak{t}_2$. It follows that the resulting quantum theory does not depend on the choice of a triangulation in an essential way.

3.2. Quantization of the trace functions. There is a simple algorithm [16, 17] for calculating the trace functions in terms of the variables $\mathcal{X}_e^{\mathfrak{t}}$ leading to Laurent polynomials of the form

$$L_\gamma^{\mathfrak{t}} \;=\; \sum_{\nu\in\mathbb{F}} C_\gamma^{\mathfrak{t}}(\nu) \prod_e (\mathcal{X}_e^{\mathfrak{t}})^{\frac{1}{2}\nu_e}\,, \tag{3.4}$$

where the summation is taken over a finite set $\mathbb{F}$ of vectors $\nu \in \mathbb{Z}^{3g-3+2n}$ with components ν_e.

In order to define an operator $\mathsf{L}_\gamma^{\mathfrak{t}}$ associated to a classical trace function L_γ it has turned out [9, 10, 53] for some pairs $(\gamma, \mathfrak{t})$ to be sufficient to simply replace $(\mathcal{X}_e^{\mathfrak{t}})^{\frac{1}{2}\nu_e}$ in (3.4) by $\exp(\sum_e \frac{1}{2}\nu_e \log \mathsf{X}_e^{\mathfrak{t}})$. Let us call such pairs $(\gamma, \mathfrak{t})$ simple. In order to define $\mathsf{L}_\gamma^{\mathfrak{t}}$ in general [53] one may use the fact that for all curves γ there exists a triangulation $\mathfrak{t}'$ such that $(\gamma, \mathfrak{t}')$ is simple, allowing us to define

$$\mathsf{L}_\gamma^{\mathfrak{t}} \;=\; \mathsf{T}_{\mathfrak{t}'\mathfrak{t}}^{-1} \cdot \mathsf{L}_\gamma^{\mathfrak{t}'} \cdot \mathsf{T}_{\mathfrak{t}'\mathfrak{t}}\,. \tag{3.5}$$

The operators $\mathsf{L}_\gamma^{\mathfrak{t}}$ defined thereby are positive self-adjoint with spectrum bounded from below by 2, as follows from the result of [37]. Two operators $\mathsf{L}_{\gamma_1}^{\mathfrak{t}}$ and $\mathsf{L}_{\gamma_2}^{\mathfrak{t}}$ commute if the intersection of γ_1 and γ_2 is empty.

It turns out that (3.5) holds in general. It follows that we may regard the algebras generated by the operators $\mathsf{L}_\gamma^{\mathfrak{t}}$ as different representations $\pi_{\mathfrak{t}}$ of an abstract algebra $\mathcal{A}_{b^2}(C) \equiv \mathrm{Fun}_{b^2}^{\rm alg}(\mathcal{M}_{\rm flat}^{\mathcal{T}}(C))$ which does not depend on the choice of a triangulation, $\mathsf{L}_\gamma^{\mathfrak{t}} \equiv \pi_{\mathfrak{t}}(L_\gamma)$

for $L_\gamma \in \mathcal{A}_{b^2}(C)$. As in the classical case one may use pants decompositions to identify convenient sets of generators for $\mathcal{A}_{b^2}(C)$ to be

$$\text{set of generators:} \quad \left\{ L_i^e,\ i \in \{s,t,u\}\,, e \in \{\text{edges of } \Gamma\}\right\}.$$

Important relations are

$$\begin{aligned} &\mathcal{P}^{(a)}_{0,4}(L^e_s, L^e_t, L^e_u; L^e_1, L^e_2, L^e_3, L^e_4) \,=\, 0\,, \qquad &&\text{if } C_e \simeq C_{0,4}\,,\\ &\mathcal{P}^{(a)}_{1,1}(L^e_s, L^e_t, L^e_u; L^e_0) \,=\, 0\,, &&\text{if } C_e \simeq C_{1,1}\,, \end{aligned} \qquad a = 2,3\,. \tag{3.6}$$

where the polynomials $\mathcal{P}^{(a)}_{0,4}$ of non-commutative variables are defined as :

$$\begin{aligned} &\mathcal{P}^{(2)}_{0,4}(L_s, L_t, L_u; L_1, L_2, L_3, L_4) := e^{\pi i b^2} L_s L_t - e^{-\pi i b^2} L_t L_s \\ &\quad - (e^{2\pi i b^2} - e^{-2\pi i b^2}) L_u - (e^{\pi i b^2} - e^{-\pi i b^2})(L_1 L_3 + L_2 L_4)\,. \end{aligned} \tag{3.7}$$

$$\begin{aligned} &\mathcal{P}^{(3)}_{0,4}(L_s, L_t, L_u; L_1, L_2, L_3, L_4) := L_1 L_2 L_3 L_4 + L_1^2 + L_2^2 + L_3^2 + L_4^2 \\ &\quad - e^{\pi i b^2} L_s L_t L_u + e^{2\pi i b^2} L_s^2 + e^{-2\pi i b^2} L_t^2 + e^{2\pi i b^2} L_u^2 - \left(2\cos \pi b^2\right)^2 \\ &\quad + e^{\pi i b^2} L_s (L_3 L_4 + L_1 L_2) + e^{-\pi i b^2} L_t (L_2 L_3 + L_1 L_4) + e^{\pi i b^2} L_u (L_1 L_3 + L_2 L_4)\big]\,. \end{aligned} \tag{3.8}$$

In the case $C_e \simeq C_{1,1}$ we have

$$\mathcal{P}^{(2)}_{1,1}(L_s, L_t, L_u; L_0) := e^{\frac{\pi i}{2} b^2} L_s L_t - e^{-\frac{\pi i}{2} b^2} L_t L_s \,-\, (e^{\pi i b^2} - e^{-\pi i b^2}) L_u\,, \tag{3.9}$$

$$\mathcal{P}^{(3)}_{1,1}(L_s, L_t, L_u; L_0) := e^{\pi i b^2}(L_s^2 + e^{-2\pi i b^2} L_t^2 + L_u^2) - e^{\frac{\pi i}{2} b^2} L_s L_t L_u + L_0 - 2\cos \pi b^2. \tag{3.10}$$

The quadratic relations $\mathcal{P}^{(2)}_{g,n} = 0$ represent the deformation of the Poisson bracket (2.12), while the cubic[5] relations $\mathcal{P}^{(3)}_{g,n} = 0$ are deformations of the relations (2.8). One furthermore finds quantum analogs of the skein relations [9, 10].

3.3. Representations associated to pants decompositions. The operators $\mathsf{L}^{\mathfrak{t}}_\gamma$ and $\mathsf{L}^{\mathfrak{t}}_{\gamma'}$ associated to non-intersecting curves γ and γ' commute. It is therefore possible to diagonalise simultaneously the quantised trace functions associated to a maximal set $\gamma = \{\gamma_1, \ldots, \gamma_h\}$ of non-intersecting closed curves characterising a pants decomposition. This can be done by constructing operators $\mathsf{R}_{\sigma|\mathfrak{t}}$ which map the operators $\mathsf{L}^{\mathfrak{t}}_{\gamma_e}$ associated to the curves γ_e, $e = 1, \ldots, h$, to the operators of multiplication by $2\cosh(l_e/2)$, respectively [53, 57]. The states in the image $\mathcal{H}_\sigma$ of $\mathsf{R}_{\sigma|\mathfrak{t}}$ can be represented by functions $\psi(l)$, $l = (l_1, \ldots, l_h)$ depending on the variables $l_e \in \mathbb{R}^+$ which parameterise the eigenvalues of $\mathsf{L}^{\mathfrak{t}}_{\gamma_e}$. The operators $\mathsf{R}_{\sigma|\mathfrak{t}}$ define a new family of representations π_σ of $\mathcal{A}_{b^2}(C)$ via

$$\pi_\sigma(L_\gamma) := \mathsf{R}_{\sigma|\mathfrak{t}} \cdot \pi_{\mathfrak{t}}(L_\gamma) \cdot (\mathsf{R}_{\sigma|\mathfrak{t}})^{-1}\,. \tag{3.11}$$

The representations π_σ are naturally labelled by pants decompositions $\sigma = (\gamma, \Gamma)$. The unitary operators $\mathsf{R}_{\sigma|\mathfrak{t}} : \mathcal{H}_{\mathfrak{t}} \to \mathcal{H}_\sigma$ were constructed explicitly in [53].

[5] Relations cubic in L_s, L_t, L_u.

The operators $\pi_\sigma(\mathsf{L}_\gamma)$ were calculated explicitly for the generators of $\mathcal{A}_{b^2}(C)$ in [57]. When σ corresponds to the pants decomposition of $C = C_{0,4}$ depicted on the left of Figure 2.2 one finds, for example, $\mathsf{L}_s := 2\cosh(\mathsf{l}/2)$,

$$\mathsf{L}_t := \frac{1}{2(\cosh \mathsf{l}_s - \cos 2\pi b^2)} \Big(2\cos \pi b^2 (L_2 L_3 + L_1 L_4) + \mathsf{L}_s (L_1 L_3 + L_2 L_4) \Big) \qquad (3.12)$$
$$+ \sum_{\epsilon = \pm 1} \frac{1}{\sqrt{2\sinh(\mathsf{l}/2)}} e^{\epsilon \mathsf{k}/2} \frac{\sqrt{c_{12}(\mathsf{L}_s) c_{34}(\mathsf{L}_s)}}{2\sinh(\mathsf{l}/2)} e^{\epsilon \mathsf{k}/2} \frac{1}{\sqrt{2\sinh(\mathsf{l}/2)}} ,$$

with operators l and k defined as $\mathsf{l}\,\psi_\sigma(l) = l\,\psi_\sigma(l)$, $\mathsf{k}\,\psi_\sigma(l) = -4\pi \mathrm{i} b^2\, \partial_l \psi(l)$, respectively, while $c_{ij}(L_s)$ is defined as $c_{ij}(L_s) = L_s^2 + L_i^2 + L_j^2 + L_s L_i L_j - 4$. L_u is given by a similar expression [57]. The operators l_s and k_s can be identified as quantum counterparts of the Fenchel-Nielsen coordinates. In the general case one may use pants decompositions to reduce the description of the operators $\pi_\sigma(\mathsf{L}_i^e)$, $i \in \{s,t,u\}$, $e \in \{\text{edges of } \Gamma\}$ to the cases $C_e \simeq C_{0,4}$ and $C_e \simeq C_{1,1}$.

The operators $\pi_\sigma(L_\gamma)$ are unbounded. The maximal domain of definition of $\pi_\sigma(\mathcal{A}_{b^2}(C))$ defines a natural subspace $\mathcal{S}_\sigma \subset \mathcal{H}_\sigma$ with topology given by the family of semi-norms $\|\pi_\sigma(\mathcal{O})\|$, $\mathcal{O} \in \mathcal{A}_{b^2}(C)$. The topological dual $\mathcal{D}_\sigma$ of $\mathcal{S}_\sigma$ is a space of distributions canonically associated to $(\mathcal{A}_{b^2}(C), \pi_\sigma)$ such that $\mathcal{S}_\sigma \subset \mathcal{H}_\sigma \subset \mathcal{D}_\sigma$.

3.4. Changes of pants decomposition. The passage between the representations π_{σ_1} and π_{σ_2} associated to two different pants decompositions can be described by operators of the form

$$\mathsf{U}_{\sigma_2\sigma_1} := \mathsf{R}_{\sigma_2|\mathsf{t}_{\sigma_2}} \cdot \mathsf{T}_{\mathsf{t}_{\sigma_2}\mathsf{t}_{\sigma_1}} \cdot (\mathsf{R}_{\sigma_1|\mathsf{t}_{\sigma_1}})^{-1} . \qquad (3.13)$$

The passage between two pants decompositions σ_1 and σ_2 can always be decomposed into elementary "moves" called F-, S-, B- and Z- moves localized in subsurfaces with $3g - 3 + n \leq 1$ [5, 17, 39]. We refer to [5, 17] for precise descriptions of the full set of generators. For future reference we have depicted the F- and S- moves in Figures 2.2 and 2.3, respectively. It is useful to formalize the resulting structure by introducing the notion of the Moore-Seiberg groupoid: The path groupoid of the two-dimensional CW-complex which has vertices identified with pants decompositions σ, edges ("generators") called F-, S-, B- and Z-moves, and faces ("relations") being certain edge-paths localized in subsurfaces with $3g - 3 + n \leq 2$ listed in [5, 17, 39].

The unitary operators $\mathsf{U}_{\sigma_2\sigma_1}$ intertwine the representation π_{σ_1} and π_{σ_2},

$$\pi_{\sigma_2}(L_\gamma) \cdot \mathsf{U}_{\sigma_2\sigma_1} \;=\; \mathsf{U}_{\sigma_2\sigma_1} \cdot \pi_{\sigma_1}(L_\gamma) . \qquad (3.14)$$

Explicit representations for the operators $\mathsf{U}_{\sigma_2\sigma_1}$ have been calculated in [41, 57] for pairs $[\sigma_2, \sigma_1]$ related by the generators of the Moore-Seiberg groupoid. The B-move is represented as

$$\mathsf{B}\psi \;=\; B^{l_3}_{l_2 l_1}\, \psi , \qquad B^{l_3}_{l_2 l_1} \;=\; e^{\pi i (\Delta_{l_3} - \Delta_{l_2} - \Delta_{l_1})} , \qquad (3.15)$$

where $\Delta_l = (1+b^2)/4b + (l/4\pi b)^2$, and ψ is a generator for the one-dimensional space associated to $C_{0,3}$. The F-move is represented in terms of an integral transformation of the form

$$\psi_s(l_s) \;\equiv\; (\mathsf{F}\psi_t)(l_s) \;=\; \int_{\mathbb{R}^+} dl_t \; F_{l_s l_t} \left[\begin{smallmatrix} l_3 & l_2 \\ l_4 & l_1 \end{smallmatrix} \right] \psi_t(l_t) . \qquad (3.16)$$

A similar formula exists for the S-move. The explicit formulae are given in [57].

The operators $\mathsf{U}_{\sigma_2\sigma_1}$ generate a projective unitary representation of the Moore-Seiberg groupoid,

$$\mathsf{U}_{\sigma_3\sigma_2} \cdot \mathsf{U}_{\sigma_2\sigma_1} = \zeta_{\sigma_3\sigma_2\sigma_1}\mathsf{U}_{\sigma_3\sigma_2}\,, \tag{3.17}$$

where $\zeta_{\sigma_3\sigma_2\sigma_1} \in \mathbb{C}$, $|\zeta_{\sigma_3\sigma_2\sigma_1}| = 1$. The explicit formulae for the relations of the Moore-Seiberg groupoid in the quantization of $\mathcal{M}^{\mathcal{T}}_{\rm flat}(C)$ are listed in [57]. The operators $\mathsf{U}_{\sigma_2\sigma_1}$ allow us to identify the spaces $\mathcal{S}_\sigma \subset \mathcal{H}_\sigma \subset \mathcal{D}_\sigma$ as different representatives of abstract spaces $\mathcal{S}(C) \subset \mathcal{H}(C) \subset \mathcal{D}(C)$ associated to C.

Having a representation of the Moore-Seiberg groupoid induces a representation of the mapping class group $\mathrm{MCG}(C)$. Elements μ of $\mathrm{MCG}(C)$ can be represented by diffeomorphisms of the surface C not isotopic to the identity, and therefore map any pants decomposition σ to another one denoted $\mu.\sigma$. Note that the Hilbert spaces $\mathcal{H}_\sigma$ and $\mathcal{H}_{\mu.\sigma}$ are canonically isomorphic, depending only on the combinatorics of the graphs σ, but not on their embedding into C. We may therefore define an operator $\mathsf{M}_\sigma(\mu) : \mathcal{H}_\sigma \to \mathcal{H}_\sigma$ as

$$\mathsf{M}_\sigma(\mu) := \mathsf{U}_{\mu.\sigma,\sigma}\,. \tag{3.18}$$

It is automatic that the operators $\mathsf{M}(\mu)$ define a projective unitary representation of $\mathrm{MCG}(C)$ on $\mathcal{H}_\sigma$.

3.5. An analog of a modular functor. The description using representations associated to pants decompositions has the advantage to make manifest that we are dealing with an analog of a modular functor. This means in particular that the representations of the mapping class group associated to Riemann surfaces of varying topological type $C_{g,n}$ restrict to, and are generated by, the representations associated to embedded subsurfaces of simple topological type $C_{0,3}$, $C_{0,4}$ and $C_{1,1}$. This property can be seen as a locality property that is essential for having relations with conformal field theory. However, we are not dealing with a modular functor in the strict sense axiomatised in the mathematical literature (see e.g. [6, 58]): The definition is restricted to stable surfaces ($2g-2+n > 0$), and the vector spaces associated to such surfaces are infinite-dimensional in general. However, the theory described above still exhibits the most essential features of a modular functor, it is in many respects as close to a modular functor as it can be in cases where the vector spaces associated to surfaces $C_{g,n}$ are infinite-dimensional.

It would interesting to develop a generalised notion of modular functor that encompasses the quantum Teichmüller theory and the many conceivable generalizations. Some suggestions in this direction were made in [54].

4. Conformal field theory

4.1. Definition of the conformal blocks. The Virasoro algebra Vir_c has generators L_n, $n \in \mathbb{Z}$, satisfying the relations

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}. \tag{4.1}$$

We will consider irreducible highest weight representations $\mathcal{V}_\alpha$ of Vir_c with $c > 1$ generated from vectors e_α annihilated by all L_n, $n > 0$, having L_0-eigenvalue $\alpha(Q-\alpha)$ if c is parameterised as $c = 1 + 6Q^2$.

Let $C \equiv C_{g,n}$ be a Riemann surface with genus g, n marked points $P_1, \ldots, P_n$, and choices of local coordinates t_r, $r = 1, \ldots, n$ vanishing at P_r, respectively. It will be convenient to assume that the local coordinates t_r are part of an atlas of local holomorphic coordinates on C with transition functions represented by Möbius-transformations. Such an atlas defines a projective structure on C.

We associate highest weight representations $\mathcal{V}_r \equiv \mathcal{V}_{\alpha_r}$, of Vir_c to P_r, $r = 1, \ldots, n$. The conformal blocks are linear functionals $\mathcal{F} : \otimes_{r=1}^n \mathcal{V}_r \to \mathbb{C}$ satisfying the invariance property

$$\mathcal{F}(\rho_\chi v) \, = 0 \, , \quad \forall v \in \otimes_{r=1}^n \mathcal{V}_r, \quad \forall \chi \in \mathfrak{V}_{\rm out}(C) \, , \tag{4.2}$$

where the notation $\mathfrak{V}_{\rm out}(C)$ is used for the Lie algebra of meromorphic differential operators on C which may have poles only at $P_1, \ldots, P_n$. The representation ρ of $\mathfrak{V}_{\rm out}(C)$ is defined on $\otimes_{r=1}^n \mathcal{V}_r$ via

$$\rho_\chi = \sum_{r=1}^{n} \sum_{k \in \mathbb{Z}} \chi_k^{(r)} L_k^{(r)} \, , \qquad L_k^{(r)} := \mathrm{id} \otimes \ldots \otimes \underset{\rm (r-th)}{L_k} \otimes \ldots \otimes \mathrm{id} \, , \tag{4.3}$$

where the $\chi_k^{(r)}$ are the Laurent coefficients of χ at P_r, $\chi(t_r) \, = \, \sum_{k \in \mathbb{Z}} \chi_k^{(r)} \, t_r^{k+1} \, \partial_{t_r} \, \in \mathbb{C}(\!(t_r)\!) \partial_{t_r}$. We may refer e.g. to [4] for more details.

The vector space $\mathcal{CB}(C, \rho)$ of conformal blocks associated to the Riemann surface C is the space of solutions to the defining invariance conditions (4.2). The space $\mathcal{CB}(C, \rho)$ is infinite-dimensional in general, being isomorphic to the space of *formal* power series in $3g - 3 + n$ variables.

4.1.1. Example. Let $n = 1$. Using the Weierstrass gap theorem it is straightforward to show that the defining condition (4.2) allows us to express the values $\mathcal{F}(v)$ for any $v \in \mathcal{V}_0$ in terms of the collection of complex numbers $\mathcal{F}\big(L_{-h}^{n_h} \ldots L_{-1}^{n_1} e_{\alpha_1}\big)$, $n_k \in \mathbb{Z}^{\geq 0}$, $k = 1, \ldots, h$, where $h := 3g - 3 + 1$.

To any conformal block $\mathcal{F}$, let us associate the *chiral partition function* defined as the value

$$\mathcal{Z}(\mathcal{F}) := \, \mathcal{F}(e) \, , \qquad e := \otimes_{r=1}^n e_{\alpha_r} . \tag{4.4}$$

The vacuum representation $\mathcal{V}_0$ corresponding to $\alpha = 0$ plays a distinguished role. It can be shown that the spaces of conformal blocks with and without insertions of the vacuum representation are canonically isomorphic, see e.g. [4] for a proof. Let the surface C' be obtained from C by introducing an additional marked marked point P_0. Let ρ and ρ' be the representations of $\mathfrak{V}_{\rm out}(C)$ and $\mathfrak{V}_{\rm out}(C')$, defined above on $\otimes_{r=1}^n \mathcal{V}_r$ and $\mathcal{V}_0 \otimes (\otimes_{r=1}^n \mathcal{V}_r)$, respectively. To each $\mathcal{F}' \in \mathcal{CB}(C', \rho')$ one may then associate a conformal block $\mathcal{F} \in \mathcal{CB}(C, \rho)$ such that

$$\mathcal{F}'(e_0 \otimes v) \equiv \mathcal{F}(v) \, . \tag{4.5}$$

for all $v \in \otimes_{r=1}^n \mathcal{V}_r$. This fact is often referred to as the "propagation of vacua".

4.2. Deformations of the complex structure of C. We shall now discuss the dependence of the spaces of conformal blocks on the choice of the Riemann surface C. The definition above defines sheaves of conformal blocks over $\mathcal{M}_{g,n}$, the moduli space of complex structures on surfaces of genus g and n punctures. Let us consider a local patch $\mathcal{U} \subset \mathcal{M}_{g,n}$ parameterised by local complex analytic coordinates $q = (q_1, \ldots, q_{3g-3+n})$, and represented by families C_q of Riemann surfaces with holomorphically varying projective structures.

A basic observation concerning the dependence of the space of conformal blocks on the complex structure is the existence of a canonical connection on the sheaves of conformal blocks over $\mathcal{M}_{g,n}$. Let us define the infinitesimal variations

$$\delta_\chi \mathcal{F}(v) := \mathcal{F}(\rho_\chi v)\,, \tag{4.6}$$

with ρ_χ being defined via (4.3) for arbitrary $\chi \in \oplus_{k=1}^n \mathbb{C}(\!(t_k)\!)\partial_{t_k}$. The "Virasoro uniformization theorem" (see e.g. [4] for a proof) implies that the Teichmüller space, being the tangent space $T\mathcal{M}_{g,n}$ to the space of complex structures $\mathcal{M}(C)$ at C is isomorphic to the double quotient

$$\mathcal{T}(C) \simeq \mathfrak{V}_{\rm out}(C) \setminus \oplus_{k=1}^n \mathbb{C}(\!(t_k)\!)\partial_{t_k} \,/\, \mathfrak{V}_{\rm in}(C)\,; \tag{4.7}$$

$\mathbb{C}(\!(t_k)\!)$ denotes the space of finite Laurent series, while $\mathfrak{V}_{\rm in}(C) := \oplus_{k=1}^n \mathbb{C}[[t_k]]\partial_k$, with $\mathbb{C}[[t_k]]$ being the space of finite Taylor series in the variable t_k. Assuming temporarily $\alpha_r = 0$, $r = 1, \ldots, n$, it follows from (4.7) that (4.6) relates the values $\mathcal{F}(\rho_\chi e)$ to derivatives of the chiral partition functions $\mathcal{Z}(\mathcal{F})$ with respect to the complex structure moduli of C. More general cases for the parameters α_r can be treated similarly. Using the propagation of vacua one may use (4.6), (4.7) to define a differential operator $\mathsf{T}(z_0)$ on $\mathcal{T}(C)$ such that

$$\mathsf{T}(z_0)\mathcal{F}(v) \;=\; \mathcal{F}'(L_{-2}e_0 \otimes v)\,. \tag{4.8}$$

The defining conditions (4.2), (4.6) imply that the conformal blocks $\mathcal{F}$ are fully characterised by the collection of all multiple derivatives of $\mathcal{Z}(\mathcal{F})$.

There are two obstacles to the integration of the canonical connection on $\mathcal{CB}(C,\rho)$, in general. The first problem is that the connection defined by (4.6) is not flat, but only projectively flat. One may, however, trivialize the curvature at least locally, opening the possibility to integrate (4.6) at least in open subsets $\mathcal{U} \subset \mathcal{M}_{g,n}$. We will later define sections horizontal with respect to the canonical connection using the gluing construction of conformal blocks.

The other problem is that $\mathcal{CB}(C,\rho)$ is simply way too big, multiple derivatives defined via (4.6) may grow without bound. However, there exist interesting subspaces of $\mathcal{CB}(C,\rho)$ on which the canonical connection may be integrated. Let $\mathcal{CB}^{\rm an}(C,\rho)$ be the subspace of $\mathcal{CB}(C,\rho)$ such that $\mathcal{Z}(\mathcal{F}_{C_q}) \equiv \mathcal{Z}(\mathcal{F}_q)$ can be analytically continued over all of $\mathcal{T}(C)$. Projective flatness of the canonical connection implies that we may in this way define a projective representation of the mapping class group on $\mathcal{CB}^{\rm an}(C,\rho)$. We will later briefly describe nontrivial evidence for the existence of a Hilbert-subspace $\mathcal{H}_{\rm CFT}(C,\rho)$ of $\mathcal{CB}^{\rm an}(C,\rho)$ which is invariant under this action. The projective representation of the mapping class group on $\mathcal{H}_{\rm CFT}(C,\rho)$ will then define an infinite-dimensional unitary projective local system $\mathcal{W}(C)$ over $\mathcal{M}(C)$. This seems to be the best possible scenario one can hope for when the spaces of conformal blocks are infinite-dimensional.

It is known [19] that the projectiveness of the local systems originating from the canonical connection on spaces of conformal blocks can be removed by tensoring with the projective line bundle $\mathcal{E}_c = (\lambda_H)^{\frac{c}{2}}$, where λ_H is the Hodge line bundle. It follows that $\mathcal{V}(C) := \mathcal{W}(C) \otimes \mathcal{E}_c$ is an ordinary holomorphic vector bundle over $\mathcal{M}(C)$. We are next going to describe how to construct *global* sections of $\mathcal{V}(C)$ by means of the gluing construction.

4.3. Gluing construction of conformal blocks. We are now going explain how to construct large families of conformal blocks by means of the gluing construction.

4.3.1. Gluing Riemann surfaces. Let C be a possibly disconnected Riemann surface, $q \in \mathbb{C}$ with $|q| < 1$, and $D_i(q) := \{P \in C; |z_i(P)| < |q|^{-\frac{1}{2}}\}$, $i = 1, 2$ be non-intersecting discs with local coordinate $z_i(P)$ vanishing at points $P_{0,i}$, for $i = 1, 2$, respectively. Let us then define a new Riemann surface $C^\sharp$ by identifying the annuli $A_i(q) := \{P \in C; |q|^{\frac{1}{2}} < |z_i(P)| < |q|^{-\frac{1}{2}}\}$ iff the coordinates $z_i(Q_i)$ of points $Q_i \in A_i$ satisfy $z_1(Q_1)z_2(Q_2) = q$. The gluing parameter q becomes part of the complex structure moduli of $C^\sharp$. By iterating this construction one may build Riemann surfaces $C_{g,n}$ of arbitrary genus g and arbitrary number n of punctures from three-punctured spheres $C_{0,3}$.

The surfaces $C_{g,n}$ obtained in this way come with a collection of embedded annuli $A_r(q_r)$, $r = 1, \ldots, h$, $h := 3g-3+n$. As the complex structure on $C_{0,3} \simeq \mathbb{P}^1 \setminus \{0, 1, \infty\}$ is unique, one may use $q = (q_1, \ldots, q_h)$ as local coordinates for $\mathcal{M}_{g,n}$ in a multi-disc centered around the boundary component in the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of $\mathcal{M}_{g,n}$ represented by the nodal surface obtained in the limit $(q_r) = 0$, $r = 1, \ldots, h$. It is possible to cover $\overline{\mathcal{M}}_{g,n}$ by local charts corresponding to the pants decompositions of C [31]. In order to get local coordinates for the Teichmüller spaces $\mathcal{T}(C)$ one may parameterise $q_r = e^{2\pi i \tau_r}$. Different local charts $\mathcal{U}_\sigma \subset \mathcal{T}(C)$ defined by the gluing construction can be labelled by the pairs $\sigma = (\gamma, \Gamma)$ introduced in Section 2.3.2.

Using the coordinates around the punctures of $C_{0,3}$ coming from the representation as $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ in the gluing construction one gets an atlas on C with transition functions represented by Möbius-transformations defining a projective structure on $C_{g,n}$. By varying the gluing parameters q_r one gets local holomorphic sections of the affine bundle $\mathcal{P}(C)$ of projective structures over $\mathcal{U}_\sigma$.

4.3.2. Gluing conformal blocks. Let us first consider Riemann surfaces $C_2 \sharp C_1$ obtained by gluing two surfaces C_i with $n_i + 1$, $i = 1, 2$ boundary components, respectively. Let $n = n_1 + n_2$, and let I_1, I_2 be sets such that $I_1 \cup I_2 = \{1, \ldots, n\}$. Let $\mathcal{F}_{C_i} \in \mathcal{CB}(C_i, \rho_i)$, $i = 1, 2$ be conformal blocks with ρ_i acting on $\mathcal{V}_i^{[n_i]} = \mathcal{V}_\beta \otimes (\otimes_{r \in I_i} \mathcal{V}_r)$ for $i = 1, 2$, respectively. Let $\langle .,. \rangle_{\mathcal{V}_\beta}$ be the Vir_c-invariant bilinear form on $\mathcal{V}_\beta$, and $\{v_e; e \in \mathbb{I}_\beta\}$, $\{\check{v}_e; e \in \mathbb{I}_\beta\}$ be dual bases for $\mathcal{V}_\beta$ satisfying $\langle v_e, \check{v}_{e'} \rangle_{\mathcal{V}_\beta} = \delta_{e,e'}$. For given $v_i \in \otimes_{r \in I_i} \mathcal{V}_r$ let $V_i(v_i)$ be the vectors in $\mathcal{V}_\beta$ defined by

$$V_i(v_i) := \sum_{e \in \mathbb{I}(\mathcal{V}_\beta)} \check{v}_e \, \mathcal{F}_{C_i}(v_e \otimes v_i), \tag{4.9}$$

A conformal block associated to the surface $C_2 \sharp C_1$ can then be constructed as

$$\mathcal{F}^\beta_{C_2 \sharp C_1}(v_2 \otimes v_1) := \left\langle V_2(v_2) \, , \, q^{L_0} V_1(v_1) \right\rangle_{\mathcal{V}_\beta} . \tag{4.10}$$

An operation representing the gluing of two boundary components of a single Riemann surface can be defined in a very similar way.

4.3.3. Gluing from pairs of pants. One can construct any Riemann surface C by gluing pairs of pants. Different ways of doing this are labelled by pants decompositions σ. The building blocks, the conformal blocks associated to $C_{0,3}$, are uniquely defined by the invariance property (4.2) up to the value of $\mathcal{F}_{C_{0,3}}$ on the product of highest weight vectors

$$N(\alpha_3, \alpha_2, \alpha_1) := \mathcal{F}_{C_{0,3}}(e_{\alpha_3} \otimes e_{\alpha_2} \otimes e_{\alpha_1}) . \tag{4.11}$$

Using the gluing construction recursively leads to the definition of a family of conformal blocks $\mathcal{F}^\sigma_{\beta,q}$ depending on the choice of pants decomposition $\sigma = (\gamma, \Gamma)$, the coordinate

q for $\mathcal{U}_\sigma \subset \mathcal{T}(C)$ defined by the gluing construction, and an assignment $\beta : e \mapsto \beta_e \in \mathbb{C}$ of complex numbers to the edges e of Γ. The parameters β_e determine the Virasoro representations $\mathcal{V}_{\beta_e}$ to be used in the gluing construction.

The partition functions $\mathcal{Z}_\sigma(\beta, q)$ defined from $\mathcal{F}^\sigma_{\beta,q}$ via (4.4) represent local sections of $\mathcal{V}(C)$ which are horizontal with respect to the canonical connection defined in Section 4.2.

4.3.4. Change of pants decomposition. It turns out that the partition functions $\mathcal{Z}_{\sigma_1}(\beta, q)$ constructed by the gluing construction in a neighborhood of the asymptotic region of $\mathcal{T}(C)$ that is determined by σ_1 have an analytic continuation to the asymptotic region of $\mathcal{T}(C)$ determined by a second pants decomposition σ_2. Based on [50, 51] it was proposed in [57] that the analytically continued partition functions $\mathcal{Z}_{\sigma_1}(\beta_1, q)$ are related to the functions $\mathcal{Z}_{\sigma_2}(\beta_2, q)$ by linear transformations of the form

$$\mathcal{Z}_{\sigma_1}(\beta_1, q) \;=\; E_{\sigma_1\sigma_2}(q) \int d\mu(\beta_2)\; W_{\sigma_1\sigma_2}(\beta_1, \beta_2)\, \mathcal{Z}_{\sigma_2}(\beta_2, q)\,. \tag{4.12}$$

The transformations (4.12) define the infinite-dimensional vector bundle $\mathcal{V}(C) = \mathcal{E}_c \otimes \mathcal{W}(C)$ over $\mathcal{M}_{g,n}$. The constant kernels $W_{\sigma_1\sigma_2}(\beta_1, \beta_2)$ represent the transition functions of the projective local system $\mathcal{W}(C)$, while the pre-factors $E_{\sigma_1\sigma_2}(q)$ can be identified as transition functions of the projective line bundle $\mathcal{E}_c$.

It is enough to establish (4.12) for the cases $C = C_{0,4}$ and $C_{1,1}$ since the Moore-Seiberg groupoid is generated from the F-, S-, B- and Z-moves. A partly conjectural[6] argument was proposed in [50, 51] suggesting that the F-move can be realised by an integral transformation of the form

$$\mathcal{Z}_{\sigma_s}(\beta_1, q) \;=\; \int_{\mathbb{S}} d\beta_2\; F_{\beta_1\beta_2}\left[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix}\right] \mathcal{Z}_{\sigma_t}(\beta_2, q)\,; \tag{4.13}$$

where $\mathbb{S} := \frac{Q}{2} + i\mathbb{R}^+$. The relevant pants decompositions σ_s and σ_t are depicted on the left and right half of Figure 2.2, respectively. We assume that $\beta_1 \in \mathbb{S}$, and that the parameters $\alpha_i \in \mathbb{S}$, $i = 1, 2, 3, 4$ label the representations assigned to the boundary components of $C_{0,4}$ according to the labelling indicated in Figure 2.2.

It was shown in [28] that (4.13) implies the following realisation of the S-move

$$\mathcal{Z}_{\sigma_s}(\beta_1, q) \;=\; e^{\pi i \frac{c}{12}(\tau + 1/\tau)} \int_{\mathbb{S}} d\beta_2\; S_{\beta_1\beta_2}(\alpha_0)\, \mathcal{Z}_{\sigma_t}(\beta_2, q)\,, \tag{4.14}$$

The pants decompositions σ_s and σ_t for $C = C_{1,1}$ are depicted in Figure 2.3.

5. Comparison with the quantization of the moduli spaces of flat connections

One may now compare the representation of the Moore-Seiberg groupoid obtained from the quantisation of $\mathcal{M}^{\mathcal{T}}_{\rm flat}(C)$ to the one from conformal field theory. It turns out that one finds exact agreement if (i) the representation parameters are identified as

$$\beta_e = \frac{Q}{2} + i\,\frac{l_e}{4\pi b}\,, \qquad \alpha_r = \frac{Q}{2} + i\,\frac{l_r}{4\pi b}\,, \qquad Q = b + b^{-1}\,, \tag{5.1}$$

[6] The main conjecture is the integrability of the representation of the algebra [50, equation (201)], equivalent to the validity of the representation [50, equation (202)].

where $r = 1, \ldots, n$, respectively, and if (ii) a suitable normalisation constant $N(\alpha_3, \alpha_2, \alpha_1)$ is chosen[7] in (4.11). This implies that there are natural Hilbert-subspaces $\mathcal{H}_{\mathrm{CFT}}(C, \rho)$ of the spaces of conformal blocks $\mathcal{CB}(C, \rho)$ on which the mapping class group action is unitary. These subspaces have (distributional) bases generated by the conformal blocks $\mathcal{F}^{\sigma}_{\beta,q}$ constructed by the gluing construction with $\beta_e \in \mathbb{S}$ for all edges e of σ. The Hilbert spaces $\mathcal{H}_{\mathrm{CFT}}(C, \rho)$ are isomorphic as representations of the Moore-Seiberg groupoid to the Hilbert spaces constructed in the quantisation of $\mathcal{M}^{\mathcal{T}}_{\mathrm{flat}}(C)$ in Section 3.

In the rest of this section we will compare the representations of *two* algebras of operators that arise naturally in the two cases, respectively: The first is the algebra $\mathcal{A}_{b^2}(C)$ generated by the quantised trace functions. This algebra will be realised naturally on spaces of conformal blocks in terms of the so-called Verlinde loop operators [2, 14]. The second algebra of operators is the algebra of holomorphic differential operators on the Teichmüller spaces $\mathcal{T}(C)$. This algebra is naturally realised on the conformal blocks via (4.6). We will briefly discuss, following [57], how a natural realisation is motivated from the point of view of the quantisation of $\mathcal{M}^{\mathcal{T}}_{\mathrm{flat}}(C)$.

5.1. Verlinde line operators. We shall now define a family of operators L_γ called Verlinde line operators labelled by closed curves γ on C acting on spaces of conformal blocks. It will turn out that the operators L_γ generate a representation of the algebra $\mathcal{A}_{b^2}(C)$ on the spaces of conformal blocks isomorphic to the one from the quantisation of $\mathcal{M}^{\mathcal{T}}_{\mathrm{flat}}(C)$. To define the operators L_γ we will need a few preparations of interest in their own right.

5.1.1. Analytic continuation. The kernels $W_{\sigma_1\sigma_2}(\beta_1, \beta_2)$ representing the transformations have remarkable analytic properties both with respect to the variables β_2, β_1, and with respect to the parameters α_r, $r = 1, \ldots, n$ of the representations assigned to the marked points [57]. An argument has furthermore been put forward in [51] indicating the absolute convergence of the series representing $\mathcal{Z}_\sigma(\beta, q)$. If the normalisation constant in (4.11) is chosen to be $N(\alpha_3, \alpha_2, \alpha_1) \equiv 1$, one may then show that the functions $\mathcal{Z}_\sigma(\beta, q)$ are entire in the variables α_r, and meromorphic in the variables β_e, having poles only if $\beta_e \in \mathbb{D}$, where $\mathbb{D} := -\frac{b}{2}\mathbb{Z}^{\geq 0} - \frac{1}{2b}\mathbb{Z}^{\geq 0}$.

This suggests that one may embed the space $\mathcal{H}_{\mathrm{CFT}}(C, \rho)$ into a larger space $\mathcal{D}_{\mathrm{CFT}}(C, \rho)$ which contains in particular the conformal blocks constructed using the gluing construction for generic *complex* $\beta_e \notin \mathbb{D} := -\frac{b}{2}\mathbb{Z}^{\geq 0} - \frac{1}{2b}\mathbb{Z}^{\geq 0}$. We will later characterise the spaces $\mathcal{D}_{\mathrm{CFT}}(C, \rho)$ more precisely. We may note, however, that the analytic properties of $W_{\sigma_1\sigma_2}(\beta_1, \beta_2)$ and $\mathcal{Z}_\sigma(\beta, q)$ ensure that the relations (4.12) can be analytically continued. The resulting relations characterise the realisation of the Moore-Seiberg groupoid on the spaces $\mathcal{D}_{\mathrm{CFT}}(C, \rho)$.

5.1.2. Degenerate punctures. The representations $\mathcal{V}_\alpha$ with $\alpha \in \mathbb{D}$ are called degenerate expressing the fact that the vectors in $\mathcal{V}_\alpha$ satisfy additional relations. Most basic are the cases where $\alpha = 0$, and $\alpha = -b^{\pm 1}/2$. In the first case one has $L_{-1}e_0 = 0$, in the second case $(L_{-1}^2 + b^{\pm 2}L_{-2})e_{-b^{\pm}/2} = 0$.

Let C' be obtained from C by introducing an additional marked point $z_0 \in C$. Analytically continuing conformal blocks with respect to the parameters α_r, $r = 0, \ldots, n$ allows one, in particular, to consider the cases where, for example, $\alpha_0 \in \mathbb{D}$. If $\alpha_0 = 0 \in \mathbb{D}$, it turns

[7] $N(\alpha_3, \alpha_2, \alpha_1) = (C(Q - \alpha_3, \alpha_2, \alpha_1))^{\frac{1}{2}}$, where $C(\alpha_3, \alpha_2, \alpha_1)$ is the function defined in [59].

out that $\mathcal{D}_{\rm CFT}(C',\rho') \simeq \mathcal{D}_{\rm CFT}(C,\rho)$, as required by the propagation of vacua. In the cases $\alpha_0 = -b^{\pm 1}/2$ it can be shown that the partition functions $\mathcal{Z}(\mathcal{F}'_q)$ for $\mathcal{F}'_q \in \mathcal{D}_{\rm CFT}(C',\rho')$ satisfy equations of the form

$$\big[\,\partial_{z_0}^2 + b^{\pm 2}\mathsf{T}(z_0)\big]\,\mathcal{Z}(\mathcal{F}'_q) \;=\; 0\,, \tag{5.2}$$

where $\mathsf{T}(z_0)$ is a certain first order differential operator that transforms under changes of local coordinates on C as a quadratic differential[8] . We will refer to these equations as the Belavin-Polyakov-Zamolodchikov (BPZ-) equations. It follows in particular that $\mathcal{D}_{\rm CFT}(C',\rho')$ $\simeq \mathbb{C}^2 \otimes \mathcal{D}_{\rm CFT}(C,\rho)$, with the two linearly independent solutions of (5.2) corresponding to the two elements of a basis for $\mathbb{C}^2$.

5.1.3. Definition of the Verlinde line operators. Consideration of multiple degenerate punctures reveals some interesting phenomena. If, for example, C'' is obtained from C by introducing *two* additional punctures at z_0 and z_{-1} with $\alpha_0 = \alpha_{-1} = -b/2$ one finds a subspace of $\mathcal{D}_{\rm CFT}(C'',\rho'') \simeq \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathcal{D}_{\rm CFT}(C,\rho)$ naturally isomorphic to $\mathcal{D}_{\rm CFT}(C,\rho)$. This is similar (in fact related) to the fact that the tensor product of two two-dimensional representations of $\mathfrak{sl}_2$ contains a one-dimensional representation. This phenomenon allows us to define natural embeddings and projections

$$\begin{aligned} \imath &: \mathcal{D}_{\rm CFT}(C,\rho) \;\hookrightarrow\; \mathcal{D}_{\rm CFT}(C'',\rho'')\,, \\ \wp &: \mathcal{D}_{\rm CFT}(C'',\rho'') \;\to\; \mathcal{D}_{\rm CFT}(C,\rho)\,. \end{aligned} \tag{5.3}$$

Note furthermore that the mapping class group $\mathrm{MCG}(C'')$ contains elements μ_γ labelled by closed curves γ on C, corresponding to the variation of the position of z_0 along γ. This allows us to define a natural family of operators on the spaces $\mathcal{D}_{\rm CFT}(C,\rho)$ as

$$\mathsf{L}_\gamma := \wp \circ \mathsf{M}(\mu_\gamma) \circ \imath\,, \tag{5.4}$$

where $\mathsf{M}(\mu)$ is the operator representing μ on $\mathcal{D}_{\rm CFT}(C'',\rho'')$. The operators L_γ are called Verlinde line operators.

Comparing the explicit formulae for the Verlinde line operators calculated in [2, 14] with the formulae for the operators $\pi_\sigma(L_\gamma)$ found in [57] (see Section 3.3 above) one finds a precise match. This means that there is a natural action of the algebra $\mathcal{A}_{b^2}(C)$ of quantised trace functions on spaces of conformal blocks. This action naturally defines dense subspaces $\mathcal{S}_{\rm CFT}(C,\rho) \subset \mathcal{H}_{\rm CFT}(C,\rho)$ as maximal domains of definition for $\mathcal{A}_{b^2}(C)$ such that $\mathcal{D}_{\rm CFT}(C,\rho)$ is the dual space of distributions forming a so-called Gelfand-triple $\mathcal{S}_{\rm CFT}(C,\rho) \subset \mathcal{H}_{\rm CFT}(C,\rho) \subset \mathcal{D}_{\rm CFT}(C,\rho)$. The spaces $\mathcal{S}_{\rm CFT}(C,\rho)$ and $\mathcal{D}_{\rm CFT}(C,\rho)$ are isomorphic as $\mathcal{A}_{b^2}(C)$-modules to the spaces $\mathcal{S}(C)$ and $\mathcal{D}(C)$ introduced in Section 3.4, respectively.

5.2. Kähler quantization of $\mathcal{T}(C)$. The relation between conformal field theory and the quantisation of $\mathcal{T}(C)$ can be tightened considerably by considering an alternative quantisation scheme for $\mathcal{T}(C)$ [52, 57] that we shall now discuss. Teichmüller theory allows one to equip $\mathcal{T}(C)$ with natural complex and symplectic structures. The natural symplectic form $\Omega_{\rm WP}$ on $\mathcal{T}(C)$ coincides with the restriction of the symplectic form $\Omega_{\rm AB}$ on $\mathcal{M}_{\rm flat}(C)$ to the Teichmüller component $\mathcal{M}^{\mathcal{T}}_{\rm flat}(C)$. Natural functions on $\mathcal{T}(C)$ are given by the values

[8] Remember that we had fixed a family of reference projective structures in the very beginning.

of the quadratic differential $t(y) \equiv t(y|q,\bar{q})$ defined from the metric of constant negative curvature $e^{2\varphi}dyd\bar{y}$ on C as $t(y) = -(\partial\varphi)^2 + \partial^2\varphi$. One may find a basis $\{\vartheta_r; r = 1,\dots,h\}$ for the space $H^0(C,K^2)$ of holomorphic quadratic differentials on C such that the functions $H_r \equiv H_r(q,\bar{q})$ on $\mathcal{T}(C)$ defined via $t(y) = \sum_{r=1}^h \vartheta_r(y)H_r$, are canonically conjugate to the complex analytic coordinates q_r on $\mathcal{T}(C)$ in the sense that $\{H_r, q_s\} = \delta_{r,s}$ [47, 48].

In the corresponding quantum theory it is natural to realize the operators H_r corresponding to H_r as differential operators $b^2\partial_{q_r}$, and to represent states by holomorphic wave-"functions" $\Psi^\sigma(q)$ [57][9] . The operator corresponding to the quadratic differential $t(y)$ will be a differential operator $\mathsf{T}(y)$. This operator coincides with the operator defined in (4.8). Recall that the space of conformal blocks $\mathcal{CB}^{\rm an}(C,\rho)$ can be identified with the space of holomorphic functions on $\mathcal{T}(C)$. These observations suggest us to identify the space of states in the quantum theory of $\mathcal{T}(C)$ with suitable subspaces of $\mathcal{CB}^{\rm an}(C,\rho)$.

It is natural to require that the mapping class group is represented on the wave-"functions" $\Psi^\sigma(q)$ as deck-transformations $(\mathsf{M}(\mu)\Psi^\sigma)(q) = \Psi^\sigma(\mu.q)$, where $\mu.q$ is the image of the point q in $\mathcal{T}(C)$ under μ. One may then show that [52, 57]

$$\mathcal{Z}_\sigma(\beta,q) \;=\; \Psi_l^\sigma(q)\,, \tag{5.5}$$

where $\sigma = (\gamma,\Gamma)$, $\gamma = (\gamma_1,\dots,\gamma_h)$, $\Psi_l^\sigma(q)$ is the wave-function of an eigenstate of the operators L_{γ_e}, $e = 1,\dots,h$, and the variables are related via (5.1), respectively.

The observations made in this section indicate that conformal field theory is nothing but another language for describing the quantum theories obtained by quantisation of $\mathcal{M}_{\rm flat}^{\mathcal{T}}(C)$.

6. Further connections

The theory outlined above generalises and unifies various themes of mathematical research. As an outlook we shall now briefly mention some of these connections, some of which offer interesting perspectives for future research.

6.1. Relation with non-compact quantum groups. There is an interesting non-compact quantum group called modular double of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ [15] which is on the algebraic level isomorphic to $\mathcal{U}_q(\mathfrak{sl}_2)$, and has a set of unitary irreducible representations $\mathcal{P}_s$, $s \in \mathbb{R}^+$ characterised by a remarkable self-duality property: They are simultaneously representations of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ and $\mathcal{U}_{\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ with $\tilde{q} = e^{\pi i/b^2}$ if $q = e^{\pi i b^2}$ [15, 45]. This family of representations is closed under tensor products [41, 46], and there exists a non-compact quantum group $\mathrm{SL}_q^+(2,\mathbb{R})$ deforming a certain subspace of the space of functions on $SL(2,\mathbb{R})$ which has a Plancherel-decomposition into the representations $\mathcal{P}_s$, $s \in \mathbb{R}^+$, [33, 45].

There exists strong evidence[10] for an equivalence of braided tensor categories of Kazhdan-Lusztig type [38] between the category of unitary representations of the Virasoro algebra having simple objects $\mathcal{V}_\alpha$, $\alpha \in \mathbb{S}$, with the category having the representations $\mathcal{P}_s$, $s \in \mathbb{R}^+$ of the modular double as simple objects. The kernel representing the F-move coincides with the 6j-symbols of the modular double of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ [50]. The complex numbers numbers representing the B-move coincide with the eigenvalues of the R-matrix of the modular

[9] More precisely holomorphic sections of the projective line bundle $\mathcal{E}_c$. $\Psi^\sigma(q)$ depends on the choice of a pants decomposition as the definition of the coordinates q depends on it.

[10] The main open problem is the issue pointed out in Section 4.3.4.

double [7].

The results of [57] furthermore imply that the braided tensor category of unitary representations of the modular double has a natural extension to a *modular* tensor category.

6.2. Relations to three-dimensional hyperbolic geometry. The Teichmüller theory has numerous relations to hyperbolic geometry in three dimensions. Let us consider, for example, the Fenchel-Nielsen coordinates (l_s, κ_s) and (l_t, κ_t) associated to the pants decompositions on the left and on the right of Figure 3, respectively. It was observed in [40] that the generating function $\mathcal{W}(l_s, l_t)$ for the change of Darboux coordinates $(l_s, \kappa_s) \leftrightarrow (l_t, \kappa_t)$ for $\mathcal{T}(C)$, defined by the relations

$$\kappa_s = \frac{\partial \mathcal{W}}{\partial l_s}\,, \qquad \kappa_t = -\frac{\partial \mathcal{W}}{\partial l_t}\,, \tag{6.1}$$

coincides with the volume $\mathrm{Vol}_T(l)$ of the hyperbolic tetrahedron with edge lengths $l = (l_1, l_2, l_3, l_4, l_s, l_t)$, with l_i, $i = 1, 2, 3, 4$ being the hyperbolic lengths of the boundaries of $C_{0,4}$.

It is therefore not unexpected to find relations to hyperbolic geometry encoded within quantum Teichmüller theory. Considering the limit $b \to 0$ of the kernel

$$F(l_s, l_t) := F_{l_s l_t}\left[\begin{smallmatrix} l_3 & l_2 \\ l_4 & l_1 \end{smallmatrix}\right]$$

appearing in (3.16) one may show that $\lim_{b\to 0} b^2 \log F(l_s, l_t)$ is equal to the volume $\mathrm{Vol}_T(l)$ of the hyperbolic tetrahedron considered above. This follows from the fact that (3.14) reduces to (6.1) in the limit $b \to 0$. A closely related result was found in [56] by direct calculation.

Braided tensor categories of representations of compact quantum groups can be used to construct invariants of three-manifolds [6, 58]. It should be interesting to investigate similar constructions using the modular tensor category associated to the modular double. It seems quite possible the resulting invariants are related to the invariants constructed in [3, 12, 29]. If so, one would get an interesting perspective on the variants of the volume conjecture formulated in [3, 12, 29]: It could be a consequence of the relations between quantum Teichmüller theory and hyperbolic geometry pointed out above, which are natural consequences of known relations between Teichmüller theory and three-dimensional hyperbolic geometry.

6.3. Relations with integrable models. There are several connections between the mathematics reviewed in this article and the theory of integrable models. We will here describe some connections to the theory of isomonodromic deformations of certain ordinary differential equations, for $g = 0$ closely related to the equations studied by Painlevé, Schlesinger and Garnier. Further connections are described in [8, 55].

6.3.1. Relations with isomonodromic deformations I. The limit $b \to 0$ of the BPZ-equations (5.2) is related to isomonodromic deformations [55].

Let us consider the case of a Riemann surface $\hat{C} \equiv C_{g,n+d+1}$ with $n + d + 1$ marked points $z_1, \ldots, z_n, u_1, \ldots, u_d$ and y. For convenience let us assume that $u_1, \ldots, u_d$ and y lie in a single chart of the surface C obtained from $\hat{C}$ by filling $u_1, \ldots, u_d$ and y. The resulting loss of generality will not be very essential. We associate representations with generic value of the parameter α_r to z_r for $r = 1, \ldots, n$, degenerate representations with parameter $-1/2b$ to the points $u_1, \ldots, u_d$, and a degenerate representation with parameter $-b/2$ to the point

y. The partition functions $\mathcal{Z}(q) \equiv \mathcal{Z}(\hat{\mathcal{F}}_q)$, $\hat{\mathcal{F}}_q \equiv \mathcal{F}_{\hat{C}_q}$ will then satisfy a system of $d+1$ partial differential equations of the form,

$$\big[\, b^{+2}\partial_{u_k}^2 + \mathsf{T}_k(u_k)\big]\mathcal{Z}(q) \;=\; 0\,, \quad k = 1, \dots, d\,, \tag{6.2a}$$

$$\big[\, b^{-2}\partial_y^2 + \mathsf{T}_0(y)\big]\mathcal{Z}(q) \;=\; 0\,. \tag{6.2b}$$

In the limit $b \to 0$ one may solve this system of partial differential equation with an ansatz of the form $\mathcal{Z}(q) = \exp(\frac{1}{b^2}\mathcal{W}(q'))\psi(y)(1+\mathcal{O}(b^2))$, where $\mathcal{W}(q')$ does not depend on y. Equation (6.2b) implies that $\psi(y)$ satisfies $(\partial_y^2 + t(y))\psi(y) = 0$, where $t(y) = \lim\limits_{b\to 0} b^2\mathcal{Z}^{-1}\mathsf{T}_0(y)\mathcal{Z}$. Equations (6.2) imply that $v_k := \partial_{u_k}\mathcal{W}$ satisfy

$$v_k^2 + t_{k,2} = t_{k,1}^2 + t_{k,2} = 0\,, \qquad k = 1, \dots, d\,, \tag{6.3}$$

with $t_{k,l}$ defined from $t(y) = \sum_{l=0}^{\infty} t_{k,l}(y-u_k)^{l-2}$. It follows that $\partial_y^2 + t(y)$ has d apparent singularities at $y = u_k$. Let $\vartheta_k(y)(dy)^2$ be a basis for $H^0(C, K^2)$, and let us define H_k via $t(y) = \sum_{k=1}^{d} H_k\vartheta_k(y)$. In the case $d = 3g-3+n$ one has enough equations (6.3) to determine the $H_k \equiv H_k(u,v)$ as functions of $u = (u_1, \dots, u_d)$ and $v = (v_1, \dots, v_d)$.

It is automatic that the monodromy of $\partial_y^2 + t(y)$ will be unchanged under variations of the complex structure of C, which is equivalent to [34, 43]

$$\frac{\partial u_k}{\partial q_r} \;=\; \frac{\partial H_r}{\partial v_k}\,, \qquad \frac{\partial v_k}{\partial q_r} \;=\; -\frac{\partial H_r}{\partial u_k}\,, \tag{6.4}$$

$\{\frac{\partial}{\partial q_k}; k = 1, \dots, d\}$ being the basis for $T\mathcal{T}(C)$ dual to the basis $\{\vartheta_k, k = 1, \dots, d\}$ for $H^0(C, K^2) \simeq T^*\mathcal{T}(C)$. It follows that the system of BPZ-equations (6.2) describes a quantisation of the isomonodromic deformation problem [55].

6.3.2. Relations with isomonodromic deformations II. A somewhat unexpected relation between conformal blocks and the isomonodromic deformation problem arises in the limit $c \to 1$. A precise relation between the tau-function for Painlevé VI and Virasoro conformal blocks was proposed in [23]. A proof of this relation, together with its generalization to the tau-functions of the Schlesinger system was given in [32]. The relations established in [32] are

$$\tau(\lambda, \kappa; q) \;=\; \sum_{m \in \mathbb{Z}^N} e^{i\kappa\cdot m}\mathcal{Z}_\sigma(\lambda + m, q)\,, \tag{6.5}$$

where $N = n-3$, $\mathcal{Z}_\sigma(\beta, q)$ are the chiral partition functions associated to the conformal blocks defined using the gluing construction in the case $C = C_{0,n}$, and $\tau(\lambda, \kappa; q)$ is the isomonodromic tau-function, defined by $H_r = -\partial_{q_r}\tau(\lambda, \kappa; q)$, here considered as a function of the monodromy data parameterised in terms of Darboux coordinates (λ, κ) for $\mathcal{M}^{\mathbb{C}}_{\rm char}(C)$ closely related to the coordinates used in [40].

In order to prove (6.5), the authors of [32] consider partition functions $\mathcal{Z}(\mathcal{F}''_q)$ of conformal blocks $\mathcal{F}''_q \in \mathcal{D}_{\rm CFT}(C'', \rho'')$ with two additional degenerate punctures as in Section 5.1. Recall that one gets an action of $\pi_1(C)$ on $\mathcal{D}_{\rm CFT}(C'', \rho'')$ from monodromies of one of the degenerate punctures. The isomorphism $\mathcal{D}_{\rm CFT}(C'', \rho'') \simeq \mathbb{C}^4 \otimes \mathcal{D}_{\rm CFT}(C, \rho)$ allows us to represent the action of $\pi_1(C)$ on $\mathcal{D}_{\rm CFT}(C'', \rho'')$ in terms of matrices having elements which are difference operators acting on $\mathcal{D}_{\rm CFT}(C, \rho)$. The remarkable fact observed in [32] is that the appearing difference operators can be diagonalised simultaneously by a generalised Fourier-transformation similar to (6.5) (provided that $g = 0$ and $c = 1$). This observation yields in

particular a new and more effectively computable way to solve the classical Riemann-Hilbert problem [32].

7. Outlook: Harmonic analysis on Diff(S^1)?

Let Π_j be a unitary irreducible representation $\Pi_j : G \to \mathrm{End}(\mathcal{V}_j)$ of a finite-dimensional Lie group on a Hilbert space $\mathcal{V}_j$ with scalar product $(.,.) : \mathcal{V}_j \otimes \mathcal{V}_j \to \mathbb{C}$, j being a label for elements in the set of irreducible unitary representations of G. Matrix elements such as $(v_2, \Pi_j(g)v_1)$, $v_i \in \mathcal{V}_j$ for $i = 1, 2$, play a fundamental role in the harmonic analysis of the Lie group G. They allow us to realise the abstract Plancherel decomposition $L^2(G) \simeq \int_{\mathbb{U}}^{\oplus} d\mu(j)\, \mathcal{V}_j \otimes \mathcal{V}_j^{\dagger}$ as a generalised Fourier-transformation

$$f(g) = \int_{\mathbb{U}} d\mu(j) \sum_{\imath,\imath' \in \mathbb{I}_j} (v_\imath, \Pi_j(g) v_{\imath'})\, \tilde{f}_{\imath\imath'}(j)\,, \tag{7.1}$$

with $\{v_\imath; \imath \in \mathbb{I}_j\}$ being an orthonormal basis for $\mathcal{V}_j$. If the representations $\mathcal{V}_j$ contain unique vectors v_2^j, v_1^j invariant under subgroups H_2 and H_1, respectively, one may similarly represent functions on the double quotients $H_2\backslash G/H_1$, as

$$f(g) = \int_{\mathbb{U}} d\mu(j)\, (v_2^j\,,\, \Pi_j(g) v_1^j)\, \tilde{f}(j)\,. \tag{7.2}$$

The functions $\mathcal{Y}(j,g) := (v_2^j\,,\, \Pi_j(g)v_1^j)$ are called spherical or Whittaker functions depending on the type of subgroups H_2 and H_1 under consideration. Equation (7.2) expresses the completeness of the functions $\mathcal{Y}(j,g)$ within $L^2(H_2\backslash G/H_1)$.

Turning back to conformal field theory let us consider the conformal blocks constructed by the gluing construction as described in Section 4.3.2. The partition function $\mathcal{Z}(\beta, q)$ can be represented as a matrix element in the form $\mathcal{Z}(\beta, q) = \langle V_2, q^{L_0} V_1\rangle$. We could consider, more generally

$$\mathcal{Z}(\beta, g) \;=\; \langle V_2\,,\, \Pi_\beta(g)\, V_1\rangle\,, \tag{7.3}$$

where $g \in \mathrm{Diff}^+(S^1)$, and Π_β is the projective unitary representation of $\mathrm{Diff}^+(S^1)$ related to the representation $\mathcal{V}_\beta$ of the Virasoro algebra by exponentiation, $\Pi_\beta(e^f) = e^{i\pi_\beta(T[f])}$, for $f(\sigma)\partial_\sigma = \sum_{n\in\mathbb{Z}} f_n e^{in\sigma}\partial_\sigma$ being a real smooth vector field on S^1, $T[f] = \sum_{n\in\mathbb{Z}} f_n L_n$. Equation (7.3) will define a function on $\mathrm{Diff}^+(S^1)$ that has an analytic continuation to the natural complexfication of $\mathrm{Diff}^+(S^1)$, the semi-group of annuli $\mathfrak{An}$ defined in [49]. One should note, however, that the states V_2 and V_1 will be annihilated by large sub-semigroups $\mathfrak{An}_2$ and $\mathfrak{An}_1$ of $\mathfrak{An}$, obtained by exponentiation of the Lie-subalgebras of the Virasoro algebra generated by vector fields on S^1 that extend holomorphically to $(C_i \setminus D_i) \cup A_i$, for $i = 1, 2$, respectively. This means that $\mathcal{Z}(\beta, g)$ will be a function on the double coset $\mathfrak{An}_2\backslash\mathfrak{An}/\mathfrak{An}_1$ which can be identified with an open subset of the Teichmüller space $\mathcal{T}(C)$.

This suggests to view the functions $\mathcal{Z}(\beta, g)$ as analogs of spherical or Whittaker functions. By taking certain collision limits where the punctures of $C_{0,4}$ collide in pairs one may even construct honest Whittaker vectors of the Virasoro algebra from the states V_i, $i = 1, 2$ [27], making the analogy even more close. From this point of view it is intriguing to compare formula (4.13) with (7.2). It is tempting to view formula (4.13) as an expression of the possible completeness of the functions within a - yet to be defined - space of "square-integrable"

functions on $\mathcal{T}(C)$, which in turn is related to a certain coset of the semigroup $\mathfrak{An}$ according to the discussion above.

These remarks suggest that the relations between conformal field theory and the quantisation of the moduli spaces of flat $\mathrm{PSL}(2,\mathbb{R})$-connections observed in Section 5 should ultimately be understood as results of "quantisation commutes with reduction"-type. Quantisation of (a space containing) T^*G, $G = \mathrm{Diff}^+(S^1)$ should produce an infinite-dimensional picture close to conformal field theory. The reduction to the finite-dimensional quantum theory of the Teichmüller spaces is a consequence of the invariances of the vectors V_i, $i = 1, 2$.

References

[1] L.F. Alday, D. Gaiotto, and Y. Tachikawa, *Liouville Correlation Functions from Four-dimensional Gauge Theories*, Lett. Math. Phys. **91** (2010), 167–197.

[2] L.F. Alday, D. Gaiotto, S. Gukov, Y. Tachikawa, and H. Verlinde, *Loop and surface operators in $\mathcal{N} = 2$ gauge theory and Liouville modular geometry*, J. High Energy Phys. **1001** (2010), 113.

[3] J.E. Andersen and R. Kashaev, *A TQFT from quantum Teichmüller theory*, Preprint arXiv:1109.6295.

[4] E. Frenkel and D. Ben-Zvi, *Vertex algebras and algebraic curves.* Second edition. Mathematical Surveys and Monographs, 88. American Mathematical Society, Providence, RI, 2004.

[5] B. Bakalov and A. Kirillov, Jr., *On the Lego-Teichmüller game.* Transform. Groups **5** (2000), no. 3, 207–244.

[6] ———, *Lectures on tensor categories and modular functors.* University Lecture Series, 21. American Mathematical Society, Providence, RI, 2001.

[7] A.G. Bytsko and J. Teschner, *R-operator, co-product and Haar-measure for the modular double of $U_q(\mathfrak{sl}(2,\mathbb{R}))$*, Comm. Math. Phys. **240** (2003), 171–196.

[8] ———, *The integrable structure of nonrational conformal field theory*, Adv. Theor. Math. Phys. **17** (2013), no. 4, 701–740.

[9] L.O. Chekhov and V. Fock, *A quantum Teichmüller space*, Theor. Math. Phys. **120** (1999), 1245–1259 (Preprint arXiv:math/9908165)

[10] ———, *Quantum modular transformations, the pentagon relation, and geodesics*, Proc. Steklov Inst. Math. **226** (1999), 149–163

[11] M. Dehn, *On curve systems on two-sided surfaces, with application to the mapping problem,* in: Papers on group theory and topology, pp. 234–252. Springer-Verlag, New York, 1987. Translated from the German by John Stillwell.

[12] T. Dimofte, S. Gukov, J. Lenells, and Don Zagier, *Exact Results for Perturbative Chern-Simons Theory with Complex Gauge Group*, Commun. Num. Theor. Phys. **3** (2009), 363–443.

[13] N. Drukker, D.R. Morrison, and T. Okuda, *Loop operators and S-duality from curves on Riemann surfaces*, JHEP 0909:031, 2009

[14] N. Drukker, J. Gomis, T. Okuda, and J. Teschner, *Gauge Theory Loop Operators and Liouville Theory*, J. High Energy Phys. **1002** (2010) 057.

[15] L. D. Faddeev, *Modular double of a quantum group*, Conference Mosh Flato 1999, Vol. I (Dijon), 149–156, Math. Phys. Stud., 21, Kluwer Acad. Publ., Dordrecht, 2000, arXiv:math/9912078.

[16] V. Fock, *Dual Teichmüller spaces* Preprint arXiv:dg-ga/9702018

[17] V.V. Fock and A. Goncharov, *Moduli spaces of local systems and higher Teichmüller theory.* Publ. Math. Inst. Hautes Études Sci. **103** (2006), 1–211.

[18] ———, *The quantum dilogarithm and representations of quantum cluster varieties.* Invent. Math. **175** (2009), 223–286.

[19] D. Friedan and S. Shenker, *The Analytic Geometry of Two-Dimensional Conformal Field Theory*, Nucl. Phys. **B281** (1987), 509–545

[20] L. Funar and R. Gelca, *On the groupoid of transformations of rigid structures on surfaces.* J. Math. Sci. Univ. Tokyo **6** (1999), 599–646.

[21] D. Gaiotto, G. Moore, and A. Neitzke, *Wall-crossing, Hitchin systems, and the WKB approximation.* Adv. Math. **234** (2013), 239–403.

[22] ———, *Framed BPS States.* ATMP **17** (2013), 241–397.

[23] O. Gamayun, N. Iorgov, and O. Lisovyy, *Conformal field theory of Painlevé VI* JHEP 10 (2012), 038

[24] W. Goldman, *Invariant functions on Lie groups and Hamiltonian flows of surface group representations.* Invent. Math. **85** (1986), 263–302.

[25] W. Goldman, *Topological components of spaces of representations.* Invent. Math. **93** (1988), 557–607.

[26] W. Goldman, *Trace Coordinates on Fricke spaces of some simple hyperbolic surfaces*, Handbook of Teichmüller theory. Vol. II, 611–684, IRMA Lect. Math. Theor. Phys. 13, Eur. Math. Soc., Zürich, 2009

[27] D. Gaiotto and J. Teschner, *Irregular singularities in Liouville theory and Argyres-Douglas type gauge theories*, JHEP 1212 (2012) 050

[28] L. Hadasz, Z. Jaskolski, and P. Suchanek, *Modular bootstrap in Liouville field theory*, Phys. Lett. **B685** (2010), 79–85.

[29] K. Hikami, *Generalized Volume Conjecture and the A-Polynomials—the Neumann-Zagier Potential Function as a Classical Limit of Quantum Invariant*, J. Geom. Phys. **57** (2007), 1895–1940.

[30] N. Hitchin, *The self-duality equations on a Riemann surface*, Proc. London Math. Soc. **55** (3) (1987), 59–126.

[31] V. Hinich and A. Vaintrob, *Augmented Teichmüller spaces and orbifolds.* Selecta Math. (N.S.) **16** (2010), 533–629.

[32] N. Iorgov, O. Lisovyy, and J. Teschner, *Isomonodromic tau-functions from Liouville conformal blocks*, Preprint arXiv:1401.6104.

[33] I.C.H. Ip, *Representation of the Quantum Plane, its Quantum Double, and Harmonic Analysis on $GL_q^+(2,R)$*, Selecta Mathematica **19** (2013), 987–1082.

[34] K. Iwasaki, *Moduli and deformations of Fuchsian differential equations on Riemann surfaces,* J. Fac. Sci. Univ. Tokyo (Sect. IA, Math) **38** (1991), 431–531.

[35] R.M. Kashaev, *Quantization of Teichmüller spaces and the quantum dilogarithm,* Lett. Math. Phys. **43** (1998) , 105–115, q-alg/9705021.

[36] ———, *Liouville central charge in quantum Teichmuller theory*, Proc. of the Steklov Inst. of Math. **226** (1999), 63–71, arXiv:hep-th/9811203.

[37] ———, *The quantum dilogarithm and Dehn twists in quantum Teichmüller theory.* Integrable structures of exactly solvable two-dimensional models of quantum field theory (Kiev, 2000), 211–221, NATO Sci. Ser. II Math. Phys. Chem., 35, Kluwer Acad. Publ., Dordrecht, 2001.

[38] D. Kazhdan and G. Lusztig, *Tensor structures arising from affine Lie algebras*, Parts I, II: J. Amer. Math. Soc. **6** (1993), no. 4, 905–947, 949–1011, Parts III, IV: J. Amer. Math. Soc. **7** (1994), no. 2, 335–381, 383–453.

[39] G. Moore and N. Seiberg, *Classical and quantum conformal field theory*, Comm. Math. Phys. **123** (1989), 177–254

[40] N. Nekrasov, A. Rosly, and S. Shatashvili, *Darboux coordinates, Yang-Yang functional, and gauge theory*, Nucl. Phys. Proc. Suppl. **216** (2011), 69–93.

[41] I. Nidaeiev and J. Teschner, *On the relation between the modular double of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ and the quantum Teichmüller theory*, Preprint arXiv:1302.3454

[42] N. Nekrasov and E. Witten, *The Omega Deformation, Branes, Integrability, and Liouville Theory*, JHEP 2010, no. 9, 092

[43] K. Okamoto, *Isomonodromic and Painlevé equations, and the Garnier system*, J. Fac. Sci. Univ. Tokyo, Sect. IA Math **33** (1986), 575–618

[44] R.C. Penner, *The decorated Teichmüller space of punctured surfaces,* Comm. Math. Phys. **113** (1987), 299–339.

[45] B. Ponsot and J. Teschner, *Liouville bootstrap via harmonic analysis on a noncompact quantum group*, arXiv:hep-th/9911110.

[46] ———, *Clebsch-Gordan and Racah-Wigner coefficients for a continuous series of representations of $U_q(sl(2,\mathbb{R}))$*, Commun. Math. Phys. **224** (2001), 613–655.

[47] P.G. Zograf and L.A. Takhtajan, *On the uniformization of Riemann surfaces and on the Weil–Petersson metric on the Teichmüller and Schottky spaces*, Math. USSR-Sb. **60** (1988), 297–313.

[48] L. Takhtajan and L.P. Teo, *Liouville action and Weil-Petersson metric on deformation spaces, global Kleinian reciprocity and holography.* Comm. Math. Phys. **239** (2003), 183– 240

[49] G. Segal, *The definition of conformal field theory.* Topology, geometry and quantum field theory, 421–577, London Math. Soc. Lecture Note Ser., 308, Cambridge Univ. Press, Cambridge, 2004.

[50] J. Teschner, *Liouville theory revisited*, Class. Quant. Grav. **18** (2001), R153–R222.

[51] ———, *A lecture on the Liouville vertex operators*, Int. J. Mod. Phys. **A19S2** (2004), 436–458

[52] ———, *From Liouville theory to the quantum geometry of Riemann surfaces.* Prospects in mathematical physics, 231–246, Contemp. Math. **437**, Amer. Math. Soc., Providence, RI, 2007

[53] ———, *An analog of a modular functor from quantized Teichmüller theory*, "Handbook of Teichmüller theory", (A. Papadopoulos, ed.) Volume I, EMS Publishing House, Zürich 2007, 685–760.

[54] ———, *Nonrational conformal field theory*, "New Trends in Mathematical Physics" (Selected contributions of the XVth ICMP), Vladas Sidoravicius (ed.), Springer Science and Business Media B.V. 2009.

[55] ———, *Quantization of the Hitchin moduli spaces, Liouville theory, and the geometric Langlands correspondence I.* Adv. Theor. Math. Phys. **15** (2011), 471–564.

[56] J. Teschner and G. S. Vartanov, *6j symbols for the modular double, quantum hyperbolic geometry, and supersymmetric gauge theories*. Preprint arXiv:1202.4698

[57] ———, *Supersymmetric gauge theories, quantization of moduli spaces of flat connections, and conformal field theory*, Lett. Math. Phys. **104** (2014), 527-551.

[58] V.G. Turaev, *Quantum invariants of knots and 3-manifolds.* de Gruyter Studies in Mathematics 18. Walter de Gruyter & Co., Berlin, 1994.

[59] A.B. Zamolodchikov and Al.B. Zamolodchikov, *Structure Constants and Conformal Bootstrap in Liouville Field Theory.* Nucl. Phys. **B477** (1996), 577–605

DESY Theory, Notkestr. 85, 22603 Hamburg, Germany
E-mail: joerg.teschner@desy.de

Author Index